# ESSAYS ON FINANCIAL ECONOMICS

by

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1

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## ABSTRACT

This thesis comprises three essays on human studies and financial economics. The first two essays explore the implications of Kahneman-Tversky's prospect theory on investors' trading behavior, equilibrium relations among asset returns, equilibrium relations between asset prices and trading volume, and Pareto optimal rules of risk sharing. Based on experimental findings, the third essay develops a revised game paradigm for the finitely repeated prisoners' dilemma and applies this new paradigm to study the interactions between debt maturity and firms' product market competition.

The first essay shows that the trading volume tends to be higher when the asset price rises than when it falls, if traders have preferences described by prospect theory. This prediction is well supported by empirical evidence. The second essay shows that the Sharpe-Lintner CAPM can be rebuilt in a two-period economy, if traders' preferences are described by prospect theory and two fund separation holds. It also provides a complete characterization of Pareto optimal sharing rules and rebuilds a representative agent theorem. Conditions for a competitive equilibrium to exist are derived.

The third essay shows that when an industry has a low concentration ratio and firms share industry-wide information which is unavailable to investors, short-term debt tends to help firms reduce competition. Collusive firms are shown to have high debt-equity ratios, a prediction which is consistent with empirical evidence.

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2

# Acknowledgements

# Contents

# Chapter 1

# A Price-Volume Formation Model

### Abstract of Chapter 1[1]

This paper explores the implications of Kahneman-Tversky's prospect theory, especially the feature of the *isolation effect*, on the equilibrium relations between price changes and trading volume using a *price-volume formation model*, which comprises the *ex ante* primary market stage (a single auction), the *interim* secondary market stage (a double auction) and the *ex post* liquidation stage. We show that, as implied by the prospect theory, traders' interim preferences depend on their pre-trade positions. That traders differ in their interim preferences generates trades. A perfect Bayesian equilibrium exists and in equilibrium, both the price and volume convey useful information. More precisely, it takes the price and the volume together to identify the values and the ranks of the signals recovered from the market price. As to traders' equilibrium bidding strategies, we show that regardless of their pre-trade positions, traders with the same *ex ante* preferences and interim information must agree on the predicted direction of the price change. That is, they always submit bids unanimously higher or unanimously lower than the price in the previous period. A zero-position trader, however, bids more (less) aggressively than his long-position counterpart whenever they feel optimistic (pessimistic) about the outlook of the market. These findings about traders' bidding behavior are used to establish two equilibrium relations between price changes and the trading volume:

1. Given any distribution of interim information, the trading volume is higher when the price rises than when it falls.

2. The trading volume is either extremely high or extremely low if the magnitude of the price change is large and traders always have similar information.

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## 1.1 Introduction

The purpose of this paper is two-fold. First, it provides a *price-volume formation model* where the formation of the asset price and trading volume is explicitly modeled and in equilibrium, the price and volume convey different and useful information to traders. Second, this price-volume formation model is used to study the implications of Kahneman-Tversky's (1979) prospect theory on the equilibrium relations between price changes and trading volume.

### 1.1.1 The Price-Volume Formation Model

The Rational Expectations Equilibrium (REE) models have replaced the Walrasian models in most studies of financial markets with differently informed traders. Because REE models do not explicitly model the process through which asset prices and trading volumes are formed, they generate several well-known paradoxes; see for example Milgrom (1981a), Bray and Kreps (1987), and Dubey, Geanakopolos and Shubik (1987). In view of these problems, the *price formation models* have been proposed as an alternative way of modeling financial markets with differently informed agents (Milgrom (1981a)). In a typical price formation model, there is a single auction where a seller is determined to sell a certain quantity of some asset. The trade is commonly known as being Pareto efficient. In general, the equilibrium volume is simply the total supply of the asset. In the real-world financial markets, however, sellers as well as buyers may choose not to clear their positions in one transaction. The trading volume is in general a random variable. Therefore a simple price formation model with fixed volume does not provide the right formulation for transactions of most financial assets.

What is needed is a double auction model where both sellers and buyers can determine their optimal quanitities of supply anddemand. The Milgrom-Stokey's no trade theorem implies that trades cannot take place unless traders differ in non-informational aspects. Thus the needed auction model is actually an *asymmetric double auction*, where participants differ in informational as well as non-informational aspects such as their preferences. In this paper, we shall provide an equilibrium asymmetric double auction model where both the price and volume are *ex ante* random variables and they convey useful information to traders *ex post*. The information conveyed by the price and volume has nice interpretations as follows.

After trades occur, when a rational trader observes the equilibrium price, he knows that some other traders have submitted a bid equal to the observed price. Since in equilibrium, all traders' bidding strategies are correctly expected, one can recover from the price the private information received by the traders whose bid equals the price. But, without knowing how many traders have received information more favorable than this particualr signai, the latter itself is not very useful. For

that information, we need volume. Trading volume tells traders the rank of the signal recovered from the price in the entire distribution of signals received by all traders. Thus it takes the price and volume together to convey useful information to traders.

This price-volume formation model will be used to study the implications of Kahneman-Tversky's prospect theory on the equilibrium relations between price changes and trading volume.

## 1.1.2 Prospect Theory and Price-Volume Relations

It has been well documented that the trading volume and price changes of an asset generally exhibit certain empirical relations. For example, the empirical evidence from the stock market and the bond market suggests that the trading volume is generally higher when the asset price rises than when it falls; see e.g. Ying (1966), Morgan (1976), Harris (1984,1986), Smirlock and Starks (1985), Wood, McInish and Ord (1985), Jain and Joh (1986), Richardson, Sefcik and Thompson (1987). Also, the trading volume tends to be positively correlated with the magnitude of the price change; see e.g. Ying (1966), Crouch (1970), Epps and Epps (1976), West-erfield (1977), Harris (1983, 1984, 1986), Wood, McInish and Ord (1985), Jain and Joh (1988), Richardson, Sefcik and Thompson (1987). Recently, there has been a growing literature which accounts for the price-volume relations using models with differently informed traders. For example, in Kyle (1985) and Admati and Pfleiderer (1988), when a less informed market maker observes a large sell order, he infers that there are better informed traders camouflaged behind and thus lowers his bid price. Similarly, when he observes a large buy order, he raises his ask price. The high volume is, in this sense, the cause of significant price changes. In Wang (1991), the less informed investors trade against the more informed investors who are known to trade also for non-informational reasons. Information asymmetry implies a positive correlation between the trading volume and the magnitude of the excess return. Based on the assumption that some market participants are strictly less informed than the others, these models are able to generate predictions about the relations between the magnitude of the price change (or excess return) and the trading volume, but fail to explain why the *direction* of the price change also relates to the trading volume. In other models, e.g. Varian (1987) and Hindy (1989), the trading volume is high if traders have diverse opinions about the outlook of the market. If the trading volume changes because of changes in the traders' opinions, then why do traders tend to disagree about the outlook of the market more when the price rises than when it falls? Since this phenomenon is pervasive, we shall look for reasons which are more fundamental than certain special assumptions about information or

market imperpections.[2] The explanation we shall propose is about traders' preferences. Assuming traders' preferences are of Kahneman-Tversky's type, we show that the relation between the trading volume and the direction of the price change can arise naturally under various information structures, including the case where the information is always public and every trader interprets the information identically.

Prospect theory was proposed as an alternative to expected utility theory, when the latter was found by psychologists to be inconsistent with the human behavior under uncertanty (Kahneman and Tversky (1979)). The development of prospect theory has been based on compelling experimental findings; see for example Tversky (1972, 1979), Tversky and Kahneman (1974) and Kahneman and Tversky (1979). Its predictions are found to be consistent with findings of subsequent research in psychology and consumer behavior; see for example Tversky and Kahneman (1981) and Thaler (1985). According to the prospect theory, when facing a random payoff, an individual first compares the payoff to some reference point or bench mark. The difference between the random payoff and the reference point defines the random gain and loss. An individual's preferences are then represented by a *value function* defined on the space of gains and losses. A value function is concave on the positive orthant, convex on the negative orthant and is everywhere increasing. The von Neumann-Morgenstern (VNM) concave utility functions defined on the space of consumption or wealth may be considered as a special class of Kahneman-Tversky's value functions with zero reference point.[3] Prospect theory is a *descriptive* theory in the sense that its purpose is to accurately describe agents' preferences under uncertainty. The expected utility theory, on the other hand, is *normative* in the sense that it is derived from a set of axioms which state the basic patterns an individual's perferences should exhibit. The approach to develop the prospect theory is *inductive*, in the sense that it is driven by compelling evidence and findings, which contrasts with the *deductive* approach taken by the expected utility theory, which is a necessary condition of its postulated axioms.

There are three major components of the prospect theory, the *certainty effect*, the *reflection effect* and the *isolation effect*, which describe how the reference point in an individual's value function may vary with the context in which the individual

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[2]Presumably, an asymmetric structure of tansaction costs may produce an asymmetric equilibrium outcome. For example, assuming that it is more costly to short sell the risky asset than to borrow at the riskless rate, or that the transaction costs of buying and selling an asset are different, one may be able to account for the relation between the trading volume and the direction of the price change; see e.g. Jennings, Starks and Fellingham (1981). However, an asymmetry assumption as such can be arbitrary. Even if it is consistent with the reality, whether its effects are significant enough to give rise to the pervasive phenomenon is dubious.

[3]In chapter 2, I investigate the equilibrium relations among asset returns and the properties of the optimal risk sharing rules, assuming that agents maximize expected Kahneman-Tversky value functions. The obtained results indeed include those for the VNM utility functions as a special case.

makes his decisions. In this study, we introduce the feature of the *isolation effect* to describe traders' preferences. We shall show that, under general information structures, this preferences assumption generates implications on traders' bidding behavior and equilibrium price-volume relations that are consistent with empirical findings.

### 1.1.3 The Isolation Effect

**Example 1** Consider the following pairs of lotteries:[4]

$$A = (4000, 0.8), \quad B = (3000, 1.0),$$

and

$$C = (4000, 0.2), \quad D = (3000, 0.25),$$

where a lottery $(M, p)$ is one that pays $M$ with probability $p$ and 0 with probability $1 - p$. The experimental findings show that most subjects prefer $B$ to $A$ and $C$ to $D$. This is a version of the famous Allais' paradox. The results show that the independence axiom postulated in the expected utility theory is violated.[5] Now consider a two stage game where in the first stage with 0.75 probability the game ends and the player gets nothing, while with $0.25$ probability the game moves on to the second stage, where the player gets the chance to make a choice between $A$ and $B$. The player is asked to commit his choice, $A$ or $B$, at the beginning of the game. The experimental findings show that most subjects prefer $B$ to $A$.

Note that at the beginning of the two stage game, in terms of the payoffs in final states, the player actually faces a choice between $C$ and $D$. However, the two lotteries have been restructured to contain a common aspect, i.e. the probability 0.25 with which the game moves on to the second stage. Because of this restructuring, most subjects make differenct choices. The experimental results suggest that the subjects have ignored the common aspect of $C$ and $D$, and regarded the choice as one between $A$ and $B$. As is consistent with the results in earlier experiments, most subjects choose $B$.

**Example 2** Let us consider another example of the *isolation effect*, in which the context is closer to the model we shall introduce in the sequel. In an experiment, subjects are asked to make a choice between

$$A = (1000, 0.5)$$

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[4]The following examples for the *isolation effect* are borrowed from Kahneman and Tversky (1979).

[5]The experimental findings also suggest that most subjects exhibit the *certainty effect*, which means that the subjects put much more weight on a sure event than an event which is very probable. The *certainty effect* suggests that in comparing $A$ to $B$, an individual may have a reference point which is different from that in comparing $C$ to $D$.

and

$$B = (500, 1.0).$$

Every subject who made the choice also receives a sure payoff of 1000. Most subjects choose $B$ over $A$. The same subjects are asked to make a choice between

$$C = (-1000, 0.5)$$

and

$$D = (-500, 1.0).$$

This time, every subject who made the choice can receive a sure payoff of 2000. Most subjects choose $C$ over $D$.[6] Notice that, in terms of total wealth, $A$ is identical to $C$ and $B$ is identical to $D$. If, for instance, the choice of $B$ over $A$ is consistent with an agent's concern about random wealth, then the choice of $C$ over $D$ must violate this concern.

These observations about the *isolation effect* give the following messages:

1. The isolation effect gives rise to certain patterns in human behavior which may be interpreted as inconsistencies or irrationalities by economists who believe in the expected utility theory.

2. On the other hand, if agents' preferences are accurately described by the prospect theory, these patterns of human behavior only indicate that agents are individually rational, i.e. the decisions agents make are indeed optimal with respect to their preferences.

3. The fact that value functions are monotonic in gains and losses implies that, when frictionless financial markets are in equilibrium, there will be no arbitrage opportunities.[7]

Although prospect theory has been supported by numerous experimental findings in human studies at the individual level, its implications on economic activities at the aggregate level remain unknown. In view of this, this paper introduces the *isolation effect* to describe traders' preferences and use the price-volume formation model to study its implications on equilibrium price-volume relations. Before economists reach concensus about which preference theory best captures the real-life agents' behavior, learning the implications of various preference theories on economic activities should be helpful. Thus it is *not* our intention to argue that the prospect theory

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[6]In two other experiments subjects are asked choose between $A$ and $B$ and between $C$ and $D$ respectively, without the sure payoffs. Most subjects prefer $B$ to $A$ and $C$ to $D$. This phenomenon is termed the *reflection effect*, which means that subjects exhibit risk-averse (risk-seeking) behavior when the comparison involves only gains (only losses).

[7]Even if markets are imperfect, as long as every agent is subject to the same market imperfections, there can be no arbitrage opportunities.

is superior to the expected utility theory. The purpose is to understand what we should expect to see in terms of the equilibrium price-volume relations, *if* prospect theory does offer an accurate description of traders' preferences.

## 1.1.4 Organization of the Paper

In section 1.2, we layout the three-stage model. At time 0, a firm issues a security to a continuum of identical traders. The equilibrium price of the security in the primary market is determined by a single auction. At time 1, traders first receive some private and public signals, which are correlated with the time-2 random dividend, and then they trade in the secondary market. The secondary market is modeled as a double auction. At time 2, the security pays out a one-time liquidation dividend to security holders and the economy ends. We show that there exists a symmetric perfect Bayesian equilibrium for this economy. In equilibrium, at time 1, we show that traders who differ only in the pre-trade security holding positions must agree on the predicted direction of the price change. That is, they will submit bids which are either unanimously higher or unanimously lower than the asset price at time 0. However, a zero-position trader's bid is higher (lower) than that of his long-position counterpart, whenever they feel optimisitc (pessimistic) about the outlook of the market. This follows from the isolation effect, which, in the current context, says that traders today will ignore the gains or losses from trade up to yesterday, because those gains or losses have realized and are common to all their feasible bidding strategies today. It follows that a trader whose preference exhibits the isolation effect is eager to capture gains from trade immediately and always tries to delay the loss from trade. For example, when the price is likely to fall, if a long-position trader ends up selling the asset at the low price, he experiences a loss from trade right away. On the other hand, if he carries the asset into the next period, knowing that tomorrow he will not care about the loss up to today, psychologically he feels that he can avoid the loss from today. Thus, in the event that the price is likely to go down, a long-position trader has incentives to submit a higher bid than his zero-position counterpart.

*Ex post*, after trades take place in the secondary market, both the price and volume are shown to convey useful information. More precisely, it takes the price and volume together to identify the values and the ranks of the signals recovered from the market price. Contrary to Milgrom (1981a) and Hindy (1989), traders' equilibrium bidding strategies do not possess the *no-regret property*.[8] It is shown that traders are *ex post* better off in some states of nature while regret their bids in other states. In fact, since the trading volume in a double auction is a random

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[8]By this is meant that in equilibrium, after observing the price and volume, each trader regrets neither his bid nor his post-trade position.

variable, but traders are still not allowed to make their bids contingent on the volume, the equilibrium generally does not possess the no-regret property.

In section 1.3, based on the findings about traders' equilibrium bidding behavior, we establish two equilibrium price-volume relations in theorems 1 and 2. In theorem 1, we show that given any distribution of traders' private signals, the trading volume is higher when the price rises than when it falls. The intuition follows directly from the observation that when the price rises, every winning trader must have submitted a bid higher than the previous asset price. But in this case, whenever a long-position trader won the bid, his zero-position counterpart must have won the bid too. This means that there are a lot of zero-position traders who won the bid, and so the volume is high.[9] On the other hand, when the price falls, among those winning traders who submitted bids lower than the previous asset price, a long-position trader had more chance to win the bid than his zero-position counterparts. The trading volume in this case cannot exceed that when the price rises.

In theorem 2, we show that the trading volume is either extremely high or extremely low, if the magnitude of the price change is large and traders always have similar information. The intuition behind this theorem is the following. According-ing to the prospect theory, the long-position traders and the zero-position traders have different bidding behavior, and this difference becomes significant when the magnitude of the price change becomes large. On the other hand, if traders always have similar information, then traders with the same pre-trade positions will have similar bidding behavior. Now suppose that the price always changes significantly and traders always have similar information. Consider all the long-position traders' bids as a group and all the zero-position traders' bids as another group. Then, the *within-group variation* will be small but the *between-group variation* will be large. This means:

- When the price rises, and a long-position trader tends to lose the bid, so do other long-position traders. The trading volume is therefore extremely high.

- When the price falls, and a long-position trader tends to win the bid, so do other long-position traders. In this case, the trading volume is extremely low.

We conclude chapter 1 in section 1.4.

## 1.2 The Price-Volume Formation Model

In this section, we layout the price-volume formation model and solve for a perfect Bayesian equilibrium.

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[9]We assume every limit order has a unit quantity.

13

## 1.2.1 The Basic Model

**Timing and The Mechanism of Asset Price Formation** We consider a three-period economy. At time 0, a firm issues a security to a continuum of identical investors with total population of $2S$, where $S$ is the total supply of the security. The quantity of the security which an investor is allowed to hold is represented by a density, which can be either 0 or 1.[10] At time 2, each unit of the security pays a one-time random liquidation dividend $P_2$, whose support is a finite closed interval $[0, K]$. At time 0, in the primary market, the price of the security, $P_0$, is determined by a single auction. At time 1, in the secondary market, the price of the security, $P_1$, is determined by a *balanced* double auction.[11] Specifically, $P_0$ and $P_1$ are respectively the suprema at times 0 and 1 of such bids that traders whose bids are higher than or equal to them have Lebesgue measures $S$. All transactions at time $t$, for $t = 1, 2$, are consummated at the equilibrium price $P_t$. For simplicity, we shall assume that short-sale is forbidden. It follows that the trading volume is always bounded above by $S$. In the following, *ex ante* refers to the time when the trading in primary market has not begun, *interim* refers to the time when the trading at time 0 is over and the trading at time 1 has not begun, and *ex post* refers to the time when the secondary market trading is finished.

**Information Structure** At time 1, prior to the trade in the secondary market, there are in total $S$ long-position traders and $S$ zero-position traders in the economy. Before trading begins in the secondary market, every trader is assumed to observe some private and public signals about $P_2$. The public information is represented by a random variable $Y$ which is correlated with $P_2$ and has the finite closed interval $[y_*, y^*]$ as its support. A zero-position and a long-position traders respectively receive private signals $X_Z$ and $X_L$, of which, conditional on each realization of $P_2$, the common distribution is $F(x|P_2)$,[12] an absolutely continuous, strictly increasing function on the support $[x_*, x^*]$. Denote the common conditional density of $X_Z$ and $X_L$ by $f(x|P_2)$. For the public signal $Y$, we shall also assume that its conditional distribution function, $H(y|P_2)$, is absolutely continuous and has conditional density $\iota(y|P_2)$. Two assumptions will be made regarding the relations between signals $X$ and $Y$ and the liquidation dividend $P_2$:

**Assumption 1 Strict Monotone Likelihood Ratio Property (MLRP)** *The*

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[10]Later on, we show that this density is integrable over traders.

[11]By balanced is meant that the Lebesgue measure of the potential sellers equals that of the potential buyers in the double auction.

[12]That traders' private signals are identically distributed leads to the fact that the impacts of other traders' bidding strategies on a particular trader's welfare are symmetric, if traders' bidding strategies are symmetric. This in turn makes the formulation an *anonymous game* (Jovanovic and Rosenthal; 1988).

*likelihood ratio*

$$\frac{f(x|P_2)}{f(x|P_2')}$$

*is strictly decreasing in $x$ for $P_2' > P_2$ and strictly increasing in $x$ for $P_2 > P_2'$. Also, the likelihood ratio*

$$\frac{\iota(y|P_2)}{\iota(y|P_2')}$$

*is strictly decreasing in $y$ for $P_2' > P_2$ and strictly increasing in $y$ for $P_2 > P_2'$.*

**Assumption 2 No Aggregate Uncertainty** *Conditional on $P_2$, the Lebesgue measure of traders who receive private signals lower than or equal to $x$ is*

$$2S \cdot F(x|P_2).$$

**Remark 1** *Some remarks are in order regarding assumptions 1 and 2:*

1. *Assumption 1 states that signals are always comparable in the terminology of Milgrom (1981b). The comparability of signals follows from the assumed strict MLRP. More precisely, the strict MLRP implies that an increase in the private or public signal leads to a first-order stochastic increase in $P_2$. Moreover, the two assumptions together imply that any $\nu$-order statistic[13] of a randomly selected section from the private signals received by the entire economy also has strict MLRP; see Hindy (1989). proposition 5.*

2. *In the above, we have assumed that all the information a trader may have, prior to the trade in the secondary market, can be summarized by the realizations of two random variables (real-valued signals), $X$ and $Y$. There are two implicit assumptions about this representation of a trader's information. First, the two signals must be sufficient statistics for all the information the trader has concerning $P_2$. Second, they must also adequately summarize the trader's information regarding the signals received by other traders; see Milgrom and Weber (1982).*

3. *The no-aggregate uncertainty property is assumed so that agents' private signals are identical but not independent. Feldman and Gilles (1985) show that it is not correct to postulate that the family of random variables observed by a continuum of agents is a continuum of independent and identically distributed random variables. The no-aggregate uncertainty formulation is one remedy for this situation proposed by Feldman and Gilles. The no-aggregate uncertainty turns out to be one form of the law of large number in the following sense: Following the no-aggregate uncertainty assumption and the strict MLRP of private*

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[13]I.e., the signals higher than or equal to this statistic have a measure equal to $\nu$.

*signals, the realization of $P_2$ can be instantly revealed if, hypothetically, every agent in this economy announces his private signal to the public.*

As will be shown in the sequel, without loss of generality, we can assume that interim, the zero-position traders and the long-position traders are respectively represented by the two closed intervals $[0, S]$ and $[S, 2S]$.[14] Now, conditional on $P_2$, let the private signals received by the zero-position and the long-position traders be represented by two continuous and strictly increasing functions $G_Z(x|P_2)$ and $G_L(x|P_2)$, with

$$G_L(x|P_2) + G_Z(x|P_2) = 2S \cdot F(x|P_2),$$

$$G_Z(x_*|P_2) = G_L(x_*|P_2) = 0,$$

and

$$G_L(x^*|P_2) = G_Z(x^*|P_2) = S.$$

Note that conditional on $P_2$, $G_Z(x|P_2)$ and $G_L(x|P_2)$ denote respectively the Lebesgue measures of the zero-position and the long-position traders whose private signals are lower than or equal to $x$. Given $P_2$, the two functions $G_Z(\cdot|P_2)$ and $G_L(\cdot|P_2)$ represent a random decomposition of $2S \cdot F(\cdot|P_2)$, which is the distribution of the private signals received by the entire economy. We have assumed that the realizations of the two random functions $G_Z(\cdot|P_2)$ and $G_L(\cdot|P_2)$ have certain nice properties: they are continuous and strictly increasing.

In the following, we shall call any two such functions $G_Z(x|P_2)$ and $G_L(x|P_2)$ a (realized) *distribution of private signals* or a *distribution of traders' types*. This terminology is adopted because *ex ante* traders are identical, and thus receiving private signals is equivalent to receiving traders' types. Since Harsanyi (1967; 1968), the *type* of a player in a Bayesian game has been used to refer to the private information that the player has.

From the informational point of view, the security market gathers individual traders' private signals and transmits them into the equilibrium price and the trading volume. Rational traders, when determining their bidding strategies, must take into account how other traders' private signals are incorporated in and revealed by the market price and the trading volume.

**Traders' Preferences and the Motives for Trade** Traders are assumed to have preferences described by Kahneman-Tversky's *prospect theory*. We shall show that the prospect theory implies that traders' interim preferences are (pre-trade) position-dependent. That traders differ in their interim preferences generates trades. This is why liquidity or noise trading is not needed in this model.[15] According to

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[14]It will be shown that the identical traders at time 0 are indifferent about taking a long or zero position in equilibrium. We can thus assume that at time 0, the security is sold to the traders in the interval $[S, 2S]$.

[15]See Kyle (1985) and Admati and Pfleiderer (1988,1989) for passive behavior of liquidity traders and Wang (1991) for a derivation of liquidity trading from traders' optimizing behavior.

the prospect theory, when facing a random payoff, an agent first compares the random payoff to some *reference point*. The difference between the random payoff and the reference point, depending on whether it is positive or negative, is defined as respectively the *gain and loss*. An agent's preferences about random payoffs is represented by a Kahneman-Tversky's *value function* defined on the space of gains and losses. The value function is real-valued, strictly increasing, continuous, passing through the origin, and is concave on the positive orthant and convex on the negative orthant.[16] We shall assume that traders care about accounting gains and losses from trade. That is, a trader buying the security at time $s$ and selling it at time $t > s$ experiences a gain or loss of $P_t - P_s$.

One major feature of the prospect theory is the *isolation effect*, by which is meant that agents when making decisions tend to disregard the common aspects of all feasible actions.[17] In our model, when a trader determines his optimal bidding strategy at time $t$ for an asset purchased at time $s < t$, the part of gain or loss represented by $P_{t-1} - P_s$ has already been realized and is common for all feasible bidding strategies at time $t$. The isolation effect then asserts that this part of gain or loss be disregarded. We shall assume:

**Assumption 3** *Traders' preferences exhibit a major feature of the prospect theory, the* **isolation effect**, *in the sense that traders at time $t$ will disregard the realized gains or losses from trade up to time $t - 1$.*

Following assumption 3, a zero-position trader at time 1, after observing the public signal $y$ and private signal $a$, maximizes over feasible bids $Z$

$$E(1_{[Z \geq P_1]}V(P_2 - P_1)|X_Z = a, Y = y, P_0),$$

where $V(\cdot)$ is some Kahneman-Tversky's value function and $1_A$ equals 1 if the event $A$ occurs and 0 otherwise. Note that in the above maximization problem, $P_1$ is deduced from other traders' bidding strategies[18] and the common prior beliefs about signals $X$ and $Y$. Denote the solution to the above zero-position trader's problem by $Z^*(a, y, P_0)$, whenever it exists. Thus, $Z^*(a, y, P_0)$ is the reaction function of a zero-position trader with private signal $a$ and public signal $y$, which depends on other traders' bidding strategies. Now define

$$J_Z(a, y, P_0) \equiv E(1_{[Z^*(a,y,P_0) \geq P_1]}V(P_2 - P_1)|X_Z = a, Y = y, P_0).$$

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[16]In this study, we only use the monotonicity of the value function.

[17]Tversky (1972) contains the experimental results about the isolation effect. In an experiment regarding agents' gambling behavior, about 81% of the subjects exhibit the isolation effect. In another experiment on selecting applicants for the admission to a school, about 94% of the subjects exhibit the isolation effect. See also Kahneman and Tversky (1979). For a discussion about the connection of the isolation effect to the independence axiom in the expected utility theory, see appendix B.

[18]Rational expectations imply that other players' strategies are correctly expected.

Then $J_Z$, whenever exists, denotes the time-1 indirect value function of a zero-position trader of type $a$. At time 1, a long-position trader after observing the public signal $y$ and private signal $a$, maximizes over feasible bids $L$

$$E(1_{[L \geq P_1]}V(P_2 - P_1) + 1_{[L < P_1]}V(P_1 - P_0)|X_L = a, Y = y, P_0),$$

where note that at time 2, the part of gain or loss $P_1 - P_0$ has been ignored by the long-position trader. Also note that the above long-position trader's maximization problem differs from that of his zero-position counterpart. That is, traders' interim preferences differ because of their pre-trade positions, although they have identical *ex ante* preferences and interim information. As being fully rational, *ex ante* a trader knows perfectly how his interim preferences will depend on his pre-trade position. This is an implication of the prospect theory.

Now denote the solution to the above long-position trader's maximization problem by $L^*(a, y, P_0)$, whenever it exists. Again, $L^*(a, y, P_0)$ is the long-position trader's reaction function given other traders' bidding strategies. Define

$$J_L(a, y, P_0) \equiv E(1_{[L^*(a,y,P_0) \geq P_1]}V(P_2 - P_1) + 1_{[L^*(a,y,P_0) < P_1]}V(P_1 - P_0)|X_L = a, Y = y, P_0).$$

Then $J_L$, whenever exists, denotes the time-1 indirect value function of a long-position trader of type $a$. In the above formulation, we have assumed that participation in trades is budget feasible for each trader.

**Equilibrium Concept** In the above formulation, we have revealed our intention to only focus on symmetric Bayesian equilibria for the secondary market. A symmetric Bayesian equilibrium is composed of a set of common prior beliefs about signals and $P_2$, and a pair of bidding strategies $(Z^*, L^*)$, which map the private signal, the public signal and the previous asset price into real numbers, such that, given that all other long-position traders adopt strategy $L^*$ and all other zero-position traders adopt strategy $Z^*$, a zero-position (respectively, long-position) trader finds it optimal to adopt $Z^*$ (respectively, $L^*$) as well. This is a natural equilibrium concept, considering *ex ante*, all traders are identical. Our auction model is one of competitive bidding, in the sense that no single trader considers the effect of his own bid on market price when solving his optimal bidding problem. Under the assumption of a continuum of traders, competitive bidding is not only economically intuitive but also mathematically reasonable. In summary, we shall model the price formation mechanism in the secondary market at time 1 using a common value double auction, and look for a symmetric Bayesian equilibrium.

At time 0, in the primary market, define $P_0^*$ as the time-0 security price which makes the identical traders at time 0 indifferent about taking a long or a zero position, i.e. $P_0^*$ solves

$$E(J_L(X_L, Y, P_0)) = E(J_Z(X_Z, Y, P_0)),$$

18

where $P_0$ is the unknown. Note that $P_0^*$, if it exists, must depend on traders' equilibrium bidding strategies in the double auction at time 1. In the sequel, we shall show that both $E(J_L)$ and $E(J_Z)$ are continuous in $P_0$ in equilibrium. This, together with some other facts, shows that the $P_0^*$ so defined exists generally. Given traders' equilibrium bidding strategies at time 1, $P_0^*$ defines a symmetric equilibrium bidding strategy at time 0.[19] Our task now is prove that a symmetric Bayesian equilibrium exists for the double auction at time 1 and that a perfect Bayesian equilibrium exists for the entire trading economy.

## 1.2.2 A Symmetric Perfect Bayesian Equilibrium

We now proceed to show the existence of a symmetric Bayesian equilibrium for the double auction at time 1.[20]

Note that any two realizations of the random functions $G_Z(x|P_2)$ and $G_L(x|P_2)$, which represent a realized distribution of traders' types, generate two continuous, strictly decreasing inverse functions $\xi_Z(s|P_2)$ and $\xi_L(s|P_2)$ which satisfy, for all $P_2$,

$$\xi_Z(0|P_2) = \xi_L(0|P_2) = x^*$$

and

$$\xi_Z(S|P_2) = \xi_L(S|P_2) = x_*.$$

Note that $\xi_Z(s|P_2)$ is the private signal such that the Lebesgue measure of the zero-position traders whose private signals are higher than or equal to $\xi_Z(s|P_2)$ is $s$. The interpretation of $\xi_L(s|P_2)$ is similar. Note that $\xi_Z(s|P_2)$ and $\xi_L(s|P_2)$ are two random functions which depend on $P_2$ as well as the random decomposition of $2S \cdot F(x|P_2)$ into $G_Z(x|P_2)$ and $G_L(x|P_2)$. To ease notation, from now on, we shall suppress the dependence of $\xi_Z(s)$ and $\xi_L(s)$ on $P_2$.

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[19]To see this, note that given that all other traders submit $P_0^*$, a trader cannot win the security by submitting a bid lower than $P_0^*$, but at the same time, submitting a bid strictly higher than $P_0^*$ does not increase the trader's welfare. Note that the $P_0^*$ so defined is actually the Nash equilibrium price in a Bertrand competition model. Since at time 0, the primary market is characterized by a continuum of identical traders who bid competitively in a single auction, the Bertrand price $P_0^*$ is a natural candidate for the time-0 equilibrium price.

[20]The major difficulty involved is that the double auction is *asymmetric*; see Hindy (1989). By an asymmetric double auction is meant that prior to the trade, traders differ in the private information as well as other non-informational aspects. Unlike in symmetric auction models, the proof of the existence of a symmetric equilibrium in asymmetric auction models is generally difficult. The difficulty first arises in recovering private signals from the price and volume without knowing which classes of traders have submitted the bid which equals the market price. The complexity leads some researchers to a study of boundedly rational trading behavior (Hindy (1989)). Our treatment for this problem is, however, relatively simple. The idea is to make sure that whatever the market price is, there are always traders of each class who submit the bid equal to the market price. We shall remark on how this treatment resolves the problem.

Let $\nu$ be the trading volume at time 1. Then, $\nu$ is the Lebesgue measure of the pre-trade zero-position traders who win the bid in the double auction. Following the short-sale constraint, $0 \le \nu \le S$. A symmetric Bayesian equilibrium at time 1 is composed of a set of common prior beliefs about the (private and public) signals and $P_2$, and a pair of bidding strategies $Z^*(X_Z, Y, P_0)$ and $L^*(X_L, Y, P_0)$, such that given the common prior beliefs and that all zero-position traders adopt $Z^*(X_Z, Y, P_0)$ and all long-position traders adopt $L^*(X_L, Y, P_0)$ as respectively their bidding strategies, no single trader wants to deviate unilaterally. When such an equilibrium exists, we should have

$$P_1 = Z^*(\xi_Z(\nu), y, P_0) = L^*(\xi_L(S - \nu), y, P_0),$$

provided that in equilibrium there are both zero-position and long-position traders who submit the bid $P_1$ when the volume is $\nu$ and the public signal is $y$. In this case, the above equality says that in a symmetric Bayesian equilibrium, after observing the equilibrium price $P_1$, traders learn both the private signals $\xi_Z(\nu)$ and $\xi_L(S - \nu)$. Indeed, the above equality gives the relation between $\xi_Z(\nu)$ and $\xi_L(S - \nu)$. It follows from proposition 5 of Hindy (1989) that given any $\nu$, both $\xi_L(S - \nu)$ and $\xi_Z(\nu)$ have strict MLRP.

Our strategy of showing the existence of a symmetric Bayesian equilibrium is to, given the common prior beliefs about signals and $P_2$, construct two bidding strategies, one for zero-position traders and the other for long-position traders, and then verify that the two prescribed strategies are actually equilibrium strategies according to the prior beliefs. To begin, consider the following two equations,

$$E(V(P_2 - z)|X_Z = a, Y = y) = 0$$

and

$$E(V(P_2 - l)|X_L = a, Y = y) = V(l - P_0),$$

where $z$ and $l$ are respectively unknowns in the two equations. Denote the solutions to the above two equations by respectively $z(a, y)$ and $l(a, y, P_0)$. Because $V(\cdot)$ is strictly increasing, $z(a, y)$ and $l(a, y, P_0)$ exist uniquely. Moreover, $z(a, y)$ and $l(a, y, P_0)$ are both non-negative, because $V(P_2)$ is non-negative.

Note that $z(a, y)$ and $l(a, y, P_0)$ are the pre-trade reservation values for the security of respectively the zero-position and the long-position traders whose private signal is $a$ and public signal is $y$. That is, if it is known that trading will not reveal any new information regarding $P_2$,[21] $z(a, y)$ and $l(a, y, P_0)$ are respectively the maximum amounts of money the zero-position and the long-position traders are willing to pay to acquire the security. The following lemma follows from the strict MLRP of signals and the unique existence of $z(a, y)$ and $l(a, y, P_0)$ for each private signal $a$, public signal $y$ and the previous market price $P_0$:

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[21]In this case, the auction model becomes effectively a private valuation model.

**Lemma 1** *Both functions $z(a,y)$ and $l(a,y,P_0)$ are strictly increasing in the first two arguments. Moreover, either $z(a,y) \leq l(a,y,P_0) \leq P_0$ or $z(a,y) \geq l(a,y,P_0) \geq P_0$.*

**Proof**
See appendix A. $\square$

Proposition 1 below shows the existence of a symmetric Bayesian equilibrium if either

$$z(x^*,y) < l(x_*,y,P_0)$$

or

$$z(x_*,y) > l(x^*,y,P_0)$$

is true.[22] The case where neither of the above inequalities holds will be dealt with in proposition 2.[23]

**Proposition 1** *If either $z(x^*,y) < l(x_*,y,P_0)$ or $z(x_*,y) > l(x^*,y,P_0)$, then $Z^*(a,y,P_0) \equiv z(a,y)$ and $L^*(a,y,P_0) \equiv l(a,y,P_0)$ define a symmetric Bayesian equilibrium for the double auction at time 1. In equilibrium, depending on the realization of $Y$, $P_0$ is either higher or lower than all traders' bids at time 1 and correspondingly, $\nu$ is either $0$ or $S$. Moreover, $P_1$ is always non-negative.*

**Proof**
See appendix A. $\square$

Now, let us consider the case where

$$z(x^*,y) \geq l(x_*,y,P_0)$$

and

$$z(x_*,y) \leq l(x^*,y,P_0).$$

Our strategy is, again, to first construct two bidding strategies, one for the zero-position traders and the other for the long-position traders, and then verify that the two prescribed strategies are indeed equilibrium strategies. For this purpose, given $\epsilon \in [0, \frac{x^*-x_*}{2}]$, let us first consider two continuous and strictly increasing real

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[22]I.e., either that the most optimistic zero-position traders have a reservation value for the security lower than that of the most pessimistic long-position traders, or that the most pessimistic zero-position traders have a reservation value for the security higher than that of the most optimistic long-position traders.

[23]The idea of proposition 1 is that if trading does not reveal any new information, every trader should simply submit the bid equal to his pre-trade reservation value for the security. In the above equilibrium, $P_1$ equals either $Z^*(x_*,y,P_0)$ or $L^*(x_*,y,P_0)$. Thus, by observing the price $P_1$, traders learn either $\xi_Z(S) = x_*$ or $\xi_L(S) = x_*$. But these have been contained in every trader's pre-trade information set already. In this case, bidding the pre-trade reservation value for the security is optimal.

functions $\zeta(\cdot)$ and $\lambda(\cdot)$ which map $[x_* + \epsilon, x^* - \epsilon]$ into $[0, K]$. Given $y$, $P_0$ and the common prior beliefs about the signals and $P_2$, for all $a, b \in [x_* + \epsilon, x^* - \epsilon]$, define $Z_\epsilon(a, y, P_0)$ and $L_\epsilon(b, y, P_0)$ as respectively the solutions to the following two equations:

$$E(V(P_2 - z)|X_Z = a, \xi_Z(\nu) = a, \xi_L(S - \nu) = \lambda^{-1}(\zeta(a))) = 0,$$

$$E(V(P_2 - l)|X_L = b, \xi_L(S - \nu) = b, \xi_Z(\nu) = \zeta^{-1}(\lambda(b))) = V(l - P_0),$$

where note that we have suppressed the conditioning of the expectations on $y$ and $P_0$ to ease notation, and that $z$ and $l$ are respectively unknowns in the above two equations. We shall define

$$\lambda^{-1}(\zeta(a)) = x^* - \epsilon,$$

in case that $\zeta(a) \geq \lambda(x^* - \epsilon)$. Similarly, define

$$\lambda^{-1}(\zeta(a)) = x_* + \epsilon,$$

in case that $\zeta(a) \leq \lambda(x_* + \epsilon)$. Also,

$$\zeta^{-1}(\lambda(a)) = x^* - \epsilon,$$

in case that $\lambda(a) \geq \zeta(x^* - \epsilon)$ and

$$\zeta^{-1}(\lambda(a)) = x_* + \epsilon,$$

in case that $\lambda(a) \leq \zeta(x_* + \epsilon)$. Given $y$, $P_0$ and the common beliefs about signals and $P_2$, the above two equations are well defined for all $a, b \in [x_* + \epsilon, x^* - \epsilon]$. Because $V(\cdot)$ is strictly increasing, $Z_\epsilon(a, y, P_0)$ and $L_\epsilon(b, y, P_0)$ both exist uniquely. Moreover, they are both positive since $V(P_2)$ is positive.

**Lemma 2** *Given $y$, $P_0$ and the common beliefs about signals, for each $\epsilon \in [0, \frac{x^* - x_*}{2}]$, there exist a pair of functions $(\zeta, \lambda)$ such that the correspondingly defined $(Z_\epsilon, L_\epsilon)$ turn out to be the functions $(\zeta, \lambda)$ themselves. Moreover, for each $a \in (x_*, x^*)$, define*

$$Z(a, y, P_0) \equiv \lim_{\epsilon \to 0} Z_\epsilon(a, y, P_0)$$

*and*

$$L(b, y, P_0) \equiv \lim_{\epsilon \to 0} L_\epsilon(a, y, P_0).$$

*Then, both limits exist.*

**Proof**
See appendix A. □

Note that $Z(a, y, P_0)$ and $L(b, y, P_0)$ are respectively the solutions to the following two equations:

$$E(V(P_2 - z)|X_Z = a, \xi_Z(\nu) = a, \xi_L(S - \nu) = L^{-1}(Z(a, y, P_0); y, P_0)) = 0,$$

$$E(V(P_2 - l)|X_L = b, \xi_L(S - \nu) = b, \xi_Z(\nu) = Z^{-1}(L(b, y, P_0); y, P_0)) = V(l - P_0),$$

where $z$ and $l$ are respectively unknowns. Now, define

$$Z(x^*, y, P_0) = \lim_{a \to x^*} Z(a, y, P_0),$$

$$Z(x_*, y, P_0) = \lim_{a \to x_*} Z(a, y, P_0),$$

$$L(x^*, y, P_0) = \lim_{a \to x^*} L(a, y, P_0),$$

and

$$L(x_*, y, P_0) = \lim_{a \to x_*} L(a, y, P_0).$$

That the above limits exist follows from the strict monotonicity of $Z(a, y, P_0)$ and $L(a, y, P_0)$ in $a$ for all $a \in (x_*, x^*)$.[24]

We shall show in proposition 2 that given $y$, $P_0$ and the common prior beliefs about signals and $P_2$, $Z(a, y, P_0)$ and $L(a, y, P_0)$ constitute a symmetric Bayesian equilibrium. First we claim:

**Lemma 3** *Given $y$, $P_0$ and the common beliefs about signals and $P_2$, the following inequalities hold:*

$$Z(x^*, y, P_0) \geq z(x^*, y),$$

$$Z(x_*, y, P_0) \leq z(x_*, y),$$

$$L(x^*, y, P_0) \geq l(x^*, y, P_0),$$

*and*

$$L(x_*, y, P_0) \leq l(x_*, y, P_0).$$

**Proof**
See appendix A. $\square$

Following lemma 3 and the assumption that

$$z(x^*, y) \geq l(x_*, y, P_0)$$

and

$$z(x_*, y) \leq l(x^*, y, P_0),$$

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[24]Note that $Z(a, y, P_0)$ is the reservation value for the security of a zero-position trader with private signal $a$, public signal $y$ and the information that $P_1 = Z(a, y, P_0)$ and that other traders respectively adopt $Z(a, y, P_0)$ and $L(a, y, P_0)$ as their bidding strategies. The interpretation of $L(a, y, P_0)$ is similar.

there must be both zero-position and long-position traders who submit the bid $P_1$, if given the common prior beliefs about signals and $P_2$, all zero-position and all long-position traders respectively adopt $Z^*(a, y, P_0) \equiv Z(a, y, P_0)$ and $L^*(a, y, P_0) \equiv L(a, y, P_0)$ as their bidding strategies. We need to show that given other traders adopt the prescribed strategies, no single trader wants to unilaterally deviate. This is accomplished in the next proposition:

**Proposition 2** *If* $z(x^*, y) \geq l(x_*, y, P_0)$ *and* $z(x_*, y) \leq l(x^*, y, P_0)$, *then* $Z^*(a, y, P_0) \equiv Z(a, y, P_0)$ *and* $L^*(a, y, P_0) \equiv L(a, y, P_0)$, *together with the the common prior beliefs about signals and* $P_2$, *define a symmetric Bayesian equilibrium for the double auction at time 1. In equilibrium, each trader's objective function has a unique peak, which gives the trader's optimal bid. Moreover,* $P_1$ *is always non-negative.*

**Proof**
See appendix A. $\square$

Given that a symmetric Bayesian equilibrium exists for the double auction at time 1, the next proposition shows the existence of the time-0 equilibrium price $P_0^*$, which makes identical traders at time 0 indifferent about taking a long or a zero position. By definition, this proves the existence of a symmetric perfect Bayesian equilibrium for the entire trading economy:

**Proposition 3** *There exists a symmetric perfect Bayesian equilibrium for the trading economy. In equilibrium,* $P_0^*$ *is non-negative.*

**Proof**
See appendix A. $\square$

We have verified in the proof of proposition 3 that participation in trade is individually rational for each trader.[25]

**Remark 2 Existence of equilibrium for asymmetric double auction** *A major difficulty facing an auction model with multiple classes of traders, i.e. traders who differ in non-informational aspects such as preferences, is in showing the existence of an equilibrium. Hindy (1989) bypasses this difficulty by introducing a notion of bounded rationality. Traders are assumed to play a different trading game in which each trader reacts not to the real bidding strategies of other traders but to some artificial market average trading strategy. Our treatment can be viewed as a general resolution to the equilibrium existence problem. The moral is that, if there are always traders in each class who submit the bid equal to the market price, then it is possible to recover information from the equilibrium price and volume, as we*

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[25]This follows from the observation that *interim* a zero-position trader can always guarantee himself a 0 utility. At time 0, in equilibrium, since all traders are indifferent about taking a long or a zero position, participation in trade must be *ex ante* individually rational for everyone.

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*showed in proposition 2. For this to be possible, we have assumed that the private signal is a continuous random variable and the economy has a continuum of traders.*

**Remark 3 The no-regret property** *What mainly distinguishes a price formation model from a rational expectations equilibrium (REE) model is that in the former traders submit bids simultaneously and must disregard the market price (Milgrom 1981a). However, when determining their optimal bidding strategies, traders in a price formation model also take into account the information to be revealed by the market price and the trading volume. Usually, after observing the market price and the trading volume, traders may wish they could have submitted different bids. But the fact they could not may not matter at all. For example, in the single auction model of Milgrom (1981a) and the double auction model of Hindy (1989), after observing the market price and the trading volume, winning traders' revised reservation values for the asset are still higher than the market price and losing traders' revised reservation values for the asset are still lower than the market price. Since all transactions are made at market price, traders regret neither their post-trade positions nor the bids they submitted. This is called the no-regret property.*

*The symmetric Bayesian equilibrium obtained in proposition 2 does not possess the no regret property. Ex post, after observing the market price and the trading volume, a trader on average will not regret his bid. (Traders will not regret their bids if ex post they are not allowed to observe trading volume.) Note that unlike in a single auction (cf. Milgrom (1981b)), the trading volume in a double auction is in general a random variable.[26] With an increase in the dimension of uncertainty, it is quite intuitive that ex post traders, who are still allowed to submit single bids only, may begin to regret their bids in certain states of nature. However, in other states of nature traders can experience gains from their bids. On average, traders do not regret.*

**Remark 4** *In the equilibrium obtained in proposition 2, both the market price and the trading volume convey useful information about $P_2$. More precisely, it takes the price and volume together to determine the values and the ranks of the private signals recovered from the equilibrium price. Note that an increase (decrease) in price with constant volume is certainly good (bad) news. On the other hand, without further information about the functions $G_Z(\cdot|P_2)$ and $G_L(\cdot|P_2)$, the implication of a higher volume with price kept constant is ambiguous.[27]*

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[26] It is interesting to note that boundedly rational traders in Hindy (1989) do not utilize information of volume.

[27] Presumably, informational stories can be told to give implications of a change in the volume, since long-position traders, having been security holders for one period, may have a different distribution of private signals from that of the outside investors (Rajan 1990). Discussions about such informational stories and their implications are out of the scope of the current paper.

Now we investigate traders' equilibrium bidding behavior at time 1. According to propositions 1 and 2, a trader's equilibrium bid is always an interior solution to his optimal bidding problem at time 1. This implies that the first order condition for the trader's optimization problem must hold for his equilibrium bid. Precisely, this means

$$D_Z(Z^*) \equiv E(V(P_2 - Z^*)|P_1 = Z^*, X_Z = a, Y = y, P_0) = 0$$

and

$$D_L(L^*) \equiv E(V(P_2 - L^*)|P_1 = L^*, X_L = a, Y = y, P_0) - V(L^* - P_0) = 0,$$

where $D_Z(\cdot)$ and $D_L(\cdot)$ are the first derivatives of objective functions of respectively zero-position and long-position traders whose private signal is $a$ and public signal is $y$. A close inspection of the above two first order conditions gives the following result:[28]

**Proposition 4** *For all public signal $y$ and private signal $a$, either $Z^*(a, y, P_0) \geq L^*(a, y, P_0) \geq P_0$ or $Z^*(a, y, P_0) \leq L^*(a, y, P_0) \leq P_0$.*

**Proof**
See appendix A. □

**Remark 5** *Proposition 4 gives the relation between the bidding behavior of a long-position trader and his zero-position counterpart, which says that traders with the same preferences and private information must agree on the predicted direction of price change, but a long-position trader bids more (less) aggressively than his zero-position counterpart when they are both pessimistic (optimistic) about the outlook of the market. This fact will be used to establish theorem 1 below.*

For the following two special cases, we can say more about the equilibrium price and the volume at time 1:

**Proposition 5** *Consider the following two cases where either the public signal $Y$ or the private signal $X$ is absent:*

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[28]The proof of proposition 4 has used the fact that at time 1, after losing the bid, a zero-position trader is guaranteed by a 0 utility, while a long-position trader's welfare depends on whether the price has risen or fallen. As a consequence, before trading begins at time 1, if the price is very likely to rise, a long-position trader knows that he may be well off even if he loses the bid. Thus at time 1, a long-position trader's willingness to hold the asset for one more period is lower when the price is likely to rise than when the price is likely to fall. This fact translates into the long-position trader's bidding behavior and results in the relation stated in proposition 4.

*1. Suppose $y_* = y^*$. That is, the public signal is absent. Then, in the equilibrium at time 1, there are always zero-position and long-position traders who submit the bid $P_1$. Moreover, volume is always strictly positive and less than $S$.*

*2. Suppose $x_* = x^*$. That is, the private signal is absent. Then, in the equilibrium at time 1, volume is either 0 or $S$.*

**Proof**
See appendix A. □

## 1.3 Equilibrium Relations Between Price Changes and Volume

Now we proceed to establish the equilibrium relations between the price changes and the trading volume for the double auction at time 1. Note that at time 1, given a distribution of traders' types, i.e. some realizations of the two random functions $G_Z(x|P_2)$ and $G_L(x|P_2)$, the price change $P_1 - P_0$ is completely determined by the public information $Y$. The next is our first major result of this paper:

**Theorem 1** *Given any distribution of traders' types, the trading volume $\nu$ at time 1 is higher when $P_1 \geq P_0$ than when $P_1 < P_0$.*

**Proof**
Since both $Z^*(a, y, P_0)$ and $L^*(a, y, P_0)$ are strictly increasing in the trader's private signal $a$, for traders in the same pre-trade positions, a trader of a higher type always has more chances to win the bid than one of a lower type. Given a distribution of traders' types, we want to show that there are more zero-position traders who win the bid when the price rises than when it falls. To this end, first note that when the price rises, i.e. $P_1 \geq P_0$, all winning bids are higher than $P_0$, while some winning bids are lower than $P_0$ when the price falls. Our assertion then follows from proposition 4. □

A close inspection of the equations

$$D_L(L^*) = D_Z(Z^*) = 0$$

reveals that

$$Z^*(a, y, P_0) = L^*(a, y, Z^*(a, y, P_0)).$$

Let us call the above **relation E**, which indicates that the long-position trader's bidding strategy also contains information about the zero-position trader's bidding strategy. To setablish our second result, we shall first construct a measure for the magnitude of the price change, without specifying in detail the distribution of the

signals. Given each public signal $y$, define the following random variable $M(y)$, which depends on $P_0$ as well as the distribution of traders' private signals:

$$M(y) = \begin{cases} Z^*(x^*, y, P_0) - P_0 & \text{if } P_1 \geq P_0, \\ P_0 - Z^*(x_*, y, P_0) & \text{otherwise.} \end{cases}$$

It follows from propositions 1, 2, 4 and 5 that $M(y)$ is a strictly positive random variable. Following from the symmetry of traders' equilibrium bidding strategies, a stochastic increase in $M(y)$ implies that $|P_1 - P_0|$ is also big. Thus, without specifying in detail the distribution of signals, a lower bound of $M(y)$ gives a sense of how big the magnitude of price change can be. In this sense, $M$ defined below measures traders' *perceived* price variability at time 0, before they observe the public signal $y$:

$$M = \inf_{y \in [y_*, y^*]} \inf_{G_L, G_Z} M(y).$$

Define the interim *informational* heterogeneity of traders by $\delta \equiv x^* - x_*$. As $\delta$ tends to zero, the economy approaches an informationally homogeneous one: all the information traders have becomes public. The following is our second major result, which shows that when traders are interim highly homogeneous, i.e. when $\delta$ is small, but the magnitude of the price change is perceived to be large, i.e. $M$ is large, then the trading volume is either extremely high or extremely low.

**Theorem 2** *The trading volume is either extremely high or extremely low if the magnitude of the price change is perceived to be large and traders are highly homogeneous. More formally, for each $M$, there is a $\delta(M)$ such that for all $\delta \leq \delta(M)$, $\nu$ is zero whenever the price falls and $\nu$ is $S$ whenever the price rises. When $M$ increases, the required $\delta(M)$ increases.*

**Proof**
By the continuity of $Z^*$ in the private signal, for each $\kappa, M > 0$, there exists a $\delta(M)$ so small that if $\delta \leq \delta(M)$, for all $y \in [y_*, y^*]$ and $P_0$,

$$Z^*(x^*, y, P_0) - \kappa M \leq Z^*(x_*, y, P_0).$$

We shall call the above inequality **relation F**. Now, when the price rises, using mean value theorem, for some $\eta \in [\min(P_0, Z^*(x^*, y, P_0)), \max(P_0, Z^*(x^*, y, P_0))]$, we have

$$L^*(x^*, y, P_0) = L^*(x^*, y, Z^*(x^*, y, P_0)) + \frac{\partial L^*(x^*, y, \eta)}{\partial P_0}(P_0 - Z^*(x^*, y, P_0)),$$

which, using **relation E**, is equal to

$$Z^*(x^*, y, P_0) + \frac{\partial L^*(x^*, y, \eta)}{\partial P_0}(P_0 - Z^*(x^*, y, P_0)).$$

Note that when the price rises,[29]

$$L^*(x^*, y, P_0) < Z^*(x^*, y, P_0),$$

and this implies that

$$\kappa(y | P_1 \geq P_0) \equiv \frac{\partial L^*(x^*, y, \eta)}{\partial P_0} > 0.$$

We now have

$$L^*(x^*, y, P_0) \leq Z^*(x^*, y, P_0) - \kappa(y)M.$$

Similarly, when the price falls, for some $\eta \in [\min(P_0, Z^*(x_*, y, P_0)), \max(P_0, Z^*(x_*, y, P_0))]$, we have

$$L^*(x_*, y, P_0) = L^*(x_*, y, Z^*(x_*, y, P_0)) + \frac{\partial L^*(x_*, y, \eta)}{\partial P_0}(P_0 - Z^*(x_*, y, P_0)),$$

which, using **relation E**, is equal to

$$Z^*(x_*, y, P_0) + \frac{\partial L^*(x_*, y, \eta)}{\partial P_0}(P_0 - Z^*(x_*, y, P_0)).$$

Again,

$$\kappa(y | P_1 < P_0) \equiv \frac{\partial L^*(x_*, y, \eta)}{\partial P_0} > 0.$$

We have

$$L^*(x_*, y, P_0) \geq Z^*(x_*, y, P_0) + \kappa(y)M.$$

Now, define $\kappa$ as the infemum of all $\kappa(y)$. When the price rises, we have

$$L^*(x^*, y, P_0) \leq Z^*(x^*, y, P_0) - \kappa M \leq Z^*(x_*, y, P_0),$$

where the last inequality follows from **relation F**. In this case, the trading volume $\nu$ is $S$. When the price falls, we have

$$L^*(x_*, y, P_0) \geq Z^*(x_*, y, P_0) + \kappa M \geq Z^*(x^*, y, P_0),$$

where the last inequality follows from **relation F**. In this case, the trading volume $\nu$ is 0. That $\delta(M)$ is increasing in $M$ is obvious.□

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[29]The following inequality is strict. Otherwise, **relation E** implies that all bids at time 1 is lower than $P_0$, contradicting the assumption that the price rises.

**Remark 6** *Let us briefly remark on the intuition behind theorem 2. Note that the prospect theory implies that a long-position trader's bidding behavior differs from that of his zero-position counterpart, because of his concern about his welfare in the case where he loses the bid, which depends on $P_0$. Thus, compared to his zero-position counterpart, a long-position trader is* historically emotional *in the following sense: When the magnitude of the price change is perceived to be large, the emotional factor becomes significant and consequently a long-position trader's bid differs a lot from that of his zero-position counterpart, in spite of the fact that they have the same information about $P_2$. On the other hand, if traders are interim highly homogeneous, which means that at time 1 the range of the private signal is small, then all traders in the same pre-trade positions behave similarly. Now, if the magnitude of the price change is large and traders are also interim highly homogeneous, consider all the long-position traders' bids as a group and all the zero-position traders' bids as another group. The* **within-group variation** *will be small and the* **between-group variation** *will be large. This means:*

- *When the price rises, it tends to rise much. Because traders are highly homogeneous, when a long-position trader tends to lose the bid, so do other long-position traders. Volume is therefore extremely high.*

- *When the price falls, it tends to fall much. Because traders are highly homogeneous, when a long-position trader tends to win the bid, so do other long-position traders. Volume is therefore extremely low.*

## 1.4  Conclusions

In this paper, we provide an equilibrium model where the price-volume formation process is explicitly modeled and both the price and volume convey information in equilibrium. The price-volume formation model is used to study the implications of Kahneman-Tversky's preference theory on the equilibrium relations between price and volume. We obtain two equilibrium relations between price and volume, that the volume is in general higher when the price rises than when it falls, and that the volume is either extremely high or extremely low if traders have highly homogeneous information and the price is volatile.

Although prosepct theory is supported by numerous experimental findings at the individual level and has been widely accepted by marketing researchers (Thaler (1985)), its implications on economic activities at the aggregate level remain unknown. This paper, to the author's knowledge, is the first attempt in this direction. The preference assumptions are always crucial in leading to the final predictions of theoretical research in economics. Learning the implications of various preference theories on economic activities is useful in settling the disputes among pref-

erence theorists and it helps economists to reach concensus. The final verdict of Kahneman-Tversky's theory, of course, has to rely on more empirical evidence and further studies in human behavior.

# Appendix A: Proofs of Lemmas and Propositions

**Proof of Lemma 1**

The first assertion follows from the strict MLRP of signals. To show the second assertion, suppose that $z(a, y) \leq P_0$. This means

$$E(V(P_2 - P_0)|X_Z = a, Y = y) \leq 0.$$

But, then

$$E(V(P_2 - P_0)|X_L = a, Y = y) - V(P_0 - P_0)$$
$$= E(V(P_2 - P_0)|X_Z = a, Y = y) \leq 0.$$

This means $l(a, y, P_0) \leq P_0$. Also,

$$E(V(P_2 - z(a, y))|X_L = a, Y = y) - V(z(a, y) - P_0)$$

$$= E(V(P_2 - z(a, y))|X_Z = a, Y = y) - V(z(a, y) - P_0)$$

$$= 0 - V(z(a, y) - P_0) \geq 0,$$

which means $l(a, y, P_0) \geq z(a, y)$. We conclude that, in this case, $z(a, y) \leq l(a, y, P_0) \leq P_0$. The case where $z(a, y) \geq l(a, y, P_0) \geq P_0$ is similar. $\square$

**Proof of Lemma 2**

Given $\epsilon$, let $\mathcal{H}$ be the collection of all continuous and strictly increasing functions which map $[x_* + \epsilon, x^* - \epsilon]$ into $[0, K]$. Endow $\mathcal{H}$ a partial ordering $\succeq$ as follows. Two elements of $\mathcal{H}$, $h_1$ and $h_2$ are such that $h_1 \succeq h_2$, if for all $a \in [x_* + \epsilon, x^* - \epsilon]$, $h_1(a) \geq h_2(a)$. First note that, given that $\zeta$ and $\lambda$ are two elements of $\mathcal{H}$, $Z_\epsilon$ and $L_\epsilon$ are also elements of $\mathcal{H}$. Given $\lambda$, an increase in $\zeta$ in the sense of $\succeq$ leads to an increase in $Z_\epsilon$ for all $a$. This means that an increase in $\zeta$ in the sense of $\succeq$ leads to an increase in $Z_\epsilon$ in the sense of $\succeq$ too. The Tarski's theorem (Tarski 1955) therefore implies that for each $\lambda$, there is a $\zeta^*(\lambda)$ such that $Z_\epsilon(\zeta^*(\lambda), \lambda) = \zeta^*(\lambda)$. Similarly, one can show that for each $\zeta$ there is a $\lambda^*(\zeta)$ such that $L_\epsilon(\zeta, \lambda^*(\zeta)) = \lambda^*(\zeta)$. Now observe that $\lambda^*(\zeta^*(\lambda))$ is an increasing mapping from the complete lattice $(\mathcal{H}, \succeq)$ into itself. The Tarski's theorem then implies the existence of a fixed point $\lambda^{**}$ such that $\lambda^*(\zeta^*(\lambda^{**})) = \lambda^{**}$. Thus, $\lambda^{**}$ and $\zeta^{**} \equiv \zeta^*(\lambda^{**})$ are the two asserted functions, corresponding to which $Z_\epsilon(\zeta^{**}, \lambda^{**}) = \zeta^{**}$ and $L_\epsilon(\zeta^{**}, \lambda^{**}) = \lambda^{**}$. Now we consider the last assertion. Note that by construction, $Z_\epsilon(a)$ and $L_\epsilon(a)$ are both continuous in $\epsilon$ for each $k \in [0, \frac{x^* - x_*}{2}]$, $\epsilon \in [0, k]$ and $a \in [x_* + k, x^* - k]$. Moreover, they are both non-negative and bounded above by $K$. It follows that they must both be uniformly continuous in $\epsilon$. A mathematic result (theorem 13-D of Simmons (1963)) says that the image of a Cauchy sequence under a uniformly continuous mapping must also be a Cauchy sequence. Now we pick a sequence of $\{\epsilon_n\}$ converging to zero. It follows that $Z_{\epsilon_n}(a)$ and $L_{\epsilon_n}(a)$ are both Cauchy sequences for all $a \in (x_*, x^*)$. $\square$

**Proof of Lemma 3**

Notice that given the common prior beliefs about signals and $P_2$, for very small positive number $\epsilon$, $z(x^* - \epsilon, y)$ is a zero-position trader's reservation value conditioned on his private signal $x^* - \epsilon$ and the public signal $y$, while $Z(x^* - \epsilon, y, P_0)$ is the same trader's reservation value conditioned on his private signal $x^* - \epsilon$, the public signal $y$, *as well as* the event that information conveyed by the market price is very favorable, i.e.,

$$E_a(E(V(P_2 - z(x^* - \epsilon, y))|$$
$$X_Z = x^* - \epsilon, \xi_Z(\nu) = a, \xi_L(S - \nu) = L^{-1}(Z(a, y, P_0), y, P_0), Y = y, P_0)) = 0,$$

and

$$E(V(P_2 - Z(x^* - \epsilon, y, P_0))|$$
$$X_Z = x^* - \epsilon, \xi_Z(\nu) = x^* - \epsilon, \xi_L(S - \nu) = L^{-1}(Z(x^* - \epsilon, y, P_0), y, P_0), Y = y, P_0)) = 0,$$

where in the first equation, $E_a(\cdot)$ indicates that the expectation is taken over all possible realizations of $a$. Thus, following the strict MLRP of signals, for $\epsilon$ small enough we should have

$$Z(x^* - \epsilon, y, P_0) \geq z(x^* - \epsilon, y).$$

Now note that

$$Z(x^*, y, P_0) = \lim_{\epsilon \to 0} Z(x^* - \epsilon, y, P_0)$$

and

$$z(x^*, y) = \lim_{\epsilon \to 0} z(x^* - \epsilon, y).$$

This shows that $Z(x^*, y, P_0) \geq z(x^*, y)$. The remaining inequalities can be analogously proved. □

**Proof of Proposition 1**

We consider only the case where $z(x^*, y) < l(x_*, y, P_0)$. The case where $z(x_*, y) > l(x^*, y, P_0)$ can be analogously proved. First note that, when $z(x^*, y) < l(x_*, y, P_0)$, if traders adopt $z(a, y)$ and $l(a, y, P_0)$ as their bidding strategies, we know for sure that $P_1 = l(x_*, y, P_0)$. No new information is revealed after the trade. Thus, it suffices to show that, conditional on a trader's pre-trade information, $z(a, y)$ and $l(a, y, P_0)$ are his optimal bidding strategies respectively when his pre-trade position is zero and when it is long. Suppose the pre-trade position is zero. The objective function of a zero-position trader with private signal $a$ and public signal $y$ is

$$E(1_{[Z \geq P_1]}V(P_2 - P_1)|X_Z = a, Y = y)$$

$$= 1_{[Z \geq l(x_*, y, P_0)]}E(V(P_2 - l(x_*, y, P_0))|X_Z = a, Y = y)$$

$$\leq 1_{[Z \geq l(x_*, y, P_0)]}E(V(P_2 - z(a, y))|X_Z = a, Y = y) = 0.$$

The maximum, 0, is reached at any $Z < l(x_*, y, P_0)$. In this case, $z(a, y)$ is (weakly) optimal for the zero-position trader. Similarly, we can show that $l(a, y, P_0)$ is optimal for a long-position trader with private signal $a$ and public signal $y$. Now the last assertion follows from lemma 1, which says that given any private signal $a$ and public signal $y$, either $z(a, y) \leq l(a, y, P_0) \leq P_0$ or $z(a, y) \geq l(a, y, P_0) \geq P_0$. $\square$

**Proof of Proposition 2**

In the following, for simplicity, conditioning a random variable on $\xi_Z(\nu) = b$ actually means conditioning it on $\xi_Z(\nu) = b$ as well as $\xi_L(S - \nu) = L^{-1}(Z(b; y, P_0); y, P_0)$, following the hypothesis that there are always zero-position and long-position traders who submit the bid $P_1$ in equilibrium. The conditioning on $\xi_L(S - \nu) = b$ is interpreted similarly.

Given $y$, $P_0$, the common prior beliefs about the signals and $P_2$, and that other traders adopt respectively the two prescribed bidding strategies, a zero-position trader with private signal $a$ and public signal $y$ has the objective function

$$E(1_{[Z \geq Z(b, y, P_0)]} V(P_2 - Z(b, y, P_0)) | X_Z = a, Y = y, P_0)$$

$$= \int_{-\infty}^{Z^{-1}(Z; y, P_0)} E(V(P_2 - Z(b, y, P_0)) | X_Z = a, Y = y, \xi_Z(\nu) = b) h(b | X_Z = a, Y = y, P_0) db,$$

where $Z(b, y, P_0) = P_1$ and $h(\cdot | a, y, P_0)$ is the density of $\xi_Z(\nu)$ conditional on the pre-trade signals $X_Z = a$, $Y = y$ and the price $P_0$. In the above integral, in those states that $b > a$, the integrand

$$E(V(P_2 - Z(b, y, P_0)) | X_Z = a, \xi_Z(\nu) = b, Y = y)$$

$$< E(V(P_2 - Z(b, y, P_0)) | X_Z = b, \xi_Z(\nu) = b, Y = y) = 0,$$

and similarly, in those states that $b \leq a$,

$$E(V(P_2 - Z(b, y, P_0)) | X_Z = a, \xi_Z(\nu) = b, Y = y) \geq 0.$$

Therefore, the optimal bid $Z^*(a, y, P_0)$ should be such that the trader wins the asset only in such states that $b \leq a$. This means that we want $Z^*(a, y, P_0) \geq Z(b, y, P_0)$ if and only if $b \leq a$, or, $Z^*(a, y, P_0) = Z(a, y, P_0)$. This shows that $Z(a, y, P_0)$ is indeed the optimal bidding strategy of zero-position traders.

Similary, given $y$, $P_0$, the common prior beliefs about the signals and $P_2$, and that other traders adopt respectively the two prescribed bidding strategies, a long-position trader with private signal $a$ and public signal $y$ has the objective function

$$E(1_{[L \geq P_1]} V(P_2 - P_1) + 1_{[L < P_1]} V(P_1 - P_0) | X_L = a, Y = y, P_0),$$

$$= E(1_{[L \geq P_1]}(V(P_2 - P_1) - V(P_1 - P_0)) | X_L = a, Y = y, P_0)$$

$$+ E(V(P_1 - P_0) | X_L = a, Y = y, P_0),$$

which is equivalent to

$$E(1_{[L \geq L(b,y,P_0)]}(V(P_2 - L(b,y,P_0)) - V(L(b,y,P_0) - P_0))|X_L = a, Y = y, P_0)$$

$$= \int_{-\infty}^{L^{-1}(L;y,P_0)} E((V(P_2 - L(b,y,P_0)) - V(L(b,y,P_0) - P_0))|X_L = a, Y = y, \xi_L(S-\nu) = b)$$

$$\eta(b|X_L = a, Y = y, P_0)db,$$

where $L(b,y,P_0) = P_1$ and $\eta(\cdot|a,y,P_0)$ is the density of $\xi_L(S-\nu)$ conditional on the pre-trade signals $X_L = a$, $Y = y$ and the price $P_0$. In the above integral, in those states that $b > a$, the integrand

$$E((V(P_2 - L(b,y,P_0)) - V(L(b,y,P_0) - P_0))|X_L = a, \xi_L(S-\nu) = b, Y = y) < 0,$$

and in those states that $b \leq a$,

$$E((V(P_2 - L(b,y,P_0)) - V(L(b,y,P_0) - P_0))|X_L = a, \xi_L(S-\nu) = b, Y = y) \geq 0.$$

Therefore, the optimal bid $L^*(a,y,P_0)$ should be such that the trader continues to hold the asset only in such states that $b \leq a$. This means that we want $L^*(a,y,P_0) \geq L(b,y,P_0)$ if and only if $b \leq a$, or, $L^*(a,y,P_0) = L(a,y,P_0)$. This proves that $L(a,y,P_0)$ is indeed the optimal bidding strategy of long-position traders. $\square$

**Proof of Proposition 3**

We must show that at time 0 the equilibrium price $P_0^*$ exists. First note that, because the signals are continuous random variables, both $E(J_L)$ and $E(J_Z)$ are continuous in $P_0$. It then suffices to find two feasible $P_0$'s, say $P_{0A}$ and $P_{0B}$, such that under $P_{0A}$, traders at time 0 all prefer a long position to a zero position and under $P_{0B}$, traders at time 0 all prefer a zero position to a long position. Consider that $P_{0A} = 0$. Recall that $P_1$ is non-negative. In this case, all traders at time 0 prefer a long position. To find $P_{0B}$, note that by the definition of $l(a,y,P_0)$, we know that

$$\lim_{P_0 \to \infty} l(a,y,P_0) = \infty.$$

Therefore, we can find some $P_0'$ such that $l(x_*,y_*,P_0') \geq z(x^*,y^*)$. We claim that at time 0, every trader prefers a zero position to a long position, if $P_{0B} = P_0'$ is the security price at time 0. To prove this claim, note that if $P_0'$ is the security price at time 0, there will be no trade at time 1, and all zero-position traders are guaranteed by a zero utility. We shall show that, under $P_{0B} = P_0'$, taking a long position at time 0 results in a negative utility. Given that $P_0'$ is the security price at time 0 and that there will be no trade at time 1, a long-position trader has an expected utility at time 0 equal to

$$E(V(P_2 - l(x_*,y,P_0'))) \leq E(V(P_2 - z(x^*,y^*))) \leq 0,$$

35

where the last inequality follows from the definition of $z(a, y)$ and the strict MLRP of signals. In this case, traders at time 0 all prefer a zero position to a long position. The proof is now complete. □

**Proof of Proposition 4**

Since this assertion has been proved for the case of proposition 1, we only need to consider the case of proposition 2. In this case, recall that a trader's equilibrium bid is always an interior solution to his optimal bidding problem at time 1. This implies that the first-order condition for a trader's optimization problem must hold for his equilibrium bid. Precisely, this means

$$D_Z(Z^*) \equiv E(V(P_2 - Z^*)|P_1 = Z^*, X_Z = a, Y = y, P_0) = 0$$

and

$$D_L(L^*) \equiv E(V(P_2 - L^*)|P_1 = L^*, X_L = a, Y = y, P_0) - V(L^* - P_0) = 0,$$

where $D_Z(\cdot)$ and $D_L(\cdot)$ are the first derivatives of objective functions of respectively zero-position and long-position traders whose private signal is $a$ and public signal is $y$, and $Z^* = Z^*(a, y, P_0)$ and $L^* = L^*(a, y, P_0)$ are respectively zero-position and long-position traders' equilibrium bidding strategies. The assertion now follows from the fact that both objective functions have unique peaks, according to proposition 2. To see this, suppose $Z^*(a, y, P_0) \geq P_0$. Then,

$$D_L(P_0) = D_Z(P_0) \geq 0.$$

Moreover,

$$D_L(Z^*) = -V(Z^* - P_0) \leq 0.$$

This means $Z^*(a, y, P_0) \geq L^*(a, y, P_0) \geq P_0$. The case where $Z^*(a, y, P_0) \leq P_0$ is similar. □

**Proof of Proposition 5**

Consider the first assertion. Suppose it is not true that there are both zero-position and long-position traders who submit the bid $P_1$. Then it follows from proposition 4 that $P_0$ is either higher or lower than everyone's bid at time 1. This means that there is either a sure gain or a sure loss from taking a long position at time 0, a contradiction to the fact that $P_0$ is the time-0 equilibrium price that makes traders indifferent between taking a long or zero position. Now because $Z^*(x^*) \geq L^*(x^*, P_0) > L^*(x_*, P_0)$, the volume must be strictly positive. Similarly, we can conclude that the volume is strictly less than $S$. The second assertion follows from proposition 1. □

36

# Appendix B: The Isolation Effect and the Independence Axiom

The isolation effect, also known as the *elimination by aspects*, is originally reported by Tversky (1972). In this appendix, we shall briefly discuss the connection of the isolation effect to the independence axiom in the expected utility theory.

Consider an abstract decision making context, where an agent is considering taking one of the two alternatives, $X$ and $Y$.The independence axiom says that the agent's preferences about $X$ and $Y$ should be independent of other alternatives, e.g. $Z$. That is, if an agent prefers $X$ to $Y$ when $Z$ is unavailable, he still does when $Z$ becomes available. However, the experimental findings strongly suggest the opposite: The agent's preferences about $X$ and $Y$ are highly dependent on whether other alternatives are available, and what other alternatives are.Tversky (1972) reports how an agent's preferences about $X$ and $Y$ depend on other alternatives. The major finding is that, agents tend to regard an alternative as a bundle of relevant aspects. To be specific, let us consider the following example, where the three alternatives $X$, $Y$ and $Z$ are expressed by their aspects:

$$X = \alpha\beta\gamma,$$

$$Y = \alpha\theta\mu,$$

$$Z = \alpha\theta\beta.$$

Note that $\alpha$ is a common aspect of all alternatives. Similarly, $\beta$ and $\theta$ are respectively the common aspects of $X$ and $Z$, and $Y$ and $Z$. The isolation effect asserts that $\alpha$ is irrelevant to the agent's decision. By eliminating aspect $\alpha$, we have four relevant aspects left. Let the importance of each aspect be denoted by a strictly positive number. Respectively, the importances of the four aspects are measured by $U(\beta), U(\gamma), U(\theta)$ and $U(\mu)$, with the sum of the four numbers normalized to one. Then, without the presence of alternative $Z$, the probability that $X$ is chosen over $Y$ is

$$P_{XY} = U(\beta) + U(\gamma),$$

and the probability that $Y$ is chosen over $X$ is

$$P_{YX} = U(\theta) + U(\mu),$$

where note that the two probabilities sum to one, and that $P_{XY}$ denotes the probability that $X$ is chosen over $Y$ when $X$ and $Y$ are the only two alternatives available. When $Z$ is also available, the probability that $X$ is chosen over $Y$ and $Z$ is

$$U(\beta)P_{XZ} + U(\gamma),$$

the probability that $Y$ is chosen over $X$ and $Z$ is

$$U(\theta)P_{YZ} + U(\mu),$$

and finally, the probability that $Z$ is chosen over $X$ and $Y$ is

$$U(\theta)P_{ZY} + U(\beta)P_{ZX}.$$

Again, the three probabilities sum to one. Let us compare the probability that $X$ is chosen to the probability that $Y$ is chosen. We have the following findings:

1. Introducing a competing alternative $Z$ reduces the probability that either $X$ or $Y$ will be chosen. This is quite intuitive.

2. $X$ is now more likely to be chosen than $Y$, if the newly introduced alternative $Z$ has more common aspects with $Y$ than with $X$.

As a conclusion, Tversky (1972) points out that the independence axiom has been violated in various contexts. Agents' preferences about two alternatives $X$ and $Y$ highly depend on whether other alternatives are also available, and what other alternatives are. In any case, agents tend to disregard the common aspects of all alternatives.

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# Chapter 2

# Prospect Theory, Equilibrium Asset Returns and Optimal Risk Sharing

### Abstract of Chapter 2[1]

This paper explores the implications of Kahneman-Tversky's prospect theory on equilibrium relations among asset returns and Pareto optimal risk sharing rules. It produces the following results:

1. The Sharpe-Lintner CAPM is rebuilt in a two-period economy where two-fund separation holds in equilibrium and agents have preferences described by prospect theory. In equilibrium, an agent's optimal portfolio is mean-variance as well as *mean-third moment* efficient. Merton's Intertemporal CAPM is derived in a continuous-time homogeneous economy due to Cox, Ingersoll and Ross (1985). These results suggest that Kahneman-Tversky's preference theory is compatible with existing asset pricing theories.

2. Thaler's optimal rules for segregating and integrating different prospects are extended to the uncertainty case.

3. A complete characterization of optimal risk sharing rules is provided. Prospect theory implies that, under optimal sharing rules, there exists an agent who bears all the losses whenever they occur. Other agents are fully insured against losses. The sharing rules are in general not monotonic, and hence non-linear. Conditions on agents' preferences so that the optimal sharing rules are piecewise linear are given.

42

4. Sufficient conditions for the existence of a competitive equilibrium are derived. For a class of economies, a competitive equilibrium always exists and the equilibrium allocations are Pareto optimal.

5. When Pareto optimal sharing rules are implemented in a competitive equilibrium of complete asset markets, there exists a representative agent, whose preference is described by prospect theory, such that after a translation of both agents' common beliefs and the Arrow-Debreu state prices which support the original competitive equilibrium, the *translated* beliefs and state prices support the representative agent's optimal consumption plan, which prescribes that the representative agent consumes the aggregate consumption in every state of the world.

## 2.1   Introduction

The development of prospect theory by Kahneman and Tversky (1979) has been based on compelling experimental findings. According to prospect theory, when an individual faces some random payoff, he first compares the payoff to some reference point, or bench mark. The difference between the payoff and the reference point is defined as either a gain or a loss depending on whether it is positive or negative. The individual's perference is then represented by a value function defined on the space of gains and losses. A value function is everywhere increasing, concave on the positive orthant, and convex on the negative orthant. The traditional concave von Neumann-Morgenstern utility functions may be considered as one special class of value functions whose reference points are zero and prospects only involve random gains.

In spite that prospect theory has been supported by subsequent experimental findings in human studies and wide acceptance of marketing researchers (Thaler (1985)), it has not attracted much attention of economists for several reasons. First, some economists suspect that an agent with preference described by prospect theory may be irrational, i.e. the agent may take actions to reduce his own wealth. Second, the indeterminacy of the reference point in the agent's value function is disturbing. In two different decision-making contexts, an agent may make completely opposite decisions because different reference points are chosen. Finally, and most importantly, prospect theory has not been developed into a powerful analytic tool like expected utility theory, although it was originally proposed to replace the latter, when the latter was found by psychologists to be inconsistent with human behavior under uncertainty (Kahneman and Tversky (1979)).

First, as to irrationality, given any reference point in an agent's value function, that the value function is increasing implies that the agent's preference is monotonic in wealth. Thus rationality simply requires that, given each decision-making context,

the reference point be properly chosen such that the agent's preference is increasing in his wealth. Rationality of a prospect-theory agent has been subject to suspicion, probably because of misinterpretations of some well-known experimental findings. Here is an example. In two separate experiments the same subjects are asked to choose one lottery out of two. In the first experiment, a subject chose lottery $A$ over lottery $B$. In the second experiment, the same subject chose lottery $C$ over lottery $D$. It turns out that the payoffs to the portfolio $B + D$ first-order stochastically dominate those to the portfolio $A + C$. It was therefore argued that this subject is irrational. Note that the subject was not allowed to choose among portfolios in the experiments, nor did he know the existence of the second experiment when he participated in the first one. If a subject had been allowed to choose among portfolios $A + C$, $A + D$, $B + C$, and $B + D$, should he not have used the payoffs to one of these portfolios as a reference point? But in that case, he would never choose the dominated portfolio $A + C$, as his preference is monotonic on the space of gains and losses!

The above example shows how the reference point is sensitive to the context in which the agent makes decisions. In the comparisons between $A$ and $B$ and that between $C$ and $D$, a subject probably have two different reference points. Moreover, neither of the two reference points is used in the new context where the subject is allowed to choose among portfolios. In any case, a rational prospect-theory agent who fully understands the context will always choose the reference point properly so that irrational decisions are never made.

Second, regarding how the reference point may adjust with the decision-making context, we currently have only a few clues provided by Kahneman and Tversky. For example, the *isolation effect* says that, before a decision is made, a decision maker tends to choose a reference point such that the common aspects of all feasible alternatives are ignored; see Chen (1992).

Finally, in view of the fact that prospect theory has not been analytically useful to economists, this paper studies some analytic properties of the Kahneman-Tversky value function and its implications on financial economics. The issues we choose to deal with include optimal rules for creating the *framing effect* (Thaler (1985)), the equilibrium relations among asset returns, and the Pareto optimal rules of risk sharing among a finite number of agents.

In section 2.2, we analyze the basic properties of a value function. Thaler's proposition (Thaler (1985)) about how different prospects should be optimally integrated or segregated is extended to the uncertainty case.

In section 2.3, we explore the implications of value functions on equilibrium asset returns. We first give equivalent conditions for first-order and second-order stochastic dominances. Assuming two fund separation holds in equilibrium, we re-derive the Sharpe-Lintner CAPM. An agent's optimal portfolio is mean-variance as well as *mean-third moment* efficient.

In section 2.4, we characterize the optimal risk sharing rules among N agents. It is shown that under the optimal sharing rules, there exists one agent who bears losses whenever they occur. All other agents are fully insured against losses. The optimal sharing rules are not always implementable in competitive asset markets equilibrium, even if markets are complete. Arrow (1964) shows that for the optimal sharing rules to be always implementable in competitive asset markets equilibrium, the value functions must be everywhere concave. Thus for Kahneman-Tversky value functions, joint conditions on value functions and return distributions are needed to guarantee the implementability. Sufficient conditions for the existence of competitive equilibria are derived. A class of economies always have competitive equilibria and moreover, the equilibrium allocations are Pareto optimal.

A *representative agent theorem* is established when Pareto optimal sharing rules are implemented in a competitive equilibrium of complete asset markets. We show that the representative agent's preference has expected value function representation. After a translation in both the agents' common beliefs and the Arrow-Debreu state prices which support the original competitive equilibrium, the translated state prices and beliefs support the representative agent's optimal consumption plan, which prescribes that the representative agent consumes the aggregate consumption in every state of the world.

The optimal sharing rules are generally not monotonic, and therefore non-linear. We obtain conditions on agents' value functions such that the optimal sharing rules are piece-wise linear.[2] The conditions require agents' value functions also be piece-wise. Each piece is a so-called HARA (hyperbolic absolute risk aversion) function. We conclude in section 2.5.

## 2.2 Basic Properties of Value Function and Framing Effect

A prospect in static environments is a random variable with its realization representing a gain if it is positive and a loss if it is negative. A prospect is said to be *regular*, if with strictly positive probabilities a gain and a loss can respectively occur. An increasing continuous real function $V(\cdot)$ defined on the real line is a Kahneman-Tversky value function (henceforth *value function*) if the following three conditions are satisfied:

1. $V(0) = 0$;

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[2]This analysis is meaningful, because a simple capital structure composed of debt and equity implies piece-wise linear sharing rules for claimants on the firm. From capital market's point of view, piece-wise linear sharing rules can be implemented by a small set of assets, such like a riskless asset, a risky asset with payoff proportional to the aggregate consumption and a European call option written on the risky asset.

2. For all $x \in R$, let $V'_-(x)$ be the left-hand derivative at $x$. Then, for all $x \geq 0$, $V'_-(-x) \geq V'_-(x)$;

3. For $x \geq 0$, $V(x)$ is concave; for $x \leq 0$, $V(x)$ is convex.

Clearly, an expected value function is unique up to a multiplication by any strictly positive number. It follows from condition 3 that $V'_-(x)$ has global maximum at 0. It follows immediately from the first two conditions that for all $x > 0$, $-V(-x) \geq V(x)$. This says that a consumer suffers from a particular amount of loss more than he can benefit from the same amount of gain. Intuition then suggests that if a consumer is indifferent about taking a prospect with only two possible realizations, the prospect should have expected gain. This indeed is true and can be easily verified. For general regular prospects with more than two outcomes, however, this is not true. A consumer is said to be *risk averse*, if he never takes *fair gambles*, i.e. prospects with zero mean. An expected value function maximizer in general may take a fair gamble while reject another. Hence his risk attitude cannot be defined on the entire space of prospects.

An interesting issue raised in prospect theory by Kahneman and Tversky is the so-called *framing effect*. As a tradition, economists have assumed that money has no labels. A consumer never distinguishes money incomes from different sources. Experimental findings about human behavior (Thaler 1985), however, suggest that consumers seem to have some *mental accounting systems*. Incomes from different sources are recorded in different *mental accounts*.[3] The belief that framing effect exists leads to the analysis of how different prospects can be optimally integrated or segregated. The following proposition is due to Thaler (1985), which concerns sure gains and sure losses:

**Proposition 1** *Let $a$ and $b$ be two real numbers.*

1. *Two sure gains should be segregated. That is, if $a, b > 0$, then $V(a + b) < V(a) + V(b)$.*

2. *Two sure losses should be integrated. That is, if $a, b < 0$, then $V(a + b) > V(a) + V(b)$.*

3. *A sure gain and a sure loss should be integrated if integration creates a sure gain. That is, if $ab < 0$, and $a + b > 0$, then $V(a + b) > V(a) + V(b)$.*

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[3]This observation is important, for example, for marketing researchers. Retailers offering short-term price deals instead of permanent price cuts may believe that only the short-term price deals are regarded as gains by consumers, while a permanent price change is considered a change of reference point which yields no gain.

The proof for the above proposition is obvious. Note that if $V(x)$ is three-time differentiable at 0, $V''(0) = 0$ and $V'''(0) \leq 0$. Generally, $V''''(x)$ need not exist. Under the assumption that for all $x$, $V''''(x)$ exists and is negative, which says that a consumer becomes more risk averse when the prospect involves higher gains, we can rebuild Thaler's proposition for the case of uncertainty:

**Proposition 2** *Assume that for all $y$, $V'''(y) < 0$.*

1. *Let $x$ be a random variable with $E(x) \geq 0$. Let $z$ be a strictly positive number. Then, $E(V(x + z)) < E(V(x)) + V(z)$. That is, the two prospects $x$ and $z$ should be segregated.*

2. *Let $x$ be a random variable with $E(x) \leq 0$. Let $z$ be a strictly negative number. Then, $E(V(x + z)) > E(V(x)) + V(z)$. That is, the two prospects $x$ and $z$ should be integrated.*

3. *Let $x$ be a random variable and $z$ be a strictly positive number with $E(x) \geq z$. Then, $E(V(x - z)) > E(V(x)) + V(-z)$. That is, the two prospects $x$ and $-z$ should be integrated.*

**Proof**
First we prove assertion 1. Define $h(z) = E(V(x + z)) - E(V(x)) - V(z)$. Then, $h(0) = 0$. We have

$$h'(z) = E(V'(x + z)) - V'(z) < V'(E(x) + z) - V'(z) \leq 0,$$

where the second inequality follows from concavity of $V(x)$ on positive orthant. Therefore, for all $z > 0$, $h(z) < 0$.

Now we consider the second assertion. Note that in this case $h(0) = 0$ and

$$h'(z) = E(V'(x + z)) - V'(z) < V'(E(x) + z) - V'(z) \leq 0,$$

where the second inequality follows from convexity of $V(x)$ on negative orthant. Therefore, for all $z < 0$, $h(z) > 0$.

Now the last assertion. Define $H(x) = V(x - z) - V(x) - V(-z)$. We have

$$H''(x) = V''(x - z) - V''(x) > 0,$$

where the inequality follows from the assumption that $V'''(x) < 0$. Then $H(x)$ is globally convex. We have

$$E(H(x)) > V(E(x) - z) - V(E(x)) - V(-z) > 0,$$

where the last inequality follows from assertion 3 of proposition 2.$\square$

We now turn to section 2.3 for a discussion of multi-asset return relations.

## 2.3 Stochastic Dominance, CAPM and Intertemporal CAPM

A prospect $A$ is said to first-order stochastically dominate prospect $B$ if $A$ is preferred to $B$ by all agents who maximize expected *increasing and continuous* value functions. A prospect $A$ is said to second-order stochastically dominate prospect $B$ if $A$ is preferred to $B$ by all agents who maximize value functions *which are concave on the positive orthant, convex on the negative orthant, and continuous at 0*.[4] In this section we will investigate the conditions for the first- and second-order stochastic dominances and derive an equilibrium relation among asset returns under the assumption that two-fund separation holds. The following equivalent condition for first-order stochastic dominance is identical to that in the case of expected utility function.

**Proposition 3** *Let $A$ and $B$ be two regular prospects with the closed interval $[a, b]$ being the union of their supports. Let $F_A(x)$ and $F_B(x)$ be the distribution functions for respectively prospects $A$ and $B$. Then, $A$ first-order stochastically dominates $B$ if and only if $F_A(x) - F_B(x) \leq 0$ for all $x \in [a, b]$.*

The proof for the above proposition is identical to that for the case of expected utility function. See, for example, Huang and Litzenberger (1988). The following equivalent condition for second-order stochastic dominance is a direct extention of that in the case of expected utility function.

**Proposition 4** *For all $x \in [a, b]$, define $S(x)$ by $\int_a^x (F_A(z) - F_B(z))dz$. Then, a regular prospect $A$ second-order stochastically dominates a regular prospect $B$, if and only if $A$ and $B$ have the same expected values and $S(x) \geq 0$ for all $x \in [a, 0]$ and $S(x) \leq 0$ for all $x \in [0, b]$. Necessarily, $S(0) = 0$.*

**Proof**
The proof is similar to that for the case of expected utility function. First consider the sufficiency part. Using integration by parts, one can show that

$$E(V(x_A)) - E(V(x_B)) = \int_a^b S(x)dV'(x),$$

which is positive by the assumption about $S(x)$, the convexity of $V(x)$ on the negative orthant, and the concavity of $V(x)$ on the positive orthant. Next consider the necessity part. Since $S(x)$ is absolutely continuous, it is continuous. If for some $x \geq 0$, $S(x) > 0$, then there exists a strictly positive number $\xi$, such that for all

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[4]If a function is concave or convex on an interval, the function is continuous on that interval.

$y \in [\max(0, x - \xi), x + \xi]$, $S(y) > 0$. Then we can find a value function which is linear outside the interval $[\max(0, x - \xi), x + \xi]$, and strictly concave on the interval. For this particular value function, $E(V(x_A)) - E(V(x_B))$ is negative, contradicting the definition of second-order stochastic dominance. If for some $x \leq 0$, $S(x) < 0$, an analogous contradiction can be established. This finishes the necessity part. $\square$

Following Rothschild and Stiglitz (1970), there is another equivalent condition for the second-order stochastic dominance: prospect $A$ second-order stochastically dominates prospect $B$ if and only if there are two prospects $u_A$ and $u_B$ such that $x_A + u_A$ and $x_B + u_B$ are equal in distribution,

$$E(u_A | x_A) = 0, \quad E(u_B | x_B) = 0,$$

$u_A = 0$ if $x_A \leq 0$, and $u_B = 0$ if $x_B \geq 0$. The following is an example:

**Example** There are four equally likely states $\omega_i, i = 1, 2, 3, 4$. The following table summarizes the information about $x_A, x_B, u_A$, and $u_B$.

| Prospects/states | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
|---|---|---|---|---|
| A | -2 | -1 | 1.5 | 1.5 |
| B | -1.5 | -1.5 | 1 | 2 |
| $u_A$ | 0 | 0 | -0.5 | 0.5 |
| $u_B$ | -0.5 | 0.5 | 0 | 0 |

Note that prospect $A$ second-order stochastically dominates prospect $B$.

The three functions $V_1(x) = x, V_2(x) = -x$ and $V_3(x) = -x^3$ are all concave on the positive orthant and convex on the negative orthant. Thus, if a prospect $A$ second-order stochastically dominates prospect $B$, we should have $E(x_A) \geq E(x_B)$, $E(-x_A) \geq E(-x_B)$, and $E(-x_A^3) \geq E(-x_B^3)$. This means that the two prospects should have the same expected value, and the dominating prospect should have smaller third moment.

An $N$-vector of prospects $x' = (x_1, x_2, x_3, \cdots, x_N)$, where $'$ denotes transpose, is said to exhibit *two fund separation* if there exist two $N$-vectors $\alpha$ and $\beta$ such that $\alpha'\iota = \beta'\iota = 1$, where $\iota$ is an $N$-vector of one's, and for any other $N$-vector $\gamma$ with $\gamma'\iota = 1$, there exists a real number $\lambda(\gamma)$ such that

$$E(V(\lambda(\gamma)\alpha'x + (1 - \lambda(\gamma))\beta'x)) \geq E(V(\gamma'x)),$$

for all $V(x)$ which are concave on the positive orthant and convex on the negative orthant. That is, two fund separation holds if there exist two fixed distinct portfolios constructed from the $N$ prospects, such that every feasible portfolio constructed from the same set of prospects is second-order stochastically dominated by some portfolio of the two fixed portfolios. If one of the two separating portfolios is a riskless asset, the phenomenon is called two fund *monetary* separation (Cass and Stiglitz 1970).

We shall show that, if two-fund separation holds in the equilibrium of a two-period economy, then the Sharpe-Lintner CAPM holds. Assume that there is a riskless asset with rate of return denoted by $r_f$. There are $N$ risky assets with rates of return denoted by the $N$-vector $r_f \iota + x$ and expected rates of return denoted by the $N$-vector $r_f \iota + e$. Agents trade at date 0 and assets pay off at date 1. Denote by the $N$-vector $a$ the proportions of initial date-0 wealth invested in respectively the $N$ risky assets. Thus the random gain or loss per dollar generated by strategy $a$ is $a'x$.

Assume two fund separation holds in equilibrium. Then given that $a'e$ equals a constant $\mu$, the dominating portfolio constructed from the two separating funds should have minimum third moment. The existence of this minimum is implied by the assumption that two fund separation holds. Thus the dominating portfolio $a$ is a solution to the following minimization program:

$$\min_a E((a'x)^3)$$

subject to $a'e = \mu$. Forming the Lagrangian, and denoting the Lagrangian multiplier by $3\pi$, we are minimizing

$$E((a'e)^3) - 3\pi(a'e - \mu).$$

For the solution to exist, it is necessary that the first-order conditions hold. That is, for all $i = 1, 2, \ldots, N$,

$$a'A_i a = \pi e_i,$$

where

$$A_i = E(x_i x x').$$

Note that for all $i, j \in \{1, 2, \ldots, N\}$, the two matrices $A_i$ and $A_j$ are both symmetric and they commute. That is, for all $i, j \in \{1, 2, \ldots, N\}$, $A_i = A_i'$ and

$$A_i A_j = A_j A_i.$$

It follows from a well-known fact in linear algebra that there exist an orthogonal matrix $C$ and $N$ diagonal matrices $D_i$ for $i = 1, 2, \ldots, N$, such that for all $i = 1, 2, \ldots, N$,

$$A_i = C' D_i C.$$

Substituting this result into the $N$ first-order conditions, we have for all $i = 1, 2, \ldots, N$,

$$b' D_i b = \pi e_i,$$

where

$$b = C'a.$$

The above conditions can be rewritten as

$$Db^2 = \pi\epsilon,$$

where the $N \times N$ matrix $D$ is so constructed that its $i$-th row is the major diagonal elements of matrix $D_i$, for all $i = 1, 2, \ldots, N$, and the $N$-vector $b^n$ is so constructed that its $i$-th element is the $i$-th element of $b$ raised to the $n$-th power. We will assume that matrix $D$ is of full rank. This assumption corresponds to the full rank assumption about the variance-covariance matrix of asset returns in the derivation of traditional CAPM. We now have

$$C'a = b = (D^{-1}\pi\epsilon)^{\frac{1}{2}},$$

where the vector $(D^{-1}\epsilon)^{\frac{1}{2}}$ is such that its $i$-th element raised to the second power is equal to the $i$-th element of $D^{-1}\epsilon$. Certainly $(D^{-1}\epsilon)^{\frac{1}{2}}$ so defined is not unique. But under the assumption that two fund separation holds, for all $\mu$, the optimal solution $a$ *must* correspond to the same vector $(D^{-1}\epsilon)^{\frac{1}{2}}$. It follows that

$$\mu = \epsilon'a = \epsilon'C''(D^{-1}\pi\epsilon)^{\frac{1}{2}},$$

or,

$$\pi = \mu^2(\epsilon'C''(D^{-1}\epsilon)^{\frac{1}{2}})^{-2}.$$

So $\pi$ is strictly positive for all $\mu > 0$. Note that from the first-order conditions, the minimum third moment is equal to

$$E((a'x)^3) = \pi\mu.$$

Now we can characterize the efficient frontier and the two separating funds.

**Proposition 5** *The following are a characterization of the first moment-third moment efficient frontier and the two separating funds.*

1. *(**Efficient Frontier**) The mean-third moment efficient frontier is the following polynomial:*
$$E((a'x)^3) = \mu^3(\epsilon'C''(D^{-1}\epsilon)^{\frac{1}{2}})^{-2}.$$

2. *(**Separating Funds**) The two separating funds are the riskless asset and the risky portfolio which has the proportions of wealth invested in the $N$ risky assets denoted by the following $N$-vector*
$$C''(D^{-1}e)^{\frac{1}{2}}(\iota'C''(D^{-1}e)^{\frac{1}{2}})^{-1}.$$

*This portfolio contains no riskless asset and will be termed the* **market portfolio** *and denoted by $m$. The expected excess rate of return on the market portfolio is*

$$\frac{e'C''(D^{-1}e)^{\frac{1}{2}}}{\iota'C''(D^{-1}e)^{\frac{1}{2}}}.$$

*A frontier portfolio $a$ with $a'e = \mu$ is generated by investing the proportion*

$$\mu \frac{\iota'C''(D^{-1}e)^{\frac{1}{2}}}{e'C''(D^{-1}e)^{\frac{1}{2}}}$$

*of one's initial wealth in the above risky portfolio and investing the rest of his wealth in the riskless asset.*

**Proof**
Assertion 1 is clear, following the above discussion. To see that assertion 2 is true, note that the frontier portfolio $a$ with $a'e = \mu$ is

$$a = \mu C''(D^{-1}e)^{\frac{1}{2}}(e'C''(D^{-1}e)^{\frac{1}{2}})^{-1}.\square$$

Since one of the two separating funds is the riskless asset, we have two-fund monetary separation. Now we are ready to show the equilibrium relation among all asset returns. Let $j$ be any portfolio and denote its rate of return by

$$r_j = r_f + j'x.$$

Given the asset structure $x$, define the *beta* of portfolio $j$ with respect to market portfolio $m$ by

$$\beta_j \equiv \frac{E((m'x)^2(j'x))}{E((m'x)^3)}.$$

We use the same *beta* as the one in the traditional CAPM to denote the measure for systematic risk, because it is indeed equal to the *beta* in the traditional CAPM, a fact to be established later. The following theorem rebuilds the CAPM:

**Theorem 1** *Assume two fund separation holds. Then for any feasible portfolio $j$, its expected rate of return $E(r_j)$ must satisfy the following relation:*

$$E(r_j) = r_f + \beta_j(E(r_m) - r_f).$$

**Proof**
Let the parameter $\mu$ be equal to

$$\mu(m) \equiv m'e.$$

52

Then the first-order condition for minimizing the third moment, after being multiplied by some feasilbe portfolio $j'$, is

$$E((m'x)^2(j'x)) = \pi\mu(j),$$

where $\pi$ (actually $3\pi$) is the Lagrangian multiplier corresponding to the above specified parameter $\mu(m)$, and $\mu(j)$ denotes $j'e$. Then it follows that

$$\mu(j) = \frac{E((m'x)^2(j'x))}{\pi},$$

which, using the definition of $\beta_j$ and the fact that

$$\pi\mu(m) = E((m'x)^3),$$

is equal to

$$\beta_j\mu(m).$$

Now, using the definition of $\mu(j)$, we have for every feasible protfolio $j$,

$$E(r_j) = r_f + \beta_j(E(r_m) - r_f).\square$$

Now we state one equivalent condition for two fund separation.

**Proposition 6** *Two fund separation holds if and only if*

1. *$x_i$'s, the excess rates of return on risky asset $i$, for all $i = 1, 2, \cdots, N - 1$ is related to some random variable $m$ in the following way:*

$$x_i = \beta_i m + \epsilon_i,$$

*where $\beta_i$'s are constants and $\epsilon_i$ is such that $E(\epsilon_i|m) = 0$ and $\epsilon_i = 0$ if $m \le 0$; and*

2. *$x_N$ is such that $m$ is some weighted average of $x_1, \cdots, x_N$.*

**Proof**
This directly follows from the necessary and sufficient condition for second-order stochastic dominance. The only difference here is that it is impossible for every dominated portfolio to be less risky on the negative orthant than the dominating portfolio. $\square$

Following the above proposition, the CAPM derived in theorem 1 is in fact the Sharpe-Lintner CAPM, which one can easily verify.

Our last task of this section is to rebuild Merton's Intertemporal CAPM using a third-degree polynomial, a counterpart to the quadratic utility function in the

expected utility theory. Just like quadratic function, satiation may occur for the third-degree polynomials.

We consider a continuous-time homogeneous economy due to Cox, Ingersoll and Ross (1985), CIR henceforth. We assume that each agent only cares about the gains from continuous trades. Everyone maximizes an expected third-degree polynomial value function.[5] Following CIR, assume there are $n$ real assets with prices denoted by the $n$-vector diffusion process $S_t$:

$$dS_t = I_S(\mu(S_t, t)dt + \sigma(S_t, t)dW_t,$$

where $I_S$ is an $n \times n$ diagonal matrix with the $i$-th real asset's price $S_{it}$ being its $i$-th major diagonal element; $\mu(S_t, t)$, $\sigma(S_t, t)$ and $W_t$ stand for respectively the $n$-vector mean returns, the $n \times (n + k)$ instantaneous variance-covariance matrix and the $n + k$-vector Brownian motions which are the primitive uncertainties. There are $k$ contingent claims with prices $\eta_t$, which have the following dynamics:

$$d\eta_t = I_\eta(a_t dt + b_t dW_t),$$

where the $k \times k$ diagonal matrix $I_\eta$ is defined similarly to $I_S$; $a_t$ and $b_t$ are to be determined in equilibrium. A borrowing and lending rate $r_t$ is also available and is to be determined in equilibrium.[6] At time $t$, a generic agent in this economy maximizes

$$E(\int_t^T e^{-\delta(\tau-t)}(dmX_\tau - dX_\tau^3)|\mathcal{F}_t),$$

where note that the economy ends at time $T$, that the agent has a value function parameter $m$, a time preference parameter $\delta$ and personal wealth $X_t$ at time $t$ and that the time-$t$ information set $\mathcal{F}_t$ is common for everyone. Let $n$-vector $\alpha_t$ and $k$-vector $\beta_t$ denote respectively the numbers of shares of the real and financial assets held by the individual at time $t$. The usual self-financing constraint implies a dynamics for $X_t$:

$$dX_t = X_t(r_t + \alpha_t'(\mu_t - r_t \iota_n) + \beta_t'(a_t - r_t \iota_k))dt + X_t(\alpha_t'\sigma_t + \beta_t'b_t)dW_t,$$

where $\iota_j$ is a $j$-vector of ones. To ease notation, in the above we have suppressed the dependence of $\mu$ and $\sigma$ on $S_t$. Assuming that

$$E(\int_t^T e^{-\delta(\tau-t)}(mX_\tau - 3X_\tau^3)(\alpha_\tau'\sigma_\tau + \beta_\tau'b_\tau)dW_\tau|\mathcal{F}_t)$$

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[5]To generate meaningful results, some assets should have positive supplies. If all assets are in zero supply, as in the economy of Duffie and Zame (1987), since aggregate wealth is unchanged, homogeneity among consumers implies that everyone has a zero indirect value function over time.

[6]We have been very imprecise in describing the primitives of the economy. The probability space, information filtration, technical conditions on parameters in the price dynamics as well as the boundary conditions for contingent claim prices have all been left out. For these we refer readers to the original construction of CIR. For now, it is important to mention that $\sigma(S_t, t)$ is always of rank $n$.

is zero, the objective function is reduced to

$$E(\int_t^T e^{-\delta(\tau-t)}[(mX_\tau-3X_\tau^3)(r_\tau+\alpha_\tau'(\mu_\tau-r_\tau\iota_n)+\beta_\tau'(a_\tau-r_\tau\iota_k))d\tau-3X_\tau^3(\alpha_\tau'\sigma_\tau+\beta_\tau'b_\tau)(\alpha_\tau'\sigma_\tau+\beta_\tau'b_\tau)']d\tau$$

The temporal value function is now concave in controls $\alpha_t$ and $\beta_t$. Standard dynamic programming techniques are used to solve for this problem.

Define $J(S, X, t)$ as the indirect value function at time $t$ with current real asset prices $S$ and personal wealth $X$. Define

$$J_X^* = J_X - e^{-\delta t}(3X^2 - m),$$

where subscripts denote partial derivatives. The next proposition summarizes the equilibrium outcomes. The derivations of these results follow the original treatment of CIR and thus the proofs are omitted.

**Proposition 7**    1. In equilibrium, there is no borrowing or lending, $\beta_t = 0, \forall t$, and

$$\alpha_t = (\sigma_t\sigma_t')^{-1}\mu_t\frac{-J_X^*}{XJ_{XX}^*} + \frac{I_S J_{XS}^*}{XJ_{XX}^*},$$

where the two terms on the right-hand side are respectively the mean-variance efficient portfolio and the hedging demand at time t.

2. The instantaneous expected excess rate of return on the optimal portfolio at time t is

$$\frac{-XJ_{XX}^*}{J_X^*}\frac{Var(dX)}{X^2} + \frac{-XJ_{XS}^*}{J_X^*}\frac{Cov(dX, dS)}{X^2}.$$

3. Define

$$\phi_X \equiv \frac{J_{XX}^*}{J_X^*}Var(dX) + \frac{J_{XS}^*}{J_X^*}Cov(dX, dS)$$

and

$$\phi_S \equiv Var(dS)\frac{J_{XS}^*}{J_X^*} + \frac{J_{XX}^*}{J_X^*}Cov(dX, dS).$$

Then,

$$I_n(a_t - r_t\iota_k) = \eta_X\phi_X + \eta_S\phi_S,$$

which is version of Merton's Intertemporal CAPM.

As agents are assumed to derive satisfaction completely from gains of trade, we do not have implications about Breeden's Consumption-based CAPM. When real asset prices follow geometric Brownian motions, clearly two-fund separation holds.

## 2.4 Optimal Risk Sharing Rules

There are three major objectives in this section: to offer a complete characterization of optimal sharing rules, to establish a representative agent theorem and to give conditions on agents' preferences such that the optimal sharing rules are piece-wise linear.

### 2.4.1 Characterization of Optimal Sharing Rules

We start with the two-agent case. Let regular prospect $x$ be the aggregate gain or loss to be shared by a principal and an agent endowed with respectively value functions $V(\cdot)$ and $U(\cdot)$. Let $V^*$ be the agent's reservation value. An optimal risk sharing rule is a compensation schedule for the agent, $s(x)$, which solves the following Lagrangian:

$$L(s(x), \pi) = \max_{s(x)} E(U(x - s(x))) + \pi(E(V(s(x))) - V^*),$$

where $\pi$ is the Lagrangian multiplier for the agent's individual rationality condition. Let $s(x)$ be the optimal sharing rule and $f(x)$ be any function which is not identically zero. Define the functional $J(a, f(x), L(s(x), \pi))$ ( for brevity, $J(a)$) by

$$J(a) = L(s(x) + af(x), \pi).$$

For $s(x)$ to be the optimal sharing rule, it is necessary that for all function $f(x)$, the following two conditions hold:

$$J'(a)|_{a=0} = 0,$$

and

$$J''(a)|_{a=0} \leq 0.$$

The above first condition yields the famous Borch rule: for all realizations of $x$,

$$\frac{U'(x - s(x))}{V'(s(x))} = \pi.$$

We shall call these equalities **condition 1**. The above second condition yields the following inequalities, which we shall call **condition 2**: for all realizations of $x$,

$$U''(x - s(x)) + \pi V''(s(x)) \leq 0.$$

Condition 2 says that even when the partnership is suffering from an aggregate loss, one agent still enjoys a gain. The other agent must bear the entire loss. Now we

assume that the optimal sharing rule is continuous and almost everywhere differentiable. Differentiating condition 1 with respect to $x$ and using condition 2, we have

$$\frac{1}{1 - s'(x)} \pi V''(s(x)) \leq 0.$$

This says that almost everywhere $s'(x) \geq 1$ if and only if $s(x) \leq 0$. Similarly,

$$\frac{1}{s'(x)} U''(x - s(x)) \leq 0,$$

which says that almost everywhere $s(x)$ is increasing if and only if $x \geq s(x)$.

Now we have a complete description of the optimal sharing rules for the two-agent case. One agent has an everywhere increasing sharing rule, while the other's sharing rule decreases first and then increases again. The agent whose sharing rule is everywhere increasing bears the entire loss whenever a loss occurs. The other agent is fully insured against losses.[7] Let us call the agent who may bear the loss "agent 1". The following are the major features of the two sharing rules:

- When an aggregate loss occurs, the slope of agent 1's sharing rule is greater than 1, while at the same time, the slope of the other agent's sharing rule is negative.

- When an aggregate gain occurs but is very small, agent 1 still bears some loss. The entire gain is enjoyed by the other agent.

- When an aggregate gain occurs and is big, both agents share the gain. In this case the slope of each agent is between 0 and 1.

We now turn to the $N$-agent case. One can think of the case as being one where one principal and $N-1$ agents, with respectively value functions $U(\cdot), V_1(\cdot), V_2(\cdot), \cdots, V_{N-1}(\cdot)$, are looking for optimal compensation schemes for the agents. For convenience, sometimes we call the principal "agent $N$" and replace the notation $U(\cdot)$ by $V_N(\cdot)$. The optimal rules are $N - 1$ functions of aggregate gain or loss, $s_1(x), s_2(x), \cdots, s_{N-1}(x)$ representing the compensation schedules for the $N - 1$ agents. In the following, sometimes we use the notation $s_N(x)$ to mean $x - \sum_{i=1}^{N-1} s_i(x)$. Let $V_1^*, V_2^*, \cdots, V_{N-1}^*$ be respectively the reservation values of the $N - 1$ agents. The optimal sharing rules are solutions to the following Lagrangian:

$$\max_{s_i(x), i=1, \ldots, N-1} L(s_i(x); \pi_i; i = 1, \cdots, N-1) = E(U(x - \sum_{i=1}^{N-1} s_i(x))) + \sum_{i=1}^{N-1} \pi_i (E(V_i(s_i(x))) - V_i^*),$$

---

[7]Certainly the agent who bears losses may be compensated in other states of the world where aggregate gains are available.

where $\pi_i$ is the Lagrangian multiplier for agent $i$'s individual rationality condition. Given $N - 1$ functions $f_1(x), f_2(x), \cdots, f_{N-1}(x)$, none of which are identically zero, define a functional $J(a_1, a_2, \cdots, a_{N-1})$ by

$$J(a_1, a_2, \cdots, a_{N-1}) = L(a_i f_i(x) + s_i(x); \pi_i; i = 1, \cdots, N - 1).$$

The following are two sets of necessary conditions for $(s_i(x); i = 1, 2, \ldots, N - 1)$ to be the optimal sharing rules: For all functions $f_1(x), \cdots, f_{N-1}(x)$,

1. **(condition 1)** for all $i = 1, 2, \ldots, N-1$, the partial derivative of $J$ at $(0, \ldots, 0)$ with respect to $a_i$ is zero. That is, $J_i(a)|_{a=0} = 0$; and

2. **(condition 2)** for all $i = 1, 2, \ldots, N - 1$, the Hessian of $J$ at $(0, \ldots, 0)$ is negative semi-definite.

Condition 1, once again, gives the Borch rule. Condition 2 implies that in every state of the world there is at least one person who has a gain. The following proposition shows that there is at most one person who bears losses in each state of the world.

**Proposition 8** *Suppose a set of optimal sharing rules exist. For all realizations of $x$, there is at most one person who bears losses.*

**Proof**
Note that if $(s_i(x); i = 1, 2, \ldots, N - 1)$ are the optimal sharing rules, then any two people cannot increase their welfare by re-allocating their gains and losses between themselves. Consider any two people $i$ and $j$. We must have $s_i(x) = s_i^*(s_i(x) + s_j(x))$ and $s_j(x) = s_j^*(s_i(x) + s_j(x))$, where $(s_k^*(x); k = i, j)$ are one set of optimal sharing rules between agents $i$ and $j$ with aggregate gain or loss being $s_i(x) + s_j(x)$ and either agent $i$ or agent $j$'s individual rationality condition is satisfied. Following our discussion for the two-agent case, there is at most one person between $i$ and $j$ who bears the losses. Considering each two-person risk sharing problem leads to the conclusion that in every state of the world, under the optimal sharing rules $(s_i(x); i = 1, 2, \ldots, N-1)$, at most one person bears losses. To see this, suppose there are two persons both bearing losses in some state of the world. Then, the sharing rules between the two persons are not optimal, contradicting the assumption that $(s_i(x); i = 1, 2, \ldots, N - 1)$ are the optimal sharing rules. $\square$

Since at least one person must bear losses when they occur, the above proposition says that *exactly one* person bears the loss in each state of the world where a loss occurs. One question comes to mind: if there are more than one state of the world where losses can occur, under the optimal sharing rules, can different people bear losses in different states?

The answer is clearly negative for $N = 2$. We thus assume $N > 2$. Suppose there are two people $i$ and $j$ who bear losses in two different states of the world

under the optimal sharing rules $s_i(x)$ and $s_j(x)$, which are both continuous and almost everywhere differentiable. Then consider the two-agent risk sharing problem with the aggregate gain and loss being $s_i(x) + s_j(x)$. Denote the two-agent optimal sharing rules by $s_i^*(y)$ and $s_j^*(y)$. Assume two distinct states

$$y_1 = s_i(x_1) + s_j(x_1) > s_i(x_2) + s_j(x_2) = y_2$$

are such that

$$s_i(x_1) < 0, s_j(x_2) < 0.$$

Then,

$$s_i^*(y_2) = s_i(x_2) > 0 > s_i(x_1) = s_i^*(y_1).$$

This cannot be true, by the continuity of $s_i^*(y)$ and that $s_i^*(y)$ is increasing whenever $s_i^*(y) \leq 0$. We thus conclude that there is *one* agent who bears losses whenever they occur.

The next proposition is part of the characterization of the optimal sharing rules.

**Proposition 9** *Let agent 1 be the one who bears losses when they occur. Let $y = x - s_1(x)$. Then, for all agents $i = 2, \ldots, N$, where agent $N$ is the principal, $0 \leq \frac{ds_i(x)}{dy} \leq 1$.*

**Proof**
Let $(s_i^*(y); i = 2, \ldots, N)$ be one set of optimal sharing rules for agents $2, \ldots, N$ given that the aggregate gain and loss to be shared is $y$ and the individual rationality conditions for agents $i = 2, \ldots, N - 1$ are satisfied. Differentiating the Borch rule with respect to $y$ gives

$$\pi_i V_i''(s_i^*(y)) \frac{ds_i^*(y)}{dy} = \pi_j V_j''(s_j^*(y)) \frac{ds_j^*(y)}{dy},$$

for all $i, j \in \{2, \ldots, N\}$. Since for all $i \in \{2, \ldots, N\}$, $s_i^*(y)$ is positive and that $\frac{ds_i^*(y)}{dy}$ sum to 1, we have for all $i \in \{2, \ldots, N\}$, $0 \leq \frac{ds_i^*(y)}{dy} \leq 1$. $\square$

The above proof has used the fact that $s_i^*(y) = s_i(x)$, for all $i = 2, \ldots, N$. Note that $y = x - s_1(x)$ is a general positive prospect. If for some $i \in \{2, \ldots, N\}$, $s_i^*(y) < 0$ with strictly positive probability, then the original sharing rules $(s_i(x); i = 1, \ldots, N - 1)$ are not optimal, for there are two people bearing losses in some states of the world. This seems to suggest that when aggregate losses never occur, under the optimal sharing rules, everyone should be fully insured against losses. From now on, to be comparable to the case of expected utility function, we shall impose this intuitive restriction on optimal sharing rules:

**Assumption 1** *If aggregate losses never occur, no agents ever bear losses.*

It can be shown that if for all $i = 1, \ldots, N$, $V_i''(0) = \infty$, then there exist optimal sharing rules satisfying this restriction. In this case, everyone's sharing rule is everywhere increasing with slope between 0 and 1.

We now have a complete characterization of the optimal sharing rules for the $N$-agent case.

**Theorem 2** *Assume the aggregate gain and loss is a regular prospect. If the optimal sharing rules exist and are continuous and almost everywhere differentiable, then under the optimal sharing rules, the following assertions are true.*

1. *There is exactly one person who bears losses whenever losses occur.*

2. *Except for the person who may bear losses, all other people are fully insured against losses. The sharing rule for the person who bears losses is increasing.*

3. *If the agent who may bear losses has a positive reservation value, then there exists an aggregate gain $K$ such that when the aggregate gain $K$ occurs, this agent bears no loss and shares no gain. That is, without loss of generality, assuming the agent who bears losses is agent 1, we have*

$$U'(K - \sum_{i=2}^{N-1} s_i(K)) = \pi_1 V_1'(0).$$

*Then, except for the person who bears losses, everyone's sharing rule takes a minimum at $K$.*

**Proof**
Assertion 1 follows from the previous discussion. For assertion 2, we now show that agent 1, who bears losses when they occur, has an increasing sharing rule. Note that for the principal, i.e. agent $N$, and agents $2, \ldots, N-1$, we have $s_i^*(x - s_1(x)) = s_i(x)$, for $i = 2, \ldots, N$, where $(s_i^*(x - s_1(x)); i = 2, \ldots, N)$ are one set of optimal sharing rules among agents $2, \ldots, N$, when aggregate gain and loss is $x - s_1(x)$ and the individual rationality conditions for agents $2, \ldots, N-1$ are satisfied. Then

$$\max_{\{s_i(x); i=1,\ldots,N-1\}} E(U(x - \sum_{i=1}^{N-1} s_i(x))) + \sum_{i=1}^{N-1} \pi_i(E(V_i(s_i(x))) - V_i^*)$$

is equivalent to

$$\max_{\{s_1(x)\}} \pi_1(E(V_1(s_1(x))) - V_1^*) + E(V(x - s_1(x))) - \sum_{i=2}^{N-1} \pi_i V_i^*,$$

60

where $\pi_i$, for all $i = 2, \ldots, N - 1$, are part of the solutions to the original problem and treated as constants here, and

$$V(y) = U(y - \sum_{i=2}^{N-1} s_i^*(y)) + \sum_{i=2}^{N-1} \pi_i V_i(s_i^*(y)),$$

for all $y \geq 0$ and $V(y)$ is some increasing and convex function for $y < 0$ with $\lim_{y \nearrow 0} V(y) = 0$. Thus $V(y)$ is a well-defined value function:

1. When $y = 0$, since for all $i = 2, \ldots, N$, $s_i^*(y)$ are non-negative and sum to $y$, they are all zero. Thus $V(0) = 0$;

2. For all $y \geq 0$, $V'(y) = U'(y - \sum_{i=2}^{N-1} s_i^*(y)) \geq 0$; and

3. For all $y \geq 0$, $V''(y) = U''(y - \sum_{i=2}^{N-1} s_i^*(y))(1 - \sum_{i=2}^{N-1} \frac{ds_i^*(y)}{dy})$, which by the previous proposition, is negative.

The monotonicity of $s_1(x)$ now follows from our previous discussions about optimal sharing rules in the two-agent case.

Now we prove the last assertion. Maintain that the agent who may bear losses is agent 1. If agent 1 has positive reservation value, he must have a gain in some state of the world. Since the aggregate gain and loss is a regular prospect, agent 1 must bear a loss in some state of the world. Since agent 1's sharing rule is increasing and continuous, there exists a constant $K$ such that when $x = K$, he has no gain or loss. For $x < K$, monotonicity of agent 1's sharing rule implies that he has a loss. Note that Borch rule implies that when agent 1 is bearing a loss, the slope of his sharing rule has an opposite sign to those of other agents' sharing rules; and that when agent 1 has a gain, the slope of everyone's sharing rule is increasing. Since agent 1's sharing rule is increasing everywhere, this means that all other agents' sharing rules take minima at $K$. $\square$

For the case of expected utility function, where only positive prospects are involved, all sharing rules are increasing with slopes between 0 and 1. In this case, the common minimum $K = 0$.

## 2.4.2 Existence of Competitive Equilibrium

Because prospect theory implies non-convex preferences, fixed point theorems based on continuity do not apply. This means competitive equilibrium may not exist for a given economy. In this section, we shall give conditions on preferences so that a competitive equilibrium can exist for any endowments except those confined in a finte region. For simplicity, we shall consider the two-state static economy. Let us start with the single agent economy. Suppose two states 1 and 2 may take place with respectively probabilities $\pi$ and $1 - \pi$. The agent with value function $V(\cdot)$ is

endowed with $e_1$ and $e_2$ respectively in the two states. Because $V(\cdot)$ is increasing, indifference curves are decreasing.

The idea is to require that an indifference curve have positive second derivatives except on a finite interval.[8] In this case, except in a finite region, every point on the indifference curve can be sustained by a set of equilibrium prices. Denote the indifference curve by

$$x_2 = f(x_1 | \pi V(x_1) + (1 - \pi)V(x_2) \geq v).$$

Then,

$$f''(x_1) = -\frac{\pi V'(x_1)^2}{(1 - \pi)V'(x_2)^2} \left( \frac{V''(x_1)}{V'(x_1)} \frac{V'(x_2)}{V'(x_1)} + \frac{\pi}{1 - \pi} \frac{V''(x_2)}{V'(x_2)} \right).$$

Thus a sufficient condition for the indifference curve to be locally convex except on a finite interval is as follows. For each $v$, there exists a positive number $M(v)$ such that whenever $x_1 \geq M(v)$,

$$\frac{V''(x_1)}{V'(x_1)} \frac{V'(V^{-1}(\frac{v - \pi V(x_1)}{1 - \pi}))}{V'(x_1)} + \frac{\pi}{1 - \pi} \frac{V''(V^{-1}(\frac{v - \pi V(x_1)}{1 - \pi}))}{V'(V^{-1}(\frac{v - \pi V(x_1)}{1 - \pi}))} \leq 0.$$

**Example: Piece-wise CARA Value Functions** Consider the following value function: $V(x) = 1 - e^{-Ax}$ if $x \geq 0$ and $V(x) = e^{Bx} - 1$ if otherwise. One can show that every indifference curve with $v \geq 2\pi - 1$ is convex. Thus given that the single agent has the piece-wise CARA value function, any endowments $e_1$ and $e_2$ with $\pi V(e_1) + (1 - \pi)V(e_2) \geq 2\pi - 1$ are sustained by some competitive equilibrium. Correspondingly, in a multi-agent economy, the equilibrium allocations will be Pareto optimal if everyone's preference is represented by a piece-wise CARA value function and he has a reservation utility higher than $2\pi - 1$.

## 2.4.3 A Representative Agent

Assume there is a single consumption good. Assume there are $N$ agents respectively endowed with value functions $(V_i(\cdot); i = 1, \ldots, N)$. There are two dates, 0 and 1. For simplicity, agents only consume at date 1. Assume that agents have a common reference point $R$ for date-1 random consumption $c_i$. Define $z_i = c_i - R$, $Z = \sum_{i=1}^{N} z_i$ and $C = \sum_{i=1}^{N} c_i$. Then, $Z = C - NR$. Assume $Z$ is a continuous random variable with probability density function $\pi(Z)$. For simplicity, assume each realization of $Z$ completely describes the state of the world. Assume there exists an Arrow-Debreu security for each realization of $Z$ with price $\phi(Z)$, which pays out $R + 1$ units

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[8]The upper contour set $\{(x_1, x_2) : \pi V(x_1) + (1 - \pi)V(x_2) \geq v\}$ can be convexified if the indifference curve has positive second derivatives except on a finite interval.

62

of consumption at date 1 if the realization $Z$ occurs and $R$ units of consumption otherwise. A consumer $i$ is endowed with consumption $R + e_i(Z)$ in state $Z$.

Agent $i$ has the following maximization program:

$$\max_{z_i(Z)} E(V_i(z_i(Z)))s.t. \int_{-NR}^{\infty} \phi(Z)(z_i(Z) - e_i(Z))dZ = 0.$$

Assuming that every agent $i$ consumes a positive amount of consumption in every state of the world and that a competitive equilibrium exists. Then, the first-order conditions are necessary: for every two distinct states of the world $Z$ and $Z'$,

$$\frac{\pi(Z)V_i'(z_i(Z))}{\pi(Z')V_i'(z_i(Z'))} = \frac{\phi(Z)}{\phi(Z')}.$$

We assume that the equilibrium implements the optimal sharing rules $(z_i(Z); i = 1, \ldots, N)$ which, given the set of strictly positive weights $\lambda_i$ for each agent $i$, solve the following Lagrangian:

$$\max_{z_i(Z); i=1,\ldots,N;} L = \sum_{i=1}^{N} \lambda_i E(V_i(z_i)) + \int_{-NR}^{\infty} \theta(Z)(Z - \sum_{i=1}^{N} z_i(Z))dZ,$$

where $\theta(Z)$ is the Lagrangian multiplier for the resource constraint in state $Z$.

**Theorem 3** *Maintain the above assumptions. Since $Z$ is a regular prospect, among the $N$ agents there is exactly one agent who bears losses whenever they occur. Without loss of generality, let him be agent 1. That is, if $Z < 0$, then $z_1(Z) < 0$. Define $K$ as such that for some $i \neq 1$,*

$$\lambda_i V_i'(z_i(K)) = \lambda_1 V_1'(0).$$

*Thus, $K$ depends on the weights $\lambda_i$. Following Borch rule, this definition is unique for all $i \neq 1$. Define $V(Z) = \sum_{i=1}^{N} \lambda_i V_i(z_i(Z))$. Define $W(Z) = V(Z + K) - V(K)$. Then, $W(Z)$ is a value function satisfying the three postulated conditions at the beginning of section 2. The representative agent is assumed to have preference represented by value function $W(\cdot)$.*

**Proof**
First note that $W(0) = 0$ and

$$W'(Z) = V'(Z + K) = \sum_{i=1}^{N} \lambda_i V_i'(z_i(Z + K))z_i'(Z + K) = \lambda_1 V_1'(z_1(Z + K)) \geq 0,$$

where the last equality follows from Borch rule. So, W(Z) is monotonic. Note that

$$W''(Z) = V''(Z + K) = \lambda_1 V_1''(z_1(Z + K))z_i'(Z + K),$$

63

which, by the definition of $K$ and the monotonicity of agent 1's sharing rule, is negative if and only if $Z \geq 0$. $\square$

The relation between the representative agent and the equilibrium state prices is stated in the next theorem:

**Theorem 4** *Consider a representative agent with preference represented by the value function $W(\cdot)$ defined in the previous theorem. Define the translations of $\pi(Z)$ and $\phi(Z)$ by*

$$\pi^*(Z) = \frac{\pi(Z + K)}{\int_{K-NR}^{\infty} \pi(s)ds},$$

*and*

$$\phi^*(Z) = \phi(Z + K).$$

*Then, the translated belief $\pi^*(Z)$ and the translated state prices $\phi^*(Z)$ support the optimal consumption plan of the representative agent, which prescribes the representative agent to consume the aggregate consumption in every state of the world.*

**Proof**
Note that for every two distinct states $Z$ and $Z'$,

$$\frac{\pi^*(Z)W'(Z)}{\pi^*(Z')W'(Z')} = \frac{\pi(Z + K)V_1'(z_1(Z + K))}{\pi(Z' + K)V_1'(z_1(Z' + K))} = \frac{\phi(Z + K)}{\phi(Z' + K)} = \frac{\phi^*(Z)}{\phi^*(Z')}. \square$$

The state prices in the representative agent economy are translated from those of the original competitive economy. For the equilibrium state prices in the representative agent economy to equal the state prices in the original competitive equilibrium, either the translation degenerates, i.e. $K = 0$, or the function

$$f(x, y) = \frac{V_1'(z_1(x))}{V_1'(z_1(y))}$$

is jointly periodic, i.e. given the constant $K$, for all $x$ and $y$ in the support of $Z$,

$$f(x, y) = f(x + K, y + K).$$

One can show that $f(\cdot, \cdot)$ is never jointly periodic. We are left with the first possibility. In view of the representative agent proposition, we can consider the two-agent case. Suppose there are two agents with value functions $U(\cdot)$ and $V(\cdot)$, with the latter being the one who bears losses when they occur. We can normalize the weight for the first agent to 1 and denote the weight for the second agent by $\lambda$. The necessary condition for $K = 0$ is that

$$\lambda = \frac{U'(0)}{V'(0)}.$$

This is a joint condition on the weights distribution and agents' preferences. This condition says that given agents' preferences, very few weights distributions allow the equilibrium state prices in the representative agent economy to be identical to those in the original competitive economy.

Note that in the case of expected utility function, everyone's sharing rule takes minimum at $K = 0$, for all weights distributions. This says that, for the case of expected utility function, the equilibrium state prices in the representative agent economy are always identical to those in the original competitive economy. Also, when $K = 0$, $W(\cdot) = V(\cdot)$, which gives the familiar construction of the representative agent for the case of expected utility function.

## 2.4.4 Piece-Wise Linear Sharing Rules

We show in the previous section that optimal sharing rules, if exist, are in general not monotonic when the aggregate gain and loss is a regular prospect. This implies that optimal sharing rules are generally non-linear. We now investigate the conditions on agents' value functions such that the optimal sharing rules are piece-wise linear.[9] Since the derivation of the conditions is based on Borch rule and is independent of the number of agents, we will consider the two-agent case for the ease of demonstration.

Let the two agents be $U$ and $V$. First we consider the necessary condition on the two agents' value functions, $U(\cdot)$ and $V(\cdot)$, for the sharing rules to be non-linear. Normalize the weight for agent $U$ to 1 and denote the weight for agent $V$ by $\lambda$. Suppose the sharing rules are

$$s_U(Z) = a + bZ + cMax(K, Z)$$

and

$$s_V(Z) = d + eZ + fMax(K, Z),$$

where $s_U(Z)$ and $s_V(Z)$ are respectively agents $U$'s and $V$'s optimal sharing rules and $a, b, c, d, e, f$ and $K$ are all differentiable functions of $\lambda$. Necessarily, $a = -d$, $c = -f$ and

$$b + e = 1.$$

We therefore rewrite agent $V$'s sharing rule as

$$s_V(Z) = -a + (1 - b)Z - cMax(K, Z).$$

By Borch rule, we have for all $Z$

$$U'(s_U) = \lambda V'(s_V),$$

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[9]A continuous piece-wise linear sharing rule is almost everywhere differentiable.

65

$$\begin{cases} U''(s_U)b = \lambda V''(s_V)(1-b) & \text{if } Z < K \\ U''(s_U)(c+b) = \lambda V''(s_V)(1-b-c) & \text{if } Z > K \end{cases}$$

and

$$\begin{cases} U''(s_U)(a' + c'K + cK' + b'Z) = \lambda V''(s_V)(-a' - c'K + -cK' + (-b')Z) + V'(s_V) & \text{if } Z < K \\ U''(s_U)(a' + (c' + b')Z) = \lambda V''(s_V)(-a' + (-b' - c')Z) + V'(s_V) & \text{if } Z > K, \end{cases}$$

where $'$ denotes differentiation with respect to $\lambda$. Under the assumption that $Z$ is a regular prospect, one of the two agents must have an everywhere increasing sharing rule and the other is fully insured against all losses. Let agent $U$ be the one who shares losses when they occur. Using the above three conditions, we have for $Z < K$, i.e. $s_U \leq 0$,

$$-\frac{U'(s_U)}{U''(s_U)} = A(\lambda) + B(\lambda)s_U,$$

where

$$A(\lambda) = -\lambda(a' + c'K + cK' - \frac{ab'}{b} - \frac{cb'K}{b}),$$

and

$$B(\lambda) = -\frac{\lambda b'}{b};$$

and for $Z > K$, i.e. $s_U \geq 0$,

$$-\frac{U'(s_U)}{U''(s_U)} = C'(\lambda) + D(\lambda)s_U,$$

where

$$C'(\lambda) = -\lambda(a' - \frac{a(b' + c')}{b + c}),$$

and

$$D(\lambda) = -\frac{\lambda(b' + c')}{(b + c)}.$$

Similarly for agent $V$, we have for all $s_V \geq 0$,

$$-\frac{V'(s_V)}{V''(s_V)} = E(\lambda) + B(\lambda)s_V,$$

where

$$E(\lambda) = -\lambda(1 + \frac{a'(1 - b)}{b} + \frac{ab'}{b} + \frac{c'K(1 - b)}{b} + \frac{cK'(1 - b)}{b} + \frac{b'cK}{b})$$

and $B(\lambda)$ is defined above. There is no restriction on the negative orthant of $V(\cdot)$.

The above necessary conditions say that the two agents $U$ and $V$ must have piecewise HARA (hyperbolic absolute risk aversion) value functions with the negative

piece of agent $U$'s value function having the same Arrow-Pratt measure of risk cautiousness[10] as that of the positive piece of agent $V$'s value function. A HARA function $h(x)$ is a solution to the ordinary differential equation $-\frac{h'(x)}{h''(x)} = a + bx$, for some constants $a$ and $b$. An example of piece-wise HARA functions is the following piece-wise CARA (constant absolute risk aversion) function:

$$V(x) = \begin{cases} A(1 - \exp^{-\frac{x}{A}}) & \text{if } x > 0 \\ -B(1 - \exp^{\frac{x}{B}}) & \text{otherwise,} \end{cases}$$

where $A$ and $B$ are two positive constants. One can show that if the two agents $U$ and $V$ have well-defined piece-wise HARA value functions with their Arrow-Pratt measures of risk cautiousness satisfying the above condition, then the optimal sharing rules, whenever exist, are piece-wise linear. Thus given that optimal sharing rules exist, these are equivalent conditions for optimal sharing rules to be piece-wise linear.

## 2.5   Concluding Remarks

We have analyzed the implications of Kahneman-Tversky's prospect theory on equilibrium asset returns and Pareto optimal risk sharing rules. We showed that prospect theory is compatible with existing asset pricing theories such as CAPM and intertemporal CAPM. Since expected utility function can be regarded as a special case of expected value function, we have verified that the results we obtain include familiar results for expected utility function as special cases.

Most relevant issues in financial economics appear in a dynamic economy. We expect the implications of prospect theory on dynamic financial economics to be abundant and useful for understanding important issues like asset pricing and consumers' optimal intertemporal consumption problems. These constitute a direction of future research.

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[10]The definition of Arrow-Pratt measure of risk cautiousness for a value function $h(x)$ at $x$ is
$\frac{d(-\frac{h'(x)}{h''(x)})}{dx}$.

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# Chapter 3

# Debt Maturity and Product Market Competition: Cooperation in the Finitely Repeated Prisoners' Dilemma

**Abstract of Chapter 3[1]**

This paper provides a revised paradigm for the *finitely repeated prisoners' dilemma* where rational cooperation between the two prisoners arises in equilibrium. The major feature of this new paradigm is that, unlike the Kreps-Milgrom-Roberts-Wilson paradigm, it does not assume information asymmetry between the two prisoners. We apply this paradigm to study the interactions between debt maturity and firms' product market competition. We show that short-term debt reduces firms' product market competition, an opposite prediction to that of the *long-purse theory* in the industrial organization literature. We also show that collusive firms tend to have high debt-equity ratios.

## 3.1 Introduction

The purpose of this paper is two-fold. First, at the game theoretical level, it provides a revised paradigm for the *finitely repeated prisoners' dilemma*, where rational cooperation between the two prisoners arises in equilibrium. Unlike the Kreps-Milgrom-Roberts-Wilson paradigm (1982), this new paradigm retains the symmetry between

the two prisoners. Second, at its application level, this paradigm is used to study the interactions between debt maturity and firms' product market competition.

Several paradigms created by game theorists have proven to be very useful in modeling economic affairs. One of the most famous game paradigms is the *prisoners' dilemma*; see Rapoport and Chammah (1965). In the traditional story, the two prisoners have jointly committed some crime and are being held separately. If each stays quiet (cooperates), they are both convicted of a minor charge and receive a one-year sentence. If just one confesses and agrees to testify against the other (defects), he goes free while the other receives a ten-year sentence. If both confess, they both receive a five-year sentence. This game is interesting because confessing is the dominating strategy: it pays to confess no matter what the other person does. The assumptions of rationality and self-interest therefore yield the prediction that people playing a game with this structure will defect. Moreover, when this game is repeated several times, to defect is still the unique equilibrium strategy.

This prediction, however, has been found to be violated in a large number of experiments with multiple trials on the prisoners' dilemma; see Axelrod (1981), Smale (1980) and Thaler (1992). A common pattern in these experiments is that subjects cooperate in several trials but the cooperation decays over time. To reconcile the theoretical prediction and the experimental findings, Kreps, Milgrom, Roberts, and Wilson (1982) provide a revised paradigm for the finitely repeated prisoners' dilemma, where one rational prisoner is uncertain about whether his partner is fully rational. An irrational prisoner is assumed to play the *tit-for-tat* strategy, which requires the player to begin by cooperating and then to choose the same response the other player has made on the previous trial. What Kreps et al. show is that, if you are playing against an opponent who you think may be irrational, then it may be rational to cooperate early in the game in order to induce your irrational opponent to cooperate too (Thaler (1992)).

The Kreps-Milgrom-Roberts-Wilson paradigm is able to generate rational cooperation in the revised finitely repeated prisoners' dilemma at a decaying rate, but the symmetry between the two prisoners is destroyed. In view of this, this paper provides an alternative way of revising the original prisoners' dilemma, in which the symmetry between the prisoners is retained and rational cooperation arises in equilibrium at a decaying rate. This new paradigm can be used to study a lot of economic activities in different contexts.

The idea is to embed the original prisoners' dilemma in an enlarged game, into which a third party is introduced. The third party is to provide punishment to both prisoners whenever defection occurs. To be concrete, we consider two firms competing in the product market for a finite number of periods. The two firms are identical and each has a 50% market share. Each firm can make some investment to change his and his rival's current market shares while the total size of the market

remains unchanged.[2] If only one firm invests, his market share increases while his rival's market share decreases by a commensurate magnitude. If both invest, with equal probability each may win the campaign. This is a version of the prisoners' dilemma. The unique equilibrium is one where both firms make the inefficient investments (defect) and end up with the same market share.

To introduce a third party into this game, we assume that neither of the two firms has sufficient funds to operate in the product market. They therefore must raise money from investors (the third party).[3] Investors, however, have incomplete industry-wide information. In particular, although investors realize that there are two firms in the industry, they cannot tell whether the two firms are local monopolists or competitive firms.[4] Under standard debt contracts, investors' payoffs will be concave in the firm's profit. As a consequence, even if investors are risk neutral, they prefer local monopolists to competitive firms, because competition increases the business risk of a firm. In the presence of short-term debt, the two competitive firms thus have incentives to avoid competition in earlier periods, in order to pretend to be local monopolists and obtain better terms of refinancing.

The above story leads to the prediction that short-term debt can be used to reduce firms' product market competition in certain contexts. The traditional *long-purse* theory, however, predicts that short-term debt generally results in more competition; see for example Bolton and Scharfstein (1988). The long-purse story goes roughly as follows. There are two firms, one with a long-purse and the other with a shallow pocket. That is, one firm has abundant internal fund and does not need external financing, while the other has to raise money from outside investors. The usual moral hazard argument implies that refinancing will become very costly for the shallow-pocket firm, if his performance in earlier periods is not good enough. This fact encourges the long-purse firm to compete more aggressively in order to make sure that his shallow-pocket rival gets punishment from the investors. As a conclusion, short-term debt should lead to more competition.

Our prediction does *not* contradict the long-purse story. In fact, the two different stories provide useful insights in two different contexts. The moral is that,

1. if *both* firms need external financing and reducing competition can jointly produce a favorable signal to the investors, then short-term debt leads to more cooperation and less competition;

2. if *only one* firm needs external financing, and the two firms do not share some industry-wide information which they want to conceal from the investors, then

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[2]Think of advertising campaign as an example.

[3]Here I have abused the terminology a little. Investors in this model do not make collective decisions.

[4]This depends on, for example, consumers' loyalty to the products of the two firms. But this information is in general unavailable to the investors.

short-term debt generally leads to more competition.[5]

If, other things being equal, small firms need external financing more than large firms do, then the above predictions say that for industries with high (respectively, low) concentration ratios, high competition is likely to be accompanied by short (respectively, long) debt maturity.

This model also generates predictions about the capital structures of collusive firms. We show that collusive firms tend to have high debt-equity ratios. The reasoning is the following. Note that the two firms cannot sustain collusion by themselves in the original prisoners' dilemma. A higher debt-equity ratio means a higher involvement of the outside investors (the third party) and therefore the latter have more incentives to provide punishment whenever firms defect. This makes collusion more likely to occur. Thus, to facilitate collusion, the firms should pay out *free cash flows* (Jensen (1986)), lever up and commit to the financial market.[6]

We also show that the likelihood of collusion is decreasing in the industry-wide profitability. There are two reasons for this result. First, the immediate gain of deviation from collusion increases with the industry-wide profitability. Second, when the entire industry is more profitable, other things being equal, firms become less risky. Accordingly, after a deviation takes place, firms only receive lighter financial punishment from investors. This again makes deviation more attractive.

The rest of this paper is organized as follows. In section 3.2, we layout the revised prisoners' dilemma. The game proceeds at three dates, 1, 2, and 3. Financial contracts are determined at date 1 and possibly renewed at date 2 if they are short-term. Firms compete at dates 2 and 3. We show that for a set of parameters, this game has a pooling perfect Bayesian equilibrium for where both local monopolists and competitive firms choose short-term debt at date 1 and competitive firms collude at date 2. Namely, competitive firms cooperate in the first round of their product market competition. In section 3.3, we show that the likelihood of collusion is increasing in the two firms' debt-equity ratios. In section 3.4, we relate our model to recent literature on debt maturity and product market competition. Section 3.5 gives concluding remarks.

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[5]The information structure is at least as important as the number of firms who need external financing in leading to these results. In Rotemberg and Scharfstein (1990), two firms both need external financing, but because of different information structure, short-term debt does not lead to less competition.

[6]In addition to debt maturity, forward market transactions may also be used as commitments to reduce competition in the product market. I thank Jean-Luc Vila for pointing out this to me.

## 3.2  The Model

We consider an industry with two identical firms who live for three dates, 1, 2 and 3. Firms operate in the product market at dates 2 and 3, which will be referred to as the *first-period* and the *second-period* competitions respectively. At dates 1 and 2, each firm has to spend \$1 to initiate respectively the first-period and the second-period operations. For the time being, assume every firm has sufficient funds to cover the one-dollar cost per period. There are two possible **types of industry**, called M and C. The firms in M industry, henceforth M firms, are local monopolists who cannot taking actions to affect their rivals' profits. The per-period profit for an M firm is $Y > 1$.

### 3.2.1  Competition in C Industry: A Prisoners' Dilemma

On the other hand, the firms in C industry, henceforth C firms, are competitors in the sense that they can simultaneously[7] make some investments, e.g. on advertising campaigns, to change their market shares and thus current-period profits. The investment costs $\theta \in [0, 1)$ per period. If neither spends $\theta$ (stays quiet), each C firm has profit $Y$ per period. If just one firm spends $\theta$ (defects), then his current-period profit becomes

$$\Pi \equiv (1 + \theta)Y - \theta,$$

and at the same time his rival's profit reduces to

$$\pi \equiv (1 - \theta)Y.$$

Namely, the investment has constant return to scale in terms of market share. If both spend $\theta$, because the two firms are equal in competitive power, with 50% chance each may win the campaign and get the winner's profit $\Pi$.[8] The following bimatrix summarizes the payoffs to the two competitive firms in each possible outcome:

| Firm 1/Firm 2 | Cooperate | Defect |
|---|---|---|
| Cooperate | $(Y, Y)$ | $(\pi, \Pi)$ |
| Defect | $(\Pi, \pi)$ | $(Y - \theta, Y - \theta)$ |

Since defecting is the dominating strategy, both C firms spend $\theta$ in the unique Nash equilibrium and end up with the same market shares.

In the prisoners' dilemma presented above, C firms defect in equilibrium, if they both have sufficient internal funds to operate in the product market. Our task is to show that, if the two C firms have paid out all free cash flows prior to date 1,

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[7]This is not an assumption about the timing of the game. Rather, it is about players' information sets when they take actions. Here a competitive firm has to make the investment decision before he sees his rival's investment decision.

[8]Note that when $\theta$ is zero, M firms and C firms are identical.

and must raise money from outside investors, then short-term debt contracts can serve as a commitment, which allows C firms to reach a Pareto improved equilibrium where neither of them defects in the first-period competition.

Before we start to introduce investors, we need to impose some restrictions on the parameters $\theta$ and $Y$ to rule out irregular and uninteresting cases:

**Assumption 1**

$$0 < \theta < \frac{1}{5}$$

*and*

$$\frac{1 + \sqrt{1 + 8\theta}}{2} < Y < \frac{1 + \theta}{1 - \theta}.$$

The above conditions imply that:

1. **Non-triviality:** $\pi < 1$.

2. **Positive NPV's:** $Y - \theta > 1$.

3. **Necessity of refinancing:** $\Pi - 1 < 1$.

Note that

$$\Pi > Y > Y - \theta > 1 > \pi > 0.$$

## 3.2.2 Introducing A Third Party: The Investors

From now on, we shall assume that both M firms and C firms need to raise money from outside investors at dates 1 and 2, to initiate their first-period and second-period operations in the product market. Investors are competitve, risk neutral, and have no time preferences. Firms are also risk neutral and have no time preferences. Investors realize that there are two firms in the industry, but fail to distinguish C firms from M firms.

**Assumption 2** *Assume for simplicity that:*

1. *Renegotiation of long-term financial contracts are prohibited.*

2. *If a long-term financinal contract is signed at date 1, no cash payments can be distributed at date 2.*

3. *Profits are observable but not verifiable.*

4. *Liquidation is costless.*

74

Since profits cannot be verified in court,[9] the repayments of a financial contract cannot be contingent on (and thus must be invariant to) the borrower-firm's profit realizations. It follows that we should consider only simple debt contracts. A simple debt contract specifies respectively the amount of money the firm borrows now and that to be repaid at the maturity date. It also allows the creditors to liquidate the firm when the latter fails to fully repay the debt. In the current framework, a firm borrowing at date 1 has to repay at either date 3 (long-term debt) or date 2 (short-term debt). In the latter case, the firm has to renew the short-term contract at date 2 in order to continue his second-period business.

We now fix some notation:

1. $m$: At date 1, investors commonly believe that with probability $m$ the borrowers are M firms.

2. $m'$: The off-equilibrium path belief corresponding to the equilibrium belief $m$.

3. $L(m)$: The long-term debt repayment when the equilibrium belief is $m$.

4. $S(m)$: The short-term debt repayment when the equilibrium belief is $m$.

Since profits are observable and investors are given the right to liquidate the firm when default takes place, it is always incentive compatible for a borrower-firm to repay the smaller amount between the required repayment of debt and his total profit. Bankruptcy thus never occurs in equilibrium.

**Lemma 1** *For all* $m \in [0,1]$,
$$1 \leq S(m) < Y,$$

*and*
$$2 \leq L(m) < 2(Y - \theta).$$

**Proof**
See appendix.

According to the previous lemma, we can solve for $S(m)$ and $L(m)$ from the following equations:

$$\frac{1-m}{2}\pi + \frac{1-m}{2}S(m) + mS(m) = 1,$$

$$\frac{1-m}{4}2\pi + (1 - \frac{1-m}{4})L(m) = 2.$$

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[9]Alternatively, we can assume that profits are not even observable without affecting our major results. In this case, however, bankruptcy may occur and makes the model more complicated.

We have

$$S(m) = \frac{2 - (1 - m)\pi}{1 + m},$$

$$L(m) = \frac{8 - 2(1 - m)\pi}{3 + m}.$$

Now we can solve for the product market equilibria at dates 2 and 3, given each possible financial arrangements of firms at date 1.

### 3.2.3 Product Market Equilibria

In this model, M firms are rather passive in the product market stage. Our current analysis thus focuses on C firms. At date 3, it is clear that C firms will compete and yield an expected second-period profit $Y - \theta$.

Now we consider C firms' competitive behavior at date 2. The following arguments are straightforward:

- If long-term debt has been used by C firms at date 1, without the chance for refinancing, they simply compete at date 2.

- If there has been a separating equilibrium at date 1, then C firms compete at date 2.

- Suppose both M firms and C firms have used short-term debt at date 1. That C firms compete at date 2 *always* forms an equilibrium at date 2.

The next lemma shows the existence of a date-2 equilibrium where C firms cooperate:

**Lemma 2** *Suppose both M firms and C firms have used short-term debt at date 1. If $\theta$ and $Y$ satisfy conditions in assumption 1,*

$$Y < \frac{1 + 3\theta}{1 + \theta},$$

*and*

$$m \geq \frac{(Y - 1)\theta}{(Y - 1)\theta + (3\theta + 1 - (1 + \theta)Y)},$$

*then at date 2 there is a pooling equilibrium where C firms choose not to make the investments. The first-period profit is $Y$ for an M firm as well as a C firm.*

**Proof**
See appendix.

The above lemma states an upper bound on $Y$ and a lower bound on $m$ in order for the cooperative equilibrium to arise at date 2. To see the intuitions behind these

bounds, we examine C firms' incentives of deviating from collusion. At date 2, given that his rival chooses to cooperate, a C firm gets the following continuation payoff by defecting:

$$\Pi - S(1) + \frac{1}{2}(\Pi - S(0)) + \frac{1}{2} \cdot 0,$$

where the first-period repayment is $S(1)$ because in equilibrium C firms pool with M firms at date 2, and the second-period repayment is $S(0)$ because C firms reveal their identities after competing.

On the other hand, if the C firm chooses to cooperate too, his continuation payoff will be

$$Y - S(1) + \frac{1}{2}(\Pi - S(m)) + \frac{1}{2} \cdot 0,$$

where the second-period repayment is $S(m)$ because in a pooling equilibrium the posterior belief is the same as the prior. By inspection, we see that

1. The higher $Y$ is, the more tempting for a C firm to unilaterally deviate from the collusion. Therefore, for the cooperative equilibrium to arise, $Y$ cannot be too high.

2. The higher $m$ is, i.e. the more likely the borrowers are M firms, the more costly the competition between C firms becomes. Thus $m$ should be high enough to prevent C firms from competing with each other.

## 3.2.4  A Pooling Perfect Bayesian Equilibrium

Now we are back to date 1. Standard backward induction procedure yields the following result:

**Proposition 1** *If $\theta$ and $Y$ satisfy conditions in assumption 1,*

$$Y < \frac{1 + 3\theta}{1 + \theta},$$

*and*

$$m \geq \max\left(\frac{(Y-1)\theta}{(Y-1)\theta + (3\theta + 1 - (1+\theta)Y}, \frac{3 - L(0) - \pi}{L(0) - 1 - \pi}\right),$$

*then this game has a pooling perfect Bayesian equilibrium, in which both M firms and C firms use short-term debt at date 1, and C firms collude at date 2. The supporting off-equilibrium belief is any*

$$m' < \frac{8 - 2\pi - 3(S(m) + 1)}{S(m) + 1 - 2\pi}.$$

**Proof**
See appendix.

## 3.3 Likelihood of Collusion

In the previous section, we have shown that introducing a third party into the original prisoners' dilemma generates rational cooperation between the two players. In the specific context we have been dealing with, C firms collude at date 2. In this section, we continue to study the factors that affect the likelihood of cooperation.

### 3.3.1 Industry-wide Profitability

The first factor is the industry-wide profitability, measured by $Y$. The incentive compatibility condition for a C firm to cooperate given that his rival cooperates is

$$J(Y) \equiv \{Y - S(1) + \frac{1}{2}(\Pi - S(m))\}$$

$$-\{\Pi - S(1) + \frac{1}{2}(\Pi - S(0))\} \geq 0.$$

We have

$$J'(Y) = -\theta - \frac{m}{1+m}(1 - \theta) < 0.$$

This means:

**Proposition 2** *The likelihood of collusion is decreasing in the industry-wide profitability.*

Note that the increase in C firms' incentives of deviation comes from two sources:

1. The immediate gain from deviation,

$$\theta(Y - 1),$$

increases in $Y$.

2. When $Y$ gets higher, other things being equal, C firms become less risky. Accordingly, after deviation takes place, C firms receive only lighter financial punishment from investors. This again makes deviation more attractive.

### 3.3.2 Debt-Equity Ratio

As in the original prisoners' dilemma, C firms fail to reach the cooperative equilibrium by themselves, because they have sufficient internal funds to operate in the product market. One thus may expect the likelihood of cooperation to be relate to the capital structures of the firms. We now confirm this conjecture.

Assume that at dates 1 and 2, every firm has internal fund $\alpha < 1$ generated by other business segments of the firm. The firm thus needs to raise only $1 - \alpha$ in each period. The *accounting debt-equity ratio* is defined as

$$\frac{1 - \alpha}{\alpha},$$

which is strictly decreasing in $\alpha$.

C firms' incentives to unilaterally deviate from the collusion are now measured by

$$J(Y, \alpha) = -\theta(Y - 1) - \frac{m}{1 + m}(\pi - 1 + \alpha).$$

We have

$$J_\alpha < 0.$$

This means:

**Proposition 3** *The likelihood of collusion is increasing in the debt-equity ratio of the firms.*

The idea is that, it is optimal for collusive firms to pay out free cash flows (Jensen (1986)), lever up and commit to the financial market. By doing so, a third party, i.e. the outside investors, is introduced into the game and provides credible punishment to the firms whenever deviation from the collusion is detected. The above proposition is consistent with the empirical evidence documented by Phillips (1990).

# 3.4 Discussions

This section relates our model to several issues in corporate finance and product market competition.

## 3.4.1 Productive Efficiency vs. Informational Efficiency

Efficient market hypothesis has been one of the most important hypotheses in finance theory. The traditional wisdom is that, if financial markets are efficient, then assets are fairly priced. This helps to reach productive efficiency. The current model, however, presents a couterexample to this argument.

In the pooling equilibrium obtained in proposition 1, markets are *not* strong form efficient at date 2, because C firms do not reveal their identities at date 2.[10] However, compared to an equilibrium where C firms compete at date 2, this pooling

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[10]Markets are strong form efficient, if all private information is revealed in equilibrium.

equilibrium is indeed more productively efficient. The point is that C firms can avoid the inefficient competition in the product market, only if the financial market provides them correct incentives. If the financial market allows them to conseal their identities, they will not engage in wasteful competition.

In general, if the distribution of information across investors is independent of the firms' productive decisions, then informational efficiency is consistent with productive efficiency. Otherwise, there may be trade-off between the two, as our model shows.

### 3.4.2 The Contracts-Separating Equilibrium

In the current model, financial contracts are never informative. Namely, only pooling equilibria may arise at date 1.[11]

One possible extention of the current model is to assume that the investment made by C firms may not be socially wasteful. For example, an effective advertising campaign may increase the total market demand, or in an declining industry, prevent demand from going down too fast. With such a revision, contracts-separating equilibrium may appear at date 1. One can show that, whenever a contracts-separating equilibrium exists at date 1, firms in the high-quality industries will use long-term debt.[12] This result is very close to the prediction of Diamond (1989). In Diamond (1989), investors cannot distinguish a high-quality firm from a low-quality one. In the presence of short-term debt, prior to refinancing, some noisy information arrives. Based on the noisy information, a high-quality firm may be mistakenly liquidated. To avoid being jeopardized by the low-quality firm and being inefficiently liquidated, a high-quality firm considers using the costly long-term debt at the very beginning to distinguish himself from the low-quality mimicker. Similarly, in our model, firms in the high-quality industry use long-term debt to distinguish themselves from those in the low-quality industry in a contracts-separating equilibrium. We can conclude that in general, using long-term debt is good news to investors.

### 3.4.3 The Efficiency of Short-term Debt

In general, is short-term debt more efficient than long-term debt? Previous research provides the following insights:

1. First, a short maturity enables the short-term creditors to reduce the agency problems associated with debt in several ways. By threatening to cut off the credit, short-term creditors can provide the firm with the incentives to take the right investments (Stiglitz and Weiss (1983)). Short-term creditors can also

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[11]This is because the Spence-Mirrlee's sorting condition fails in this case.
[12]The proofs are available upon request.

screen prospective clients (Diamond (1989a)). As a result of reduced adverse selection as well as moral hazard problems, short-term debt appears to be more efficient than long-term debt.

2. Rajan (1990) points out that the firm's *ex ante* investment decisions may be distorted because of the information monopoly that a bank (a short-term creditor) may enjoy *ex post*. The idea is that, compared to the public investors, a bank that has business relations with a firm may be able to acquire superior information about the firm's quality. The bank's privileged information together with its ability to control the firm implies that *ex post* it has strong bargaining power against the firm. The *ex post* bank opportunism leads to distortions in the firm's *ex ante* production plans. The distortion may be so big that short-term debt becomes less efficient than long-term debt.

Unlike the previous literature, we considered the efficiency of short-term debt in the context of product market competition. The major feature of short-term debt is in this model is that it can result in different pooling equilibria in product market. Depending on the efficiency in these pooling equilibria, short-term debt may and may not be more efficient than long-term debt. To be more specific, we consider the following two cases:

1. In the basic model presented in section 2, short-term debt leads to higher productive efficiency than long-term debt, because the wasteful competition between C firms are avoided in the pooling equilibrium resulting from the short-term debt.

2. Alternatively, let us assume that (i) profits are unobservable and the investments are observable, (ii) with some probability the second-period demand in both M and C industries may vanish, and (iii) the investments of C firms can prevent the second-period demand in C industry from declining. One can show that,[13] in this case, for a set of parameters, the game has a pooling equilibrium where both M firms and C firms use short-term debt at date 1 and both M firms and C firms make the investments at date 2. Recall that according to our previous assumption, these investments have no effects on M firms. Thus the investments made by M firms are completely wasteful. This is a reverse story to that presented in section 2. The idea is that, by spending on useless projects, M firms can pool with C firms and obtain better terms of refinancing. Short-term debt leads to double inefficiencies in this pooling equilibrium: First, M firms spend on useless projects. Second, M firms may obtain refinancing even if their operations in the second period are socially wasteful

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[13]The proof is available upon request.

(negative NPV). In this case, short-term debt, that leads to this inefficient pooling equilibrium, is less efficient than long-term debt.

As a conclusion, in the context of product market competition, short-term debt may and may not be more efficient than long-term debt. The relative efficiency of short-term debt to long-term debt depends on the efficiency of the pooling equilibrium resulting from the short-term debt.

## 3.5 Concluding Remarks

We have provided in this paper a revised paradigm for the finitely repeated prisoners' dilemma and shows that rational cooperation between the two prisoners arise in equilibrium. As an application, we have used this paradigm to study the relations between debt maturity and firms' product market competition and generated several testable predictions. In particular, if other things being equal, small firms need external financing more than large firms do, then for an industry with low concentration ratio, i.e. the firms are mostly small and need external financing, short maturity is likely to be accompanied by low competition. On the other hand, the long-purse theory predicts that, for an industry with high concentration ratio, i.e. long-purse big firms and shallow-pocket small firms co-exist, short maturity only encourages big firms to predate small ones, leading to high competition.

The model also predicts that the collusion is more likely to arise if the industry has relatively low profitability. At the corporate level, collusive firms tend to commit to high debt-equity ratios.

This game paradigm can also be used to study other economic problems with similar structures. For example, it can be used to study the optimal compensation scheme for two employees who share the private information about their working ability, which is unavailable to the employer. A conjecture following our previous analysis is that short-term labor contract may lead to more cooperation between the two employees.

Like most recent theoretical literature in corporate finance, this model has assumed specific information and competitive structures. The predictions of this paper are quite sensitive to the information and competitive structures assumed at the very beginning. Applications of these predictions must be cautious, as these predictions work well only if the real-world data exhibit similar structures. A unifying theory regarding the relations between debt maturity and product market competition is still in order.

# Appendix

**Proof of Lemma 1**

The proof follows directly from assumption 1 and the solutions for $S(m)$ and $L(m)$. □

**Proof of Lemma 1**

The proof directly follows from the incentive compatibility condition under which a C firm cooperates given that his rival cooperates. □

**Proof of Proposition 1**

One can easily verify that under the conditions on $\theta$, $Y$, $m$ and $m'$ stated in the proposition, neither C firms nor M firms want to deviate. The derivation of these conditions, though straightforward, is quite tedious. It is available upon request. □

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