ESSAYS ON OPTIMAL ECONOMIC GROWTH

by

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B.A., Hebrew University, Jerusalem
(1959)

M.A., Hebrew University, Jerusalem
(1961)

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June, 1964

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CAMBRIDGE 39, MASSACHUSETTS

May 15, 1964

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On behalf of the department, I am happy to give you permission to have your thesis printed instead of typewritten, in accordance with the Institute regulation on page 47 of the Graduate Student Manual (September, 1963).

Sincerely yours,

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Robert L. Bishop
Professor of Economics;
Head, Department of Economics
and Social Science

RLB:ah

19 Kilsyth Road
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Dear Professor Bishop:

In partial fulfillment of the requirements for the degree of Doctor of Philosophy in Economics, I hereby submit the following thesis entitled:

"Essays on Optimal Economic Growth".

Respectfully yours,

David Levhari

A B S T R A C T

ESSAYS ON OPTIMAL ECONOMIC GROWTH

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David Levhari

In the first chapter we show that among all exponentials paths in capital-embodied labor-augmenting technical change, with a homogeneous of first order production function, the path in which saving is the same as profit or interest the same as the rate of growth, has maximum per capita consumption. In Chapter II we prove the same theorem for the case of fixed coefficients. The third chapter deals with the model discussed by Arrow, in which, unlike the capital-embodied models of Chapter I and II, there is divergence between social and private returns. Here we calculate the social return and the subsidy required to bring social and private return to equality. We then show that exponential growth is stable in this model, and that among all exponential paths, we again have a dominant one in which saving is equal to virtual profit, the profit that capitalists would have had they also received remuneration for the external effects of their investments.

Chapter IV deals with a problem presented by Solow and Tobin on the determination of the social rate of return and the rate of interest in the capital-embodied model. Chapter V shows that the Kaldor-Mirrlees model is not much different from the neoclassical models. It is practically impossible to distinguish between this model and those of Solow and Phelps.

Chapter VI presents a proof of Samuelson's nonsubstitution theorem in a Leontief model with no joint product and one primary input. Then we show the impossibility of Ruth Cohen's curiosum with the whole base of products. Chapter VII indicates possible applications of optimal control theory of Pontryagin and others to problems discussed with the classical calculus of variations by Samuelson, Solow, and others.

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ACKNOWLEDGMENTS

I wish to thank the members of my committee, Professors R. M. Solow, P. A. Samuelson, and F. M. Fisher, for their assistance and comments in the choosing of topics for these essays and in their preparation.

I would also like to thank M.I.T. and the Department of Economics for financial support during the academic years 1961-62 through 1963-64.

I acknowledge also the editing and typing of draft and final text by Mrs. Joyce W. Bradley and the preparation of figures by Eytan Sheshinski.

David Levhari

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CHAPTER I

THE GOLDEN AGE AND GOLDEN RULES IN VINTAGE CAPITAL MODELS

In recent years there have been frequent discussions on what Professor Phelps [1] called the golden rule of accumulation. As Professor Solow [2] commented, the fact that among all exponential paths sustainable consumption is the highest under equality of the own rate of interest and the rate of growth, is purely technological.

The value of the golden rule for optimization over time of the Ramsey type has been recently discussed by Koopmanns [3].

Here we shall prove that even in a more complex technological situation, i.e., exponential labor augmenting embodied technological change, still the rules are the same: that among all exponential paths, the path with highest consumption is the path where the rate of saving and the share of profits are the same, and so are the rate of interest under stationary expectation and the rate of growth [4]. We shall give separate

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1. Phelps, E.S., "A Golden Rule of Accumulation", American Economic Review, September, 1961.

2. Solow, Robert M., Review of Economic Studies, Comments, June, 1962.

3. Koopmanns, T.C., "On the Concept of Optimal Economic Growth", Cowles Foundation discussion paper 163, 1963 (unpublished).

4. The change has to be Harrod neutral to allow an exponential solution to exist. can easily see that in the case of disembodied technological change where

$$Q_o e^{gt} = F(e^{\mu t} K e^{gt}, e^{\lambda t} L_o e^{nt}) .$$

The capital-output ratio can remain constant only if $\mu = 0$. (Or in the case of Cobb-Douglas, it can be put into this form.) I mention all this since Phelps on one occasion (American Economic Review, September, 1961) asserts that his golden rule path holds for more general cases. See Paul A. Samuelson, Review of Economic Studies, June, 1962, Comments, p. 254.

proofs for the case of finite marginal product of labor at $L = 0$ and infinite marginal product of labor at $L = 0$. In the first case we get finite life of capital; as time goes on, we shift labor from old machines to new machines and at a certain age, we completely discard the machines. In the second case, obsolescence takes the shape of shifting labor from the old machines to the new machines, but we never completely discard any capital.

An example of a production function of type I is the c.e.s. production function with elasticity of substitution less than 1, $\rho > 0$, $\sigma < 1$.

$$Q(v,t) = \gamma[\delta I^{-\rho}(v) + (1-\delta)(e^{\lambda v} L(v,t))^{-\rho}]^{-\frac{1}{\rho}},$$

$$\frac{\partial Q(v,t)}{\partial L(v,t)} = \gamma(1-\delta)[\delta I^{-\rho} L(v,t) + (1-\delta)\, e^{-\lambda \rho v}]^{-\frac{1+\rho}{\rho}} e^{-\lambda \rho v},$$

as $L(v,t) \to 0$, $\frac{\partial Q(r,t)}{\partial L(v,t)} \to \gamma(1-\delta)^{-\frac{1}{\rho}} e^{\lambda v}$.

An example of the second kind (type II) is the c.e.s. production function ($\sigma \geq 1$, $\rho \geq 0$). In the case of equality, we get the known case of Cobb-Douglas. With $\sigma > 1$, since we can produce without labor, it is clear that no piece of capital is going to be completely discarded.

We shall start with the probably more realistic case of production functions of type I. (This turns out to be the more complicated case.) Then we shall prove shortly a similar theorem for type II.

We shall ignore physical decay even though there is no difficulty in bringing radioactive depreciation into the model.

Let $F(K,L)$ be homogeneous of first degree production function. $\frac{\partial F}{\partial L}$ is assumed finite at $L = 0$. Assume labor augmenting capital-embodied technological change of the form

$$Q(v,t) = F(I(v), e^{\lambda v} L(v,t)) .$$

$Q(v,t)$ is the output at time t of capital vintage v . $I(v)$ is capital vintage v , $L(v,t)$ is labor allocated to capital vintage v at time t . Labor is shiftable, so that competition will bring equality of marginal product of labor on all vintages. The supply of labor at time t , $L(t)$, is exogenously given. The wage at time t , $w(t)$, is the marginal product of labor on the oldest machines. If $h(o) = \frac{\partial F}{\partial L}$ at $L = 0$, then $w(t) = h(o)\, e^{\lambda(t-T)}$ where $T(t)$ is the age of the oldest capital used at time t .

$$\frac{\partial F(I(v), e^{\lambda v} L(v,t))}{\partial L(v,t)} = w(t)$$

implies

$$\frac{\partial F(I(v), e^{\lambda v} L(v,t))}{\partial (e^{\lambda v} L(v,t))} e^{\lambda v} = w(t) .$$

Since F is homogeneous of first degree, it can be written in the form

$$h\left(\frac{e^{\lambda v} L(v,t)}{I(v)}\right) = e^{-\lambda v} w(t) ,$$

where $h(x)$ is a monotonic decreasing function.

$$\frac{e^{\lambda v} L(v,t)}{I(v)} = h^{-1}(e^{-\lambda v} w(t)) .$$

$$\frac{L(v,t)}{I(v)} = e^{-\lambda v} h^{-1}(e^{-\lambda v} w(t)) .$$

The foregoing is for vintages for which

$$\frac{\partial F(I(v),\ e^{\lambda v} L(v,t)}{\partial L(r,t)} = w(t) ;$$

for all other vintages

$$\frac{\partial F(I(v),\ e^{\lambda v} L(r,t)}{\partial L(v,t)} < w(t) ,$$

$$L(v,t)=0$$

and we allocate no labor to them. The function h^{-1} is an increasing function of v , $e^{-\lambda v}$ is a decreasing function, and it is not clear whether at t we allocate more labor or less labor to higher vintages. It is obvious that in efficiency units we allocate more labor per unit of new machines, but in the natural units, this may turn out, as it does in the c.e.s. production function, to be less labor per unit of machines.

$$Q(v,t) = F[I(v),\ h^{-1}(e^{-\lambda v} w(t))\ I(v)]$$

$$= I(v)\ F(1,\ h^{-1}(e^{-\lambda v} w(t)) \quad .$$

We get the pair of integral equations describing the system:

$$L(t) = \int_{t-T(t)}^{t} e^{-\lambda v}\ h^{-1}(e^{-\lambda v}\ h(o)\ e^{\lambda(t-T)})\ I(v)\ dv , \qquad (1,1)$$

$$Q(t) = \int_{t-T(t)}^{t} F(1,\ h^{-1}(e^{-\lambda v}\ h(o)\ e^{\lambda(t-T)})\ I(v)\ dv . \qquad (1,2)$$

Given the history $I(v)$ for $v \leq t$ and labor force $L(t)$, we solve equation (1.1) for T. It is easy to prove that the right-hand side of (1.1) is a monotonic increasing function of T. It is easy to see this intuitively, since by increasing T we must allocate more labor to this vintage so as to bring it to equality with the lower marginal product of labor on the oldest machines with zero rentals, and moreover, we get some new machines to which we now allocate labor, which we did not do before.

After solving (1.1) for $T(t)$ we solve (1.2) for $Q(t)$. Assuming now some saving behavior, we get the behavior of the system along time. Assuming constant rate of saving s,

$$L(t) = \int_{t-T}^{t} e^{-\lambda v} h^{-1}(e^{-\lambda v} h(o) e^{\lambda(t-T)}) I(v) dv , \qquad (1.1')$$

$$I(t) = s \int_{t-T}^{t} F(1, h^{-1}(e^{-\lambda v} h(o) e^{\lambda(t-T)}) I(v) dv .$$

It is not easy to deal analytically with these two integral equations. However, we shall mainly be interested in a particular exponential solution. Let $L(t) = L_o e^{nt}$, $Q(t) = Q_o e^{gt}$, and $I(t) = sQ_o e^{gt}$.

$$L_o e^{nt} = sQ_o \int_{t-T}^{t} e^{-\lambda v} h^{-1}(e^{-\lambda(v-t+T)} h(o)) e^{gv} dv.$$

$$Q_o e^{gt} = sQ_o \int_{t-T}^{t} F(1, h^{-1}(e^{-\lambda(v-t+T)} h(o)) e^{gv} dv.$$

Substitute now $u = v - t + T$.

$$L_o e^{nt} = sQ_o e^{(g-\lambda)(t-T)} \int_0^T e^{(g-\lambda)u} h^{-1}(h(o)\, e^{-\lambda u})\, du \, .$$

$$e^{gt} = se^{g(t-T} \int_0^T e^{gu} F[1, h^{-1}(e^{-\lambda u} h(o)\,]\, du \, .$$

The exponential solution is admissible only if $n = g - \lambda$ or $g = n + \lambda$, i.e., the rate of balanced growth is the rate of increase of the labor force in efficiency units. Then

$$L_o = sQ_o \, e^{-(g-\lambda)T} \int_0^T e^{(g-\lambda)u} h^{-1}(h(o)\, e^{-\lambda u})\, du \, , \qquad (1.1")$$

$$1 = se^{-gT} \int_0^T e^{gu} F(1, h^{-1}(e^{-\lambda u} h(o)))\, du \, . \qquad (1.2")$$

We see from equations (1.1") and (1.2") that the life of capital T must be a constant independent of t . (The L.H.S. of these equations is independent of t ; hence, the R.H.S. must be also.) These two equations are algebraic equations for Q_o and T in terms of s and the other parameters of the system. It is not hard to see by implicit differentiation, which we shall perform later, that $Q_o'(s) > 0$ and $\frac{dT}{ds} < 0$; the level of output is larger and the life of machines is shorter, i.e., obsolescence is faster, as the rate of saving goes up.

The share of profit in the economy is

$$\pi = 1 - \frac{w(t)\ L(t)}{Q(t)} = 1 - \frac{h(o)\ e^{\lambda(t-T)}\ L_o e^{nt}}{Q_o e^{gt}} = 1 - \frac{h(o)\ e^{-\lambda T}\ L_o}{Q_o} .$$

$$\pi = 1 - \frac{w(o)\ L_o}{Q_o} .$$

As one might expect on the exponential path, this distribution is not changing and remains the same as at $t = 0$. π is again, of course, a function of s , the saving rate ,

$$\pi = 1 - \frac{g(o)\ e^{-\lambda T(s)}\ L_o}{Q_o(s)} . \tag{1.3}$$

We would like to find the path of maximum sustainable consumption. For this, let us maximize $(1-s)\ Q_o(s)$.

$$-Q_o(s) + (1-s)Q_o'(s) = 0 .$$

$$\frac{Q_o'(s)}{Q_o(s)} = \frac{1}{1-s} . \tag{1.4}$$

<u>Theorem 1:</u> If (1.1"), (1.2"), and (1.4) hold, they imply $s = \pi$.

Let us differentiate implicitly equations (1.1") and (1.2") with respect to s .

$$0 = \frac{L_o}{s} + \frac{Q_o'(s)}{Q_o(s)} L_o - (g-\lambda) L_o \frac{dT}{ds} + sQ_o \, g^{-1}(e^{-\lambda T} h(o)) \frac{dT}{ds} .$$

$$0 = \frac{1}{s} - g \frac{dT}{ds} + sF[1, g^{-1}(e^{-\lambda T} h(o))] \frac{dT}{ds} .$$

$$\frac{dT}{ds} = \frac{1}{s} \frac{1}{g - sF(1, h^{-1}(e^{-\lambda T} h(o))} .$$

$$\frac{Q_o'(s)}{Q_o(s)} = \frac{1}{s} [\frac{(g-\lambda) - s \frac{Q_o}{L_o} h^{-1}(e^{-\lambda T} h(o))}{g - s \, F(1,h^{-1}(e^{-\lambda T} h(o))} - 1] .$$

By definition of h^{-1} , $h^{-1}(e^{-\lambda T} h(o)) = \frac{L(o,o)}{I(o)}$ and $F[1, h^{-1}(e^{-\lambda T}h(o))]$ $= F(1, \frac{L(o,o)}{I(o)}) = \frac{Q(o,o)}{I(o)}$.

$$\frac{Q_o'(s)}{Q_o(s)} = \frac{1}{s} \frac{(g-\lambda) - s \frac{Q_o}{L_o} \frac{L(o,o)}{I(o)}}{g - s \frac{Q(o,o)}{I(o)}} .$$

But $I(o) = sQ_o$.

$$\frac{Q_o'(s)}{Q_o(s)} = \frac{1}{s} [\frac{nL_o - L(o,o)}{gQ_o - Q(o,o)} \frac{Q_o}{L_o} - 1] .$$

We must interpret the term

$$\frac{gQ_o - Q(o,o)}{L(o,o) - nL_o} .$$

Remembering that

$$L_o e^{nt} = \int_{t-T}^{t} L(v,t)\, dv ,$$

let us take derivatives of both sides with respect to t .

$$nL_o e^{nt} = L(t,t) - L(t-T,t) + \int_{t-T}^{t} \frac{\partial L(v,t)}{\partial t}\, b .$$

$$nL_o e^{nt} = L(t,t) + \int_{t-T}^{t} \frac{\partial L(v,t)}{\partial t}\, dv .$$

Performing the same operations on

$$Q_o e^{gt} = \int_{t-T}^{t} F(I(v), L(r,t))\, dv ,$$

we obtain

$$gQ_o e^{gt} = F(I(t), L(t,t) + \int_{t-T}^{t} \frac{\partial F(I(v), L(v,t))}{\partial L(v,t)} \frac{\partial L(v,t)}{\partial t}\, dv .$$

Remembering now that marginal product of labor for all vintages is the labor wage, we get:

$$gQ_o e^{gt} = Q(t,t) + w(t) \int_{t-T}^{t} \frac{\partial L(v,t)}{\partial t} \, dv ,$$

$$gQ_o e^{gt} = Q(t,t) + w(t)[nL_o e^{nt} - L(t,t)] .$$

$$w(t) = \frac{gQ_o e^{gt} - Q(t,t)}{nL_o e^{nt} - L(t,t)} ,$$

and

$$w(o) = \frac{gQ_o - Q(o,o)}{nL_o - L(o,o)} .$$

So

$$\frac{Q_o'(s)}{Q_o(s)} = \frac{1}{s} [\frac{Q_o}{w_o L_o} - 1] ,$$

$$\frac{Q_o'(s)}{Q_o(s)} = \frac{1}{s} [\frac{1}{1-\pi} - 1] > 0 \quad [5] .$$

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5. To see that $\frac{dT}{ds} < 0$,

$$\frac{dT}{ds} = \frac{1}{s} \frac{1}{g-s\ F(1,\ h^{-1}(e^{-\lambda T}\ h(o))} ,$$

$$\frac{dT}{ds} = \frac{1}{s} \frac{1}{g-s\ \frac{Q(o,o)}{I(o)}} = \frac{1}{s} \frac{1}{g - \frac{Q(o,o)}{Q(o)}} .$$

Since $gQ_o - Q(o,o) = \int_{-T}^{o} \frac{\partial L(v,t)}{\partial t}$, it is easy to see that decreasing returns imply for all v:

$$\frac{\partial L(v,t)}{\partial t} < 0 \qquad \text{and} \qquad g - \frac{Q(o,o)}{Q(o)} < 0 .$$

On the maximum sustainable path of consumption

$$\frac{Q_o'(s)}{Q_o(s)} = \frac{1}{1-s} \quad ;$$

$$\frac{1}{1-s} = \frac{1}{s}\left(\frac{\pi}{1-\pi}\right) , \qquad \frac{s}{1-s} = \frac{\pi}{1-\pi} , \qquad s = \pi$$

Q.E.D.

Let us now prove the following theorem.

Theorem 2: Theorem 1 implies that under stationary expectations the rate of interest and the rate of growth are the same on the maximum sustainable path of consumption.

Look at investment made at v ; the rental at time t is

$$r(v,t) = \frac{Q(r,t)}{I(v)} - w(t)\,\frac{L(v,t)}{I(v)} .$$

The rate of interest ρ should satisfy the equations

$$1 = \int_v^{v+T} \left\{ \frac{Q(v,t)}{I(v)} - w(t)\,\frac{L(v,t)}{I(v)} \right\} e^{-\rho(t-v)}\, dt \quad ,$$

$$1 = \int_v^{v+T} \Big\{ F[1,\ h^{-1}(e^{-\lambda v}\, h(o)\, e^{\lambda(t-T)}] - h(o)e^{\lambda(t-T)}\, e^{-\lambda v}\, h^{-1}\, [e^{-\lambda v}\, h(o)\, e^{\lambda(t-T)}] \Big\}\, e^{-\rho(t-v)}\, dt \ .$$

Substituting $u = t - v$,

$$1 = \int_{o}^{T} e^{-\rho u} \left\{ F[1, h^{-1}(e^{\lambda(u-T)} h(o))] - h(o)\, e^{-\lambda T}\, e^{\lambda u}\, h^{-1}[h(o) e^{\lambda(u-T)}] \right\} du \quad .$$

The expression in brackets in the integrand is obviously $r(o,u)$, the rental of capital vintage o at u .

Our theorem says that on the maximum sustainable consumption path, the solution to this equation is g . We have to prove that

$$1 = \int_{o}^{T} e^{-gu} \left\{ F[1,\ h^{-1}(e^{\lambda(u-T)}\ h(o))] - (h(o)\ e^{-\lambda T})\ e^{\lambda u}\ h^{-1}(h(o)\ e^{\lambda(u-T)}) \right\} du \quad .$$

Taking the derivative of the right-hand expression with respect to ρ , we find that $-\rho \int_{o}^{T} e^{-\rho u}\, r(o,u)\, du < 0$. We see that if g is a solution, it is the only solution. Let us denote by $p(v,t)$ profit of capital vintage v at t . Total profit on this path according to theorem 1 is $sQ_o e^{gt}$.

$$p(v,t) = F[1, h^{-1}(e^{-\lambda(v-t+T)} h(o))] I(v) - w(t) e^{-\lambda v} h^{-1}(e^{-\lambda(v-t+T)} h(o)) I(v) \quad .$$

$$sQ_o e^{gt} = sQ_o \int_{t-T}^{t} e^{gv} \left\{ F[1,\ h^{-1}(e^{-\lambda(v-t+T)}\ h(o))] \right.$$

$$\left. - \ h(o)\ e^{\lambda(t-T)}\ e^{-\lambda v}\ h^{-1}(e^{-\lambda(v-t+T)}\ h(o)) \right\} dv \quad .$$

Substituting $w = v - t + T$ we find:

$$e^{gt} = \int_0^T \left\{F[1,\ h^{-1}(e^{-\lambda w}\ h(o))] - h(o)\ e^{-\lambda w}\ h^{-1}(e^{-\lambda w}\ h(o))\right\} e^{g(t-T)} e^{gw}\ dw \quad ,$$

$$1 = \int_0^T \left\{F[1,\ h^{-1}(e^{-\lambda w}\ h(o))] - h(o)\ e^{-\lambda w}\ h^{-1}(e^{-\lambda w}\ h(o))\right\} \ e^{g(w-T)}\ dw \quad .$$

Now substituting $u = -w + T$,

$$1 = \int_0^T \left\{F[1,\ h^{-1}(e^{\lambda(u-T)}h(o))] - h(o)\ e^{\lambda(u-T)}h^{-1}(h(o)e^{\lambda(u-T)})\right\} e^{-gu}\ du \quad .$$

So $\rho(s)$ on the golden rule path satisfies $\rho(s) = g$.

Q.E.D.

We shall now prove the analogous theorem for production functions of type II, production functions which have infinite marginal product of labor at $L = 0$, and for which all vintages of capital are always employed.

Using the same notation as before, we find that

$$h\left(\frac{e^{\lambda v}\ L(v,t)}{I(v)} \right) = e^{-\lambda v}\ w(o) \ .$$

Calling $e^{\lambda v}\dfrac{L(v,t)}{I(v)} = x$, $h(x) = e^{-\lambda v}\ w(o)$. Differentiating with respect to $w(o)$, we get $h'(x)\dfrac{dx}{dw} = e^{-\lambda v}$, $\dfrac{dx}{dw} = \dfrac{e^{-\lambda v}}{h'(x)}$.

$$L_o e^{nt} = \int_{-\infty}^{t} e^{-\lambda v} h^{-1} (e^{-\lambda v} w(t)) I(v) dv \ .$$

$$Q_o e^{nt} = \int_{-\infty}^{t} F[1, h^{-1}(e^{-\lambda V} w(t))] I(v) dv \ .$$

$$L_o e^{nt} = sQ_o \int_{-\infty}^{t} e^{(g-\lambda)v} h^{-1}(e^{-\lambda(v-t)} w(o)) \ I(v) dv \ .$$

$$Q_o e^{gt} = sQ_o \int_{-\infty}^{t} e^{gv} F[1, h^{-1}(e^{-\lambda(v-t)} w(o))] I(v) dv \ .$$

Substituting $u = v - t$:

$$L_o = sQ_o \int_{-\infty}^{o} e^{(g-\lambda)u} \ h^{-1}(e^{-\lambda u} w(o)) \ du \ .$$

$$1 = s \int_{-\infty}^{t} e^{gu} F[1, h^{-1} (e^{-\lambda u} w(o))] du \ .$$

These are two equations for $Q_o(s)$ and $w(o)$. Again, it is possible to show that $Q_o'(s) > 0$, $\frac{dw(o)}{ds} > 0$.

Using the notation $x = e^{\lambda v} \frac{L(v,t)}{I(v)} = h^{-1}(e^{-\lambda v}\, w(t))$,

$$w(t) = \frac{\partial\, I(v)\, F[1,\, e^{\lambda v} \frac{L(v,t)}{I(v)}]}{\partial L(v,t)} = \frac{\partial F(1,x)}{\partial x}\, e^{\lambda v} .$$

$$\frac{\partial\, F(1,x)}{\partial\, x} = e^{-\lambda v}\, w(t) .$$

Differentiating implicitly with respect to s :

$$0 = \frac{L_o}{s} + \frac{L_o}{Q_o(s)} Q_o'(s) + sQ_o(s) \frac{dw(o)}{ds} \int_{-\infty}^{o} e^{(g-\lambda)u} \frac{e^{-\lambda u}}{h'(x)}\, du .$$

$$0 = \frac{1}{s} + s\int_{-\infty}^{o} e^{gu} \frac{\partial\, F(1,x)}{\partial\, x} \frac{dx}{dw(o)} \frac{dw(o)}{ds}\, du .$$

The second of these equations takes the form:

$$0 = \frac{1}{s} + sw(o) \int_{-\infty}^{o} e^{(g-\lambda)u} \frac{e^{-\lambda u}}{h'(x)}\, du .$$

Let $B = \frac{dw(o)}{ds} \int_{-\infty}^{o} e^{(g-\lambda)u} \frac{e^{-\lambda u}}{h'(x)}\, du$.

$$\frac{1}{s} + sw(o)\, B = 0 , \qquad B = \frac{-1}{w(o)\, s^2} .$$

On the maximum sustainable consumption path,

$$\frac{Q_o'(s)}{Q_o(s)} = \frac{1}{1 - s} ,$$

$$L_o \left(\frac{1}{s} + \frac{1}{1-s}\right) + sQ_oB = 0 ,$$

$$L_o \frac{1}{s(1-s)} = \frac{sQ_o}{w(o)\ s^2} ,$$

$$\frac{L_o}{1-s} = \frac{Q_o}{w(o)} ,$$

$$s = 1 - \frac{w(o)\ L_o}{Q_o} .$$

Q.E.D.

By methods similar to those used for production function type I , it is possible to find that again on this path the rate of interest and rate of growth are the same.

Next we shall turn to proving similar theorems for the case of fixed coefficients.

CHAPTER II

THE GOLDEN RULE IN FIXED-COEFFICIENTS, CAPITAL-EMBODIED, HARROD-NEUTRAL TECHNOLOGICAL CHANGE

Let v = the date of birth of capital,

t = current time,

$Q(v,t)$ = gross output at t using capital vintage v ,

$I(v)$ = investment at v ,

$L(v,t)$ = labor allocated at time t , to capital of vintage v ,

s = saving rate.

Assume $Q(v,t) = aI(v) = be^{\lambda v} L(v,t) = \min (aI(v), be^{\lambda v} L(v,t))$. Using capital vintage v , we need for a unit of production $\frac{1}{a}$ capital and $\frac{1}{be^{\lambda v}}$ labor. We have a process of automation in which the capital-output ratio remains constant while the capital-labor ratio, or output-labor ratio, declines exponentially. Assume exponential growth in which investment grows exponentially $I(v) = sQ_o e^{gv}$. Generally,

$$I(t) = s \int_{t-m(t)}^{t} e^{-\lambda v} I(v)\, dv \quad , \tag{2.1}$$

$$L_o e^{nt} = \frac{a}{b} \int_{t-m(t)}^{t} e^{-\lambda v} I(v)\, dv \quad , \tag{2.2}$$

and with the exponential profile of investment,

$$L_o e^{nt} = \frac{a}{b} \int_{t-m(t)}^{t} sQ_o e^{(g-\lambda)v}\, dv \quad ,$$

$$L_o e^{nt} = \frac{a}{b} \frac{sQ_o}{g-\lambda} e^{(g-\lambda)t} [1 - e^{-(g-\lambda) m(t)}]$$

So if $g = n + \lambda$ we are on the exponential path with constant life of capital, $m(t) \equiv m$. To find the effective life of capital, we use

$$Q_o e^{gt} = s \int_{t-m}^{t} aQ_o e^{gt}\, dt \quad ,$$

$$Q_o e^{gt} = \frac{sa}{g} Q_o e^{gt} (1-e^{-gm}),$$

$$1 = \frac{sa}{g} (1 - e^{-gm}) \ ,$$

$$m = \frac{1}{g} \log (1- \frac{g}{as}) = - \frac{1}{n+\lambda} \log (1- \frac{n+\lambda}{as}) \ [1] \ . \qquad (2.3)$$

We find that the effective life of capital is a decreasing function of the saving rate. We have a lower bound for the effective life of capital; we get $s = 1$, $m = \frac{1}{n+\lambda} \log (1 - \frac{n+\lambda}{a})$. All this, of course, is for $\frac{n+\lambda}{as} < 1$, i.e., $\frac{n+\lambda}{a} \leq s \leq 1$.

Let us now use equation (2.2) to determine Q_o, or the level of the economy as a function of the saving rate. On the exponential path,

---

1. If $s < \frac{n+\lambda}{a}$ then the rate of saving is too small to maintain steady growth at rate $n + \lambda$ with constant lifetime, and so $m \rightarrow \infty$.

$$L_o e^{nt} = \frac{a}{b} \int_{t-m}^{t} se^{-\lambda v} Q_o e^{(n+\lambda)v} dv \quad ,$$

$$L_o = \frac{as}{bn} Q_o(1 - e^{-nm}) \quad ,$$

$$L_o = \frac{a}{bn} sQ_o(1 - e^{\frac{n}{n+\lambda} \log (1- \frac{n+\lambda}{as})}) \quad ,$$

$$L_o = \frac{a}{bn} sQ_o[1- (1- \frac{n+\lambda}{as})^{\frac{n}{n+\lambda}}] \quad ,$$

$$Q_o(s) = \frac{nL_o \frac{b}{a}\frac{1}{s}}{1-(1- \frac{n+\lambda}{as})^{\frac{n}{n+\lambda}}} \quad .$$

Now, among all the exponential paths, we want to find the one with sustainable more consumption. We want to maximize with respect to the saving rate,

$$(1-s)\, Q_o(s) = \frac{nL_o \frac{b}{a}\frac{1}{s}(1-s)}{1-(1- \frac{n+\lambda}{as})^{\frac{n}{n+\lambda}}} \quad ;$$

$$\underset{\frac{n+\lambda}{a} \leq s \leq 1}{\text{Max}} \quad \frac{1-s}{s} \frac{1}{1-(1- \frac{n+\lambda}{as})^{\frac{n}{n+\lambda}}} \quad .$$

Call the function to be maximized $\phi(s)$. We shall now show that $\phi(s)$ cannot reach a maximum [2] in the boundaries $\phi(s) > 0$ for $a \leq 1$

---

2. All this is very similar to what I have shown in the discussion of Arrow's model in Chapter III.

and $\phi(1) = 0$, so at 1 we cannot find a maximum. Now let us calculate $\phi'(s)$ at $\frac{n+\lambda}{a}$:

$$\phi'(s) = \frac{1}{[1-(1-\frac{n+\lambda}{as})^{\frac{n}{n+\lambda}}]^2}\left\{-\frac{1}{s^2}[1-(1-\frac{n+\lambda}{as})^{\frac{n}{n+\lambda}}] + \frac{n}{a}(1-\frac{n+\lambda}{as})^{\frac{n}{n+\lambda}-1}\frac{1}{s^2}\frac{1-s}{s}\right\},$$

and it is not hard to see that $\phi'(\frac{n+\lambda}{a}) = \infty$. So the maximum is attained at an interior point, and it must be the solution of $\phi'(s) = 0$.

$$-[1-(1-\frac{n+\lambda}{as})^{\frac{n}{n+\lambda}}] + \frac{n}{a}(1-\frac{n+\lambda}{as})^{\frac{n}{n+\lambda}-1}\frac{1-s}{s} = 0 ,$$

or

$$(1-\frac{n+\lambda}{as})^{\frac{n}{n+\lambda}} + \frac{n}{a}(1-\frac{n+\lambda}{as})^{\frac{n}{n+\lambda}-1}\frac{1-s}{s} = 0 . \qquad (2.4)$$

It is not possible to solve the equation for s , but in the following we show that the solution of this equation will be such that the share of profit with the saving rate that satisfies this equation is equal to the saving rate. Moreover, with this saving rate, the rate of growth and the rate of return are the same.

Denote the rental of capital vintage v at t by $r(v,t)$.

$$r(t-m,t) = 0 , \quad 1 - W(t)\frac{1}{be^{\lambda(t-m)}} = 0 , \quad W(t) = be^{\lambda(t-m)} .$$

$$r(v,t) = a - W(t)\frac{a}{be^{\lambda v}} = a - be^{\lambda(t-m)}\frac{a}{be^{\lambda v}} = a(1-e^{\lambda(t-m-v)}) .$$

Let π denote the share of gross profit in the economy.

$$\pi = \frac{\int_{t-m}^{t} r(v,t)\; I(v)\; dv}{a \int_{t-m}^{t} I(v)\; dv} \quad ,$$

$$\pi = \frac{\int_{t-m}^{t} (1 - e^{\lambda(t-m-v)})\; e^{(n+\lambda)v}\; dv}{\int_{t-m}^{t} e^{(n+\lambda)v}\; dv} \quad ,$$

$$\pi = 1 - \frac{n+\lambda}{n} \; \frac{e^{-\lambda m}\,(1 - e^{-nm})}{1 - e^{-(n+\lambda)\,m}} \quad .$$

Using the equation for m ,

$$e^{-nm} = (1- \frac{n+\lambda}{as})^{\frac{n}{n+\lambda}} \quad ,$$

$$e^{-\lambda m} = (1- \frac{n+\lambda}{as})^{\frac{\lambda}{n+\lambda}} \quad .$$

$$\pi = 1- \frac{n+\lambda}{n} \; \frac{(1- \frac{n+\lambda}{as})^{\frac{\lambda}{n+\lambda}} - (1- \frac{n+\lambda}{as})}{1- (1 + \frac{n+\lambda}{as})} \quad .$$

$$\pi(s) = 1- \frac{as}{n} \left[(1- \frac{n+\lambda}{as})^{\frac{\lambda}{n+\lambda}} - (1 - \frac{n+\lambda}{as}) \right] \quad .$$

Take our equation for maximum consumption,

$$(1-\frac{n+\lambda}{as})^{\frac{n}{n+\lambda}} + \frac{n}{a}\frac{1-s}{s}(1-\frac{n+\lambda}{as})^{-\frac{\lambda}{n+\lambda}} = 1 \quad ,$$

multiply both sides by $(1-\frac{n+\lambda}{as})^{\frac{\lambda}{n+\lambda}}$, and we find

$$\frac{n}{a}\frac{1-s}{s} = (1-\frac{n+\lambda}{as})^{\frac{\lambda}{n+\lambda}} - (1-\frac{n+\lambda}{as}) \quad ,$$

$$\pi(s_{max}) = 1-\frac{as}{n}\frac{n}{a}\frac{1-s}{s} = s \quad .$$

So with the optimal saving rate, the share of profits and saving rate are the same.

Now let us show that in this situation also the rate of interest and the rate of growth are the same. With perfect foresight, the cost of producing a new capital unit and its discounted value should be the same.

$$a\int_{t}^{t+m} (1 - e^{\lambda(u-m-t)})\, e^{-r(u-t)}\, du = 1 \quad [3] \quad ,$$

$$\frac{a}{r}(1 - e^{-rm}) - a\,\frac{e^{-rm} - e^{-\lambda m}}{\lambda - r} = 1 \quad ,$$

3. Professor Solow has shown that under the assumption of perfect foresight the only possible rate of interest in this situation is a constant rate.

$$\frac{a}{r}\left[1-\left(1-\frac{n+\lambda}{as}\right)^{\frac{r}{n+\lambda}}\right]-\frac{a}{\lambda-r}\left[\left(1-\frac{n+\lambda}{as}\right)^{\frac{r}{n+\lambda}}-\left(1-\frac{n+\lambda}{as}\right)^{\frac{\lambda}{n+\lambda}}\right]=1 .$$

Now let us try the rate of growth $r = n + \lambda$ as a solution.

$$\frac{a}{n+\lambda}\left[1-\left(1-\frac{n+\lambda}{a\,s}\right)\right]-\frac{a}{n}\left[\left(1-\frac{n+\lambda}{as}\right)^{\frac{\lambda}{n+\lambda}}-\left(1-\frac{n+\lambda}{as}\right)\right]=1 ,$$

for the optimizing

$$s\left(1-\frac{n+\lambda}{as}\right)^{\frac{\lambda}{n+\lambda}}-\left(1-\frac{n+\lambda}{as}\right)=\frac{n}{a}\,\frac{1-s}{s} ,$$

and

$$\frac{1}{s}-\frac{a}{n}\left[\frac{n}{a}\,\frac{1-s}{s}\right]=1 .$$

So with the s of the golden rule, the rate of growth and the rate of interest are the same, $n + \lambda$ [4].

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4. It is easy to see that $n + \lambda$ is the unique solution of this equation for r .

CHAPTER III

FURTHER IMPLICATIONS OF LEARNING BY DOING [1]

To make this discussion more or less self-contained, we shall describe briefly the main features of Arrow's model. Arrow's basic assumption is that productivity is related to cumulative gross investment. Instead of having productivity increasing as a function of time, say $A(t)$ with $\dot{A}(t) > 0$, we assume that it is a function of cumulative gross investment G [2]. The basic assumptions of the model are the following:

1. Learning depends on gross investment.
2. Technical progress is fully embodied.
3. Fixed coefficients.
4. Fixed physical lifetime of a machine.

We shall use the following notation:

G	- cumulative gross investment. $G = \int_0^t I(s)d(s)$.
$\gamma(G) = a$	- output capacity of a machine with serial number $G=a$.
$\lambda(G) = bG^{-n}$	- labor requirement per unit of time for operating a machine with serial number G .
L	- total employment.
x	- total output of final products.

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1. K. J. Arrow, "The Economic Implication of Learning by Doing", Review of Economic Studies, June, 1962.

2. Generally, of course, $\dot{G}(t) > 0$, so that instead of having technological change as a function of time, we get technological change as a function of some monotonic transformation of time. There are many similarities between the Arrow model and a model of fixed-coefficients embodied technological change.

We shall assume in what follows that the economic life of a machine is shorter than its physical life.

G' - the serial number of the oldest machines used at a certain time.

Total output $x = \int_{G'}^{G} \gamma(G)\, dG = aG - aG'$,

$$L = \int_{G'}^{G} \lambda(G)\, dG = b \int_{G'}^{G} G^{-n}\, dG = \frac{b}{1-n} [G^{1-n} - G'^{1-n}] , \qquad n \neq 1 ,$$

$$= b \log \frac{G}{G'} , \qquad n=1 .$$

By solving for G' and substituting this solution, we find

$$x = aG[1-(1-\frac{L}{CG^{1-n}})^{\frac{1}{1-n}}] , \qquad n \neq 1 ,$$

where $C = \frac{b}{1-n}$,

$$x = aG(1 - e^{-L/b}) , \qquad n=1 .$$

If wage w is measured in units of x , then the oldest machines must earn zero quasi rents, so that

$$a - wbG'^{-n} = 0 ,$$

$$G' = (w \frac{b}{a})^{1/n} .$$

Arrow pays special attention to the case of exponential growth, in which labor is growing at rate σ and accumulated gross investment and output are both growing at the rate of $\frac{\sigma}{1-n}$; he proves that with stationary expectation, the interest rate is constant, r .

Social and Private Returns

Let us say that society saves an extra h units at 0 when the serial number of capital is $G(0)$. The product we produce with the new capital is ah , but we need to shift labor from the oldest capital, so that we lose product in the amount az , where z is the amount of capital scrapped.

$$z = [(G+h)^{1-n} - \frac{L}{C}]^{\frac{1}{1-n}} - (G^{1-n} - \frac{L}{C})^{\frac{1}{1-n}} ,$$

$$\triangle x = ah - a\left\{[(G+h)^{1-n} - \frac{L}{C}]^{\frac{1}{1-n}} - (G^{1-n} - \frac{L}{C})^{\frac{1}{1-n}}\right\} .$$

Calculating $\frac{dx}{dh} = \lim_{\Delta h \to 0} \frac{\Delta x}{\Delta h}$, we find

$$\frac{dx}{dh} = a[1-(G^{1-n} - \frac{L}{C})^{\frac{n}{1-n}} G^{-n}] = a[1-(\frac{G'}{G})^{n}]. \qquad (3.1)$$

Thus at a time when the serial number is G , society has an extra product in the amount of $a(1-(\frac{G'}{G})^{n})$. It is clear that we could have found the marginal social product of capital by calculating

$$\frac{\partial x(G,L)}{\partial G} = a[1-(\frac{G'}{G})^{n}] .$$

On the other hand, let us now calculate the private rental of an extra unit of capital investment when the capital serial number is $G(0)$. Quasi rent at time t for a unit of capital invested at 0 is $a-w(t)bG^{-n}(0)$, and $w(t) = \frac{a}{b} G'^{n}(t)$. So the quasi rent of capital invested when $t = 0$ is $a[1-(\frac{G'(t)}{G(0)})^n]$. $G'(t)$ is increasing and at some $t>0$, $G'(t) = G(0)$, and this capital is scrapped.

It is obvious that uniformly with t,

$$a[1-(\frac{G'(t)}{G(t)})^n] \geq a[1-(\frac{G'(t)}{G(0)})^n] ,$$

with equality holding only at $t = 0$. After $G'(t) \geq G(0)$, the private rental is 0.

It is easy to find out what happens in the exponential world described by Arrow. Here $\frac{G'}{G}$ is constant, and if m is the length of life of capital,

$$\frac{G'}{G} = \frac{G_o e^{\frac{\sigma}{1-n}(t-m)}}{G_o e^{\frac{\sigma}{1-n} t}} = e^{-\frac{\sigma}{1-n} m} .$$

The marginal social product is then $a(1-e^{\frac{\sigma}{1-n} nm})$, which is a constant independent of time. Quasi rent is

$$a[1-(\frac{G'(t)}{G(0)})^n] = a[1-e^{\frac{\sigma}{1-n} n(t-m)}] .$$

At $t = 0$ it is the same as the marginal social product $a(1-e^{-gnm})$, and then it declines and at $t = m$ it is zero.

For the case $n = 1$ [3], $x = aG(1-e^{-L/b})$,

$$\frac{\partial x}{\partial G} = a(1-e^{-L/b}) = a(1-\frac{G'}{G}) \quad ;$$

and $MSP = a(1-(\frac{G'}{G})^n)$ holds for all $n \leq 1$. Let us denote by ρ_s the social rate of return and by ρ_p the private rate of interest. In the exponential world in which Arrow has shown that the rate of interest under perfect expectation or stationary expectation is constant, it is clear from the dominance of marginal social product over private rentals that $\rho_s > \rho_p$. Using the identity $\int_t^\infty f(u)\, e^{-\int_e^u f(x)dx} du = 1$, if $\int_t^\infty f(x)dx$ is divergent, we find that at each t the instantaneous social rate of return is $\rho_s(t) = a(1-(\frac{G'(t)}{G(t)})^n)$. The only case of constant social rate of return is of constant G'/G , which occurs in the exponential case. In the case of "quickening", when G'/G is an increasing function of time, the instantaneous rate of social return is decreasing; this is clear intuitively, since we transfer labor from "not very old" capital to new capital. As an example, if $\frac{G'}{G} = e^{-\alpha t}$, $\alpha > 0$, then $\rho_s(t) = a(1-e^{-\alpha n t})$. We find $\frac{G'}{G}$ by $\frac{G'}{G} = (1-\frac{L}{CG^{1-n}})^{\frac{1}{1-n}}$ and by knowing the profiles of L and G (or saving profile). All pricing processes and distributional characteristics of the model can be expressed in terms of G'/G .

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3. For this case, R. Solow reached the same result by a different approach (unpublished lectures, 1962).

It is more difficult to follow the pattern of private rate of return. If $m(v)$ is the economic life of machines born at v, and assuming perfect foresight, $\rho_p(v)$ should satisfy the functional equations

$$a \int_v^{v+m(v)} e^{-\int_v^t \rho_p(v)du} \left[1-\left(\frac{G'(t)}{G(v)}\right)^n\right] dt = 1$$

and

$$G'(v+m(v)) = G(v) .$$

In the exponential case where the life of capital is constant m, the equation takes the form

$$\int_v^{v+m} a\left(1-e^{n\frac{\sigma}{1-n}(t-v-m)}\right) e^{-\int_v^t \rho_p(v)\,du} \, dt = 1 ,$$

where $\frac{\sigma}{1-n}$ is the rate of growth of output and m is the length of life of capital. Arrow has shown that this equation possesses one and only one constant solution. It is possible to prove that this solution is the only solution of this functional equation [4].

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4. To prove that the functional equation for $\rho_p(t)$ can have only a constant for a solution, change the variables to $\tau = t - v$ and then

$$\int_0^m a\left(1-e^{\frac{n}{1-n}\sigma(\tau-m)}\right) e^{-\int_v^{v+t} \rho_p(v)du} \, d\tau = 1 .$$

The right-hand side is independent of v and the left-hand side must be also. Denote

$$\int_0^v \rho_p(v)\, du = R(v) .$$

$R(v+\tau) - R(v)$ must be independent of v for all τ. The only function that satisfies this is a linear function, $R(v) = p+rv$, and since $R(o) = 0$, $R(v) = rv$ and $\rho_p = r$.

<u>Shares of Capital and Labor</u>

The pseudo production function

$$x = aG\left[1-\left(1-\frac{L}{CG^{1-n}}\right)^{\frac{1}{1-n}}\right]$$

is of increasing returns to scale, and it is clear that both capital and labor cannot get their marginal social productivities. However, let us calculate

$$\frac{\partial x}{\partial L} = \frac{1}{1-n}\,\frac{a}{c}\,G^{n}\left(1-\frac{L}{CG^{1-n}}\right)^{\frac{n}{1-n}}, \qquad (3.2)$$

$$\frac{L\frac{\partial x}{\partial L}}{x} = \frac{\left[\left(\frac{L}{G^{1-n}}\right)^{\frac{1-n}{n}} - \frac{1}{c}\left(\frac{1}{G^{1+n}}\right)^{\frac{1}{n}}\right]^{\frac{n}{1-n}}}{b\left[1-\left(1-\frac{L}{CG^{1-n}}\right)^{\frac{1}{1-n}}\right]},$$

which is the same as the result Arrow gets for $\frac{wL}{x}$. So in spite of the fact that all the income of labor in this model is rent, since labor offers itself inelastically, labor gets its social product, which clearly means that capital cannot get its marginal social product.

Let us assume that capital had received its marginal social product; then the share of capital would have been

$$\frac{G\frac{\partial x}{\partial G}}{x} = \frac{\left[1-\left(G^{1-n}-\frac{L}{C}\right)G^{-n}\right]^{\frac{n}{1-n}}}{b\left[1-\left(1-\frac{L}{CG^{1-n}}\right)^{\frac{1}{1-n}}\right]},$$

$$\frac{\frac{\partial x}{\partial G} G}{x} = \frac{1 - (\frac{G'}{G})^n}{1 - \frac{G'}{G}} .$$

Assume $0 < n < 1$ and $0 < \frac{1 - (\frac{G'}{G})}{1 - \frac{G'}{G}} < 1$, using $(\frac{G'}{G}) < 1$ and $(\frac{G'}{G})^n > \frac{G'}{G}$. The implied labor share is

$$1 - \frac{\frac{\partial x}{\partial G} G}{x} = \frac{(\frac{G'}{G})^n - (\frac{G'}{G})}{1 - (\frac{G'}{G})} .$$

Calculating on the other hand what labor really gets, we find

$$\frac{wL}{x} = \frac{c}{b} \frac{(F^{1-n} - G'^{1-n}) G'^n}{G - G'} = \frac{1}{1-n} \frac{(\frac{G'}{G})^n - (\frac{G'}{G})}{1 - (\frac{G'}{G})} .$$

So we get the simple relationship between actual wages and what the wages would have been had capital received its marginal product. The labor share is inflated by a factor $\frac{1}{1-n}$ and $\frac{1}{1-n} > 1$ for $0 < n < 1$. Capital share is

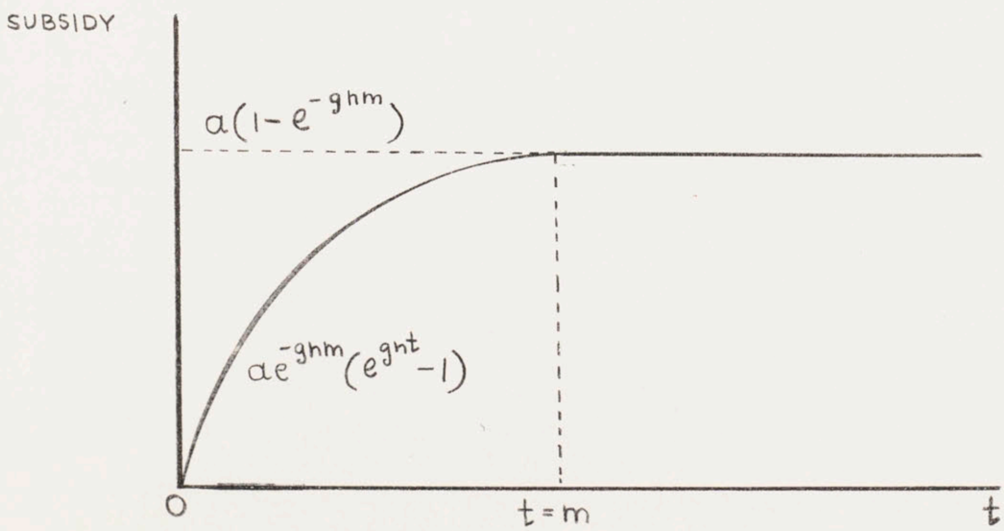
$$1 - \frac{wL}{x} = 1 - \frac{1}{1-n} \frac{(\frac{G'}{G})^n - (\frac{G'}{G})}{1 - \frac{G'}{G}} = \frac{1}{1-n} \left[\frac{1 - (\frac{G'}{G})^n}{1 - \frac{G'}{G}} - n\right] \quad [5] .$$

---

5. As a matter of fact, why not pay capital above its marginal social product if it increases savings and we are in a situation where more saving increases total social welfare. This is true in the no-time-discounting case, at least so long as the saving rate is below the "golden rule" saving rate, which will be discussed in this section.

The income of labor is rent; for efficient allocation over time we must pay capital its marginal social product without regard to labor wages. Notice that in this model capital returns are also rent. Saving, or allocation through time, is the only economic allocation problem in the model. The government can impose a tax on wages at a rate n , and then transfer the proceeds to all people who once invested, regardless of whether the capital they invested is being used. The subsidy will increase and will eventually achieve the level of the marginal social product of capital. Eventually the capital will be scrapped, but marginal social product remains the same, since it increased the serial number.

Thus in the exponential case, where marginal social product of capital is $(g = \frac{\sigma}{1-n})$ $a(1-e^{-gnm})$ and rental of capital at t is $a(1-e^{ng(t-m)})$, each unit of capital gets a subsidy of $a(1-e^{-gnm}) - a(1-e^{ng(t-m)}) = ae^{-gnm}(e^{ngt}-1)$ for $0 \leq t \leq m$ and $a(1-e^{-gnm})$ for $t \geq m$. It is clear that in the non-exponential case, where marginal social product, private return, and life of capital vary, we shall get a rather complicated subsidy system, and what is more important, the subsidy payments are not connected to the use of capital. Capital may be long "dead", but since it added to serial number of today, it continues to have social product. In the exponential case the subsidy is of type



For the general case as well as for the exponential case, a simpler system of subsidy which would accomplish the same result is the following: tax all profits and proportion n of wages; then give back a subsidy to all investors so that each receives payment according to the proportion of his accumulated gross investment in total accumulated gross investment up to this date. It is clear according to our shares calculation that in this way capital would get its marginal social product.

To find the welfare effect of our subsidy, we must assume some welfare function of the type used by Arrow and then assume some functional relationships between saving and rate of return. In the case described by Arrow, this type of subsidy guarantees $\rho_s = \beta$.

$$a[1 - (\frac{G'}{G})^n] = \beta \quad .$$

Using Arrow's (40), $(\frac{G'}{G})^n = (1 - \frac{1}{\alpha\mu})^n$ where $\mu = \frac{G}{x}$:

$$1-(1- \frac{1}{\alpha\mu})^n = \frac{\beta}{\alpha} \quad .$$

$$1- \frac{\beta}{\alpha} = (1- \frac{1}{\alpha\mu})^n = W \quad ,$$

using Arrow's notation $(1- \frac{\beta}{\alpha}) = \gamma$, $\gamma = W$. Arrow is showing that $\gamma = (1- \frac{x}{ab})^n$ on the optimal path, which implies that the "capital"-output ratio G/x is the same on the optimal path and on the competitive path with the subsidies.

The influence of the increase of private return on allocation over time depends on how saving changes with the rate of return and on our objective function. We have here followed all of Arrow's assumptions in his last section (p. 171), especially the section before formula (61). Individuals have a rate of time preference of β ; the supply of capital is infinitely elastic at private return of β . Society will take all investments at a rate above β , none at a rate below β . So the private return must be β in a case where some, but not all, income is saved. With the subsidy, there is no divergence between the private and the social return. Social rate of return = $\frac{\partial x}{\partial G}$ = private rate of return, and so $\frac{\partial x}{\partial G} = \beta$. We shall show later that $\frac{\partial x}{\partial G} = \beta$ implies that we are on the path calculated by Arrow.

In the exponential world,

$$L_o e^{\sigma t} = C[G_o^{1-n} e^{\sigma t} - G_o^{1-n} e^{\sigma(t-m)}] \quad ,$$

$$L_o = CG_o^{1-n} (1 - e^{-\sigma m}) \quad ,$$

$$e^{-\sigma m} = 1 - \frac{L_o}{CG_o^{1-n}} = \left(\frac{G'}{G}\right)^{1-n} \quad ,$$

$$m = -\frac{1}{\sigma} \log \left(1 - \frac{L_o}{CG_o^{1-n}}\right) = -\frac{1}{\frac{\sigma}{1-n}} \log \left(\frac{G'}{G}\right) \quad .$$

G_o , as we shall now show, is a function of the rate of saving s . Then $\frac{G'}{G}$ is a function of the rate of saving s ; since all distributional characteristics can be expressed as functions of $\frac{G'}{G}$, the system is determined by giving its rate of saving.

Let use Arrow's (23), labor cost per unit of output,

$$W(v) = w(v) \frac{\lambda[G(v)]}{\gamma[G(v)]} = w(v) \frac{bG^{-n}}{a} = \left(\frac{G'}{G}\right)^n \quad .$$

Arrow's (39),

$$\frac{g}{x} = s = \mu \frac{\sigma}{1-n} \; ;$$

$$\left(\frac{G'}{G}\right)^n = \left(1 - \frac{1}{a\mu}\right)^n = \left(1 - \frac{\frac{\sigma}{1-n}}{as}\right)^2 , \quad \frac{G'}{G} = 1 - \frac{\frac{\sigma}{1-n}}{as} \; ,$$

$$m = -\frac{1}{\frac{\sigma}{1-n}} \log \left(\frac{G'}{G}\right) = -\frac{1}{\frac{\sigma}{1-n}} \log \left(1 - \frac{\frac{\sigma}{1-n}}{as}\right) , \qquad (3.5)$$

$$\frac{dm}{ds} < 0 \; .$$

The first thing that we notice from this is that to get exponential growth, a necessary requirement is $\frac{\frac{\sigma}{1-n}}{as} < 1$, $\frac{\frac{\sigma}{1-n}}{s} < a$, which clearly implies $\frac{\sigma}{1-n} < a$. This condition is brought up by Arrow in his section on optimal consumption, where he requires (52) $\beta > \frac{\sigma}{1-n}$, (56) $a > \beta$, implying $a > \frac{\sigma}{1-n}$. For $(\frac{G'}{G})$ to be less than one (i.e., for $W(v) > 0$ real) we must have $\frac{\sigma}{1-n} < a$ and $\frac{1}{s} \frac{\sigma}{1-n} < a$, a condition which we will use in the discussion of an optimal saving program. Secondly, we can solve for G_o as a function of s:

$$\left(1 - \frac{L_o}{CG_o^{1-n}}\right)^{\frac{1}{1-n}} = \left(1 - \frac{\frac{\sigma}{1-n}}{as}\right) \; ,$$

---

6. If $s < \frac{\frac{\sigma}{1-n}}{a}$, the rate of saving is too small to maintain steady growth at rate $\frac{\sigma}{1-n}$ with constant lifetime.

$$1 - \frac{L_o}{CG_o^{1-n}} = \left(1 - \frac{\frac{\sigma}{1-n}}{as}\right)^{1-n} ,$$

$$1 - \left(1 - \frac{\frac{\sigma}{1-n}}{as}\right)^{1-n} = \frac{L_o}{C} \frac{1}{G_o^{1-n}} ,$$

$$G_o(s) = \frac{\left(\frac{L_o}{C}\right)^{\frac{1}{1-n}}}{\left[1 - \left(1 - \frac{\frac{\sigma}{1-n}}{as}\right)^{1-n}\right]^{\frac{1}{1-n}}} .$$

We could get this directly from the exponential solution to the differential equation $\dot{G} = sx$.

As s goes up, $\frac{G'}{G}$ is increased, i.e., m is decreased. So by increasing the rate of saving, we get a quickening, i.e., shortening of the effective life of capital. If $s = 1$, i.e., with no consumption, $\frac{G'}{G} = 1 - \frac{\frac{\sigma}{1-n}}{a}$ and $m = -\frac{1}{\frac{\sigma}{1-n}} \log \left(1 - \frac{\frac{\sigma}{1-n}}{a}\right)$, which is the lower bound on the effective life of capital.

Stability of Exponential Growth

One of the basic questions which Arrow does not discuss is the stability of exponential growth. Let us assume a constant saving rate s . The motion of the system is described by

$$\dot{G} = sx(t) = sa\ G(t)\left[1 - \left(1 - \frac{L_o e^{\sigma t}}{CG^{1-n}}\right)^{\frac{1}{1-n}}\right] .$$

Trying the exponential solution $G(t) = G_o e^{\frac{\sigma}{1-n} t}$, we find

$$\frac{\sigma}{1-n} = sa[1-(1-\frac{L_o}{CG_o^{1-n}})^{\frac{1}{1-n}}] .$$

Solving for G_o in terms of saving rate s we get the previous expression for $G_o(s)$, which is a monotonic increasing function of s. This is the only exponential solution of the system. Writing the equation in the form

$$\frac{\dot{G}}{G} = sa[1-(1-\frac{L_o e^{\sigma t}}{CG^{1-n}})^{\frac{1}{1-n}}].$$

To prove that $G(t) \rightarrow G_o e^{\frac{\sigma}{1-n} t}$, we must prove that $\frac{\dot{G}}{G} \rightarrow \frac{\sigma}{1-n}$. Assume that at 0 we start with $\frac{\dot{G}}{G} > \frac{\sigma}{1-n}$. Let

$$\frac{d \log G}{dt} = \frac{\dot{G}}{G} = \frac{\sigma}{1-n} + \phi(t).$$

$\phi(t)$ is a positive function, since if $\phi(t)$ reached zero, then it would stay on the exponential path.

$$\log G = C_o + \frac{\sigma}{1-n} t + \int_0^t \phi(t) \ dt .$$

Let us now look at the function $\zeta(t) = \int_0^t \phi(t)\, dt$, which is monotonic since $\dot{\zeta}(t) = \phi(t) > 0$. There are two possibilities: either $\zeta(t) \rightarrow C_1$, a constant, as $t \rightarrow \infty$, $\log G \underset{t \rightarrow \infty}{\rightarrow} (C_o + C_1) + \frac{\sigma}{1-n} t$, and

and $G \to G_0 e^{\frac{\sigma}{1-n}t}$, as we want to prove; or else $\varsigma(t) \underset{t\to\infty}{\longrightarrow} \infty$. We shall rule out the second possibility.

$$G = e^{C_0} e^{\frac{\sigma}{1-n}t} e^{\varsigma(t)}$$

and

$$\frac{\dot{G}}{G} = sa[1-(1-\frac{L_0 e^{\sigma t}}{Ce^{C_0(1-n)} e^{\sigma t} e^{(1-n)\varsigma(t)}})^{\frac{1}{1-n}}] ,$$

$$\frac{\dot{G}}{G} = sa[1-(1-\frac{L_0}{C\ e^{(1-n)C_0} e^{(1-n)\varsigma(t)}})^{\frac{1}{1-n}}] .$$

If $\varsigma(t) \underset{t\to\infty}{\longrightarrow} \infty$, the right-hand side tends to zero, a contradiction to $\frac{\dot{G}}{G} > \frac{\sigma}{1-n}$. Similar considerations show that if $\frac{\dot{G}}{G} < \frac{\sigma}{1-n}$ at 0 , it must tend to $\frac{\sigma}{1-n}$, and G tends to $G_0 e^{\frac{\sigma}{1-n}t}$.

Optimal Saving in the Arrow Model

We should now like to bring up a few interpretations of Arrow's optimal growth path. First of all, let us point out that Arrow does not take into account that the initial serial number $G(0)$ is given. So in (49), $U = U_1 - \lim_{t\to\infty} e^{-\beta t} G(t) + G(0)$, but then in the optimal behavior he derives optimal capital expansion of the form $G(t) = \bar{G}e^{\frac{\sigma}{1-n}t}$, where $\bar{G}$ is the constant maximizing the expression

$$\bar{G}[a-\beta-a(1-\frac{L_0}{CG^{1-n}})^{\frac{1}{1-n}}] .$$

But for $t = 0$ we find $G(0) = \bar{G}$, so that at $t = 0$ we have a discontinuity unless $\bar{G} = G(0)$. Since we want maximize

$$= \int_0^\infty e^{-\beta t} C(t)\, dt \quad ,$$

intuitively it seems that so long as $\frac{\partial x}{\partial G}$, or rather, $\frac{\partial \dot{G}}{\partial G}$, the own rate of interest, is above β , we should invest everything; then when marginal product of capital is below β , we should consume everything. Eventually we reach a path where $\frac{\partial x}{\partial G} = \beta$:

$$a[1 - (\frac{G'}{G})^n] = \beta \ ,$$

$$(\frac{G'}{G})^n = 1 - \frac{\beta}{a} = \gamma \ .$$

But on the exponential path,

$$(\frac{G'}{G})^n = (1 - \frac{L_o}{C\bar{G}^{1-n}})^{\frac{n}{1-n}} .$$

(Arrow denotes this term by ν .) The optimal path is obtained by $\nu = \gamma$, which is the solution derived by Arrow. Thus we see that the solution is of the following type: if $\frac{\partial x}{\partial G} > \beta$, invest everything until the path $\frac{\partial x}{\partial G} = \beta$ is reached. Now as labor is continuing to grow, invest exactly the amount required to remain on $\frac{\partial x}{\partial G} = \beta$ [7].

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7. We know that the path is exponential, since only on the exponential path is $\frac{\partial x}{\partial G}$ a constant.

Alternatively, let us apply optimal control theory,

$$x(G,L) = aG[1-(1-\frac{1}{CG^{1-n}})^{\frac{1}{1-n}}] \quad ;$$

s(t), the saving rate, is our control; $\dot{G}(t) = s(t)\,X(G,L)$; $0 \leq s(t) \leq 1$. Now applying the Pontryagin maximizing principle, we define a Hamiltonian:

$$\begin{aligned} H &= e^{-\beta t}\,[1-s(t)]\,X(G,L) + \psi(t)\,s(t)\,X(G,L) \\ &= e^{-\beta t}\,X(G,L) + [\psi(t) - e^{-\beta t}]\,X(G,L)\,s(t), \end{aligned}$$

$$\dot{\psi}(t) = -\frac{\partial H}{\partial G} = -\,e^{-\beta t}\,\frac{\partial x}{\partial G}\,[\psi(t) - e^{-\beta t})\,\frac{\partial x}{\partial G}\,s(t) \quad .$$

Now if

$$\psi(t) > e^{-\beta t}\,, \qquad s(t) = 1\;;$$

$$\psi(t) < e^{-\beta t}\,, \qquad s(t) = 0\;.$$

In the third case, where $\psi(t) = e^{-\beta t}$, the maximizing principle does not help us in finding the control. It is clear that most of the time the path would be in the situation $\psi(t) = e^{-\beta t}$, $\dot{\psi}(t) = -\beta\, e^{-\beta t}$, and $\dot{\psi}(t) = -e^{-\beta x}\frac{\partial X}{\partial G}$ implies $\frac{\partial x}{\partial G} = \beta$.

$\psi(t)$ can be interpreted as the price of the capital good in terms of the consumer good. If it so happens that $\psi(t) > e^{-\beta t}$, saving everything; $\dot{G}(t) = X(G,L)$ and G will be growing in the exponential case faster than L until we reach $\frac{\partial X}{\partial G} = \beta$, where there is in the exponential case only one rate of s which will keep us on the path.

It is obvious from $W = v = \gamma = 1 - \frac{\beta}{\alpha}$ that if β is higher, γ is lower, and so from $v = (\frac{G'}{G})^n = (1 - \frac{1}{a\mu})$ it is clear that $\mu = \frac{G}{x}$ is lower, or that the saving rate is lower. (Obviously $0 < v < 1$, or $0 < \beta < a$.) If $\beta = 0$, i.e., if we do not apply any pure rate of time preference, the functional is now convergent unless we apply a different utility function. But we can still find among all exponential patterns one which dominates the others and which we can call the "gold rule path" [8]. We shall find the rate of saving such that among all exponential patterns we get more consumption.

As we have already shown on the exponential path,

$$G_o(s) = \frac{(\frac{L_o}{C})^{\frac{1}{1-n}}}{[1-(1-\frac{1}{sa}\frac{\sigma}{1-n})^{1-n}]^{\frac{1}{1-n}}} .$$

Production is growing like $x(o)\, e^{\frac{\sigma}{1-n}t}$, and consumption like $(1-s)\, X_o e^{\frac{\sigma}{1-n}t}$. We want to maximize $x_o(1-s)$:

$$\dot{G}(o) = sx_o , \qquad G_o \frac{\sigma}{1-n} = sx_o$$

implies

$$x_o(s) = \frac{G_o \frac{\sigma}{1-n}}{s} .$$

So we want to maximize $x_o(1-s) = \frac{\sigma}{1-n} \frac{G_o(s)}{s} (1-s) = \psi(s)$:

---

8. It is not exactly the ordinary golden rule, since the pseudo-production function is not homogeneous of first degree.

$$\underset{\frac{\frac{\sigma}{1-n}}{a} \leq s \leq 1}{\text{Max}} \frac{1-s}{s} \frac{1}{\left[1-\left(1-\frac{1}{sa}\frac{\sigma}{1-n}\right)^{1-n}\right]^{\frac{1}{1-n}}} = \text{Max}\ \psi(s) \quad .$$

On the boundaries, the function $\psi(s)$ gets the values $\psi(1) = 0$, $\psi\left(\frac{\frac{\sigma}{1-n}}{a}\right) = \frac{a}{\frac{\sigma}{1-n}} - 1 > 0$; so if the function has one extremum, it must be the maximum or inflexion point.

$$\log \psi(s) = \log \frac{1-s}{s} + \log G(s) \quad ;$$

$$\frac{d \log \psi(s)}{ds} = (1-n)\frac{s}{1-s}\left(-\frac{1}{s^2}\right) + \frac{1}{1-n} \frac{\left[1-\frac{1}{sa}\frac{\sigma}{1-n}\right]^{-n} \frac{1}{s^2}}{\left[1-\left(1-\frac{1}{sa}\frac{\sigma}{1-n}\right)^{1-n}\right]} \quad ;$$

$$\left.\frac{d \log \psi(s)}{ds}\right|_{s = \frac{\frac{\sigma}{1-n}}{n}} = +\infty \quad .$$

Thus we have a compact set and the maximum is not attained at the bounaries; the extremum must therefore be a maximum if it is unique:

$$\underset{\frac{\frac{\sigma}{1-n}}{a} \leq s \leq 1}{\text{Max}} \left(\frac{1-a}{s}\right)^{1-n} \frac{1}{\left[1-\left(1-\frac{1}{sa}\frac{\sigma}{1-n}\right)^{1-n}\right]^{\frac{1}{1-n}}} \quad ,$$

$$\underset{\frac{\frac{\sigma}{1-n}}{a} \leq s \leq 1}{\text{Max}} \frac{1}{\left(\frac{1}{1-s}\right)^{1-n} - \left(\frac{s}{1-s} - \frac{1-s}{a}\frac{\sigma}{1-n}\right)^{1-n}} \quad ,$$

or a minimum of the reciprocal:

$$\underset{\frac{\frac{\sigma}{1-n}}{a} \leq s \leq 1}{\text{Min}} \left(\frac{s}{1-s}\right)^{1-n} - \left(\frac{s}{1-s} - \frac{1-s}{a}\,\frac{\sigma}{1-n}\right)^{1-n} = \Omega(s),$$

$$\Omega'(s) = (1-n)\left(\frac{s}{1-s}\right)^{-n}\frac{1}{(1-s)^2} - (1-n)\left(\frac{s}{1-s} - \frac{1}{1-s}\,\frac{1}{a}\,\frac{\sigma}{1-n}\right)^{-n}\left[\frac{1}{(1-s)^2} - \frac{1}{(1-s)^2}\,\frac{\sigma}{1-n}\,\frac{1}{a}\right] = 0 ,$$

$$\left(\frac{s}{1-s}\right)^{-n} = \left[\frac{s}{1-s} - \frac{1}{1-s}\,\frac{\sigma}{a(1-n)}\right]^{-n}\left[1 - \frac{\sigma}{a(1-n)}\right] ,$$

$$\frac{s}{1-s} = \frac{s}{1-s}\left[1 - \frac{\sigma}{a(1-n)}\right]^{-\frac{1}{n}} - \frac{1}{1-s}\left[1 - \frac{\sigma}{a(1-n)}\right]^{-\frac{1}{n}}\frac{\sigma}{a(1-n)} .$$

Substitute

$$\frac{s}{1-s} = y , \qquad \frac{1}{1-s} = \frac{1 - s + s}{1 - s} = 1 + y .$$

The equations for y are:

$$y = y\left[1 - \frac{\sigma}{a(1-n)}\right]^{-\frac{1}{n}} - \left[1 - \frac{\sigma}{a(1-n)}\right]^{-\frac{1}{n}}\frac{\sigma}{a(1-n)}\,y - \left[1 - \frac{\sigma}{a(1-n)}\right]^{-\frac{1}{n}}\frac{\sigma}{a(1-n)} ,$$

$$y\left\{1 - \left[1 - \frac{\sigma}{a(1-n)}\right]^{-\frac{1}{n}} + \frac{\sigma}{a(1-n)}\left[1 - \frac{\sigma}{a(1-n)}\right]^{-\frac{1}{n}}\right\} = -\left[1 - \frac{\sigma}{a(1-n)}\right]^{-\frac{1}{n}}\frac{\sigma}{a(1-n)} ,$$

$$y\left\{1 - \left[1 - \frac{\sigma}{a(1-n)}\right]^{\frac{n-1}{n}}\right\} = -\left(1 - \frac{\sigma}{1-n}\right)^{-\frac{1}{n}}\frac{\sigma}{a(1-n)} ,$$

$$y = \frac{\left[1 - \frac{\sigma}{a(1-n)}\right]^{-\frac{1}{n}}\frac{\sigma}{a(1-n)}}{\left[1 - \frac{\sigma}{a(1-n)}\right]^{\frac{n-1}{n}} - 1} = \frac{\frac{\sigma}{a(1-n)}}{\left[1 - \frac{\sigma}{a(1-n)}\right] - \left[1 - \frac{\sigma}{a(1-n)}\right]^{\frac{1}{n}}} ;$$

$$s = \frac{y}{1+y} = \frac{\frac{\sigma}{a(1-n)}}{1-[1-\frac{\sigma}{a(1-n)}]^{\frac{1}{n}}} \quad . \tag{3.7}$$

Now we must check that $\frac{\frac{\sigma}{1-n}}{a} \leq s \leq 1$.

$$s = \frac{1-[1-\frac{\sigma}{a(1-n)}]}{1-[1-\frac{\sigma}{a(1-n)}]^{\frac{1}{n}}} \quad ,$$

$0 < n < 1$ and $0 < \frac{\sigma}{a(1-n)} < 1$, implies

$$[1-\frac{\sigma}{a(1-n)}]^{\frac{1}{n}} < [1-\frac{\sigma}{a(1-n)}] \quad ,$$

which implies $s < 1$. The other inequality,

$$\frac{\frac{\sigma}{a(1-n)}}{1-[1-\frac{\sigma}{a(1-n)}]^{\frac{1}{n}}} > \frac{\sigma}{a(1-n)} \, ,$$

$$1 > \left\{1-[1-\frac{\sigma}{a(1-n)}]^{\frac{1}{n}}\right\} \, ,$$

$$0 > -[1-\frac{\sigma}{a(1-n)}]^{\frac{1}{n}} \quad ,$$

So

$$0 < \frac{\sigma}{a(1-n)} < s < 1 \quad .$$

We note that optimal saving is independent of L_o, b, which gives only the sale of the economy, and it depends only on $\frac{\frac{\sigma}{1-n}}{a}$.

Let us now show that on the golden rule path the rate of growth and the social rate of return are the same. For this, we shall use Arrow's (39), which says

$$s = \frac{g}{x} = \mu \frac{\sigma}{1-n} \quad ,$$

and (40),

$$\left(\frac{G'}{G}\right)^n = W = \left(1 - \frac{x}{aG}\right)^n = \left(1 - \frac{1}{a\mu}\right)^n \quad .$$

Assuming that the social rate of return is the same as the rate of growth,

$$\frac{\partial x}{\partial G} = a\left[1 - \left(\frac{G'}{G}\right)^n\right] = a\left\{1 - \left[1 - \frac{1}{a\mu}\right]^n\right\} = \frac{\frac{\sigma}{1-n}}{a} \quad ,$$

Solving for μ,

$$\mu = \frac{1/a}{1 - \left(1 - \frac{\frac{\sigma}{1-n}}{a}\right)^{\frac{1}{n}}} \quad ,$$

and the corresponding saving rate,

$$s = \mu \frac{\sigma}{1-n} = \frac{\frac{\sigma}{a(1-n)}}{1 - \left(1 - \frac{\frac{\sigma}{1-n}}{a}\right)^{\frac{1}{n}}} \quad ,$$

which is exactly the saving rate that we got by maximizing [9]. If capital, through the subsidy system, receives its marginal product, the share of profit is

$$\frac{\frac{\partial x}{\partial G} G}{X} = \frac{\partial x}{\partial G} \mu = \frac{\sigma}{1-n} \mu = s .$$

Thus we see that on the golden rule path, the share of profit is equal to the rate of saving.

It is clear that Arrow's exponential path for $\beta > a > 0$ is below the golden rule path; society, because of its time preference, is not ready to do the saving to get to the higher path. If society is above Arrow's path, it prefers the short outburst of consumption and a return to the lower path.

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9. Clearly we could reverse the derivation and from

$$s = \frac{\frac{\sigma}{a(1-n)}}{1-(1-\frac{\frac{\sigma}{1-n}}{a})^{\frac{1}{n}}} ,$$

derive

$$\frac{\partial x}{\partial G} = \frac{\frac{\sigma}{1-n}}{a} .$$

CHAPTER IV

THE RATE OF RETURN AND THE RATE OF INTEREST IN CAPITAL-EMBODIED TECHNOLOGICAL CHANGE

Let $F(K,L)$ be the production function of the economy. There is no technological change, and we assume perfect foresight and for simplicity ignore depreciation [1]. The labor force is changing through population growth, and capital is accumulated through saving. The instantaneous social rate of return or force of interest in the competitive model is $F_K[K(t), L(t)] = r(t)$ [2]. The instantaneous rental of capital at t is also $r(t)$. The discounted value of rentals at any t is 1, i.e., the present value of the rentals stream is equal to the cost of production of capital. To show this, we have to prove

$$\int_t^{\infty} r(u) e^{-\int_t^u r(v)dv} du = 1 .$$

Substitute $w = \int_t^u r(u)dv$, $\frac{dw}{du} = r(u)$, and assume $\int_t^{\infty} r(u)av = \infty$; we then find

$$\int_t^{\infty} r(u) e^{-\int_t^u r(v)dv} du = \int_t^{\infty} e^{-w} dw = 1 .$$

---

1. There is no difficulty in bringing into the picture radioactive decay in rate δ . It is easy to find that this will reduce the social rate of return or the rate of interest by δ . There are difficulties with other types of depreciation.

2. For the following, see Paul A. Samuelson, "The Evaluation of 'Social Income'; Capital Formation and Wealth, in The Theory of Capital, Lutz and Hague, eds. (London, Macmillan, 1961), p. 42.

It is easy to include in this analysis disembodied technological change also. Again the instantaneous rate of return at t is the marginal product of capital at t. As an example, on the exponential path with a rate of saving s and Cobb-Douglas production function $F(K,L) = e^{\lambda t} K^{1-\alpha} L^{\alpha}$, $r(t) = (1-\alpha) \frac{\lambda + \alpha n}{\alpha s}$ [3,4]. Recent discussions of Tobin and Solow [5] reveal the fact that the situation is quite different in the embodied technological change. The social rate of return and, since there are no externalities, the rate of interest turn out to be below the rental or marginal product of new capital. Solow and Tobin dealt mainly with the capital-augmenting case, and we shall in the following try to prove that under general conditions this will be the case.

Let $F[I(v), L(v,t), v]$ be our production function and $r(t,u)$ the rental of capital vintage t at u. If $m(t)$ is the effective life of capital, i.e., $r[t, t+m(t)] = 0$, then the functional equation which should be satisfied by the rate of interest or rate of return, $r(t)$, is [6]

$$1 = \int_{t}^{t+m(t)} r(t,u)\, e^{-\int_{t}^{u} r(x) d\lambda}\, du$$

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3. If $s = 1 - \alpha$, $r = n + \frac{\lambda}{\alpha}$, the rate of growth is as expected.

4. In an unpublished note, Solow calculated in the disembodied case the maximum sustainable increase in consumption by one unit of extra saving today.

5. Robert M. Solow, Capital Theory and the Rate of Return (Amsterdam, North-Holland Publishing Co., 1963), p. 56.

6. Here we use an unpublished theorem of C. Von Weizäcker which shows that in capital-embodied models of this type, the rate of interest and the social rate of return are the same.

We do not rule out the case $m(t) = \infty$. By the nature of technological change, we must have $0 \leq r(t,u) < r(u,u)$ for all $u > t$. According to the theorem proved,

$$\int_t^\infty r(u,u) e^{-\int_t^u r(x,x)dx} du = 1 ,$$

and hence,

$$\int_t^{t+m(t)} r(t,u) e^{-\int_t^u r(x,x)dx} du < 1 .$$

So generally $r(x,x)$ should be greater than the rate of interest. We could not show pointwise dominance, but the instantaneous rate of interest cannot be the same as the rental of new capital, and generally it is below it. We have to take into account obsolescence, or in the case of social return, the fact that new capital is more productive.

As an example let us calculate the social rate of return in Cobb-Douglas embodied technological change with constant saving rate and a golden-age path [7].

Let the labor force be $L(t) = L_o e^{nt}$ and investment $I(t) = sQ(t)$. In the golden age, $Q(t) = Q_o e^{gt}$, $Q(t) = B\, L(t)^{\alpha}\, J(t)^{1-\alpha}$, while

$$J(t) = \int_{-\infty}^{t} e^{\frac{\lambda}{1-\alpha} v}\, I(v)\, dv ,$$

$$Q_o e^{gt} = BL_o^{\alpha}\, e^{\alpha nt}\, s^{1-\alpha}\, Q_o^{1-\alpha} \left[\int_o^t e^{\frac{\lambda}{1-\alpha} v}\, I(v)\, dv\right]^{1-\alpha} ,$$

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7. See Solow, "Investment and Technical Progress", in Arrow, Carlin and Suppes, eds., Mathematical Methods in the Social Sciences (Stanford University Press, 1960). We shall use the same notation as Solow.

$$g = n + \frac{\lambda}{\alpha} ,$$

and

$$Q_o = \frac{B^{\frac{1}{\alpha}} L_o s^{\frac{1-\alpha}{\alpha}}}{[\frac{\lambda}{1-\alpha} + g]^{\frac{1-\alpha}{\alpha}}} .$$

Using the fact that we reshuffle the labor so as to equalize marginal product, one can get (Solow's equation 9)

$$L_o e^{nt} = h(t) \; sQ_o \int_{-\infty}^{t} e^{(\frac{\lambda}{1-\alpha} + g)v} \, du ,$$

and

$$L_o = h(t) \; sQ_o \; e^{\frac{\lambda}{\alpha(1-\alpha)} t} ,$$

$$h(t) = \frac{(\frac{\lambda}{1-\alpha} + g) \; L_o \; e^{\frac{\lambda}{-\alpha(1-\alpha)} t}}{sQ_o} .$$

We find for the marginal product of labor at t , $m(t)$,

$$m(t) = \alpha B \; h(t)^{\alpha-1} = \alpha B \left[\frac{(\frac{\lambda}{1-\alpha} + g) \; L_o}{sQ_o}\right]^{\alpha-1} e^{\frac{\lambda}{\alpha} t} .$$

Using Solow's equation (13), the quasirent of capital vintage v at t is:

$$r(v,t) = (1-\alpha) \left[\frac{\alpha^{\alpha} \; B\alpha^{-\alpha} \; (\frac{\lambda}{1-\alpha} + g)^{\alpha(1-\alpha)} \; L_o^{\alpha(1-\alpha)}}{s^{\alpha(1-\alpha)} \; Q_o^{\alpha(1-\alpha)}} \; e^{\lambda(v-t)}\right]^{\frac{1}{1-\alpha}} .$$

Substituting for Q_o , we finally find

$$r(v,t) = \frac{1-\alpha}{s} \ \frac{\lambda}{1-\alpha} + g) \ e^{\frac{\lambda}{1-\alpha}(v-t)}$$

and

$$r(t,t) = \frac{1-\alpha}{s} \left(\frac{\lambda}{1-\alpha} + g\right) .$$

Now, the rate of interest should satisfy the functional equation

$$1 = \int_t^{\infty} r(t,n) \ e^{-\int_t^v r(x)dx} \ du \quad ,$$

or

$$1 = \frac{1-\alpha}{s} \left(\frac{\lambda}{1-\alpha} + g\right) \int_t^u e^{-\frac{\lambda}{1-\alpha}(u-t)} \ e^{-\int_t^u r(x)dx} \ du \quad .$$

Differentiating both sides with respect to t ,

$$\frac{1-\alpha}{s} \left(\frac{\lambda}{1-\alpha} + g\right) \int_t^{\infty} e^{-\frac{\lambda}{1-\alpha}(u-t)} \ e^{-\int_t^u r(x)dx} \left(\frac{\lambda}{1-\alpha} + r(t)\right) - \frac{1-\alpha}{s} \left(\frac{\lambda}{1-\alpha} + g\right) = 0 \ .$$

Using the original functional equation,

$$\frac{\lambda}{1-\alpha} + r(t) = \frac{1-\alpha}{s} \left(\frac{\lambda}{1-\alpha} + g\right) ,$$

$$r(t) = \frac{1-\alpha}{s} \left(\frac{\lambda}{1-\alpha} + g\right) - \frac{\lambda}{1-\alpha} = r(t,t) - \frac{\lambda}{1-\alpha} \ [8] \ .$$

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8. Again for $s = 1 - \alpha$, $r(t) = g$.

$\frac{\lambda}{1-\alpha}$ is the rate of capital augmenting, and the rate of return is the rental or maginal product on new capital minus the rate of capital augmenting [9].

Remembering that $g = n + \frac{\lambda}{\alpha}$, we find that

$$r(t) = \frac{1 - \alpha(1-s)}{s(1-\alpha)\alpha}\lambda + \frac{n(1-\alpha)}{s} .$$

Solow [10] claims that unlike the disembodied case, it is quite possible in the embodied case that the rate of return will be a decreasing function of λ and may, though it is not plausible, reach negative values. We see that in our case, $r(t)$ is instead increasing linearly in λ. It is inversely related to s and has a lower bound for $s = 1$:

$$r(t) = \frac{1}{(1-\alpha)\alpha} + n(1-\alpha).$$

If μ is the rate of capital augmenting, $r(t,t)$ is generally an increasing function of μ, say $\phi(\mu,t)$. Tobin and Solow showed that in this case $r(t) = \phi(\mu,t) - \mu$, and so if $\frac{\partial \phi}{\partial \mu} \geq 1$ for all t, the rate of return will be, as in our case, a monotonic increasing function of μ.

We could carry on similar discussions on the determination of the social rate of return in different golden-age situations of the type discussed in the first chapter of this work. In all these cases, $r(t,t)$

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9. It is easy to include radioactive decay of capital at a rate of δ ; we then find by the same methods $r(t) = r(t,t) - \delta - \frac{\lambda}{1-\alpha}$.

10. Solow, Robert M., Capital Theory and the Rate of Return, op. cit.

and $r(t)$ are constants, and using the theorem we proved $r(t) < r(t,t)$. It is hard in these cases to compute explicitly the rate of interest as a function of the saving rate and the rest of the parameters of the system. We always get equations of the type

$$\int_{t}^{t+m} e^{-r(u-t)} r(t,u) \, dn = 1 \quad ,$$

and because of the monotonicity with respect to r and the fact that the range of this function is from zero to infinity, we get a unique solution. It seems hopeless to solve these equations in the case of changing rate of interest.

CHAPTER V

NOTES ON MALDOR-MIRRLEES' TECHNICAL PROGRESS FUNCTION [1]

Despite all "non-neo-neoclassical" pretensions, the Kaldor-Mirrlees model is not very different from the classical model. It is especially similar to Solow's model of embodied technological change and to Solow's notion of ex ante substitutability and ex post fixed proportions [2]. It is especially close to Phelps' model, "Substitution, Fixed Proportion, Growth and Distribution" [3].

Let us use Kaldor-Mirrlees notation. p_t denotes output per worker on new machines, i_t investment per labor on new machines. For simplicity, let us ignore physical wear and tear. Let $L(t,t)$ be labor using the new equipment, $I(t)$ the investment in new equipment. Assume with Kaldor and Mirrlees or Phelps that $L(v,t) = L(v,v)$ for all $t \geq v$ as long as capital vintage v is used. Kaldor and Mirrlees implicitly assume that after $I(t)$ is invested, the capital-labor ratio is fixed; we cannot alter the labor necessary to operate the equipment after the equipment is produced. There is an implicit assumption that capitalists have some freedom to choose their technique of production before the machines are produced, but then machines are either fully operated or not operated at all afterwards. (Kaldor and Mirrlees make their choice of technique by bringing in their institutionally determined horizon h, that is, the businessmen want to get back their investment after h periods.)

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1. Kaldor, N., and J. A. Mirrlees, "A New Model of Economic Growth", Review of Economic Studies, June, 1962.

2. Solow, Robert M., "Substitution and Fixed Proportions in the Theory of Capital", Review of Economic Studies, June, 1962.

3. Phelps, E.S., International Economic Review, September, 1963.

Kaldor and Mirrlees assume the existence of a "technical progress function" f such that:

$$\frac{\dot{p}(t)}{p(t)} = f\left(\frac{\dot{i}}{i}\right) , \qquad f(o) > 0 , \qquad f' > 0 , \qquad f'' < 0 .$$

For $t > 0$ we find $p(t) = C_o e^{\int_o^t f\left(\frac{\dot{i}}{i}\right) du}$, so if capital vintage v is operated at $t > v$,

$$\frac{Q(v,t)}{L(v,t)} = C_o e^{\int_o^v f\left(\frac{\dot{i}}{i}\right) du} ,$$

$$Q(v,t) = C_o L(v,t) e^{\int_o^v f\left(\frac{\dot{i}}{i}\right) du} ,$$

On the assumption that $\frac{\dot{i}}{i} \geq 0$,

$$Q(t) = \int_{t-m(t)}^{t} Q(v,t) dv = \int_{t-m(t)}^{t} C_o \, L(v,t) \, e^{\int_o^v f\left(\frac{\dot{i}}{i}\right) du} du$$

$$= C_o \int_{t-m(t)}^{o} L(v,v) \, e^{\int_o^v f\left(\frac{\dot{i}}{i}\right) du} dv ;$$

$$L(t) = \int_{t-m(t)}^{t} L(v,t) \, dv = \int_{t-m(t)}^{t} L(v,v) \, dv .$$

Now, having the histories of investment $I(t)$ and employment $L(t,t)$, we can determine $m(t)$, the age of the oldest capital, and $Q(t)$. By assuming some saving behavior, either that all profits are being saved or rather that $I(t) = sQ(t)$, and making some behavioristic assumptions on the way entrepreneurs choose their techniques of production, we get the way the system behaves along time. We can of course perform in this system computer experiments similar to the type proposed by Solow. We can, for example, assume as Phelps did that entrepreneurs assume that the rate of interest remains the same while the wage rises exponentially. There may be some problems of divergence of social and private return if the individual investor does not get the proceeds of the beneficial influences of his investment over total productivity, but in the Kaldor-Mirrlees model this is not necessarily so, as we shall try to show.

First of all, on the exponential growth path to which Kaldor and Mirrlees pay special attention, $\frac{\dot{i}}{i} = \beta$ - a constant and, say, $\lambda = f(\beta)$, $Q(v,t) = C_o e^{\lambda v} L(v,t)$, and with constant capital-output ratio $Q(v,t) = aI(v)$. So we get exactly the same results as those of Tobin and Solow in unpublished notes on fixed coefficients and Harrod-neutral embodied technological change, which are discussed in Chapter II of this work.

Of course the causation is different, but insofar as "positive economics" is concerned, we cannot distinguish between the two models -- the capital-labor ratio remains the same and the capital-labor ratio is increasing exponentially. Kaldor and Mirrlees try to bring out some heuristic arguments as to why exponential growth is a stable path so that if we are not too far from the path of exponential growth, we will get roughly the same results, and indeed, it seems that nature did not perform an experiment with enough variance to distinguish between these models.

If we have small variations in $\dot{i}$ but variations that can be approximated uniformly quite well by a linear function $f(\frac{\dot{i}}{i}) = \alpha + \beta(\frac{\dot{i}}{i})$ [4] ,

$$\frac{\dot{p}}{p} = \alpha + \beta(\frac{\dot{i}}{i}) \quad ,$$

$$p = C\, e^{\alpha t}\, i^{\beta} \quad ,$$

$$Q(v,t) = C\, e^{\alpha v}\, I(v)^{\beta}\, L(v,v)^{1-\beta} \quad ,$$

$$Q(t) = C \int_{t-m(t)}^{t} e^{\alpha v}\, I(v)^{\beta}\, L(v,v)^{1-\beta}\, dv \quad ,$$

$$L(t) = \int_{t-m(t)}^{t} L(v,v)\, dv \quad ,$$

The similarity between this kind of model and that of Phelps is obvious. We could have started with embodied neutral technological change in a Cobb-Douglas production function. Before investing, capitalists can choose any capital-labor ratio, but beyond this point capital is congealed. We can even be a little more general and assume that the producer faces <u>ex ante</u> any first-order homogeneous production function $Q = F(K,L)$, and with Hicks-neutral technological change, we get

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4. Assume as an example that the technical progress function has the shape $\log\,(A + \gamma\,\frac{\dot{i}}{i}) = \log A + \log\,(1 + \frac{\gamma}{A}\frac{\dot{i}}{i})$, and if variations in $\frac{\dot{i}}{i}$ are small, this can be approximated by $\log A + \frac{\gamma}{A}\frac{\dot{i}}{i}$.

$$Q(t,t) = e^{\gamma t} F[I(t), L(t,t)] \quad ,$$

which implies

$$p_t = e^{\gamma t} g(i_t) \quad .$$

Where $g^{(K/L)} = F(\frac{K}{L}, 1)$,

$$\frac{\dot{p}_t}{p_t} = \gamma + (\frac{i_t \, g'(i_t)}{i_t})(\frac{\dot{i}}{i}) = \theta(i)(\frac{\dot{i}}{i}) + \gamma \quad .$$

$\theta(i)$ is the elasticity of the production function with respect to K; in the Cobb-Douglas case it is a constant. So if we have technical progress function $\phi(\dot{i},i) = \gamma + \theta(i)(\frac{\dot{i}}{i})$, $0 < \theta(i) < 1$, we can still give our model the interpretation, as Phelps does, that the producer faces an <u>ex ante</u> neoclassical production function; but once investment is made according to corresponding expectations, everything is congealed. Once the history of the capital-labor ratio $i(v)$ and investment $I(v)$ are given, we get

$$L(t) = \int_{t-m(t)}^{t} i(v) \, I(v) \, dv \quad ,$$

$$Q(t) = \int_{t-m(t)}^{t} e^{\gamma v} \, g(k(v)) \, \frac{I(v)}{k(v)} dv.$$

This set of equations gives us $m(t)$ and $Q(t)$. The wage rate is given by the requirement

$$e^{\gamma(t-m)} \, g(k(t-m)) = w(t) \quad .$$

$$\dot{Q}(t) = \frac{g(k(t)) \, e^{\gamma t} \, I(t)}{k(t)} - \frac{g(k(t-m)) \, e^{\gamma(t-m(t))} \, I(t-m(t))(I-m'(t))}{k(t-m(t))} \; ;$$

$$\dot{L}(t) = \frac{I(A)}{k(t)} - \frac{I(t-m(t))(1 - m'(t))}{k(t-m(t))} \quad .$$

By substitution we get

$$\dot{Q}(t) = \frac{K(t)}{k(t)} \, [g(k(t)) \, e^{\gamma t} - g(k(t-m(t)) \, e^{\gamma(t-m(t))}] + g(k(t-m(t))e^{\lambda(t-m(t))}\dot{L}(t).$$

By investing today an extra unit of capital, society might have today

$$\frac{1}{k(t)} \, [g(k(t)) \, e^{\gamma t} - g(k(t-m(t)) \, e^{\gamma(t-m(t))}]$$

extra units of output, since by transferring one unit of labor, we lose $g(k(t-m)) \, e^{\gamma(t-m(t))}$ units of output on the oldest vintage and gain $g(k(t)) \, e^{\gamma t}$ on the newest vintage, and by having one more unit of new vintage capital, we have to transfer $\frac{1}{k(t)}$ units of labor.

At time t , society gets from investment made at v (if $v > t - m(t)$) extra product of

$$\frac{1}{k(v)} \, [g(k(v)) \, e^{\gamma v} - g(k(t-m(t)) \, e^{\gamma(t-m(t))}] \; .$$

(When $v = t-m(t)$, it is reduced to zero.) The individual investor, on the other hand, gets rentals of

$$\frac{e^{\gamma v}}{k(v)}\, g(k(v)) - \frac{w(t)}{k(v)} \quad ,$$

$$w(t) = e^{\gamma(t-m(t))}\, g[k(t-m(t))] \quad ,$$

$$\frac{e^{\lambda v}}{k(v)}\, g(k(v)) - \frac{e^{\gamma(t-m(t))}\, g[k(t-m(t))]}{k(v)} \quad ,$$

and there is no divergence between social and private return.

The individual investor wants to maximize

$$\int_{t}^{t+m(t)} \frac{e^{\gamma t}\, g(k(t)) - w(v)}{k(t)}\, e^{-5(v-t)}\, dv \quad ,$$

where ρ is the expected force of interest and $m(t)$ the expected life of capital, i.e.,

$$e^{\gamma t}\, g(k(t)) = w(t+m(t)) \quad .$$

We can again do some analysis with exponential expectation [5].

The main novelty in the Kaldor-Mirrlees discussion is of course whenever $f(\frac{\dot{i}}{i})$ cannot be derived or very well approximated by ordinary neo-classical production functions. In all these cases we cannot integrate explicitly $\log p(t) = C + \int_{0}^{t} f(\frac{\dot{i}}{i})\, dt$; the whole history counts. It is hard to do analytical analysis with integral equations of this sort.

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5. This seems to be the only consistent expectation with a constant rate of interest.

Performing computer experiments of the type proposed by Solow may give us some feeling for how the system is really different from neoclassical analysis and the divergence between social and private capital [6]. Kaldor and Mirrlees do not offer us any clues with respect to this problem [7].

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6. All the empirical calculations of Kaldor and Mirrlees are on the exponential path where their model gives results no different from those of ten other models, as we have argued before.

7. However, it does seem very surprising that Kaldor in his comments claims that golden rule savings in this model are below the share of profits.

CHAPTER VI

A NONSUBSTITUTION THEOREM AND SWITCHING OF TECHNIQUES [1]

Samuelson, in his papers in Åkerman Festschrift [2] and in his article on the surrogate production function [3], presents the following theorem:

With one primary input, say labor, and with no joint products, the price pattern at any profit rate is independent of final demand (and the NNNP [4] is linear).

We shall give in the following proof of this nonsubstitution theorem in a generalized Leontief model.

Let there be k_1 activities which can be used to produce good 1, $a^{11}\ a^{12} \ldots\ a^{1k_1}$; k_2 activities to produce good 2 , ... , and k_n for good n . Each activity is composed of a column of $n + 1$ elements. The first element (which we shall denote by 0) gives the labor requirements, and the remaining n components give the requirements of inputs of goods to produce one unit of gross output of the given commodity. There is only circular capital in our model; each year's capital is used to

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1. The following theorems have been proved or suggested by P. A. Samuelson. Here I shall present complete proof for his theorems.

2. Samuelson, P.A., "A New Theorem on Nonsubstitution", in 'Money, Growth and Methodology' and Other Essays in Economics in Honor of Johan Åkerman, H. Hegeland, ed., (Stockholm, CWK Gleerup Lund, 1961).

3. Samuelson, P.A., "Parable and Realism in Capital Theory: The Surrogate Production Function", Review of Economic Studies, June, 1962.

4. NNNP is the set of baskets of consumption that are open to labor; we use here the terminology of Samuelson in reference [2].

produce the next year's output [5]. Altogether we have $\prod_{i=1}^{n} k_i$ input-output Leontief matrices; let us denote them by a, b, c, ... Assume all these matrices to be non-negative indecomposable matrices. In the following, we discuss only stationary states in which prices and the rate of interest do not change. We normalize our prices by assuming that the wage rate, paid at the beginning of the period when the labor is supplied, is unity.

Let a rate of profit of r be given. If a has been the only possible matrix, with a_o its correspondent labor requirements, then the prices in a stationary state satisfy $p_i = a_{oj}(1+r) + \sum_{i=1}^{n} (1+r)\, a_{ij}\, p_i$, or in vector notation, $p = a_o(1+r) + (1+r)pa$ and $p = a_o(1+r)(I-(1+r)a)^{-1}$. This of course says that in a stationary state, competition drives the price of a good to its cost of production. r has to be below the maximum sustainable rate of interest r^*; in other words, $\lambda = \frac{1}{1+r}$ should be greater than the Frobenius root of a, $\lambda^* = \frac{1}{1+r^*}$. Then the inverse $[I-(1+r)a]^{-1}$ is composed of all positive elements [6].

Each p_i is an increasing function of r; to see this it is easiest to expand $[I-(1+r)a]^{-1}$ and to get

$$p = a_o(1+r)[I+(1+r)a + (1+r)^2 a^2 + \ldots + (1+r)^n a^n + \ldots] \quad [7].$$

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5. In all this we follow closely Piero Sraffa, Production of Commodities by Means of Commodities (Cambridge University Press, 1960).

6. See F. R. Gantmacher, Applications of the Theory of Matrices, Vol. II, (New York, Chelsea Publishing Co., 1959), pp. 61-69.

7. By our assumption that $r < r^*$, the series is convergent to $[I-(1+r)a]^{1}$; see Gantmacher, ibid.

So $p_i(r)$, with all a elements non-negative, is a monotonic increasing function of r ; moreover, $p_i(r) \longrightarrow \infty$ as $r \longrightarrow r^*$. Alternatively, if we look on the factor-price frontier $\frac{1}{p_i(r)}$ we get a decreasing function which reaches zero at $r = r^*$. The prices can also be written in the form $p = a_o(\lambda I - a)^{-1}$, $\lambda = \frac{1}{1+r}$, and all p_i are monotonic decreasing functions of λ , each with a pole at the Frobenius root $\lambda^* = \frac{1}{1+r^*}$.

If we have alternative techniques for producing some of the goods, the "invisible hand" will produce a situation in which we find maximum real wages for a given rate of interest [8]. One may be inclined to ask: real wages in terms of what good? Part of the result of Samuelson's non-substitution theorem will be that this does not matter. We shall get the maximum real wage in terms of any good or any combination of goods; we shall be on the outer envelop of the price-factor frontiers for all i .

<u>Theorem:</u> Let λ be given. Among the $\prod_{i=1}^{n} k_i$ matrices there exists one $\begin{bmatrix} a_o^* \\ a^* \end{bmatrix}$ for this λ which minimizes all the elements of the vector $p = a_o^*(\lambda I - a^*)^{-1}$.

Let us start with some matrix a and find its prices $p_a = a_o(\lambda I - a)^{-1}$ ($r \geq \frac{1}{1+r_a^*}$; otherwise a cannot be used). Use these prices, p_a , to evaluate the costs of using alternative activities. If for some alternative activity, say activity (b_{01}, b_1) , used for producing good 1 we find

$$\tilde{p}_1 = b_{01}(1+r) + b_{11}(1+r)\, p_{1a} + b_{21}\, p_{2a}(1+r) + \ldots + b_{n1}(1+r)\, p_{na}$$

8. See P. A. Samuelson, "Wages and Interest: A Modern Dissection of Marxian Economic Models", <u>American Economic Review</u>, December, 1957.

and $\tilde{p}_1 < p_{1a}$, we then introduce this activity instead of the activity that has been used for producing good 1 . Taking the new lower price, $\tilde{p}_1$, into unit costs of activities 2, ..., n we get lower prices for 2, ..., n; we then must take the new lower prices' feedback, and so on [9]. This is essentially a process that we expect the "invisible hand" to produce. After introducing the new process, we get a new matrix b with labor requirements b_o . The iterative process described will settle on new stationary prices, $p_b = b_o(\lambda I - b)^{-1}$. By construction of b , using the indecomposability of the matrices, we find that $p_b < p_a$. Continue the process with b in the same way. Since p is strictly declining during the process, we cannot return to an "old" matrix, since the prices in the stationary state will be again as in the old situation. So we have no cycles and we have a finite number of alternative matrices, and eventually the process will end. Let us use again a_o, a to denote a matrix with the property that if we take the prices generated - p_a and use them for evaluating alternative processes, no process can produce one unit of i with cost smaller than p_{ai} . So for any alternative process b_i of producing good i ,

$$\lambda p_{ai} \leq \lambda \tilde{p}_{bi} = b_{oi} + p_a b^i ,$$

$$\lambda p_{ai} - p_a b^i \leq b_{oi} ,$$

or in matrix notation, for any alternative matrix b ,

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9. The indecomposability guarantees that eventually, taking all feedbacks (or even n feedbacks are enough), all prices will decline. We use the fact that if a is indecomposable, then

$$I + a + a^2 + \ldots + a^{n-1} > 0 .$$

$$p_a(\lambda I-b) \leq b_o \quad .$$

Let us multiply both sides by the matrix of positive elements $(\lambda I-b)^{-1}$; we get

$$p_a \leq b_o(\lambda I-b)^{-1} = p_b \quad .$$

So p_a is minimum and not even one component may have a lower value with other matrices. We should show that this is minimum not only among the matrices which are composed of pure activities but also among those that are a convex combination of pure activities. For this, it is enough to prove that if we take a convex combination of the minimizing a_o, a and any other b_o, b , the prices generated will be no less than p_a :

$$p_a = a_o(\lambda I-a)^{-1} \quad ,$$

$$p_b = b_o(\lambda I-b)^{-1} \quad .$$

The prices generated by convex combination $0 < \mu < 1$ are the solution of

$$\pi[\lambda I - \mu a - (1-\mu)b] = \mu a_o + (1-\mu)\, b_o \quad .$$

By construction of a , $p_a(\lambda I-b) \leq b_o$;

$$\begin{aligned} \pi(\lambda I - \mu a - (1-\mu)\, b] &\geqq \mu a_o + (1-\mu)\, p_a\, (\lambda I-b) \\ &= \mu p_a\, (\lambda I-a) + (1-\mu)\, p_a\, (\lambda I-b) \\ &= p_a\, [\lambda I - \mu a - (1-\mu)\, b] \; . \end{aligned}$$

Multiplying both sides by the matrix of positive elements $[\lambda I - \mu a - (1-b)b]^{-1}$, we get

$$\pi \geqq p_a .$$

So we have proved that in each rate of interest r there exists a matrix a composed of pure activities which will minimize prices in terms of wages, or rather, maximize real wages.

As we change the rate of interest r or the rate of discount λ , we may of course switch from one matrix to another. Looking at some good i , at a certain rate of interest we use activity a_{oi}, a^i ; then we may switch to another activity b_{oi}, b , and so on [10]. If $p(\lambda)$ is the transformed factor-price frontiers, each component $p_i(\lambda)$ is a monotonic decreasing pricewise differentiable function. For b_o, b to be used, we need $\lambda p_i(\lambda) - p(\lambda)\, b^i \geq b^o$. Now the function $\lambda p_i(\lambda) - p(\lambda)\, b^i$ is not necessarily monotonic in the relevant domain and we may get few changes of the sense of the inequality. It is quite possible to get Ruth Cohen's curiosum [11] that some activity is used at a certain rate of interest, and as we reduce the rate of profit, we switch to another activity, but eventually as we reduce it further we return to the old technique. In the following we prove that though it is quite possible that we could use some

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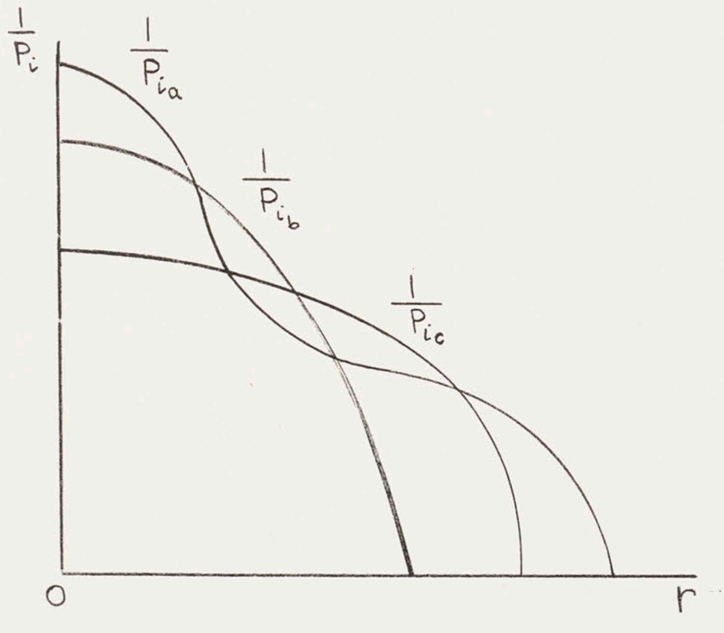
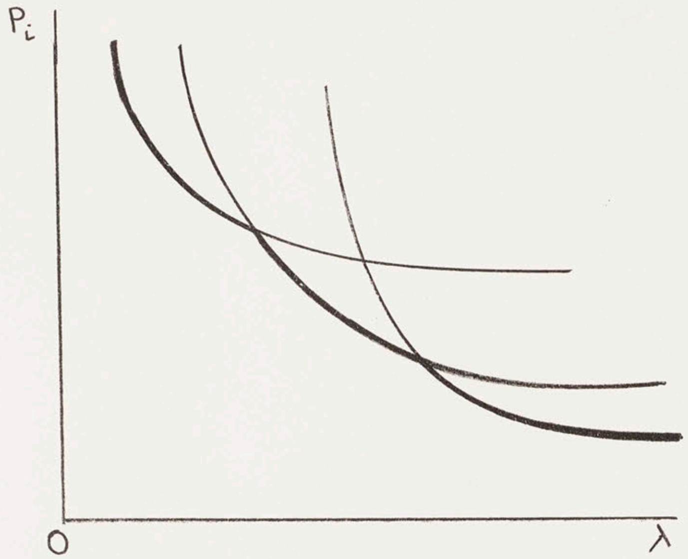
10. See the discussions of:
Robinson, Joan, The Accumulation of Capital, 1956;
Sraffa, Piero, Production of Commodities by Means of Commodities, 1960;
McManus, M., "Process Switching in the Theory of Capital", Economica, May, 1963.

11. Robinson, Joan, op. cit., (London, Macmillan, 1956).

technique of producing good i at a high rate of interest, then switch to another technique and, as we reduce it further, return to the old technique, this cannot happen with the whole matrix. We cannot have curiosum with the whole base. If at a certain rate of interest we use a_o, a and as we reduce interest we switch to b, b_o, we cannot as we reduce it further return to the old matrix. In other words, for each i, if the factor-price frontier of matrix a dominates all other factor-price frontiers, and then at a certain rate of interest there is a switch and another matrix b has a factor-price frontier which dominates, and then c, d, ..., etc., we cannot have a sequence of the type a, b, c, ..., a. We cannot have the situation represented in the figure below, in which we use a at a high rate of interest, then b at a lower rate, then c, and then return to a.



Alternatively, we can carry the discussion in the space of λ and p_i. The "invisible hand" will minimize p_i for given λ, so we shall be on the inner envelop.



[$p_{ia}(\lambda)$ or $p_{ib}(\lambda)$ have poles at $\frac{1}{1+r_a^*}$ and $\frac{1}{1+r_b^*}$ respectively.]

If a is to be used with certain λ we must get

$$a_o(\lambda i - a)^{-i} \ (\lambda I - b) \leq b_o \quad ,$$

$$a_o(\lambda I - a)^{-1} \ (\lambda I - a + a - b) \leq b_o \quad ,$$

$$a_o + a_o(\lambda I - a)^{-1} \ (a - b) \leq b_o \quad ,$$

$$a_o(\lambda I - a)^{-1} \ (a - b) \leq b_o - a_o \quad .$$

We have shown already that all the elements of $a_o(\lambda I - a)^{-1}$ are monotonic decreasing functions of λ. To show it in another way, we can use the following fact: if the elements of the matrix are functions of parameter $\theta, (a_{ij}(\theta))$, then taking the derivative of the identity $a(\theta)a^{-1}(\theta) = I$ we find

$$\frac{da(\theta)}{d\theta} a^{-1}(\theta) + a(\theta) \frac{d}{d\theta} a^{-1}(\theta) = 0$$

and

$$\frac{d}{d\theta} a^{-1}(\theta) = -a^{-1}(\theta) \frac{d}{d\theta} a(\theta)\, a^{-1}(\theta) \quad .$$

Using this for $a_o(\lambda I - a)^{-1}$ we get

$$\frac{d}{d\lambda} a_o (\lambda I - a)^{-1} = -a_1(\lambda I - a)^{-2} < 0 \quad .$$

Now, for two positive indecomposable matrices there exists a semipositive vector x such that either $(a-b)x > 0$, $(a-b)x < 0$, or $(a-b)x = 0$. The condition has some economic meaning; there exists some activity level x such that we need more circular capital of <u>all</u> goods either with a or with b , or we are indifferent. To show this we can observe that a and b can be looked upon as the output and input matrices of a Von Neumann model. As we know, there exists a maximum rate of expansion α and activity level x^* such that $ax^* = \alpha\, bx^* > 0$, $x^* \geq 0$; if $\alpha > 1$, then $ax^* - bx^* > 0$; if $\alpha < 1$, $bx^* = \frac{1}{\alpha} ax^*$, $bx^* - ax^* > 0$; if $\alpha = 1$, then $ax^* - bx^* = 0$. As we have seen, for a to be used we must have

$$a_o(\lambda I - a)^{-1} (a-b) \leq b_o - a_o \quad ,$$

or if we multiply both sides by the semipositive column vector x^* , we find that it is necessary that

$$a_o(\lambda I - a)^{-1} (a-b)\, x^* \leq (b_o - a_o)\, x^* .$$

Let us call the scalar function on the left $\psi(\lambda)$. $\psi'(\lambda) = -a_o(\lambda I - a)^{-2}(a-b)x^*$, $a_o(\lambda I - a)^{-2} > 0$. $\psi'(\lambda)$ is therefore either monotonic increasing or decreasing, according to whether $(a-b)x^* > 0$ or $(a-b)x^* < 0$, or it may be a constant if $(a-b)x^* = 0$. In a case in which a has been used and we switch with higher λ to another matrix b, $\psi'(\lambda)$ must be negative at the switch point. So by the fact that if it is negative for certain λ it is always negative, a cannot return. The case $\psi'(\lambda) > 0$ prevents symmetrically the return of b if it has been replaced by b. If $\psi'(\lambda) = 0$, $\psi(\lambda) = \text{Const.}$, we get either identity or dominance of the factor-price frontier. If Const. $< (b_o - a_o)x^*$, a is always preferred over b since if b at certain λ had dominant factor-price frontiers, by the indecomposability this should also hold for the basket ax^* (or bx^*) and Const. $> (b_o - a_o)x^*$. If Const. $> (b_o - a_o)x^*$, a is never used. If Const. $= (b_o - a_o)x^*$, both factor-price frontiers are identical. Let us now take the matrices a, b, ..., . We can order them according to their Frobenius roots $\lambda_a \leq \lambda_b \leq \ldots$. For corresponding high rates of interest, we use matrix a. This is obvious for the range $\lambda_a < \lambda \leq \lambda_b$. For $\lambda_b < \lambda$ we may eventually switch to b or c, and so on. If we switch, we can disregard as we increase λ all the matrices that have been replaced once. As λ is increased to 1, rate of interest to zero, we return to the static situation under which we have the known nonsubstitution theorem. In this situation the pipes are so filled that labor is the only thing to be saved and the "invisible hand" will minimize labor usage for each specified final demand.

CHAPTER VII

APPLICATIONS OF VARIATIONAL METHODS TO OPTIMAL GROWTH POLICIES

In recent years there have been intensive developments in variational methods. We shall attempt to throw some light on problems of economic growth discussed by Ramsey and, in multisectoral economy, by Samuelson and Solow [1]. We shall start by applying the Pontryagin [2] Maximum Principle to Ramsey's discussion and then to those of Samuelson and Solow. Later in this chapter we shall apply ordinary calculus for a solution of discrete analogues of some of these problems.

The Ramsey Model as an Optimal Control Process

Let $C(t)$ = consumption,

$K(t)$ = capital good,

$f(K,t)$ = production function with some form of technological change, and

$u(C,t)$ = instantaneous utility with some form of time preference.

With $K(0)$ and $K(t_1)$ given, maximize

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1. Ramsey, F.P., "A Mathematical Theory of Savings", Economic Journal, Vol. 38, December, 1928, p. 543.
   Samuelson, P.A., and R.M. Solow, "A Complete Capital Model Involving Heterogeneous Capital Goods", Quarterly Journal of Economics, November, 1956, p. 537.
   Samuelson, P.A., "Efficient Paths of Capital Accumulation in Terms of Calculus of Variations", in Arrow, Karlin and Suppes, ed., Mathematical Methods in the Social Sciences (Stanford University Press, 1960).

2. Pontryagin, L.S., V.G. Boltyanskii, R.V. Gamkrelidze and E.F. Mischenko, The Mathematical Theory of Optimal Control Processes (New York, Interscience, John Wiley and Sons, 1962).

$$\int_0^{t_1} u[C(t),t)dt$$

under the constraint $\dot{K} = f(K,t) - C(t)$.

Let us form the Hamiltonian H :

$$H = - u(C,t) + \mu(t)[f(K,t) - C(t)] \quad ,$$

where $\mu(t)$ is an auxiliary variable (or Lagrange multiplier). Our control is $C(t)$.

Using the Maximum Principle, the necessary conditions for the optimal path are:

$$\frac{\partial H}{\partial C} = 0 \ , \qquad - \frac{\partial u}{\partial C} - \mu(t) = 0 \ , \qquad - \mu(t) = \frac{\partial u(C,t)}{\partial C} \quad ,$$

$$\dot{\mu}(t) = - \frac{\partial H}{\partial K} = -\mu(t) \ f'(K,t) \quad .$$

$$- \frac{\dot{\mu}(t)}{\mu(t)} = f'(K,t) \quad ,$$

or

$$- \frac{\frac{d}{dt} \frac{\partial u}{\partial C}}{\frac{\partial u}{\partial C}} = f'(K,t) \quad .$$

We get the usual solution. However, since the equation is not autonomous (time appears explicitly), we cannot get the first integral. $-\mu(t)$ can be interpreted as the price of capital at time t ; $\mu(t) = \frac{\partial u}{\partial C}$) shows that at time t on the optimal path, the producer has the alternative to consume and get $\frac{\partial u}{\partial C}$ extra utility, or to save and get a value of $\mu(t)$ of capital. The condition

$$-\frac{\frac{d\mu(t)}{dt}}{\mu(t)} = f'(K,t)$$

can be interpreted as the condition of zero profitability on the optimal path. $f'(K,t)$ is the own rate of interest at t, denoted by r. We find

$$r + \frac{\frac{d(-\mu(t))}{dt}}{-\mu(t)} = 0 \quad .$$

So on the optimal path the profit on capital is zero, i.e., the decline of the price of capital caused by accumulation or by approaching the horizon exactly compensates the income-derived r.

It seems that the dual price interpretation given by Bliss [3] can be easily extended to models with time preference and disembodied technological change.

An alternative formulation of the Ramsey problem is not to fix the terminal condition in the form of $K(t_1)$, but to assume that the present generation has some utility function defined by the bequest it leaves to future generations at t_1. We have $\phi[K(t_1)]$, the utility of the present generation or the present "planning commission" defined by terminal capital. Again, of course, the Euler equation holds, since whatever $K(t_1)$ we choose, the path $K(0) - K(t_1)$ should be optimal in the previous sense.

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3. Bliss, C., "Duality and the Ramsey Model", unpublished notes, 1963.

$$\text{Max:} \int_0^{t_1} u\,(C(t))\,dt + \phi(K(t_1))\ ;$$

$$\text{Max:} \int_0^{t} u(f(K)-\dot{K})dt + \phi(K(t_1))\ .$$

In addition to the Euler equation we get a condition to determine the constant of integration, which takes the form

$$u_{\dot{K}}(f(K)-\dot{K})\ \delta K + \frac{\partial \phi}{\partial K}\,\delta K \Big|_{t=t_1} = 0\ ,$$

or, since δK_1 is arbitrary,

$$u_{\dot{K}}(f(K)-\dot{K}) + \frac{\partial f}{\partial K} = 0\ , \quad t = t_1\ ,$$

$$-\frac{\partial u}{\partial C}\ \frac{\partial C}{\partial \dot{K}} + \frac{\partial f}{\partial K} = 0\ .$$

We get the obvious condition

$$\frac{\partial u}{\partial C} = \frac{\partial f}{\partial K_1}\ , \quad t = t_1\ .$$

The marginal utility from extra consumption should be the same as the marginal utility of an extra unit of terminal capital.

It is easy to see that in the control formulation we get

$$-\mu(t_1) = \frac{\partial \phi}{\partial K_1}\ ,$$

i.e., the price of capital at the terminal point should equal the marginal utility of terminal capital.

If we have the price of the capital at t_1, $p_K(t_1)$, i.e., $\phi[K(t_1)] = p_K(t_1)\, K(t_1)$, we set in the control formulation the transversality condition $-\mu(t_1) = p_K(t_1)$. We solve the equations

$$\dot{\mu}(t) = -\frac{\partial H}{\partial K}, \qquad \mu(t) = \frac{\partial u}{\partial C}, \qquad \dot{K} + C = f(K),$$

with $K(0)$ given; and $\mu(t_1) = -p(t_1)$.

A Formulation of Samuelson and Samuelson-Solow Articles as Optimal Control Processes [4]

There is an advantage in formulating efficient paths in terms of the Pontryagin Maximum Principle even in cases which can be handled as well by means of the classical calculus of variations, particularly since we get natural price valuation of capital goods. Moreover, we get exactly the same equations describing the optimal paths for the dual problems of maximum terminal capital S_1 (with given $S_2, \ldots, S_n$) in fixed time or maximum time for given S_1.

From instantaneous efficiency conditions we get the transformation locus

$$\dot{S}_1 = f[S_1(t), \ldots, S_n(t), \quad \dot{S}_2, \ldots, \dot{S}_n).$$

Let our controls be:

$$\dot{S}_2 = \mu_2, \ldots, \dot{S}_n = \mu_n .$$

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4. Samuelson, P.A.; P. A. Samuelson and R.M. Solow, op. cit., reference [1].

Maximize

$$S_1(t_1) = \int_0^{t_1} f(S_1, \ldots, S_n, \dot{S}_2, \ldots, \dot{S}_n)dt$$

with given initial capital goods $S_1(0), \ldots, S_n(0)$,

terminal capital goods $S_2(t_1), \ldots, S_n(t_1)$.

Let us form the Hamiltonian:

$$H = \psi_1(t)\, f(S;\mu) + \sum_{i=2}^{n} \psi_i\, \mu_i \quad .$$

The necessary conditions for the optimal path:

$$\frac{\partial H}{\partial \mu_j} = 0 \ , \qquad j = 2, \ldots, n \ ,$$

or

$$\left.\begin{array}{l} \psi_1(t) \dfrac{\partial f}{\partial \dot{S}_2} + \psi_2(t) = 0 \\ \vdots \\ \psi_1(t) \dfrac{\partial f}{\partial \dot{S}_n} + \psi_n(t) = 0 \end{array}\right\}$$

$$\psi_2(t) = -\psi_1(t) \frac{\partial f}{\partial \dot{S}_2} \quad .$$

If we give $-\psi_1(t), -\psi_2(t), \ldots, -\psi_n(t)$ a price-path interpretation (for the corresponding capital goods), we have an ordinary value-of-marginal-product = price-of-factor relationship. The auxiliary variables should satisfy the equations:

$$\dot{\psi}_1(t) = - \frac{\partial H}{\partial S_1} = -\psi_1(t) \frac{\partial f}{\partial S_1}$$

$$\dot{\psi}_2(t) = - \frac{\partial H}{\partial S_2} = -\psi_1(t) \frac{\partial f}{\partial S_2}$$

$$\vdots$$

$$\dot{\psi}_n(t) = - \frac{\partial H}{\partial S_n} = -\psi_1(t) \frac{\partial f}{\partial S_n} \ .$$

From the equation

$$\psi_j(t) = -\psi_1(t) \frac{\partial f}{\partial \dot{S}_j} \ ,$$

we get by differentiation:

$$\dot{\psi}_j(t) = -\dot{\psi}_1(T) \frac{\partial f}{\partial \dot{S}_j} - \psi_1(t) \frac{d}{dt} \frac{\partial f}{\partial \dot{S}_j} = \psi_1(t) \frac{\partial f}{\partial S_j} \frac{\partial f}{\partial \dot{S}_j} - \psi_1(t) \frac{d}{dt} \frac{\partial f}{\partial \dot{S}_j}$$

$$- \psi_1(t) \quad \frac{\partial f}{\partial S_j} = \psi_1(t) \frac{\partial f}{\partial S_j} \frac{\partial f}{\partial \dot{S}_j} - \psi_1(t) \frac{d}{dt} \frac{\partial f}{\partial \dot{S}_j} \ .$$

We then get

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{S}_j} = \frac{\partial f}{\partial S_j} \frac{\partial f}{\partial \dot{S}_j} + \frac{\partial f}{\partial S_j} , \quad j = 2, \ldots, n \ ,$$

which is the fundamental efficiency equation (equation 5) of Samuelson.

We have a system of 2n differential equations of first degree, one equation for each state variable S_j , and one for each auxiliary variable ψ_j . In general we need 2n conditions. We have n initial conditions $S_1(0), \ldots, S_n(0)$, (n-1) terminal conditions $S_2(0), \ldots, S_n(t_1)$, and we get an additional transversality condition $-\psi_1(t_1)=1$. (Our prices are normalized in this way.)

Solving the system we get the optimal program of capital accumulation and the dual price relations $-\psi_1(t) \ldots -\psi_n(t)$.

Had we given these prices to maximizing producers, we would have gotten the optimal growth path. Instead of giving the target Max $S_1(t_1)$, we could have Max $\sum C_i S_i(t_1)$, the equation of optimal path remaining the same, but instead of terminal conditions for $S_2(t_1) \ldots S_n(t_1)$ we have the transversality conditions $-\psi_j(t_1) = C_j$.

The own rate of interest of good 1 is $\frac{\partial f}{\partial S_1} = r_1 = \frac{\dot{\psi}_1(t)}{\psi_1(t)}$. For good 2 it is $\frac{\partial \dot{S}_2}{\partial S_2} = - \frac{\dot{\psi}_2(t)}{\psi_2(t)} = r_2$, and so on, and for the price ratio, we find $\frac{\partial \dot{S}_1}{\partial \dot{S}_2} = - \frac{\psi_2(t)}{\psi_1(t)}$. From the fundamental relation, we find

$$\frac{\frac{d}{dt}\left(\frac{\psi_2}{\psi_1}\right)}{\frac{\psi_2(t)}{\psi_1(t)}} = r_1 - r_2 \ .$$

Denoting $p_2 = \frac{\psi_2}{\psi_1}$,

$$r_2 = r_1 + \frac{1}{p_2} \frac{dp_2}{dt} \ .$$

Moreover, we see that just as in the Ramsey model, we get a condition of zero profitability:

$$r_1 + \frac{\dot{\psi}_1}{\psi_1} = 0 \quad ,$$

$$r_2 + \frac{\dot{\psi}_2}{\psi_2} = 0 \, ,$$

and so on, i.e., own rate of interest exactly compensates for the decline of the price of the capital good.

Generalization of the Ramsey problem to the n-goods case is

$$\text{Max} \int_0^{t_1} u[C_1(t) \ldots C_n(t)] \, dt \, ,$$

subject to

$$C_1 + \dot{S}_1 = f(S_1 \ldots S_n, \; C_2 + \dot{S}_2, \ldots, C_n + \dot{S}_n) \quad ,$$

and the same initial and terminal conditions as before,

$$\dot{S}_2 = \mu_2, \ldots, \; \dot{S}_n = \mu_n \quad .$$

Our controls are

$$C_1 \ldots C_n \quad ,$$

$$\mu_2 \ldots \mu_n \quad .$$

Form the Hamiltonian:

$$H = - u(C_1(t), \ldots, C_n(t) + \psi_1(t)[f(S_1, C+\mu)-C_1] + \psi_2(t) \, \mu_2 + \ldots + \psi_n(t)\mu_n \; ;$$

$$\frac{\partial H}{\partial C_1} = - \frac{\partial u}{\partial C_1} - \psi_1(t) = 0$$

$$- \psi_1(t) = \frac{\partial u}{\partial C_1} ,$$

$$\frac{\partial H}{\partial C_2} = - \frac{\partial u}{\partial C_2} + \psi_1(t) \frac{\partial f}{\partial (C_2 + \dot{S}_2)} = 0$$

$$\frac{\partial H}{\partial \mu_2} = \frac{\partial H}{\partial \dot{S}_2} = \psi_1(t) \frac{\partial f}{\partial (C_2 + \dot{S}_2} + \psi_2(t) = 0$$

— — — — — — — — — — — —

$$\frac{\partial H}{\partial \mu_n} = \frac{\partial H}{\partial \dot{S}_n} = \psi_1 \frac{\partial f}{\partial (C_n + \dot{S}_n)} + \psi_n(t) = 0 .$$

We get

$$\frac{\partial u}{\partial C_2} = - \frac{\partial u}{\partial C_1} \frac{\partial f}{\partial (C_2 + \dot{S}_2)} ,$$

$$\dot{\psi}_1 = - \frac{\partial H}{\partial S_1} = - \psi_1(t) \frac{\partial f}{\partial S_1} ,$$

$$\frac{d}{dt} \frac{\partial u}{\partial C_1} = - \frac{\partial u}{\partial C_1} \frac{\partial f}{\partial S_1} ,$$

$$\dot{\psi}_2 = - \frac{\partial H}{\partial S_2} = - \psi_1(t) \frac{\partial f}{\partial S_2} .$$

We finally find:

$$\frac{\partial f}{\partial (C_2 + \dot{S}_2)} \frac{d}{dt} \frac{\partial u}{\partial C_1} = \frac{\partial u}{\partial C_1} \frac{\partial f}{\partial S_2} ,$$

and for any i ,

$$\frac{\partial f}{\partial (C_i + \dot{S}_i)} \frac{d}{dt} \frac{\partial u}{\partial C_1} = \frac{\partial u}{\partial C_1} \frac{\partial f}{\partial S_i} , \quad i = 1, 2, \ldots, n ,$$

which are the differential equations which Solow and Samuelson get. Initial conditions are $S_1(0) \ldots S_n(0)$. Terminal conditions are $S_1(t_1) \ldots S_n(t_1)$.

The part of the system:

$$\frac{\partial H}{\partial \mu_2} = \psi_1(t) \frac{\partial f}{\partial \dot{S}_2} + \psi_2(t) = 0$$
$$\vdots$$
$$\frac{\partial H}{\partial \mu_n} = \psi_1(t) \frac{\partial f}{\partial \dot{S}_n} + \psi_n(t) = 0$$

$$\dot{\psi}_1 = - \psi_1(t) \frac{\partial f}{\partial S_1}$$
$$\vdots$$
$$\dot{\psi}_n = - \psi_1(t) \frac{\partial f}{\partial S_n}$$

is exactly the same as in the problem discussed by Samuelson, and from this we can easily get the fundamental efficiency relationship in the same way that we did earlier. It is clear intuitively that this should be the case for whatever program of consumption we decide on; for given $S_2(t), \ldots, S_n(t)$ at any point in time, we must have $\max S_1(t)$.

Again, we could have a problem in which the terminal conditions are not given, but instead we have some valuation of the future. In the simplest form this may be given in the form of prices of capital goods at t_1 :

$$p_1(t_1) \ldots p_n(t_1) \quad .$$

Then we try to maximize

$$\int_0^{t_1} u(C_1 \ldots C_n)\, dt + \sum p_i(t_1)\, S_i(t_1) \quad .$$

Instead of the terminal conditions we get transversality conditions

$$- \psi_j(t_1) = p_j(t_1) \quad .$$

Of course, again we have zero profitability on the optimal path.

It is of some interest to note that if we interpret $-\psi_j(t)$ as the price of good j and $-H$ as NNP , then according to the Maximum Principle, $-H$ is constant. So if we use the optimal program, the decline of capital goods prices is such that it exactly compensates the increase in utility $u(C_1 \ldots C_n)$.

A case which can be dealt with computationally as well as analytically is that of a quadratic utility function and linear input-output relationships. Since a quadratic utility function implies negative marginal utilities for big enough C , we shall assume that we use this function only as an approximation in certain ranges.

Let $x(t)$ denote the vector of n capital goods and $C(t)$ the vector of consumption. Let B be the $n \times n$ matrix and d' the row vector $(1 \times n)$.

$$u[C(t)] = C' BC + d'C \quad ,$$

$$\text{Max} \int_{0}^{t_1} (C' BC + d'C)dt,$$

subject to $C + \frac{dx}{dt} = Ax$ [5]. The initial conditions are $x(0)$ and the terminal conditions are $x(t_1)$.

$$H = -(C, BC + d) + Ax, \psi) - (C, \psi) \quad ,$$

where (a,b) denotes the scalar product of the vectors a, b.

We transform by similarity transformation to new variables y. We assume for simplicity that all roots of matrix A are distinct so that A is similar to a diagonal matrix. (It may happen, of course, that this will be true only in the complex field.)

$$\mathring{y} = \Lambda y(t) + FC(t) .$$

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5. Assume that we get dynamic Leontief input-output system $c = x - ax - b\mathring{x}$ if b is not singular:

$$\mathring{x} = b^{-1}(I-a)x - b^{-1} c \quad ,$$

$$b^{-1}(I-a) = A \quad ,$$

and call the vector of transformed consumption

$$C = b^{-1} c \quad .$$

Let $r(t)$ be the auxiliary variables:

$$H = -(C,BC+d) + (\Lambda y,r) + (FC(t),r)$$

$$\frac{\partial H}{\partial C} = -(2BC+d) + F'r(t) = 0$$

$$2BC + d = F'r(t)$$

$$C(t) = \frac{1}{2} B^{-1} F'r(t) - \frac{1}{2} B^{-1}_{d}$$

$$\dot{r}(t) = -\frac{\partial H}{\partial y} = -\Lambda' r(t)$$

$$\dot{r}_i(t) = -\lambda_i r_i , \quad r_i(t) = e^{-\lambda_i t} p_i , \quad p_i = r_i(0) .$$

The solution of this system of equations:

$$r(t) = e^{-\Lambda' t} p$$

$$c(t) = \frac{1}{2} B^{-1} F' e^{-\Lambda t} p - \frac{1}{2} B^{-1} d$$

$$y(t) = e^{\Lambda t} y(0) + e^{\Lambda t} \int_{0}^{t_1} e^{-\Lambda \tau} FC(\tau) d\tau .$$

Hence:

$$y(t) = e^{\Lambda t} y(0) + e^{\Lambda t} \int_{0}^{t_1} (\frac{1}{2} e^{-\Lambda\tau} FB^{-1} F^{1} e^{-\Lambda\tau} p - \frac{1}{2} C'd) dt .$$

Call $M = F^{-1} CF$:

$$e^{-\Lambda t} = \begin{bmatrix} e^{-\lambda_1 \tau} & & 0 \\ & \ddots & \\ 0 & & e^{-\lambda_n \tau} \end{bmatrix}$$

$$e^{-\Lambda \tau} M e^{-\Lambda \tau} = \left[m_{ij} \, e^{-(\lambda_i + \lambda_j) \tau} \right] .$$

Using the fact that $A(t)\, a\, dt = A(t)\, dt\, a$, where $A(t)$ is the matrix, a the vector of constants, we get:

$$w_{ij}(t) = \frac{m_{ij}}{\lambda_i + \lambda_j} (1 - e^{-(\lambda_i + \lambda_j)t} ,$$

$$y(t) = e^{\Lambda t} y(0) + \frac{1}{2} e^{\Lambda t} w(t) \rho - \frac{1}{2} B^{-1} dt ,$$

$$y(t_1) = e^{\Lambda t_1} [y(0) + w(t_1)\rho) - \frac{1}{2} B^{-1} dt .$$

We solve for ρ and we have a complete optimal program in terms of y and r . We transform back to $x(t)$ and $\psi(t)$ and we have the optimal consumption path $C(t)$ and optimal capital expansion $x(t)$. In this case of a Leontief type of production and quadratic utility function, the actual calculation seems quite simple with present computational facilities. The main problem is of course to find characteristic roots of A . It seems that this simple example can serve also for inquiry into sensitivity and other properties of the actual optimal path.

Application to Fiscal Policy

Problems of a different sort to which these techniques may be applied is the problem of optimal fiscal and monetary policies [6]. Econometric models generally have the structure

$$x(t) = Ax(t) + Bx(t-1) + Cz(t) \quad ,$$

with $x(t)$ = endogenous variables,

$z(t)$ = exogenous variables.

After estimation of matrices A, B, C, we solve for the endogenous variables in terms of the predetermined variables:

$$(I-A)\,x(t) = Bx(t-1) + Cz(t)$$

$$x(t) = (I-A)^{-1}\,Bx(t-1) + (I-A)^{-1}\,Cz(t) \quad .$$

We get a system of difference equations describing the path of the endogenous variables under the forcing functions $z(t)$. Let us assume that the functions $z(t)$ are controllable. (Clearly we have ignored the random elements, and moreover, $z_j(t)$ may be erratically uncontrollable, but we assume that $z(t)$ is composed of elements such as government expenditures, taxation, money supply, etc.) We shall use a continuous analog of the system of different equations:

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6. Theil, H., Economic Forecasts and Policy, 2nd ed. (Amsterdam, North-Holland Publishing Co., 1961).

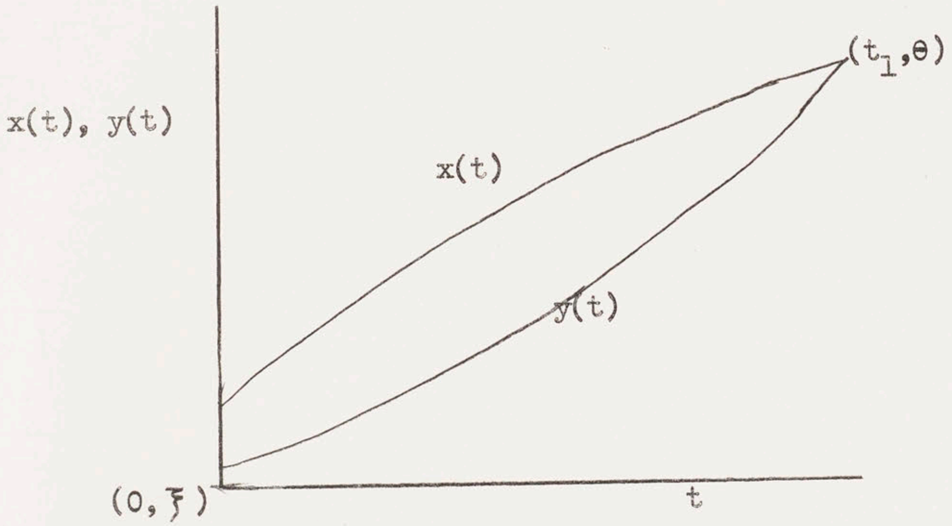
   Holt, Charles C., Linear Decision Rule for Economic Stabilization and Growth (Pittsburgh, Graduate School of Industrial Administration, Carnegie Institute of Technology, 1960).

$$\dot{X}(t) = Mx(t) + Nz(t) \quad .$$

Assume that the policymaker has some functions $y(t)$ $(n \times 1$ vector) such that $y(0) = \xi \; y(t_1) = \theta$ and that he wants to minimize:

$$\int_0^{t_1} \sum g_i[x_i(t) - y_i(t)]^2 \, dt + \int_0^{t_1} \sum c_i z_i^2 \, dt \quad ,$$

with terminal condition $x(t_1) = \theta$.



The first integral shows that the policymaker wants to minimize the weighted squares of the discrepancies between the actual path of the economy $x(t)$ and some designated path $y(t)$. The second integral implies that the policymaker tries to minimize the square of discrepancy $x(t) - y(t)$ without using too much control because of some disutility that society attaches to control. (We include, of course, the special case $C_i = 0$.)

Denote as (a, b) the scalar product of vectors a and b :

$$G = \begin{bmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_n \end{bmatrix} \qquad C = \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_r \end{bmatrix}$$

$$\text{Min:} \int_0^{t_1} [x(t)-y(t),\ Gx(t)-y(t)] + (z,Cz)\ dt \quad .$$

Let us assume for simplicity no constraints of z . We form the Hamiltonian H and use the Pontryagin theorem to find the optimal controls and the behavior of the system under these controls:

$$H = \{x(t)-y(t),\ G[x(t)-y(t)]\} + (z,Cz) + (Mx,\psi) + (Nz,\psi)$$

$$\frac{\partial H}{\partial z} = 2Cz + N'\psi$$

$$z(t) = -\frac{1}{2} C^{-1} N^1 \psi(t)$$

$$\mathring{X}(t) = Mx(t) - \frac{1}{2} NC^{-1} N^1 \psi(t)$$

$$\mathring{\psi}(t) = -\frac{\partial H}{\partial X} = -M^1 \psi(t) - 2G(x-y)$$

$$\mathring{\psi}(t) = -M^1 \psi(t) - 2Gx(t) + 2Gy(t) \quad .$$

In matrix form we get the system:

$$\begin{pmatrix} \dot{x} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} M & -\frac{1}{2} NC^{-1} N' \\ -2G & -M' \end{pmatrix} \begin{pmatrix} x \\ \psi \end{pmatrix} + \begin{pmatrix} 0 \\ 2Gy \end{pmatrix}$$

We solve the linear system with $2n$ equations and $2n$ boundary conditions $x(0) = \xi_1$, $x(t_1) = \theta$.

Since boundary conditions are at two different points, we set $\psi(0) = \pi$ and eventually solve for π . $2Gy$ are the forcing functions of this system.

Let $\phi(t)$ be the $2n \times 2n$ fundamental matrix of solutions:

$$\begin{pmatrix} x(t) \\ \psi(t) \end{pmatrix} = \phi(t) \binom{\xi}{\pi} + \phi(t) \int_0^t \phi^{-1}(\tau)\, 2Gy(\tau) d\tau$$

$$\phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} \qquad \phi^{-1}(t) = \begin{bmatrix} \mathcal{X}_{11}(t) & \mathcal{X}_{12}(t) \\ \mathcal{X}_{21}(t) & \mathcal{X}_{22}(t) \end{bmatrix}$$

$$x(t) = \phi_{11}(t)\xi_1 + \phi_{12}\pi + 2\phi_{11}(t) \int_0^t \mathcal{X}_{12}(\tau)Gy(\tau)d\tau + 2\phi_{12}(t) \int_0^t \mathcal{X}_{22}(\tau)Gy(\tau)d\tau$$

$$\psi(t) = \phi_{12}(t)\xi + \phi_{22}\pi + 2\phi_{12}(t) \int_0^t \mathcal{X}_{12}(\tau)Gy(\tau)d\tau + 2\phi_{22}(t) \int_0^t \mathcal{X}_{22}(\tau)Gy(\tau)d\tau .$$

Now we can solve for π (in terms of the initial conditions):

$$\theta = x(t_1) = \phi_{11}(t_1)\xi + \phi_{12}(t_1)\pi + 2\phi_{11}(t_1)\int_0^{t_1} \chi_{12}(\tau)Gy(\tau)d\tau$$

$$+ 2\phi_{12}(t_1)\int_0^{t} \chi_{22}(\tau)Gy(\tau)d\tau$$

$$\pi = \phi_{12}^{-1}(t_1)[\theta - \phi_{11}(t_1)\xi - 2\phi_{11}(t_1)\int_0^{t_1} \chi_{12}(\tau)Gy(\tau)d\tau - 2\phi_{12}(t_1)\int_0^{t_1} \chi_{22}(\tau)Gy(\tau)d\tau]$$

(all this, of course, provided $\phi_{12}^{-1}(t_1)$ exists).

Introducing the value of π, we get the optimal control $z(t)$, and the optimal path of the economy under this control $x(t)$.

Note on Inequalities

The Maximum Principle is specially designed to extend to the case of inequalities in our controls. Let $f_1, \ldots, f_n$ be n production functions of the economy and d_{ij} be the proportion of good j allocated to the production of i, β_i, the amount saved out of the production of good i.

$$\dot{S}_1 = \beta_1 f_n(\alpha_{11}S_1, \alpha_{12}S_2, \ldots, \alpha_{1n}S_n)$$

$$\vdots$$

$$\dot{S}_n = \beta_n f_n(\alpha_{n1}S_1, \alpha_{n2}S_2, \ldots, \alpha_{nn}S_n) ,$$

$$-\delta \leq \beta_i \leq 1 , \qquad \sum_{j=1}^{n} \alpha_{ij} \leq 1 .$$

Instantaneous utility is a function of consumptions $u[(1-\beta_1)C_1, \ldots, (1-\beta_n)C_n)$.

$$\text{Max} \int_0^T u[(1-\beta)C_1, \ldots, (1-\beta_n)C_n]\, dt \quad ,$$

subject to $S(0)$ and $S(T)$.

Again we can form the Hamiltonian,

$$H = - u[(1-\beta)C] + \sum \psi_i(s,t)\, \beta_i f_i \quad ,$$

and our system of equations is:

$$S_i = \frac{\partial H}{\partial \psi_i} , \qquad \dot{\psi}_i = - \frac{\partial H}{\partial S_i} , \qquad \underset{[d_{ij},\, \beta_i]}{\text{Max}}\, H \; .$$

But to explicitly solve even quite simple cases seems impossible, and the only route that may be open is to try computer experiments with all the difficulties of two-boundary problems. The few attempts to use this principle in economics have been in linear systems [7], both in production and in utility, which may be realistic in engineering but not in economics, and our intuition tends to reject bang-bang type policies in economics.

Discrete Maximizations over Time

Much of the theory of the Maximum Principle and similar methods and their application to economics can be employed as well in the discrete case [8].

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7. See, for example, Lionel G. Stoleru, "An Optimal Policy for Economic Growth", paper presented at the autumn, 1963, meeting of the Econometric Society, Boston, Massachusetts.

8. As a matter of fact, historically the approach of Euler was such that he got his differential equation as the limits of the difference equations.

The optimal path will be described by a system of difference equations of the state variables and the Lagrange multipliers. The controls as functions of the state variables and the Lagrange multipliers will again be determined by maximiation of some function H [9].

One of the main differences between the continuous and the discrete case is that in the continuous case we have a cannonical system of differential equations:

$$\dot{x} = \frac{\partial H}{\partial \psi} , \qquad \dot{\psi} = - \frac{\partial H}{\partial x} .$$

This system does not seem to carry over to the discrete case.

The analogy between the discrete and the continuous case is especially simple in the case of the Ramsey-type problem [10]. Consider the sum:

$$\phi = \sum_{t=0}^{n-1} f(t,y_t,p_t) ,$$

where y is a function of t, and $p_t = y_{t+1} - y_t$. We get the necessary conditions for maximizing or minimizing ϕ with respect to y_t by differentiation:

$$\frac{\partial \phi}{\partial y_t} = f_{y_t}(t,y_t,p_t) - f_{p_t}(t,y_t,p_t) + f_{p_t}(t-1, y_{t-1}, p_{t-1}) = 0 .$$

This we may write as $f_y(t) - \Delta f_p(t-1) = 0$ for $0 < t \leq n-1$.

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9. This has been done by this author in an unpublished note.

10. Here we shall follow closely Fort, Thomlinson, Finite Differences and Difference Equations in the Real Domain (Oxford, Clarendon Press, 1948), Chapter 8.

If y_o, y_n is not given from outside, we get "natural boundary conditions" $f_y(o) - f_p(o) = 0$ and $f_p(n-1) = 0$. The similarity of this to the Euler equation is obvious.

For sufficiency, let us denote by a_{ij} $\frac{\partial^2 \phi}{\partial y_i \partial y_j}$ and let us check whether the matrix (a_{ij}) is possitive definite (or negative definite). If $|i-j| > 1$, $a_{ij} = 0$;

$$a_{ii} = \phi_{y_i y_i} = f p_i p_i(i) + f_p(i-1) + 2f_{yp}(i) + f_{yy}(i) \quad ,$$

$$a_{i\,i-1} = f_{yp}(i-1) - f_{pp}(i-1) \quad .$$

We denote by D_j the principal minor composed of the first j rows:

$$D_j = \begin{bmatrix} a_{11} & a_{12} & & & \\ a_{12} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & a_{jj-1} & a_{j} & 0 \end{bmatrix}$$

and define $D_o = 1$. We easily find that:

$$D_j = a_{jj} D_{j-1} - a^2_{jj-1} D_{j-2} \quad .$$

If this second-order difference equation, subject to initial conditions $D_o = 1$, $D(1) = a_n = \phi_{y_1 y_1}$, has a positive solution, we get a minimum. If D_j alternates, we get a maximum.

As an example, let our instantaneous utility function, $-\frac{1}{C^n}$, be given, as are $K(o), K(T)$ and the production conditions, $\dot{K} + C = rK$. We want to maximize

$$- \int_o^T \frac{1}{C^n} \, dt \quad .$$

The Euler equation is

$$\ddot{K} - \frac{n+2}{n+1} r\dot{K} + \frac{r^2}{n+1} K = 0 \quad ,$$

$$K(t) = A_1 e^{rt} + A_2 e^{\frac{t}{n+1}} \quad ,$$

where

$$A_1 = \frac{K(T) - K(o) e^{\frac{r}{n+1}t}}{e^{rT} - e^{\frac{r}{n+1}t}} \quad ,$$

$$A_2 = \frac{K(o) e^{rT} - K(T)}{e^{rT} - e^{\frac{r}{n+1}t}} \quad .$$

Extending the horizon to infinity, i.e., if $T \rightarrow \infty$, we find

$$K(t) = K(o) e^{\frac{r}{n+1}t} \quad ,$$

$$C(t) = r \frac{n}{n+1} K(o) e^{\frac{rn}{n+1}t} \quad ,$$

$$\dot{K} = \frac{r}{n+1} K(t) \quad .$$

The discrete case analogue production obeys $rK_t = C_t + K_{t+1} - K_t$, and our aim is:

$$\underset{[K_t]}{\text{Max}} \sum \frac{1}{[rK_t - (K_{t+1} - K_t)]^n} \quad .$$

Differentiating with respect to K_t we find:

$$K_{t+1} - [(1+r)^{\frac{1}{n+1}} + (1+r)] K_t + (1+r)^{\frac{1}{1+n}} (1+r) K_{t-1} = 0 \quad ,$$

$$K_t = A_1(1+r)^t + A_2(1+r)^{\frac{t}{n+1}} \quad ,$$

$$A_1 = \frac{K(T) - K(o)(1+r)^{\frac{T}{n+1}}}{(1+r)^T - (1+r)^{\frac{T}{n+1}}} \quad ,$$

$$A_2 = \frac{K(o)(1+r)^T - K(T)}{(1+r)^T - (1+r)^{\frac{T}{n+1}}} \quad .$$

If we open the horizon and $T \rightarrow \infty$ we find

$$K_t = K(o)(1+r)^{\frac{t}{n+1}} \quad ,$$

$$C_t = K(o) [(1+r)^{\frac{n}{n+1}} - 1](1+r)^{\frac{t+1}{n+1}} \quad ,$$

$$\Delta K_t = K(o) [(1+r)^{\frac{n}{n+1}} - 1](1+r)^{\frac{t+1}{n+1}} \quad .$$

We see the complete analogy between this system and the previous continuous system. Nevertheless, there remain a few unanswered puzzles. In the continuous autonomous case, when t does not appear explicitly, it is possible, as is well-known, to get the first integral of the system. This is what enables Ramsey to give us policy in the form $\dot{K} = \psi(K)$; knowing K at t we invest $\psi(K)$.

There does not seem to be a discrete case analogous to the first integral of the Euler equation in the autonomous case. We have tried a few examples with different functionals, and while the difference equations we get are analogous to the Euler equations and have the same form, we did not find anything analogous to the first integral. We looked at the difference equation which is analogous to the first integral of the continuous case, but the solution to the second-order "Euler" difference equation did not satisfy this equation. It seems that it is impossible to extend Hamilton theory to the discrete case; and we cannot, in analogy to Ramsey, give our policy rate in the form $K_{t+1} = \psi(K_t)$.

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