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0-CYCLES ON GRASSMANNIANS AS REPRESENTATIONS OF PROJECTIVE GROUPS

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Abstract. Let F be an infinite division ring, V be a left F-vector space, $r \geq 1$ be an integer. We study the structure of the representation of the linear group $\mathrm{GL}_F(V)$ in the vector space of formal finite linear combinations of r-dimensional vector subspaces of V with coefficients in a field.

This gives a series of natural examples of irreducible infinite-dimensional representations of projective groups. These representations are non-smooth if F is locally compact and non-discrete.

Let F be a division ring (a.k.a. a skew field), and V be a left F-vector space. Define multiplication in the associative unital 'matrix' ring $\mathrm{End}_F(V)$ so that V becomes a *left* $\mathrm{End}_F(V)$-module. In particular, $\mathrm{End}_F(V)$ is opposite to F if $\dim V = 1$.

Assume that $\dim V = r + r' > 1$ for a pair of cardinals $\mathbf{r} = (r, r')$. Denote by $\mathrm{Gr}(\mathbf{r}, V)$ the set of all F-vector subspaces of V of dimension r and of codimension r' ($\mathbf{r}$-*subspaces* for brevity). If $r < \dim V + 1$ we set $\mathrm{Gr}(r, V) := \mathrm{Gr}(\mathbf{r}, V)$ for $r' = \dim V - r$. For instance, $\mathrm{Gr}(1, V)$ is the projective space $\mathbb{P}(V) := F^\times\backslash(V \setminus \{0\})$; $\mathrm{Gr}(0, V)$ and $\mathrm{Gr}((\dim V, 0), V)$ are points.

For any associative ring A, denote by $A[\mathrm{Gr}(\mathbf{r}, V)]$ the set of all finite formal linear combinations $\sum_{j=1}^N a_j[L_j]$ with coefficients a_j in A of F-vector subspaces L_j of V in $\mathrm{Gr}(\mathbf{r}, V)$.

The set $A[\mathrm{Gr}(\mathbf{r}, V)]$ carries a natural structure of an A-bimodule: $a\cdot(\sum_i a_i[L_i])\cdot a' := \sum_i aa_ia'[L_i]$.

Let $G := \mathrm{GL}_F(V) := \mathrm{Aut}_F(V)$ be the group of invertible elements of $\mathrm{End}_F(V)$. The natural G-action on $\mathrm{Gr}(\mathbf{r}, V)$ is transitive and gives rise to an A-linear G-action on $A[\mathrm{Gr}(\mathbf{r}, V)]$.

Obviously, the module $A[\mathrm{Gr}(\mathbf{r}, V)]$ admits a proper submodule $A[\mathrm{Gr}(\mathbf{r}, V)]^\circ$ formed by all finite formal linear combinations $\sum_j a_j[L_j]$ with $\sum_j a_j = 0$, which is nonzero if $r \neq 0$ and $r' \neq 0$.

Our goal is to describe, for any coefficient field K, the structure of the $K[G]$-module $K[\mathrm{Gr}(\mathbf{r}, V)]$. Namely, for any infinite F, we show (in Theorem 4.3) that a canonical nonzero submodule M_1 (constructed in Lemma 1.1) of $\mathbb{Z}[\mathrm{Gr}(\mathbf{r}, V)]^\circ$ has the property that $K \otimes M_1$ is the only simple submodule of each nonzero $K[G]$-submodule of $K[\mathrm{Gr}(\mathbf{r}, V)]$.

This irreducibility result is deduced from the case of $\dim V = 2$ (§3). It is also shown in Lemma 3.3 the irreducibility of the representation of $\mathrm{SL}_2(\mathbb{Q})$ induced by any non-trivial one-dimensional representation of a proper parabolic subgroup in $\mathrm{SL}_2(\mathbb{Q})$.

The module $K[\mathrm{Gr}(\mathbf{r}, V)]^\circ$ coincides with $K \otimes M_1$ if and only if either r or r' is finite.

Several remarks on the case of finite F are collected in §5.

When F is a local field, one usually studies either unitary or *smooth* (i.e. with open stabilizers; they are called *algebraic* in [3]) representations, while the representations considered here are non-smooth. However, the latter representations are smooth if F is discrete and r is finite; for a field F, they arise as direct summands of "restrictions" of certain geometrically meaningful representations of automorphisms groups of universal domains over F, cf. [5, §4].

1. Generators of $A[\mathrm{Gr}(r, V)]^\circ$ for an integer r

For any ring A and any set Γ, denote by $A[\Gamma]$ the set of all finite formal linear combinations $\sum_{j=1}^N a_j[g_j]$ with coefficients a_j in A of elements $g_j \in \Gamma$.

If Γ is a group, we consider $A[\Gamma]$ as associative ring with evident relations $[g][g'] = [gg']$, $a[g] = [g]a$ for all $g, g' \in \Gamma$ and $a \in A$. The element $[1]$ is the unit of the ring.

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The $A[G]$-module $A[\mathrm{Gr}(\mathbf{r},V)] = A\otimes\mathbb{Z}[\mathrm{Gr}(\mathbf{r},V)]$ is generated by $[L]$ for any $L\in\mathrm{Gr}(\mathbf{r},V)$.

The following lemma shows that, for an integer r, the $A[\mathrm{GL}_F(V)]$-module $A[\mathrm{Gr}(r,V)]^\circ = A\otimes\mathbb{Z}[\mathrm{Gr}(r,V)]^\circ$ is generated by $[L]-[L']$ for any $L,L'\in\mathrm{Gr}(r,V)$ with $\dim(L\cap L') = r-1$.

Lemma 1.1. *Let $\mathbf{r}$ be a pair of cardinals. Let L,L' be $\mathbf{r}$-subspaces in V with $\dim(L/L\cap L') = \dim(L'/L\cap L') = 1$. Then the G-submodule $M_1 = M_1(\mathbf{r},V)$ of $\mathbb{Z}[\mathrm{Gr}(\mathbf{r},V)]^\circ$ generated by the difference $[L]-[L']$ contains all differences $[L_0]-[L_1]$ of $\mathbf{r}$-subspaces with $\dim(L_0/L_0\cap L_1) = \dim(L_1/L_0\cap L_1)<\infty$, but does not contain differences $[L_0]-[L_1]$ of other pairs of $\mathbf{r}$-subspaces.*

In particular, M_1 coincides with $\mathbb{Z}[\mathrm{Gr}(\mathbf{r},V)]^\circ$ if and only if at least one of r and r' is finite.

Proof. Let $c=\dim(L_0/L_0\cap L_1)$. Fix complete flags $E_0=0\subset E_1\subset E_2\subset\cdots\subset E_c = L_0/(L_0\cap L_1)$ and $F_0=0\subset F_1\subset F_2\subset\cdots\subset F_c = L_1/(L_0\cap L_1)$ and set $L_i' = \tilde{E}_{c-i}+\tilde{F}_i$, where $\tilde{E}$ denotes the preimage of a subspace $E\subseteq V/(L_0\cap L_1)$ under the projection $V\to V/(L_0\cap L_1)$. Then $L_0',L_1',\dots,L_c'$ are $\mathbf{r}$-subspaces, while $L_{i-1}'\cap L_i'$ is a hyperplane in both L_{i-1}' and L_i' for each i, $1\le i\le c$.

As G acts transitively on the set of pairs (S,S') of $\mathbf{r}$-subspaces of V with $\dim(S/S\cap S') = \dim(S'/S\cap S')=1$, all $[L_i']-[L_{i+1}']$ belong to the G-orbit of $[L]-[L']$. As $[L_0]-[L_1] = ([L_0']-[L_1'])+([L_1']-[L_2'])+\cdots+([L_{c-2}']-[L_{c-1}'])+([L_{c-1}']-[L_c'])$, we see that M_1 contains $[L_0]-[L_1]$.

If either of r and r' is finite then for any pair L_0,L_1 of $\mathbf{r}$-subspaces of V one has $\dim(L_0/L_0\cap L_1) = \dim(L_1/L_0\cap L_1)<\infty$, so it is clear from the above that $[L]-[L']$ generates the G-module $\mathbb{Z}[\mathrm{Gr}(\mathbf{r},V)]^\circ$.

If $[L_0]-[L_1]\in M_1$, i.e., $[L_0]-[L_1] = \sum_{i=1}^N a_i([L_i']-[L_i''])$ with $\dim(L_i'/L_i'\cap L_i'') = \dim(L_i''/L_i'\cap L_i'')=1$, then rename L_i' and L_i'' in a way to get a sequence $L_0 = L_0',L_1',\dots,L_{n-1}',L_n' = L_1$ with $\dim(L_{i-1}'/L_{i-1}'\cap L_i') = \dim(L_i'/L_{i-1}'\cap L_i') = 1$. Then $\dim(L_0'/\bigcap_{i=0}^n L_i')\le n$, and thus, $\dim(L_0/L_0\cap L_1)$ is finite.

In particular, if $[L]-[L']$ generates $\mathbb{Z}[\mathrm{Gr}(\mathbf{r},V)]^\circ$ then $L_0/(L_0\cap L_1)$ is finite-dimensional for any pair L_0,L_1 of $\mathbf{r}$-subspaces of V, so at least one of r and r' should be finite. ☐

2. (Endo)morphisms and decomposability

Let F be a division ring, V be a left F-vector space, $\mathbf{r}_0=(r_0,r_0')$, $\mathbf{r}_1=(r_1,r_1')$ be two pairs of cardinals such that $r_0+r_0' = r_1+r_1' = \dim V$; we may omit r_i' if $r_i<\dim V+1$. For an $\mathbf{r}_0$-subspace L in V, denote by $\mathrm{St}_{[L]}$ the stabilizer of the point $L\in\mathrm{Gr}(\mathbf{r}_0,V)$ in the group $G:=\mathrm{GL}_F(V)$.

It is easy to see that the G-orbit of a F-vector subspace L in V is determined by the pair of cardinals $(\dim L,\dim V/L)$; the G-orbit of a pair of F-vector subspaces L,L' in V is determined by the quintuple of cardinals $(\dim(L\cap L'),\dim L/(L\cap L'),\dim L'/(L\cap L'),\dim V/L,\dim V/L')$.

Let A be an associative unital ring. For each triple of cardinals $\mathbf{s}=(s,s',s'')$ with $s+s'=r_0$ and $s+s''=r_1$ (so s' and s'' may be omitted if $s<\min(r_0,r_1)+1$), let

$$\eta_{\mathbf{s}}^{\mathbf{r}_0,\mathbf{r}_1} : A[\mathrm{Gr}(\mathbf{r}_0,V)]\to A[\mathrm{Gr}(\mathbf{r}_1,V)]$$

be the $A[G]$-morphism given by $[L]\mapsto\sum_{L'}[L']$ if the latter sum is finite and non-empty, where L' runs over all $\mathbf{r}_1$-subspaces in V such that $\dim(L\cap L')=s$, $\dim L/(L\cap L')=s'$ and $\dim L'/(L\cap L')=s''$. Such L''s form an $\mathrm{St}_{[L]}$-orbit.

It is clear that $\eta_{(r,0,0)}^{\mathbf{r},\mathbf{r}}$ is identical on $A[\mathrm{Gr}(\mathbf{r},V)]$ and $\ker\eta_0^{\mathbf{r},0} = \ker\eta_{(r,0,r')}^{\mathbf{r},(\dim V,0)} = A[\mathrm{Gr}(\mathbf{r},V)]^\circ$.

Lemma 2.1. *Set $R:=A[G]$. Let $\mathbf{r}_0,\mathbf{r}_1$ be two pairs of cardinals with $r_0+r_0' = r_1+r_1' = \dim_F V$.*

Then two structures of right A-module on $\mathrm{Hom}_R(A[\mathrm{Gr}(\mathbf{r}_0,V)],A[\mathrm{Gr}(\mathbf{r}_1,V)])$ coincide and it is freely generated by

$\eta_{(0,r_0,0)}^{\mathbf{r}_0,\mathbf{r}_1}$	*if* $\mathbf{r}_1=(0,\dim V)$;
$\eta_{(r_0,0,r_0')}^{\mathbf{r}_0,\mathbf{r}_1}$	*if* $\mathbf{r}_1=(\dim V,0)$;
$\eta_{(r_0,0,r_0'-r_1')}^{\mathbf{r}_0,\mathbf{r}_1}$	*if V is infinite, while F and $r_0'\ge r_1'$ are finite;*
$\eta_{(r_1,r_0-r_1,0)}^{\mathbf{r}_0,\mathbf{r}_1}$	*if V is infinite, while F and $r_0\ge r_1$ are finite;*
the identity $\eta_{(r_0,0,0)}^{\mathbf{r}_0,\mathbf{r}_0} = id_{A[\mathrm{Gr}(\mathbf{r}_0,V)]}$	*if V is infinite and* $\mathbf{r}_0=\mathbf{r}_1$,
$\eta_s^{r_0,r_1}$ *for all s,* $\sigma\le s\le\min(r_0,r_1)$,	*if V is finite, where* $\sigma:=\max(0,r_0-r_1')$,
0	*otherwise.*

The ring $\mathrm{End}_{\mathbb{Z}[G]}(\mathbb{Z}[\mathrm{Gr}(\mathbf{r},V)])$ *is commutative. If* F *is infinite and* $\mathbf{r}_0\neq\mathbf{r}_1$ *then, in notation of §1,* $\mathrm{Hom}_R(A\otimes M_1(\mathbf{r}_0,V),A\otimes M_1(\mathbf{r}_1,V))=0$*, while* $\mathrm{End}_R(A\otimes M_1(\mathbf{r}_0,V))=A$ *if* $r_0\neq 0$ *and* $r_0'\neq 0$*, so the* R*-modules* $A[\mathrm{Gr}(\mathbf{r}_0,V)]$ *and* $A\otimes M_1(\mathbf{r}_0,V)$ *are indecomposable if* A *is a field.*

Proof. The cases $\mathbf{r}_1\in\{(0,\dim V),(\dim V,0)\}$ are trivial, since then $\mathrm{Gr}(\mathbf{r}_1,V)$ reduces to a single point, so we may further assume $\mathbf{r}_1\notin\{(0,\dim V),(\dim V,0)\}$.

Fix some $L\in\mathrm{Gr}(\mathbf{r}_0,V)$, and suppose that the $\mathrm{St}_{[L]}$-orbit of a point $L'\in\mathrm{Gr}(\mathbf{r}_1,V)$ is finite.

We, thus, assume that $L'\neq 0$ and $L'\neq V$.

- If $L\cong F^{\oplus r_0}$ is infinite then either (i) $L\subseteq L'$ or (ii) $L\cap L'=0$. In the case (ii), $L'\subseteq L$, since adding different elements of L to a basis element of L' one gets different L''s, and therefore, $L'=0$. If, in the case (i), $V/L\cong F^{\oplus r_0'}$ is finite then all L' containing L form a single finite orbit; if $V/L\cong F^{\oplus r_0'}$ is infinite and $L'\neq V$ then $L'=L$.
- If $V/L\cong F^{\oplus r_0'}$ is infinite then the image of L' in V/L should be either 0 or V/L, i.e., either (i) $L'\subseteq L$ or (ii) $V=L+L'$. In the case (i), either (a) $L'=L$, or (b) $L\cong F^{\oplus r_0}$ is finite and then all L' contained in L form a single finite orbit. In the case (ii), either (a) $L'\supseteq L$, or (b) $L\cap L'\neq L$. (iia): $L'=V$, which is excluded. (iib): any orbit is infinite. Namely, choose a vector $v\in L\setminus L'$ and a collection $\{e_i\}_{i\in I}$ presenting a basis of V/L; then the subspaces $L_j:=L\cap L'+\langle e_i^j\mid i\in I\rangle_F$, where $e_i^j=e_i$ if $i\neq j$, while $e_i^i=e_i+v$, are paiwise distinct.
- If V is finite then all orbits are finite and they are parametrized by $s=\dim(L\cap L')$, where $\sigma:=\max(0,r_0-r_1')\leq s\leq\min(r_0,r_1)$.

As the R-module $A[\mathrm{Gr}(\mathbf{r}_0,V)]$ is generated by $[L]$, any R-module morphism $A[\mathrm{Gr}(\mathbf{r}_0,V)]\to A[\mathrm{Gr}(\mathbf{r}_1,V)]$ is determined by the image of $[L]$, which in turn is an element of $A[\mathrm{Gr}(\mathbf{r}_1,V)]^{\mathrm{St}_{[L]}}$, i.e., a linear combination of sums of the elements of several finite $\mathrm{St}_{[L]}$-orbits in $\mathrm{Gr}(\mathbf{r}_1,V)$.

One has $\eta_{s'}^{r',r''}\eta_s^{r,r'}[L]=\sum_{L'}\eta_{s'}^{r',r''}[L']=\sum_{L'}\sum_{L''}[L'']=\sum_{L''\in\mathrm{Gr}(r'',V)}N_{L,L''}[L'']$, where

$$N_{L,L''}=|\{L'\in\mathrm{Gr}(r',V)\mid \dim(L\cap L')=s,\ \dim(L'\cap L'')=s'\}|.$$

It follows that $\eta_{s'}^{r,r}\eta_s^{r,r}=\eta_s^{r,r}\eta_{s'}^{r,r}$. In other words, the algebra $\mathrm{End}_R(A[\mathrm{Gr}(r,V)])$ is commutative if V is finite, as soon as so is A. If V is infinite then $\mathrm{End}_R(A[\mathrm{Gr}(\mathbf{r},V)])=A$.

The R-module $A\otimes M_1(\mathbf{r}_0,V)$ is generated by $[L]-[L']$ for any $L,L'\in\mathrm{Gr}(\mathbf{r}_0,V)$ with $\dim L/(L\cap L')=\dim L'/(L\cap L')=1$, so any morphism φ from the R-module $A\otimes M_1(\mathbf{r}_0,V)$ is determined by the image of $[L]-[L']$, which in turn is an element of $(A[\mathrm{Gr}(\mathbf{r}_1,V)]^\circ)^{\mathrm{St}_{[L]}\cap\mathrm{St}_{[L']}}$.

If F is infinite then the only proper subspaces in V fixed by $\mathrm{St}_{[L]}\cap\mathrm{St}_{[L']}$ are $L,L',L\cap L'$, while the $\mathrm{St}_{[L]}\cap\mathrm{St}_{[L']}$-orbits of other proper subspaces are infinite. This means that $\varphi([L]-[L'])=a[L]+b[L']+c[L\cap L']$ for some $a,b,c\in A$. Consider $g\in G$ such that $g(L)\subset L+L'$, $g(L)\notin\{L,L'\}$, $g(L\cap L')=L\cap L'$ and $g(L')=L'$. Then $\varphi([L]-[g(L)])=\varphi([L]-[L'])-g\varphi([L]-[L'])=a([L]-[g(L)])$. As $\dim L/(L\cap g(L))=\dim g(L)/(L\cap g(L))=1$, the element $[L]-[g(L)]$ is another generator of $A\otimes M_1$, so we get $\mathrm{End}_R(A\otimes M_1(\mathbf{r}_0,V))=A$ and the required vanishing for $\mathbf{r}_0\neq\mathbf{r}_1$. □

Remark 2.2. If V is finite then the morphism $\eta_s^{r,r'}$ of §2 is dual to the morphism $\eta_s^{r',r}$ under the non-degenerate symmetric bilinear pairing on $K[\mathrm{Gr}(\bullet,V)]$, given by $([L],[L])=1$ and $([L],[L'])=0$ if $L\neq L'$: $((\eta_s^{r,r'})^*[L'],[L])=([L'],\eta_s^{r,r'}[L])$ is 1 if $\dim(L\cap L')=s$, and is 0 otherwise.

3. The one-dimensional case

Lemma 3.1. *Let* F *and* K *be fields, and* V *be a two-dimensional* F*-vector space. If* $|F|\leq 3$ *assume in addition that* F *and* K *are of the same characteristic. Let* P *be a subgroup in* $\mathrm{SL}(V)$ *and* $\rho:P\to K^\times$ *be a character. Suppose that the* $K[\mathrm{SL}(V)]$*-module* $W_\rho:=K[\mathrm{SL}(V)]\otimes_{K[P]}\rho$ *admits a submodule* W *such that* $\dim_K(W_\rho/W)=1$*. Then* $\rho=1$ *and* $W=K[\mathrm{SL}(V)/P]^\circ$*.*

Proof. As index of commutator subgroup of $\mathrm{SL}(V)$ is 1 if $|F|>3$, and it is $|F|$ if $|F|\leq 3$, any one-dimensional K-representation of $\mathrm{SL}(V)$ is trivial. On the other hand, $\mathrm{Hom}_{K[\mathrm{SL}(V)]}(W_\rho,K)=\mathrm{Hom}_{K[P]}(\rho,K)$, so the $K[\mathrm{SL}(V)]$-module W_ρ admits a one-dimensional quotient if and only if $\rho=1$, while W is the kernel of the degree morphism. □

Remark 3.2. We are particularly interested in the case of a proper parabolic subgroup P.

1. Denote by ∞ the point of $\mathbb{P}(V)$ fixed by P. Then $\mathrm{SL}(V)/P \to \mathbb{P}(V)$, $[g] \mapsto g\infty$ is an isomorphism of $\mathrm{SL}(V)$-sets (so that $[P/P] \mapsto \infty$) inducing an isomorphism of $K[\mathrm{SL}(V)]$-modules $W_\rho \xrightarrow{\sim} K[\mathbb{P}(V)]$ if $\rho = 1$.

2. Note that (i) $P \cong F^\times \ltimes^2 F$ where $F^\times$ acts on F by squares: $[a : b \mapsto a^2 b]$; (ii) $[P, P] = P^u$ if $|F| > 3$ and $[P, P] = 1$ if $|F| \le 3$, where $P^u \cong F$ is the unipotent radical of P. This shows that ρ factors through P/P^u.

If considered as $K[P]$-module, W_ρ splits as $K[\mathrm{SL}(V) \setminus P] \otimes_{K[P]} \rho \oplus \rho$, where $K[\mathrm{SL}(V) \setminus P] \otimes_{K[P]} \rho$ is a free module over the group algebra $K[P^u]$ of P^u. Namely, as the affine line $\mathbb{P}(V) \setminus \{\infty\}$ is a principal homogeneous space over P^u, a choice of an element $O \in \mathrm{SL}(V) \setminus P$ gives rise to an isomorphisms of left $K[P^u]$-modules $\iota_P : K[P^u] \otimes_K \rho \xrightarrow{\sim} K[\mathrm{SL}(V) \setminus P] \otimes_{K[P]} \rho$, $[u] \otimes e \mapsto [uO] \otimes e$.

Let $T \cong P/P^u$ be the torus in P fixing the class o of O on the affine line $\mathbb{P}(V) \setminus \{\infty\}$. The set $\{g \in OP \ : \ g^2 = -1\}$ consists of all elements of $\mathrm{SL}(V)$ interchanging the points o and ∞ of the projective line $\mathrm{SL}(V)/P$. This is a principal homogeneous space over T. As it is non-empty, we may further assume that $O^2 = -1$. As $(O\mathbf{t})^2 = -1$ for any $\mathbf{t} \in T$, we get $\mathbf{t}O = O\mathbf{t}^{-1}$, so the T-action on the target of ι_P corresponds to the T-action on $K[P^u] \otimes_K \rho$ given by $\mathbf{t} : [u] \otimes e \mapsto [\mathbf{t}u\mathbf{t}^{-1}] \otimes \mathbf{t}^{-1}e$, since $\mathbf{t} \circ \iota_P : [u] \otimes e \mapsto [\mathbf{t}uO] \otimes e = [(\mathbf{t}u\mathbf{t}^{-1})\mathbf{t}O] \otimes e = [(\mathbf{t}u\mathbf{t}^{-1})O] \otimes \mathbf{t}^{-1}e$.

This means that the $K[P]$-submodules of $K[\mathrm{SL}(V) \setminus P] \otimes_{K[P]} \rho$ correspond to the T-invariant ideals in $K[P^u]$.

3.1. **The case of** $\mathrm{SL}_2(\mathbb{Q})$**.**

Lemma 3.3. *Let V be a two-dimensional $\mathbb{Q}$-vector space, K be a field. Let P be a proper parabolic subgroup in $\mathrm{SL}(V)$ and $\rho : P \to K^\times$ be a character.*

Then (i) *the $K[\mathrm{SL}(V)]$-module $W_\rho := K[\mathrm{SL}(V)] \otimes_{K[P]} \rho$ is simple if $\rho \neq 1$,* (ii) *$K[\mathbb{P}(V)]^\circ$ is a unique simple submodule of $K[\mathbb{P}(V)] = W_1$.*

Proof. By Lemma 3.1, it suffices to show that any nonzero $K[\mathrm{SL}(V)]$-submodule W in W_ρ has K-codimension ≤ 1. We keep the setting of Remark 3.2 with $F = \mathbb{Q}$. We identify $P^u \cong \mathbb{Q}$ with the rational powers $X^{\mathbb{Q}}$ of an indeterminate X, so that $K[P^u] \cong \bigcup_{N \ge 1} K[X^{1/N}, X^{-1}]$.

Any ideal in $K[P^u]$ is determined by its intersections with each subalgebra $K[X^{1/N}, X^{-1}]$, and thus, is generated by a collection of polynomials of minimal degree $P_N(X^{1/N})$ for all $N \ge 1$ such that $P_N(0) = 1$ and $P_N(X) | P_M(X^{N/M})$ in $K[X]$ if $M|N$.

As $P/P^u \cong \mathbb{Q}^\times$ acts evidently on $P^u \cong X^{\mathbb{Q}}$, any P/P^u-invariant proper ideal $I = (P_N(X^{1/N}))_{N \ge 1}$ in $\bigcup_{N \ge 1} K[X^{1/N}, X^{-1}]$ contains $P_N(X^{M_1^2/(M_2^2 N)})$ for all $M_1, M_2, N \ge 1$. In particular, $P_N(X)$ divides $P_N(X^{M^2})$ for all $M, N \ge 1$, which implies that if $P_N(\alpha) = 0$ then $P_N(\alpha^{M^2}) = 0$ for all $M \ge 1$, so the set $\{\alpha^{M^2}\}_{M \ge 1}$ is finite, i.e., α is a root of unity, say $\alpha^{M_N} = 1$. Then $P_N(X) | (X^{M_N} - 1)^{m_N}$ for some $M_N, m_N \ge 1$, and thus, I contains $(X^{M_N/N} - 1)^{m_N}$, and therefore, I contains $(X^{M_N^2/N} - 1)^{m_N} \in ((X^{M_N/N} - 1)^{m_N})$, and consequently, $(X^{1/N} - 1)^{m_N} \in I$ for all $N \ge 1$. Let $s \ge 1$ be such that $m_s \le m_N$ for all $N \ge 1$. Then $(X^{1/(N^2 s)} - 1)^{m_s} \in I$ for all $N \ge 1$, and therefore, $(X^{1/N} - 1)^{m_s} \in ((X^{1/(N^2 s)} - 1)^{m_s}) \subset I$ for all $N \ge 1$.

Let $J \subset \bigcup_{N \ge 1} K[X^{1/N}, X^{-1}]$ be the ideal generated by $X^{1/N} - 1$ for all $N \ge 1$. As J^n/J^{n+1} is a K-vector space of dimension $\max(\delta_{n,0}, \delta_{\mathrm{char}(K),0}) \le 1$ with the group P/P^u acting by n-th powers, we conclude that I coincides with J^n for some $n \ge 0$.

The splitting $W_\rho = \rho \dotplus K[\mathrm{SL}(V) \setminus P] \otimes_{K[P]} \rho$ induces two projections $\pi_\infty : W \to \rho$ and $\pi_{\mathrm{aff}} : W \to K[\mathrm{SL}(V) \setminus P] \otimes_{K[P]} \rho$. As $\mathrm{SL}(V)$ is transitive on $\mathbb{P}(V)$, the projection π_∞ is surjective. As $\mathrm{Hom}_{K[P]}(W_\rho, \rho) = K$, if π_∞ splits as a morphism of $K[P]$-modules then W contains $[1] \otimes \rho$, and therefore, $W = W_\rho$.

Assume now that the projection π_∞ does not split, so then (i) $0 \to J^m \otimes_{K[P]} \rho \to W \to \rho \to 0$, (ii) the projection π_{aff} is injective, i.e., the $K[P]$-module W is isomorphic to a power of J. Since the quotient of the $K[P]$-module W by J^m is ρ, we get $m = 1$, so $W \cong K[\mathrm{SL}(V) \setminus P] \otimes_{K[P]} \rho$ as $K[P]$-module, and thus, the $K[\mathrm{SL}(V)]$-module W contains the linear combination $([1] + a[O]) \otimes e$

for some $a \in K$ and $O \in \mathrm{SL}(V) \setminus P$. Then W contains $([1]+a[uO]) \otimes e$ for all $u \in P^u$, and therefore, $\dim_K(W_\rho/W) \leq 1$. □

3.2. **The case of $\mathrm{SL}_2(F)$ in equal positive characteristics.**

Lemma 3.4. *Let F and K be fields of characteristic $p > 0$, and V be a two-dimensional F-vector space. Then the $K[\mathrm{PSL}(V)]$-module $K[\mathbb{P}(V)]^\circ$ is simple.*

Proof. Let $M \neq 0$ be a $K[\mathrm{PSL}(V)]$-submodule in $K[\mathbb{P}(V)]^\circ$, and $\alpha = \sum_{i=1}^n a_i[x_i] \in M$ for some $n \geq 1$, pairwise distinct $x_i \in \mathbb{P}(V)$ and some $a_i \in K^\times$ with $\sum_{i=1}^n a_i = 0$. Let $P \cong F^{\times 2} \ltimes F$ be the stabilizer of x_1 in $\mathrm{PSL}(V)$.

For a choice of a point $O \in \mathbb{P}(V) \setminus \{x_1\}$, denote by $\beta = \sum_{i=2}^n a_i[x_i - O]$ of $K[P^u]$ such that $\beta O = \alpha - a_1[x_1]$. Then $\beta^{p-1}\alpha = a_1^p \cdot ([x_1] - [O])$. As O can be chosen to be an arbitrary point of $\mathbb{P}(V) \setminus \{x_1\}$, α generates $K[\mathbb{P}(V)]^\circ$ as $K[\mathrm{PSL}(V)]$-module. □

4. The case of an infinite base skew field

Lemma 4.1. *Let V be a left vector space over a division ring F, and λ be an F-linear functional on V. Then, for any $w \in V$, the endomorphism $1 - \lambda t w \in \mathrm{End}_F(V)$, $x \mapsto x - \lambda(x)tw$, is not invertible for at most one value of $t \in F$.*

Proof. Let $\mu := \lambda(w)$. Then, for any $t \in F$ such that $\mu t \neq 1$, one has $(1-\lambda t w)(1+\lambda t(1-\mu t)^{-1}w) = 1 - \lambda t w + \lambda t(1-\mu t)^{-1}(1-\mu t)w = 1$. □

Lemma 4.2. *For each integer $s, N \geq 1$, let $D_s(N) := \sum_{m \equiv s \bmod p} (-1)^m \binom{N}{m} \in \mathbb{Z}$. Then the matrix*

$$\Delta = \begin{bmatrix} D_1(N) & \cdots & D_{p-1}(N) \\ \cdots & \cdots & \cdots \\ D_1(N+p-2) & \cdots & D_{p-1}(N+p-2) \end{bmatrix}$$

is invertible over $\mathbb{Z}[1/p]$.

Proof. Denote by μ_p the set of complex p-th roots of unity. Then

$$D_s(N) = \frac{1}{p} \sum_{\zeta \in \mu_p} \zeta^{-s}(1-\zeta)^N.$$

Fix a primitive $\zeta \in \mu_p$. Then the matrix $p\Delta$ coincides with the product

$$\begin{bmatrix} (1-\zeta)^N & (1-\zeta^2)^N & \cdots & (1-\zeta^{p-1})^N \\ (1-\zeta)^{N+1} & (1-\zeta^2)^{N+1} & \cdots & (1-\zeta^{p-1})^{N+1} \\ \cdots & \cdots & \cdots & \cdots \\ (1-\zeta)^{N+p-2} & (1-\zeta^2)^{N+p-2} & \cdots & (1-\zeta^{p-1})^{N+p-2} \end{bmatrix} \begin{bmatrix} \zeta^{-1} & \zeta^{-2} & \cdots & \zeta^{-(p-1)} \\ \zeta^{-2} & \zeta^{-4} & \cdots & \zeta^{-2(p-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \zeta^{-(p-1)} & \zeta^{-2(p-1)} & \cdots & \zeta^{-(p-1)(p-1)} \end{bmatrix},$$

so $\det(p\Delta)$ is product of Vandermonde determinants:

$$\det(p\Delta) = \prod_{j=1}^{p-1}(1-\zeta^j)^N \prod_{1 \leq j < s \leq p-1} (\zeta^j - \zeta^s) \prod_{j=1}^{p-1} \zeta^{-j} \prod_{1 \leq j < s \leq p-1} (\zeta^s - \zeta^j).$$

As the norm in the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ of the element $1 - \zeta^j$ is p, we see that $\det \Delta = \pm p^m$ for an integer $m \geq 0$, so $\det \Delta$ is invertible in $\mathbb{Z}[1/p]$. □

4.1. **A filtration.** For a left vector space V over a division ring F, let $G := \mathrm{GL}_F(V) := \mathrm{Aut}_F(V)$.

Theorem 4.3. *Let $\mathbf{r} = (r, r')$ be a pair of cardinals ≥ 1, F be an infinite division ring of characteristic $p \geq 0$, V be a left F-vector space of dimension $r + r'$. Set $M_0 := \mathbb{Z}[\mathrm{Gr}(\mathbf{r}, V)]^\circ$ and $G := \mathrm{GL}_F(V)$. Then there is a sequence of nonzero G-submodules $M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$ such that*

1. *M_0/M_1 is a free abelian group, vanishing if and only if at least one of r and r' is finite;*
2. *if $p > 0$ then the natural map $\mathbb{Z}[1/p] \otimes M_n \to \mathbb{Z}[1/p] \otimes M_1$ is surjective for any $n \geq 1$;*
3. *for any $n \geq 1$ and any field K, the natural map $K \otimes M_n \to K \otimes M_1$ is surjective;*

(4) *if, for an associative unital ring A, a $A[G]$-submodule M of $A\otimes M_0$ contains $\alpha=\sum_{i=0}^N a_i[L_i]$, where $L_0\not\subseteq\bigcup_{i=1}^N L_i$, then M contains $Aa_0\otimes M_n$ for some n depending on N.*

Proof. Fix an arbitrary subspace $U\subset V$ of dimension $r-1$ and of codimension $r'+1$. Fix some $e_0,e_1\in V$ that are F-linearly independent in V/U. For each sequence $(t_i)_{i\ge1}$ in $F^\times$ and all $n\ge1$, define $\gamma_n((t_i)_{i\ge1}):=\sum_{I\subseteq\{1,\dots,n\}}(-1)^{|I|}[U+F(e_0+(\sum_{i\in I}t_i)e_1)]\in M_0$.

Note that (i) the G-orbit of $\gamma_n((t_i)_{i\ge1})$ is independent of a particular choice of U,e_0,e_1, (ii) $\gamma_{n+1}((t_i)_{i\ge1})=(1-\xi)\gamma_n((t_i)_{i\ge1})$ for any $\xi\in G$ identical on $U+F\cdot e_1$ and such that $\xi e_0=e_0+t_{n+1}e_1$, (iii) all $\gamma_n((t_i)_{i\ge1})$ are nonzero if, e.g., $t_1,t_2,\dots$ are either linearly independent over the prime subfield, or all equal to 1 if $p=0$.

For each $n\ge1$, let M_n be the G-submodule in M_0 generated by the elements $\gamma_n((t_i)_{i\ge1})$ for all sequences $(t_i)_{i\ge1}$ in $F^\times$. In particular, we have inclusions $M_n\supseteq M_{n+1}$ for all n.

Then (1) follows from Lemma 1.1: M_0/M_1 is the group of formal finite linear combinations $\sum_i a_i[L_i]'$, where $a_i\in\mathbb{Z}$ and $[L]'$ are classes of 'commensurable' $\mathbf{r}$-subspaces, i.e. $L_0\sim L_1$ if $\dim(L_0/L_0\cap L_1)=\dim(L_1/L_0\cap L_1)<\infty$.

Set $\gamma_n:=\gamma_n(1,1,1,\dots)=\sum_{s=0}^n(-1)^s\binom{n}{s}[U+F(e_0+se_1)]$. As any nonzero $\gamma_1((t_i)_{i\ge1})$ belongs to the G-orbit of γ_1, the G-module M_1 is generated by γ_1.

If $p>0$, it follows from Lemma 4.2 that the $\mathbb{Z}[1/p]$-submodule in $\mathbb{Z}[1/p]\otimes M_0$ generated by $\gamma_n=\sum_{s=0}^{p-1}D_s(n)[U+F(e_0+se_1)],\gamma_{n+1},\dots,\gamma_{n+p-2}$ contains γ_1, which implies (2).

If $\ell:=\mathrm{char}(K)\neq p$ is a prime then $\gamma_{\ell^N}\equiv[U+F\cdot e_0]-[U+F(e_0+\ell^Ne_1)]=g_N\gamma_1 \pmod{\ell M_1}$ for any integer $N\ge1$ and some $g_N\in G$, so $M_n+\ell M_1$ contains γ_1 for any $n\ge1$, which proves (3) for $K=\mathbb{Z}/\ell$, and thus, for all fields K of characteristic ℓ.

By Lemmas 3.3 and 3.4, if characteristic of F is 0, or if characteristics of F and K coincide, then the $K[G]$-submodule in $K\otimes M_0$ generated by γ_n coincides with $K\otimes M_1$ for any $n\ge1$.

As (2) implies the case of $\mathrm{char}(K)\neq p>0$, this completes the proof of (3).

For (4), we are going to show that, together with $\alpha=\sum_{i=0}^N a_i[L_i]$, any $A[G]$-submodule M of $A\otimes M_0$ contains $a_0\otimes\gamma_n$ for some n if L_0 is not contained in L_i, $1\le i\le N$.

It is a folklore result that a vector space over an infinite division ring cannot be a finite union of proper linear subspaces, see e.g. [4, Theorem 1.2]. Fix some $v\in L_0\setminus\bigcup_{i=1}^N L_i$, some $w\in V\setminus L_0$. For each $1\le i\le N$, fix some F-linear morphism $\xi_i:V\to F\cdot w\subset V$ vanishing on L_i but not on v.

For each subset $I\subseteq\{1,\dots,N\}$, the image of the endomorphism $\xi_I:=\sum_{i\in I}\xi_i$ of V is contained in $F\cdot w$, i.e., $\xi_I=\lambda_I\cdot w$ for a linear functional $\lambda_I:=\sum_{i\in I}\lambda_i$ on V, so by Lemma 4.1 there is at most one value of t such that $1+\lambda_Itw$ is not invertible. As F is infinite, we may therefore replace w with a nonzero multiple, so that $1+\xi_I=1+\lambda_I\cdot w$ become invertible for all subsets $I\subseteq\{1,\dots,N\}$.

Then the element $\Xi:=\sum_{I\subseteq\{1,\dots,N\}}(-1)^{|I|}[1+\xi_I]\in\mathbb{Z}[G]$ annihilates all $[L_i]$ for $1\le i\le N$, and

$$\Xi[L_0]=\sum_{I\subseteq\{1,\dots,N\}}(-1)^{|I|}\left[L_0\cap\ker\xi_I\dotplus F\cdot\left(v+(\sum_{i\in I}\lambda_i(v))w\right)\right].$$

In particular, (i) if $r=1$ and $\lambda_i=t_i$ for all $1\le i\le N$ then $g\Xi\alpha=a_0\otimes\gamma_N((t_i)_{i\ge1})$ for any $g\in G$ such that $gv=e_0$ and $gw=e_1$; (ii) if $\sum_{i\in I}\lambda_i\neq0$ for any non-empty I then $\Xi[L_0]=[L_0]+\sum_{n=1}^M b_n[L_n']$ for some hyperplanes $L_n'\neq L_0$ in $V':=L_0\dotplus F\cdot w$ and some $b_n\in\mathbb{Z}$.

Set $V'':=V'/\bigcap_{n=1}^N\ker\xi_n$, which is of dimension $\le N$. Denote by $V''^\vee$ the dual space of V''. Consider the canonical identification $\psi:\mathrm{Gr}(\dim V''-1,V'')\xrightarrow{\sim}\mathbb{P}(V''^\vee)$, sending each hyperplane L in V'' to the line of all linear functionals vanishing on L. Then $\psi(g^{-1}[L])=g^*\psi([L])$ for all $g\in\mathrm{GL}_F(V'')$ (or their lifts in $\mathrm{GL}_F(V'')$), where $(g^*\lambda)(x):=\lambda(gx)$ for all $\lambda\in V''^\vee$ and $g\in\mathrm{GL}_F(V'')$.

Then ψ identifies the image of $\Xi[L_0]=[L_0]+\sum_{n=1}^M b_n[L_n']$ in $\mathbb{Z}[\mathrm{Gr}(\dim V''-1,V'')]^\circ$ with $[q_0]+\sum_{n=1}^M b_n[q_n]\in\mathbb{Z}[\mathbb{P}(V''^\vee)]^\circ$. As we have just seen in the case $r=1$ (with $V''^\vee$ instead of V), for any sequence $(t_i)_{i\ge1}$ in $F^\times$, there is an element $\beta\in\mathbb{Z}[\mathrm{GL}_F(V'')]$ (similar to the element Ξ) such that $\beta([q_0]+\sum_{n=1}^M b_n[q_n])=\sum_{I\subseteq\{1,\dots,M\}}(-1)^{|I|}[F\cdot(e_0''+(\sum_{i\in I}t_i)e_1'')]$ for some F-linearly independent $e_0'',e_1''\in V''$. Denote by β^* the image of β under the anti-involution $\mathbb{Z}[\mathrm{GL}_F(V'')]\to\mathbb{Z}[\mathrm{GL}_F(V'')]$,

$[g] \mapsto [g^{-1}]$. Then, for any linear combination β' of elements of G identical on $\mathbb{P}(\bigcap_{n=1}^N \ker \xi_n)$ and extending β^*, one has $\beta'(\Xi[L_0]) = \gamma_M((t_i)_{i\geq 1})$, and thus, $\beta'(\Xi\alpha) = a_0 \otimes \gamma_M((t_i)_{i\geq 1})$. □

Corollary 4.4. *Let $\mathbf{r} = (r, r')$ be a pair of cardinals ≥ 1, F be an infinite division ring, K be a field, V be a left F-vector space of dimension $r + r'$.*

Then any nonzero K-subrepresentation of $\mathrm{GL}_F(V)$ in $K[\mathrm{Gr}(\mathbf{r}, V)]$ contains $K \otimes M_1$.

In particular, $K \otimes M_1$ is the only irreducible K-subrepresentation of $\mathrm{GL}_F(V)$ in $K[\mathrm{Gr}(\mathbf{r}, V)]$.

The following conditions are equivalent: (i) *at least one of r and r' is finite,* (ii) $M_1 = \mathbb{Z}[\mathrm{Gr}(\mathbf{r}, V)]^\circ$, (iii) $K \otimes M_1 = K[\mathrm{Gr}(r, V)]^\circ$, (iv) $K[\mathrm{Gr}(\mathbf{r}, V)]^\circ$ *is irreducible,* (v) $K[\mathrm{Gr}(\mathbf{r}, V)]/K \otimes M_1$ *is irreducible,* (vi) $K[\mathrm{Gr}(\mathbf{r}, V)]/K \otimes M_1 \cong K$.

5. Some remarks on the case of a finite base field

There is an extensive literature on representations of finite Chevalley groups, see e.g. [1, 2]. For this reason we do not treat in detail the case where F is a finite field.

5.1. The case of characteristic 0 coefficient field.

Proposition 5.1. *Let $F = \mathbb{F}_q$ be a finite field, V be a finite F-vector space, K be a field where $q^n \neq q$ for $2 \leq n \leq \dim V + 1$ (e.g., $\mathrm{char}(K) = 0$), $r \geq 1$ be an integer. Then the $K[\mathrm{PGL}(V)]$-module $K[\mathrm{Gr}(r, V)]$ is a sum of $\min(r, \dim V - r) + 1$ pairwise distinct simple submodules.*

Proof. As the $K[\mathrm{PGL}(V)]$-modules are semisimple, it suffices to apply Lemma 2.1 asserting that the algebra $\mathrm{End}_{K[\mathrm{PGL}(V)]}(K[\mathrm{Gr}(r, V)])$ is commutative and of dimension $\min(r, \dim V - r) + 1$ as K-vector space. □

Proposition 5.2. *Let F be a finite field, V be an infinite F-vector space, K be a field of characteristic 0, $r \geq 1$ be an integer. Then both $K[\mathrm{PGL}(V)]$-modules, $K[\mathrm{Gr}(r, V)]$ and $K[\mathrm{Gr}((\dim V, r), V)]$, admit unique composition series, both of length $r + 1$.*

This follows from a description of the smooth K-representations of $\mathrm{GL}(V)$ for a countable F-vector space V given in [5, Theorem A.17]. The corresponding composition series are

$$0 \subset K \otimes \ker \eta_{r-1}^{r,r-1} \subset K \otimes \ker \eta_{r-2}^{r,r-2} \subset \cdots \subset K \otimes \ker \eta_1^{r,1} \subset K \otimes \ker \eta_0^{r,0} \subset K[\mathrm{Gr}(r, V)] \quad \text{and}$$

$0 \subset \Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_{r-1} \subset \Phi_r \subset K[\mathrm{Gr}((\dim V, r), V)]$, where $\Phi_n = K \otimes \ker \eta_{(\dim V, 0, n)}^{(\dim V, r), (\dim V, r-n)}$.

5.2. The case of positive characteristic coefficient field.

Proposition 5.3. *Let K be a field of characteristic ℓ. Let F be a union of finite fields.*

(1) *Suppose that V is finite and either $\dim \mathbb{P}(V) = 1$ or ℓ is not characteristic of F.*
   - *If ℓ does not divide $|\mathbb{P}(V)|$ then $K[\mathbb{P}(V)] = K[\mathbb{P}(V)]^\circ \oplus K \cdot \sum_{x \in \mathbb{P}(V)} [x]$ is the sum of two simple $K[\mathrm{PGL}(V)]$-submodules.*
   - *If ℓ divides $|\mathbb{P}(V)|$ then $K \cdot \sum_{x \in \mathbb{P}(V)} [x]$ is the only simple submodule of $K[\mathbb{P}(V)]$, while $K[\mathbb{P}(V)]^\circ / K \cdot \sum_{x \in \mathbb{P}(V)} [x]$ is the only simple submodule of $K[\mathbb{P}(V)] / K \cdot \sum_{x \in \mathbb{P}(V)} [x]$.*

(2) *If $\mathbb{P}(V)$ is infinite and either $\dim V = 2$ or characteristic of F is not ℓ then $K[\mathbb{P}(V)]^\circ$ is the only simple $K[\mathrm{PGL}(V)]$-submodule of $K[\mathbb{P}(V)]$.*

Proof. We have to show that any $\alpha = \sum_{x \in \mathbb{P}(V)} a_x [x] \in K[\mathbb{P}(V)]$ generates a $K[\mathrm{PGL}(V)]$-submodule containing $K[\mathbb{P}(V)]^\circ$, whenever not all a_x are equal. As $\mathrm{PGL}(V)$ is 2-transitive on $\mathbb{P}(V)$, it suffices to show that the $K[\mathrm{PGL}(V)]$-submodule generated by α contains a difference of two distinct points.

For each x with $a_x \neq 0$ fix its lift $\tilde{x} \in V$. Choose a maximal subset B consisting of F-linearly independent elements among $\tilde{x}$'s. We replace F with the subfield of F generated by the coefficients of the elements $\tilde{x}$ in the base B, and replace $\mathbb{P}(V)$ with the projectivization of the space spanned by the $\tilde{x}$'s over the new F. We thus assume that $\mathbb{P}(V)$ is finite. Then we proceed by induction of the dimension n of $\mathbb{P}(V)$.

For each hyperplane $H \subset \mathbb{P}(V)$, let $U_H \subset \mathrm{PGL}(V)$ be the translation group of the affine space $\mathbb{P}(V) \setminus H$. Then $(\sum_{h \in U_H} h)\alpha = (\sum_{x \notin H} a_x) \sum_{x \notin H} [x] + q^{\dim \mathbb{P}(V)} \sum_{x \in H} a_x [x]$, where q is order of F.

For the induction step in the case $\ell \not| q$ (and $\dim \mathbb{P}(V) > 1$), fix some hyperplane H containg points y, z with $a_y \neq a_z$ and fix some $\eta \in \mathrm{PGL}(V)$ such that $\eta(H) = H$, $\eta(y) = z$ and $\eta(u) = u$ for some $u \in H$. Then $(\eta-1)(\sum_{h\in U_H} h)\alpha = q^{\dim \mathbb{P}(V)} \sum_{x\in H}(a_{\eta^{-1}x}-a_x)[x] = q^{\dim \mathbb{P}(V)}(\cdots+(a_y-a_z)[z]+0[u])$, so we are reduced to the case of dimension $n-1$.

Assume now that $\dim \mathbb{P}(V) = 1$.

(1) If ℓ does not divide $q+1$ then there is $y \in \mathbb{P}(V)$ such that $(q+1)a_y \neq \sum_{x\in\mathbb{P}(V)} a_x$. If ℓ divides $q+1$ then fix an arbitrary $y \in \mathbb{P}(V)$ with $a_y \neq 0$ (so that $(q+1)a_y = 0 \neq \sum_{x\in\mathbb{P}(V)} a_x$). Fix some involution $\xi \in \mathrm{PGL}(V)$ such that $\xi y \neq y$. Then $(\xi-1)(\sum_{h\in U_{\{y\}}} h)\alpha = (qa_y - \sum_{x\neq y} a_x)([\xi y]-[y]) = ((q+1)a_y - \sum_{x\in\mathbb{P}(V)} a_x)([\xi y]-[y])$. Thus, assuming that either ℓ does not divide $q+1$ or $\sum_{x\in\mathbb{P}(V)} a_x \neq 0$, the $K[\mathrm{PGL}(V)]$-submodule generated by α contains $[\xi y]-[y]$, and therefore, it contains $K[\mathbb{P}(V)]^\circ$.

Assume now that ℓ divides $q+1$ and $\sum_{x\in\mathbb{P}(V)} a_x = 0$. Then $-a_y^{-1}(\sum_{h\in U_{\{y\}}} h)\alpha = \sum_{x\in\mathbb{P}(V)}[x]$.

Assuming in addition that $\alpha \notin K\cdot\sum_{x\in\mathbb{P}(V)}[x]$, fix some $z \in \mathbb{P}(V)$ with $a_z \neq a_y$. Let $T \subset \mathrm{PGL}(V)$ be the torus fixing y and z. Then $(\sum_{h\in T} h)\alpha = (\sum_{x\neq y,z} a_x)\sum_{x\neq y,z}[x] + (q-1)a_y[y] + (q-1)a_z[z] = -(a_y+a_z)\sum_{x\neq y,z}[x] - 2a_y[y] - 2a_z[z]$.

If $\ell = 2$, fix some $\xi \in \mathrm{PGL}(V)$ such that $\xi y = z$ and $\xi z \neq y$. Then $(\sum_{h\in T} h)\alpha = -(a_y+a_z)\sum_{x\in\mathbb{P}(V)}[x] + (a_y+a_z)[y] + (a_y+a_z)[z]$, so $(\xi-1)(\sum_{h\in T} h)\alpha = (a_y+a_z)([\xi z]-[y])$, and thus, the $K[\mathrm{PGL}(V)]$-submodule generated by α contains $[\xi z]-[y]$, and therefore, it contains $K[\mathbb{P}(V)]^\circ$.

If $\ell \neq 2$, fix some involution $\xi \in \mathrm{PGL}(V)$ such that $\xi y = z$. Then $(\xi-1)(\sum_{h\in T} h)\alpha = 2(a_y-a_z)([y]-[z])$, and thus, the $K[\mathrm{PGL}(V)]$-submodule generated by α contains $[y]-[z]$, and therefore, it contains $K[\mathbb{P}(V)]^\circ$.

(2) Any nonzero element of $K[\mathbb{P}(V)]$ can be considered as an element $\alpha = \sum_{x\in\mathbb{P}^1(\mathbb{F}_q)} a_x[x]$ for a finite subfield $\mathbb{F}_q \subset F$. Extending $\mathbb{F}_q$ in F if necessary, we may assume that $a_x = 0$ for at least one x. Then it follows from (1) that any $K[\mathrm{PGL}_2(\mathbb{F}_q)]$-submodule containing α contains $K[\mathbb{P}^1(\mathbb{F}_q)]^\circ$, and thus, any $K[\mathrm{PGL}(V)]$-submodule containing α contains $K[\mathbb{P}(V)]^\circ$. □

Proposition 5.4. *Let $F = \mathbb{F}_q$ be a finite field, V be an F-vector space and K be a field extension of F. Then the $K[\mathrm{PGL}(V)]$-module $K[\mathbb{P}(V)]^\circ$ is simple if and only if* $\dim V = 2$.

Proof. Let $\mathrm{Sym}^n_F V := (V^{\otimes^n_F})_{\mathfrak{S}_n}$ be the n-th symmetric power of V, so $\dim_F \mathrm{Sym}^n_F V = \binom{\dim V+n-1}{\dim V-1}$. The natural morphism of $K[\mathrm{PGL}(V)]$-modules $K[\mathbb{P}(V)]^\circ = K[(V\setminus\{0\})/F^\times]^\circ \to K\otimes_F \mathrm{Sym}^{q-1}_F V$, $\sum_x a_x[x] \mapsto \sum_x a_x\tilde{x}^{q-1}$, is nonzero. One has $\dim_K K[\mathbb{P}(V)]^\circ = |\mathbb{P}(V)|-1 = (q^{\dim V}-q)/(q-1)$.

If $\dim V = 2$ then $\dim_K K[\mathbb{P}(V)]^\circ = \dim_K K\otimes_F \mathrm{Sym}^{q-1}_F V = q$.

Assuming that $(q^n-q)/(q-1) \geq \binom{n+q-2}{n-1}$ for some $n \geq 2$, let us show that $(q^{n+1}-q)/(q-1) > \binom{n+q-1}{n}$. Indeed, $\binom{n+q-1}{n} = \binom{n+q-2}{n-1}(n+q-1)/n < \binom{n+q-2}{n-1}q \leq q(q^n-q)/(q-1) < (q^{n+1}-q)/(q-1)$.

This implies that $\dim K[\mathbb{P}(V)]^\circ > \dim K\otimes_F \mathrm{Sym}^{q-1}_F V$ if $\dim V > 2$, and thus, the above morphism is not injective, so $K[\mathbb{P}(V)]^\circ$ is not simple.

The simplicity in the case $\dim V = 2$ is shown in Lemma 3.4. □

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