Performance of Phase Noisy Optical Systems with Frequency Stabilisation

by

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Abstract

When frequency division multiplexing is used to divide the bandwidth of a single mode optical fiber between many users, the stability and control of the source frequencies becomes an important issue. Frequency stabilisation is required to prevent channel collisions, which occur when frequency drift causes two or more channels to interfere over a range of overlapping frequencies. In addition the phase noise of the laser will result in an increase in the receiver probability of error.

This thesis aims to analyse the effect of frequency feedback stabilisation when it is applied to a phase noisy laser. We are interested in the effect of the feedback on the receiver for FSK modulation. The analysis involves finding the statistics of the phase noise process when feedback is applied. This is then used to find an approximation of the receiver statistics. The probability of error at the receiver can then be calculated as a function of the received signal to noise ratio and the characteristics of the applied frequency feedback.

Thesis Supervisor: Pierre A. Humblet
Title: Professor of Electrical Engineering and Computer Science

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Above all though I wish to thank my Family. My sister for being a friend as we grew up and my parents for their love and support that I can never repay. If I can give back a small measure of what you have given to me I will be happy. Mum Dad this is for you.

Contents

6 Conclusions and Further Research

List of Figures

Chapter 1

Introduction

1.1 Background

The advent of low loss single mode optical fibers coupled with improvements in the performance of semiconductor laser diodes has opened up the field of optical communications. This new technology has already provided gigahertz transmission rates, a factor of ten improvement over electronic communications. The impetus behind this field of research is the still greater data rates which may be achieved. Single mode optical fibers have a low loss low dispersion window that is terahertz wide. This could provide virtually unlimited bandwidth for most applications we consider today. However before we can exploit this tremendous potential there are many problems that must still be addressed.

High data rate electronic signals occupy little more than gigahertz of bandwidth, so there is a need for an efficient bandwidth sharing scheme. At present frequency division multiplexing FDM is an attractive scheme to utilise the transmission bandwidth of an optical fiber. To implement (FDM) there are two major technologies emerging:

1. Direct detection. This uses narrow band optical filters to provide the receiver sensitivity required to pick out a particular signal in the terahertz optical spectrum. If it is to provide high receiver sensitivity, it must also have high gain optical amplifiers to place the received signal above the thermal noise limited performance. [Lec 91].

2. Coherent detection. This is analogous to heterodyne reception in radio. The incoming signal is mixed with the output of a laser at the receiver and the combined signal is incident on the photo detector. This gives an electronic intermediate signal which we filter electronically to to select the signal of interest. A large enough local oscillator power will ensure the received signal is in the shot noise quantum limited regime.

At present direct detection schemes have been implemented [Lil 84] using existing technologies, but coherent detection schemes have a potential to provide increased performance

8

when technology advances [Yam 81]. When FDM is implemented using either direct detection or coherent detection both schemes suffer from laser instability and phase noise. Two main problems arise from the frequency noise of a laser:

1. Drifting of the laser center frequency. In a FDM system each channel occupies a certain bandwidth. If the center frequencies of the transmitting lasers drift with time, channel collisions occur when two different signals occupy the same bandwidth. This makes accurate reception impossible.

2. The associated phase noise. The random phase of the optical carrier broadens the spectrum of the transmitted signal. To receive all the transmitted signal power the detection filters must have a broader bandwidth. This increase the additive noise from the detection process. In addition the phase noise will corrupt data carried in the phase of the transmitted optical signal.

Extensive work has been done on both these problems. To reduce the laser drift various schemes have been proposed. They all require frequency locking of the transmitting lasers.

1. Lock the laser to an external reference frequency. The laser is locked to the emission frequency of some atom or molecule. It has the advantage of being an absolute frequency reference, therefore we are assured that the frequencies of remotely distributed lasers will never coincide. However atomic resonances are not found at uniformly spaced intervals. This means that full use of the entire transmission window will not be possible.

2. Lock the center frequencies to the resonances of a Fabry Perot filter or some other resonator. This provides a uniform spacing between the channels, however the center frequencies will be time varying due to changes in the temperature or stress of the Fabry Perot.

Work on these stabilisation techniques has been done by Glance et al. [Gla 87], [Gla 88] and Li [Li 91] among others. The phase noise problem has also been the subject of many papers and it is the basis of this thesis. Therefore it is discussed in greater depth in the next two sections.

1.2   Phase Noise of a Laser

The line spectrum of a laser has been examined experimentally and found to be a Lorentzian about the center frequency of the laser [Sai 81]. This experimental observation corresponds to the laser having a white frequency noise. Henry [Hen 82] was the first to derive a suitable theory for the white frequency noise of a semiconductor laser. In his model Henry proposes

9

two mechanisms that change the instantaneous phase of the electric field inside the laser. They are the random phase of spontaneously emitted photons, and the change in refractive index of the cavity in response to a change in the electric field density. The phase can now be modeled as a random walk that in the limit of small step size becomes a Wiener process. The frequency noise is the derivative of the phase noise and is a zero mean white Gaussian process. To see how this results in a Lorentzian lineshape of the laser we write the output as

$$s(t) = A\cos(2\pi f_c t + \theta(t) + \phi)$$

$\theta(t)$ is the laser phase noise, f_c is the nominal center frequency of the laser and ϕ is a random initial phase. $\theta(t)$ is given by:

$$\theta(t) = 2\pi \int_0^t n(t)dt$$

where $n(t)$ is the zero mean white Gaussian frequency noise that has power spectral density height σ^2. $\theta(t)$ then has variance:

$$var[\theta(t)] = (2\pi)^2\sigma^2 t$$

To find the line shape of the laser we first require the correlation function of its output. This is given by:

$$R_s(\tau) = \frac{A^2}{2}\cos(2\pi f_c t)e^{-2\pi^2\sigma^2|\tau|}$$

Taking the Fourier transform with respect to τ gives the power spectral density:

$$S_s(f) = \frac{A^2\sigma^2}{4}\left[\frac{1}{(f-f_c)^2 + (\pi\sigma^2)^2} + \frac{1}{(f+f_c)^2 + (\pi\sigma^2)^2}\right]$$

The positive frequency part of this power spectral density is a Lorentzian centered at f_c. It has a 3dB bandwidth equal to $2\pi\sigma^2$. This 3dB bandwidth is called the linewidth of the laser and it is an important parameter to describe the performance of the laser:

$$\beta \stackrel{\triangle}{=} 2\pi\sigma^2$$

β is the unit of choice for experimentalists as it can be directly measured using a spectrum analyser, unlike the height of the PSD of the white frequency noise.

In fact this is a simplified model of the phase noise of the laser. A more accurate analysis would include a $1/f$ noise in the laser power spectral density, and also a carrier relaxation resonance peak around 20 gigahertz. We ignore the $1/f$ noise in our analysis as it has been demonstrated by Kaufmann [Kau 82] that this low frequency noise can be removed

10

by tracking of the laser output. The resonance peak is ignored as a white frequency noise model is inaccurate for large frequencies anyway.

We have seen that the phase noise of the laser causes a broadening of the power spectral density of the laser from an impulse to a Lorentzian. The phase noise will also cause a broadening of the modulated output of the laser. Therefore it is worth noting the different stabilisation techniques that have been proposed to combat the phase noise problem.

1. Double cavity lasers. The second cavity can be implemented with an external reflector to form the second cavity. The reflector will be either a grating [Saito 82] or a plane mirror [Taylor 82]. Light from the laser's output is fed back to the laser cavity using a reflective device, either the plane mirror or a grating.

2. Phase feedback. This has been proposed in a paper by Ho [Ho 86]. A phase discriminator is required to extract phase information from the laser output and a means of changing the output phase of the laser in response to the phase error signal. Anything that we can achieve with a frequency feedback loop can be done with phase feedback, using a different filter in the feedback loop. The difficulty is finding a suitable phase discriminator. In the method proposed by Ho et al. a scanning Fabry Perot interferometer is used. The bandwidth of the feedback loop in this system is limited by the scanning rate of the Fabry Perot.

3. Frequency feedback. This has been proposed and implemented by Glance et al. for FSK modulation [Gla 87], [Gla 88]. A Fabry Perot is used as a frequency discriminator. This produces an error signal proportional to the error in the laser's output frequency that is fed back through an electronic feedback loop. This will provide high pass filtering of the laser frequency noise. The system implemented by Glance et al. performs as a frequency feedback loop when it is analysed using a quasi static analysis. To achieve this performance in the physical system, we are restricted to using low data rates. Other systems which could be used include one proposed by Swanson and Alexander [Swa 91]. Ten percent of the output power of the laser is incident on an optical etalon , and the reflected light acts as a wide band optical frequency discriminator. The reflected light is detected by a photodetector, and electronic compensation removes the FM modulation of the laser. The compensated frequency noise is then fed back to the laser to form a frequency feedback loop.

4. The same improvement in overall system performance can be achieved using a phase locked loop at the receiver. This is discussed in a paper by Kazovsky [Kaz 86]. The phase locked loop can be implemented electronically after photodetection of the received signal, or alternatively with coherent detection we could implement it using a feedback signal to the local oscillator laser.

11
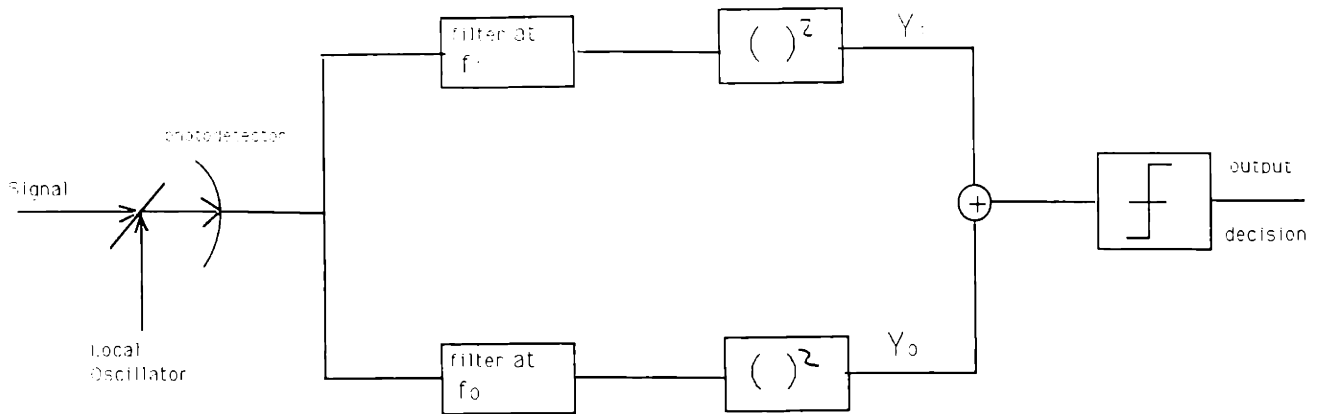
Figure 1-1: Incoherent detection of phase noisy optical signals

1.3 Receiver Model for FSK Detection

The noise statistics for direct detection schemes and coherent detection schemes are different. For direct detection the photodetector output is a Poisson random process with a rate parameter that depends on the optical power incident on the photodetector. This is described in a paper by Personick [Per 73]. The probability of error for such a system using optical amplifiers to achieve shot noise limited performance is given in a paper by Humblet and Azizoglu [Hum 91]. The performance of this when phase noise is introduced would be an interesting problem to analyse. Instead we treat the simpler problem of coherent detection of phase noisy signals. The large local oscillator power in this case allows us to model the the photodetector output as signal plus additive white Gaussian noise [Lec 91].

The optimal receiver structure for coherent detection of phase noisy signals does not lie within the scope of this thesis. Instead we consider the receiver structure shown in figure 1.1 that performs incoherent detection of the intermediate frequency signal.

We assume the filters are far enough apart that when a "one" is transmitted there is no signal component in Y_0 and vice versa. The model of the signal incident on the photodetector is :

$$s(t) = A \cos(2\pi f_c t + \theta(t)) + n(t)$$

where $n(t)$ is additive white Gaussian noise with a two sided power spectral density height $N_0/2$. When a "zero" is transmitted Y_0 and Y_1 will be given by:

$$Y_0 = |s(t) \otimes h(t) + n_{ci} + j n_{so}|^2$$

12

$$Y_1 = |n_{ci} + jn_{so}|^2$$

where n_{ci} and n_{so} are independent identically distributed Gaussian random variables with zero mean and variance:

$$\sigma_{nco}^2 = \sigma_{nsi}^2 = \frac{N_0}{4} \int_0^T h(t)dt$$

Then defining

$$Y \triangleq \left| \int_0^T s(\tau)h(T - \tau)d\tau \right|$$

the statistics of Y_0 and Y_1 conditioned on Y are:

$$P_{Y_0/Y}(y_0/Y) = \frac{1}{2\sigma^2} e^{-(y_0 + Y^2)/2\sigma^2} I_0 \left(\frac{Y\sqrt{y_0}}{\sigma^2} \right)$$

and:

$$P_{Y_1}(y_1) = \frac{1}{2\sigma^2} e^{-y_1/2\sigma^2}$$

Thererfore if we wish to evaluate the received probability of error it is necessary to find the statistics of the random variable Y which depends on the phase noise of the transmitted signal. Various methods have been proposed to solve this problem. The original work was done by Kazovsky and while it is not as rigorous as later work it provides a useful introduction to the problem [Kaz 86], [Kaz 87]. A more accurate method that we hoped would give a suitable analysis of our problem is given by Garrett and Jacobsen [Gar 85], [Gar 86], [Gar 86]. It is discussed in the next chapter but we did not pursue it to a conclusion as it was not accurate enough. The solution we finally adopted is given by Foschini et al. in two papers [Fos 87], [Fos 88].

1.4   Thesis Overview

Here we give an outline of how this thesis is organised.

- **Chapter 2** This gives the system model. We introduce the work of Glance et al. which provides the early motivation for the thesis. We then describe the early work done on the problem and explain why it was not pursued further. In the final section the work of Foschini et al. is introduced as an accurate description of phase noise for the case of $\theta(t)$ a Wiener process.

- **Chapter 3** This presents a derivation of the phase noise statistics for a system with frequency feedback stabilisation applied. We show how the results of Foschini et al. can be extended to cover the more general phase noise problem and derive a set of sufficient conditions on the covariance function of the phase noise process that will lead to a simplification of the later analysis.

13

- **Chapter** 4 Having chosen a filter that leads to a phase noise process satisfying the conditions of chapter 3 we present the solution of the problem. In the final section we introduce four different approximations to the phase noisy random variables that we will use to find the probability of error of the receiver.

- **Chapter** 5 This gives the results for a FSK transmission scheme using frequency feedback stabilisation. Probability of error curves are given for the four different approximations used, and in the final section we give a comparison of the four approximations.

- **Chapter** 6 This is a conclusions chapter and it also includes possible extensions to further research.

Chapter 2

System Model

In this chapter we will outline a method proposed and implemented by Glance [Glance 86] that will provide a frequency stabilisation loop for a semiconductor laser diode. The model that we give for this system was derived by Ho [Ho 90] in her Master's thesis, and the ideas in section 2 show our early attempts to find a framework that would enable us to continue her work. The final section introduces the techniques we use to solve the phase noise problem throughout the remainder of the thesis.

2.1   Description of a Frequncy Stabilisation System

Glance achieves frequency stabilisation of a FSK modulated laser by using a Fabry Perot filter as a frequency discriminator. The basic form of a frequency feedback loop is given in figure 2.1(a), and a block diagram of the system built by Glance is given in figure 2.1(b). For the single user case Glance's model reduces to the form of figure 1.1. For the N user case it differs from the case of N different feedback loops because a single Fabry Perot filter is used to stabilise all N users. By doing this Glance has ensured that the relative frequency spacings between the N channels do not change. The price that is paid for this stability is an increase in the phase noise of each individual laser. For a large number of users the resulting spectral broadening of each laser will result in an increase in the received BER.

There are two problems that require further analysis in Glance's system.

1. The improvement in the received probability of error for a single user system with frequency feedback.

2. The degradation in system performance as the number of stabilised users increases.

The next two sections give a fuller description of the system proposed by Glance. This description along with figure 2.1 is taken from the master's thesis done by Ho [Ho 90]. We start with a heuristic description of the single user system. If we neglect the frequency noise
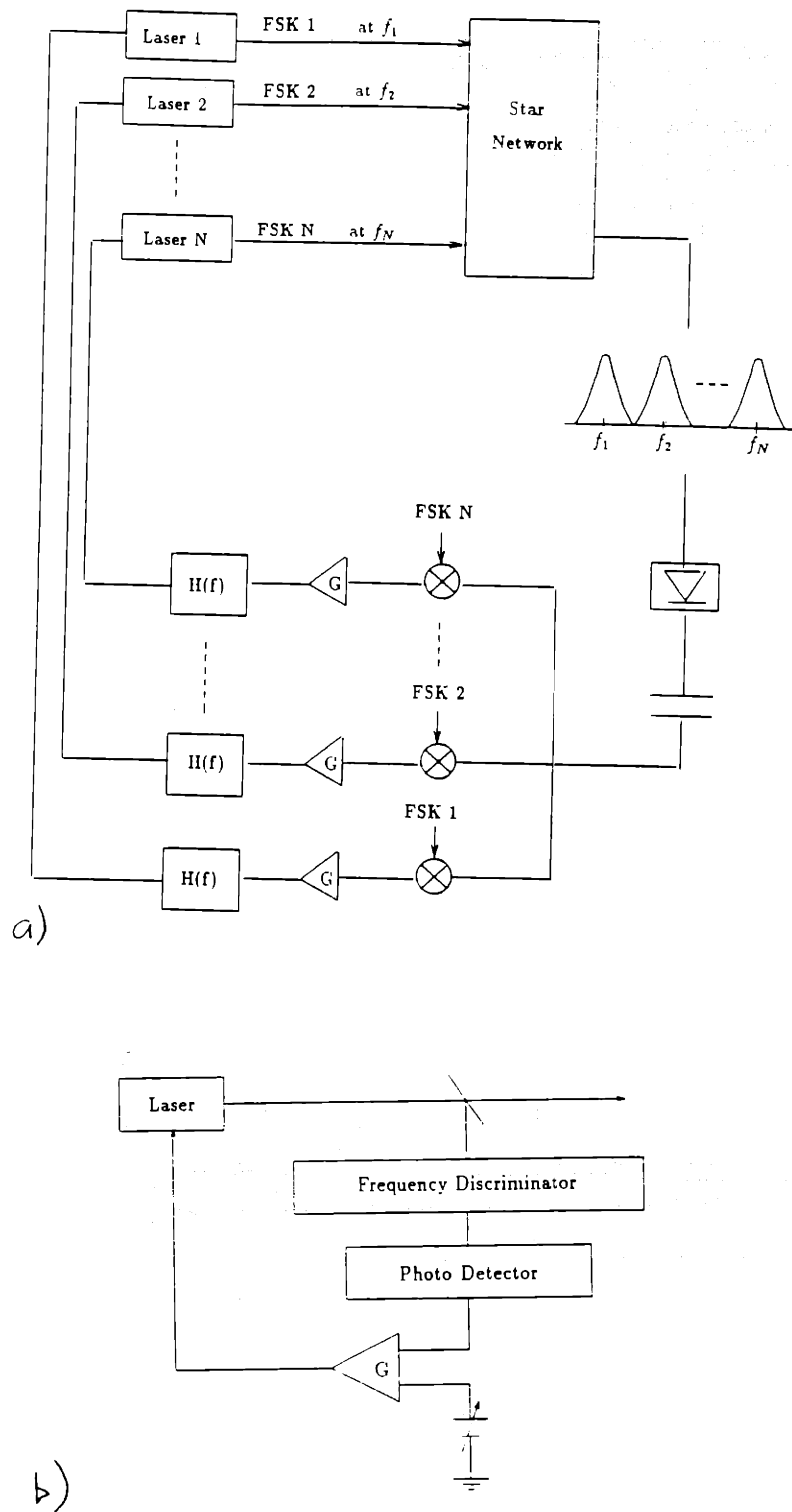
15

Figure 2-1: (a) Glance's system model. (b) A typical frequency feedback loop

of the laser, the output can be written as:

$$s(t) = A\cos(2\pi(f_c \pm \Delta f)t)$$

where f_c is the center frequency of the laser with no external modulation applied, $f_c + \Delta f$ is the frequency the laser transmits to indicate a "one" is being sent and $(f_c - \Delta f)$ is the frequency transmitted to indicate a "zero". The laser's center frequency will coincide with one of the resonance peaks of the Fabry Perot filter . As a result, the laser output will be symmetrically placed at $\pm\Delta f$ from this resonance peak. The symmetry of the Fabry Perot's transmission curve, will ensure that the transmitted power is independent of a one or a zero being sent. Thus for the case of no phase noise no error signal is generated and the system is in a position of stable equilibrium.

When the frequency noise of the laser is introduced the laser's output can be written as

$$s(t) = A\cos(2\pi(f_c \pm \Delta f)t + \theta(t))$$

where $\theta(t)$ is the phase noise of the laser

$$\theta(t) = \int_o^t n(\tau)d\tau$$

and $n(t)$ is a zero mean white Gaussian noise of spectral height σ^2. The frequency noise $n(t)$ changes the instantaneous frequency of the laser output. This changes the power transmitted by the Fabry Perot and an error signal will be produced. The details of this feedback loop will depend on a one or a zero being sent by the laser, but the overall effect is unchanged. For this reason we now consider only the case of a zero being sent.

A positive fluctuation in the frequency noise will give an increase in the power transmitted by the Fabry Perot. This is detected by the photodetector and multiplied by the sign of the transmitted FSK bit which is negative. Therefore a negative signal will be fed back to the laser. In this manner we created a negative feedback loop that will stabilise the output frequency of the laser. The argument can be followed through for a one being sent to verify the system works as we claim.

In the single user case the output from the photodetector is correlated with the user bit stream to produce a feedback signal with the correct sign. In the multiuser system, the photodetector output is a sum of the error signals from all the lasers. Therefore the feedback signal to a particular laser is the FSK bit stream of that laser multiplied by the sum of all the error signals. The worst case occurs when all the data streams are identical. The feedback signal to any laser will now be the sum of all the error signals produced by all the lasers in the system. This problem might at least be analytically tractable and give

a worst case performance bound. The major concern in doing this is that the model is unstable while the original system is stable. A more accurate analysis would include the decorrelation that occurs between the feedback terms when the data streams are statistically independent. Investigating the stability of this system as the number of users N increases is an interesting problem to look at.

We now turn again to the work done by Ho who developed an accurate model describing the effect of the multiuser feedback in terms of a matrix equation. A brief outline of this work and the resulting model is now given.

We again start with the single user case. The power transmission curve of the Fabry Perot is linerarised about $(f_c \pm (\Delta f))$ and has a slope of $\pm A$ (Watts/hertz). We assume the photodetector has an efficiency of B (amps/Watt). The Fabry Perot and the photodetector combined have a response AB (amps/Hertz). The output frequency of a semiconductor laser responds nonlinearly to changes in the input current, and it has been proposed as a method of producing FSK modulation [Yamamoto 81]. We will linearise this effect and use it to adjust the frequency output of the laser. The slope will be C (Hertz/amp). Finally we introduce an electronic amplifier into the feedback loop with gain D (volts/volt). This gives us a closed loop transfer function for the system with gain ABCD. By chosing D correctly we can ensure that ABCD = 1, this normalises everything and the frequency discriminator behaves as though it is linear with slope 1.

The major problems in this analysis are the linearisations we have performed. These will limit the range of feedback frequencies for which our model is valid. We performed two linearisations, one for the power transmission curve of the Fabry Perot, the other for the response of the laser's frequency to changes in input current. Of the two our linearisation ot the Fabry perot will probably be the most critical. Electronic equalisers can be introduced to extend the linear response of the output frequency of the laser to its input current. However the Fabry Perot has a periodic power transmission curve, so its maximum linear range will be equal to this periodicity while in practice it will be much less. This is a concern that we should be aware of in the later work.

When we assume that we remain in the linear range of the model we get the single user system shown in figure 2.2. In this model $x(t)$ is the original frequency noise of the laser, $n(t)$ is the additive white Gaussian noise of the photodetector and $w(t)$ is the output frequency noise of the laser after the feedback loop has been formed. When we consider the multi user system proposed by Glance the system model becomes too complicated too draw. Instead we give the governing equation in table 1.

The next section describes our early attemps to solve this problem.
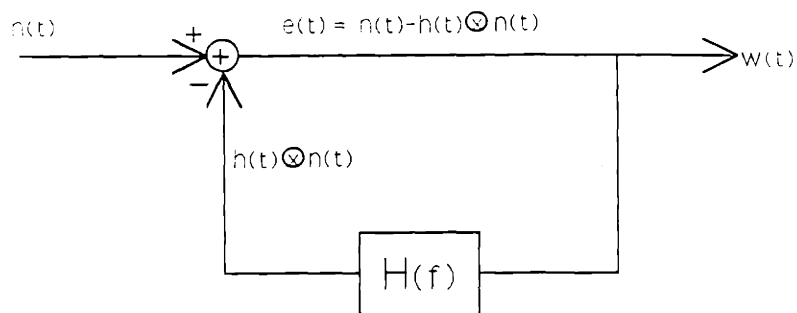
Figure 2-2: Linear feedback model for a single user

$$
\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} * K \longrightarrow \begin{bmatrix} 0 & d_1 & d_1 & \cdots \\ d_2 & 0 & d_2 & \cdots \\ \vdots & & \ddots & \\ d_n & d_n & \cdots & 0 \end{bmatrix} \begin{bmatrix} e_1 d_1 \\ e_2 d_2 \\ \vdots \\ e_N d_N \end{bmatrix} * L
$$

Table 2.1: Equation for N user feedback loop

19

2.2 The Early Work

Our initial aim was to extend the work done by Ho on the single user system to the multiuser problem. In her thesis Ho used the mean squared error (MSE) of the feedback system as a performance criterion to describe how well the feedback loop at the transmitter is working. When we try and do this for the many user system we find that it is not possible to solve the problem directly. The nonlinear terms in the equation make it impossible to decouple the problem into a set of scalar equations that could be solved independently. This problem is inherent to the continuous time nature of the original problem. Therefore first effort was to reduce the complexity of the problem by discretising. This gave the problem of finding the optimal discrete time filter to minimise

$$\min E\left[e_i^2\right] \tag{2.1}$$

where e_i is the sampled feedback error of the i'th user.

We used the bit time of the FSK signals as the basic unit of time. This oversimplified the problem, because it ensured the sampled errors of different filters are independent. Therefore the approach is likely to be very inaccurate but could be improved by making the discretisation time less than the bit time. However, we did not persue this work to this stage because the minimum mean squared error did not give an accurate assesment of the systems performance. It deals only with the transmitter, while in communications problems we are really concerned with the performance from the transmitter to the receiver. Therefore we decided that a more realistic indicator of how well the frequency feedback is working is the probability of error at the receiver.

The drawback in this approach is the difficulty in finding the received probability of error in the presence of phase noise. This applies to the case of modeling $\theta(t)$ as a simple Brownian motion process and we are dealing with a more complicated and as yet unknown phase noise process. In addition we still cannot find the exact statistics of the phase noise for the N user case. Taking this into consideration our next attempt to solve the problem was based on the work done by Garrett and Jacobsen in [Garrett 87] . This seemed to be idealy suited because they solve for the received probability of error by discretising the phase noise This coincides with what we have done to reduce the complexity of the frequency stabilisation loop.

In the introduction we have already looked at the problem of the reception of a phase noisy signal in the presence of additive white Gaussian noise. The statistics of interest are still:

$$Y_0 = |s(t) \otimes h(t) + n_{ci} + jn_{so}|^2$$

and

$$Y_1 = |n_{ci} + jn_{so}|^2$$

To solve for the received probability of error it is necessary to find the statistics of Y, with:

$$Y = |s(t) \otimes h(t)|$$

This will enable us to remove the conditioning of Y_0. Once we have the unconditional densities for Y_1 and Y_0 we can find P_e. Garrett and Jacobsen find an approximation to the statistics of Y by assuming that we can model the frequency noise as constant over one bit time. To relate this constant to the phase noise process of the laser they define it as;

$$\Delta f_\phi = \frac{\theta(T) - \theta(0)}{2\pi T} \tag{2.2}$$

They are assuming the frequency noise in a bit time is the constant value that will give the change in phase measured in the bit time. To find the statistics of Y consider

$$Y = \int_0^T h(t - \tau) A \cos(2\pi f_c t + \theta(t)) d\tau$$

When we replace $\theta(t)$ by $2\pi\Delta f_\phi t$ then Y becomes the amplitude response of the chosen filter $h(t)$ to a sinusoid at frequency $(f_c + f_\phi)$. This gives us a link between the statistics of f_ϕ, which we know from the phase noise statistics, and the statistics of Y. Therefore if we believe the approximation the problem can be solved. In addition the discretisation of the phase noise has made it possible to solve the multiuser case of Glances' frequency stabilisation problem.

However after some consideration we decided that the statistics of

$$Y = \int_0^T h(t - \tau) A \cos(2\pi(f_c + f_\phi)\tau) d\tau$$

do not accurately model the statistics of the actual random variable of interest. To get a feel for this consider the case when $h(t)$ is an integrate and dump filter. This choice of filter gives a great deal of physical intuition. The actual random variable of interest is now given by:

$$y_{actual} = \int_0^T \left| e^{j\theta(i/T)t} dt \right| \tag{2.3}$$

while Garrett and Jacobsen use the approximation:

$$y_{approx} = \left| \int_0^T e^{j\frac{\theta(T)-\theta(0)}{T}t} dt \right| \tag{2.4}$$

$$y_{\text{approx}} = \left| \frac{T \sin([\theta(T) - \theta(0)]/2)}{[\theta(T) - \theta(0)]} \right| \tag{2.5}$$

Visualising what these two equations represent, y_{actual} is the time average of the complex random variable $e^{j\theta(t)}$. As $\theta(t)$ varies $e^{j\theta(t)}$ will move around the unit circle, in a random manner, and the random variable of interest will depend on the path this takes on the unit circle. For the approximation given by Garrett and Jacobsen for y_{approx} they assume that the statistics of the random variable depend only on the end points of the motion and ignore the differences that occur when different paths lead to the same end points. It is worth while trying to see when this approach may be valid and when it will lead to difficulties. The arguments are vague but they illustrate the basic features of the real phase noise problem.

When the frequency noise is white, which is our starting model for the laser's phase noise, $\theta(t)$ is a Brownian motion which is a continuous process with probability one. For the case of small phase noise the values of $\theta(0)$ and $\theta(T)$ will be highly correlated and we would expect that in going from one point to the other, $\theta(t)$ has followed a fairly regular path (this is because large fluctuations in $\theta(t)$ are unlikely). Therefore in this case the approximation made by Garrett and Jacobsen should not be too gross a simplification. However in the case of large phase noise, with large probability the path taken between two end points $\theta(0)$ and $\theta(T)$ will be very different for different trials. Therefore in this case the approximation used will be very poor. The variance of $[\theta(0) - \theta(T)]$ is $2\beta T$ where β is the laser's linewidth, so when βT is large $\theta(0)$ and $\theta(T)$ will not be highly correlated, and the approximation will be poor. Unfortunately it is the case of relatively large phase noise that we are most interested in, therefore there is little merit in considering this problem much further.

This conclusion led us to try and find a more accurate description of the effect of phase noise on the performance of the single user system. Once this has been accuratly modeled, it would be a nice problem to extend it to the multiuser system proposed by Glance. However, for the single user system, the performance does not depend on the feedback structure that we use so the systems proposed by Glance or by Swanson and Alexander are equally valid implementations.

2.3 The System Model to be Solved

To analyse the effect that phase noise has on the probability of error at the receiver we decided to adopt the approach taken by Foschini et al. [Foschini 88a] ,[Foschini 88b]. In this work Foschini et al. use the receiver structure shown in figure 2.3 for FSK reception. As before the received IF signal is the phase noisy sinusoid corrupted by additive white Gaussian noise.

$$s(t) = A\cos(2\pi f_i t + \theta(t)) + n(t), \quad i = 0, 1 \tag{2.6}$$
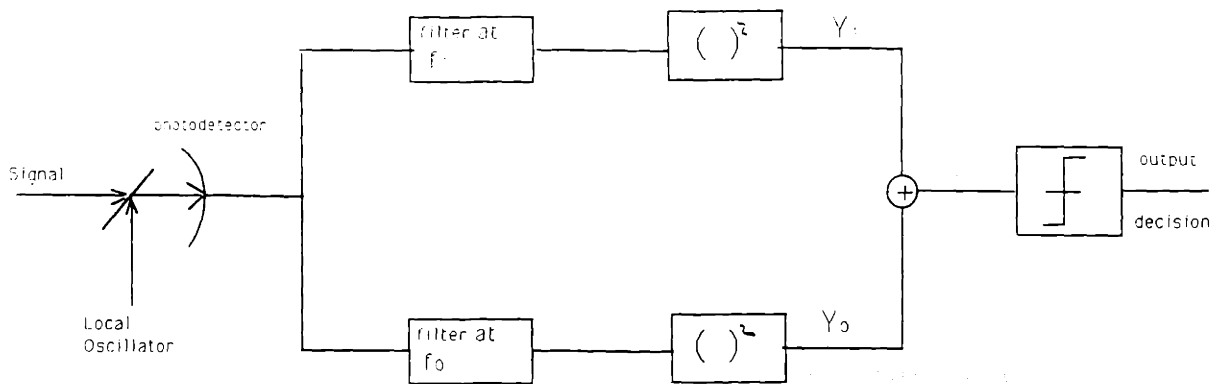
Figure 2-3: Receiver for phase noise problem

and f_i is the received frequency when a zero or a one is transmitted. In the work done by Foschini

$$\theta(t) = \int_0^t n(t)dt$$

where $n(t)$ is a zero mean white Gaussian noise process with PSD height $\beta/2\pi$. Thus $\theta(t)$ is a zero mean Brownian motion with variance $2\pi\beta t$. The additive Gaussian noise is a result of the detection process [Lec 91],[Personick 72] and it has PSD height $N_0/2$. To find an expression for the sampled outputs at time T it is assumed that the filters centered at f_0 and f_1 are far enough apart that when a one is sent the output of the filter at f_0 contains no signal component and vice versa. The system is fully symmetric with respect to a one or a zero being transmitted so to find the probability of error we need only consider the case of a zero being sent. The sampled outputs become:

$$Y_0 = \left| \frac{A}{2} \int_0^T e^{j\theta(t)}dt + n_{ci} + jn_{so} \right|^2 \tag{2.7}$$

$$Y_1 = |n_{ci} + jn_{so}|^2 \tag{2.8}$$

n_{ci} and n_{so} are i.i.d. Gaussian random variables associated with the additive Gaussian noise. They are zero mean and have variance

$$\sigma^2 = N_0 T/4$$

The statistics of Y_0 and Y_1 conditioned on

$$Y = \left| \frac{A}{2} \int_0^T e^{j\theta(t)} dt \right|$$

are

$$P_{Y_0}(y_0/X) = \frac{1}{2\sigma^2} e^{-(y_0+Y^2)/2\sigma^2} I_0 \left(\frac{Y\sqrt{y_0}}{\sigma^2} \right) \tag{2.9}$$

$$P_{Y_1}(y_1) = \frac{1}{2\sigma^2} e^{-y_1/2\sigma^2} \tag{2.10}$$

Foschini et al. find an approximation to the statistics of Y that enables them to find the density of Y_0, $P_{Y_0}(y_0)$:

$$P_{Y_0}(y_0) = \int_{-\infty}^{\infty} P_{Y_0/Y}(y_0/y) P_Y(y) dy$$

The probability of error is then given by:

$$P_e = Pr(Y_0 < Y_1) = Pr(Z \leq 0)$$

with $Z = Y_0 - Y_1$,

Y_1 and Y_0 are statistically independent given the transmitted data therefore the moment generating function of Z is

$$G_Z(\omega) = G_0(\omega) G_1^*(\omega)$$

Where $G_1(\omega)$ is the characteristic function of Y_1 and $G_1(\omega)$ is the characteristic function of Y_0, both of which are known. The probability density function of Z is evaluated from $G_Z(\omega)$ using an inverse fast Fourier and the received probability of error is calculated as:

$$Pe = \int_{-\infty}^0 P_Z(z) dz$$

We do not follow this approach.

Instead we turn to a paper by Azizoglu and Humblet [Azizoglu 90]. This gives a simpler method of finding Pe once the statistics for Y have been found . Again we start with

$$Pe = P_r(Y_0 \leq Y_1)$$

but now Azizoglu and Humblet get a conditional error probability based on Y.

$$Pe(Y) = \frac{1}{2} e^{-Y^2/2\sigma^2} \tag{2.11}$$

24

They put this in its simplest form with a series of renormalistions:

$$Y = \frac{AT}{2} \left| \int_0^1 e^{j\theta(Tu)} du \right|$$ (2.12)

and as $\theta(Tu)$ has variance $2\pi\beta Tu$ we get with $\gamma = 2\pi\beta T$:

$$Y = \frac{AT}{2} \left| \int_0^1 e^{j\sqrt{\gamma}\psi(t)} dt \right|$$ (2.13)

By defining a new random variable $X(\gamma)$ as

$$X(\gamma) = \left| \int_0^1 e^{j\sqrt{\gamma}\psi(t)} dt \right|^2$$ (2.14)

then

$$Pe(X(\gamma)) = \frac{1}{2} e^{-\zeta X(\gamma)/2}$$ (2.15)

where $\zeta = A^2 T / 2N_0$ is the received IF signal to noise ratio, and we have

$$Pe = E_{X(\gamma)} \left[\frac{1}{2} e^{-\zeta X(\gamma)/2} \right]$$ (2.16)

To get the statistics of $X(\gamma)$ to remove this conditioning we will use Foschini's perturbation expansion. This result is presented in the next chapter along with the extension to the more general phase noise problem which results from applying frequency feedback stabilisation.

Chapter 3

Phase Noise Statistics

In this chapter, we look at the different methods used by Foschini et al. and Azizoglu and Humblet to find the statistics of $X = \left| \int_0^1 e^{j\sqrt{\gamma}\psi(t)}dt \right|^2$. We discuss the extension of these results when the phase noise is no longer a Brownian motion process. To simplify the analysis we introduce a new stochastic process. This is related to the phase noise process of interest, and it yields the same statistical results as the phase noise process. By using the new process, we will get a solution for the statistics of $X(\gamma)$ in terms of a single summation of independent, squared Gaussian random variables.

3.1 Phase Noise as a Brownian Motion Process

Insight to the general phase noise problem can be developed by considering the case treated by Foschini et al. with a Brownian motion phase noise, $\psi(t)$ that has a variance $E(\psi^2(t)) = t$. We wish to find the statistics of $X = \left| \int_0^1 e^{j\sqrt{\gamma}\psi(t)}dt \right|^2$. Foschini et al. expand the complex exponential into a power series, take its magnitude and retains the first order powers of γ.

$$X(\gamma) \simeq X_L = 1 - \gamma \left[\int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt \right)^2 \right] \tag{3.1}$$

Because the process $\psi(t)$ is Gaussian a 2nd moment model will give a full statistical description of the system. This enables us to use a series expansion to represent $\psi(t)$. The basis functions are all orthogonal on the interval $[0, 1]$ thus:

$$\psi(t) = \sum_{i=0}^{\infty} x_i \phi_i(t)$$

$$\int_0^1 \phi_i(t)\phi_j(t)dt = \delta_{ij} \tag{3.2}$$

gives as the series expansion, [Van Trees 67].

$$\psi(t) = \sum_{i=0}^{\infty} x_i \phi_i(t) \tag{3.3}$$

where x_i is zero mean, Gaussian with variance $E[x_i x_j] = \frac{\delta_{ij}}{(i-1/2)^2 \pi^2}$ and:

$$\phi_i(t) = \sqrt{2} \sin[(n - 1/2)\pi t], \quad 0 \le t \le 1$$

In terms of the original problem using a generalisation of Parseval's theorem would give:

$$\int_0^1 \psi^2(t) dt = \sum_{i=0}^{\infty} x_i^2$$

However when we look at the second term in the bracket of equation 3.1 we do not get so simple a result.

$$\int_0^1 \psi(t) dt = \sum_{i=0}^{\infty} x_i \int_0^1 \psi(t) dt$$

so

$$\left(\int_0^1 \psi(t) dt \right)^2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_i x_j \left(\int_0^1 \psi_i(t) dt \right) \left(\int_0^1 \psi_j(t) dt \right) \tag{3.4}$$

This double summation is not convenient, we would like to write X_L as a single summation of squared Gaussian random variables:

$$X_L = 1 - \gamma \sum_{i=1}^{\infty} x_i^2$$

We will see later how this form helps simplify our final analysis. Foschini et al. now find a remarkable result. They realise that the underlying frequency noise is a white Gaussian process for which any orthogonal basis will form the Karhuen Loeve expansion. Using this they chose a sine series expansion for the white noise, giving;

$$
\begin{aligned}
w(t) &= \sum_{i=1}^{\infty} x_i \sin i\pi t \\
E[x_i x_j] &= \delta_{ij}
\end{aligned}
\tag{3.5}
$$

and it integrates up to give the phase noise process;

$$\psi(t) = \sum_{i=1}^{\infty} \frac{x_i}{i\pi} \cos(i\pi t) + \text{constant} \tag{3.6}$$

This cosine basis is not a complete orthogonal basis unless it is augmented by 1. When this is done all the coefficients are independent by construction, except for the coefficient of 1. This is the price paid for not using the Karhuen Loeve expansion. However if we now look at the random variable of interest we see why Foschini at al. chose to use this basis.

$$
\begin{aligned}
X_L &= 1 - \gamma \left[\int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt \right)^2 \right] \\
&= 1 - \gamma \left[x_0^2 + \sum_{i=1}^{\infty} \frac{x_i^2}{(i\pi)^2} - (x_0)^2 \right] \\
&= 1 - \gamma \left[\sum_{i=1}^{\infty} \frac{x_1^2}{(i\pi)^2} \right]
\end{aligned}
\tag{3.7}
$$

Because in the cosine basis the constant function is actually $\phi_0(t)$, the integral, $\int_0^1 \psi(t)dt$ gives the zero'th coefficient of $\psi(t)$. Therefore from $\int_0^1 \psi(t)dt$ we have a single term x_0^2 which cancels with the x_0^2 from $\int_0^1 \psi^2(t)dt$. X_L can now be written as a single infinite sum of iid Gaussian random variables rather than as a double summation.

Once we have the expression in the form;

$$
X_L = 1 - \gamma \sum_{i=1}^{\infty} \frac{x_i^2}{(i\pi)^2} \ , \quad E[x_i x_j] = \delta_{ij}.
\tag{3.8}
$$

We notice that the random nature of X_L comes only from the sum of iid Gaussian random variables. To find the statistics of this process we first find the characteristic function of $\sum_{i=1}^{\infty} x_i^2/(i\pi)^2$. An aid to this is the more compact notation ;

$$
\sum_{i=1}^{\infty} \frac{x_i^2}{(i\pi)^2} = X^T D X
\tag{3.9}
$$

X is a an infinite dimensional vector of iid Gaussian random variables $x_i \sim N[0,1]$. D is an infinite dimentional diagonal matrix with diagonal entries $D_{ij} = 1/(i\pi)^2$. Using this notation we find the characteristic function of this random variable as;

$$
\begin{aligned}
\mathcal{L}(s) &= E \left[\exp -s \left(\int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt \right)^2 \right) \right] \\
&= E \left[\exp -s \left(X^T D X \right) \right] \\
&= \lim_{N \to \infty} \left(\frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 ... dx_N e^{-1/2(1/2(X^T(I+2sD)X))}
\end{aligned}
\tag{3.10}
$$

$$= \lim_{N \to \infty} N \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \ldots dx_N e^{-1/2(1/2(X^T(I+2sD)X))} \qquad (3.10)$$

Which is solved by completion of the square using;

$$\int_{-\infty}^{\infty} dx \frac{1}{\sqrt{N_0 \pi}} e^{-(x-A)^2/2N_0} = 1 \qquad (3.11)$$

This gives

$$\mathcal{L}(s) = \left| I + 2sD^2 \right|^{-1/2} \qquad (3.12)$$

Where $|M|$ is the determinant of a a general matrix M. Now that we have the characteristic function it is a simple step to find the density using an Inverse Fast Fourier Transform routine. This gives the density of X_L from which we can calculate the received probability of error. Before we extend the result to the more general phase noise problem, it is worth while looking a bit more closely at the approximation used by Foschini et al.

$$X(\gamma) = \left| \int_0^1 e^{j\sqrt{\gamma}\psi(t)} dt \right|^2$$

$$X_L = 1 - \gamma \left[\int_0^1 \psi^2(t) dt - \left(\int_0^1 \psi(t) dt \right)^2 \right]$$

$$X_L = 1 - \gamma X^T D^2 X \qquad (3.13)$$

The first thing to notice is that the perturbation expansion of $X(\gamma)$ using the complex exponential form is the same as approximating $\cos(z)$ in the interval $[0,1]$ by $\cos(z) \geq (1 - z^2/2)$ [Foschini 89]. Therefeore X_L will provide an upper bound to the received probability of error. The next thing to realise is that $X(\gamma)$ will be bounded by $0 \leq X(\gamma) \leq 1$, while $-\infty \leq X_L \leq 1$. The negative tail of X_L will seriously affect the accuracy of our results in the region of small Pe. We see this most clearly from the conditional probability of error expression $Pe(X) = 1/2e^{-\zeta X(\gamma)}$. When X_L goes negative, Pe(X) will become exponentially large. We have two techniques to improve the accuracy of the results when X_L is being used.

1. We truncate X_L to lie in the region $[0,1]$ and simply ignore the negative tail. This is a simple technique, but it cannot be proven to give an upperbound on the received probability of error.

2. We can remove the negative tail and lump its probability to form an impulse at the

29

tion introduced by Azizoglu and Humblet in their paper. They refer to it as an "exponential" approximation, and it is given by;

$$
\begin{aligned}
X_E &= \exp\left\{-\left[\int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt\right)^2\right]\right\} \\
&= \exp\left\{-\gamma[X^T D^2 X]\right\}
\end{aligned}
\tag{3.14}
$$

This has the advantage that X_E is bounded by $[0,1]$ and therefore lies in the same range as the original random variable, but it does not provide a bound to the received probability of error. It is likely to be a good approximation however as Azizoglu and Humblet calculated the first, second and fourth moments of X and X a function of γ, and found them to be in good agreement.

3.2 Extension to a General Phase Noise Problem

In this section we find the statistics of $X = \left|\int_0^1 e^{j\sqrt{\gamma}\psi(t)}dt\right|^2$ when we have applied frequency feedback to the laser. We no longer have a white frequency noise, so the approach used by Foschini et al. to derive the cosine series expansion is no longer valid. However we still wish to find X_L in terms of a single summation rather than a double summation. We have already seen that for the Brownian motion problem, if we use a Karhen Loeve expansion for $\psi(t)$ it will lead to an expression for for X_L which has a double summation. This occurs because of the nature of the second integral;

$$
\int_0^1 \psi(t)dt = \sum_{i=0}^\infty x_i \int_0^1 \phi_i(t)dt
\tag{3.15}
$$

Some thought on this leads us to the conclusion that the orthogonal expansion used for $\psi(t)$ must include the function $\phi_0(t) = 1$. When this is the case

$$
\int_0^1 \phi_i(t)dt = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}
\tag{3.16}
$$

and we will automatically get a single series expansion for X_L. The problem is finding the appropriate basis for a general phase noise process. Our first attempt to solve this problem involved the cosine expansion used by Foschini et al. It is the white frequency noise that enabled them to use this basis, therefore we checked if $\sin(i\pi t)$ can be used as a Karhuen Loeve expansion of any other frequency noise process. We did this by going in reverse to form a new noise process from the series expansion:

$$
n(t) = \sum_i x_i \sin(i\pi t),
\tag{3.17}
$$

where x_i are independent Gaussian random variables with zero mean and variance $E[x_i^2] = \lambda_i$ with λ_i not yet specified. The covariance function of this noise process is then;

$$
\begin{aligned}
E[n(t)n(s)] &= E\left[\sum_i x_i \sin(i\pi t) \sum_j x_j \sin(j\pi s)\right] \\
&= E\left[\sum_i \sum_j j x_i x_j \cos(i\pi t) \cos(j\pi s)\right] \\
&= \sum_i \sum_j E[x_i x_j] \sin(i\pi t) \sin(j\pi s) \\
&= \sum_i \lambda_i \sin(i\pi t) \sin(i\pi s)
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
K_{nn}(t,s) &= \sum_i \lambda_i \sin(i\pi t) \sin(i\pi s) \\
&= \frac{1}{2} \sum_i \lambda_i [\cos(i\pi(t-s)) - \cos(i\pi(t+s))]
\end{aligned}
\tag{3.19}
$$

We see that the sin expansion will represent a noise process which has both a wide sense stationary component (function of $(t-s)$), and a non-wide sense stationary component (function of $(t+s)$). In the physical system of interest we are start with a WSS white noise which has $K_{ww}(t,s) = \delta(t-s)$ and perform linear feedback operations on it. This will give another WSS process and we will not be able to use the $\sin(n\pi t)$ expansion to represent it. This does not explain how the expansion works for the white noise problem treated by Foschini et al. They start with a WSS process which has a covariance function $K_{ww}(t,s) = \delta(t-s)$ and they use a non WSS process to represent it on the interval of interest $[0,1]$. This provided an interesting problem. Any orthogonal basis can be used to represent a white noise process, however when we use an orthogonal cosine expansion, $K_{nn}(t,s)$ has a non wide sense stationary component, therefore the question of how a known WSS white noise can be represented by a non WSS process must be considered. To see how this apparent paradox can be resolved it is necessary to look at the form of the covariance function on the t,s plane. For the white noise process ;

$$
\begin{aligned}
K_{nn}(t,s) &= \frac{1}{2} \sum_i [\cos(i\pi(t-s)) - \cos(i\pi(t+s))] \\
&= \delta(t-s) - \delta(t+s)
\end{aligned}
\tag{3.20}
$$

Looking at the t,s plane in figure (3.1) $\delta(t+s)$ exists only the line $t+s = 0$ therefore it never enters into the region of interest, $[0 \leq t,s \leq 1]$, which is shaded. Therefore we are able to use the $\sin(n\pi t)$ expansion for the white noise process on the region of interest but
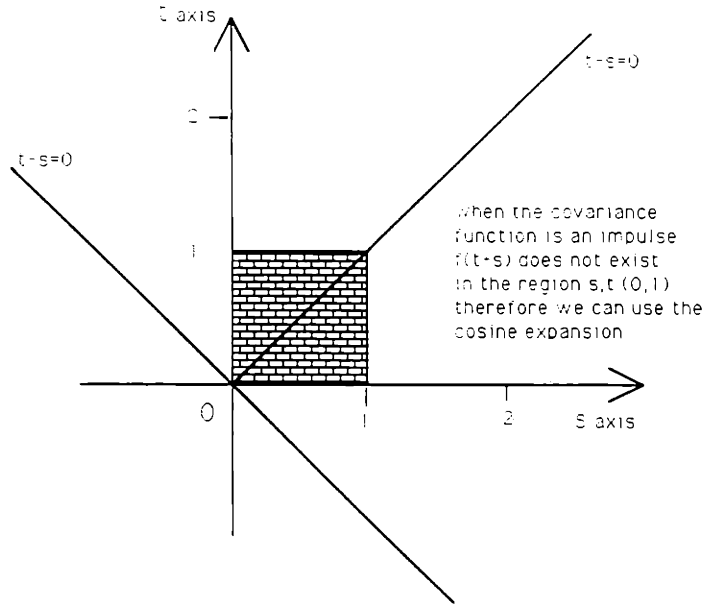
31

t axis

t-s=0

2 —

t-s=0

when the covariance
function is an impulse
f(t-s) does not exist
in the region s,t (0,1)
therefore we can use the
cosine expansion

O   1   2   s axis

Figure 3-1: $K_{nn}(t,s)$ on the t,s plane

it will clearly not work for any other noise process. Having found this last result we began to wonder if there is some sort of magical property of the white noise process that enables Foschini et al. to use the cosine expansion for the Brownian motion process. In fact it is a magical property of the approximation which makes this possible for any frequency noise process, as we will now show.

Once we perform the perturbation expansion, the random variable of interest becomes;

$$\Psi = \int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt\right)^2 \tag{3.21}$$

Suppose that we add a random variable a to the process $\psi(t)$. We observe that as far as Ψ is concerned we have not changed the problem. Defining a new process $y(t)$ to be $y(t) = \psi(t) + a$. Then:

$$\int_0^1 y^2(t)dt - \left(\int_0^1 y(t)dt\right)^2 = \int_0^1 (\psi(t)+a)^2 dt - \left(\int_0^1 (\psi(t)+a)dt\right)^2$$

$$= \int_0^1 (\psi^2(t) + 2a\psi(t) + a^2)dt - \left(\int_0^1 \psi(t)dt\right)^2$$

$$- 2a\int_0^1 \psi(t)dt - a^2$$

$$- \ 2a \int_0^1 \psi(t)dt - a^2$$

$$= \ \int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt \right)^2$$

$$= \ \Psi \qquad\qquad (3.22)$$

Therefore we can augment the process $\psi(t)$ by a random variable and solve the Karhuen Loeve expansion for this new process $y(t)$. The statistics of the series expansion of

$$\int_0^1 y^2(t)dt - \left(\int_0^1 y(t)dt \right)^2$$

will then be exactly the same as the statistics of a series expansion of

$$\int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt \right)^2$$

As a clue to chosing a consider writing;

$$\int_0^1 \psi(t)^2 dt - \left(\int_0^1 \psi(t)dt \right)^2 = \int_0^1 \left(\psi(t) - \int_0^1 \psi(u)du \right)^2$$

This shows that the underlying process of interest is in fact:

$$\psi - \int_0^1 \psi(u)du$$

If we chose a to be

$$a = - \int_0^1 \psi(u)du$$

then $y(t)$ is equal to the underlying process and the Karhuen Loeve expansion for $y(t)$ will give us the desired basis [Per 91]. To find the Karhuen Loeve expansion for the new process we first have to find its covariance function, this is very simple to do.

$$
\begin{aligned}
K_{yy}(t,s) &= E\left[\left(\psi(t) - \int_0^1 \psi(u)du \right) \left(\psi(s) - \int_0^1 \psi(v)dv \right) \right] \\
&= E\left[\psi(t)\psi(s) - \psi(t) \int_0^1 \psi(v)dv - \psi(s) \int_0^1 \psi(u)du \right. \\
&\quad + \left. \int_0^1 \psi(u)du \int_0^1 \psi(v)dv \right] \\
&= K_{\psi\psi}(t,s) - \int_0^1 K\psi\psi(s,u)du - \int_0^1 K_{\psi\psi}(t,v)dv \\
&\quad + \int_0^1 K\psi\psi(u,v)dudv \qquad\qquad (3.23)
\end{aligned}
$$

33

eigenfunction solutions of the integral equation given below: [Van Trees 67]

$$\int_0^1 K_{yy}(t,s)\phi(s)ds = \lambda\phi(t) \tag{3.24}$$

substituting in for $K_{yy}(t,s)$ gives;

$$\int_0^1 \left(K_{\psi\psi}(t,s) - \int_0^1 K_{\psi\psi}(s,u)du - \int_0^1 K_{\psi\psi}(t,v)dv + \int_0^1 K_{\psi\psi}(u,v)dudv \right) \phi(s)ds = \lambda\phi(t)$$

When $\phi(t) = \phi(s) = 1$ we find;

$$\int_0^1 K_{\psi\psi}(t,s)ds - \int_0^1 \int_0^1 K_{\psi\psi}(s,u)duds \quad - \quad \int_0^1 K_{\psi\psi}(t,v)dv$$
$$+ \quad \int_0^1 \int_0^1 K_{\psi\psi}(u,v)dudv = \lambda \tag{3.25}$$

This implies

$$\lambda = 0$$

Therefore the constant $\psi(t) = 1$ is an eigenfunction corresponding to an eigenvalue zero. We thus achieve the desired property for the orthogonal expansion of $y(t)$, it has 1 as a member of the orthogonal basis so the solution for Ψ will be a single summation, without a contribution from x_0^2. Therefore to obtain the full solution to the problem, we need only find the eigenfunctions which are orthogonal to one and the corresponding eigenvalues. This will lead to a simplification of the integral equation which we must solve. The remaining eigenfunctions are all orthogonal to one. We use this property to write:

$$\int_0^1 \left(\int_0^1 \int_0^1 K_{\psi\psi}(u,v)dudv \right) \psi(s)ds = 0$$

$$\int_0^1 \left(\int_0^1 K_{\psi\psi}(t,v)dv \right) \Phi(s)ds = 0 \tag{3.26}$$

Therefore the integral equation that we must solve for the remaining eigenvalues/eigenfunctions reduces to:

$$\int_0^1 K_{\psi\psi}(t,s)\phi(s)ds - \int_0^1 \int_0^1 K_{\psi\psi}(s,u)du\phi(s)ds = \lambda\phi(t) \tag{3.27}$$

The appropriate check of this result is to see if it gives us the correct cosine expansion for the case when $\psi(t)$ is a Brownian motion process. $K_{\psi\psi}(t,s)$ is equal to $\min(t,s)$ and we get;

$$\int_0^t s\phi(s)ds + \int_t^1 t\phi(s)ds - \int_0^1 du \left(\int_0^u s\phi(s)ds + \int_u^1 u\phi(s)ds \right) = \lambda\phi(t) \tag{3.28}$$

34

Differenciation w.r.t. t leads to the second order differential equation;

$$\frac{d^2\phi(t)}{dt^2} = -\frac{1}{\lambda}\phi(t) \tag{3.29}$$

with the solution:

$$\phi(t) = A\cos(\beta t) + B\sin(\beta t); \quad \beta = \sqrt{\frac{1}{\lambda}} \tag{3.30}$$

When this is substituted back into the original integral equation to satisfy the boundary conditions we see that the solution must satisfy the equation;

$$\begin{bmatrix} \sin\beta & -\cos\beta \\ -\frac{\sin\beta}{\beta^2} - \frac{\sin\beta}{2} & \frac{-1}{\beta^2} + \frac{\cos\beta}{\beta^2} + \frac{\cos\beta}{2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

which implies;

$$\frac{\sin(\beta)}{\beta^2} = 0 \quad ; \beta = n\pi, \quad n \neq 0 \tag{3.31}$$

Thus we get the solution:

$$\psi(t) = \sum_{i=1}^{\infty} z_i \cos(n\pi) + z_0 \tag{3.32}$$

with all the $z_i, i \neq 0$ i.i.d. Gaussian random variables $z_i \sim N[0, 1/(n\pi)^2]$ and;

$$\Psi = \sum_{i=1}^{\infty} z_1^2 = \sum_{i=1}^{\infty} \frac{1}{i^2\pi^2} x_i \quad \text{for} \quad x_i \sim N[0,1] \tag{3.33}$$

So as we would expect use of this new random process $y(t)$ leads us to the same solution as the one found by Foschini. We can now extend this to get the same desired of results for any phase noise process.

3.3 Independent Increment Processes

The variance of the Brownian motion phase noise increases linearly with time, $\sigma^2(t) = t$. As a result we might expect the statistics of $X = \left| \int_{(k-1)}^{k} e^{j\psi(t)} dt \right|^2$ to be a function of k. Foschini et al. use the fact that $\psi(t)$ is an independent increments process to show that the statistics of X do not change with k.

When we apply stabilising feedback to the frequency noise we will automatically introduce memory to both the frequency noise and the associated phase noise. Therefore the phase noise is no longer an independent increment process. However, the statistics of X do not depend on k for our chosen feedback filter. To show this, we take a result from chapter

4, section 4.3 . X can be written as:

$$X = \left| \int_k^{k+1} \int_k^{k+1} e^{j\sqrt{\gamma}[\psi(t) - \psi(s)]} dt ds \right|$$

The variance of $[\psi(t) - \psi(s)]$ is a function of t, s:

$$E[(\psi(t) - \psi(s))^2] = \frac{1}{2r} \left[1 - e^{-r|t-s|} \right]$$

thus $E[(\psi(t+k) - \psi(s+k))^2] = E[(\psi(t) - \psi(s))^2]$, and because it is a Gaussian random process, we conclude the statistics of X are independent of k.

However because we are using an approximation rather than X actually require the statistics of our approximation change with time as well as considering the original random variable. We will be concerned how the statistics of $\Psi_k = \left[\int_{k-1}^k \psi^2(t) dt - \left(\int_{k-1}^k \psi(t) dt \right)^2 \right]$ vary with the time index k. We find the statistics of Ψ_k as a series expansion in two different ways.

1. We can solve the Karhuen Loeve expansion for $y(t)$ in the interval $[k-1, k]$. This would mean solving the integral equation;

$$\int_{k-1}^k \left(K_{\psi\psi}(t,s) - \int_{k-1}^k K_{\psi\psi}(t,s) dt \right) \phi(s) ds = \lambda \phi(t), \quad t, s \in [k-1, k]$$

This is not the favored approach because it is difficult to compare the result to for a general k unless we solve for the density function explicitly. Our difficulty is the solutions apply to different intervals so cannot be directly compared.

2. The alternative to the above procedure is to find the Karhuen Loeve expansion of the boundary value problem in the same interval. To do this we must modify the covariance function of the basic stochastic process we are considering. If we expand $K_{\psi\psi}(t+k, s+k)$ in the interval $[0, 1]$, it is the same as expanding $K_{\psi\psi}(t,s)$ in the interval $[k-1, k]$.

In general

$$K_{\psi\psi}(t,s)_k = f(t+k, s+k)$$

and we require:

$$K_{yy}(t,s)_k = K_{\psi\psi}(t,s)_k - \int_0^1 K_{\psi\psi}(t,s)_k dt - \int_0^1 K_{\psi\psi}(t,s)_k ds$$

36

$$+ \quad \int_0^1 \int_0^1 K_{\psi\psi}(t,s)dsdt \qquad\qquad (3.34)$$

to be independent of k. It is simple to write four functions of t, s that satisfy this condition.

1. constant

2. function of $|t - s|$

3. function of t only

4. function of s only

Thus we can say that if $k_{\psi\psi}(t,s)$ is of the form;

$$K_{\psi\psi}(t,s) = constant + g_1(|t - s|) + g_2(t) + g_3(s)$$

then $K_{yy}(t,s)_k$ is independent of k and we have as a result a solution Ψ_k that is independent of k. Therefore even when $\psi(t)$ is not an independent increments process, when $K_{\psi\psi}(t,s)$ satisfies the form of the above equation, the statistics of Ψ will be independent of k.

Chapter 4

Solution for the General Phase Noise Problem

4.1 Possible Implementations

Two coherent detection schemes that we propose are show in figure 4.1. Of these two methods the simplest to follow is shown in Figure 4.1(a). It achieves frequency stabilisation of the two lasers in the system independently. The FSK signal modulation is performed by an external modulator to avoid feedback filtering of the transmitted FSK signal, which would be filtered if we performed the modulation inside the frequency feedback loop. The phase of the two lasers are independent random variables. $\theta_s(t)$ is the phase of the transmitter, and $\theta_{LO}(t)$ is the phase of the local oscillator laser. The received IF signal is:

$$A \cos(2\pi f_c t + \theta_s(t) + \theta_{LO}(t)) = A \cos(2\pi f_c t + \theta(t))$$

where $\theta(t) = \theta_s(t) + \theta_{LO}(t)$

Because of the independence of $\theta_s(t)$ and $\theta_{LO}(t)$ then:

$$
\begin{aligned}
K_{\theta\theta}(t,s) &= K_{\theta_s\theta_s}(t,s) + K_{\theta_{LO}\theta_{LO}}(t,s) \\
&= 2K_{\theta_s\theta_s}(t,s)
\end{aligned}
$$

when the lasers have the same linewidth and applied feedback. Therefore the phase noise process we are interested in is the sum of the phase noise processes of the local oscillator and the transmitting laser.

In figure 4.1(b) we implement frequency locking through frequency feedback to the local oscillator. The feedback occurs after photodetection so we need to use frequency discriminator for the IF signal. This limits the bandwidth of the feedback loop but it is simpler to find a frequency discriminator for IF signals than it is to make one for optical signals.

If we briefly consider the optical frequency discriminator that could be used to implement the system of figure 4.1(a), the feedback loop proposed by Glance et al. is a possibility but
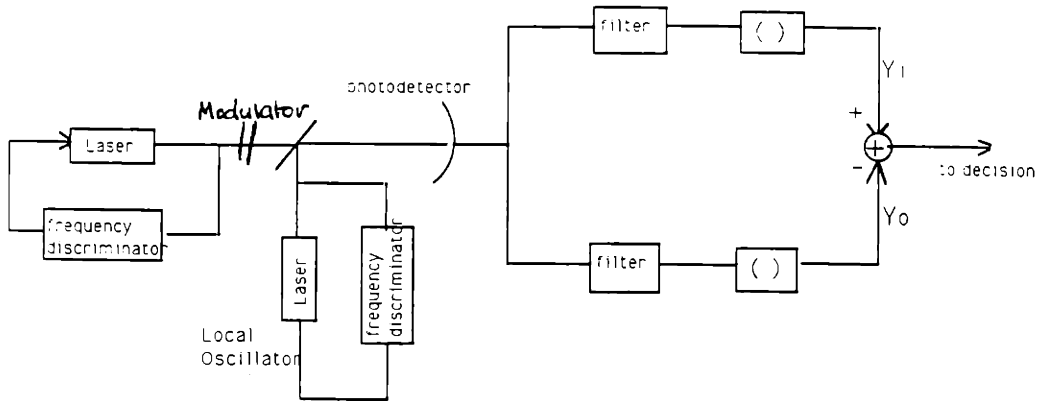
a)
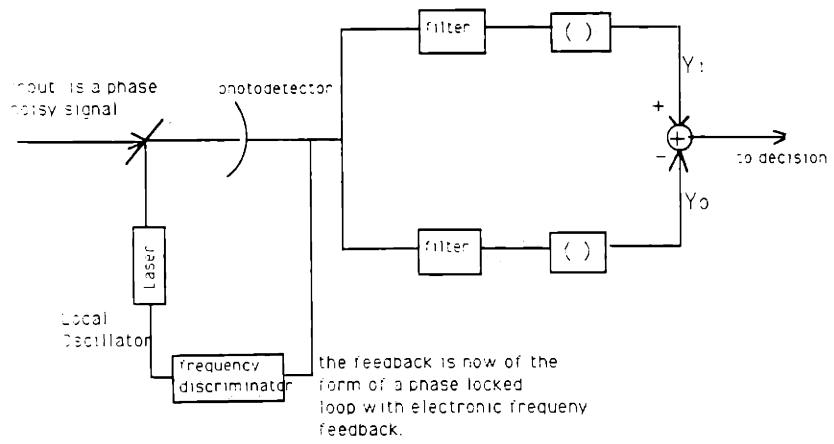
b)

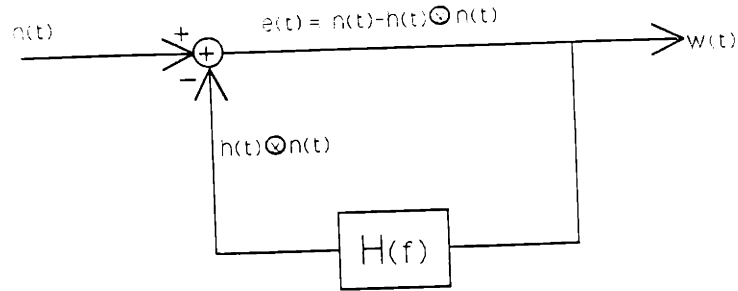Figure 4-1: Implementation schemes for frequency stabilisation

Figure 4-2: Frequency feedback model

it is narrowband. Therefore a more suitable method may be that proposed by Swanson and Alexander [Swa 91]. Either of the two models we have proposed will lead to the same form of solution as that proposed by Foschini et al.

4.2   Phase Noise processes with Frequency Feedback

Having implemented the model of figure 4.1(a) we use the linearised frequency feedback model shown in figure 3.2

The output frequency noise of the laser has the form:

$$w(t) = n(t) - h(t) \otimes w(t)$$

which implies:

$$w(t) = n(t) \otimes \mathcal{F}\mathcal{T}^{-1}\left(\frac{1}{1 + H(f)}\right)$$

If we write

$$G(f) = \left(\frac{1}{1 + H(f)}\right)$$

then

$$w(t) = n(t) \otimes g(t)$$

The feedback filter $h(t)$ is electronic so it is obviously a low pass filter. we assume this removes the linearity restriction of our model because with a narrow enough low pass filter we are unlikely to move into the nonlinear region of the frequency feedback loop. $G(f)$ will

40

now be a high pass filter. We chose $G(f)$ to be a single pole high pass filter:

$$G(f) = \frac{j2\pi f}{b + j2\pi f} = 1 - \frac{b}{b + j2\pi f} \tag{4.1}$$

which corresponds to $H(f)$ as a simple integrator with gain b. The feedback loop formed is now identical to a first order phase locked loop, except that it is applied to the frequency noise of the laser rather than the phase noise of the laser [Mess 90]. The gain determines the position of the pole $G(f)$.

$$H(f) = \frac{b}{j2\pi f}$$

Because we have an underlying Gausian noise process and we perform only linear operations on it, $w(t)$ will be a zero mean Gaussian random process and it will be specified only by its covariance function, which we write in terms of the filter $g(t)$ as:

$$
\begin{aligned}
K_{ww}(t,s) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\alpha \, d\beta \, K_{nn}(\alpha, \beta) g(t - \alpha) g(s - \beta) \\
&= \frac{\beta}{2\pi} \int_{-\infty}^{\infty} g(t - \alpha) g(s - \alpha) d\alpha
\end{aligned} \tag{4.2}
$$

From equation (4.1) we can write $g(t)$ in a more suitable manner

$$g(t) = \delta(t) - p(t)$$

where;

$$p(t) = be^{-bt} u(t)$$

with $u(t)$ the unit step function. Substituting this into equation 4.2 we get :

$$
\begin{aligned}
K_{ww}(t,s) &= \frac{\beta}{2\pi} \int_{-\infty}^{\infty} d\alpha \left[(\delta(t - \alpha) - p(t - \alpha))(\delta(s - \alpha) - p(s - \alpha)) \right] \\
&= \frac{\beta}{2\pi} \left[\delta(t - s) - p(s - t) - p(t - s) + \int_{-\infty}^{\infty} d\alpha p(t - \alpha) p(s - \alpha) \right]
\end{aligned} \tag{4.3}
$$

It is not the frequency noise which we are interested in however but the phase noise:

$$\theta(t) = 2\pi \int_0^t w(\tau) d\tau$$

$$K_{\theta\theta}(t,s) = (2\pi)^2 \int_0^t \int_o^s K_{ww}(\tau, \tau\prime) d\tau d\tau\prime \tag{4.4}$$

41

Substituting from equation (4.3) into equation (4.4) we get;

$$K_{\theta\theta}(t,s) = (2\pi\beta) \int_0^t \int_0^s \left\{ \delta(\tau - \tau\prime) - p(\tau - \tau\prime) - p(\tau\prime - \tau) + \int_{-\infty}^{\infty} d\alpha\, p(\tau - \alpha)p(\tau\prime) \right\} d\tau d\tau\prime$$

To solve this for $K_{\theta\theta}(t,s)$ when $p(t) = be^{bt}u(t)$ we split the integral into four separate components and solve these individually. This gives:

$$\int_0^t \int_0^s \delta(\tau - \tau\prime) d\tau d\tau\prime = \min(t,s)$$

$$\int_0^t \int_0^s -p(\tau - \tau\prime) d\tau d\tau\prime = -\frac{1}{b} \left\{ b\min(t,s) + e^{-b\max(t,s)} - e^{b|t-s|} \right\}$$

$$\int_0^t \int_0^s -p(\tau\prime - \tau) d\tau d\tau\prime = -\frac{1}{b} \left\{ b\min(t,s) + e^{-b\max(t,s)} - e^{b|t-s|} \right\}$$

$$\int_0^t \int_0^s p(\tau\prime - \tau)p(\tau - \tau\prime) d\tau d\tau\prime = \frac{1}{2b} \left\{ 2b\min(t,s) + e^{-b\min(t,s)} + e^{\max(t,s)} - 1 - e^{-b|t-s|} \right\}$$

therefore:

$$K_{\theta\theta}(t,s) = \frac{2\pi\beta}{2b} \left\{ 1 + e^{-b|t-s|} - e^{-bt} - e^{-bs} \right\} \tag{4.5}$$

We will now normalise this to get the covariance function of the normalised phase noise process $\psi(t)$. Starting with $K_{\theta\theta}(t_1, t_2)$ we define $\theta_1(t) = \sqrt{\gamma}\theta(t)$ with $\gamma = 2\pi\beta T$, thus:

$$K_{\theta_1\theta_1}(\tau_1, \tau_2) = \frac{1}{2\pi\beta T} K_{\theta\theta}(t_1, t_2)$$

substituting $\psi(\tau) = \theta_1(\tau T)$ gives;

$$K_{\psi\psi}(\tau_1, \tau_2) = \frac{1}{2\pi\beta} K_{\theta\theta}(\tau_1 T, \tau_2 T)$$

therefore :

$$
\begin{aligned}
K_{\psi\psi}(t,s) &= \frac{1}{2bT} \left\{ 1 + e^{-bT|t-s|} - e^{-bTs} - e^{-bTt} \right\} \\
&= \frac{1}{2r} \left\{ 1 + e^{-r|t-s|} - e^{-rt} - e^{-rs} \right\}
\end{aligned}
\tag{4.6}
$$

Where r is a dimensionless parameter equal to bT that indicates the product of the filter bandwidth and the integration time T at the receiver. It is interesting to check this result in the limit of $r \to 0$. This should reduce to the Brownian motion case with $K_{\psi\psi}(t,s) = \min(t,s)$. We expand the exponential into a power series $e^x = 1 + x + x^2 + \cdots$ and neglect terms of order higher than r^2.

42

$$
\begin{aligned}
K_{\psi\psi}(t,s) &= \lim_{r \to 0} \frac{1}{2r} \{1 + [1 - r(\max(t,s) - \min(t,s))] - (1 - r\max(t,s)) - (1 - r\min(t,s))\} \\
&= \lim_{r \to 0} \frac{1}{2r} [2r \min(t,s)] \\
&= \min(t,s)
\end{aligned}
$$

The covariance function has a very surprising property. Unlike the original phase noise process for which the variance is a linearly increasing function of time:

$$
\sigma^2(t) = 2\pi\beta t
$$

The covariance function with feedback is:

$$
K_{\theta\theta}(t,s) = \frac{2\pi\beta}{2b} \left[1 + e^{-b|t-s|} - e^{-bt} - e^{-bs}\right]
$$

corresponding to a variance:

$$
\sigma_\theta^2(t) = \frac{2\pi\beta}{2b} \left[1 - e^{-bt}\right]
$$

This no longer increases grows linearly with time. Instead it tends asymptotically to a value of $2\pi\beta/b$. This has interesting implications for a system which uses a DPSK modulation format. To understand this better, we first look at the problem with a Brownian motion phase noise which is treated by Azizoglu [Aziz 91].

In DPSK the bit stream to be transmitted is differentially encoded and then impressed on the phase output of the laser. This gives a received signal of the form:

$$
s(t) = A \cos(2\pi f_c t + \theta(t) + b_n \pi)(n-1)T \le t \le nT
$$

b_n is the n'th differentially encoded bit, and the actual bit is decoded from b_n by:

$$
a_n = b_n \oplus b_{n+1}
$$

with $\oplus$ the exlusive OR operation. The receiver must decide from the phase information of the signal if a zero or a one has been encoded. The simplest model assumes the total phase information can be extracted exactly (no additive noise at the receiver). Given that the total phase is known, a decision can be made by:

$$
\overline{\Delta\Phi} = \frac{1}{T} \int_{(n-1)T}^{nT} [\theta_{total}(t) - \theta_{total}(t-T)]dt
$$

43

$$= \frac{1}{T} \int_{(n-1)T}^{nT} [\theta(t) - \theta(t-T)]dt + a_n \pi$$

The error floor of this system (the probability of error that exists as the signal to noise ratio tends to infinity) is due to the random variable $1/T \int_{(n-1)T}^{nT} [\theta(t) - \theta(t-T)]dt$.

When fequency feedback stabilisation is applied to the transmitting laser, we can try to use the feedback parameter b to reduce the variance of $1/T \int_{(n-1)T}^{nT} [\theta(t) - \theta(t-T)]dt$. This is likley to improve the system performance. Therefore an interesting problem is to see if frequency feedback stabilisation of the laser will give a greater performance gain for DPSK than it does for FSK.

However in this thesis weare interested in using the modified covariance function to find the statistics of $X = \left| \int_0^1 e^{j\sqrt{\gamma}\psi(t)} dt \right|^2$, to analyse the system performance for FSK.

4.3   Solution of series expansion

We are going to find the statistics of Foschini's approximation:

$$X_L = 1 - \gamma \left[\int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt \right)^2 \right]$$

To do this we use the results of sections 3.2 and 3.3. The covariance function we have satisfies the conditions of section 3.3 namely that $K_{\psi\psi}(t,s)$ includes only functions of t, s that involve a constant, a function of $|t - s|$,a function of t only and a function of s only. Therefore the statistics of

$$\Psi_k = \int_{k-1}^k \psi^2(t)dt - \left(\int_{k-1}^k \psi(t)dt \right)^2$$

do not depend on k and we need only consider the random variable

$$\Psi_0 = \int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt \right)^2$$

To find the statistics of Ψ in terms of a single summation we apply the results of section 3.2 and find the Karhuen Loeve expansion of the new random process

$$y(t) = \psi(t) - \int_0^1 \psi(u)du$$

Following the analysis we had in section 3.3 we conclude that we need only solve the integral

equation:

$$\int_0^1 \left[K_{\psi\psi}(t, s) - \int_0^1 K_{\psi\psi}(t, s)dt \right] \phi(s)ds = \lambda\phi(t) \qquad (4.7)$$

For the given phase noise process we have to solve:

$$\int_0^1 \frac{1}{2r} \left[1 - e^{-rt} - e^{-rs} + e^{-r|t-s|} \right] \phi(s)ds$$
$$\int_0^1 \int_0^1 \frac{1}{2r} \left[1 - e^{-rt} - e^{-rs} + e^{-r|t-s|} \right] dt\phi(s)ds = \lambda\phi(t) \qquad (4.8)$$

Differentiating this three times with respect to t leads to the differential equation;

$$\frac{d^3\phi(t)}{dt^3} = \left(r^2 - \frac{1}{\lambda} \right) \frac{d\phi(t)}{dt}$$

This has a general solution:

$$\phi(t) = A + B \sin\beta t + C \cos\beta t$$

where

$$\beta = \sqrt{\left(\frac{1}{\lambda} - r^2 \right)} \qquad (4.9)$$

This is substituted back into equation (3.7) to satisfy the required boundary conditions, a procedure that we performed using Maple, a symbolic algebra package. The details are given in the appendix and the final equation that β must satisfy is:

$$-2\beta^2 r - 2r^3 + (-2\beta r^2 - \beta^3 r - 2\beta^3 + r^3\beta) \sin\beta$$
$$+ (2r\beta^2 + 2r^2\beta^2 + 2r^3) \cos\beta = 0 \qquad (4.10)$$

It is interesting to look at the solution to this equation for small values of r to gain insight to the change in the eigenvalues as a function of r. For small r the roots of equation 4.10 will have the form:

$$\beta_i = i\pi + \epsilon_i$$

when ϵ_i is a small perturbation:

$$\sin(i\pi + \epsilon_i) \simeq \epsilon_i$$

$$\cos(i\pi + \epsilon_i) \simeq 1$$

Thus:

$$-2(i\pi + \epsilon_i)^2 r - 2r^3 + [-2(i\pi + \epsilon_i)r^2 - (i\pi + \epsilon_i)^3 r - 2(i\pi + \epsilon_i)^3 + (i\pi + \epsilon_i)r]$$

45

$$+ \quad [2r(i\pi + \epsilon_i)^2 + 2r^2(i\pi + \epsilon_i)^2 + 2r^3] \simeq 0$$

$$\epsilon_i[-2r^2(i\pi) - r(i\pi)^3 - 2(i\pi)^3 + r^3(i\pi)] + 2r^2[(i\pi)^2 + 2(i\pi)\epsilon_i] \simeq 0$$

$$
\begin{aligned}
\epsilon_i &= \frac{-2r^2(i\pi)^2}{4(i\pi)r^2 - 2r^2(i\pi) - r(i\pi)^3 - 2(i\pi)^3 + r^3(i\pi)} \\
&\simeq \frac{-2r^2(i\pi)^2}{-2(i\pi)^3 - r(i\pi)^3} \\
&\simeq \frac{r^2}{i\pi(1 + r/2)} \\
&\simeq \frac{r^2 - r^3/2}{i\pi}
\end{aligned}
$$

and for λ_i we see:

$$
\begin{aligned}
\lambda_i &= \frac{1}{r^2 + (i\pi + \epsilon_i)^2} \\
&= \frac{1}{r^2 + (i\pi)^2 + 2r^2 - r^3}
\end{aligned}
$$

For small values of r the decrease in λ_i is due equally to the increase in β_i and the introduction of r^2 into the denominator of the above equation. A plot of the first two eigenvalues as a function of r is given in the section on results.

We solve this equation fully for large values of r using Matlab and from equation (3.9) we will find the eigenvalues λ which satisfy the original integral equation (3.7). This gives us our statistical description for X the form :

$$
\begin{aligned}
\Psi &= \int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt\right)^2 \\
&= \int_0^1 y^2(t)dt - \left(\int_0^1 y(t)dt\right)^2 \\
&= \sum_{i=0}^{\infty} y_i^2 \quad - y_0^2 \\
&= \sum_{i=1}^{\infty} y_i^2
\end{aligned}
$$

Where the y_i's are zero mean independent Gaussian random variables with $E(y_i^2) = \lambda_i$. Thus we can give the result in terms of a set of independent identically distributed zero

mean Gaussian random variables of unit variance $x_i \sim N[0, 1]$.

$$\Psi = \sum_{i=1}^{\infty} \frac{1}{r^2 + \beta^2} x_i^2$$

$$= \sum_{i=1}^{\infty} \lambda_i x_i^2$$

The characteristic function of this random variable is:

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} P_\Psi(\psi) e^{j2\pi f \psi} d\psi$$

$$= \prod_{i=1}^{\infty} \frac{1}{(1 - j4\pi f \lambda_i)^{1/2}} \qquad (4.11)$$

The associated density function is found for different eigenvalue solutions using an Inverse Fast Fourier Transform routine.

4.4 Calculations for the Receiver Probability of Error

When we take the inverse fast Fourier transform of equation (4.11) we improve the performance of the routine by using a set of tilted random variables. Consider a time process $y(t)$ and a Fourier transform $Y(f)$. If we take the Fourier transform of $y(t)e^{at}$ with a in the region of convergence it gives:

$$y(t) \leftrightarrow Y(f)$$

$$y(t)e^{at} \leftrightarrow Y(f - \frac{a}{j2\pi})$$

Therefore we get the actual density of the random variable Ψ by;

$$P_\Psi(\psi) = \mathcal{F}\mathcal{T}^{-1} \left[\prod_{i=1}^{\infty} \frac{1}{(1 - j4\pi f \lambda_i)}^{1/2} \right]$$

$$= e^{-\lambda_1 t/2.5} \mathcal{F}\mathcal{T}^{-1} \left[\prod_{i=1}^{\infty} \frac{1}{(1 - j4\pi(f - \lambda_1/j5\pi)\lambda_i)}^{1/2} \right]$$

The improved results are because we are evaluating the inverse Fourier transform for a smoother function of t, therefore the Gibbs ripple at the edges of the transform will be reduced. The choice of $a = \lambda_1/2.5$ came from considering the form of $P_\Psi(\psi)$.

$$\Psi = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} x_i^2$$

is a sum of independent chi-squared variates. The chi-square distribution has a tail density

of the form $\sim e^{-t\lambda_i/2}$ and $P_\Psi(\psi)$ is the convolution of an infinite number of these densities. It will be dominated by the variate with the sharpest tail distribution, which corresponds to the largest eigenvalue λ_i. By chosing $a = \lambda_1/2.5$ we reduce the effect of the largest eigenvalue on $P_\Psi(\psi)$. We use $\lambda_1/2.5$ rather than $\lambda_1/2$ because the data points for the IFFT run from $f = -10.2$ to 10.2 at intervals of 0.1 and includes the point $f = 0$. If we used $a = \lambda_1/2$ in the function that is being sampled:

$$\prod_{i=1}^{\infty} \frac{1}{(1 + j4\pi(f - \lambda_1/j4\pi)\lambda_i)^{1/2}}$$

is undefined at $f - 0$ and so $a = \lambda_1/2.5$ is chosen as it gave the best performance when we evaluated the IFFT routine.

4.4.1 Solution for the Linear Approximation

The linear approximation is when we use X_L instead of the actual random variable X, where:

$$\begin{aligned} X_L &= 1 - \gamma \left[\int_0^1 \psi^2(t)dt - \left(\int_0^1 \psi(t)dt \right)^2 \right] \\ &= 1 - \gamma\Psi \end{aligned}$$

rather than

$$X = \left| \int_0^1 e^{j\sqrt{\gamma}\psi(t)}dt \right|^2$$

We have the density of Ψ for different values of r from the last section and Ψ takes values in the range $-\infty \le \psi \le 1$ rather than the actual range of X which is the interval $[0, 1]$. Using X_L we improve the approximation in two separate ways . Both begin by truncating the density of X_L so that it will lie in the correct interval. The simplest way to do this is to set the density of P_{X_L} to zero for all values of X_L less than zero. While this is a simple result it will lack somewhat in physical significance as this density will not give an upper bound on the received probability of error. The alternative approach retains the upper bound nature of the original approximation. We set the density to zero for values of X_L less than zero and then lump the removed probability together as an impulse at the origin. This will give us two different densities for the random variable X_L: once we have these we use it to remove the conditioning of the received probability of error.

$$Pe = E_{X_L}\left[\frac{1}{2}e^{-\zeta X_L/2} \right]$$

The results of these two different approximations are given in the next chapter.

4.4.2 Results for the Exponential Approximation

This uses an approximation proposed by Azizoglu and Humblet, which they call their "exponential" approximation. They chose it to fall in the same interval as the original random variable. The definition is :

$$X_E = e^{-\gamma \Psi}$$

We do not find the received probability of error by evaluating the statistics of X_E through a transformation of the random variable Ψ and then using the conditional error probability in terms of of X_E. Instead we use

$$Pe(X_E) = \frac{1}{2} e^{-\zeta X_E/2}$$

from which:

$$Pe = E_\Psi \left[\frac{1}{2} \exp(-\zeta/2(\exp(-\gamma \Psi))) \right]$$

When we use this approximation we do not get a bound on Pe. However we might expect it to lie closer to the actual Pe than the linear estimate simply because it originally lies in the correct range $[0, 1]$.

4.4.3 Jensen's Bound

This is another idea to find an approximation for the probability of error that is used by Azizoglu and Humblet in their paper. The conditional probability of error is a convex function of X in the interval $[0, 1]$

$$Pe(X) = \frac{1}{2} e^{(-\zeta X/2)}$$

therefore the average error is bounded by :

$$Pe \geq \frac{1}{2} e^{-\zeta \overline{X}/2}$$

We find $\overline{X}$ from the covariance function of the phase noise process.

$$
\begin{aligned}
\overline{X} &= E \left| \int_0^1 e^{j\sqrt{\gamma}\psi(t)} dt \right|^2 \\
&= E \left| \int_0^1 e^{j\sqrt{\gamma}\psi(t)} dt \int_0^1 e^{j\sqrt{\gamma}\psi(s)} ds \right| \\
&= E \left| \int_0^1 \int_0^1 e^{j\sqrt{\gamma}[\psi(t)-\psi(s)]} dt ds \right|
\end{aligned}
$$

$[\psi(t) - \psi(s)]$ is a zero mean Gaussian random variable that has a variance which depends on t, s. If we define

$$f(t,s) \stackrel{\triangle}{=} E[(\psi(t) - \psi(s))^2]$$

and then take the expectation inside the integrals we get :

$$\overline{X} = \int_0^1 \int_0^1 E \left| e^{j\sqrt{\gamma}[\psi(t) - \psi(s)]} \right| dtds$$

and $E[\exp(j\sqrt{\gamma}[\psi(t) - \psi(s)])]$ is just the characteristic function of a zero mean Gaussian random variable with variance $f(t,s)$ that is evaluated at $\sqrt{\gamma}$. Looking it up in [Drake 87] we get for $\overline{X}$

$$\overline{X} = \int_0^1 \int_0^1 dtds e^{-\gamma f(t,s)} dtds$$

This is a general result for an Gaussian Phase noise. To specialise it to the case of interest we must evaluate $f(t,s)$.

$$
\begin{aligned}
f(t,s) &= E[(\psi(t) - \psi(s))^2] \\
&= K_{\psi\psi(t,t)} + K_{\psi\psi(s,s)} - 2K_{\psi\psi(t,s)} \\
&= \frac{1}{2r}\left[2 - 2e^{-rt} + 2 - 2e^{-rs} - 2e^{-r|t-s|} + 2e^{-rt} + 2e^{-rs} - 2\right] \\
&= \frac{1}{r}\left[1 - e^{-r|t-s|}\right]
\end{aligned}
$$

This gives :

$$
\begin{aligned}
\overline{X} &= \int_0^1 \int_0^1 \exp(-\gamma/r(1 - \exp(-r|t-s|)))dtds \\
&= e^{-r/\gamma} \int_0^1 \int_0^1 \exp(\frac{\gamma}{r}\exp(-r|t-s|))dtds \\
&= 2e^{-r/\gamma} \int_{s=0}^1 \int_{t=s}^1 \exp(\frac{\gamma}{r}\exp(-r(t-s)))dtds \quad\quad (4.12)
\end{aligned}
$$

The value of $\overline{X}$ was evaluated for the same values of γ and r as the other approximations and substituted back into the conditional probability of error equation, to give a lower bound on Pe. Figure 4.3 compares the actual mean of X to $\overline{X_L}$, the mean of the linear approximation and $\overline{X_E}$, the mean of the exponential approximation. The closeness of $\overline{X_E}$ to $\overline{X}$ indicates that for large values of γ X_E will be a good approximation.

In the next chapter on results, we use the calculated density to find the performance curves for our frequency stabilised optical system.
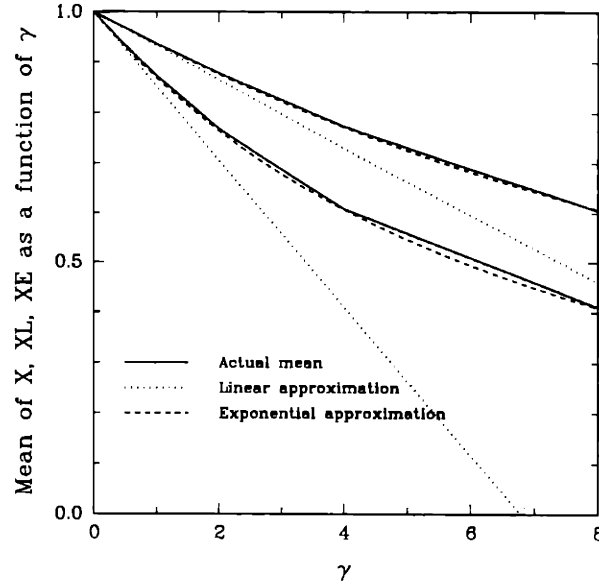
Figure 4-3: Mean of statistics $\overline{X}$, $\overline{X_L}$, and $\overline{X_E}$ for $r = 1$, $r = 5$.

Appendix

4.A   Derivation of Equation 4.10

The integral equation (4.7) must be solved:

$$\int_0^1 \left[K_{\theta\theta}(t,s) - \int_0^1 K_{\theta\theta}(t,s)dt \right] \phi(s)ds = \lambda\phi(t) \tag{4.13}$$

when $K_{\theta\theta}(t,s)$ is given by:

$$K_{\theta\theta}(t,s) = \frac{1}{2r} \left[1 - e^{-rt} - e^{-rs} + e^{-r|t-s|} \right] \tag{4.14}$$

The integral equation is differenciated with respect to t using Liebzitz' rule for differenciating under an integral. To do this we realise that

$$\int_0^1 K_{\theta\theta}(t,s)dt$$

is not a function of t, therefore differenciating this term once will give zero and it may be neglected. The remaining term includes a function of $|t - s|$ and to treat it we use the

standard trick of splitting the integral into two parts. Therefore we will differentiate:

$$\int_0^1 \frac{1}{2r}\left[1 - e^{-rt} - e^{-rs}\right]\phi(s)ds + \int_0^t \frac{1}{2r}e^{-r(t-s)}\phi(s)ds + \int_t^1 \frac{1}{2r}e^{-r(s-t)}\phi(s)ds = \lambda\phi(t)$$

$$(4.15)$$

After three differenciations this produces the equation:

$$\frac{d^3\phi(t)}{dt^3} = \left(r^2 + \frac{1}{\lambda}\right)\frac{d\phi(t)}{dt} \tag{4.16}$$

which has solution

$$\phi(t) \;=\; a + B\sin\beta t + C\cos\beta t$$
$$\beta \;=\; \sqrt{r^2 - \frac{1}{\lambda}} \tag{4.17}$$

To satisfy the initial boundary conditions this is substituted back into the integral equation. Therefore for all t we must satisfy:

$$\lambda(A + B\sin\beta t + C\cos\beta t) = \int_0^1\left[K_{\theta\theta}(t,s) - \int_0^1 K_{\theta\theta}(t,s)dt\right](A + B\sin\beta t + C\cos\beta t)ds$$

$$(4.18)$$

We get three expressions one for each coefficient A, B, C. They are:

$$\begin{aligned}
\lambda A \;=\; & A\left[\frac{1}{2r^2} + \frac{1}{2r} + e^{-r}\frac{1}{2r^2}\right]\\
& + \; Ae^{-rt}\left[-\frac{1}{2r} - \frac{1}{2r^2}\right]\\
& + \; Ae^{rt}\left[-e^{-r}\frac{1}{2r^2}\right]
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
B\sin\beta t \;=\; & B\frac{\sin\beta t}{(r^2 + \beta^2)}\\
& + \; B\left[\frac{-r\cos\beta}{2\beta(r^2+\beta^2)}\frac{-\beta\cos\beta}{2r(r^2+\beta^2)} + \frac{-\beta e^{-r}\cos\beta}{2r(r^2+\beta^2)} + \frac{e^{-r}\sin\beta}{2(r^2+\beta^2)} + \frac{-r}{2\beta(r^2+\beta^2)}\right]\\
& + \; Be^{-rt}\left[\frac{r\cos\beta}{2\beta(r^2+\beta^2)} + \frac{\beta\cos\beta}{2r(r^2+\beta^2)} - \frac{r}{2\beta(r^2+\beta^2)}\right]\\
& + \; Be^{rt}\left[-\frac{\beta\cos\beta e^{-r}}{2r(r^2+\beta^2)} - \frac{\sin\beta e^{-r}}{2(r^2+\beta^2)}\right]
\end{aligned} \tag{4.20}$$

$$C\cos\beta t \;=\; C\frac{\cos\beta t}{(r^2+\beta^2)}$$

$$+ \quad C \left[\frac{r \sin\beta}{2\beta(r^2 + \beta^2)} \frac{\beta \sin\beta}{2r(r^2 + \beta^2)} + \frac{e^{-r} \cos\beta}{2(r^2 + \beta^2)} + \frac{\beta e^{-r} \sin\beta}{2r(r^2 + \beta^2)} + \frac{-1}{2(r^2 + \beta^2)} \right]$$

$$+ \quad Ce^{-rt} \left[\frac{-r \sin\beta}{2\beta(r^2 + \beta^2)} - \frac{b \sin\beta}{2r(r^2 + \beta^2)} - \frac{1}{2(r^2 + \beta^2)} \right]$$

$$+ \quad Ce^{rt} \left[-\frac{cosbe^{-r}}{2(r^2 + \beta^2)} + \frac{\beta \sin\beta e^{-r}}{2r(r^2 + \beta^2)} \right] \tag{4.21}$$

For the boundary conditions that have been satisfied, A, B, C must ensure that the right hand side of the the equation always equals the left hand side. We form a matrix equation in terms of A, B, C where the first row is the coefficient of 1, the second row is the coefficient of e^{-rt} and the third row is the coefficient of e^{rt}. These coefficients must be equal to zero for all times, therefore we generate a determinant equation. After a few stages of simplification this gives:

$$\begin{vmatrix} -\lambda & 0 & \frac{1}{(r^2+\beta^2)} \\ -\frac{1}{2r} - \frac{1}{2r^2} & \frac{r\cos\beta}{2\beta(r^2+\beta^2)} + \frac{\beta\cos\beta}{2r(r^2+\beta^2)} - \frac{r}{2\beta(r^2+\beta^2)} & -\frac{r\sin\beta}{2\beta(r^2+\beta^2)} - \frac{\beta\sin\beta}{2r(r^2+\beta^2)} - \frac{1}{2(r^2+\beta^2)} \\ \frac{-e^{-r}}{2r^2} & \frac{-\beta\cos\beta e^{-r}}{2r(r^2+\beta^2)} - \frac{\sin\beta e^{-r}}{2(r^2+\beta^2)} & \frac{-\cos\beta e^{-r}}{2(r^2+\beta^2)} + \frac{\beta\sin\beta e^{-r}}{2r(r^2+\beta^2)} \end{vmatrix}$$

Solving this gives us equation 4.22

$$-2\beta^2 r - 2r^3 \quad + \quad (-2\beta r^2 - \beta^3 r - 2\beta^3 + r^3\beta)\sin\beta$$
$$+ \quad (2r\beta^2 + 2r^2\beta^2 + 2r^3)\cos\beta = 0 \tag{4.22}$$

This gives the solution for β for any strenght of the applied feedback. From β we can calculate the eigenvalues using :

$$\lambda = \frac{1}{\beta^2 + r^2}$$

It should be noted that we have multipled through by β^3 to simplify the analysis. Therefore $\beta = 0$ is not a solution of the actual eigenvalue equation.

Chapter 5

Results

In this chapter the results obtained for four different approximations to the statistics of the phase noisy envelope of the received signal are presented . The receiver that we use is repeated in figure $5-1$ to emphasise that this is the problem that has been solved. The first section gives the statistics of Ψ.

$$\Psi = \int_0^1 \psi(t)dt - \left(\int_0^1 \psi(t)dt\right)^2$$

on which the first three approximations depend. Once we have these statistics we present the results of using the exponential approximation of Azizoglu and Humblet, followed by the results of the linear approximation introduced by Foschini et al. . The final section gives the results obtained using Jensen's bound for the conditional received probability of error.
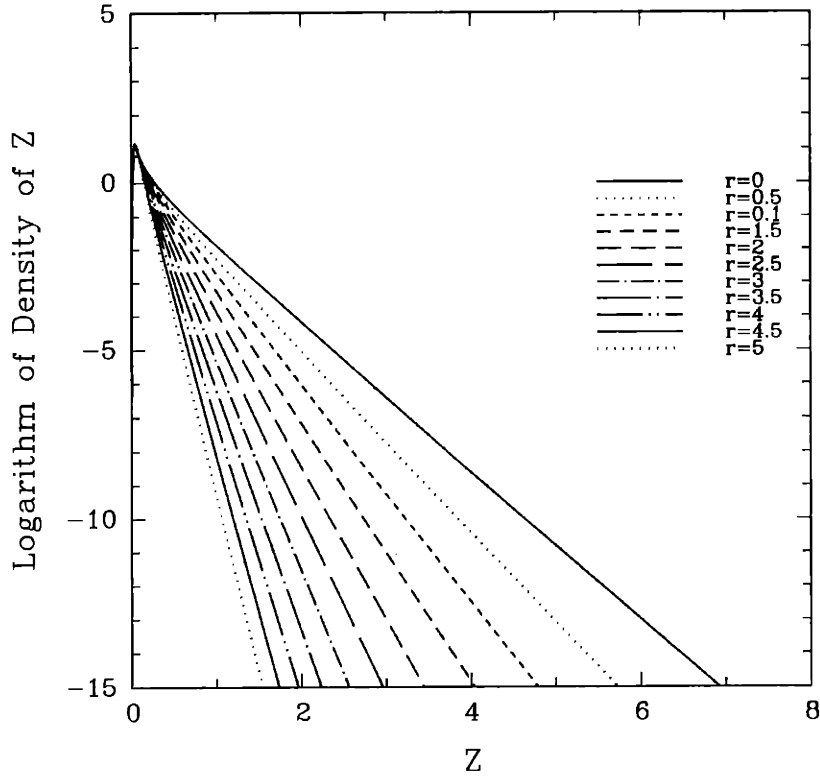
Figure 5-1: Probability density function of Ψ as a function of r and ψ

5.1 Density function of Ψ

When we look at the characteristic function of $P_\Psi(\psi)$ where we defined

$$\Psi = \int_0^1 \psi(t)^2 dt - \left(\int_0^1 \psi(t)dt\right)^2$$

$$\mathcal{FT}P_\Psi(t) = \prod_1^\infty \frac{1}{(1 - j4\pi f\lambda)}^{1/2}$$

It corresponds to a convolution of an infinite number of independent chi-squared random variables, therefore we would expect the tail distribution to be dominated by the largest eigenvalue. This is can be verified to be the case by looking at the form of a chi-square distribution. It has:

$$1/(1 - j4\pi f\lambda)^{1/2} \leftrightarrow \frac{t^{-1/2}e^{-t/2\lambda}}{2^{1/2}\lambda\Gamma(1/2)}$$

which gives a tail distribution:

$$P_\Psi(t) \simeq e^{-t/2\lambda}$$
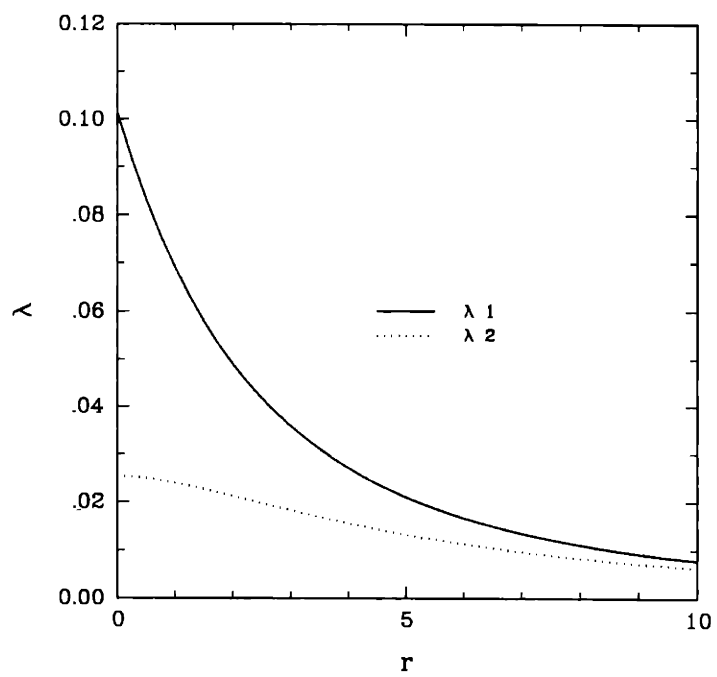
First two values of λ as a function of r

Figure 5-2: First two eigenvalues as a function of r

taking the logarithm of this gives:

$$\log_{10}(P_\Psi(\psi)) \simeq log_{10}(e)/2\lambda$$

The tail density behaves in this manner, therefore the plot of λ_1 as a function of r gives a good indication of how the density function of ψ will behave.

The next three sections all apply the density function of Ψ to different approximations to the received probability of error.

5.2 Exponential Approximation

In the following sections, four different graphs are given. Three show how the received probability of error varies with r, the feedback parameter for a given value of γ which was defined as $\gamma \stackrel{\triangle}{=} 2\pi\beta T$ and gives the original linewidth of the laser before frequency feedback stabilisation is applied. The other graph shows how the received probability of error behaves as a function of γ when r is held constant.
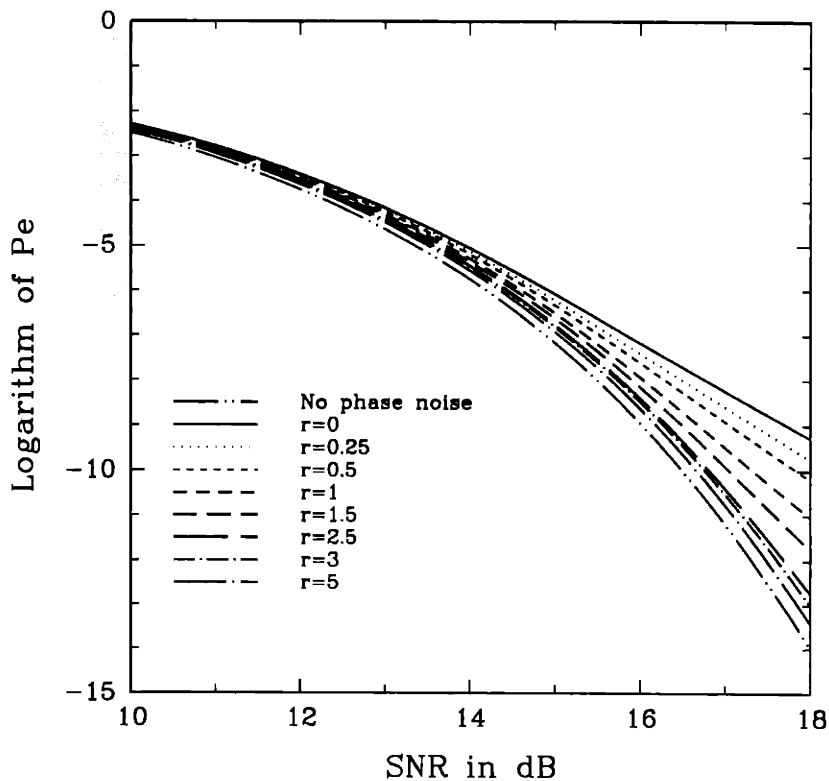
Figure 5-3: Results for different values of r with γ equal to 0.5 using the exponential approximation

The results were calculated using the conditional probability of error:

$$P_e = E_X[1/2e^{-\zeta X/2}]$$

and we use the statistics of X_E to approximate X where:

$$E_X = e^{\gamma \Psi}$$

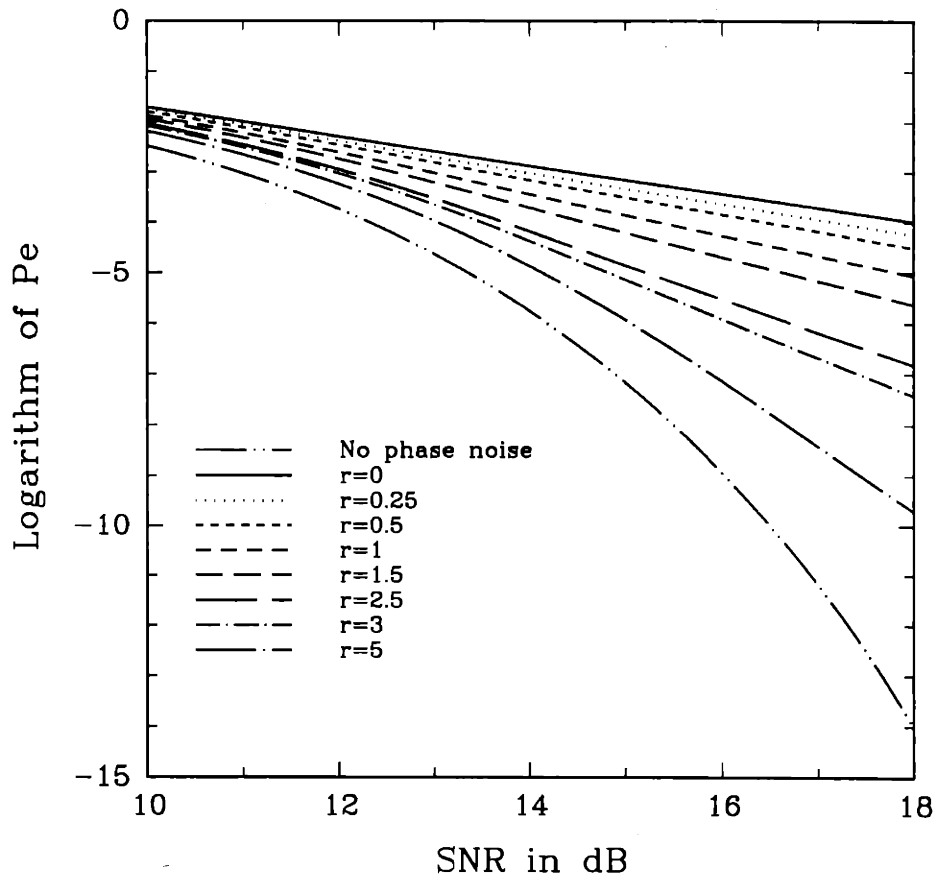The graph clearly shows the improvement in received probability of error as r increases. The gain in performance increases for larger values of ζ, the IF signal to noise ratio. If we extend the curve further we would see a flattening off of the probability of error curves. This is a numerical effect as our probability density function is inaccurate below values of 10^{-15}. In the region shown we expect the results to be accurate.
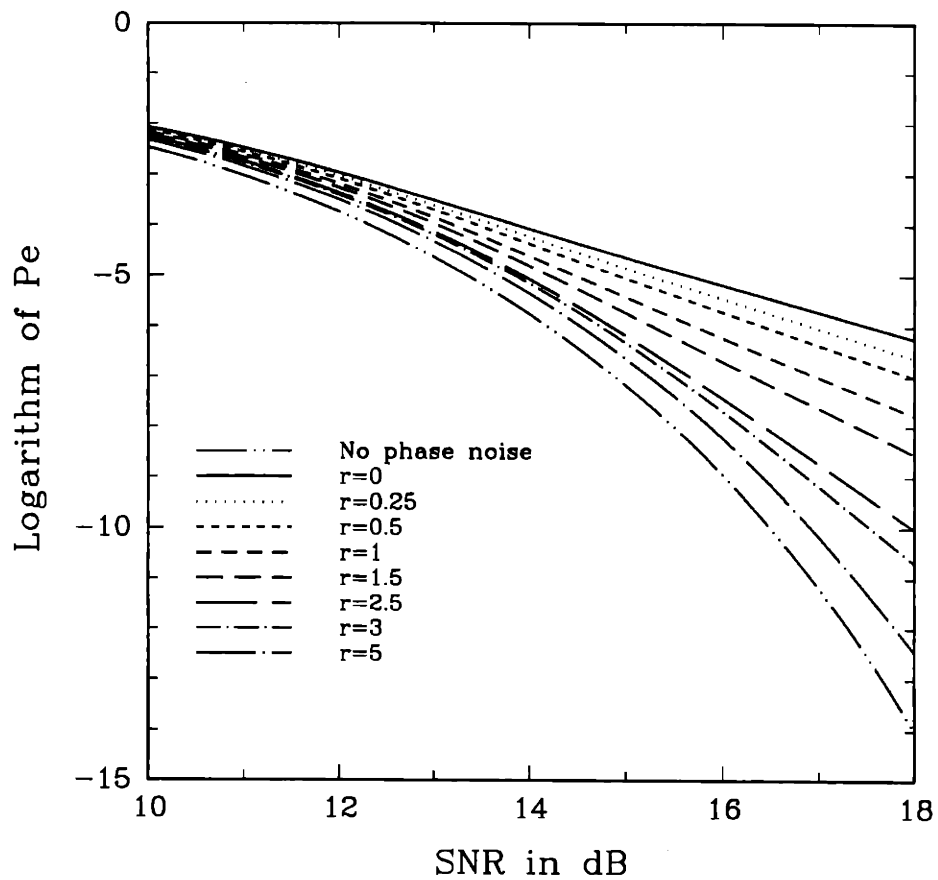
Figure 5-4: Results for different values of r with γ equal to 1 using the exponential approximation

This graph again shows the improvement in received probability of error as r increases. The gain in performance increases for larger values of ζ, the IF signal to noise ratio.

Figure 5-5: Results for different values of r with γ equal to 2 using the exponential approximation

This graph again shows the improvement in received probability of error as r increases. The gain in performance is larger than before because the system performance is so bad with no applied feedback. To get acceptable performance we require to use values of $r = 5$.
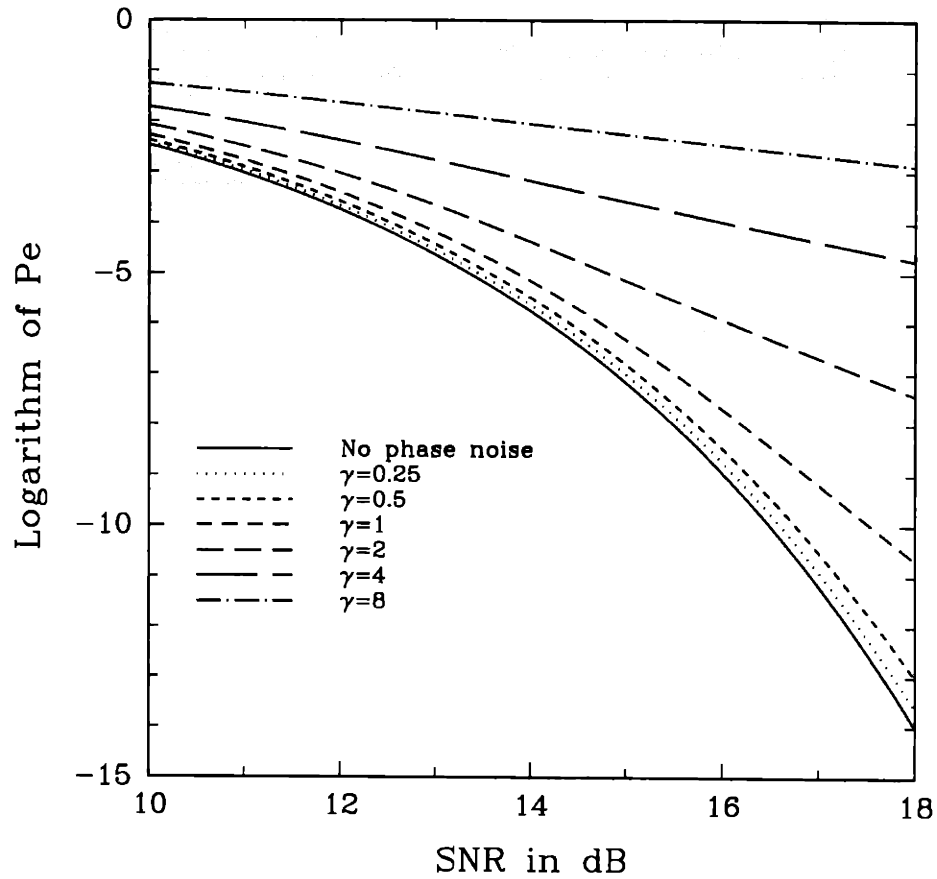
Figure 5-6: Performance for increasing γ with $r = 3$ using the exponential approximation

This figure shows clearly what we stated previously. For small values of γ we come close to the ideal system performance with moderate values of r. However for larger values of γ we have to go to unreasonably high bandwidths of the feedback loop.
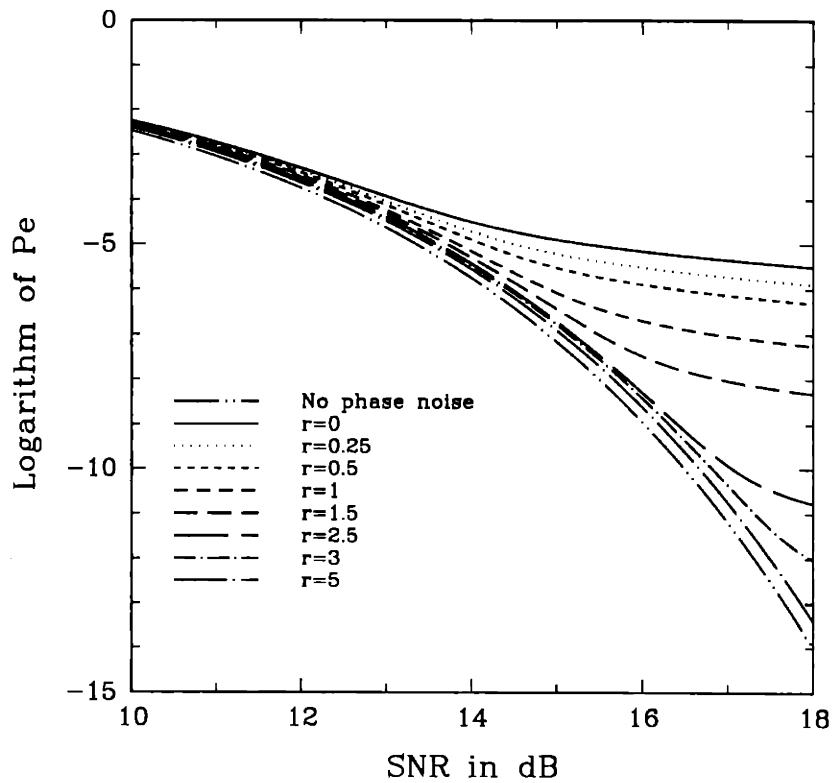
Figure 5-7: Results for different values of r with γ equal to 0.5 using the truncated linear approximation

5.3 Results for the Truncated Linear Approximation

The results of this section apply to the approximation of Foschini et al. . However we lose the upper-bound nature of their solution by throwing away the density of X_L when it becomes negative. Therefore the approximation is:

$$P_{\widehat{X_L}}(\widehat{X_L}) = \begin{cases} P_{X_L}(X_L) & X_L \geq 0 \\ 0 & \text{else} \end{cases} \tag{5.1}$$

The results show the same behaviour as the exponential approximation.
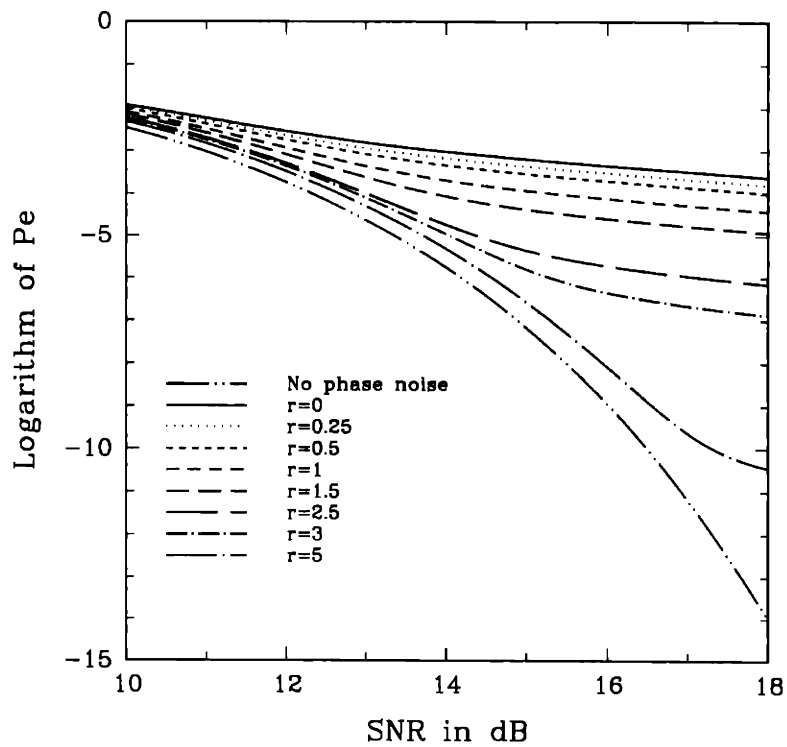
Figure 5-8: Results of P_e for different values of r when γ equals 1 using the truncated linear approximation

This shows an improvement of the received Pe with increasing r, however at large values of SNR the performance flattens off considerably. This is because the probability density function of X_L has not become zero at $X_L = 0$.
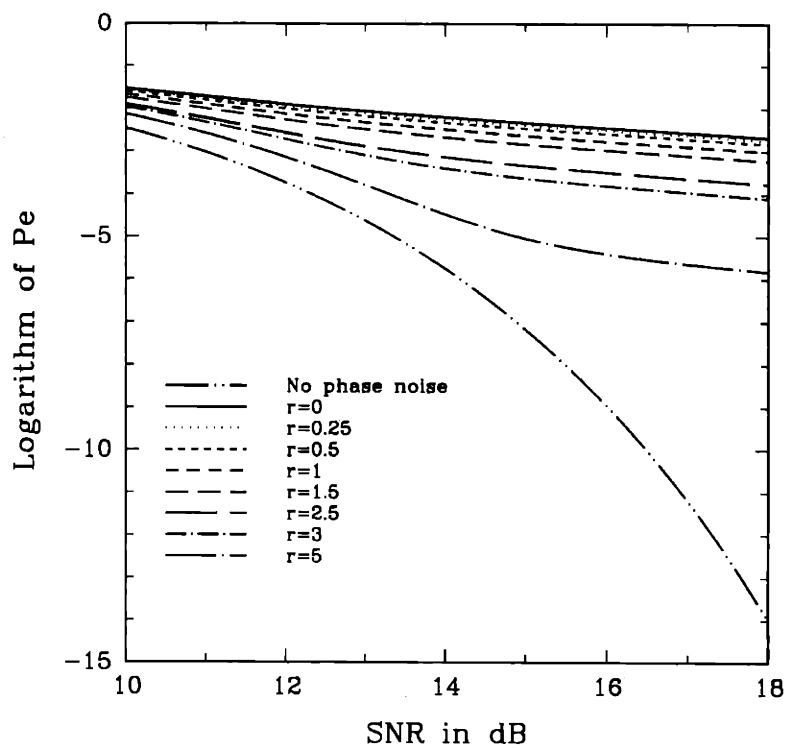
Figure 5-9: Results of P_e for different values of r when γ equals 2 using the truncated linear approximation

As before Pe flattens off. The effect is more marked at larger values of γ.
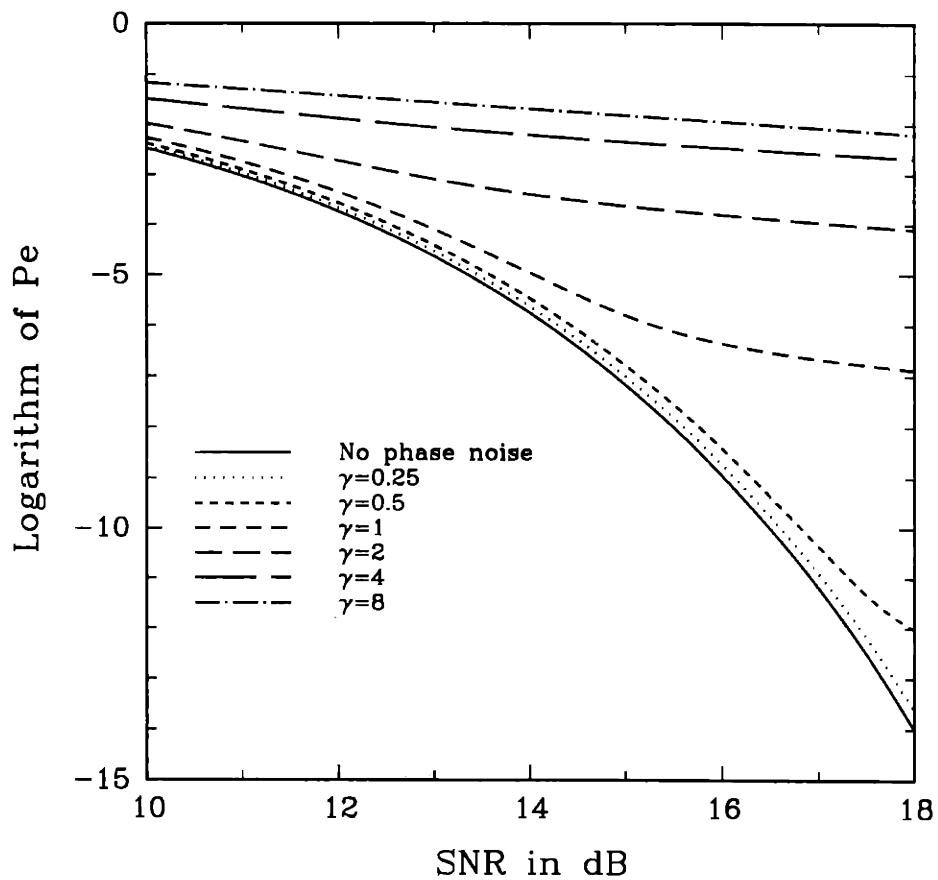
Figure 5-10: Results for different values of γ when $r = 3$ using the truncated linear approximation

The Performance at γ equal to 2 is very poor, even at $r = 3$. This implies that some further filtering ought to be implemented like the double filtering treated by Foschini et al. or Azizoglu and Humblet.
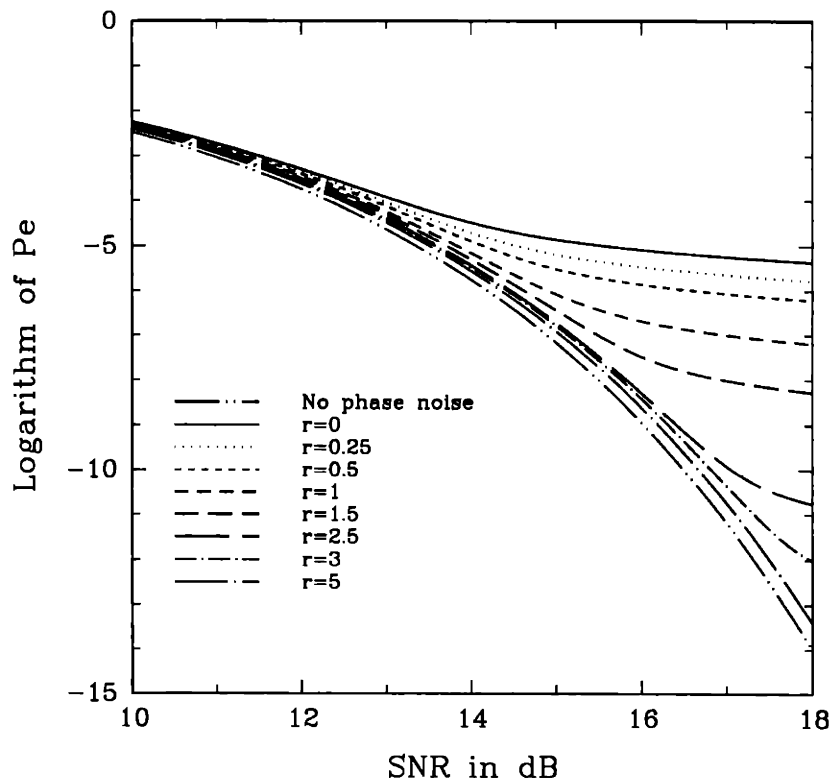
Figure 5-11: Results for different values of r with γ equal to 0.5 for the upper bound approximation

5.4   Upper bound Results Using the Linear Approximation

The results of this section apply to the approximation of Foschini et al. . To retain the upper bound nature of the approximation, the probability of $X_L \leq 0$ is placed as an impulse at the origin. Therefore :

$$\widehat{X_L} = \begin{cases} X_L & X_L \geq 0 \\ 0 & \text{else} \end{cases} \tag{5.2}$$

$$P_{\widehat{X_L}}(X) = \begin{cases} P_{X_L}(X)I(X > 0) + \delta(X)P_{X_L}(X_L \leq 0) & X_L \geq 0 \\ 0 & \text{else} \end{cases} \tag{5.3}$$

The results behave in the same manner as the last section, but all the curves lie at a higher value of received probability of error.
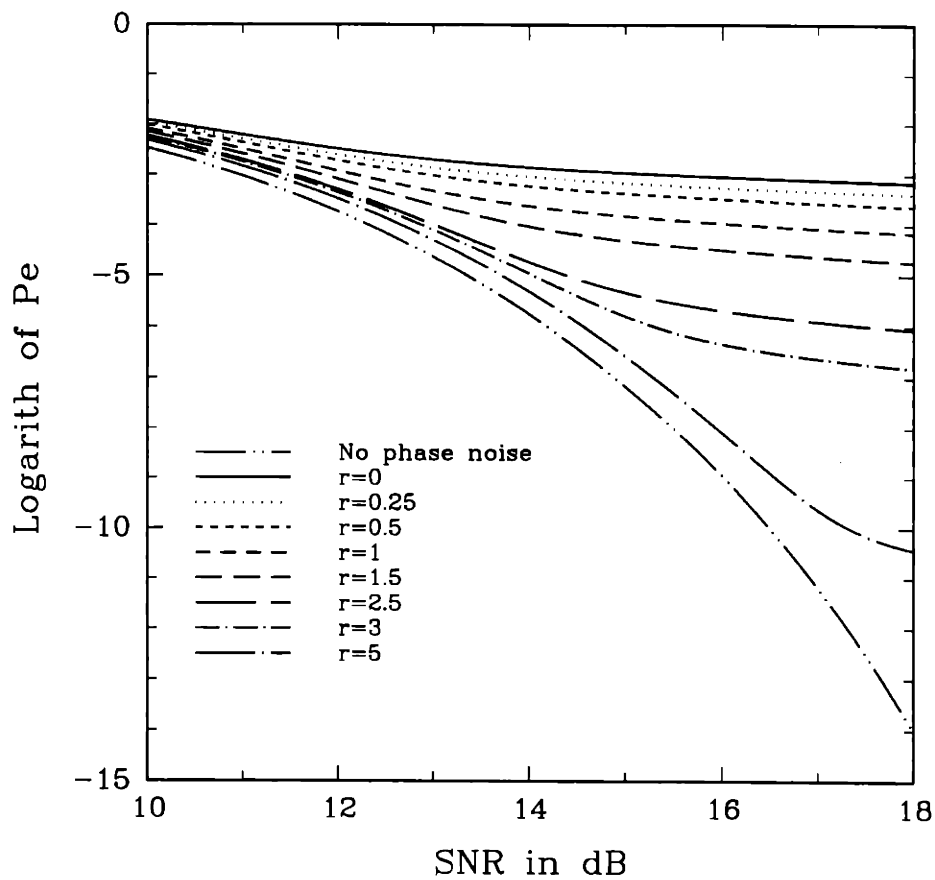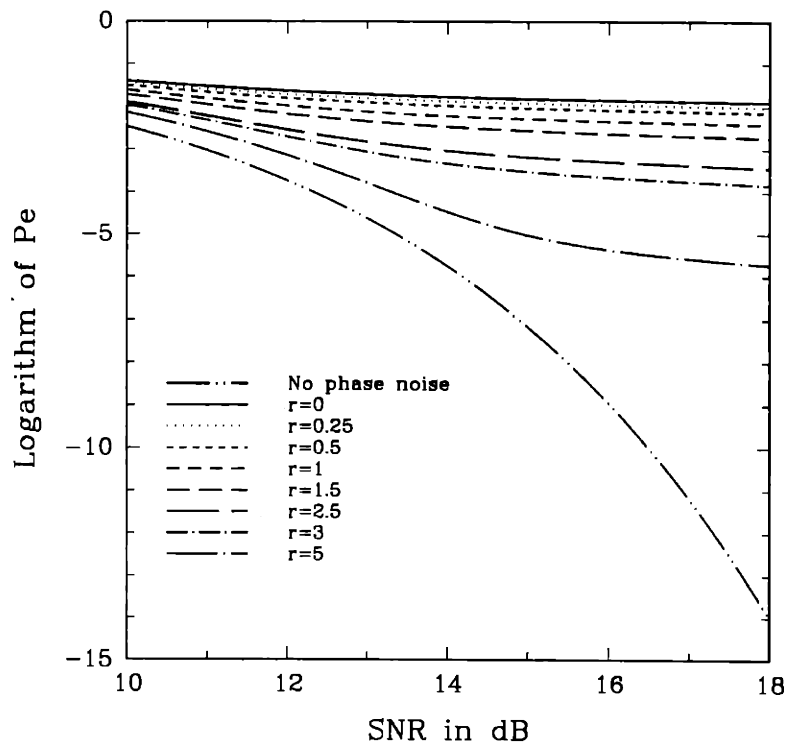
Figure 5-12: Results of P_e for different values of r when γ equals 1 for the upper bound approximation

The surprise here is how little the value of Pe has changed for the upperbound approximation. In fact this is expected as the tail density the is lumped at the origin in this upper bound is very small for small values of γ.

Figure 5-13: Results of P_e for different values of r when γ equals 2 for the upper bound approximation

This shows slightly more clearly the effect of the lumped impulse at the origin, although it is still a small effect.
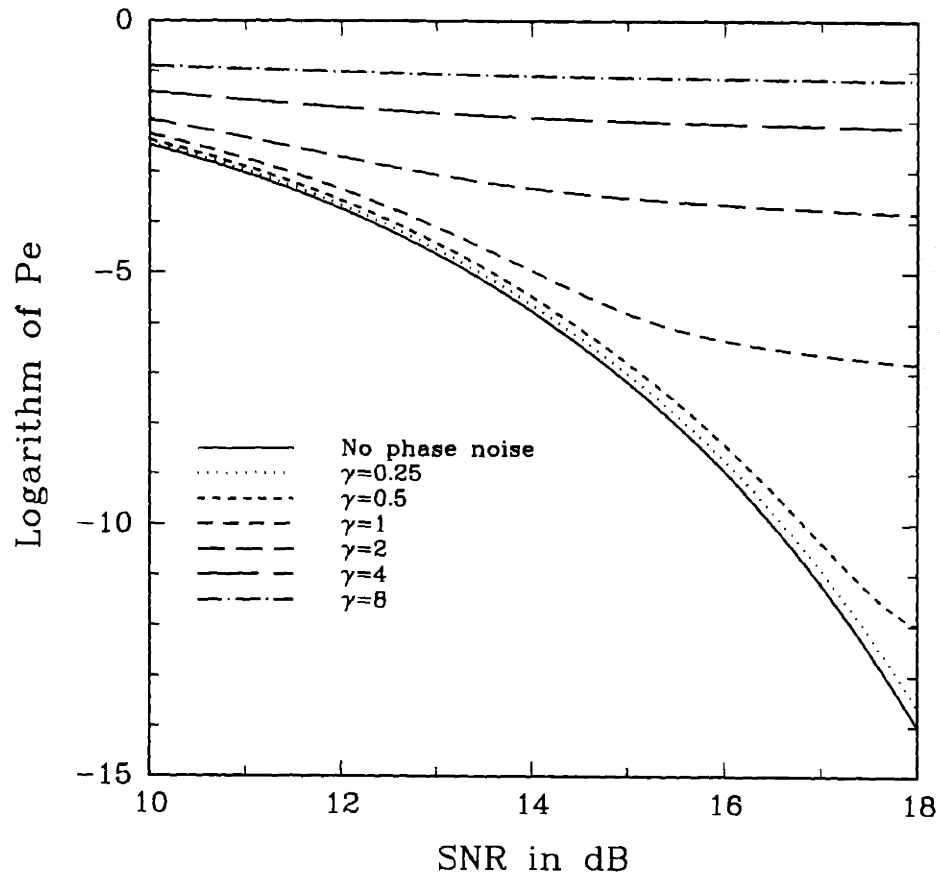
Figure 5-14: Results for different values of γ when $r = 3$ for the upper bound approximation

This just reinforces what we concluded before that as γ gets large this technique of combating phase noise is not sufficiently efficient. We can also see for the larger values of γ how the impulse placed at the origin has shifted the upperbound curves above the truncated approximation.
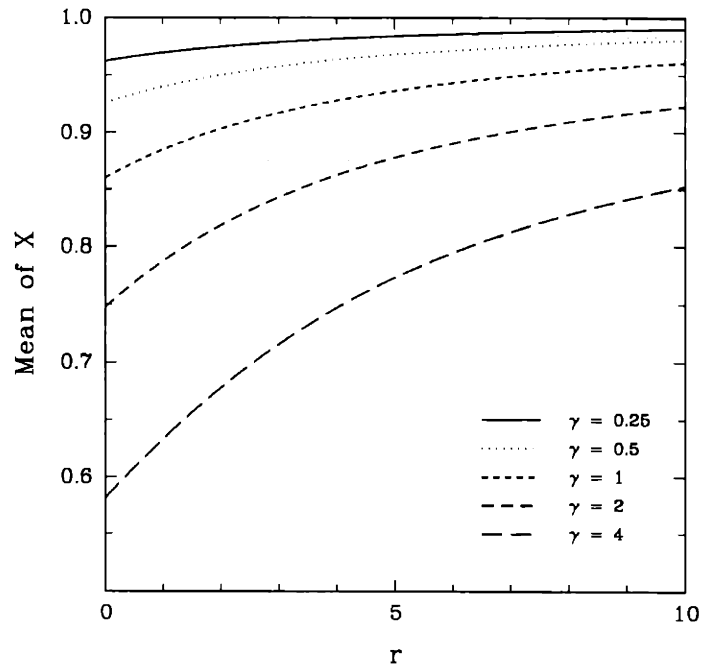
Plot of Mean of X as a function of r

Figure 5-15: $\overline{X}$ as a function of r and γ

5.5 Jensen's Bound

This result differs from those already given as it is not derived using the calculated density function of Ψ. Instead we used the convexity of the conditional error probability $P_e(X)$ in terms of X to get :

$$\overline{P_E} \geq 1/2e^{-\zeta \overline{X}/2}$$

The mean value of X

$$X = \left| \int_0^1 e^{j\sqrt{\gamma}\psi(t)}dt \right|$$

is found for different values of r and the probability of error curves plotted using the above equation. The curve for $\overline{X}$ as a function of r is given as it illustrates how this bound behaves as a function of r.
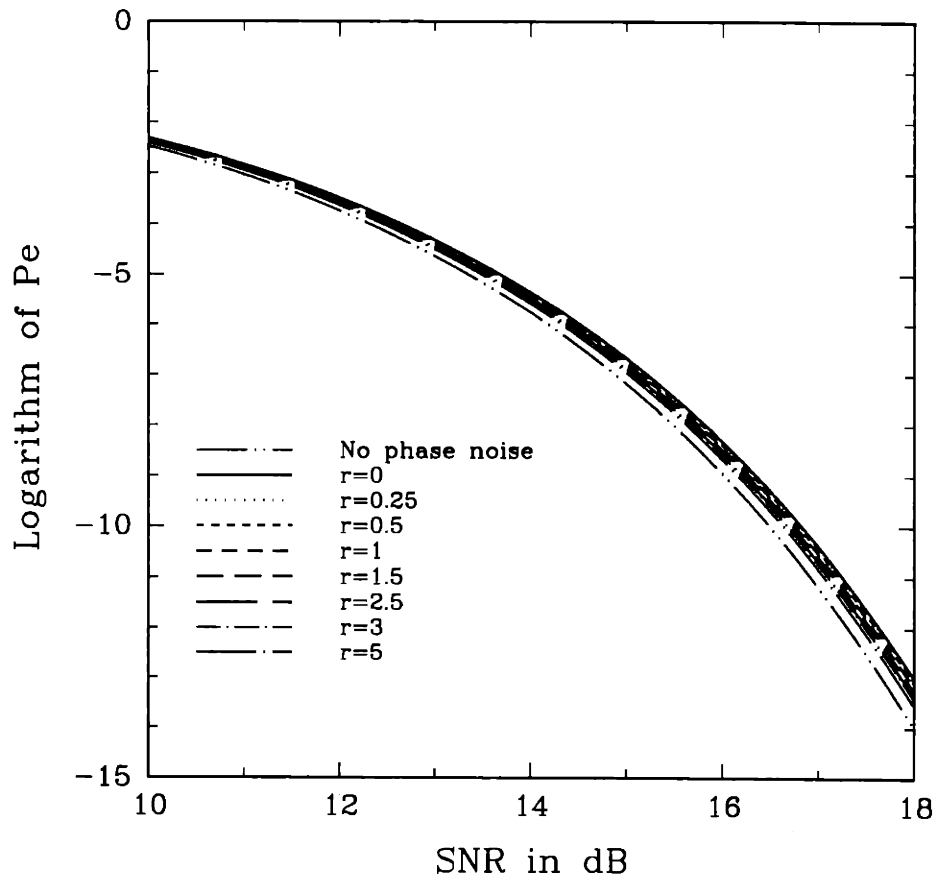
Figure 5-16: Results for different values of r with γ equal to 0.5 using the Jensen bound

The lower bound is so tight to the ideal case of FSK for this small value of γ that it does not convey much information.
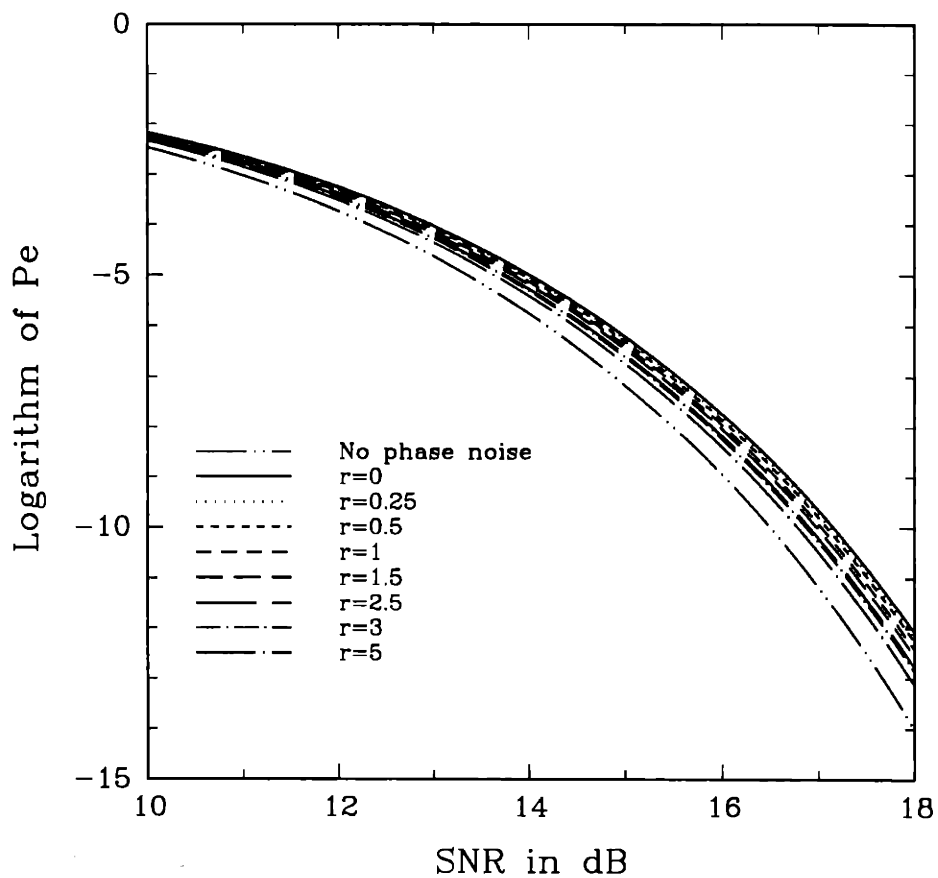
Figure 5-17: Results of P_e for different values of r when γ equals 1 using the Jensen bound

Here we have a bit more spread in the Pe curves as a function of r but they are definitely lower bound.
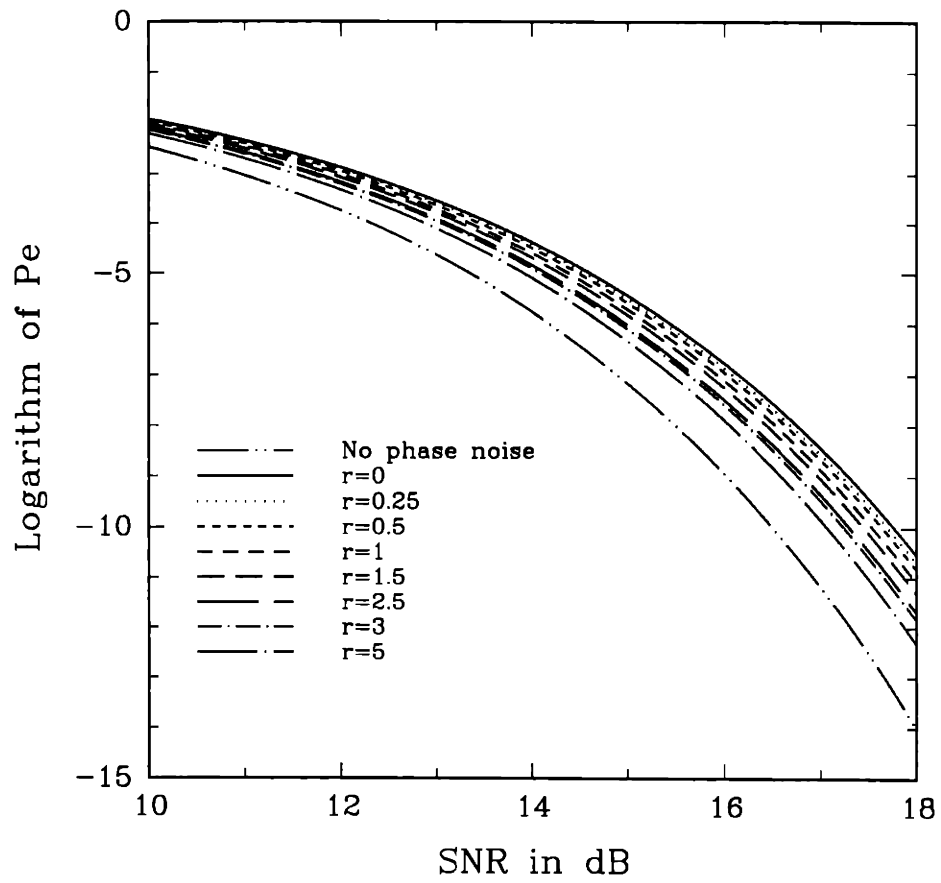
Figure 5-18: Results of P_e for different values of r when γ equals 2 using the Jensen bound
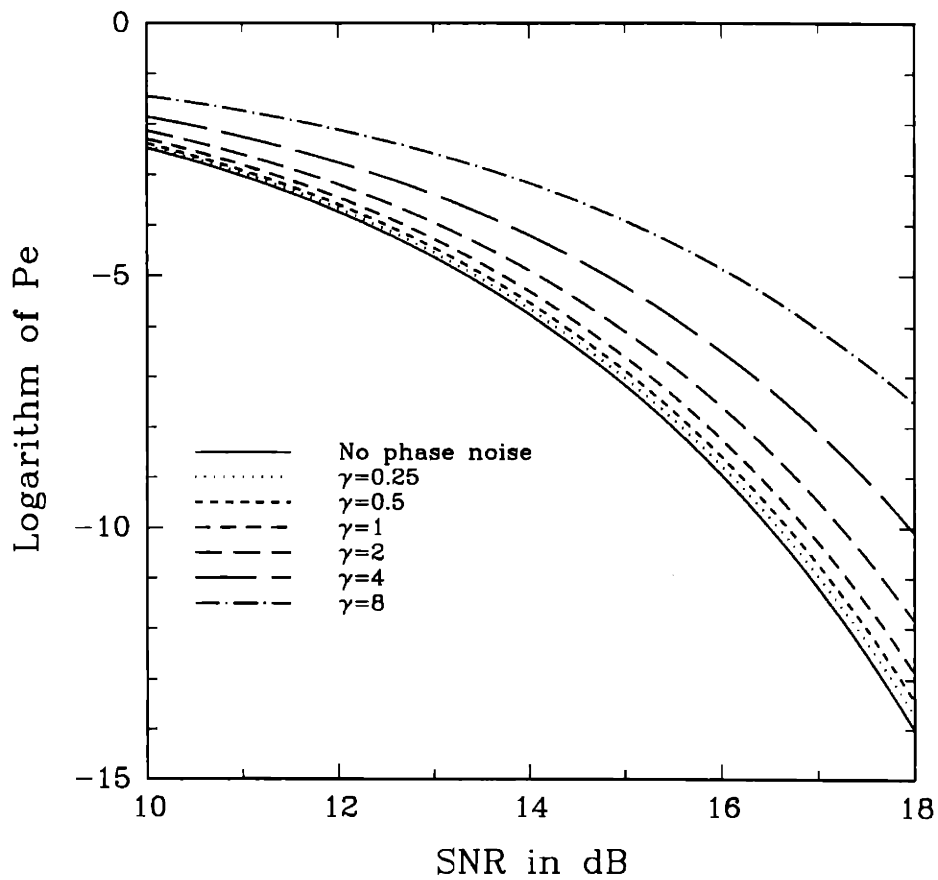
As before.

Figure 5-19: Results for different values of γ when $r = 3$ using the Jensen bound

For large values of γ this is interesting as it shows if the actual behaviour follows the lower bound, then it is still a convex function of SNR.
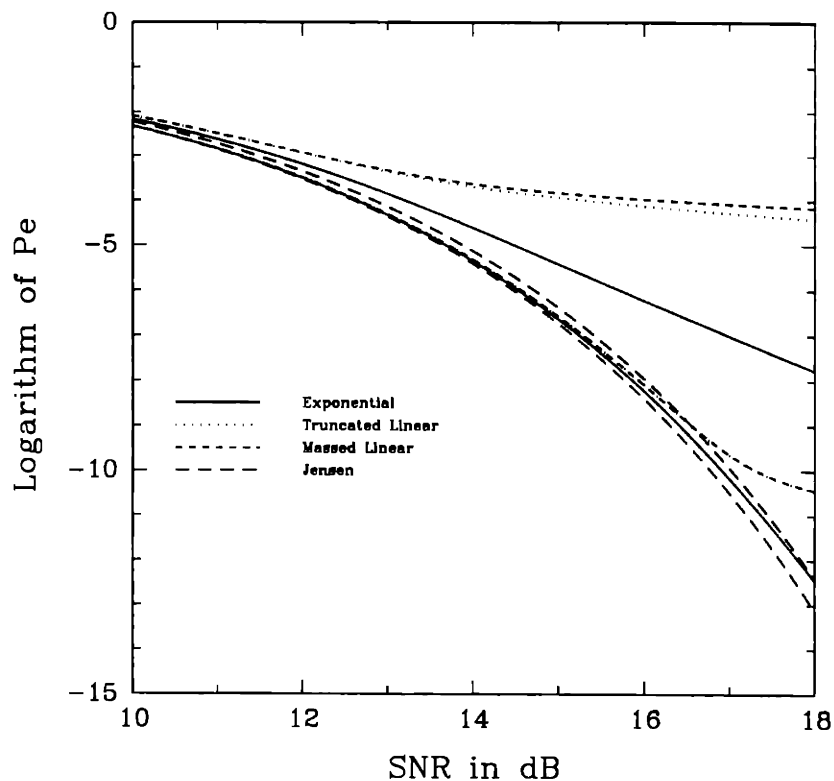
Figure 5-20: P_e for when $r = 1, 5$ and $\gamma = 1$.

5.6   Comparison of the Different Approximations Used

We give two curves to show how the results change according to which approximation is used. One for different values of r with γ fixed. The other for different values of γ with r fixed. Notice here that the two linear approximations are close together. This is because we are dealing with small values of the phase noise. We also see that the upper and lower bounds never cross, and in fact exhibit a large spread. Therefore it is not possible to tightly bound the actual performance, except at small values of SNR. Finally the exponential approximation always lies between the upper and lower bounds, thus it is likley to be a reasonable approximation to the actual performance.
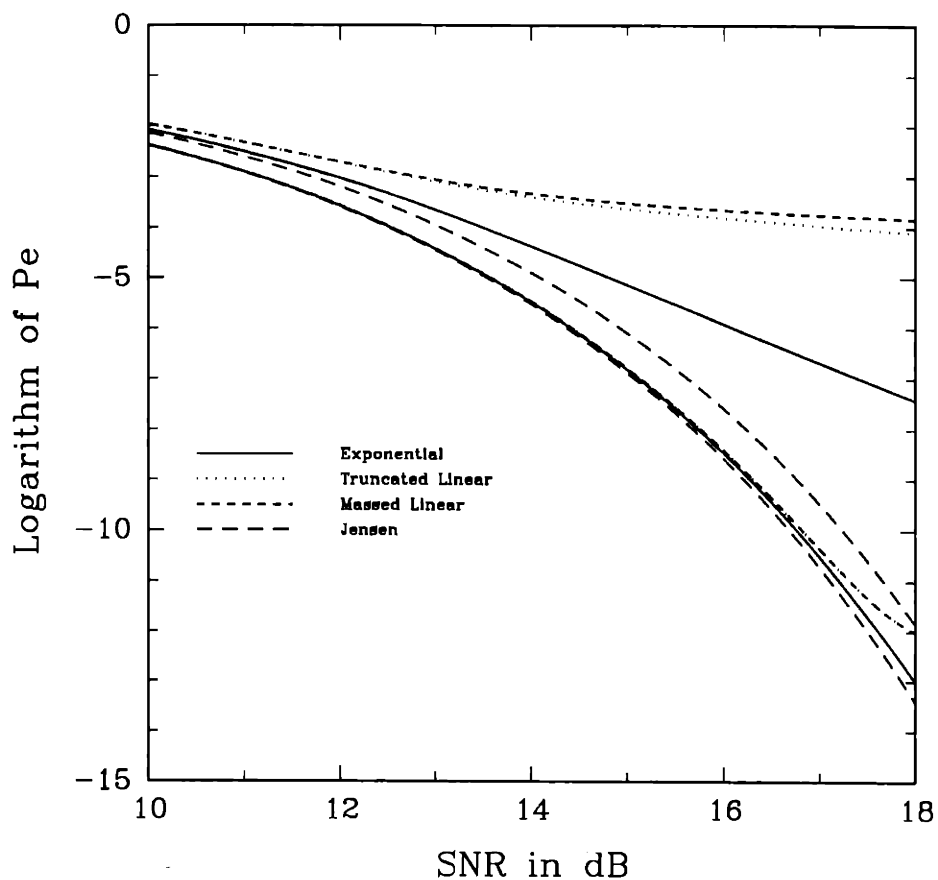
Figure 5-21: P_e for $\gamma = 0.5, 2$ and $r = 3$.

For FSK these different approximations show that for large values of γ the frequency feedback stabilisation scheme does not provide a sufficiently good improvement in the received Pe to make it a viable alternative to other schemes designed to combat phase noise. [Fos 87], [Fos 89]. However it does give gains at small values of γ which may make it attractive in some applications. It may also prove valuable for PSK or DPSK as we describe in the next chapter.

Chapter 6

Conclusions and Further Research

6.1 Summary

In this thesis we have investigated the effect of frequency feedback stabilisation on the received probability of error of incoherently detected FSK signals. In our analysis we derived the statistics of the laser phase noise process when frequency feedback has been applied. This gave a zero mean Gaussian random process with a general covariance function $K_{\theta\theta}(t, s)$. To calculate the received probability of error, we used a receiver structure proposed by Foschini et al. . The probability of error calculation necessitated finding the statistics of a random variable

$$X = \left| \int_0^T e^{j\theta(t)} dt \right|^2 \tag{6.1}$$

We followed Foschini et al. by performing a perturbation expansion of the complex exponential in equation 6.1. This led to a new random variable Ψ. This was found in terms of an infinite summation of the squares of independent Gaussian random variables. As a result we could derive an expression for the characteristic function of Ψ and take its inverse Fourier Transform to get the statistics of Ψ.

At the end of Chapter 4 we gave four different methods of finding an approximation to the received probability of error. Three of these approximations used the statistics of Ψ and the fourth used the covariance function of the phase noise process. The results of the four different approximations were then given in Chapter 5. All four sets of results showed an improvement in the probability of error at the receiver as the bandwidth of the feedback filter is increased.

This is exactly the result we would expect, however the improvement in the performance gets further away from the ideal case of no phase noise as γ the intensity of the phase noise increases. For low data rate signals this is no problem as the performance at high phase noise levels can be improved by increasing r where $r = bT$. For long bit times T we can use relatively small filter bandwidths $(b/2\pi)$ to achieve the required value of r.
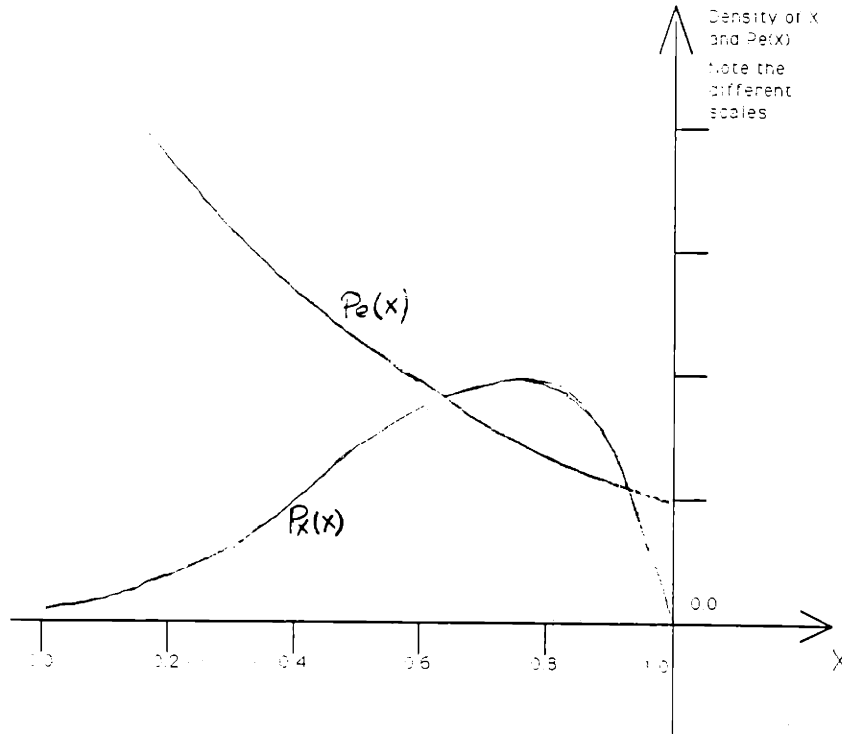
Figure 6-1: Error floor for the linear approximation

Therefore we conclude that frequency feedback stabilisation will be particularly effective for small phase noises or low data rate transmissions.

In addition to these conclusions about the system performance it is interesting to look at the results of the four different approximations. As we would expect Jensen's inequality gives a definite lower bound to the received probability of error, and it always lies below the upperbound approximation of Foschini. The interesting cases are the two linear approximations by Foschini. We would not expect the chosen receiver to have an error probability floor with increasing signal to noise ratio. In fact the flattening out of the probability of error curves is a result of the probability density function of X_L not being equal to zero when X is equal to zero. We try and show this observation in figure 6.1. $P_e(X)$ is the conditional probability of error, $P_e(X) = 1/2e^{-\zeta X/2}$ and $P_X(X)$ is the density of the truncated linear approximation of X. As the SNR increases $P_e(X)$ will tend to become an impulse at $X = 0$. For the truncated linear approximation the calculated probability of error will not decrease as the tail density is flat in the vicinity of $X = 0$. This gives an error floor for the results of this approximation. Therefore it seems likley that the most accurate approximation may well be the exponential approximation as it tends to zero at the origin.

6.2 Future Research

There are many interesting problems that can be extended from this work they include:

- Reducing the low frequency part of the frequency noise has helped only moderatly, suggesting that the high frequency part has a substantial role. Analysing the phase noise problem using a better model for high frequency effects such as the relaxation resonance would give a better idea of how best to combat the phase noise problem using feedback techniques.

- Extension of the work to PSK or DPSK. When the transmitted information is carried by the phase of the optical signal, it is more susceptible to corruption by the phase noise process. Azizoglu in his thesis [Aziz 91] points out for a Brownian motion phase noise that the variance of the phase grows linearly with time. It is this linear growth in the phase uncertainty that makes it difficult to find a good receiver structure. When we consider the phase noise process with frequency feedback stabilisation we see that the phase variance is no longer linearly growing with time. In fact

$$var^2(\theta(t)) = \frac{1}{r}\left[1 - e^{-rt}\right]$$

It now tends towards a maximum value of $1/r$. This makes the technique attractive for DPSK, but the statistical dependence of the phase noise will make it difficult to analyse for complex receiver structures.

- A return to the original problem to analyse the effect of increasing the number of users in Glance's frequency stabilisation system. Here some sort of discretisation will likely be required, and it will be difficult to successfully model the phase noise process accurately.

- Derivation of the Pe for a double filter receiver structure [Fos 87], [Aziz 90] when we have implemented frequency feedback stabilisation. Because the phase noise is no longer an independent increments process, an exact analysis may not be possible. However the correlation coefficient of

$$X_{M,k,n} = \left|\int_{k-1}^{k-1+n/M} e^{j\sqrt{\gamma}\psi(t)}dt\right|^2$$

can be found, and if it is small for some values of r the process can be treated as an independent increments process with a reasonable degree of confidence.

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