

## Correction to: Spectral Theory for Interacting Particle Systems Solvable by Coordinate Bethe Ansatz

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# ERRATUM TO “SPECTRAL THEORY FOR INTERACTING PARTICLE SYSTEMS SOLVABLE BY COORDINATE BETHE ANSATZ”

ALEXEI BORODIN, IVAN CORWIN, LEONID PETROV, AND TOMOHIRO SASAMOTO

This is a correction to Theorems 7.3 and 8.12 in [1]. These statements claimed to deduce the spatial Plancherel formula (spatial biorthogonality) of the ASEP and XXZ eigenfunctions from the corresponding statements for the eigenfunctions of the  $q$ -Hahn system. Such a reduction is wrong. We are grateful to Yier Lin for pointing this out to us.

We have updated the arXiv version of the paper with the necessary corrections [2]. Below is the summary of the issue and the steps we made to correct the presentation of the ASEP and XXZ applications of our results about the  $q$ -Hahn eigenfunctions.

**$q$ -Hahn spatial biorthogonality.** Recall that the  $q$ -Hahn left and right eigenfunctions are given by

$$\Psi_{\vec{z}}^{\ell}(\vec{n}) := \sum_{\sigma \in S(k)} \prod_{1 \leq B < A \leq k} \frac{z_{\sigma(A)} - qz_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^k \left( \frac{1 - z_{\sigma(j)}}{1 - \nu z_{\sigma(j)}} \right)^{-n_j},$$

$$\Psi_{\vec{z}}^r(\vec{n}) := (-1)^k (1 - q)^k q^{\frac{k(k-1)}{2}} m_{q,\nu}(\vec{n}) \sum_{\sigma \in S(k)} \prod_{1 \leq B < A \leq k} \frac{z_{\sigma(A)} - q^{-1}z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^k \left( \frac{1 - z_{\sigma(j)}}{1 - \nu z_{\sigma(j)}} \right)^{n_j}$$

where  $\vec{n} = (n_1 \geq \dots \geq n_k)$ . (Here and below we bring only the essential notation from the original paper [1].) Their spatial biorthogonality written in the small contour form reads [1, Corollary 3.13]

$$\sum_{\lambda \vdash k} \oint_{\gamma_k} \dots \oint_{\gamma_k} dm_{\lambda}^{(q)}(\vec{w}) \prod_{j=1}^{\ell(\lambda)} \frac{1}{(w_j; q)_{\lambda_j} (\nu w_j; q)_{\lambda_j}} \Psi_{\vec{w} \circ \lambda}^{\ell}(\vec{n}) \Psi_{\vec{w} \circ \lambda}^r(\vec{m}) = \mathbf{1}_{\vec{m}=\vec{n}}, \quad (1)$$

with all integration contours being small positively oriented circles around 1, and where

$$dm_{\lambda}^{(q)}(\vec{w}) := \frac{(1 - q)^k (-1)^k q^{-\frac{k^2}{2}}}{m_1! m_2! \dots} \det \left[ \frac{1}{w_i q^{\lambda_i} - w_j} \right]_{i,j=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} w_j^{\lambda_j} q^{\frac{\lambda_j^2}{2}} \frac{dw_j}{2\pi i}.$$

Here  $\vec{w} = (w_1, \dots, w_{\ell(\lambda)}) \in \mathbb{C}^{\ell(\lambda)}$ , and  $m_j$  is the number of components of  $\lambda$  equal to  $j$  (so that  $\lambda = 1^{m_1} 2^{m_2} \dots$ ), and

$$\vec{w} \circ \lambda := (w_1, qw_1, \dots, q^{\lambda_1-1} w_1, w_2, qw_2, \dots, q^{\lambda_2-1} w_2, \dots, w_{\ell(\lambda)}, qw_{\ell(\lambda)}, \dots, q^{\lambda_{\ell(\lambda)}-1} w_{\ell(\lambda)}) \in \mathbb{C}^k.$$

**ASEP spatial biorthogonality.** To obtain the ASEP eigenfunctions from the  $q$ -Hahn ones we set  $\nu = 1/q = 1/\tau$ , where  $\tau \in (0, 1)$  is the ASEP asymmetry parameter:

$$\Psi_{\vec{z}}^{\text{ASEP}}(x_1, \dots, x_k) = \Psi_{-\vec{z}}^{\ell}(x_k, \dots, x_1)|_{q=\nu^{-1}=\tau},$$

$$(\mathcal{R}\Psi_{\vec{z}}^{\text{ASEP}})(x_1, \dots, x_k) \cdot \mathbf{1}_{x_1 < \dots < x_k} = (\tau^{-1} - 1)^{-k} \Psi_{-\vec{z}}^r(x_k, \dots, x_1)|_{q=\nu^{-1}=\tau}.$$

Here  $x_1 < \dots < x_k$  are the ASEP spatial coordinates. The spatial biorthogonality of the ASEP eigenfunctions reads

$$\oint_{\tilde{\gamma}_{-1}} \dots \oint_{\tilde{\gamma}_{-1}} dm_{(1^k)}^{(\tau)}(\vec{z}) \prod_{j=1}^k \frac{1 - 1/\tau}{(1 + z_j)(1 + z_j/\tau)} \Psi_{\vec{z}}^{\text{ASEP}}(\vec{x}) (\mathcal{R}\Psi_{\vec{z}}^{\text{ASEP}})(\vec{y}) = \mathbf{1}_{\vec{x}=\vec{y}}, \quad (2)$$

where the integration is performed over sufficiently small positively oriented circles around  $-1$ . This biorthogonality of the ASEP eigenfunctions follows from the paper by Tracy and Widom [4], as we explain in detail in [2, Proof of Theorem 7.3]. Next we discuss the gap in our original argument.

**Why (2) does not follow from (1) as claimed.** The “proof” of ASEP spatial biorthogonality given in [1] claimed to deduce (2) by plugging  $\nu = 1/q$  into (1) before performing the integration. Indeed, identity (2) looks as if one takes the  $q$ -Hahn small contour formula (1), removes all terms corresponding to partitions  $\lambda \neq (1^k)$ , and then plugs in  $\nu = 1/q$ ,  $q = \tau$ . Formula (2) (following from [4]) *a posteriori* implies that under this specialization, the contribution of all additional terms with  $\lambda \neq (1^k)$  vanishes.

First, observe that the substitution  $\nu = 1/q$  before the integration might change the value of the integral because of the factors of the form  $\frac{1}{1-q\nu w_i}$  in the integrand for  $\lambda \neq (1^k)$ . Before the substitution  $\nu = 1/q$  the residue at  $w_i = (q\nu)^{-1}$  was not picked while after the substitution we have  $1 - q\nu w_i = 1 - w_i$ , so this factor adds an extra pole inside the integration contour.

With the agreement that the substitution  $\nu = 1/q$  occurs after the integration, the “proof” of (2) presented in [1] asserted a stronger statement: For each individual  $\lambda \neq (1^k)$  and any two permutations  $\sigma, \omega \in S(k)$  (coming from  $\Psi_z^\ell$  and  $\Psi_z^r$ , respectively) the corresponding term vanishes after setting  $\nu = 1/q$ . This assertion is wrong.

For example, take  $\vec{x} = (10, 9, 8, 7, 6, 5)$  and  $\vec{y} = (5, 4, 3, 2, 1, 0)$ . The summand in the integrand in (1) corresponding to  $\lambda = (3, 2, 1)$ , and permutations  $\sigma = 321546$  and  $\omega = 645123$  has the form (before setting  $q = 1/\nu = \tau$ ):

$$\begin{aligned} & \text{const} \cdot \frac{(1 - \nu q w_1)^7 (1 - \nu q w_2)^3}{(1 - w_1)^7 (1 - w_2)^3 (1 - w_3)} \\ & \times \frac{(q w_1 - w_2) (q^2 w_1 - w_2)^2 (q^3 w_1 - w_2) (q^2 w_1 - w_3) (q^3 w_1 - w_3) (q w_2 - w_3) (q^2 w_2 - w_3)}{(w_1 - w_2)(w_1 - w_3)(w_2 - w_3)(q w_2 - w_1)^2 (q^2 w_2 - w_1) (q w_3 - w_1) (q w_3 - w_2)} \\ & \times f_1(w_1) f_2(w_2) f_3(w_3). \end{aligned}$$

Here  $f_1(w_1)$  is independent of  $w_2, w_3$  and has no zeroes or poles at  $w_1 = 1$  and  $w_1 = 1/(q\nu)$ , and similarly for  $f_2(w_2)$  and  $f_3(w_3)$ . One can check that the residue of this term at  $w_3 = 1$ ,  $w_2 = 1$ , and  $w_1 = 1$  does not vanish when setting  $q = 1/\nu$ . (Note that the result of the integration depends on the order of taking the residues for individual summands due to the presence of the factors of the form  $w_i - w_j$  in the denominators. These factors cancel out after summing over all permutations  $\sigma, \omega$ , and each summand indexed by  $\lambda$  is independent of the order of integration because the result of the summation is a function symmetric in the  $w_i$ 's.)

Let us mention another (possibly related) subtlety in the spatial biorthogonality of the ASEP eigenfunctions as compared to the general  $q$ -Hahn case. Namely, in the  $q$ -Hahn situation the contribution of individual permutations coming from the eigenfunctions vanishes, while in the ASEP case this is not the case (see [2, Remark 7.6] for details). The proof of the ASEP statement in [4] employs nontrivial combinatorics to determine cancellations of specific combinations of permutations.

**Corrections we made in the new version [2] compared to the published version [1].** We have replaced the incorrect “proof” of Theorem 7.3 (spatial biorthogonality of the ASEP eigenfunctions) by its derivation from the earlier result of Tracy and Widom [4]. We have also removed Theorem 8.12 which claimed a spatial biorthogonality statement of the XXZ eigenfunctions based on a similar incorrect direct substitution  $\nu = \theta$ .

**The same gap in [3].** The claim similar to (1) but with more general  $\nu = q^{-I}$ , where  $I$  is an arbitrary positive integer, is made in [3, Appendix A] (by a subset of the current authors). When  $I = 1$ , this identity is correct, but does not follow from the general  $\nu \in (0, 1)$  formulas (as explained above). Moreover, for  $I \geq 2$  the claimed orthogonality does not seem to hold as stated. A separate erratum will be prepared to address the issues in the work [3].

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A. BORODIN, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, DEPARTMENT OF MATHEMATICS, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139-4307, USA, AND INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, BOLSHOY KARETNY PER. 19, MOSCOW 127994, RUSSIA

Email address: [borodin@math.mit.edu](mailto:borodin@math.mit.edu)

I. CORWIN, COLUMBIA UNIVERSITY, DEPARTMENT OF MATHEMATICS, 2990 BROADWAY, NEW YORK, NY 10027, USA

Email address: [ivan.corwin@gmail.com](mailto:ivan.corwin@gmail.com)

L. PETROV, UNIVERSITY OF VIRGINIA, DEPARTMENT OF MATHEMATICS, 141 CABELL DRIVE, KERCHOF HALL, P.O. BOX 400137, CHARLOTTESVILLE, VA 22904-4137, USA AND INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, BOLSHOY KARETNY PER. 19, MOSCOW, 127994, RUSSIA

Email address: [lenia.petrov@gmail.com](mailto:lenia.petrov@gmail.com)

T. SASAMOTO, DEPARTMENT OF PHYSICS, TOKYO INSTITUTE OF TECHNOLOGY 2-12-1 OOKAYAMA, MEGURO-KU, TOKYO, 152-8550, JAPAN

Email address: [sasamoto@phys.titech.ac.jp](mailto:sasamoto@phys.titech.ac.jp)