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### *Observational Implications of Non-Exponential Discounting*

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Title: Observational Implications of Non-Exponential Discounting

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Resume: Dans cette note, nous donnons un exemple élémentaire avec une fonction d'utilité concave où la règle d'épargne essentiellement unique et consistante est discontinue ; une telle règle d'épargne ne peut pas survenir avec une fonction d'escompte exponentielle.

Title: Observational Implications of Non-Exponential Discounting

Abstract: In this note we provide an elementary example with a concave utility function where the essentially unique consistent savings rule is discontinuous; such a savings rule could not arise with exponential discounting.

Keywords: Decision Theory, Dynamic Consistency

Mots-clefs: Théorie de la décision, Consistance dynamique

JEL Classification: D01

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## 1. INTRODUCTION

The standard assumption in economics is that individuals' preferences are dynamically consistent. That is, if a plan of action is optimal at time 0, the individual will have no incentive to revise it at future times. If that individual is maximizing a time additively separable utility function over consumption, dynamic consistency implies that the individual must discount the future with an exponential discount function. Yet all attempts to directly identify a discount function suggest that it is not exponential (see Ainslie [1992] and citing papers). In particular, individuals are prepared to accept a lower interest rate to postpone consumption from  $t$  periods hence to  $t + 1$  periods hence than to postpone consumption from today to the next period.

The issue of agents who depart from exponential discounting was first studied in the economics literature by Strotz [1956]. How should individuals with non-exponential discounting make intertemporal choices, given their dynamic inconsistency? A savings rule was said to be a "consistent" savings rule if it was optimal for an individual to follow the rule if he anticipated that his future selves would follow that rule. Strotz claimed that individuals following such consistent rules would act as if they were maximizing a utility function derived from certain exponential weights. This claim implied that individuals with non-exponential discounting would be observationally equivalent (in a stationary environment) to individuals with exponential discounting. There is a simple intuition for such a result: when an individual makes a saving decision, we learn about the relative weights he puts on the present and the future, but we learn nothing about the relative weights he puts on different periods in the future.

Strotz's particular characterization of consistent rules turned out to be false (see Pollak [1968]). Nonetheless a large literature on dynamically inconsistent individual choice and the formally equivalent problem of intergenerational altruism has built on Strotz's ideas (see Laibson [1996] and O'Donoghue and Rabin [1997]) for contributions and further references). The intergenerational altruism literature showed that the observational equivalence claim was false, in at least some settings.<sup>1</sup> In particular, if a consumer has a smooth and strictly concave utility function and discounts the future exponentially, his optimal savings rule must be differentiable. But Kohlberg [1976] described an example (with non-exponential discounting) where no *differentiable* consistent savings rule exists; Leininger [1986] and Bernheim and Ray [1987] showed the *existence* of a consistent savings rule in the same setting. Thus there is a non-constructive proof of the existence of intertemporal stationary savings problems where non-exponential discounting has observational implications (i.e., non-differentiable savings). In this note, which we first circulated in 1997, we provide (for a slightly different setting) an elementary example with a concave utility function where the essentially unique consistent savings rule is discontinuous; such a savings rule could not arise with exponential discounting. We demonstrate this savings rule by construction. The construction has the advantage that it provides simple intuition about what drives the discontinuity.<sup>2</sup>

We do not attempt to review a large and important literature on related topics in the last twenty years. However, there has been little work addressing the observable implications in

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<sup>1</sup>We are grateful to David Laibson, Wolfgang Leininger and Matthew Rabin for guidance on this literature.

<sup>2</sup>Examples making similar points were already constructed in related contexts. Laibson [1996] discussed examples with liquidity constraints and log utility where a similar phenomenon occurs: exponential discounters would have discontinuous and sometimes decreasing consumption rules. O'Donoghue and Rabin [1997] described a discrete choice example where exponential discounters would have increasing consumption rules, but non-exponential discounters have a decreasing consumption rule.

non-exponential discounting in simple savings problems.<sup>3</sup> An important recent paper of Cao and Werlang [2018] show that in a standard hyperbolic-discounting model there are plausible conditions under which all Markov equilibria display either saving or dissaving at all wealth levels, even when discontinuous.

## 2. THE INTERTEMPORAL CONSUMPTION PROBLEM

An individual has a continuous, strictly increasing, concave utility function over consumption,  $u : \mathfrak{R}_+ \rightarrow \mathfrak{R}$ , and discounts the future with discount weights  $\{\delta_\tau\}_{\tau=0}^\infty$ . Assume  $\sum_{\tau=0}^\infty \delta_\tau$  exists and without loss of generality set  $\delta_0 = 1$ . Thus the utility of the time  $t$  self from consumption stream  $\mathbf{x} = \{x_\tau\}_{\tau=0}^\infty$  is

$$v_t(\mathbf{x}) = \sum_{\tau=0}^{\infty} \delta_\tau u(x_{t+\tau}).$$

A savings rule is a function  $s : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$  with  $s(y) \leq y$  for all  $y \in \mathfrak{R}_+$ . Write  $s^0(y) \equiv y$  and  $s^{k+1}(y) \equiv s(s^k(y))$  for each  $k = 0, 1, 2, \dots$ . If the time  $t$  self expects his future selves to follow rule  $s$  and currently has wealth  $y$ , he will choose savings  $x \in [0, y]$  to maximize

$$u(y - x) + \sum_{\tau=1}^{\infty} \delta_\tau u(s^{\tau-1}(x) - s^\tau(x)).$$

Thus savings rule  $s$  is a *consistent savings rule* if for all  $y \in \mathfrak{R}_+$ ,

$$s(y) \in \arg \max_{x \in [0, y]} \left\{ u(y - x) + \sum_{\tau=1}^{\infty} \delta_\tau u(s^{\tau-1}(x) - s^\tau(x)) \right\}. \quad (2.1)$$

## 3. OBSERVATIONAL EQUIVALENCE WITH CONSTANT RISK AVERSION UTILITY FUNCTIONS

Phelps and Pollak [1968] observed that in at least some environments, it is not possible to distinguish exponential from non-exponential discounting. Consider the case where  $u$  satisfies constant relative risk aversion (CRRA); i.e., with coefficient  $\rho \in (0, \infty)$ ,

$$u(x) = \begin{cases} \frac{x^{1-\rho}-1}{1-\rho}, & \text{if } \rho \neq 1 \\ \ln(x), & \text{if } \rho = 1 \end{cases}.$$

Let  $\lambda^*$  solve

$$\frac{\lambda^\rho}{1-\lambda} = \sum_{\tau=1}^{\infty} \delta_\tau \lambda^{(1-\rho)(\tau-1)} \quad (3.1)$$

(such a  $\lambda^*$  always exists<sup>4</sup>).

<sup>3</sup>But see Morris [2002] and Krussell and Smith [2003].

<sup>4</sup>Consider what happens as  $\lambda$  increases from 0 to 1. The left hand side increases continuously from 0 to  $\infty$ . The right hand side varies continuously from either a positive number (if  $\rho \leq 1$ ) or  $\infty$  (if  $\rho > 1$ ) to  $\sum_{\tau=1}^{\infty} \delta_\tau$ .

**Lemma 1.** *There is a consistent saving rule with  $s(y) = \lambda^* y$ .*

This can be shown by substituting a linear saving rule into equation (2.1) and solving first order conditions.<sup>5</sup> This solution coincides with the solution of Phelps and Pollak [1968] in the case where, for all  $\tau \geq 1$ ,  $\delta_\tau = \beta\delta^\tau$  for some  $\beta \leq 1$  and  $\delta < 1$ .<sup>6</sup> There is observational equivalence here since observing  $\lambda^*$  does not tell the observer if the individual has exponential discount weights or not. This equivalence is an artifact of the CRRA assumption, which ensures that an individual's relative marginal weights on current and future consumption are not influenced by any linear rule allocating future consumption among different periods. In the next section, we show that the equivalence does not survive without the CRRA assumption.

#### 4. DISCONTINUOUS SAVINGS

Consider the concave, piecewise linear utility function

$$u(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 1 + \beta(x - 1), & \text{if } 1 \leq x \leq 2 \\ 1 + \beta + \gamma(x - 2), & \text{if } 2 \leq x \end{cases}$$

where  $1 > \beta > 2\gamma > 0$ . Let  $\delta_\tau = 0$  for all  $\tau \geq 2$  and  $\delta_1$  satisfy  $1 > \delta_1 > \beta > \delta_1\beta > 2\gamma > 2\delta_1\gamma$ . With commitment, the optimal consumption rule (see figure 1) would be

$$c^*(y) = \begin{cases} y, & \text{if } 0 \leq y \leq 1 \\ 1, & \text{if } 1 \leq y \leq 2 \\ y - 1, & \text{if } 2 \leq y \leq 3 \\ 2, & \text{if } 3 \leq y \leq 4 \\ y - 2, & \text{if } 4 \leq y \end{cases} .$$

Figure 1 about here

The corresponding savings rule (see figure 2) is

$$s^*(y) = \begin{cases} 0, & \text{if } 0 \leq y \leq 1 \\ y - 1, & \text{if } 1 \leq y \leq 2 \\ 1, & \text{if } 2 \leq y \leq 3 \\ y - 2, & \text{if } 3 \leq y \leq 4 \\ 2, & \text{if } 4 \leq y \end{cases} .$$

Figure 2 about here

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If  $\rho > 1$ , the right hand side is strictly decreasing, so there is a unique solution. There is also a unique solution in the case of log utility (i.e.,  $\rho = 1$ ):  $\lambda^* = (\sum_{\tau=1}^{\infty} \delta_\tau) / (1 + \sum_{\tau=1}^{\infty} \delta_\tau)$ . There may be multiple solutions if  $\rho < 1$ .

<sup>5</sup>Assuming future selves follow linear savings rule  $s(y) = \lambda y$ , utility from consuming  $x$  out of current wealth  $y$  is equal to  $\frac{1}{1-\rho} \left[ (y-x)^{1-\rho} + \sum_{\tau=1}^{\infty} \delta_\tau ((1-\lambda)\lambda^{\tau-1}x)^{1-\rho} \right]$ . This is maximized setting  $\left(\frac{x}{y-x}\right)^\rho = (1-\lambda)^{1-\rho} \sum_{\tau=1}^{\infty} \delta_\tau \lambda^{(1-\rho)(\tau-1)}$ . Setting  $x = \lambda y$  gives equation (3.1).

<sup>6</sup>Setting  $\delta_\tau = \beta\delta^\tau$  in equation (3.1), we have  $\frac{\lambda^\rho}{1-\lambda} = \frac{\beta\delta}{1-\delta\lambda^{1-\rho}}$ ; re-arranging gives  $\lambda^\rho = \delta(\beta + (1-\beta)\lambda)$ ; this is equation (46b) of Phelps and Pollak [1968] (with different notation; they also allowed for a linear production technology).

**Lemma 1.** *The essentially unique consistent savings rule  $s$  is:*

$$s(y) = \begin{cases} 0, & \text{if } 0 \leq y \leq 1 \\ y - 1, & \text{if } 1 \leq y \leq 2 \\ 1, & \text{if } 2 \leq y < 4 + \frac{\gamma}{\delta_1\beta - \gamma} \\ 1 \text{ or } 2 + \frac{\gamma}{\delta_1\beta - \gamma}, & \text{if } y = 4 + \frac{\gamma}{\delta_1\beta - \gamma} \\ y - 2, & \text{if } 4 + \frac{\gamma}{\delta_1\beta - \gamma} < y \leq 5 \\ 3, & \text{if } 5 \leq y \end{cases} .$$

See figure 3.

Figure 3 about here

**Proof.** Suppose the individual could commit to any optimal consumption rule; he would choose to consume everything this period or next. In particular, his desired consumption next period would be equal to his commitment savings,  $s^*(y)$ . Thus *any* consistent savings rule has  $s(y) = 0$  for all  $0 \leq y \leq 1$ . But then setting  $s(y) = \begin{cases} y - 1, & \text{if } 1 \leq y \leq 2 \\ 1, & \text{if } 2 \leq y \leq 3 \end{cases}$  must uniquely achieve the first best. Now let  $v(x)$  be the value to the individual of leaving savings  $x$  to the next period under savings rule  $s$ . We must have

$$v(x) = \begin{cases} \delta_1 x, & \text{if } 0 \leq x \leq 1 \\ \delta_1, & \text{if } 1 \leq x \leq 2 \\ \delta_1(1 + \beta(x - 2)), & \text{if } 2 \leq x \leq 3 \end{cases} .$$

While we have not identified  $s(y)$  for  $y \geq 3$ , we know that the marginal utility to the current self is at most  $\delta_1\gamma$  for any extra units passed on, so  $v(x) \leq \delta_1(1 + \beta + \gamma(x - 3))$  for all  $x \geq 3$ . The individual's problem becomes one of maximizing  $u(y - x) + v(x)$  (where  $v$  is *not* concave). Note that at a maximum  $x$  may never exceed 2 (where the marginal utility of savings is  $\delta_1\gamma$ ) and may never be in the interval  $(1, 2)$  (where the marginal utility of savings is 0). Thus for  $y \geq 3$ , savings must be 1 or in the interval  $[2, 3]$ . If  $3 \leq y \leq 4$ , this is maximized setting  $x = 1$ ; if  $5 \leq y$ , this is maximized setting  $x = 3$ . If  $4 \leq y \leq 5$ , local maxima have  $x = 1$  and  $x = y - 2$ . The former gives utility  $1 + \beta + \gamma(y - 3) + \delta_1$ , while the latter gives utility  $1 + \beta + \delta_1 + \delta_1\beta(y - 4)$ . The former exceeds the latter if  $y \geq 4 + \frac{\gamma}{\delta_1\beta - \gamma}$ . ■

The corresponding consumption rule (see figure 4) is

$$c(y) = \begin{cases} y, & \text{if } 0 \leq y \leq 1 \\ 1, & \text{if } 1 \leq y \leq 2 \\ y - 1, & \text{if } 2 \leq y < 4 + \frac{\gamma}{\delta_1\beta - \gamma} \\ 2 \text{ or } 3 + \frac{\gamma}{\delta_1\beta - \gamma}, & \text{if } y = 4 + \frac{\gamma}{\delta_1\beta - \gamma} \\ 2, & \text{if } 4 + \frac{\gamma}{\delta_1\beta - \gamma} < y \leq 5 \\ y - 3, & \text{if } 5 \leq y \end{cases} .$$

Figure 4 about here

Let us make a few observations about this example.

- The utility function is rather special. But it could easily be perturbed (for example, in such a way that it was continuously differentiable) without altering the qualitative features of this example. Similarly, we could allow strictly positive but small discount weights  $\delta_\tau$  for  $\tau \geq 2$  without qualitatively changing the results.
- There is a simple intuition for the discontinuity. The current self wishes to save only if the savings will be consumed by his next period self. At some point, he will refrain from saving until he has enough resources to ensure increased consumption in the next period. But then he will save a lot (and his savings will jump discontinuously).
- Following Phelps and Pollak [1968], discount weights of the form  $\delta_\tau = \beta\delta^\tau$  (for some  $\beta \leq 1$  and  $\delta < 1$ ) have been used in the literature to capture the idea of “hyperbolic” discounting (e.g., Laibson [1996]). In this case, non-exponential discounting corresponds to the assumption  $\beta < 1$ . We can similarly use piecewise linear utility functions to construct examples with discontinuous savings schedules within this class of discount weights (with non-exponential discounting). In this case, the intuition is reversed. The current self wishes to save only if the savings will be *passed on* beyond his next period self. At some point he will refrain from saving until he has enough resources to ensure increased *saving* in the next period, at which point he will save a lot.
- By focussing on consistent savings rules, we are restricting attention to stationary Markov perfect equilibria of the underlying infinite savings game (i.e., subgame perfect equilibria where strategies depend only on current wealth and not on either past payoff-irrelevant history or the calendar date). But the assumption of bounded marginal utility of consumption at zero in our example ensures that *all* subgame perfect equilibria will be essentially identical to the one we study.

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