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Convergence of The Undrained Split Iterative Scheme for Coupling Flow with Geomechanics in Heterogeneous Poroelastic Media

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Abstract

Recently an accurate coupling between subsurface flow and reservoir geomechanics has received more attention in both academia and industry. This stems from the fact that incorporating a geomechanics model into upstream flow simulation is critical for accurately predicting wellbore instabilities and hydraulic fracturing processes. One of the recently introduced iterative coupling algorithms to couple flow with geomechanics is the undrained split iterative coupling algorithm [26, 28]. The convergence of this scheme is established in [28] for the single rate iterative coupling algorithm, and in [26] for the multirate iterative coupling algorithm, in which the flow takes multiple finer time steps within one coarse mechanics time step. All previously established results study the convergence of the scheme in homogeneous poroelastic media. In this work, following the approach in [5], we extend these results to the case of heterogeneous poro-elastic media, in which each grid cell is associated with its own set of flow and mechanics parameters for both the single rate and multirate schemes. Second, following the approach in [6], we establish a priori error estimates for the single rate case of the scheme in homogeneous poro-elastic media. To the best of our knowledge, this is the first rigorous and complete mathematical analysis of the undrained split iterative coupling scheme in heterogeneous poro-elastic media.

Keywords. poroelasticity; Biot system; undrained-split iterative coupling; contraction mapping; a priori error estimates; heterogeneous poroelastic media

1 Introduction

Solving a coupled flow and geomechanics problem is of high importance to the field of petroleum engineering. This is a direct consequence of the fact that several physical processes cannot be modeled correctly without incorporating a geomechanical model into the underlying physical model. Important examples include reservoir deformation, surface subsidence, pore collapse and well-bore stability, fault activation, and hydraulic fracturing (see e.g., [1,7,20,23,24,36] and references therein). These processes are of great economic importance. Such a coupling can greatly enhance upstream related operations including drilling simulation, basin modeling, hydraulic fracturing, and reservoir simulation.

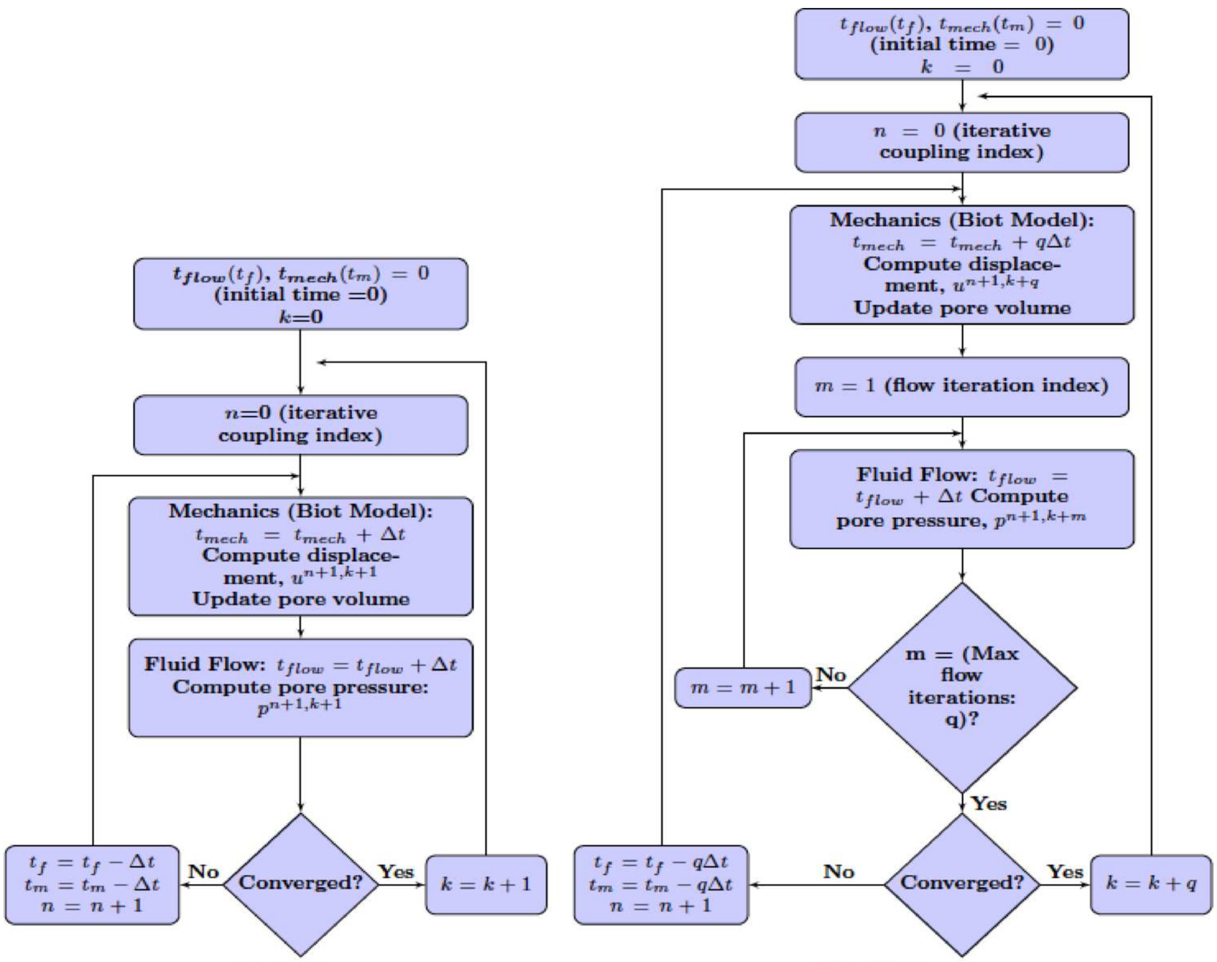
In practice, there are three different ways to solve a coupled flow and geomechanics problem. They are known as the fully implicit method (or the simultaneously coupling approach), the explicit method (or the loosely coupled approach), and the iterative coupling method (see e.g., [22] for a comparison of the these three different coupling approaches). The fully implicit approach solves the two problems by linearizing the full system simultaneously. This provides an unconditionally stable approach. However, The algebraic system obtained from the linearization of simultaneously coupled system is difficult to solve and often requires the use of preconditioners to decouple the two systems [17, 18]. A recent work by [13, 14, 35] formulated a fixed-stress preconditioning technique to decouple the two systems in an efficient way. On the other extreme lies the explicit coupling approach in which the two problems are decoupled, and are solved in a sequential manner without imposing any iterative coupling iteration between the two [4, 21, 36]. It provides a much simpler linear system to solve as the two equations are decoupled and we have excellent solvers to solve mechanics and flow equations separately. The disadvantage is that it is at best conditionally stable, and several stabilization techniques have been proposed in the literature to cure this issue (see e.g., [21,36]). In this scheme, since the two systems are fully decoupled, the mechanics problem can be solved at selective time steps [32]. A rigorous stability analysis of these multirate loosely coupled schemes, in which the mechanics problem is solved at selective time steps, is provided in [4]. The iterative coupling approach lies in between these two extremes: it decouples the two problems, but imposes an iterative coupling iteration between the two until convergence is obtained. We shall focus on this coupling approach in this work.

Four main flow-mechanics iterative coupling approaches exist in literature including the undrained split, the drained split, the fixed-stress split, and the fixed-strain split iterative coupling schemes [12, 13, 22–24, 27, 28]. In the undrained split scheme, the mechanics problem is solved first followed by the flow. In this scheme, the fluid mass (i.e., fluid content of the medium) is assumed to be constant during the mechanics solve [22, 25, 26, 28]. The drained split scheme solves the mechanics problem first as well (followed by flow), but assumes constant fluid pressure during the mechanics solve [22, 25]. In contrast, the fixed-stress split and fixed-strain split coupling schemes start by solving the flow problem first followed by mechanics [24, 28]. In the fixed-stress split scheme, a constant volumetric mean total stress is assumed during the flow solve, while a constant strain is assumed in the fixed-strain split scheme [3, 24, 28]. In this work, we will consider the convergence analysis of the first approach, that is, the undrained split iterative coupling approach. We note here that for the undrained split scheme to converge, sufficient fluid compressibility should be present. Subsequently, we make this more precise later (see Remark 4.6)

The convergence of the fixed-stress and undrained split iterative coupling schemes in homogeneous poroelastic media has been established in [28] for the single rate case (in which the flow and mechanics problems share the same time step), and in [2, 3, 26] for the multirate scheme in which the flow problem is solved for multiple finer time steps within one coarse mechanics time step (the single rate and multirate schemes are illustrated in Figure 1.1). In addition, the work of [23, 24] used von-Neumann type of analysis, and energy and spectral methods to analyze the stability of the aforementioned four iterative coupling approaches. Recently [5] performs convergence analysis in heterogeneous poro-elastic media where convergences of the single rate and multirate fixed-stress split iterative coupling were obtained. The extensions to the nonlinear case has been performed in [11] and to the fractured case in [19]. Multiscale extension of the fixed stress splitting is in [16]. Space time based method has been explored in the [8, 10] and the stability analysis of the discretization scheme in [31].

In this work, our main objectives are two fold; first, we establish the convergence of the single rate and multirate undrained split iterative coupling scheme in heterogeneous poroelastic media, following a similar approach as in [5]. Second, we will derive a priori error estimates for the single rate undrained split iterative coupling scheme in homogeneous poro-elastic media (see e.g., [6]). To the best of our knowledge, this is the first rigorous convergence analysis of the single rate and multirate undrained split iterative coupling schemes in heterogeneous poro-elastic media. In addition, a priori error estimates for the single rate undrained split iterative coupling scheme are derived here for the first time.

The paper is structured as follows: Section 2 contains model equations and discretizations, followed by a detailed description of the undrained split iterative coupling scheme in Section 3. Section 4 contains the convergence results for the single rate and multirate undrained split iterative coupling schemes in heterogeneous poroelastic media, followed by an a priori error estimate result for the single rate scheme (in homogeneous media) in Section 5. Finally, the conclusions are presented in Section 6.



(a) Single Rate (b) Multirate

Figure 1.1: A flowchart of the single rate and multirate undrained split iterative coupling schemes (Image Courtesy of [1]).

2 Model Equations & Discretizations

The domain Ω is assumed to be an open, connected, and bounded domain of $I\!R^d$, where $d = 1, 2$ or 3. The boundary $\partial\Omega$ is assumed to be Lipschitz continuous boundary. We also assume that Γ represents the part of the boundary with positive measure (with a Lipschitz continuous boundary for $d = 3$). We also note that $\Gamma_D \cup \Gamma_N = \Gamma$, where Γ_D is the part of the boundary with Dirichlet boundary conditions, and Γ_N is the part with Neumann boundary conditions. For our homogeneous poro-elastic media analysis, we assume a linear and isotropic medium $\Omega \subset \mathbb{R}^d$ with a slightly compressible fluid inside the reservoir. The viscosity ($\mu_f > 0$) is assumed to be constant in time, and the fluid density is a linear function of pressure. The reference density of the fluid $\rho_f > 0$, the fluid compressibility c_f, the pore volume φ^*, and other poro-elastic parameters including the Lamé coefficients $\lambda > 0$ and $G > 0$, and the Biot coefficient α are assumed to be positive. We also assume that absolute permeability tensor, $\boldsymbol{K}$, is symmetric, bounded, and uniformly positive definite in space (and constant in time). The quasi-static Biot model [9, 15] we will analyze in this work reads: Find $\boldsymbol{u}$ and p satisfying the following equations for all time $t \in]0, T[$:

$$
\begin{aligned}
\textbf{Flow Equation:} \quad & \frac{\partial}{\partial t}\Big(\big(\tfrac{1}{M} + c_f\varphi_0\big)p + \alpha\nabla\cdot\boldsymbol{u}\Big) - \nabla\cdot\Big(\tfrac{1}{\mu_f}\boldsymbol{K}\big(\nabla p - \rho_{f,r} g\nabla\eta\big)\Big) = \tilde{q} \ \text{ in } \Omega \\
& -\nabla\cdot\boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u}, p) = \boldsymbol{f} \ \text{ in } \Omega, \\
\textbf{Mechanics Equations:} \quad & \boldsymbol{\sigma}^{\text{por}}(\boldsymbol{u}, p) = \boldsymbol{\sigma}(\boldsymbol{u}) - \alpha\, p\, \boldsymbol{I} \ \text{ in } \Omega, \\
& \boldsymbol{\sigma}(\boldsymbol{u}) = \lambda(\nabla\cdot\boldsymbol{u})\boldsymbol{I} + 2\,G\boldsymbol{\varepsilon}(\boldsymbol{u}) \ \text{ in } \Omega \\
\textbf{Boundary Conditions:} \quad & \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega,\ \boldsymbol{K}(\nabla p - \rho_{f,r} g\nabla\eta)\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma_N,\ \ p = 0 \text{ on } \Gamma_D, \\
\textbf{Initial Conditions (t=0):} \quad & \Big(\big(\tfrac{1}{M} + c_f\varphi_0\big)p + \alpha\nabla\cdot\boldsymbol{u}\Big)(0) = \big(\tfrac{1}{M} + c_f\varphi_0\big)p_0 + \alpha\nabla\cdot\boldsymbol{u}_0 \ \text{ in } \Omega.
\end{aligned}
$$

We first note that the system above is linear. The flow and mechanics problems are coupled through those terms associated with the Biot coefficient α. In addition, $\rho_{f,r} > 0$ is a constant reference density (with respect to a reference pressure p_r). Moreover, g, M, φ_0, η, denotes the gravitational acceleration, the Biot modulus, the initial porosity, and the distance in the direction of gravity respectively. Furthermore, $\tilde{q} = \frac{q}{\rho_{f,r}}$, and q is a mass source or sink term.

2.1 Mixed variational formulation

Throughout our analysis in this paper, we will use a mixed finite element formulation for flow and a conformal Galerkin formulation for mechanics (for the spatial discretization). The backward-Euler scheme will be used for temporal discretization. We also note that our work can be extended to other mixed formulation approaches (for the spatial discretization, see for instance [34]). If we let $\mathfrak{T}_h$ denote a regular family of conforming elements of $\overline{\Omega}$, then using the lowest order Raviart-Thomas

(RT) spaces, the flow and mechanics discrete spaces are as follows [1]:

$$\text{Discrete Displacements:} \quad \boldsymbol{V}_h = \{\boldsymbol{v}_h \in H^1(\Omega)^d\,;\, \forall T \in \mathfrak{T}_h, \boldsymbol{v}_{h|T} \in \mathbb{P}_1{}^d, \boldsymbol{v}_{h|\partial\Omega} = \boldsymbol{0}\},$$
$$\text{Discrete Pressures:} \quad Q_h = \{p_h \in L^2(\Omega)\,;\, \forall T \in \mathfrak{T}_h, p_{h|T} \in \mathbb{P}_0,\ \ p_h = 0,\ \text{on}\ \Gamma_D\},$$
$$\text{Discrete Fluxes:} \quad \boldsymbol{Z}_h = \{\boldsymbol{q}_h \in H(\nabla\cdot;\Omega)^d\,; \forall T \in \mathfrak{T}_h, \boldsymbol{q}_{h|T} \in \mathcal{RT}_0^d,\ \boldsymbol{q}_h \cdot \boldsymbol{n} = 0\ \text{ on}\ \Gamma_N\}.$$

The single rate scheme assumes a uniform time step of size $\Delta t = t_k - t_{k-1}$ for all k. The multirate scheme assumes two different time steps: a fine time step for flow (Δt), and a coarse time step for mechanics ($q\Delta t$). Here, q denotes the number of flow fine time steps that are solved within one coarse mechanics time step. Assuming a uniform fine flow time step of size Δt, the total simulation time is given by $\mathrm{T} = \Delta t\ \mathrm{N}$, where N denotes the total number of fine flow time steps. Discrete time points are given by $t_i = i\Delta t$, $0 \leqslant i \leqslant N$.

Notation: We will assume that k denotes the time step index in the single rate scheme. For the multirate scheme, k, and m represent the coarse mechanics time step, and fine flow time step indices, respectively. If we solve q flow fine time steps within one coarse mechanics time step, then we have $1 \leq m \leq q$. We also note that q can change across coarse mechanics time steps. Moreover, for our analysis, we denote the difference between two consecutive coupling iterations by $\delta\xi^{n,k} = \xi^{n+1,k} - \xi^{n,k}$, where ξ stands for $p, \boldsymbol{u}$, or $\boldsymbol{z}$, and n represents the iterative coupling iteration index.

3 Undrained Split Iterative Coupling Algorithm

The undrained split iterative coupling scheme assumes a constant fluid mass during the mechanics solve. In this scheme, the mechanics problem is solved first followed by the flow problem. The continuous strong form of the splitting scheme is given below. The superscript n denotes the coupling iteration index (between flow and mechanics):

Step (a) [Mechanics] Given p^n and $\boldsymbol{u}^n$, we solve for $\boldsymbol{u}^{n+1}$ satisfying

$$-\nabla\cdot \boldsymbol{\sigma}^{\mathrm{por}}(\boldsymbol{u}^{n+1}, p^n) - L\,\nabla\cdot \Big(\big(\nabla\cdot \boldsymbol{u}^{n+1}\big)\, \boldsymbol{I} \Big) = \boldsymbol{f} - L\,\nabla\cdot \Big(\big(\nabla\cdot \boldsymbol{u}^{n}\big)\, \boldsymbol{I} \Big)$$
$$\boldsymbol{\sigma}^{\mathrm{por}}(\boldsymbol{u}^{n+1}, p^n) = \boldsymbol{\sigma}(\boldsymbol{u}^{n+1}) - \alpha\, p^n\, \boldsymbol{I}$$
$$\boldsymbol{\sigma}(\boldsymbol{u}^{n+1}) = \lambda(\nabla\cdot \boldsymbol{u}^{n+1})\boldsymbol{I} + 2\,G\boldsymbol{\varepsilon}(\boldsymbol{u}^{n+1})$$

Step (b) [Flow] Given $\boldsymbol{u}^{n+1}$, we solve for $p^{n+1}, \boldsymbol{z}^{n+1}$ satisfying

$$\Big(\tfrac{1}{M} + c_f\varphi_0\Big)\tfrac{\partial}{\partial t}p^{n+1} - \nabla\cdot \boldsymbol{z}^{n+1} = -\alpha\nabla\cdot \tfrac{\partial}{\partial t}\boldsymbol{u}^{n+1} + \tilde{q}$$
$$\boldsymbol{z}^{n+1} = \tfrac{1}{\mu_f}\boldsymbol{K}\big(\nabla\, p^{n+1} - \rho_{f,r} g \nabla\, \eta\big)$$

We note that the parameter L in the right and left hand sides of the mechanics equation denote a regularization term. The subsequent convergence analysis will determine its value.

4 Convergence Analysis of the Undrained Split Scheme in Heterogeneous Media

For our convergence analysis results, we will assume a heterogeneous and isotropic poro-elastic medium. In addition to the assumptions mentioned above (in the Model Equations section), for the fully discrete formulation, we recall that the domain is denoted by $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, or 3, with an external boundary (denoted by $\partial\Omega$, and $\boldsymbol{n}$ denotes its outward unit normal vector). Our spatial domain will be discretized into N_Ω conforming grid elements E_i $(\overline{\Omega} = \bigcup_{i=1}^{N_\Omega} E_i)$. Each grid element E_i will be associated with an independent set of mechanics and flow parameters: $G_i, M_i, \lambda_i, \boldsymbol{K}_i, \alpha_i, c_{f_i}, \mu_{f_i}$ and φ_{0i}. In addition, the localized permeabilities $\boldsymbol{K}_i$ include viscosities μ_{f_i} (i.e. $\boldsymbol{K}_i = \frac{\boldsymbol{K}_i}{\mu_{f_i}}$). For each grid element E_i, $\boldsymbol{n}_i$ denotes its outward normal vector such that $\boldsymbol{n}_i = -\boldsymbol{n}_{i-1}$ for every two adjacent grid elements E_i and E_{i-1} with a common boundary.

4.1 Localized Fully Discrete Weak Formulation for the Single Rate Case:

Following the same approach as outlined in [5] for cancelling the boundary terms, the localized fully discrete weak formulation for the single undrained split iterative coupling scheme is as follows:
Step (a): Given $p_h^{n,k}$ and $\boldsymbol{u}^{n,k}$ from the previous coupling iteration, find $\boldsymbol{u}_h^{n+1,k} \in \boldsymbol{V}_h$ such that,

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h\,,\ 2\sum_{i=1}^{N_\Omega}\big(G_i\boldsymbol{\varepsilon}({\boldsymbol{u}_h}^{n+1,k}),\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\big((\lambda_i+L_i)\nabla\cdot\boldsymbol{u}_h^{n+1,k},\nabla\cdot\boldsymbol{v}_h\big)_{L^2(E_i)} = \sum_{i=1}^{N_\Omega}\big(\alpha_i p_h^{n,k},\nabla\cdot\boldsymbol{v}_h\big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\big(L_i\nabla\cdot\boldsymbol{u}_h^{n,k},\nabla\cdot\boldsymbol{v}_h\big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\big(\boldsymbol{f},\boldsymbol{v}_h\big)_{L^2(E_i)} \tag{4.1}$$

Step (b): Given $\boldsymbol{u}_h^{n+1,k}$, find $p_h^{n+1,k} \in Q_h$, $\boldsymbol{z}_h^{n+1,k} \in \boldsymbol{Z}_h$ such that:

$$\forall \theta_h \in Q_h\,,\ \sum_{i=1}^{N_\Omega}\Big(\big(\frac{1}{M_i}+c_{f_i}\varphi_{0i}\big)\Big(\frac{p_h^{n+1,k}-p_h^{k-1}}{\Delta t}\Big),\theta_h\Big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\big(\nabla\cdot\boldsymbol{z}_h^{n+1,k},\theta_h\big)_{L^2(E_i)} = -\sum_{i=1}^{N_\Omega}\Big(\alpha_i\nabla\cdot\Big(\frac{\boldsymbol{u}_h^{n+1,k}-\boldsymbol{u}_h^{k-1}}{\Delta t}\Big),\theta_h\Big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\big(\tilde{q},\theta_h\big)_{L^2(E_i)} \tag{4.2}$$

$$\forall \boldsymbol{q}_h \in \boldsymbol{Z}_h\,,\ \sum_{i=1}^{N_\Omega}\big(\boldsymbol{K}_i^{-1}\boldsymbol{z}_h^{n+1,k},\boldsymbol{q}_h\big)_{L^2(E_i)} = \sum_{i=1}^{N_\Omega}(p_h^{n+1,k},\nabla\cdot\boldsymbol{q}_h)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\big(\nabla(\rho_{f,r}g\eta),\boldsymbol{q}_h\big)_{L^2(E_i)} \tag{4.3}$$

4.1.1 Proof of Contraction for the Localized Single Rate Case:

We first define the quantity to be contracted on locally for each grid cell as: $\tilde{m}^{n,k}|_{E_i} = \frac{L_i}{\gamma_i}\nabla\cdot\boldsymbol{u}^{n,k} + \frac{\alpha_i}{\gamma_i}p^{n,k}$, for each $E_i \in \Omega, 1 \le i \le N_\Omega$, iterative coupling iteration $n \ge 1$, and time step t_k. We note here that γ_i and L_i are free adjustable parameters and will be determined such that the scheme achieves contraction locally on each $\tilde{m}|_{E_i}$. We recall that $\delta\xi^{n,k} := \xi^{n,k} - \xi^{n-1,k}$ denotes the difference between two consecutive coupling iterations, where $\xi^{n,k}$ can stand for $p_h^{n,k}, \boldsymbol{u}_h^{n,k}$, or $\boldsymbol{z}_h^{n,k}$.

Remark 4.1 *A physical interpretation of the localized regularization term L_i can be given as follows: the standard undrained split scheme assumes the fluid content of the medium to be fixed during the mechanics solve. The fluid content of the medium is a function of both pore pressure and mechanical displacement. In our work, the regularization term L_i is a quantity that scales both the pore pressure and mechanical displacement such that the scheme contracts on a modified (or scaled) expression of the increment in fluid content of the medium (given by $\tilde{m}^{n,k} = m_0 + \left(\frac{L_i}{2}\right)^{(1/2)}\nabla\cdot\boldsymbol{u}^{n,k} + \left(\frac{\alpha_i^2}{2L_i}\right)^{(1/2)}p^{n,k}$, where m_0 is the initial fluid content of the medium, or initial porosity). The value of L_i will be optimized in the sense that the contraction coefficient is the sharpest (or smallest) accross all grid elements E_i.*

- **Step (1): Elasticity equation**
  Considering (4.1) for the difference between two consecutive coupling iterations and testing with $\boldsymbol{v}_h = \delta\boldsymbol{u}_h^{n+1,k+1}$, we get:

$$\begin{aligned}\sum_{i=1}^{N_\Omega} 2G_i\left\|\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_h^{n+1,k+1})\right\|_{L^2(E_i)}^2 + \sum_{i=1}^{N_\Omega}(\lambda_i + L_i)\left\|\nabla\cdot\delta\boldsymbol{u}_h^{n+1,k+1}\right\|_{L^2(E_i)}^2 &\\ = \sum_{i=1}^{N_\Omega}\left(\alpha_i\delta p_h^{n,k+1} + L_i\nabla\cdot\delta\boldsymbol{u}_h^{n,k+1}, \nabla\cdot\delta\boldsymbol{u}_h^{n+1,k+1}\right)_{L^2(E_i)} &\\ = \sum_{i=1}^{N_\Omega}\left(\gamma_i\delta\tilde{m}^{n,k+1}, \nabla\cdot\delta\boldsymbol{u}_h^{n+1,k+1}\right)_{L^2(E_i)} &\\ \le \sum_{i=1}^{N_\Omega}\frac{\varepsilon_i}{2}\left\|\nabla\cdot\delta\boldsymbol{u}^{n+1,k+1}\right\|_{L^2(E_i)}^2 + \sum_{i=1}^{N_\Omega}\frac{1}{2\varepsilon_i}\gamma_i^2\left\|\delta\tilde{m}^{n,k+1}\right\|_{L^2(E_i)}^2 &\end{aligned}$$

  by Young's inequality. For each $\varepsilon_i = \lambda_i + L_i$, we obtain,

$$\sum_{i=1}^{N_\Omega} 4G_i\left\|\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_h^{n+1,k+1})\right\|_{L^2(E_i)}^2 + \sum_{i=1}^{N_\Omega}(\lambda_i + L_i)\left\|\nabla\cdot\delta\boldsymbol{u}_h^{n+1,k+1}\right\|_{L^2(E_i)}^2 \le \sum_{i=1}^{N_\Omega}\frac{\gamma_i^2}{\lambda_i + L_i}\left\|\delta\tilde{m}^{n,k+1}\right\|_{L^2(E_i)}^2 . \tag{4.4}$$

- **Step (2): Flow equations**
  Considering (4.2) for the difference between two consecutive coupling iterations, testing with $\theta_h = \delta p_h^{n+1,k+1}$, and multiplying by Δt, we get: (recall $\beta_i = \frac{1}{M_i} + c_{f\,i}\varphi_{0i}$)

$$\sum_{i=1}^{N_\Omega} \beta_i \left\| \delta p_h^{n+1,k+1} \right\|_{L^2(E_i)}^2 + \sum_{i=1}^{N_\Omega} \Delta t \big(\nabla \cdot \delta \boldsymbol{z}_h^{n+1,k+1}, \delta p_h^{n+1,k+1}\big)_{L^2(E_i)} = -\sum_{i=1}^{N_\Omega} \alpha_i \Big(\nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+1}, \delta p_h^{n+1,k+1}\Big)_{L^2(E_i)}. \tag{4.5}$$

  Now, in a similar manner, considering (4.3) for the difference between two consecutive coupling iterations, and testing with $\boldsymbol{q}_h = \delta \boldsymbol{z}_h^{n+1,k+1}$, we get

$$\sum_{i=1}^{N_\Omega} \left\| \boldsymbol{K}_i^{-1/2} \delta \boldsymbol{z}_h^{n+1,k+1} \right\|_{L^2(E_i)}^2 = \sum_{i=1}^{N_\Omega} \Big(\delta p_h^{n+1,k+1}, \nabla \cdot \delta \boldsymbol{z}_h^{n+1,k+1}\Big)_{L^2(E_i)}. \tag{4.6}$$

  Substituting (4.6) into (4.5), we have

$$\sum_{i=1}^{N_\Omega} \beta_i \left\| \delta p_h^{n+1,k+1} \right\|_{L^2(E_i)}^2 + \sum_{i=1}^{N_\Omega} \Delta t \left\| \boldsymbol{K}_i^{-1/2} \delta \boldsymbol{z}_h^{n+1,k+1} \right\|_{L^2(E_i)}^2 + \sum_{i=1}^{N_\Omega} \alpha_i \Big(\nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+1}, \delta p_h^{n+1,k+1}\Big)_{L^2(E_i)} = 0. \tag{4.7}$$

- **Step (3): Combining Mechanics and Flow**
  Adding (4.7) to (4.4), we obtain

$$\sum_{i=1}^{N_\Omega} 4G_i \left\| \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_h^{n+1,k+1}) \right\|_{L^2(E_i)}^2 + \sum_{i=1}^{N_\Omega} \left\{ \beta_i \left\| \delta p_h^{n+1,k+1} \right\|_{L^2(E_i)}^2 + \alpha_i \Big(\nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+1}, \delta p_h^{n+1,k+1}\Big)_{L^2(E_i)} + (\lambda_i + L_i) \left\| \nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+1} \right\|_{L^2(E_i)}^2 \right\} + \sum_{i=1}^{N_\Omega} \Delta t \left\| \boldsymbol{K}_i^{-1/2} \delta \boldsymbol{z}_h^{n+1,k+1} \right\|_{L^2(E_i)}^2 \le \sum_{i=1}^{N_\Omega} \frac{\gamma_i^2}{\lambda_i + L_i} \left\| \delta \tilde{m}^{n,k+1} \right\|_{L^2(E_i)}^2. \tag{4.8}$$

- **Step (4): Identifying the parameters**
  Below, the procedure for determining the two adjustable parameters (γ and L) is illustrated. These parameters should be chosen such that the terms on the left hand side of (4.8) remain positive and the scheme contracts on $\tilde{m}$. Expanding the L^2 norm of $\tilde{m}$ for each E_i, we have:

$$\left\|\delta\tilde{m}^{n+1,k+1}\right\|^2_{L^2(E_i)} = \frac{L_i^2}{\gamma_i^2}\left\|\nabla\cdot\delta\boldsymbol{u}^{n+1,k+1}\right\|^2_{L^2(E_i)} + \frac{\alpha_i^2}{\gamma_i^2}\left\|\delta p^{n+1,k+1}\right\|^2_{L^2(E_i)} + \frac{2\alpha_i L_i}{\gamma_i^2}\left(\delta p^{n+1,k+1}, \nabla\cdot\delta\boldsymbol{u}^{n+1,k+1}\right)_{L^2(E_i)}.$$

Matching coefficients by comparing with the terms in the curly brackets in (4.8) provides us with the following conditions:

$$\frac{L_i^2}{\gamma_i^2} \le (\lambda_i + L_i), \qquad \frac{\alpha_i^2}{\gamma_i^2} \le \beta_i, \qquad \frac{2\alpha_i L_i}{\gamma_i^2} = \alpha_i.$$

The third equality gives, $L_i = \frac{\gamma_i^2}{2}$. The first inequality translate to $\lambda_i + \frac{L_i}{2} \ge 0$ which is trivially satisfied. The second inequality sets a lower bound on the value of the regularization term as: $L_i \ge \frac{\alpha_i^2}{2\beta_i}$. The upper bound on L_i comes from the contraction coefficient condition $\frac{2L_i}{\lambda_i + L_i} < 1$ as $L_i < \lambda_i$. Thus, for the scheme to be contractive, we need $\frac{\alpha_i^2}{2\beta_i} \le L_i < \lambda_i$. Therefore, we have the following theorem:

Theorem 4.2 *[Localized Single Rate Banach Contraction Estimate] For* $\beta_i = \frac{1}{M_i} + c_{f_i}\varphi_{0i}$, $\frac{{\alpha_i}^2}{2\beta_i} \le L_i < \lambda_i$, $\tilde{m}^{n,k} = \left(\frac{L_i}{2}\right)^{(1/2)}\nabla\cdot\boldsymbol{u}^{n,k} + \left(\frac{\alpha_i^2}{2L_i}\right)^{(1/2)} p^{n,k}$, *for each* $E_i \in \Omega, 1 \le i \le N_\Omega$, *the localized undrained split iterative coupling scheme is a contraction given by*

$$\sum_{i=1}^{N_\Omega} 4G_i\left\|\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_h^{n+1,k})\right\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\left(\beta_i - \frac{{\alpha_i}^2}{2L_i}\right)\left\|\delta p_h^{n+1,k}\right\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\left(\lambda_i + \frac{L_i}{2}\right)\left\|\nabla\cdot\delta\boldsymbol{u}_h^{n+1,k}\right\|^2_{L^2(E_i)}$$
$$+\Delta t\sum_{i=1}^{N_\Omega}\left\|\boldsymbol{K}_i{}^{-1/2}\delta\boldsymbol{z}_h^{n+1,k}\right\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\left\|\delta\tilde{m}^{n+1,k+1}\right\|^2_{L^2(E_i)} \le \max_{1\le i\le N_\Omega}\left(\frac{2L_i}{\lambda_i + L_i}\right)\sum_{i=1}^{N_\Omega}\left\|\delta\tilde{m}^{n,k+1}\right\|^2_{L^2(E_i)}. \tag{4.9}$$

Remark 4.3 *For* $L_i < \lambda_i$, $\frac{2L_i}{\lambda_i + L_i} < 1$ *for all grid elements* $E_i \in \Omega, 1 \le i \le N_\Omega$ *ensuring the contraction of the scheme.*

4.2 Localized Fully Discrete Weak Formulation for the Multirate Case:

Following the same approach as in the single rate case (and outlined in [5]), the localized fully discrete weak formulation for the multirate undrained split iterative coupling scheme is as follows: Step (a): Given $p_h^{n,k+q}$ and $\boldsymbol{u}_h^{n,k+q}$ from the last iterative coupling iteration, find $\boldsymbol{u}_h^{n+1,k+q} \in \boldsymbol{V}_h$ such that

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h\,,\; 2\sum_{i=1}^{N_\Omega}\big(G_i\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1,k+q}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\big((\lambda_i + L_i)\nabla\cdot\boldsymbol{u}_h^{n+1,k+q}, \nabla\cdot\boldsymbol{v}_h\big)_{L^2(E_i)} = \sum_{i=1}^{N_\Omega}\big(\alpha_i p_h^{n,k+q}, \nabla\cdot\boldsymbol{v}_h\big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\big(L_i\nabla\cdot\boldsymbol{u}_h^{n,k+q}, \nabla\cdot\boldsymbol{v}_h\big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\big(\boldsymbol{f}_h, \boldsymbol{v}_h\big)_{L^2(E_i)}. \tag{4.10}$$

Step (b): For $1 \leq m \leq q$, given $\boldsymbol{u}_h^{n+1,k+q}$, find $p_h^{n+1,m+k} \in Q_h$, and $\boldsymbol{z}_h^{n+1,m+k} \in \boldsymbol{Z}_h$ such that,

$$\forall \theta_h \in Q_h\,,\; \frac{1}{\Delta t}\sum_{i=1}^{N_\Omega}\Big(\big(\frac{1}{M_i} + c_{f\,i}\varphi_{0i}\big)\Big(p_h^{n+1,m+k} - p_h^{n+1,m-1+k}\Big), \theta_h\Big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}(\nabla\cdot\boldsymbol{z}_h^{n+1,m+k}, \theta_h)_{L^2(E_i)} = -\frac{1}{\Delta t}\sum_{i=1}^{N_\Omega}\Big(\frac{\alpha_i}{q}\nabla\cdot\Big(\boldsymbol{u}_h^{n,k+q} - \boldsymbol{u}_h^{n,k}\Big), \theta_h\Big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\Big(\tilde{q}_h, \theta_h\Big)_{L^2(E_i)}, \tag{4.11}$$

$\forall \boldsymbol{q}_h \in \boldsymbol{Z}_h$,

$$\sum_{i=1}^{N_\Omega}\Big(\boldsymbol{K}_i^{-1}\boldsymbol{z}_h^{n+1,m+k}, \boldsymbol{q}_h\Big)_{L^2(E_i)} = \sum_{i=1}^{N_\Omega}\Big(p_h^{n+1,m+k}, \nabla\cdot\boldsymbol{q}_h\Big)_{L^2(E_i)} + \sum_{i=1}^{N_\Omega}\Big(\rho_{f,r}g\nabla\eta, \boldsymbol{q}_h\Big)_{L^2(E_i)}, \tag{4.12}$$

4.2.1 Proof of Contraction for the Localized Multirate Case:

We define the localized quantity of contraction in this case as:

$$\tilde{m}_q^{n+1,k+m}|_{E_i} = \frac{L_i}{\gamma_i q}\nabla\cdot\boldsymbol{u}^{n+1,k+q} + \frac{\alpha_i}{\gamma_i}(p^{n+1,k+m} - p^{n+1,k+m-1}),\ \text{for } 1 \leq m \leq q,$$

where γ_i is an adjustable parameter to be determined such that the scheme contracts on $\tilde{m}_q|_{E_i}$ for each grid E_i.

- **Step (1): Elasticity equation**
  Considering (4.10) for the difference between two consecutive coupling iterations, and testing

with $\boldsymbol{v}_h = \delta \boldsymbol{u}_h^{n+1,k+q}$, we get:

$$\begin{aligned}
&\sum_{i=1}^{N_\Omega} 2G_i \big\| \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_h^{n+1,k+q}) \big\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega} (\lambda_i + L_i) \big\| \nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q} \big\|^2_{L^2(E_i)} \\
&\qquad = \sum_{i=1}^{N_\Omega} \big(\alpha_i \delta p_h^{n,k+q} + L_i \nabla \cdot \delta \boldsymbol{u}_h^{n,k+q}, \nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q} \big)_{L^2(E_i)} \\
&\qquad = \sum_{i=1}^{N_\Omega} \Big(\sum_{m=1}^{q} \Big(\alpha_i \big(\delta p_h^{n,m+k} - \delta p_h^{n,m-1+k} \big) + \frac{L_i}{q} \nabla \cdot \delta \boldsymbol{u}_h^{n,k+q} \Big), \nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q} \Big)_{L^2(E_i)} \\
&\qquad \le \sum_{i=1}^{N_\Omega} \frac{\varepsilon_i}{2} \big\| \nabla \cdot \delta \boldsymbol{u}^{n+1,k+q} \big\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega} \frac{1}{2\varepsilon_i} \gamma_i^2 \sum_{m=1}^{q} \big\| \delta \tilde{m}_q^{n,k+m} \big\|^2_{L^2(E_i)} .
\end{aligned}$$

by noting that $\sum_{m=1}^{q} \Big(\delta p_h^{n,m+k} - \delta p_h^{n,m-1+k} \Big) = \delta p_h^{n,k+q}$ and using Young's inequality. For each E_i, choose $\varepsilon_i = \lambda_i + L_i$ to obtain:

$$\sum_{i=1}^{N_\Omega} 4G_i \big\| \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_h^{n+1,k+q}) \big\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega} (\lambda_i + L_i) \big\| \nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q} \big\|^2_{L^2(E_i)} \le \sum_{i=1}^{N_\Omega} \frac{\gamma_i^2}{\lambda_i + L_i} \sum_{m=1}^{q} \big\| \delta \tilde{m}_q^{n,k+m} \big\|^2_{L^2(E_i)} . \tag{4.13}$$

- **Step (2): Flow equations**
Considering (4.11) for the difference between two consecutive coupling iterations, testing with $\theta_h = \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k}$, and multiplying by Δt, we get: (recall $\beta_i = \frac{1}{M_i} + c_{f\,i} \varphi_{0i}$ for each E_i)

$$\begin{aligned}
&\sum_{i=1}^{N_\Omega} \beta_i \Big\| \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \Big\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega} \Delta t \big(\nabla \cdot \delta \boldsymbol{z}_h^{n+1,m+k}, \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \big)_{L^2(E_i)} = \\
&- \sum_{i=1}^{N_\Omega} \frac{\alpha_i}{q} \Big(\nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q}, \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \Big)_{L^2(E_i)} .
\end{aligned} \tag{4.14}$$

Now, consider (4.12) for two consecutive local flow finer time steps, $t = t_{m+k}$, and $t = t_{m-1+k}$, and test with $\boldsymbol{q}_h = \delta \boldsymbol{z}_h^{n+1,m+k}$ to obtain

$$\sum_{i=1}^{N_\Omega} \Big(\boldsymbol{K}_i^{-1} \Big(\delta \boldsymbol{z}_h^{n+1,m+k} - \delta \boldsymbol{z}_h^{n+1,m-1+k} \Big), \delta \boldsymbol{z}_h^{n+1,m+k} \Big)_{L^2(E_i)} = \sum_{i=1}^{N_\Omega} \Big(\delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k}, \nabla \cdot \delta \boldsymbol{z}_h^{n+1,m+k} \Big)_{L^2(E_i)}. \tag{4.15}$$

Substituting (4.15) into (4.14), we have

$$\sum_{i=1}^{N_\Omega} \beta_i \Big\| \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \Big\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega} \Delta t \Big(\boldsymbol{K}_i^{-1} \Big(\delta \boldsymbol{z}_h^{n+1,m+k} - \delta \boldsymbol{z}_h^{n+1,m-1+k} \Big), \delta \boldsymbol{z}_h^{n+1,m+k} \Big)_{L^2(E_i)} = - \sum_{i=1}^{N_\Omega} \frac{\alpha_i}{q} \Big(\nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q}, \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \Big)_{L^2(E_i)}.$$

By Young's inequality, with further simplifications,

$$\sum_{i=1}^{N_\Omega} \beta_i \Big\| \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \Big\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega} \frac{\alpha_i}{q} \Big(\nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q}, \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \Big)_{L^2(E_i)}$$
$$\sum_{i=1}^{N_\Omega} \frac{\Delta t}{2} \Big(\Big\| \boldsymbol{K}_i^{-1/2} \delta \boldsymbol{z}_h^{n+1,m+k} \Big\|^2_{L^2(E_i)} - \Big\| \boldsymbol{K}_i^{-1/2} \delta \boldsymbol{z}_h^{n+1,m-1+k} \Big\|^2_{L^2(E_i)} + \Big\| \boldsymbol{K}_i^{-1/2} (\delta \boldsymbol{z}_h^{n+1,m+k} - \delta \boldsymbol{z}_h^{n+1,m-1+k}) \Big\|^2_{L^2(E_i)} \Big) = 0.$$

Summing for q local flow time steps and after some simplifications (telescopic cancellations together with the fact that $\delta \boldsymbol{z}_h^{n+1,k} = 0$), we get

$$\sum_{i=1}^{N_\Omega} \sum_{m=1}^{q} \Big(\beta_i \Big\| \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \Big\|^2_{L^2(E_i)} + \frac{\alpha_i}{q} \Big(\nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q}, \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \Big)_{L^2(E_i)} \Big)$$
$$\sum_{i=1}^{N_\Omega} \frac{\Delta t}{2} \Big\| \boldsymbol{K}_i^{-1/2} \delta \boldsymbol{z}_h^{n+1,k+q} \Big\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega} \frac{\Delta t}{2} \sum_{m=1}^{q} \Big\| \boldsymbol{K}_i^{-1/2} \Big(\delta \boldsymbol{z}_h^{n+1,m+k} - \delta \boldsymbol{z}_h^{n+1,m-1+k} \Big) \Big\|^2_{L^2(E_i)} = 0. \tag{4.16}$$

- **Step (3): Combining Mechanics and Flow**
  Adding (4.16) to (4.13), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N_\Omega} 4G_i \left\| \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_h^{n+1,k+q}) \right\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega} \sum_{m=1}^{q} \Bigg\{ \beta_i \left\| \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \right\|^2_{L^2(E_i)} \\
&+ \frac{\alpha_i}{q} \Big(\nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q}, \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \Big)_{L^2(E_i)} + \frac{\lambda_i + L_i}{q} \left\| \nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q} \right\|^2_{L^2(E_i)} \Bigg\} \\
&+ \sum_{i=1}^{N_\Omega} \frac{\Delta t}{2} \left\| \boldsymbol{K}_i^{-1/2} \delta \boldsymbol{z}_h^{n+1,k+q} \right\|^2_{L^2(E_i)} + \sum_{i=1}^{N_\Omega} \frac{\Delta t}{2} \sum_{m=1}^{q} \left\| \boldsymbol{K}_i^{-1/2} \Big(\delta \boldsymbol{z}_h^{n+1,m+k} - \delta \boldsymbol{z}_h^{n+1,m-1+k} \Big) \right\|^2_{L^2(E_i)} \\
&\leq \sum_{i=1}^{N_\Omega} \frac{\gamma_i^2}{\lambda_i + L_i} \sum_{m=1}^{q} \left\| \delta \tilde{m}_q^{n,k+m} \right\|^2_{L^2(E_i)} .
\end{aligned}
\tag{4.17}
$$

- **Step (4): Identifying the parameters**
  For each E_i, we should choose the parameters γ_i, and L_i such that the terms on the left hand side of (4.17) remain positive, and the scheme achieves contraction on $\tilde{m}_q|_{E_i}$. Clearly,

$$
\begin{aligned}
\left\| \delta \tilde{m}_q^{n+1,k+m} \right\|^2_{L^2(E_i)} &= \frac{L_i^2}{q^2 \gamma_i^2} \left\| \nabla \cdot \delta \boldsymbol{u}^{n+1,k+q} \right\|^2_{L^2(E_i)} + \frac{\alpha_i^2}{\gamma_i^2} \left\| (p^{n+1,k+m} - p^{n+1,k+m-1}) \right\|^2_{L^2(E_i)} \\
&+ \frac{2\alpha_i L_i}{\gamma_i^2 q} \left((p^{n+1,k+m} - p^{n+1,k+m-1}), \nabla \cdot \delta \boldsymbol{u}^{n+1,k+q} \right)_{L^2(E_i)} .
\end{aligned}
$$

  Matching coefficients by comparing with the coefficients of the terms in the curly brackets in (4.17), we obtain $\frac{L_i^2}{q^2\gamma_i^2} \leq \frac{\lambda_i + L_i}{q}$, $\frac{\alpha_i^2}{\gamma_i^2} \leq \beta_i$, and $\frac{2\alpha_i L_i}{\gamma_i^2 q} = \frac{\alpha_i}{q}$. The last equality gives $\gamma_i^2 = 2L_i$. Substituting in the first inequality, we obtain the condition $2q(\lambda_i + L_i) \geq L_i$, which is trivially satisfied for $q \geq 1, \lambda_i \geq 0$, and $L_i \geq 0$. The second inequality gives a lower bound on the value of L_i as $L_i \geq \frac{\alpha_i^2}{2\beta_i}$. Moreover, For the scheme to be contractive, we require that $\frac{\gamma_i^2}{\lambda_i + L_i} < 1$. This gives the following upper bound on L_i as $L_i < \lambda_i$. Therefore, we have the following combined condition $\frac{\alpha_i^2}{2\beta_i} \leq L_i < \lambda_i$ for each grid element E_i. Our main result summarizes the above contraction result.

Theorem 4.4 *[Localized Multirate Banach Contraction Estimate] For* $\beta_i = \frac{1}{M_i} + c_{f\,i}\varphi_{0i}$, $\frac{{\alpha_i}^2}{2\beta_i} \leq L_i < \lambda_i$, $\tilde{m}_q^{n,k+m} = \left(\frac{L_i}{2q^2}\right)^{(1/2)} \nabla \cdot \boldsymbol{u}^{n,k+q} + \left(\frac{\alpha_i^2}{2L_i}\right)^{(1/2)} (p^{n,k+m} - p^{n,k+m-1})$ *for* $1 \leq m \leq q$, *and for each* $E_i \in \Omega, 1 \leq i \leq N_\Omega$, *the localized multirate undrained split iterative coupling scheme, in which the* *flow fine time step is Δt, and the coarse mechanics time step is $q\Delta t$ (i.e., q flow fine time steps are solved within one coarse mechanics time step), is a contraction given by*

$$\begin{aligned}
&\sum_{i=1}^{N_\Omega} 4G_i \left\| \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_h^{n+1,k+q}) \right\|^2_{L^2(E_i)} + \sum_{m=1}^{q}\sum_{i=1}^{N_\Omega} \left(\beta_i - \frac{{\alpha_i}^2}{2L_i}\right) \left\| \delta p_h^{n+1,m+k} - \delta p_h^{n+1,m-1+k} \right\|^2_{L^2(E_i)} \\
&\quad + \sum_{i=1}^{N_\Omega} \left(\lambda_i + \left(\frac{2q-1}{2q}\right) L_i\right) \left\| \nabla \cdot \delta \boldsymbol{u}_h^{n+1,k+q} \right\|^2_{L^2(E_i)} + \Delta t \sum_{i=1}^{N_\Omega} \frac{1}{2} \left\| {\boldsymbol{K}_i}^{-1/2} \delta \boldsymbol{z}_h^{n+1,k+q} \right\|^2_{L^2(E_i)} \\
&\quad + \Delta t \sum_{m=1}^{q}\sum_{i=1}^{N_\Omega} \frac{1}{2} \left\| {\boldsymbol{K}_i}^{-1/2} (\delta \boldsymbol{z}_h^{n+1,m+k} - \delta \boldsymbol{z}_h^{n+1,m-1+k}) \right\|^2_{L^2(E_i)} \\
&\quad + \sum_{m=1}^{q}\sum_{i=1}^{N_\Omega} \left\| \delta \tilde{m}^{n+1,m+k} \right\|^2_{L^2(E_i)} \leq \max_{1\leq i\leq N_\Omega} \left(\frac{2L_i}{\lambda_i + L_i}\right) \sum_{m=1}^{q}\sum_{i=1}^{N_\Omega} \left\| \delta \tilde{m}^{n,m+k} \right\|^2_{L^2(E_i)}. \qquad (4.18)
\end{aligned}$$

Remark 4.5 *As in the single rate case, for $L_i < \lambda_i$, $\dfrac{2L_i}{\lambda_i + L_i} < 1$ for all grid elements $E_i \in \Omega, 1 \leq i \leq N_\Omega$, thus the scheme always contracts. Moreover, For $q = 1$, we retrieve the single rate result.*

Remark 4.6 *For both the single rate and multirate cases, the condition on L_i translates to the following condition:*

$$\frac{{\alpha_i}^2}{2\lambda_i} < \left(\frac{1}{M_i} + {c_f}_i \varphi_{0i}\right) \text{ for each } E_i. \qquad (4.19)$$

Therefore, in heterogenous poroelastic media, sufficient fluid compressibility is needed for the undrained split scheme to be convergent. This is not the case for the fixed-stress split scheme, which implies that the fixed-stress scheme is more robust. Moreover, By comparing the rates of convergence (i.e. the contraction coefficient values) of the undrained split scheme versus the fixed stress split scheme (as derived in the work of [5]), we can see that fixed stress coefficient is sharper (i.e., smaller). This is confirmed by the work of [27, 33] in which the numerical efficiency of these two schemes were compared against the Mandel problem. In this work, the fixed stress split scheme converged in fewer coupling iterations compared to the undrained split scheme (the number of iterative coupling iterations dropped down to 2 after 10 time steps for the fixed stress split scheme, whereas it took the undrained split scheme more than 5 iterative coupling iterations to converge for the first 1100 time steps).

5 A Priori Error Estimates for the Single Rate Undrained Split Iterative Coupling Scheme

In this section, we derive an a priori error estimate for the single rate undrained split coupling scheme in homogeneous poro-elastic media. The derivation of a priori error estimates in heterogeneous media is more involved and will be considered in a future work. We will use the same notation as the one used before, except that we will be de-associating viscosity μ from permeability $\boldsymbol{K}$ (as this is no longer needed for this kind of analysis).

In deriving our a priori error estimates for the undrained split scheme, we will follow a similar approach as outlined in the work of [6]. For a particular time step $t = t_k$, and a specific iterative coupling iteration $n \geq 0$, we seek an estimate of the form $\left\|\xi_h^{n,k} - \xi(t_k)\right\|$, where ξ denotes $p_h, \boldsymbol{z}_h$, or $\boldsymbol{u}_h$. The triangle inequality gives us

$$\left\|\xi_h^{n,k} - \xi(t_k)\right\| \leq \left\|\xi_h^{n,k} - \xi_h^k\right\| + \left\|\xi_h^k - \xi(t_k)\right\|$$

where ξ_h^k is the solution obtained by the fully coupled scheme (we assume that the iterative coupling scheme will converge to the solution of the simultaneously coupled scheme, also known as the fully implicit or simultaneously coupled scheme). The work of [29, 30] derived a priori error estimates for the simultaneously coupled scheme; therefore, it remains to derive an error estimate for $\left\|\xi_h^{n,k} - \xi_h^k\right\|$. Estimating this term is done in two steps:

- **Step (1):** Deriving a Banach contraction result on $\left\|\xi_h^{n,k} - \xi_h^k\right\|$.
- **Step (2):** Deriving stability estimates for the simultaneously coupled scheme $\left\|\xi_h^k - \xi_h^{k-1}\right\|$

Step 1: Banach Contraction Estimate on $\left\|\xi_h^{n,k} - \xi_h^k\right\|$
We first define the error terms as follows:

$$e_p^n = p_h^{n,k} - p_h^k, \;\; e_u^n = \boldsymbol{u}_h^{n,k} - \boldsymbol{u}_h^k, \;\; e_z^n = \boldsymbol{z}_h^{n,k} - \boldsymbol{z}_h^k.$$

The weak formulation of the fully-coupled scheme is given as follows:

Find $p_h^k \in Q_h$, $\boldsymbol{z}_h^k \in \boldsymbol{Z}_h$, and $\boldsymbol{u}_h^k \in \boldsymbol{V}_h$ such that,

$$\forall \boldsymbol{v}_h \in V_h\,,\; 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^k), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big)_{L^2(\Omega)} + \lambda\big(\nabla \cdot \boldsymbol{u}_h^k, \nabla \cdot \boldsymbol{v}_h\big)_{L^2(\Omega)} - \alpha\big(p_h^k, \nabla \cdot \boldsymbol{v}_h\big)_{L^2(\Omega)} = \big(\boldsymbol{f}_h^k, \boldsymbol{v}_h\big)_{L^2(\Omega)}, \tag{5.1}$$

$$\begin{aligned}\forall \theta_h \in Q_h\,,\; \frac{1}{\Delta t}\Big(\frac{1}{M} + c_f\varphi_0\Big)\Big(\Big(p_h^k - p_h^{k-1}\Big), \theta_h\Big)_{L^2(\Omega)} + \frac{1}{\mu_f}\Big(\nabla \cdot \boldsymbol{z}_h^k, \theta_h\Big)_{L^2(\Omega)} \\ = -\frac{\alpha}{\Delta t}\Big(\nabla \cdot \Big(\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}\Big), \theta_h\Big)_{L^2(\Omega)} + \Big(\tilde{q}_h, \theta_h\Big)_{L^2(\Omega)},\end{aligned} \tag{5.2}$$

$$\forall \boldsymbol{q}_h \in \boldsymbol{Z}_h\,,\; \Big(\boldsymbol{K}^{-1}\boldsymbol{z}_h^k, \boldsymbol{q}_h\Big)_{L^2(\Omega)} = \Big(p_h^k, \nabla \cdot \boldsymbol{q}_h\Big)_{L^2(\Omega)} + \Big(\rho_{f,r} g \nabla \eta, \boldsymbol{q}_h\Big)_{L^2(\Omega)}. \tag{5.3}$$

In addition, the weak formulation of the iteratively coupled scheme reads

Step (a): Given $p_h^{n,k}$ and $\boldsymbol{u}^{n,k}$ from the previous coupling iteration, find $\boldsymbol{u}_h^{n+1,k} \in \boldsymbol{V}_h$ such that

$$\begin{aligned}\forall \boldsymbol{v}_h \in \boldsymbol{V}_h\,,\ 2G\big(\boldsymbol{\varepsilon}({\boldsymbol{u}_h}^{n+1,k}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big)_{L^2(\Omega)} + (\lambda + L)\big(\nabla\cdot \boldsymbol{u}_h^{n+1,k}, \nabla\cdot \boldsymbol{v}_h\big)_{L^2(\Omega)} = \\ \alpha\big(p_h^{n,k}, \nabla\cdot \boldsymbol{v}_h\big)_{L^2(\Omega)} + L\big(\nabla\cdot \boldsymbol{u}_h^{n,k}, \nabla\cdot \boldsymbol{v}_h\big)_{L^2(\Omega)} + \big(\boldsymbol{f}, \boldsymbol{v}_h\big)_{L^2(\Omega)} \qquad (5.4)\end{aligned}$$

Step (b): Given $\boldsymbol{u}_h^{n+1,k}$, find $p_h^{n+1,k} \in Q_h$, $\boldsymbol{z}_h^{n+1,k} \in \boldsymbol{Z}_h$ such that:

$$\begin{aligned}\forall \theta_h \in Q_h\,,\ \big(\frac{1}{M} + c_f\varphi_0\big)\Big(\frac{p_h^{n+1,k} - p_h^{k-1}}{\Delta t}, \theta_h\Big)_{L^2(\Omega)} + \frac{1}{\mu_f}\big(\nabla\cdot \boldsymbol{z}_h^{n+1,k}, \theta_h\big)_{L^2(\Omega)} \\ = -\alpha\Big(\nabla\cdot\Big(\frac{\boldsymbol{u}_h^{n+1,k} - \boldsymbol{u}_h^{k-1}}{\Delta t}\Big), \theta_h\Big)_{L^2(\Omega)} + \big(\tilde{q}, \theta_h\big)_{L^2(\Omega)} \qquad (5.5)\end{aligned}$$

$$\forall \boldsymbol{q}_h \in \boldsymbol{Z}_h\,,\ \big(\boldsymbol{K}^{-1}\boldsymbol{z}_h^{n+1,k}, \boldsymbol{q}_h\big)_{L^2(\Omega)} = (p_h^{n+1,k}, \nabla\cdot \boldsymbol{q}_h)_{L^2(\Omega)} + \big(\nabla(\rho_{f,r} g\eta), \boldsymbol{q}_h)_{L^2(\Omega)} \qquad (5.6)$$

By taking the difference of the weak formulations for the iterative coupling scheme (equations (5.4), (5.5), and (5.6)), and the fully-coupled scheme (equations (5.1), (5.2), and (5.3)), the terms at time step t^{k-1} get cancelled as we assumed that the iteratively coupled scheme at the previous time step ($t = t^{k-1}$) will converge to the solution obtained by fully coupled scheme (i.e., the iteration is carried out to full convergence such that the error with respect to the solution obtained by the fully coupled scheme is equal to zero). Under this assumption, the weak formulation (mixed formulation for flow, CG for mechanics) in terms of the error terms defined above can be written as:

$$\forall \theta_h \in Q_h\,,\ \frac{1}{\Delta t}\big(\frac{1}{M} + c_f\varphi_0\big)\Big(e_p^{n+1}, \theta_h\Big)_{L^2(\Omega)} + \frac{1}{\mu_f}\Big(\nabla\cdot e_z^{n+1}, \theta_h\Big)_{L^2(\Omega)} = \frac{1}{\Delta t}\Big(-\alpha\nabla\cdot e_u^{n+1}, \theta_h\Big)_{L^2(\Omega)}, \qquad (5.7)$$

$$\forall \boldsymbol{q}_h \in \boldsymbol{Z}_h\,,\ \Big(\boldsymbol{K}^{-1}e_z^{n+1}, \boldsymbol{q}_h\Big)_{L^2(\Omega)} = \Big(e_p^{n+1}, \nabla\cdot \boldsymbol{q}_h\Big)_{L^2(\Omega)}, \qquad (5.8)$$

$$\begin{aligned}\forall \boldsymbol{v}_h \in V_h\,,\ 2G\big(\boldsymbol{\varepsilon}(e_u^{n+1}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big)_{L^2(\Omega)} + (\lambda + L)\big(\nabla\cdot e_u^{n+1}, \nabla\cdot \boldsymbol{v}_h\big)_{L^2(\Omega)} \\ = \alpha\big(e_p^n, \nabla\cdot \boldsymbol{v}_h\big)_{L^2(\Omega)} + L\big(\nabla\cdot e_u^{n+1}, \nabla\cdot \boldsymbol{v}_h\big)_{L^2(\Omega)}. \qquad (5.9)\end{aligned}$$

We also define the quantity of contraction as

$$e_m^n = \frac{\alpha}{\gamma}e_p^n + \frac{L}{\gamma}\nabla\cdot e_u^n$$

The parameters L and γ are adjustable parameters, which will be determined by the proof of contraction. Now, we follow a similar argument as in the work of [26] to derive a Banach contraction result as follows. First, consider the elasticity equation (5.9) and test with $\boldsymbol{v}_h = e_u^{n+1}$, to get:

$$\begin{aligned} 2G\|\boldsymbol{\varepsilon}(e_u^{n+1})\|^2_{L^2(\Omega)} + (\lambda + L)\|\nabla \cdot e_u^{n+1}\|^2_{L^2(\Omega)} &= (\alpha e_p^n + L\nabla \cdot e_u^n, \nabla \cdot e_u^{n+1})_{L^2(\Omega)} \\ &= (e_m^n, \nabla \cdot e_u^{n+1})_{L^2(\Omega)} \\ &\leq \frac{\epsilon}{2}\|\nabla \cdot e_u^{n+1}\|^2_{L^2(\Omega)} + \frac{\gamma^2}{2\epsilon}\|e_m^n\|^2_{L^2(\Omega)} \end{aligned}$$

by Young's inequality. For $\epsilon = \lambda + L$, we get:

$$\frac{4G}{\lambda + L}\|\boldsymbol{\varepsilon}(e_u^{n+1})\|^2_{L^2(\Omega)} + \|\nabla \cdot e_u^{n+1}\|^2_{L^2(\Omega)} \leq \frac{\gamma^2}{(\lambda + L)^2}\|e_m^n\|^2_{L^2(\Omega)}. \tag{5.10}$$

Next, consider the flow equation (5.7), and test with $\theta_h = e_p^{n+1}$ to get (recall that $\beta = \frac{1}{M} + c_f\varphi_0$):

$$\beta\|e_p^{n+1}\|^2_{L^2(\Omega)} + \frac{\Delta t}{\mu_f}(\nabla \cdot e_z^{n+1}, e_p^{n+1})_{L^2(\Omega)} = -\alpha(\nabla \cdot e_u^{n+1}, e_p^{n+1})_{L^2(\Omega)}. \tag{5.11}$$

In a similar way, testing (5.8) with $\boldsymbol{q}_h = e_z^{n+1}$ yields:

$$\|\boldsymbol{K}^{-1/2}e_z^{n+1}\|^2_{L^2(\Omega)} = \left(e_p^{n+1}, \nabla \cdot e_z^{n+1}\right)_{L^2(\Omega)}. \tag{5.12}$$

Combining (5.11) with (5.12) gives:

$$\beta\|e_p^{n+1}\|^2_{L^2(\Omega)} + \frac{\Delta t}{\mu_f}\|\boldsymbol{K}^{-1/2}e_z^{n+1}\|^2_{L^2(\Omega)} + \alpha(\nabla \cdot e_u^{n+1}, e_p^{n+1})_{L^2(\Omega)} = 0. \tag{5.13}$$

Multiplying the mechanics equation (5.10) by a free parameter ($c^2 > 0$) and adding it to the flow equation (5.13) yield:

$$\begin{aligned} \frac{4G}{\lambda + L}\|\boldsymbol{\varepsilon}(e_u^{n+1})\|^2_{L^2(\Omega)} + \Big\{c^2\beta\|e_p^{n+1}\|^2_{L^2(\Omega)} + \alpha c^2(\nabla \cdot e_u^{n+1}, e_p^{n+1})_{L^2(\Omega)} + \|\nabla \cdot e_u^{n+1}\|^2_{L^2(\Omega)}\Big\} \\ + \frac{c^2\Delta t}{\mu_f}\|\boldsymbol{K}^{-1/2}e_z^{n+1}\|^2_{L^2(\Omega)} \leq \frac{\gamma^2}{(\lambda + L)^2}\|e_m^n\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.14}$$

Now, recalling that $e_m^{n+1} = \frac{\alpha}{\gamma}e_p^{n+1} + \frac{L}{\gamma}\nabla \cdot e_u^{n+1}$, we have:

$$\|e_m^{n+1}\|^2_{L^2(\Omega)} = \frac{\alpha^2}{\gamma^2}\|e_p^{n+1}\|^2_{L^2(\Omega)} + \frac{2\alpha L}{\gamma^2}(e_p^{n+1}, \nabla \cdot e_u^{n+1})_{L^2(\Omega)} + \frac{L^2}{\gamma^2}\|\nabla \cdot e_u^{n+1}\|^2_{L^2(\Omega)}. \tag{5.15}$$

Matching the coefficients on the right hand side of (5.15) with the coefficients in the curly brackets on the left hand side of (5.14), we determine $L = \gamma = \frac{\alpha^2}{\beta}$, and $c^2 = \frac{2}{L}$.

With the above coefficients, we have the following Banach contraction estimate on e_m^n:

$$\Big(\frac{4G}{\lambda+L}\Big)\|\boldsymbol{\varepsilon}(e_u^{n+1})\|^2_{L^2(\Omega)} + \frac{2\beta\Delta t}{\alpha^2\mu_f}\Big\|\boldsymbol{K}^{-1/2}e_z^{n+1}\Big\|^2_{L^2(\Omega)} + \frac{\beta^2}{\alpha^2}\Big\|e_p^{n+1}\Big\|^2_{L^2(\Omega)} + \Big\|e_m^{n+1}\Big\|^2_{L^2(\Omega)} \leq \Big(\frac{L}{\lambda+L}\Big)^2\Big\|e_m^n\Big\|^2_{L^2(\Omega)}. \tag{5.16}$$

with $L = \gamma = \frac{\alpha^2}{\beta}, \beta = \frac{1}{M} + c_f\varphi_0$.

We easily derive for a coupling iteration $n > 0$:

$$\begin{aligned}\Big\|e_m^n\Big\|^2_{L^2(\Omega)} &\leq \Big(\frac{L}{\lambda+L}\Big)^{2(n)}\Big\|\frac{\alpha}{L}e_p^0 + \nabla\cdot e_u^0\Big\|^2_{L^2(\Omega)} \\ &\leq \frac{1}{L^2}\Big(\frac{L}{\lambda+L}\Big)^{2(n)}\Big(\alpha^2\Big\|e_p^0\Big\|^2_{L^2(\Omega)} + L^2\Big\|\nabla\cdot e_u^0\Big\|^2_{L^2(\Omega)} + 2\alpha L(e_p^0,\nabla\cdot e_u^0)_{L^2(\Omega)}\Big) \\ &\leq \frac{1}{L^2}\Big(\frac{L}{\lambda+L}\Big)^{2(n)}\Big(\alpha^2\Big\|e_p^0\Big\|^2_{L^2(\Omega)} + L^2\Big\|\nabla\cdot e_u^0\Big\|^2_{L^2(\Omega)} + 2\alpha L\Big(\frac{1}{2\epsilon}\Big\|e_p^0\Big\|^2_{L^2(\Omega)} + \frac{\epsilon}{2}\Big\|\nabla\cdot e_u^0\Big\|^2_{L^2(\Omega)}\Big)\Big).\end{aligned}$$

for $\epsilon > 0$. Let $\epsilon = 1$ to obtain,

$$\begin{aligned}\Big\|e_m^n\Big\|^2_{L^2(\Omega)} &\leq \frac{1}{L^2}\Big(\frac{L}{\lambda+L}\Big)^{2(n)}\bigg((\alpha^2+\alpha L)\Big\|e_p^0\Big\|^2_{L^2(\Omega)} + (L^2+\alpha L)\Big\|\nabla\cdot e_u^0\Big\|^2_{L^2(\Omega)}\bigg) \\ &\leq \Big(\frac{L}{\lambda+L}\Big)^{2(n-1)}\bigg(\frac{(\alpha^2+\alpha L)}{(\lambda+L)^2}\Big\|e_p^0\Big\|^2_{L^2(\Omega)} + \frac{(L^2+\alpha L)}{(\lambda+L)^2}\Big\|\nabla\cdot e_u^0\Big\|^2_{L^2(\Omega)}\bigg)\end{aligned}$$

Therefore, we can write:

$$\begin{aligned}&\Big\|e_p^{n+1}\Big\|^2_{L^2(\Omega)} + \|\boldsymbol{\varepsilon}(e_u^{n+1})\|^2_{L^2(\Omega)} + \Big\|\boldsymbol{K}^{-1/2}e_z^{n+1}\Big\|^2_{L^2(\Omega)} \leq \\ &\qquad \Big(\frac{L}{\lambda+L}\Big)^{(2n)}C_1\bigg(\frac{(\alpha^2+\alpha L)}{(\lambda+L)^2}\Big\|e_p^0\Big\|^2_{L^2(\Omega)} + \frac{(L^2+\alpha L)}{(\lambda+L)^2}\Big\|\nabla\cdot e_u^0\Big\|^2_{L^2(\Omega)}\bigg).\end{aligned} \tag{5.17}$$

where $C_1 = \Big[\frac{\lambda+L}{4G} + \frac{\alpha^2}{\beta^2} + \frac{\alpha^2\mu_f}{2\beta\Delta t}\Big]$.

Noting that: $e_p^0 = p_h^{0,k} - p_h^k = p_h^{k-1} - p_h^k$, $e_z^0 = \boldsymbol{z}_h^{0,k} - \boldsymbol{z}_h^k = \boldsymbol{z}_h^{k-1} - \boldsymbol{z}_h^k$, and $e_u^0 = \boldsymbol{u}_h^{0,k} - \boldsymbol{u}_h^k = \boldsymbol{u}_h^{k-1} - \boldsymbol{u}_h^k$, we finally have:

$$\left\|p_h^{n+1,k}-p_h^k\right\|^2_{L^2(\Omega)}+\left\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1,k}-\boldsymbol{u}_h^k)\right\|^2_{L^2(\Omega)}+\left\|\boldsymbol{K}^{-1/2}(\boldsymbol{z}_h^{n+1,k}-\boldsymbol{z}_h^k)\right\|^2_{L^2(\Omega)}$$
$$\leq\left(\frac{L}{\lambda+L}\right)^{(2n)}\left(\tilde{\eta}_1\left\|p_h^k-p_h^{k-1}\right\|^2_{L^2(\Omega)}+\tilde{\eta}_2\left\|\nabla\cdot\boldsymbol{u}_h^k-\boldsymbol{u}_h^{k-1}\right\|^2_{L^2(\Omega)}\right) \tag{5.18}$$

for $L=\frac{\alpha^2}{(\frac{1}{M}+c_f\varphi_0)}$, $\tilde{\eta}_1=\frac{C_1(\alpha^2+\alpha L)}{(\lambda+L)^2}$, and $\tilde{\eta}_2=\frac{C_1(L^2+\alpha L)}{(\lambda+L)^2}$.

Step 2: Stability estimate on $\left\|\xi_h^k-\xi_h^{k-1}\right\|$

Following the same approach as in [6], we derive the stability estimate for the simultaneously coupled scheme. We first recall the form of the fully coupled scheme as:

Find $p_h^k\in Q_h$, $\boldsymbol{z}_h^k\in\boldsymbol{Z}_h$, and $\boldsymbol{u}_h^k\in\boldsymbol{V}_h$ such that,

$$\forall\theta_h\in Q_h\,,\ \frac{1}{\Delta t}\Big(\big(\frac{1}{M}+c_f\varphi_0\big)\Big(p_h^k-p_h^{k-1}\Big),\theta_h\Big)_{L^2(\Omega)}+\frac{1}{\mu_f}\Big(\nabla\cdot\boldsymbol{z}_h^k,\theta_h\Big)_{L^2(\Omega)}$$
$$=-\frac{\alpha}{\Delta t}\Big(\nabla\cdot\Big(\boldsymbol{u}_h^k-\boldsymbol{u}_h^{k-1}\Big),\theta_h\Big)_{L^2(\Omega)}+\Big(\tilde{q}_h,\theta_h\Big)_{L^2(\Omega)}, \tag{5.19}$$

$$\forall\boldsymbol{q}_h\in\boldsymbol{Z}_h\,,\ \Big(\boldsymbol{K}^{-1}\boldsymbol{z}_h^k,\boldsymbol{q}_h\Big)_{L^2(\Omega)}=\Big(p_h^k,\nabla\cdot\boldsymbol{q}_h\Big)_{L^2(\Omega)}+\Big(\rho_{f,r}g\nabla\eta,\boldsymbol{q}_h\Big)_{L^2(\Omega)}, \tag{5.20}$$

$$\forall\boldsymbol{v}_h\in V_h\,,\ 2G\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^k),\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big)_{L^2(\Omega)}+\lambda\big(\nabla\cdot\boldsymbol{u}_h^k,\nabla\cdot\boldsymbol{v}_h\big)_{L^2(\Omega)}-\alpha\big(p_h^k,\nabla\cdot\boldsymbol{v}_h\big)_{L^2(\Omega)}=\big(\boldsymbol{f}_h^k,\boldsymbol{v}_h\big)_{L^2(\Omega)}. \tag{5.21}$$

Step (a), Flow Equations: Testing (5.19) with $\theta_h=p_h^k-p_h^{k-1}$, we obtain:

$$\beta\left\|p_h^k-p_h^{k-1}\right\|^2_{L^2(\Omega)}+\frac{\Delta t}{\mu_f}\Big(\nabla\cdot\boldsymbol{z}_h^k,p_h^k-p_h^{k-1}\Big)_{L^2(\Omega)}$$
$$=\alpha\Big(\nabla\cdot(\boldsymbol{u}_h^k-\boldsymbol{u}_h^{k-1}),p_h^k-p_h^{k-1}\Big)_{L^2(\Omega)}+\Delta t\Big(\tilde{q}_h,p_h^k-p_h^{k-1}\Big)_{L^2(\Omega)}. \tag{5.22}$$

Now, taking the difference of two consecutive time steps $t=t_k$ and $t=t_{k-1}$ of (5.20), and testing with $\boldsymbol{q}_h=\boldsymbol{z}_h^k$, we derive:

$$\Big(K^{-1}(\boldsymbol{z}_h^k-\boldsymbol{z}_h^{k-1}),\boldsymbol{z}_h^k\Big)_{L^2(\Omega)}=\Big(p_h^k-p_h^{k-1},\nabla\cdot\boldsymbol{z}_h^k\Big)_{L^2(\Omega)}. \tag{5.23}$$

Combining (5.22) with (5.23) (Recall the identity $a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$), we obtain:

$$\beta\left\|p_h^k - p_h^{k-1}\right\|^2_{L^2(\Omega)} + \frac{\Delta t}{2\mu_f}\left(\left\|K^{-1/2}\boldsymbol{z}_h^k\right\|^2_{L^2(\Omega)} - \left\|K^{-1/2}\boldsymbol{z}_h^{k-1}\right\|^2_{L^2(\Omega)} + \left\|K^{-1/2}(\boldsymbol{z}_h^k - \boldsymbol{z}_h^{k-1})\right\|^2_{L^2(\Omega)}\right)$$
$$= -\alpha\Big(\nabla\cdot(\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}), p_h^k - p_h^{k-1}\Big)_{L^2(\Omega)} + \Big(\tilde{q}_h, p_h^k - p_h^{k-1}\Big)_{L^2(\Omega)}. \tag{5.24}$$

Step (b), Elasticity Equation: Considering (5.21) for the difference of two consecutive time steps, $t = t_k$ and $t = t_{k-1}$, and testing with $\boldsymbol{v}_h = \boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}$, we obtain

$$2G\left\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1})\right\|^2_{L^2(\Omega)} + \lambda\left\|\nabla\cdot(\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1})\right\|^2_{L^2(\Omega)} - \alpha\Big(p_h^k - p_h^{k-1}, \nabla\cdot(\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1})\Big)_{L^2(\Omega)}$$
$$= \Big(\boldsymbol{f}_h^k - \boldsymbol{f}_h^{k-1}, \boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}\Big)_{L^2(\Omega)} \tag{5.25}$$

Step (c), Combining Flow with Elasticity:
Combining (5.24) with (5.25) yields

$$\beta\left\|p_h^k - p_h^{k-1}\right\|^2_{L^2(\Omega)} + \frac{\Delta t}{2\mu_f}\left(\left\|K^{-1/2}\boldsymbol{z}_h^k\right\|^2_{L^2(\Omega)} - \left\|K^{-1/2}\boldsymbol{z}_h^{k-1}\right\|^2_{L^2(\Omega)} + \left\|K^{-1/2}(\boldsymbol{z}_h^k - \boldsymbol{z}_h^{k-1})\right\|^2_{L^2(\Omega)}\right)$$
$$+ 2G\left\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1})\right\|^2_{L^2(\Omega)} + \lambda\left\|\nabla\cdot(\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1})\right\|^2_{L^2(\Omega)}$$
$$= \Delta t\Big(\tilde{q}_h, p_h^k - p_h^{k-1}\Big)_{L^2(\Omega)} + \Big(\boldsymbol{f}_h^k - \boldsymbol{f}_h^{k-1}, \boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}\Big)_{L^2(\Omega)}$$
$$\leq \frac{1}{2\epsilon_1}\left\|\tilde{q}_h\right\|^2 + \frac{\epsilon_1}{2}\left\|p_h^k - p_h^{k-1}\right\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon_2}\left\|\boldsymbol{f}_h^k - \boldsymbol{f}_h^{k-1}\right\|^2_{L^2(\Omega)} + \frac{\epsilon_2}{2}\left\|\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}\right\|^2_{L^2(\Omega)}$$
$$\leq \frac{1}{2\epsilon_2}\left\|\boldsymbol{f}_h^k - \boldsymbol{f}_h^{k-1}\right\|^2_{L^2(\Omega)} + \frac{\epsilon_2\mathcal{P}_\Omega^2 C_\kappa^2}{2}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1})\|^2_{L^2(\Omega)}.$$

By Poincaré, Korn, and Young inequalities, and for ϵ_1, and $\epsilon_2 > 0$. Recall that Poincaré's inequality in $H_0^1(\Omega)$ reads: there exists a constant $\mathcal{P}_\Omega$ depending only on Ω such that

$$\forall v \in H_0^1(\Omega)\,,\ \|v\|_{L^2(\Omega)} \leq \mathcal{P}_\Omega |v|_{H^1(\Omega)}. \tag{5.26}$$

In addition, Korn's first inequality in $H_0^1(\Omega)^d$ reads: there exists a constant C_κ depending only on Ω such that

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d\,,\ |\boldsymbol{v}|_{H^1(\Omega)} \leq C_\kappa \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega)}. \tag{5.27}$$

Choosing $\epsilon_1 = \frac{\beta}{\Delta t}$, and $\epsilon_2 = \frac{2G}{\mathcal{P}_\Omega^2 C_\kappa^2}$, and summing for $1 \leq k \leq N$ (recall that N denotes the total number of time steps), we derive

$$\frac{\tilde{c}_f}{2}\sum_{k=1}^{N}\left\|p_h^k-p_h^{k-1}\right\|_{L^2(\Omega)}^2+\frac{\Delta t}{2\mu_f}\left(\left\|K^{-1/2}\boldsymbol{z}_h^N\right\|_{L^2(\Omega)}^2+\sum_{k=1}^{N}\left\|K^{-1/2}(\boldsymbol{z}_h^k-\boldsymbol{z}_h^{k-1})\right\|_{L^2(\Omega)}^2\right)$$
$$+G\sum_{k=1}^{N}\left\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^k-\boldsymbol{u}_h^{k-1})\right\|_{L^2(\Omega)}^2+\lambda\sum_{k=1}^{N}\left\|\nabla\cdot(\boldsymbol{u}_h^k-\boldsymbol{u}_h^{k-1})\right\|_{L^2(\Omega)}^2\leq\frac{\Delta t}{2\mu_f}\left\|K^{-1/2}\boldsymbol{z}_h^0\right\|_{L^2(\Omega)}^2$$
$$+\frac{\Delta t^2}{2\tilde{c}_f}\sum_{k=1}^{N}\left\|\tilde{q}_h\right\|_{L^2(\Omega)}^2+\frac{\mathcal{P}_\Omega^2C_\kappa^2}{4G}\sum_{k=1}^{N}\left\|\boldsymbol{f}_h^k-\boldsymbol{f}_h^{k-1}\right\|_{L^2(\Omega)}^2. \tag{5.28}$$

This leads to

$$\sum_{k=1}^{N}\left\|p_h^k-p_h^{k-1}\right\|_{L^2(\Omega)}^2\leq$$
$$\frac{\Delta t}{\mu_f\beta}\left\|K^{-1/2}\boldsymbol{z}_h^0\right\|_{L^2(\Omega)}^2+\frac{\Delta t^2}{\beta^2}\sum_{k=1}^{N}\left\|\tilde{q}_h\right\|_{L^2(\Omega)}^2+\frac{\mathcal{P}_\Omega^2C_\kappa^2}{2G\beta}\sum_{k=1}^{N}\left\|\boldsymbol{f}_h^k-\boldsymbol{f}_h^{k-1}\right\|_{L^2(\Omega)}^2, \tag{5.29}$$
$$\sum_{k=1}^{N}\left\|\nabla\cdot(\boldsymbol{u}_h^k-\boldsymbol{u}_h^{k-1})\right\|_{L^2(\Omega)}^2\leq$$
$$\frac{\Delta t}{2\mu_f\lambda}\left\|K^{-1/2}\boldsymbol{z}_h^0\right\|_{L^2(\Omega)}^2+\frac{\Delta t^2}{2\tilde{c}_f\lambda}\sum_{k=1}^{N}\left\|\tilde{q}_h\right\|_{L^2(\Omega)}^2+\frac{\mathcal{P}_\Omega^2C_\kappa^2}{4G\lambda}\sum_{k=1}^{N}\left\|\boldsymbol{f}_h^k-\boldsymbol{f}_h^{k-1}\right\|_{L^2(\Omega)}^2. \tag{5.30}$$

Combining (5.29) with (5.30), we reach at:

$$\sum_{k=1}^{N}\left\|p_h^k-p_h^{k-1}\right\|_{L^2(\Omega)}^2+\sum_{k=1}^{N}\left\|\nabla\cdot(\boldsymbol{u}_h^k-\boldsymbol{u}_h^{k-1})\right\|_{L^2(\Omega)}^2\leq\Delta t\tilde{\eta}_3\left\|K^{-1/2}\boldsymbol{z}_h^0\right\|_{L^2(\Omega)}^2+\Delta t^2\tilde{\eta}_4\sum_{k=1}^{N}\left\|\tilde{q}_h\right\|_{L^2(\Omega)}^2$$
$$+\tilde{\eta}_5\sum_{k=1}^{N}\left\|\boldsymbol{f}_h^k-\boldsymbol{f}_h^{k-1}\right\|_{L^2(\Omega)}^2, \tag{5.31}$$

where $\tilde{\eta}_3=\frac{1}{\mu_f}C_2$, $\tilde{\eta}_4=\frac{1}{\beta}C_2$, $\tilde{\eta}_5=\frac{\mathcal{P}_\Omega^2C_\kappa^2}{2G}C_2$, and $C_2=\left(\frac{1}{\beta}+\frac{1}{2\lambda}\right)$.

Combining Step 1 & Step 2: Combining (5.18) with (5.31) , for a generic constant $C>0$ (which will be revealed later), we can write:

$$
\begin{aligned}
&\left\|p_h^{n+1,k} - p_h^k\right\|_{L^2(\Omega)}^2 + \left\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1,k} - \boldsymbol{u}_h^k)\right\|_{L^2(\Omega)}^2 + \left\|\boldsymbol{K}^{-1/2}(\boldsymbol{z}_h^{n+1,k} - \boldsymbol{z}_h^k)\right\|_{L^2(\Omega)}^2 \\
&\qquad \leq \Big(\frac{L}{\lambda + L}\Big)^{2n}\Big(\tilde{\eta}_1\left\|p_h^k - p_h^{k-1}\right\|_{L^2(\Omega)}^2 + \tilde{\eta}_2\left\|\nabla\cdot(\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1})\right\|_{L^2(\Omega)}^2\Big) \\
&\qquad \leq \Big(\frac{L}{\lambda + L}\Big)^{2n} C_3\Big(\sum_{k=1}^{N}\left\|p_h^k - p_h^{k-1}\right\|_{L^2(\Omega)}^2 + \sum_{k=1}^{N}\left\|\nabla\cdot(\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1})\right\|_{L^2(\Omega)}^2\Big) \\
&\qquad \leq \Big(\frac{L}{\lambda + L}\Big)^{2n} C_3\Big(\Delta t\left\|K^{-1/2}\boldsymbol{z}_h^0\right\|_{L^2(\Omega)}^2 + \Delta t^2\sum_{k=1}^{N}\left\|\tilde{q}_h\right\|_{L^2(\Omega)}^2 + \sum_{k=1}^{N}\left\|\boldsymbol{f}_h^k - \boldsymbol{f}_h^{k-1}\right\|_{L^2(\Omega)}^2\Big).
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
&\left\|p_h^{n+1,k} - p_h^k\right\|_{L^2(\Omega)} + \left\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1,k} - \boldsymbol{u}_h^k)\right\|_{L^2(\Omega)} + \left\|\boldsymbol{K}^{-1/2}(\boldsymbol{z}_h^{n+1,k} - \boldsymbol{z}_h^k)\right\|_{L^2(\Omega)} \\
&\qquad \leq C\Big(\frac{L}{\lambda + L}\Big)^{n}\Big(\Delta t\left\|K^{-1/2}\boldsymbol{z}_h^0\right\|_{L^2(\Omega)}^2 + \Delta t^2\sum_{k=1}^{N}\left\|\tilde{q}_h\right\|_{L^2(\Omega)}^2 + \sum_{k=1}^{N}\left\|\boldsymbol{f}_h^k - \boldsymbol{f}_h^{k-1}\right\|_{L^2(\Omega)}^2\Big)^{1/2}
\end{aligned}
\tag{5.32}
$$

To deal with: $\left\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1,k} - \boldsymbol{u}_h^k)\right\|_{L^2(\Omega)}$, we use Poincare's and Korn's first inequalities as follows:

$$
\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{n+1,k} - \boldsymbol{u}_h^k)\|_{L^2(\Omega)} \geq \frac{1}{\mathcal{P}_\Omega C_\kappa}\|\boldsymbol{u}_h^{n+1,k} - \boldsymbol{u}_h^k\|_{H^1(\Omega)}
$$

In addition, to deal with: $\left\|\boldsymbol{K}^{-1/2}(\boldsymbol{z}_h^{n+1,k} - \boldsymbol{z}_h^k)\right\|_{L^2(\Omega)}$, we assume the permeability tensor $\boldsymbol{K}$ to be uniformly bounded and uniformly elliptic. This means that there exits positive constants λ_{min}, and λ_{max}, such that

$$
\lambda_{min}\|\xi\|^2 \leq \xi^t\boldsymbol{K}(x)\xi \leq \lambda_{max}\|\xi\|^2 \;\;\Rightarrow\;\; \|\boldsymbol{K}^{-1/2}(\boldsymbol{z}_h^{n+1,k} - \boldsymbol{z}_h^k)\|_{L^2(\Omega)} \geq \frac{1}{\lambda_{max}^{1/2}}\|\boldsymbol{z}_h^{n+1,k} - \boldsymbol{z}_h^k\|_{L^2(\Omega)}.
$$

Therefore, we have (for a positive constant C):

$$
\begin{aligned}
&\left\|p_h^{n+1,k} - p_h^k\right\|_{L^2(\Omega)} + \left\|\boldsymbol{u}_h^{n+1,k} - \boldsymbol{u}_h^k\right\|_{H^1(\Omega)} + \left\|\boldsymbol{z}_h^{n+1,k} - \boldsymbol{z}_h^k\right\|_{L^2(\Omega)} \\
&\quad \leq C\Big(\frac{L}{L + \lambda}\Big)^{n}\Big(\Delta t\left\|K^{-1/2}\boldsymbol{z}_h^0\right\|_{L^2(\Omega)}^2 + \Delta t^2\sum_{k=1}^{N}\left\|\tilde{q}_h\right\|_{L^2(\Omega)}^2 + \sum_{k=1}^{N}\left\|\boldsymbol{f}_h^k - \boldsymbol{f}_h^{k-1}\right\|_{L^2(\Omega)}^2\Big)^{1/2}.
\end{aligned}
\tag{5.33}
$$

Therefore, we conclude that for every coupling iteration $n \geq 0$,

$$\begin{aligned}
&\left\|p_h^{n+1,k} - p(t_k)\right\|_{\ell^\infty(L^2)} + \left\|\boldsymbol{u}_h^{n+1,k} - \boldsymbol{u}(t_k)\right\|_{\ell^\infty(H^1)} + \left\|\boldsymbol{z}_h^{n+1,k} - \boldsymbol{z}(t_k)\right\|_{\ell^\infty(L^2)} \\
&\quad \leq \left\|p_h^{n+1,k} - p_h^k\right\|_{\ell^\infty(L^2)} + \left\|\boldsymbol{u}_h^{n+1,k} - \boldsymbol{u}_h^k\right\|_{\ell^\infty(H^1)} + \left\|\boldsymbol{z}_h^{n+1,k} - \boldsymbol{z}_h^k\right\|_{\ell^\infty(L^2)} \\
&\qquad + \left\|p_h^k - p(t_k)\right\|_{\ell^\infty(L^2)} + \left\|\boldsymbol{u}_h^k - \boldsymbol{u}(t_k)\right\|_{\ell^\infty(H^1)} + \left\|\boldsymbol{z}_h^k - \boldsymbol{z}(t_k)\right\|_{\ell^\infty(L^2)} \\
&\quad \leq C\Big(\frac{L}{\lambda + L}\Big)^n \Big(\Delta t \left\|K^{-1/2}\boldsymbol{z}_h^0\right\|^2_{L^2(\Omega)} + \Delta t^2 \sum_{k=1}^N \left\|\tilde{q}_h\right\|^2_{L^2(\Omega)} + \sum_{k=1}^N \left\|\boldsymbol{f}_h^k - \boldsymbol{f}_h^{k-1}\right\|^2_{L^2(\Omega)}\Big)^{1/2} \\
&\qquad + \left\|p_h^k - p(t_k)\right\|_{\ell^\infty(L^2)} + \left\|\boldsymbol{u}_h^k - \boldsymbol{u}(t_k)\right\|_{\ell^\infty(H^1)} + \left\|\boldsymbol{z}_h^k - \boldsymbol{z}(t_k)\right\|_{\ell^2(L^2)}
\end{aligned}$$

Now, by the work of [29, 30], we have:

$$\left\|p_h^k - p(t_k)\right\|^2_{\ell^\infty(L^2)} + \left\|\boldsymbol{u}_h^k - \boldsymbol{u}(t_k)\right\|^2_{\ell^\infty(H^1)} + \left\|\boldsymbol{z}_h^k - \boldsymbol{z}(t_k)\right\|^2_{\ell^2(L^2)} \leq C(h^{2r_1+2} + h^{2r_2}) + O(\Delta t^2)$$

for a positive constant $C > 0$ and mesh size h. r_1 and r_2 are the degree of the polynomials used in the mixed space $(Q_h, \boldsymbol{Z}_h)$. For RT_0 mixed space, $r_1 = 0$, and $r_2 = 1$.
Therefore, we are ready now to present the a priori error estimate for the single rate undrained split iterative coupling scheme as follows:

Theorem 5.1 *Assuming RT_0 spaces (the lowest order Raviart-Thomas spaces) for flow discretization, and piecewise continuous linear approximations for mechanics together with sufficient regularity in the true solution, and for a particular time step t_k, and iterative coupling iteration $n \geq 1$, and a regularization parameter $L = \frac{\alpha^2}{(\frac{1}{M} + c_f\varphi_0)}$, the following a priori error estimate (to the leading order in time) for the single rate undrained split iterative coupling scheme holds:*

$$\begin{aligned}
&\left\|p_h^{n,k} - p(t_k)\right\|_{\ell^\infty(L^2)} + \left\|\boldsymbol{u}_h^{n,k} - \boldsymbol{u}(t_k)\right\|_{\ell^\infty(H^1)} + \left\|\boldsymbol{z}_h^{n,k} - \boldsymbol{z}(t_k)\right\|_{\ell^\infty(L^2)} \\
&\leq C_1\Big(\tfrac{L}{\lambda+L}\Big)^{(n-1)}\Big(\Delta t\left\|K^{-1/2}\boldsymbol{z}_h^0\right\|^2_{L^2(\Omega)} + \Delta t^2 \textstyle\sum_{k=1}^N \left\|\tilde{q}_h\right\|^2_{L^2(\Omega)} + \sum_{k=1}^N \left\|\boldsymbol{f}_h^k - \boldsymbol{f}_h^{k-1}\right\|^2_{L^2(\Omega)}\Big)^{1/2} \\
&\quad + \Big(C_2 h^2 + O(\Delta t^2)\Big)^{1/2}
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= 3\Big(1 + \mathcal{P}_\Omega C_\kappa + \lambda_{\max}^{1/2}\Big)\Big(\max(\tilde{\eta}_1, \tilde{\eta}_2) \times \max(\tilde{\eta}_3, \tilde{\eta}_4, \tilde{\eta}_5)\Big)^2, \\
C_2 &= C_2(T, \boldsymbol{K}, M, c_f, \varphi_0, C_{k_1}, p_h^k, p_{h,t}^k, \boldsymbol{z}_h^k, \boldsymbol{u}_{h,t}^k).
\end{aligned}$$

6 Conclusions

This paper considered a rigorous convergence analysis of the undrained split iterative coupling scheme for coupling flow with geomechanics. In the undrained split iterative coupling scheme, the flow and mechanics problems are solved sequentially assuming a constant fluid mass during the mechanics solve. An iterative coupling iteration is imposed between the two problems until convergence is obtained. Our main contributions in this work are two-fold. First, we established the localized Banach contraction results for both the single rate and multirate undrained split coupling schemes in heterogeneous poro-elastic media, following a similar approach as in [5]. Second, we rigorously derived a priori error estimates for the single rate scheme in homogeneous poro-elastic media following a similar approach as in [6]. To the best of our knowledge, this is the first time localized Banach contraction results have been obtained for the single rate and multirate undrained split scheme, and a priori error estimates have been derived for the single rate scheme. In future work, we will consider deriving a priori error estimates for fractured poro-elastic media and for multirate iterative coupling schemes.

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