Essays on Optimal Auctions

by

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Abstract

This dissertation is composed of three essays on optimal auctions. In the first chapter I characterize optimal auctions of an indivisible good in the context of a general model, which incorporates the independent private values and common value models as particular cases. The problem of identifying optimal auctions is solved with a mechanism design technique. In the optimal auction the good is allocated to the bidder with the highest contribution to the seller's utility, provided a minimum announcement requirement is satisfied; if no one meets the minimum announcement requirement, the seller will keep the good. A feature of the optimal auction is that the optimal required minimum announcements are, in general, determined endogenously, as functions of the announcements.

Then, the outcome of standard auctions is compared to that of the optimal auction, and it is shown that the first price, the second price and the English auctions can be used to implement the optimal mechanism in a symmetric common value model, but the Dutch auction cannot. Moreover, the analysis of English auctions shows that allowing the seller to bids or to make a final *take it or leave it* offer may be required for the implementation of the optimal mechanism.

An application to auctions of incentive contracts with common costs is developed.

The model is generalized to a multiple unit framework in Chapter 2, where I consider the problem of a seller who has several units of a homogeneous indivisible good to be sold. Optimal selling procedures are derived in a general model with asymmetric bidders.

Implementation mechanisms are discussed in the context of a particular case of the model. Under the assumption of unit demands, it is shown that generalizations (to multiple units) of standard auctions (in particular, the uniform price auction, the discriminatory price auction, and a sequential English auction) may implement the optimal mechanism; moreover, in a sequential optimal auction, it is proven that the sequence of prices paid in each auction is a supermartingale. In a model with linear valuations I show that the seller either sells all the units or none; in particular, there are situations in which bundling all the units and auctioning them as a single indivisible lot is optimal.

To illustrate the complexity of the optimal auction in the general framework, an example of a two unit auction is fully analyzed in an Appendix.

The third essay, in Chapter 3, looks at the problem of a seller that has a divisible good to be sold. Optimal selling procedures are derived in two general models with asymmetric bidders, which include the independent private values and the common value models as special cases. I show that standard procedures (such as the discriminatory price or uniform price auctions) are

not, in general optimal. Instead, the optimal auction can be implemented through mechanisms in which the bidders announce variables that reveal their private information (such as a quantity demanded or a payment to be made) and allocations and payments are decided on the basis of these announcements.

The results provide new elements to the identification of optimal mechanisms for auctioning treasury bills.

Several examples illustrate the models. In particular, an example is provided that shows that, in a common value model, the seller may want to bundle, in the sense that no more than one bidder will receive a positive amount.

Thesis Supervisor: Jean Tirole
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Thesis Supervisor: Drew Fudenberg
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To Fátima

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Acknowledgements

This Ph.D. dissertation is the corollary of over a decade of continuous study in Economics, and I would like to acknowledge to the many people that have contributed to make it possible.

In October 1980, when I enrolled the Licenciatura em Economia of the Universidade Católica Portuguesa, I was unsure whether I would want to become an economist. Fortunately, Manuel Sebastião was in charge of the first Economics course that I took and, mainly due to his outstanding undergraduate teaching ability, at the end of my first year at the university no doubts were left. In the following years, until June 1985, I was very fortunate for having learned Economics from some of the best Portuguese economists. This thesis would not have been possible were not the solid foundations that they helped me build.

During 1986 and 1987, in the Mestrado em Economia at the Universidade Nova de Lisboa, I deepened my knowledge of Economic Theory. There I was exposed to some of the recent research in Microeconomics and Game Theory, which later revealed to be very valuable.

Several people contributed to create the conditions that made possible my coming to MIT. Luís Campos e Cunha, advised me through the process of application for Ph.D. studies in the US, work that was complemented by the help of few colleagues that had already started their Ph.D. work. Not less important was the effort of Fernando Adão da Fonseca to create the conditions that made possible for many young Portuguese economists to study abroad.

Finally at MIT, since September 1988, among many leading researchers and professors, I came to know four excellent professors, who had a decisive influence on this dissertation: Mathias Dewatripont, Oliver Hart, Drew Fudenberg and Jean Tirole.

From Mathias I learned basic Noncooperative Game Theory and Contract Theory, areas in which I have been working for the last three years. With Oliver I followed a course on the Theory of the Firm, and his comments and conversations revealed to be very helpful for the

progress of my research.

During the last four years, I have constantly been learning from Drew: in the first two years, attending his courses in Game Theory; since then, as his teaching assistant and advisee. Without his suggestions this thesis would have lacked in clarity and would not have focused some important issues.

Jean has been all that can be expected from an advisor. He was always available (even during this last year, when he was already in France) to discuss ideas and to go over the endless drafts. His comments, suggestions and new ideas shaped and strengthened the content of this dissertation. But his help goes beyond this work; he improved my research ability in many ways, a valuable preparation for my future career.

Since my first day at MIT, I found a friend in António Mello, at the time an Assistant Professor at the Sloan School of Management. During the first stages of my research, António was an informal advisor, and the joint work that we did, was very important in learning how to do research.

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But not only Professors and fellow Economics students helped me in the production of this thesis. My parents, sisters and brother, were always very supportive of my work and have their share in this dissertation.

The Luso-American Educational Commission gave me assistance during the four years, while MIT, Invotan and the Luso-American Foundation provided valuable partial financial aid, which is gratefully acknowledged.

The last word goes to the most influential contributor: my wife, Fátima. Were not for her and I would not have ever done my Ph.D. abroad. I wish there would be ways to show her how grateful I am for her support.

Biographical Notes

Born on August 24, 1963, in Seixal, Portugal, Fernando Manuel Ribeiro Branco did the first eleven years of his education at neighboring schools.

In 1980, he started his undergraduate studies in Economics at the Universidade Católica Portuguesa, in Lisbon, which he finished in June 1985. Since 1982 and until July 1988, he was a teaching assistant for over fifteen courses of the Economics and Management programs of the Universidade Católica Portuguesa.

During the same period he followed over twenty courses in the Mathematics program of the Universidade de Lisboa, which he discontinued in 1986. Fernando Branco then entered a Master's Program in Economics at the New University of Lisbon, whose course work was completed in 1987.

Between 1985 and 1988, he worked as a Consultant to the Banco Português do Atântico and as a member of the Portuguese Hermes Model (a macroeconometric model of the Portuguese Economy, supported by the EEC). From January 1987 to May 1988 Fernando Branco was also a Lecturer in Economics at the Escola Naval.

In September 1988, he enrolled the graduate program in Economics at MIT, which has led to the present dissertation. During this period, he worked as a consultant to the Portuguese Government, in the Summer of 1989, and to the Bank of Portugal, in the Summer of 1990, as a member of MIT teams headed by Professor Richard Eckaus. He was back to the Bank of Portugal in January 1991, studying the functioning of the Money Markets in Portugal.

Besides the work included in this dissertation, Fernando Branco has also authored several works among which

"The Design of Multidimensional Auctions," MIT Mimeo, March 1992.

"Why Are Wages in Portugal Lower than Elsewhere in the ECC?," (joint with António S. Mello), in João Ferreira do Amaral, Diogo Lucena and António S. Mello (Eds.), *The Portuguese Economy Towards 1992*, January 1992, pp. 131–151. Kluwer Academics Publishers: Norwell, Massachusetts.

"A Theory of Partial Sales and Underpricing in Privatizations," (joint with António S. Mello) Sloan School of Management, WP#3282-91-EFA, April 1991.

"A Continuous Time Approach to Research and Development," *Economia,* 14(3), October 1990, 387–423.

Upon finishing his Ph.D., Fernando Branco will hold a position of Assistant Professor in the Department of Economics of the Universidade Católica Portuguesa (Lisbon, Portugal).

Introduction

Auctions are probably the oldest and simplest mechanisms for the organization of multilateral trade. Milgrom and Weber (1982) mentions reports of auctions in Babylonia around the fifth century B.C., during the Roman Empire, and in the seventh century A.D. in China. Used for centuries, it might be surprising that standard mechanisms for the auctioning of a single object (the first price sealed bid auctions, the second price sealed bid auction, the English auction and the Dutch auction) have kept very simple rules. In the sealed bid auctions the bidders submit sealed bids and the object is awarded to the bidder that submitted the highest bid; in the first price auction the price paid is the highest bid, while in the second price auction the winner pays the second highest bid. In oral auctions the bidders see each others actions. In the English auction the price is increased until one bidder only is willing to buy at the current price; in the Dutch auction the process is reversed, the price starts at a very high level and decreases until a bidder announces his willingness to buy at the current price.

Certainly due to the lack of appropriate tools to analyze relationships among asymmetrically informed agents, till recently economists had paid little attention to the formal study of auctions. The development of game theory revealed to be very powerful for analysis in this area and, starting with Vickrey's seminal work (Vickrey, 1961), the situation has greatly changed in the last thirty years. Auction theory has emerged as an important and autonomous body of the economic theory.[1]

The literature has evolved in both the theoretical and the empirical directions.[2] The theoretical literature has, on one hand, studied the properties of widely used mechanisms and,

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[1] With its last revision, the *Journal of Economic Literature* classification system created a specific entry for auctions.

[2] See Engelbrecht-Wiggans (1980), McAfee and McMillan (1987) and Wilson (1990) for excellent surveys of the literature and extensive lists of references.

on the other hand, identified optimality properties of auctions. The most notable result of this research is the revenue equivalence theorem which states that if agents are risk neutral and valuations are independent realizations of the same random variable — the independent private values model —, from all the possible mechanisms, standard auctions are among those that maximize the auctioneer's expected utility.[3] The empirical literature (complemented with a large set of laboratorial experiments) has described the behavior of agents in many distinct auctions, testing the theory and suggesting new directions for theoretical research.

In real life situations it is hard to argue that the assumptions of the independent private values models are met. More generally valuations are not independent and in extreme cases, such as Treasury bill auctions, all bidders have the same (unknown) valuation — the common value model —. This created an interest for models with correlated valuations, which have been appearing in the last decade. The main result from this line of research is that the auctioneer can arbitrarily approximate the symmetric information solution.[4] In light of the empirical relevance of standard auctions, this result makes economists uncomfortable: standard auctions are largely used, yet their optimality has only be proven for the independent private values model, a very narrow class. The literature on auctions with correlated valuations has usually made the additional assumption that the bidders' private information is also correlated, which is appropriate for many situations. I argue that the joint use of the assumptions to analyze departures from the independent private values model did not let us identify the assumption that makes one cross the frontier separating models in which standard auctions are optimal from those in which the auctioneer gets the full surplus. Identifying the assumption that drives the change in the result is very important because while independent valuations are hardly acceptable in many situations, independent private information is more likely to be found.[5]

In this dissertation I study optimal auctions in models with correlated valuations and independent private information, showing that the optimality of standard auctions is not as sensitive to the assumption about the valuations as it is to the assumption about the private information.

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[3]There are minor technical assumptions underlying this result. See Myerson (1981) for a complete characterization of the result.

[4]Under symmetric information the good is efficiently allocated and the seller appropriates all the surplus generated by the transaction. See Crémer and McLean (1985) and 1988, McAfee et al. (1989), and McAfee and Reny (1991).

[5]I devote several sections of the dissertation to the discussion of examples to support that independent private information might be a reasonable assumption.

In the first chapter, I look at the sale of a single object. Assuming that all the payments must be realized at the time of the auction,[6] I characterize optimal mechanisms in a general asymmetric model and show that the seller cannot reach the symmetric information solution. Moreover, some standard auctions are among the optimal mechanisms in a symmetric model, even in a model in which the bidders have exactly the same (partially unknown) valuation — a common value model —.[7] But, there are many situations in which the seller has more than one object to be sold, case that has received less attention in the literature. A very important example are procurement auctions (e.g., auctions of drilling rights in federal land, contracts for public works). In Chapter 2, I generalize the model of the first chapter to analyze these situations. In this case, if the bidders only draw utility from the first unit, most of the generalizations of standard auctions will be optimal but, in the general model, the optimal auction is more complex. The analysis is closed in Chapter 3 with a model of a divisible good, in which optimal auctions are characterized and some optimal mechanisms (some of which resemble the standard auctions) are presented. The leading application fo the results in this chapter are Treasury bill auctions.

It is my hope that the three essays in this dissertation have positive and normative contributions. On one hand, they provide theoretical elements for the design of optimal auctions; on the other hand, they may improve our ability to understand how auctions really work. Future empirical work will ultimately judge the relevance of this work.

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[6]This is a very important assumption, which I also consider reasonable, for most applications. If the seller could use mechanisms that would involve the announcement of the winner's true value once it had been realized, he would be able to achieve full extraction of the surplus.

[7]The Dutch auction only does not allow the implementation of the optimal auction.

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Chapter 1

Optimal Auctions of an Indivisible Good: Extending the Optimality of Standard Auctions

1.1 Introduction

There is a large body of literature on optimal auctions of an indivisible good when the bidders' valuations are privately known and independently distributed, the independent private values model (e.g., Myerson, 1981, for risk neutral bidders, Maskin and Riley, 1984, for risk averse bidders). However, less is known when the valuation is the same for all bidders but no one knows it, the common value model (or intermediate forms, where each bidder may value the good differently from any other bidder, but the valuations are not independent).[1]

From the auction literature there are three works that more closely relate to the present chapter. Myerson (1981) describes a model of optimal auctions of an indivisible object. There is a seller and n potential buyers; all agents are risk neutral and have a private estimate of the value of the object, denoted by t_i ($i = 0, \ldots, n$), where the index 0 denotes the seller. The agents' true valuations may differ from the private estimates, in that, if any agent knew

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[1]Myerson (1981) allows for a particular type of correlation among the bidders' valuations. Other results, with common value, provide mechanisms for the seller to fully extract the surplus (e.g., Crémer and McLean, 1985 and 1988, and McAfee et al., 1989).

other agents' personal estimates, he would want to revise his own estimate; the revision process is such that the valuations are $v_i(t) = t_i + \sum_{j=1, j \neq i}^{n} e_j(t_j)$, for some functions $e_j(\cdot)$. In the optimal auction the good is allocated to the bidder that has the highest contribution to the seller's utility, as long as his announcement is larger than a required minimum announcement set *ex-ante*. A particular application shows that standard auctions are optimal in the case of independent and identically distributed private values.[2]

A general model of auctions is presented in Milgrom and Weber (1982). In this work, the analysis is restricted to a comparison among standard auctions. In their model there is an indivisible object to be sold, through a standard auction, to one of n bidders.[3] Each bidder's valuation depends on a vector of signals,[4] on which every bidder has some private information. They describe the equilibrium bids in standard auctions, and compare the seller's expected profit from the several mechanisms.

A third relevant work is McAfee et al. (1989), which is a representative of the recent line of research on common value auctions. They consider a model of optimal auctions of an object with a well defined, but unknown, value which is equal for every bidder. Each bidder has a private estimate of the value of the object. It is shown that, if the seller can construct an unbiased estimate of the true value from any bidder's private signal, there is a simple mechanism that lets the seller appropriate the full surplus: the seller randomly selects two bidders, asks the private information from the first, and sells the object to the second at a price equal to the expected value of the object, conditional on the announcement of the first bidder.

In this chapter, I characterize optimal auctions in a model which, while meeting the assumptions in Milgrom and Weber (1982), imposes some additional restrictions, namely: the private signals are independent, and the relations among the random variables (true valuations and private signals) are linear.[5] These two additional assumptions locate me into a model in which

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[2]I will use standard auctions to refer to first price and second price sealed bid auctions, English and Dutch oral bid auctions.

[3]In the model, the first price sealed bid auction and the Dutch oral auction are strategically equivalent, so they do not directly analyze the Dutch auction.

[4]The technical assumption is that the random variables are affiliated.

[5]While the first assumption cannot be avoided to be able to use the mechanism design technique, the second one is made for simplicity; I would only need the valuation to be linear in the bidder's private signal. The assumption that the private signals are independently distributed is inappropriate for most situations. However, the analysis is unchanged if the private signals, conditional on information commonly known, are independent. For example, if a report of an expert providing an estimate of the object's value is publicly revealed, I would

buyers receive independent private signals and make possible the application of the mechanism design technique developed in Myerson (1981) to identify optimal mechanisms.

The model can also be seen as a generalization of that in Myerson (1981). The basic difference between the two works is on the assumption about the link between private signals and valuations. In Myerson's general model each agent has a revision function, expressing the effect of others' information on the agent's valuation, which is required to be identical across all agents, including the seller. This may be an appropriate assumption in the case of a common value model which is not the type of model that he analyzes. Instead, in the present work, I want to address the issue of optimal auctions with interdependent valuations and, in particular, common values auctions. I consider linear revision functions, but let them differ across bidders. As a result, a distinct class of models is analyzed, which helps us understand some of the (so far unexplained) properties of auctions in practice.

A special case of the general model, a symmetric common value model, is discussed in some detail. The approach taken differs from that in McAfee et al. (1989), and this is why the optimal auctions differ. In simple terms, while in McAfee et al. (1989) the private signals can be seen as independent estimators of the good's value, in my model they can be seen as independent estimators of different components of the good's value. Thus, because each bidder has some private (monopolistic) information on a component, it should be no surprise that in the latter the seller cannot fully extract the surplus.

The chapter is organized as follows. First, in Section 1.2, the general model is presented and the main assumptions are discussed. I argue that this auction model may describe the market of some goods (e.g., objects of art, wine, non-durable consumption goods without resale markets) or may appear as the first stage of a dynamic game describing the markets of some other goods (e.g., oil tracts, privatization of state owned enterprises, non-durable consumption goods with resale markets). Some illustrative examples are presented in Section 1.3.

Third, in Section 1.4, the general model is analyzed and the optimal mechanism is characterized. In the optimal mechanism, bidders are ordered according to their individual contributions to the seller's expected utility, should they receive the good. If all the contributions are non-

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require that the private signals, conditional on this public information, are independently distributed; but the private signals may be correlated.

positive, the seller will keep the good; otherwise, the good is allocated to the bidder with the highest contribution, whose expected payment is equal to the expected value of the good, had he received the lowest private signal that would still allow him to get the good.

Two properties of the optimal auction are emphasized. Generalizing the characterization of optimal auctions in the independent private values case, the optimal mechanism is not, in general, implementable by any standard auction and, for heterogenous bidders, it requires the seller to discriminate in favor of the bidders that are expected to receive a low private signal. However, in contrast with the individual private values model, the seller's optimal minimum announcement policy is not, in general, to set an *ex-ante* required minimum announcement for each bidder, but to make a bidder's required minimum announcement contingent on other bidders' announcements.

In Section 1.5, standard auctions are compared to the optimal mechanism, and conditions under which the optimal mechanism can be implemented with a standard auction are identified.[6] I show that there is a symmetric common value model in which the first price, the second price or the English auctions can implement the optimal mechanism, but the Dutch auction cannot, which contrasts with results for some common value models, such as the model presented in McAfee et al. (1989). The design of the standard auctions that implement the optimal mechanism is discussed in detail, and it is shown that the Dutch auction fails to do so because it does not provide the seller with all the information he needs to compute the optimal required minimum bids.

The analysis of standard auctions also shows that some observed behavior in English auctions (e.g., the use of *phantom bids* or a final *take it or leave it* offer by the seller) can be motivated by this optimal auction approach. Moreover, I will argue that, even though a Revenue Equivalence Result exists, the simplicity of the equilibrium strategies in the English auction may help justifying its popularity. I finish the section describing a sealed bid auction, which implements the optimal mechanism, where the equilibrium bids are not sensitive to the players' beliefs about the distribution of other players' signals.

An application of the model to procurement is discussed in Section 1.6. Starting from the

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[6]The standard auctions that will be analyzed will allow the seller to set minimum bid rules, which are functions of the bidders' bids.

analysis in Laffont and Tirole (1987) of the selection, by a regulator, of a contractor to realize a particular project, I study the design of auctions of incentive contracts, when the firms' costs are correlated. The analysis shows that most of the basic results in the independent private costs case have direct analogues in the common costs case.

Section 1.7 summarizes and discusses possible extensions of the model, some of which are taken in other chapters.

1.2 The Model

I consider a model describing the sale of a single unit of an indivisible good.[7] The seller, who does not value the good, wants to sell this unit maximizing his expected profit.[8]

The following assumptions are made.

Assumption 1.1 *There are n risk neutral utility maximizing potential buyers.*

The potential buyers may either be final consumers/users of the good that is sold (e.g., auctions of art objects) or agents that will later operate in a related market (e.g., auctions of goods with resale markets). I assume risk neutrality so that the final results are directly comparable with most of the literature; in particular, when analyzing common value models, I want to show that under risk neutrality the seller may not be able to capture all the surplus.[9]

Assumption 1.2 *Each potential buyer has his own private information about the value of the good, T_i.*

A buyer's information may result from introspection, specific information he has gathered, or individual interpretation of public information.

Assumption 1.3 *Other agents do not observe the realization of T_i and treat it as a draw from a cumulative distribution $F_{T_i}(\cdot)$, with support $\mathbf{T}_i = [\underline{t}_i, \bar{t}_i]$.*

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[7]In the description of the general model, I will assume that a seller is organizing the auction. However, the model can be used to analyze situations where a buyer is designing the auction.

[8]The consideration of a nonzero seller's valuation would not change the basic results.

[9]Robert (1991) shows that this is the case with risk aversion or limited liability. In this paper, to prevent the seller for attaining the first best through the use of Maskin mechanisms, I must assume that all the payments must be made at the time of the auction, so that they cannot be contingent on the true valuation of the winner. I will return to this issue in footnote 17.

The vector of all private signals, $(T_1, \ldots, T_n)$, is denoted by T, distributed on $\mathbf{T}$; T_{-i} denotes the vector of private signals of all potential buyers but i, $(T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n)$, with support on $\mathbf{T}_{-i}$. Realizations of a random variable are denoted by its lower case letter, with or without a hat (e.g., t and $\hat{t}$ denote realizations of the random vector T).

In the general model, I allow for heterogeneity among potential buyers, i.e., the distributions and their supports may vary across potential buyers.

Assumption 1.4 *Any two variables T_i and T_j, with $j \neq i$, are independent.*

This assumption is very important, but it is not as restrictive as it may seem. In particular, if there is some public information about the value of the object, one may expect the private signals to be correlated; however, as long as the correlation is only introduced through the public information, i.e., private signals conditional on the public signal are independent, all the results are maintained.

Assumption 1.5 *The value of the good to potential buyer i, V_i, is correlated with the private signals; its conditional expectation is*

$$E\{V_i(T) | T = t\} = \sum_{j=1}^{n} \nu_{ij} \, t_j$$

where the parameters ν_{ij} $(j \neq i)$ are nonnegative, and ν_{ii} is positive.

The value of the good, $V_i(T)$, may not be the same for each bidder and it may be unknown, when the auction is realized. For example, when the rights to explore a oil field are auctioned its value is unknown, the results from exploratory drilling provide useful information about the value but no agent is fully informed; in the case of the privatization of a state owned firm buyers may not be sure about the firm's value. On the other hand, the model also incorporates cases in which the valuations are known at the time of the auction, such as the independent private values model that one gets by taking $V_i(T) = T_i$.

Because all agents are risk neutral, I only specify relations among the expected valuations and the agents' private signals. The restriction to a linear relationship is an important simplification.

A regularity assumption is imposed on the cumulative distributions of the private signals.

Assumption 1.6 *The function $t_i - [1 - F_{T_i}(t_i)]/f_{T_i}(t_i)$ is nondecreasing in t_i.*

This is just a technical assumption that ensures that random schemes will not be optimal. This assumption is made for convenience, and the essential of the analysis could be done without it.

The model can be seen as a description of auctions of very different goods, motivating several interpretations for the parameters.[10] One may consider auctions of objects of art, collectable wine, or nondurable consumption goods, where the private signal may be interpreted as a taste parameter of the bidder; auctions of treasury bills or consumption goods with secondary markets where the private information may represent the knowledge of characteristics of the buyers in the secondary market; auctions of oil and gas leases, or the privatization of a state owned firm, where the private signals are the result of prospecting or auditing activities; auctions of incentive contracts where firms have estimates of cost parameters. Some motivating examples are presented in the next section.

1.3 Illustrative Examples

In this section some examples of auctions that satisfy the assumptions of the general model described in the previous section are presented. There are two groups of examples: in the first group, the auction is modelled as a single stage game and the primitive assumptions are just interpreted in terms of the specific example; in the second group, the auction is modelled as the first stage of a dynamic game where the assumptions about the auction appear as a result of the equilibrium continuation actions.

1.3.1 Objects of Art

Objects of art are commonly traded through auctions. Here, I assume that bidders are agents that intend to keep the good (e.g., representatives of a museum).[11]

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[10]The examples presented are meant as illustrations of the general model; in particular, I do not claim that this model is the best way to describe such situations. Some of the examples (e.g., treasury bills) would require the auction to be organized as an indivisible block which is not usually the case.

[11]Ashenfelter (1989) mentions that auction houses often design rules to restrict the resale of the object.

Consider the following simple model. One painting is to be auctioned. There are n towns and one museum in each town. Each museum will send a representative to the painting's auction. The value of the object for a museum is given by the expected number of visitors that will go to the museum to see it.

Let n_i be the population of town i. Each person in town i is characterized by a taste parameter θ, uniformly distributed in $[0, \bar{\theta}_i]$, which represents the utility drawn from seeing the object. Town i's museum knows $\bar{\theta}_i$, but others only know that it is uniformly distributed in $[0.5, 1]$, and independently distributed across towns.

Let c_{ij} be the cost of a round trip between towns i and j, with $c_{ij} < 0.5$.[12] Let $m_{ij}(\bar{\theta}_i)$ be the expected number of people from town i that will be willing to visit the museum in town j. Then the valuation of town's i museum, conditional on all the private information is

$$E\{ V_i(T) | \bar{\Theta}\} = m_{ii}(\bar{\Theta}_i) + \sum_{\substack{i=1 \\ i \neq j}}^{n} m_{ji}(\bar{\Theta}_j)) \,,$$

which may vary across towns.

Let the private information of the museum located in town i, T_i, be the proportion of the town's population that are willing to visit the museum, i.e.,[13]

$$t_i = 1 - \frac{c_{ii}}{\bar{\theta}_i} \,,$$

which is the expected proportion of the population whose taste parameter exceeds the travel cost. Then, after simple manipulations, we can see that the expected proportion of town i's population that will be willing to see the painting in town j's museum is related with t_i through

$$\frac{m_{ij}}{n_i} = \left(1 - (1 - t_i) \frac{c_{ij}}{c_{ii}} \right) \,.$$

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[12]In particular, there is a cost to visit a museum in the home town, i.e., $c_{ii} > 0$. The upper bound on costs simplifies the algebra of the example. In this way, there will be a strictly positive expected number of visitors from each town to the museum that will exhibit the painting.

[13]This information is privately known, because $\bar{\theta}_i$ is privately known.

Thus, the valuation of town i's museum, given all the available information will be

$$\mathrm{E}\{V_i(T)| T = t\} = \sum_{j=1}^{n} n_j \left(1 - \frac{c_{ji}}{c_{jj}} + \frac{c_{ji}}{c_{jj}} t_j\right),$$

which is a linear function of the private signals.[14]

To verify that this example meets the assumptions of the model, it is left to show that the regularity condition on Assumption 1.6 is satisfied. From the assumptions on the distribution of $\bar{\theta}_i$ and the relation between t_i and $\bar{\theta}_i$, I can obtain the cumulative distribution function of T_i,

$$F_{T_i}(t_i) = \frac{2\,c_{ii}}{1 - t_i} - 1$$

and $t_i - [1 - F_{T_i}(t_i)]/f_{T_i}(t_i)$ equals t_i, which is nondecreasing in t_i.

1.3.2 Wine

Auctions of wine are discussed in depth in Ashenfelter (1989). In this example I assume that the wine bought may either be consumed or traded at a later date. Any bidder has a personal estimate of the utility that she will enjoy from the consumption of the wine, T_i.

Consider the following example. At date 0 the wine is auctioned and (possibly) sold to one of n bidders. With probability q the winner's taste parameter will drop to 0, and she will then have the opportunity to make a take it or leave it offer to sell the wine to one buyer. To simplify, I assume no discounting and that the wine must be drunk at date 1 (i.e., at date 2 no one will value it).

Consider that T_i is exponentially distributed in $\mathbf{R}_+$, with mean λ, and it is independent across bidders.[15] The parameter λ is unknown but, after the auction, the winner uses a maximum likelihood estimator to update her estimate of λ, i.e., the new estimate will be equal to the mean of the taste parameters revealed in the auction.

I now compute the equilibrium of the game. Suppose that at date 1 the winner's valuation drops to 0. Then, she must decide the price that maximizes her revenue, i.e., ρ that maximizes

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[14]There will be a constant that if considered in the general model would not affect the analysis.

[15]In this example the support of $F_{T_i}(\cdot)$ is not compact; however, we will see that this assumption does not change the analysis.

$\rho \, \text{Prob}\{T > \rho\}$; the solution is $\rho = \lambda$, yielding the expected revenue λe^{-1}. Thus, before the first auction, the expected valuation, conditional on all the private signals, is just

$$\text{E}\{V_i(T)| T = t\} = (1 - q)t_i + q\,e^{-1}\sum_{j=1}^{n} \frac{t_j}{n} \, ,$$

which is a linear function of the private signals.

The remaining assumptions of the general model are satisfied by construction.

1.3.3   Consumption Goods with Resale

Some nondurable consumption goods are often sold by the producer to intermediaries through an auction; then, the winner in the auction organizes a secondary market where the product is sold to the final consumers. It is also possible that the product is initially auctioned as an indivisible object, which is traded as divisible in the secondary market.

For the simplicity of the example, let me consider the market of fish. There are three groups of agents: the fisherman, the intermediaries and the consumers. There are two stages: in the first stage, the fisherman organizes an auction to sell the fish as an indivisible object, and the intermediaries bid for it; in the second stage, the winner of the first stage sells the fish to the consumers in a secondary market, where the fish is a divisible good.

The secondary market is organized in a place where many other products are offered, and even the intermediaries that did not get the fish in the first stage are present, selling other products. Each consumer prefers to buy his products from a particular intermediary, i.e., there are clienteles. Suppose that the consumers' expenditures on fish are a constant fraction of their total income. From the repeated interaction between intermediaries and consumers, each intermediary has an informed estimate about the expected amount that his clientele is willing to spend on fish, which represents his private information, T_i. If intermediary i has the fish to sell, a consumer from j's clientele will only buy with probability ν_{ij}.

According to the description of the secondary market, in the first stage every intermediary will know that his expected valuation, conditional on all information available after the auction, is given by

$$\text{E}\{V_i(T)| T = t\} = \sum_{j=1}^{n} \nu_{ij}\, t_j \, .$$

Thus, as long as the distributions of expected expenditures satisfy the additional assumptions of the model, the example is a particularization of the general model.

1.3.4 Auctions of Incentive Contracts

Laffont and Tirole (1987) study the selection by an utilitarian regulator of a contractor and its contract in a natural monopoly, assuming that a firms' cost depends on an efficiency parameter and the level of effort exerted by a manager. There are n potential candidates to the realization of the project; firm i's efficiency parameter, t_i, is privately known and independently and identically distributed among the potential contractors.

Our model provides a particular simple way to generalize their results to the correlated efficiency case. Suppose that T_i is firm i's private information, but the true efficiency parameter is some convex combination of all the private estimates, i.e., the cost functions can be written as

$$C_i = \sum_{j=1}^{n} \nu_{ij}\, t_j - e_i\,,$$

where e_i is the level of effort exerted by the manager of the firm.[16]

The results of the model could then be used to study the characteristics of the optimal contracts. This will be done in Section 1.6.

1.4 Optimal Auctions

In this section I analyze the general model described in Section 1.2, the set of feasible mechanisms is characterized and the optimal mechanism is identified.

By the Revelation Principle (Myerson, 1979), we know that attention can be restricted to direct revelation mechanisms; these require bidders to announce their private information (i.e., the realization of T_i) and the seller to allocate the good according to the bidders' reports and some prespecified rules.

The assumption of the agents' risk neutrality simplifies the characterization of a mechanism. In particular, from the seller's point of view the only relevant elements of a mechanism are the

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[16]The firms' private signals may have several interpretations: for example, information about specific stages of the production, characteristics of the technology, costs of resources.

buyers' expected payments; from a buyer's point of view the relevant elements of a mechanism are the probability that he will get the good and his expected payment. Formally, I define a mechanism (that I designate by *auction*) as follows.

Definition 1.1 *An* **auction** *is any pair of functions* (p, c), *with*

$$p : \mathbf{T} \rightarrow \left\{ \mathbf{p} \, \middle| \, \mathbf{p} \in [0,1]^n, \sum_{i=1}^{n} p_i \leq 1 \right\}$$

and

$$c_i : \mathbf{T}_i \rightarrow \mathbf{R} \, .$$

The function $p_i(\cdot)$ is the probability that the good will be given to bidder i; the function $c_i(\cdot)$ is bidder i's expected payment.[17]

Given an auction (p, c), the agents' utility may be written as:

$$U_0(p, c) = \mathrm{E}\{\textstyle\sum_{i=1}^{n} c_i(T_i)\}$$

for the seller's utility,[18] and

$$U_i(p, c, t_i) = \mathrm{E}\{V_i(T) p_i(T) | T_i = t_i\} - c_i(t_i)$$

for buyer i's utility.

To characterize the set of feasible auctions I denote bidder i's expected probability of getting the good by $Q_i(p, t_i)$, i.e.,

$$Q_i(p, t_i) = \mathrm{E}\{p_i(T) | T_i = t_i\} \, . \tag{1.1}$$

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[17] With this characterization of the mechanisms, I am ignoring the possibility of more complex mechanisms which would involve possible payments after the true valuation is realized. To illustrate the use of such a mechanism, consider, as an example, that V_i is deterministically related to T, i.e., there is no noise after t is known. In this type of environment the seller should not let the winner know the announcements of other bidders in the auction; then, after the winner had observed the realization of V_i, he would be asked to announce the realization of $\sum_{j \neq i} \nu_{ij} T_j$; if this announcement did not agree with the announcements made in the auction, the seller would impose a very large penalty on every bidder. With such a mechanism the seller would avoid untruthful announcements and would be able to appropriate the full surplus. Following the arguments of Crémer and McLean (1985, 1988), this type of mechanism should also work, even if there is uncertainty. However, I restrict the analysis to mechanisms that do not require payments posterior to the auction.

[18] The index 0 will refer to the seller; indices 1 to n refer to buyers.

When announcing his private information a bidder sends the message that maximizes his expected utility, in particular, he is not restricted to a truthful announcement. Because of the restriction to the analysis of truthful revelation mechanisms, an auction (p, c) must be designed in such a way that a bidder prefers to report the true information. The following definition formalizes this requirement.

Definition 1.2 *An auction (p, c) is* **incentive compatible** *if, for all i, t_i and $\hat{t}_i$,*

$$U_i(p, c, t_i) \geq E\left\{ V_i(T) p_i(\hat{t}_i, T_{-i}) \middle| T_i = t_i \right\} - c_i(\hat{t}_i). \tag{1.2}$$

Condition (1.2) has a clear meaning: the utility from a truthful report must be no lower than the utility from any other possible report. However, this is not the best way to write the incentive compatibility constraints. The following lemma provides a condition in terms of levels of the utility function.

Lemma 1.1 *An auction (p, c) is incentive compatible if and only if, for all i, t_i and $\hat{t}_i$,*

$$U_i(p, c, t_i) \geq U_i(p, c, \hat{t}_i) + \nu_{ii} (t_i - \hat{t}_i) Q_i(p, \hat{t}_i). \tag{1.3}$$

Proof: See Appendix A1. ∎

Condition (1.3), in the lemma, imposes a bound on the difference of the utility level for any two distinct private signals. The constraint relates to the fact that if the difference is smaller, bidders will not want to truthfully reveal their private information. This is the reason why the seller will not, in general, be able to extract the total value of the good from the winner.

A second constraint must be satisfied by an auction: any particular buyer is free to participate or not in the auction; thus, a bidder's expected utility from participation in the auction must not be lower than the expected utility from no participation. Definition 1.3 formalizes this requirement.

Definition 1.3 *An auction (p, c) is* **individually rational** *if, for all i and t_i,*

$$U_i(p, c, t_i) \geq 0. \tag{1.4}$$

I now define feasible auctions.

Definition 1.4 *An auction (p,c) is* **feasible** *if and only if it is incentive compatible and individually rational.*

In looking for the optimal auction, attention is restricted to feasible auctions. Proposition 1.1 provides a set of conditions that fully characterize feasible auctions. Because t_j is announced by player j, player i only derives a rent from t_i, his private information. Proposition 1.1 is the standard result of the independent private values model adapted to this model.

Proposition 1.1 *An auction (p,c) is feasible if and only if, for all i, t, t_i and $\hat{t}_i$,*

$$(t_i - \hat{t}_i)\, Q_i(p, t_i) \geq (t_i - \hat{t}_i)\, Q_i(p, \hat{t}_i)\,, \tag{1.5}$$

$$\frac{\partial U_i(p, c, t_i)}{\partial t_i} = \nu_{ii}\, Q_i(p, t_i) \ \textit{(almost everywhere)} \tag{1.6}$$

$$U_i(p, c, t_i) \geq 0 \tag{1.7}$$

$$\sum_{i=1}^{n} p_i(t) \leq 1 \tag{1.8}$$

$$p_i(t) \geq 0 \tag{1.9}$$

Proof: First, I prove that feasibility implies conditions (1.5) to (1.9). From Lemma 1.1, using condition (1.3) twice, incentive compatibility implies

$$\nu_{ii}\,(t_i - \hat{t}_i)\, Q_i(p, \hat{t}_i) \leq U_i(p, c, t_i) - \ \ _i(p, c, \hat{t}_i) \leq \nu_{ii}\,(t_i - \hat{t}_i)\, Q_i(p, t_i)\,, \tag{1.10}$$

which implies (1.5). Dividing the terms in (1.10) by $(t_i - \hat{t}_i)$,[19] and taking limits as $\hat{t}_i \to t_i$, in the center I get $\partial U_i(p, c, t_i)/\partial t_i$, while both bounds converge to $\nu_{ii}\, Q_i(p, t_i)$; hence, (1.6) follows. Expression (1.7) is a direct requirement of individual rationality; (1.8) and (1.9) follow from the definition of an auction.

Now, I prove that conditions (1.5) to (1.9) imply feasibility, i.e., incentive compatibility and individual rationality. To get incentive compatibility, use (1.6) and the assumption that ν_{ii} is

---

[19]If $t_i < \hat{t}_i$ the inequalities in (1.10) will be reversed, but the argument is not changed.

positive to write (1.5) as

$$(t_i - \hat{t}_i) \frac{\partial U_i(p, c, t_i)}{\partial t_i} \geq (t_i - \hat{t}_i) \frac{\partial U_i(p, c, \hat{t}_i)}{\partial t_i} . \tag{1.11}$$

Condition (1.11) states the convexity of $U_i(p, c, t_i)$ in t_i, which can also be written as

$$U_i(p, c, t_i) \geq U_i(p, c, \hat{t}_i) + (t_i - \hat{t}_i) \frac{\partial U_i(p, c, \hat{t}_i)}{\partial t_i} . \tag{1.12}$$

Using (1.6) on (1.12), one gets (1.3) which, by Lemma 1.1, implies incentive compatibility. From condition (1.6)

$$U_i(p, c, t_i) = U_i(p, c, \underline{t}_i) + \int_{\underline{t}_i}^{t_i} \nu_{ii} Q_i(p, x) \, dx \; ; \tag{1.13}$$

then, using (1.7) and the fact that $Q_i(p, x)$ is an expected probability, therefore positive, I get individual rationality. ∎

Conditions (1.5) and (1.6) express the convexity of the utility function, which is an incentive compatibility requirement; condition (1.7) is a result of individual rationality, that together with (1.5) and (1.6), yields incentive compatibility; conditions (1.8) and (1.9) ensure that the p_i's are probabilities.

Individual rationality imposes a limit to the expected payment of a bidder; incentive compatibility imposes restrictions on the relative expected payments for distinct announcements; in particular, bidders with a high private signal must be prevented from deviating and announcing a lower value, which requires the bidders with a high private signal to be given a rent. The argument does not apply to the bidder with the lowest possible private signal, who must then have zero utility. This result and the expected payment schedule in an optimal auction are described in Proposition 1.2.

Proposition 1.2 *Suppose that (p^*, c^*) is an optimal auction. Then a bidder reporting the lowest possible private signal has zero utility, i.e.,*

$$U_i(p^*, c^*, \underline{t}_i) = 0 , \tag{1.14}$$

and, for all i and t_i, the expected payment schedule satisfies

$$c_i^*(t_i) = E\left\{ V_i(T) p_i^*(T) - \nu_{ii} \int_{\underline{t}_i}^{t_i} p_i^*(x, T_{-i})\, dx \,\Big|\, T_i = t_i \right\}. \tag{1.15}$$

Proof : Write the seller's objective function as

$$
\begin{aligned}
U_0(p,c) &= E\{\textstyle\sum_{i=1}^n c_i(T_i)\} \\
&= \sum_{i=1}^n E\{V_i(T) p_i(T)\} - \sum_{i=1}^n E\{V_i(T) p_i(T) - c_i(T_i)\} \\
&= \sum_{i=1}^n E\{V_i(T) p_i(T)\} - \sum_{i=1}^n E\{U_i(p,c,T_i)\} \\
&= \sum_{i=1}^n E\{V_i(T) p_i(T)\} - \sum_{i=1}^n E\left\{ U_i(p,c,\underline{t}_i) + \nu_{ii} \int_{\underline{t}_i}^{T_i} Q_i(p,x)\, dx \right\} \\
&= \sum_{i=1}^n E\left\{ V_i(T) p_i(T) - \nu_{ii} \frac{1 - F_{T_i}(T_i)}{f_{T_i}(T_i)} p_i(T) \right\} - \sum_{i=1}^n U_i(p,c,\underline{t}_i) \\
&= \sum_{i=1}^n E\left\{ \left(V_i(T) - \nu_{ii} \frac{1 - F_{T_i}(T_i)}{f_{T_i}(T_i)} \right) p_i(T) \right\} - \sum_{i=1}^n U_i(p,c,\underline{t}_i) \tag{1.16}
\end{aligned}
$$

where the penultimate equality comes from

$$
\begin{aligned}
E\left\{ \int_{\underline{t}_i}^{T_i} Q_i(p,x)\, dx \right\} &= \int_{\mathbf{T}_i} \left(\int_{\underline{t}_i}^{t_i} Q_i(p,x)\, dx \right) f_{T_i}(t_i)\, dt_i \\
&= \int_{\underline{t}_i}^{\bar{t}_i} \left(\int_{\underline{t}_i}^{t_i} \int_{\mathbf{T}_{-i}} p_i(x, t_{-i}) f_{T_{-i}}(t_{-i})\, dt_{-i}\, dx \right) f_{T_i}(t_i)\, dt_i \\
&= \int_{\underline{t}_i}^{\bar{t}_i} \int_{\mathbf{T}_{-i}} \left(\int_x^{\bar{t}_i} p_i(x, t_{-i}) f_{T_i}(t_i)\, dt_i \right) f_{T_{-i}}(t_{-i})\, dt_{-i}\, dx \\
&= \int_{\underline{t}_i}^{\bar{t}_i} \int_{\mathbf{T}_{-i}} (1 - F_{T_i}(x)) p_i(x, t_{-i}) f_{T_{-i}}(t_{-i})\, dt_{-i}\, dx \\
&= \int_{\mathbf{T}} \frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)} p_i(t) f_T(t)\, dt \\
&= E\left\{ \frac{1 - F_{T_i}(T_i)}{f_{T_i}(T_i)} p_i(T) \right\}.
\end{aligned}
$$

The bidders' expected payments, $c(\cdot)$, only appear in $U_i(p,c,\underline{t}_i)$, so maximizing the seller's utility requires the minimization of $U_i(p,c,\underline{t}_i)$. From equation (1.13) and the definition of

$Q_i(p, t_i)$ I may write

$$U_i(p, c, t_i) = U_i(p, c, \hat{t_i}) - \nu_{ii} \int_{\hat{t_i}}^{t_i} Q_i(p, x)\, dx$$

$$= \mathrm{E}\{V_i(T) p_i(T) | T_i = t_i\} - c_i(t_i) - \nu_{ii}\, \mathrm{E}\left\{\int_{\hat{t_i}}^{t_i} p_i(x, T_{-i})\, dx\right\}. \qquad (1.17)$$

But, by (1.7), $U_i(p, c, \hat{t_i}) \geq 0$, and the best the seller can get is to make this term zero. This is done by setting $c_i(t_i)$ as in (1.15). ∎

To complete the characterization of an optimal auction the optimal allocation rules, $p(\cdot)$, must be described. The following proposition provides the problem that the seller has to solve to determine them.

Proposition 1.3 *Consider an auction (p^*, c^*) satisfying (1.15). Let p^* solve the following problem*

$$\max_{p(\cdot)} \quad E\left\{\sum_{i=1}^n \left(V_i(T) - \nu_{ii} \frac{1 - F_{T_i}(T_i)}{f_{T_i}(T_i)}\right) p_i(T)\right\}$$

$$s.t. \quad \begin{cases} (t_i - \hat{t_i})\, Q_i(p, t_i) \geq (t_i - \hat{t_i})\, Q_i(p, \hat{t_i}) \\[2mm] \sum_{i=1}^n p_i(t) \leq 1 \\[2mm] p_i(t) \geq 0\,. \end{cases} \qquad (\mathcal{P}1.1)$$

Then (p^*, c^*) is an optimal auction.

Proof : The proof follows immediately from (1.16) and Propositions 1.1 and 1.2. ∎

Optimal auctions are easily characterized from Propositions 1.2 and 1.3. The objective function in problem $(\mathcal{P}1.1)$ is linear in $p(\cdot)$; so, ignoring the constraint related to the convexity of the utility function, the solution to the problem would require, for any t, $p_i(t)$ to be equal to 1 for the term with the largest coefficient, provided this coefficient is positive, and zero for all other terms. Assumption 1.6 ensures that the functions $Q_i(\cdot, \cdot)$ associated to those $p_i(\cdot)$ will satisfy the additional constraint.[20] The precise characterization of the solution is given in

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[20]For fixed t_{-i}, if t_i increases, $\mathrm{E}\{V_i(T) | T = t\} - \nu_{ii}(1 - F_{T_i}(t_i))/f_{T_i}(t_i)$ does not decrease; hence $p_i(t)$ does not decrease. Therefore, $Q_i(p, t_i)$ does not decrease and the condition follows.

Proposition 1.4.

Before, I define some new functions. First, let me define the functions $\pi_i(t)$ as follows

$$\pi_i(t) \equiv E\{V_i(T)|T = t\} - \nu_{ii}\frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)}, \qquad (1.18)$$

which is the contribution to the seller's expected utility of allocating the good to bidder i, when the vector of signals is t.[21] Second, I define the functions $T_i^*(t_{-i})$ as

$$T_i^*(t_{-i}) \equiv \inf\left\{\hat{t}_i \;\middle|\; \pi_i(\hat{t}_i, t_{-i}) \geq \max_{j \neq i}\{0, \pi_j(\hat{t}_i, t_{-i})\}, \hat{t}_i \in T_i\right\}, \qquad (1.19)$$

which will be interpreted as bidder i's minimum winning announcement, above which he will get the good and below which he will not get it. Thus, one has the following proposition.

Proposition 1.4 *Let an auction* (p^*, c^*) *satisfy*

$$p_i^*(t) = \begin{cases} 1 & \text{if } t_i > t_i^* \\ 0 & \text{otherwise} \end{cases} \qquad (1.20)$$

$$c_i^*(t_i) = E\{V_i(T_i^*, T_{-i})p_i^*(T)|T_i = t_i\} \qquad (1.21)$$

Then (p^*, c^*) *is an optimal auction.*

Proof : With the new notation, the objective function in problem $(\mathcal{P}1.1)$ can be written as $E\{\sum_{i=1}^n \pi_i(T)p_i(T)\}$. Consider expression (1.20); for each realization of T it requires $p_i^*(t)$ to be equal to 1 for the term with the highest positive coefficient on the objective function of problem $(\mathcal{P}1.1)$, and all others $p_j^*(t)$ to be 0. This allocation rule clearly maximizes the objective function of problem $(\mathcal{P}1.1)$ subject to the two constraints on $p_i(t)$. For it to be a solution to problem $(\mathcal{P}1.1)$, it must be checked that $Q_i(p^*, t_i)$ is increasing in t_i; if $t_i < \hat{t}_i$,

$$Q_i(p^*, \hat{t}_i) = Q_i(p^*, t_i)$$
$$+\text{Prob}\left\{\pi_i(t_i, T_{-i}) \leq \max_{\{j \neq i\}} \pi_j(t_i, T_{-i}), \max_{\{j \neq i\}} \pi_j(\hat{t}_i, T_{-i}) < \pi_i(\hat{t}_i, T_{-i})\right\},$$

---

[21]Often referred in the literature as *virtual valuations.*

and $Q_i(p^*, \hat{t}_i) > Q_i(p^*, t_i)$. So (1.20) solves ($\mathcal{P}1.1$). Then, use (1.19) and (1.20) in (1.15) to get (1.21). ∎

The optimal auctions are, in general, very complex, and present properties that are not immediate extensions of those in Myerson (1981):

1. The seller sets a required minimum announcement for each bidder, $T_i^*(t_{-i})$, which is the realization of T_i such that $\pi_i(t_i^*, t_{-i}) = 0$, given t_{-i}.[22] Thus, in general, the required minimum reports are random variables, that depend on T. This result provides a theoretical justification for the use of random reservation prices in auctions, which has explicitly been used in recent empirical works (e.g., Hendricks and Porter, 1988, Hendricks et al., 1989, and Hendricks et al., 1990, for auctions of gas and oil leases, and Genesove, 1991, for auctions of cars). It contrasts with its analog in the independent private valuations model, where the optimal minimum bids are numbers that can be set before the bids are announced.

2. The good is allocated to the bidder with the highest positive $\pi_i(t)$; if no $\pi_i(t)$ is positive, the seller will keep the good. However, due to possible asymmetries in the valuation functions, even if the private signals are drawn from the same distribution, the good may not be allocated to the bidder that reports the highest private information. This result contrasts with its analog in the independent private values model, where if the signals are drawn from the same distribution, the winner must be the bidder that reported the highest signal.

3. A bidder's expected payment is equal to the good's expected value to him had his private signal been equal to his minimum winning announcement. This corresponds to a direct extension of its analog for the optimal auctions in Myerson (1981).

The results in this section provide a characterization of the optimal revelation mechanism, but does not address the relevant issue of implementation. Having characterized the optimal

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[22]This equation may have no solution. In that case, the seller sets t_i^* equal to $\underline{t}_i$, and in the optimal auction there will be no required minimum report for bidder i, in the sense that, even if he has the lowest possible signal, he has a positive probability of getting the good.

auctions in the general model, in the next section I would like to discuss the possibility of its implementation through standard auctions.

1.5 On the Optimality of Standard Auctions

It is known that under the assumption of independent and identically distributed private values the optimal mechanism can be implemented through any standard auction (see, e.g., Myerson, 1981).[23] This result is a particular case of Proposition 1.4. However, in the model, the structure of the optimal auctions suggests that there may be other environments in which the optimal mechanism can be implemented by standard auctions. In this section I present a set of symmetry conditions defining a common value model and show that the first price, the second price and the English auctions, with appropriate minimum bid requirements, are optimal, while the Dutch auction can not be optimal.[24] The main reason for the suboptimality of the Dutch auction is related to the fact that it conveys less information to the seller than any of the other three standard auctions; in particular, the seller does not get the information he needs to determine the required minimum bids.

1.5.1 A Symmetric Common Value Model

Besides the assumptions of the general model I consider the following assumptions:

Assumption 1.7 *The private signals are identically distributed, for all bidders, i.e.,* $T_i = T_j$ *and* $F_{T_i}(\cdot) = F_{T_j}(\cdot) = F(\cdot)$, *for any* i *and* j.

Assumption 1.8 *The effect of bidder* i*'s private information on bidder* j*'s value, is the same for all* i *and* j, *i.e.,* $\nu_{ji} = \nu$ *for a positive* ν.

From the results in Section 1.4 it can be seen that, if bidders are heterogeneous, the optimal auction will treat them asymmetrically. On the other hand, standard auctions treat the bidders

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[23]Provided that Assumption 1.6 is satisfied.

[24]In the analysis of standard auctions attention is restricted to the symmetric equilibria. There may exist asymmetric equilibria in which the seller's revenue is not maximized. See Milgrom (1981), Levin and Harstad (1986), Maskin and Riley (1986), Bikhchandani and Riley (1991) and Harstad (1991) on conditions for the existence and properties of asymmetric equilibria in standard auctions.

symmetrically; thus to identify conditions under which the optimal auction can be implemented by standard auctions, conditions ensuring bidders' homogeneity are needed. This is the reason for Assumptions 1.7 and 1.8.

Assumption 1.9 *The hazard rate $h_T(t) = f_T(t)/(1 - F_T(t))$ is nondecreasing in t.*

In this model, a more restrictive regularity condition on the density function is needed. The less restrictive assumption considered in Section 1.4 was only used in the proof of Proposition 1.4 to ensure that, when computing the optimal solution to problem $(\mathcal{P}1.1)$, the first constraint could be ignored, and stochastic allocation rules were not optimal. In this section, because I want to identify an environment where standard auctions are optimal, the decision to give the good to the bidder that submits the highest bid (which, because the equilibrium bids will be strictly increasing in the bidder's private signal, is equivalent to say the bidder that reports the highest private signal) must be optimal. Assumption 1.9 appears as a result of this additional requirement. Still, most of the commonly used densities satisfy this hazard rate condition.

The model of this section is a common value model, i.e., the value of the good is the same for all bidders. The analysis will show that standard auctions can implement the optimal mechanism in this model. Thus, I observe that a key assumption for the optimality of standard auctions is not the assumption of independent and identically distributed private *valuations* but the assumption of independent and identically distributed private *signals*.[25]

For the analysis of the symmetric common value model I introduce an additional definition. When deciding his bid in a standard auction, each bidder will need to form expectations about the true valuation, other bidders' private signal, and the seller's required minimum announcement. In particular, bidder i will form expectations based on the order statistics of the vector $\{T_{-i}, T_i^*\}$; let $T_{i(1)} > \cdots > T_{i(n-1)}$ denote the order statistics of T_{-i}; then, T_i^*, as defined in Section 1.4, will be the highest between $T_{i(1)}$ and T_i^*.

Next, I describe the optimal auction in the symmetric common value model, and analyze the standard auctions: first price, second price, English, and Dutch auction.

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[25]Standard auctions are optimal in the independent private values case and in the symmetric common value case. It can be shown that they are still optimal in any case in which the valuations can be written as a particular convex combination of those in the previous two cases. The question to whether they are optimal in other cases remains open.

1.5.2 Optimal Auctions

The design of the optimal auction is obtained directly from the general model's optimal auction, described in Proposition 1.4, incorporating the additional assumptions of the symmetric common value model.

Proposition 1.5 *Let (p^*, c^*) be an optimal auction in the symmetric common value model. Then*

$$p_i^*(t) = \begin{cases} 1 & \text{if } t_i > t_i^* \\ 0 & \text{otherwise} \end{cases} \tag{1.22}$$

$$c_i^*(t_i) = E\{V(T_i^*, T_{-i}) p_i^*(T) | T_i = t_i\} \tag{1.23}$$

Proof : The proof is a direct application of the results of Proposition 1.4 to the symmetric common value model. In this model,

$$\pi_i(t) = \nu \sum_{j=1}^n t_j - \nu \frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)} ,$$

so, by Assumption 1.9, the highest $\pi_i(t)$ is associated with the highest private signal and, by Proposition 1.4, the optimal $p_i(t)$ satisfies (1.22) To get (1.23) I use (1.22) and the additional assumptions in this section into (1.21). ∎

The optimal auction of the symmetric common value model, described in Proposition 1.5, has the seller setting appropriate required minimum announcements and allocating the good to the bidder reporting the highest signal, provided that it is higher than the required minimum signal. A bidder's expected payment is equal to the expected value of the good, had he received a private signal equal to his minimum winning signal.

The characterization of the optimal auction contrasts with known results in common value models, where the seller can fully extract the surplus and achieve the first best (e.g., Crémer and McLean, 1985 and 1988, and McAfee et al., 1989). The important difference between those models and this model lies on the relationship between the true value and the private signals. Those common value models take the valuation as a random variable whose realization is not

influenced by the bidders' private signals, while the model in this section considers a direct link between private signals and valuation, which is sufficient to require the seller to give rents to the winner.

In what follows it is proven that the optimal auction can be implemented by some standard auctions, with the appropriate minimum bid requirements.

1.5.3 First Price Auction

The analysis starts with the first price auction.

Definition 1.5 *In a* **first price auction** *the good is allocated to the bidder that submits the highest bid, if higher than the minimum bid set by the seller for this bidder. He pays his bid, and other bidders pay nothing.*

The following lemma has been proven, in a more general framework, by Milgrom and Weber (1982).[26]

Lemma 1.2 *Consider a first price auction with n bidders, without required minimum bids. Let bidder i's bid be*

$$b^F(t_i) = E\Big\{ V(T_{i(1)}, T_{-i}) \Big| T_{i(1)} < t_i, T_i = t_i \Big\} .$$

The vector $(b^F(t_1), \ldots, b^F(t_n))$ *forms a Bayesian equilibrium of the first price auction.*

Proof : See Appendix A1. ∎

The equilibrium strategies given in the lemma have a clear interpretation. Suppose that the equilibrium bid is increasing in the bidder's private signal; then a bidder knows that the fact that he wins means that his private signal is higher than any other private signal. Therefore, to correct for the winner's curse, he is willing to bid the expected valuation of the good, conditional on having the highest private signal, i.e., $T_i > T_{i(1)}$.

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[26]Milgrom and Weber (1982) characterize the equilibrium of standard auctions in a general model with affiliated signals. Here I use a version of their results, applied to my particular model.

An immediate consequence of Lemma 1.2 is that the winner, while paying less than the expected value of the good given his own information, in some states of the nature will pay more than the expected value of the good, given all the available information.[27]

Lemma 1.2 is useful to prove the following proposition.

Proposition 1.6 *Let the seller set the following required minimum bids*

$$\underline{b}^F(t_{-i}) = E\left\{V(T^*_i, T_{-i})|\, T_{i(1)} < \underline{t}^*_i\right\};.$$ (1.24)

The first price auction with minimum bids $\underline{b}^F(t_{-i})$ implements the optimal auction in the symmetric common value model.

Proof : Let (p^F, c^F) be the probability and expected payment functions this first price auction implies. First, note that $\underline{b}^F(t_{-i}) = b^F(\underline{t}^*_i)$, i.e., the required minimum bid can be interpreted as the bid submitted by the seller when he has private signal $\underline{t}^*_i$; and the seller can be seen as using a strategy of the type defined in Lemma 1.2. Therefore, according to Lemma 1.2, if bidder i's bidding strategy is

$$b^F(t_i) = E\{V(T^*_i, T_{-i})|\, T^*_i < t_i,\, T_i = t_i\}\,,$$

the vector $(b^F(t_1), \ldots, b^F(t_n))$ forms a Bayesian equilibrium of the first price auction with minimum bids $\underline{b}^F(t_{-i})$. Bid $b^F(t_i)$ is increasing in t_i, because $\nu > 0$, and the conditional expectation of any of the order statistics is increasing in t_i. Therefore, the highest bid will be submitted by the bidder with the highest t_i, and the first price auction, with minimum bids $\underline{b}^F_i(t_{-i})$, allocates the good optimally, i.e.,

$$p^F_i(t) = \begin{cases} 1 & \text{if } t_i > t^*_i \\ 0 & \text{otherwise.} \end{cases}$$ (1.25)

Bidder i's expected payment is

$$c^F_i(t_i) = E\left\{b^F(T_i)\, p^F_i(T)\,\middle|\, T_i = t_i\right\}$$

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[27]This type of result does not appear in the case of the independent private values model, because a bidder's bid is never larger than his own valuation.

$$= \mathrm{E}\Big\{ \mathrm{E}\{V(T_i^*, T_{-i})| T_i^* < T_i, T_i\} \, p_i^F(T) \Big| T_i = t_i \Big\}$$

$$= \mathrm{E}\Big\{ V(T_i^*, T_{-i}) \, p_i^F(T) \Big| T_i = t_i \Big\} \, .$$

where the last equality results from the definition of $p_i^F(T)$, which is 1 if and only if $T_i^* < T_i$ and 0 otherwise.

Therefore, (p^F, c^F) satisfies conditions (1.22) and (1.23), and the first price auction implements the optimal auction, in the symmetric common value model. ∎

There is an important difference between the first price auction commonly used in the literature and the first price auction described in Proposition 1.6. In the latter, the minimum bids set by the seller are endogenous and cannot be announced in advance. In fact, the seller would need to follow the specific rules to compute the minimum bids, after all bids have been submitted, committing himself to these rules. This makes the analysis of the first price auction more difficult than usual; the seller must compute the equilibrium bids and invert them to obtain t_{-i}. It also extends to the complexity of the calculations needed to determine the equilibrium bids, which are much more complex than their analogues in the independent private values model.[28] However, there are many situations where first price auctions, without preannounced required minimum bids, are used.

Having analyzed the first price auction, I move to the analysis of the second price auction.

1.5.4 Second Price Auction

In this subsection, a second price auction that implements the optimal auction of the symmetric common value model is designed. The structure of the subsection is similar to that of the previous subsection: it starts defining a second price auction, characterizes the equilibrium, and uses these results to construct the optimal second price auction.

Definition 1.6 *In a* **second price auction** *the good is allocated to the bidder that submits the highest bid, if higher than the minimum bid set by the seller for this bidder. He pays the highest between the second highest bid and his required minimum bid. Other bidders pay nothing.*

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[28] Note that in the computation a bidder needs to incorporate the rules that define the required minimum bids.

Milgrom and Weber (1982) provide a characterization of the equilibrium bids in a second price auction.

Lemma 1.3 *Consider a second price auction with n bidders, without required minimum bids. Let bidder i's bid be*

$$b^S(t_i) = E\left\{V(T)\mid T_{i(1)} = t_i, T_i = t_i\right\}.$$ (1.26)

The vector of bids $(b^S(t_1), \ldots, b^S(t_n))$ forms a Bayesian equilibrium of the second price auction.

Proof: See Theorem 6 in Milgrom and Weber (1982). ∎

The reasoning for the computation of the equilibrium bid in the second price auction is similar to that for the computation of the equilibrium bids in the first price auction. The difference is that the bidder knows that, if he wins, he will pay the second highest bid; thus, a bidder submits a bid equal to the maximum that he may be willing to pay for the good, i.e., the expected value of the good if the second highest bid is submitted by an agent with a private information equal to his own information.[29]

A first remark on the previous result is that, unlike the equilibrium bids of the second price auction in the independent private values model, the equilibrium is no longer in dominant strategies. The reason for this result lies on the fact that a bidder's true valuation depends on others' private information, so that, when deciding his bid he still needs to compute expectations over the others' private information, which does not happen in the independent private values model.

Using Lemma 1.3, I prove the following proposition.

Proposition 1.7 *Consider the second price auction with required minimum bids*

$$\underline{b}^S(t_{-i}) = E\left\{V(T)\mid T_{i(1)} = \underline{t}_i^*, T_i = \underline{t}_i^*\right\}.$$ (1.27)

The second price auction with minimum bids $\underline{b}^S(t_{-i})$ implements the optimal auction in the symmetric common value model.

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[29]From this fact, it may be clear that the winner's expected payment has to be equal to the winner's expected payment in the first price auction.

Proof: Let (p^S, c^S) be the probability and expected payment functions that this second price auction induces. First, note that $\underline{b}^S(t_{-i}) = b^S(\underline{t}_i^*)$. Thus, if the required minimum bids are interpreted as the seller's bids when he has the signal $\underline{t}_i^*$, they have the structure of the bids given in Lemma 1.3. Then, according to Lemma 1.3, in equilibrium, bidder i's bid will be given by

$$b^S(t_i) = \mathrm{E}\left\{V(T)\,\Big|\,T_{i(1)} = t_i,\, T_i = t_i\right\}.$$

Bid $b^S(t_i)$ is increasing in t_i, because $\nu > 0$, and the conditional expectation of any of the order statistics is increasing in t_i. Therefore, the highest bid will be submitted by the bidder with the highest t_i, and the second price auction, with minimum bids $\underline{b}^S(t_{-i})$, allocates the good optimally, i.e.,

$$p_i^S(t) = \begin{cases} 1 & \text{if } t_i > t_i^* \\ 0 & \text{otherwise.} \end{cases} \tag{1.28}$$

Bidder i's expected payment is

$$
\begin{aligned}
c_i^S(t_i) &= \mathrm{E}\left\{b^S(T_i^*)p_i^S(T)\,\Big|\,T_i = t_i\right\} \\
&= \mathrm{E}\left\{\mathrm{E}\left\{V(T)|T_{i(1)} = T_i^*,\, T_i = T_i^*,\, T_i^*\right\}p_i^S(T)\,\Big|\,T_i = t_i\right\} \\
&= \mathrm{E}\left\{\mathrm{E}\{V(T_i^*, T_{-i})|\,T_i^* < T_i,\, T_i\}p_i^S(T)\,\Big|\,T_i = t_i\right\} \\
&= \mathrm{E}\left\{V(T_i^*, T_{-i})p_i^S(T)\,\Big|\,T_i = t_i\right\}.
\end{aligned}
$$

where the last equality results from the definition of $p_i^S(T)$, which is equal to 1 if and only if $T_i^* < T_i$ and is equal to 0 otherwise.

Therefore, (p^S, c^S) satisfies conditions (1.22) and (1.23), and the second price auction, with minimum bids $\underline{b}^S(t_{-i})$, implements the optimal auction, in the symmetric common value model. ∎

Unlike the situation in the independent private values model, in this common value model the second price auction has some properties very close to those of the first price auction: the equilibrium strategies require similar calculations,[30] and, in some states of the nature, the

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[30]This applies not only to the bidders but to the seller as well.

winner pays more than the expected value of the good, conditional on the existing information. These results contrast with the properties of the English auction that is studied in the next subsection.

1.5.5 English Auction

The English auction is certainly the most used auction mechanism. In this subsection it is shown that the English auction can also be used to implement the optimal auction of the symmetric common value model. I will also argue that some of the characteristics of this particular auction may explain the preference that it appears to have in many situations.

Definition 1.7 *An* **English auction** *is an oral auction with increasing bids. Bids increase continuously, with all agents observing the price at which every bidder has stopped, until all but one bidder stop bidding; then, the good is allocated to the active bidder, if he pays the maximum between the last bid and the minimum bid submitted by the seller. Other bidders pay nothing.*

In fact, the analysis is based on a variant of the oral English auction, that Milgrom and Weber (1982) refer to as the Japanese English auction, where the price increases continuously and each bidder observes the price at which every bidder withdraws. These authors proved a general result that, in the context of my model, gives the following lemma.

Lemma 1.4 *Consider an English auction with n bidders and no minimum required bids. Let bidder i bid according to the following rule, that I denote by $b^E(t_i)$:*

"Remain active until all other bidders stop or the bid reaches

$$E\{V(T)|T_k = T_i \, (k \ active), \, T_j = t_j \, (j \ not \ active), \, T_i = t_i\} \, ."$$

The vector $(b^E(t_1), \ldots, b^E(t_n))$ forms a Bayesian equilibrium of the English auction.

Proof : See Theorem 10 in Milgrom and Weber (1982). ■

The reasoning behind the behavior in Lemma 1.4 is very simple. Should a bidder not stop at the prescribed price and stay in the bidding at a higher price, and all the remaining bidders

will have private signals higher than his own private signal; therefore, if he is the last active bidder, he will pay more than what the good is worth to him; on the other hand, stopping at a lower price will allow for the possibility that the bidder will not receive the good in situations in which he would have been willing to pay more than the price paid by the winner; thus, he is better off following the prescribed rules.

The rules in Lemma 1.4 do not immediately specify equilibrium strategies, which give a bidder's stopping price as a function of his private information and other bidders' observed stopping prices; however, one can invert the stopping bids to compute the bidders' private information. In this symmetric common value model, because the valuation is linear in the bidders' private signals, this inversion is very simple. The adopted notation (i.e., conditioning expectations on the private signals instead of on the observed stopping prices) makes the comparison with the analogous results in the other standard auctions easier than if expectations were conditional on stopping prices.

There is an important difference between the English auction and the previous (sealed bid) auctions. Because the English auction is an oral ascending auction and because in its equilibrium the bidder with the highest private signal wins, all the private information of the losers is revealed during the bidding process. Thus, a bidder should stop when the price is equal to the value of the good if the private information of all the remaining active bidders was equal to his own information. As a result, the actual price paid by the winner will never be higher than the expected valuation of the good, given all the existing information. This, in contrast with the previous auctions, shows that, in this model, the English auction inherits some of the properties it exhibits in the case of independent private values.

The optimality result is given in the following proposition.

Proposition 1.8 *Consider the English auction with required minimum bids, after bidder i is the only one active,*

$$\underline{b}^E(t_{-i}) = E\{V(T_i^*, T_{-i}) | T_{-i} = t_{-i}\} , \qquad (1.29)$$

The English auction with minimum bids $\underline{b}^E(t_{-i})$ implements the optimal auction in the symmetric common value model.

Proof : Consider the strategies in Lemma 1.4. Bidder i will remain active until the price reaches

$$\bar{b}^E(t_j \,(j \text{ not active}), t_i) = \mathrm{E}\{V(T) | \, T_k = T_i \,(k \text{ active}), T_j = t_j \,(j \text{ not active}), T_i = t_i\} \,.$$

First note that $\underline{b}^E(t_{-i}) = \bar{b}^E(t_{-i}, t_i^*)$, i.e., the required minimum bids may be interpreted as the seller's final bid if his private information is t_i^* and it is of the type described in Lemma 1.4. Therefore, bidder i will bid according to the strategies in Lemma 1.4. Then, the bid at which bidder i decides to stop is increasing in t_i, and the last bidder to stop is the one with the second highest private signal. After he stops, if t_i^* is higher than the private signal revealed by the last bidder, the seller still increases the bid. Let bidder i be the last one to remain active; then the bidding will stop at

$$\bar{b}^E(t_{-i}) = \mathrm{E}\{V(T_i^*, T_{-i}) | \, T_{-i} = t_{-i}\}$$

and the good is allocated if $t_i > t_i^*$. Hence, the English auction with minimum bids $\underline{b}^E(t_{-i})$ allocates the good optimally, i.e., letting (p^E, c^E) be the probability and expected payment functions it implies,

$$p_i^E(t) = \begin{cases} 1 & \text{if } t_i > t_i^* \\ 0 & \text{otherwise.} \end{cases} \tag{1.30}$$

Bidder i's expected payment is

$$\begin{aligned}
c_i^E(t_i) &= \mathrm{E}\left\{ \bar{b}^E(T_{-i}) \, p_i^E(T) \Big| T_i = t_i \right\} \\
&= \mathrm{E}\left\{ \mathrm{E}\{V(T_i^*, T_{-i}) | T_{-i}\} \, p_i^E(T) \Big| T_i = t_i \right\} \\
&= \mathrm{E}\left\{ \mathrm{E}\{V(T_i^*, T_{-i}) | T_i^* < T_i, T_i\} \, p_i^E(T) \Big| T_i = t_i \right\} \\
&= \mathrm{E}\left\{ V(T_i^*, T_{-i}) \, p_i^E(T) \Big| T_i = t_i \right\} .
\end{aligned}$$

where the last equality results from the definition of $p_i^E(T)$, which is equal to 1 if and only if $T_i^* < T_i$ and is 0 otherwise.

  Therefore, (p^E, c^E) satisfies conditions (1.22) and (1.23), and the English auction implements the optimal auction, in the symmetric common value model. ■

Even though, the conditions to define optimal strategies in an English auction were written in a more complex way than those for the sealed bid auctions, the calculations they require are much simpler.

An interesting feature of the implementation of the optimal English auction is that it suggests that the seller must have the possibility to raise the bids after all but one bidder have stopped bidding. This can be achieved in several ways. One of them is the practice of *lift lining* or *phantom bidding*, unlawful in many auctions but referred by Graham et al. (1990) as very common, where the seller would participate in the bidding process as any other agent; an alternative, mentioned in Genesove (1991), for auctions of cars, is to let the seller decide whether or not to accept the final bid, and, if not, to make a *take it or leave it* offer to the last bidder. Both these characteristics had not yet been addressed as features of optimal auctions and reasons for their use remained unclear.[31]

1.5.6 Dutch Auction

The Dutch auction is an oral descending bid auction, less used than the other standard auctions. In this subsection, it will be shown that, in the symmetric common value model, a Dutch auction cannot implement the optimal mechanism.

Definition 1.8 *A **Dutch auction** is an oral auction with decreasing bids. The auctioneer starts with a very high price, and continuously decreases it until either a minimum price is reached or a bidder announces his willingness to buy at the current price. In the former case the seller keeps the good; in the latter case, the good is allocated to the bidder that stopped the price, who pays the current price. Other bidders pay nothing.*

Suppose that the minimum bid is fixed before the auction. From the bidders' point of view, the Dutch and the first price sealed bid auctions are strategically equivalent, and there is a one-to-one correspondence between strategies and outcomes in one auction and strategies and

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[31]Graham et al. (1990) show that, in a English auction with heterogeneous bidders, phantom bidding improves the seller's utility over a policy of a fixed minimum bid. However, this is a second best result, because English auctions are not optimal with heterogeneous bidders.

The use of a final *take-it-or-leave-it* can be understood as an attempt of the seller for the extraction on a larger surplus from the winner. However, this does not seem to be its propose because if it was the seller should always use it, which is not the case.

outcomes in the other auction. This is the reason why the Dutch auction is also optimal in the independent private values model and why it does not deserve a specific treatment in Milgrom and Weber (1982).

However, in this model there is a crucial difference. The required minimum bids in an optimal auction are determined endogenously and, to compute them, the seller needs to have access to all bidders' private information, which the Dutch auction does not provide. Therefore, the Dutch auction cannot implement the optimal mechanism, even though it may correctly allocate the good among bidders.

There is a modification of the Dutch auction that would implement the optimal auction, but it is not seen in practice nor it seems to have very attractive properties. In such a modified Dutch auction the auctioneer would keep continuously decreasing the price until all bidders had announced when they wanted to stop. From these announcements the seller could then compute the optimal minimum bid and decide whether or not to allocate the good to the bidder that first announced his willingness to buy it. This auction appears to suffer from many incentive problems.[32] Even though this variant may implement the optimal mechanism, the previous reasons justify that I refrain from a more detailed characterization of the Dutch auction.

1.5.7   Average Losing Price Auction

In Subsection 1.5.4 I showed that the second price auction implements the optimal mechanism but, unlike the situation in the independent private values model, this is no longer in dominant strategies. Following Fudenberg and Tirole (1991), a motivation to look at dominant strategy implementation is that the seller may prefer to use it, because it does not depend on the players' beliefs about each other and it does not require players to go through the calculation of the Bayesian equilibrium strategies.

The structure of the symmetric common value model creates a particular problem to the determination of a dominant strategy auction. In fact, every bidder cares about the other bidders not only through their announcements in the auctions (which influence the allocation of the good and the expected payment) but also through their private information (which

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[32]It is hard to justify why would any bidder want to stay in the auction room to announce his willingness to pay after knowing that he would not receive the good.

influence the valuation).[33] Hence, in this section I construct a mechanism that implements the optimal mechanism, in which the bidders' optimal strategy is independent of their beliefs about the distribution of the others' private signals, even though they are not dominant strategies.[34]

In the auction with the previous characteristics the *ex-post* utility of any bidder that truthfully reports his signal should be larger than his *ex-post* utility from any other report. Because, in equilibrium, every bidder will report the truth, this auction satisfies incentive compatibility constraints that are stronger than (1.2), and, for all i, t, t_i and $\hat{t}_i$, can be written as

$$\mathrm{E}\{V(T)p_i(T)|T=t\} - c_i(t_i) \geq \mathrm{E}\left\{V(T)p_i(\hat{t}_i, T_{-i})\Big| T = t\right\} - c_i(\hat{t}_i). \tag{1.31}$$

I then define a simple auction, which I call the average losing price auction, that will be shown to implement the optimal mechanism and satisfies conditions (1.31).

Definition 1.9 *In an* **average losing price auction** *the good is allocated to the bidder that submits the highest bid, if higher than the minimum bid set by the seller for this bidder. He pays the average of the second highest bid (including the seller's bid) and all other bidders' bids. Other bidders pay nothing.*

There are clear reasons to justify this type of auction in the context of the symmetric common value model. First, exactly as in the independent private values case, the second highest bid must be used to establish a limit to the rents left to the winner. Second, in this common value model the valuation of the good depends on all the available private information, therefore the price paid must depend on all the submitted bids.[35]

The following lemma characterizes the equilibrium strategies in the average losing price auction, when there are no minimum bids.

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[33]It can be shown that the symmetric common value model does not belong to the class of models analyzed in Mookherjee and Reichelstein (1989), that allow the implementation of the optimal Bayesian mechanism in dominant strategies, because they restrict attention to mechanisms where each agent only cares about others announcements, in the extent that they affect the allocation. For a description of their assumptions and results see Fudenberg and Tirole (1991).

[34]The auction in this subsection is a sealed bid auction that has exactly the same outcome as the English auction described in Subsection 1.5.5.

[35]Because the incentive constraints are satisfied for every realization of types, the price must depen'. on the actual submitted bids and not expectations over others' types.

Lemma 1.5 *Consider an average losing price auction with n bidders and no required minimum bids. Let bidder i's bid be*

$$b^{AL}(t_i) = E\{V(T)|T_j = t_i\,(\forall j)\}\,. \tag{1.32}$$

The vector of bids $(b^{AL}(t_1),\dots,b^{AL}(t_n))$ forms an equilibrium of the average losing price auction.

Proof : See Appendix A1. ■

It is important to notice that the optimal bids in the average losing price auction do not require the bidders to compute expectations over the others' private signals. Any bidder will just bid the expected value of the good if all the private signals were equal to his own. Hence, the equilibrium bids are very simple and robust to changes on the bidders' beliefs about the distributions over others' private signals.

The implementation of the optimal mechanism with the average losing price auction will only require the seller to set appropriated minimum bids that ensure that the good is optimally allocated. Such an auction is presented in the following proposition.

Proposition 1.9 *Consider the average losing price auction with required minimum bids*

$$\underline{b}^{AL}(t_{-i}) = E\{V(T^*_i, T_{-i})|T_j = T^*_i\,(\forall j)\}\,. \tag{1.33}$$

The average losing price auction with minimum bids $\underline{b}^{AL}(t_{-i})$ implements the optimal auction in the symmetric common value model.

Proof : Let (p^{AL}, c^{AL}) be the probability and expected payment functions that this average losing price auction induces. First, note that $\underline{b}^{AL}(t_{-i}) = b^{AL}(\underline{t}^*_i)$. Thus, the required minimum bids can be interpreted as the seller's bids when he has the signal $\underline{t}^*_i$, and has the structure of the bids given in Lemma 1.5. Then, according to Lemma 1.5, in equilibrium, bidder i's bid will be given by

$$b^{AL}(t_i) = E\{V(T)|T_j = t_i\,(\forall j)\}\,.$$

Bid $b^{AL}(t_i)$ is increasing in t_i, because $\nu > 0$. Therefore, the highest bid will be submitted by the bidder with the highest t_i, and the average losing price auction, with minimum bids

$\underline{b}^{AL}(t_{-i})$, allocates the good optimally, i.e.,

$$p_i^{AL}(t) = \begin{cases} 1 & \text{if } t_i > t_i^* \\ 0 & \text{otherwise.} \end{cases} \qquad (1.34)$$

Bidder i's expected payment is

$$
\begin{aligned}
c_i^{AL}(t_i) &= \mathrm{E}\Big\{ \tfrac{1}{n}\Big(b^{AL}(T_i^*) + \sum_{j=1, j\neq i}^{n} b^{AL}(T_j)\Big)\, p_i^{AL}(T)\Big| T_i = t_i\Big\} \\
&= \mathrm{E}\Big\{ \tfrac{1}{n}\Big(\mathrm{E}\{V(T_i^*, T_{-i})|T_k = T_i^*\} + \sum_{j=1, j\neq i}^{n} \mathrm{E}\{V(T_j, T_{-j}|T_k = T_j\}\Big)\, p_i^{AL}(T)\Big| T_i = t_i\Big\} \\
&= \mathrm{E}\Big\{ \mathrm{E}\{V(T_i^*, T_{-i})| T_i^* < T_i, T_i\}\, p_i^{AL}(T)\Big| T_i = t_i\Big\} \\
&= \mathrm{E}\Big\{ V(T_i^*, T_{-i})\, p_i^{AL}(T)\Big| T_i = t_i\Big\}.
\end{aligned}
$$

where the last equality results from the definition of $p_i^{AL}(T)$, which is equal to 1 if and only if $T_i^* < T_i$ and is equal to 0 otherwise.

Therefore, (p^{AL}, c^{AL}) satisfy conditions (1.22) and (1.23), and the average losing price auction, with minimum bids $\underline{b}^{AL}(t_{-i})$, implements the optimal auction, in the symmetric common value model. ∎

The fact that the average losing price auction implements the optimal auction in strategies that are independent of the bidders' beliefs about the distributions of others' types ensures that it has some attractive properties. The equilibrium bids are very simple, and any bidder only needs to bid as if every one else had his signal; also, the price paid in any contingency is lower than the expected value of the good, given all the information.

1.6 Application: Auctioning Incentive Contracts

In this section, illustrating an application of the model to a procurement problem, I would like to use the example introduced in Subsection 1.3.4 and the results from Sections 1.4 and 1.5 to study the properties and implementation of an optimal auction of incentive contracts. The analysis generalizes that in Laffont and Tirole (1987) to the case where the firms' efficiency

parameter is common to all the potential contractors.[36] I show that the main properties of the optimal mechanism in the private costs case are extended to the common cost model; in particular, the dichotomy property, that the optimal incentive scheme is the same as that in the single firm case, is preserved. The intuition should be clear; because of the assumption of independent private estimates of the efficiency parameter, the competition among firms will only serve to truncate the distribution of the winner's private estimate in exactly the same way it does in the private costs model; therefore, the optimal effort will be the same as that in the single firm case; in particular, the optimal target cost will only change by the exact effect of others' private estimates of the efficiency parameter on the winner's cost function.

1.6.1 The Model

I use the model in Laffont and Tirole (1987) with the modification on the cost functions given in Subsection 1.3.4.

There are n firms that can realize a project. Firm i's cost of realizing the project is,[37]

$$C_i = \frac{1}{n} \sum_{j=1}^{n} t_j - e_i \tag{1.35}$$

where e_i is manager i's effort level and t_i is firm i's private estimate of the efficiency parameter.[38] The cost parameters are drawn independently from the same distribution with continuous density $f(\cdot)$ on $[\underline{t}, \overline{t}]$, which we assume is bounded below by a strictly positive number and exhibits a monotonic hazard rate, with $F(\cdot)/f(\cdot)$ nondecreasing.

The utility function of manager i is

$$U_i = \tau_i - \psi(e_i) \tag{1.36}$$

where τ_i is a net monetary transfer received from the regulator and $\psi(\cdot)$ is the manager's disutility of effort, with $\psi' > 0$, $\psi'' > 0$ and $\psi''' \geq 0$.

---

[36] A more complete characterization of the optimal auction could allow for some cost heterogeneity, with private and common elements.

[37] Here I only analyze the case of the symmetric common costs model. I conjecture that the main results are kept in a more general model.

[38] We can interpret t_i as firm i's private estimate of the efficiency parameter.

The social value of the project is S. For simplicity, I will assume S to be high enough so that, even if the true efficiency parameter is $\bar{t}$, it is worth doing the project.[39]

There is a social cost, associated with distortionary taxation, of raising funds by the regulator, λ.

The regulator does not observe the technological parameters nor the effort level chosen by the manager of the winning firm, but observes the firm's realized cost.

1.6.2 Optimal Auction

The regulator organizes an auction to select the firm that will realize the project.[40]

In this framework, one looks for mechanisms in which firms announce their private cost parameters, t_i, and the regulator decides on triplets of the form $(p_i(t), K_i(t), \tau_i(t_i))$, where $p_i(\cdot)$ is the probability that firm i is selected, $K_i(\cdot)$ is the cost required from firm i, if selected; and $\tau_i(\cdot)$ is the expected transfer to firm i.

Firm i's *ex-ante* expected utility can be written as

$$U_i(t_i) = \tau_i(t_i) - \mathrm{E}\left\{ p_i(T)\, \psi\left(\frac{1}{n}\sum_{j=1}^{n} T_j - K_i(T)\right) \middle| T_i = t_i \right\}.$$

It is worth noting that in the optimal auction there will be a *one-to-one* correspondence between the level of the (expected) transfer and the utility of a manager. Hence, in the decision variables of the regulator, the optimal transfer can be replaced by the utility level of the firms.

---

[39]This assumption implies that in the optimal auction the project is awarded with probability one.

[40]The analysis in this section follows very closely the analysis of Laffont and Tirole (1991) of the private cost case. A more detailed analysis is presented in Appendix A2.

In Appendix A2 I show that the regulator's problem can be written as

$$
\max_{p(\cdot),K(\cdot),U(\cdot)} \quad \mathrm{E}\left\{ \left(\sum_{i=1}^{n} p_i(T)\right) S - \lambda \sum_{i=1}^{n} U_i(T_i) \right.
$$
$$
\left. - (1+\lambda) \sum_{i=1}^{n} p_i(T) \left(K_i(T) + \psi\left(\frac{1}{n}\sum_{i=1}^{n} T_j - K_i(T)\right) \right) \right\}
$$

$$
\text{s.t.} \quad \begin{cases} \dfrac{\partial U_i(t_i)}{\partial t_i} = -\dfrac{1}{n} \mathrm{E}\left\{ p_i(T)\, \psi'\left(\tfrac{1}{n}\sum_{j=1}^{n} T_j - K_i(T)\right) \Big| T_i = t_i \right\} \\[2mm] U_i(\bar{t}) = 0 \\[2mm] \sum_{i=1}^{n} p_i(t) \leq 1 \\[2mm] p_i(t) \geq 0 \,. \end{cases} \qquad (\mathcal{P}1.2)
$$

The first step to find a solution to problem ($\mathcal{P}1.2$) is to characterize the optimal required cost. Lemma A2.1, in Appendix A2, shows that the optimal cost, $K_i^*(\cdot)$, can be written as

$$
K_i^*(t) = \frac{1}{n} k_i^*(t_i) + \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} t_j \,,
$$

for some function $k_i^*(\cdot)$. As a result, the problem of determining the optimal $K_i(\cdot)$ is equivalent to the problem of determining the optimal $k_i(\cdot)$. Consider the problem of determining the optimal $k_i(\cdot)$, for a fixed choice of $p_i(\cdot)$. The regulator's problem can be solved isolating firm i and solving

$$
\max_{k_i(\cdot)} \quad \mathrm{E}\left\{ -\lambda\, U_i(T_i) - (1+\lambda)\, Q_i(p,T_i) \left(\frac{1}{n} k_i(T_i) + \psi\left(\frac{1}{n} T_i - \frac{1}{n} k_i(T_i) \right) \right) \right\}
$$

$$
\text{s.t.} \quad \begin{cases} \dfrac{\partial U_i(t_i)}{\partial t_i} = -\dfrac{1}{n} Q_i(p,t_i)\, \psi'\left(\dfrac{1}{n} t_i - \dfrac{1}{n} k_i(t_i) \right) \\[2mm] U_i(\bar{t}) = 0 \,; \end{cases}
$$

which is exactly the problem that defines the required cost if there was a single firm (see Appendix A2). Therefore, the dichotomy property is preserved and the winner's optimal level of effort will be the same as that in the single firm case.

Note that, because of the symmetry assumptions of the model, the optimal auction will

treat the bidders symmetrically, which implies that the solution to the previous problem, $k_i^*(\cdot)$, does not depend on the firm i considered. To emphasize this result, hereafter I will just write $k^*(\cdot)$.

It can be shown that the optimal required cost is increasing in the winner's private signal. Using this property, one can characterize the regulator's problem as

$$\max_{p(\cdot),k(\cdot)} \quad E\left\{\sum_{i=1}^{n} p_i(T)\left(S + \frac{\lambda}{n(1+\lambda)}\,\psi'\left(\frac{1}{n}T_i - \frac{1}{n}k(T_i)\right)\frac{F_{T_i}(T_i)}{f_{T_i}(T_i)}\right.\right.$$
$$\left.\left. -(1+\lambda)\left(\frac{1}{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} T_j + k(T_i) + \psi\left(\frac{1}{n}T_i - \frac{1}{n}k(T_i)\right)\right)\right)\right\}$$

$$\text{s.t.} \quad \begin{cases} \dfrac{\partial U_i(t_i)}{\partial t_i} = -\dfrac{1}{n}E\left\{p_i(T)\,\psi'\left(\frac{1}{n}T_i - \frac{1}{n}k(T_i)\right)\Big|\,T_i = t_i\right\} \\[2mm] U_i(\bar{t}) = 0 \\[2mm] \sum_{i=1}^{n} p_i(t) \leq 1 \\[2mm] p_i(t) \geq 0\,. \end{cases} \qquad (\mathcal{P}1.3)$$

Problem $(\mathcal{P}1.3)$ is the analogue to problem $(\mathcal{P}1.1)$ in this particular application. The objective function is also linear in the probabilities; the first and second constraints ensure incentive compatibility and individual rationality, respectively.

To finish the analysis I would then need to specify who is the winner and the transfer schedule. Because of the properties of the optimal required cost function, proven in Appendix A2, it can be seen that the coefficient of $p_i(T)$ in the objective function is nondecreasing in T_i; hence, from the general model, we know that in the optimal auction the winner will be the firm that reports the lowest private cost parameter, thus generalizing the result in Laffont and Tirole (1987). Moreover, competition among firms serves only to truncate the distribution of the private cost parameter, in the same way as in the private values case, and reduces the lump-sum transfer made by the regulator to the winner. Using the relation between the managers' utilities and the expected transfers, I can then complete the characterization of the optimal

auction writing the expression for the expected transfer

$$\tau_i^*(t_i) = Q_i(p, t_i)\,\psi\!\left(\frac{1}{n}t_i - \frac{1}{n}k^*(t_i)\right) + \frac{1}{n}\int_{t_i}^{\bar{t}} Q_i(p, x)\,\psi'\!\left(\frac{1}{n}x - \frac{1}{n}k^*(x)\right)dx. \qquad (1.37)$$

So, like in the private costs case, the regulator uses the existence of other potential contractors to reduce the expected transfer paid. In the case of a single firm, analyzed in detail in Laffont and Tirole (1986), the expression for the expected transfer is the special case of (1.37), with $Q_i(p, x) = 1$.

1.6.3 Implementing the Optimal Auction

Having characterized the optimal auction, it is important to study simple mechanisms that implement the optimal auction. In this subsection I construct two of such mechanisms: in the first mechanism the firms bid for the target costs and the regulator decides on the winner, transfers and required costs according to the bids and some pre-specified rules; in the second mechanism the firms bid for the transfer to be paid by the regulator and the target cost, and the regulator's decisions are based on a scoring rule, which aggregates a bidder's bid into a single number.

Laffont and Tirole (1987) describe the implementation of the optimal auction through a dominant strategy mechanism. In the first part of this subsection I would like to concentrate on a simple mechanism that implements the optimal auction of the previous subsection not in dominant strategies but in strategies that are independent of players' beliefs over the distribution of types, a modification of the average losing price auction, following the analysis in Subsection 1.5.7.[41]

In Section 1.5 I showed that the average losing price auction implements the optimal auction of the symmetric common value model in strategies that do not depend on beliefs about the distribution of types. Here I use that result to construct a modified average losing price auction (which I will refer as the average losing cost auction) that implements the optimal auction of incentive contracts.

Consider the following auction. Let firms announce their target costs, which will be functions

---

[41]It is also possible to construct modifications of the standard auctions that implement the optimal mechanism.

of the firm's private estimate, $C_i^{AL}(t_i)$. The regulator selects the firm submitting the lowest cost, let it be firm i, which is required to produce at cost

$$K_i^{AL}(C_1^{AL}, \ldots, C_n^{AL}) = \frac{1}{n} k^*(C_i^{AL}) + \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} C_j^{AL},$$

where $k^*(\cdot)$ is the optimal function determined in the previous subsection; the transfer to the winner is

$$\tau_i^{AL}(C_1^{AL}, \ldots, C_n^{AL}) = \psi\left(\frac{1}{n} C_i^{AL} - \frac{1}{n} k^*(C_i^{AL})\right) + \frac{1}{n} \int_{C_i^{AL}}^{C_{i(1)}^{AL}} \psi'\left(\frac{1}{n} x - \frac{1}{n} k^*(x)\right) dx.$$

Following the analysis of the dominant strategy auction in Laffont and Tirole (1987), it can be checked that the mechanism just described implements the optimal auction in the common cost case; in particular, contingent on any other firm bidding truthfully, the dominant bid of each firm is $C^{AL}(t_i) = t_i$, i.e., the firm bids the cost that it would have if all other firms had the same private estimate.

In procurement situations it is probably more common that bidders submit bids not only for the target cost but also for the net transfer. Under this circumstances it is particularly simple to construct a mechanism based on the score of each bid, given by a single value that aggregates the two components of the bid; in such a mechanism (which is designated by a *scoring rule*) the winner would be firm whose bid had the lowest score.

Let (c_i^{SR}, C_i^{SR}) be the bid of firm i, where c_i^{SR} is the net transfer to the manager and C_i^{SR} is the target cost. As Laffont and Tirole (1991) have shown, the simplest scoring rule, which corresponds to the summation of the two elements of the bid, would not create the right incentives on the firms, and a modified scoring rule, which aggregates the bid for the net transfer with a modified cost target, must be used. In my model, there is a second reason for the consideration of a modified cost, which is the fact that the optimal required cost depends on every firms' information.

From the bids, the regulator computes bidder i's modified cost as

$$\tilde{C}_i(C^{SR}) = \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} k^{*-1}(C_j^{SR}) + \frac{1}{n} \left(\int_{\underline{t}-e^*}^{C_i^{SR}} \psi'\left(\frac{1}{n} k^{*-1}(x) - \frac{1}{n} x\right) dx + (\underline{t} - e^*) \right)$$

which is used to compute the scores, $s_i^{SR} = c_i^{SR} + \tilde{C}_i(C^{SR})$. Then the project is awarded to the firm that has the lowest score, say firm i, which is required to produce at cost

$$K_i^{SR}(C^{SR}) = \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} k^{*-1}(C_j^{SR}) + \frac{1}{n} C_i^{SR}$$

and receives a net transfer equal to the highest transfer that it could have asked for and still be the winner (so the transfer that would give it the same score as the second lowest score). Following the analysis in Laffont and Tirole (1991), for the private cost case, it can be checked that this modified scoring rule induces firms to bid a net transfer equal to the optimal disutility of effort and a target cost equal to $k^*(t_i)$, hence the optimal auction is implemented.

1.6.4 Using Linear Contracts

The implementation of the optimal auction described in the previous subsection is not robust to the addition of a random element to the realized cost. The transfer is optimal if and only if the realized cost is not random, so that the winner can be required to produce at cost $K_i^*(T)$.

Let me now consider that the cost functions are

$$C_i = \frac{1}{n} \sum_{j=1}^{n} t_j - e_i + \varepsilon_i,$$

where ε_i is a white noise independent of the private signals. In this framework, Laffont and Tirole (1987) showed that linear transfers of the form

$$\tau_i(t, K_i) = a_i(t) - b(t_i)(C_i - K_i^*(t))$$

where $a_i(t)$ is the optimal transfer in the certainty case, $b(t_i) = \psi'(\frac{1}{n} t_i - \frac{1}{n} k^*(t_i))$, C_i is the realized cost, and $K_i^*(t)$ is the target cost, are optimal.

It is possible to change the average losing cost auction of the previous subsection to be robust to the addition of noise. The firms bid for an expected target cost, C_i^{AL}. Then, the fixed transfer, $a_i(C_1^{AL}, \ldots, C_n^{AL})$, is equal to the lump-sum transfer in the average losing cost auction, $\tau_i^{AL}(C_1^{AL}, \ldots, C_n^{AL})$; the expected target cost is $K_i^{AL}(C_1^{AL}, \ldots, C_n^{AL})$; and the share of cost over-runs, $b(C_i^{AL})$, is equal to $\psi'\left(\frac{1}{n} C_i^{AL} - \frac{1}{n} k^*(C_i^{AL})\right)$.

1.7   Concluding Remarks

The model presented in this chapter sheds some light on the issue of optimal auctions of an indivisible good, covering a continuum of cases between two extreme cases: the independent private values model and the common value model.

The characterization of optimal auctions provided some interesting findings that help to understand how auctions really work. As in the case of the independent private values model, the optimal auction does not, in general, require the good to be allocated to the bidder that submits the highest bid. Also, the seller sets minimum bid requirements which, in general, are not achieved through constant minimum bids, set *ex-ante*, but are determined endogenously, as a function of the bidders' announcements.

Then I looked at a symmetric common value model where the first price, the second price and the English auction can implement the optimal mechanism, with the use of random required minimum bids. This feature explains the suboptimality of the Dutch auction. The analysis of the standard auctions also showed that *lift-lining* or a final *take it or leave it* offer by the seller may be justified in an optimal oral ascending bid auction. The simplicity of the equilibrium strategies in the English auction, relative to those in sealed bid auctions, and the fact that the price paid by the winner never exceeds the expected valuation, given the existing information, suggests elements that support its popularity.

There are some aspects, connected to this work, that are still unanswered. Keeping the assumption of indivisible goods, the issue of multiple unit auctions could be addressed as a generalization of the present work. Taking a distinct direction, a general model of optimal auctions of divisible goods would be useful to analyze some economically relevant problems (e.g., auctions of treasury bills or auctions of shares). These generalizations will, respectively, be addressed in Chapters 2 and 3 of this dissertation.

A1   Appendix: Proofs

This appendix presents proofs to some of the results in the text.

Proof of Lemma 1.1:   The right hand side of the incentive compatibility condition (1.2) can be written as

$$\mathrm{E}\big\{ V_i(T)\, p_i(\hat{t}_i, T_{-i})\big|\, T_i = t_i \big\} - c_i(\hat{t}_i)$$
$$= \mathrm{E}\Big\{ \Big(\nu_{ii} T_i + \textstyle\sum_{j \neq i} \nu_{ij} T_j \Big) p_i(\hat{t}_i, T_{-i})\Big|\, T_i = t_i \Big\} - c_i(\hat{t}_i)$$
$$= \mathrm{E}\big\{ V_i(T)\, p_i(T)\big|\, T_i = \hat{t}_i \big\} - c_i(\hat{t}_i) + \mathrm{E}\big\{ \nu_{ii}\,(t_i - \hat{t}_i)\, p_i(\hat{t}_i, T_{-i}) \big\}$$
$$= U_i(p, c, \hat{t}_i) + \nu_{ii}\,(t_i - \hat{t}_i)\, Q_i(p, \hat{t}_i)\, .$$

Therefore, (1.3) is equivalent to (1.2). ∎

Proof of Lemma 1.2:   Let

$$v(t_i, t_{i(1)}) = \mathrm{E}\big\{ V(T)\big|\, T_{i(1)} = t_{i(1)},\, T_i = t_i \big\}$$

and

$$L(x|t_i) = \frac{F_{T_{i(1)}}(x)}{F_{T_{i(1)}}(t_i)}\, ;$$

then, by Theorem 14 in Milgrom and Weber (1982), in equilibrium bidder i will bid

$$b^F(t_i) = \int_{\underline{t}_i}^{t_i} v(x, x)\, dL(x|t_i)\, .$$

Note that $L(\cdot|t_i)$ is just the cumulative distribution function of $T_{i(1)}$ conditional on $T_{i(1)} < t_i$; thus the equilibrium bid $b^F(t_i)$ can be written as in the lemma. ∎

Proof of Lemma 1.5:   I first check that (1.31) is satisfied in the average losing price auction. Rewriting the definition of the equilibrium bids one has

$$b^{AL}(t_i) = n\nu t_i\, . \tag{A1.1}$$

Let $b^{AL}_{(1)} > \cdots > b^{AL}_{(n)}$ be the order statistics of the vector of bids. The price paid by the winner is

$$\frac{1}{n}\left(b^{AL}_{(2)} + \textstyle\sum_{j=2}^{n} b^{AL}_{(j)} \right),$$

which is always lower than the expected value of the good. Suppose that a bidder bids $\hat{b}(t_i)$ greater than $b^{AL}(t_i)$; this will only affect his outcome if $\hat{b}(t_i) > b^{AL}_{(2)} > b^{AL}(t_i)$, situation in which he will get the good but paying an excess of $(b^{AL}_{(2)} - b^{AL}(t_i))/n$ over the value of the good to him. Similarly, if he bids $\hat{b}(t_i)$ lower than $b^{AL}(t_i)$; this will only affect his outcome if $b^{AL}(t_i) > b^{AL}_{(1)} > \hat{b}(t_i)$, situation in which he will not get the good while he could have had a surplus of $(b^{AL}(t_i) - b^{AL}_{(1)})/n$. Moreover, individual rationality is satisfied. Thus, it is optimal to bid as prescribed in the lemma. ∎

A2   Appendix: Auctioning Incentive Contracts (Details)

In this appendix I provide a more detailed analysis of the application presented in Section 1.6.

Let me start with the construction of the regulator's problem, described in the text as problem ($\mathcal{P}$1.2). The regulator wants to maximize the welfare which is the sum of the consumers' surplus and the firms' utilities. The consumers' surplus is given by the surplus generated by the project minus the costs

associated with it (i.e., cost of the project and transfers to the firms, taking into account the cost of raising funds); the firms' utilities were written in the text. Hence, the regulator's objective is to maximize the following function:

$$
\mathrm{E}\left\{\left(\sum_{i=1}^{n} p_i(T)\right) S - (1+\lambda)\sum_{i=1}^{n}\left(p_i(T)\,K_i(T) + \tau_i(T_i)\right)\right.
$$
$$
\left. + \sum_{i=1}^{n}\left(\tau_i(T_i) - p_i(T)\,\psi\left(\frac{1}{n}\sum_{j=1}^{n} T_j - K_i(T)\right)\right)\right\} .
$$

Simple manipulations of the previous function show that it can be written as the objective function of problem ($\mathcal{P}$1.2) given in Subsection 1.6.2.

In solving the problem I claimed that the optimal required cost, $K_i^*(\cdot)$, could be written as

$$
K_i^*(t) = \frac{1}{n}\,k_i^*(t_i) + \frac{1}{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} t_j ,
$$

for some function $k_i^*(\cdot)$. The statement and proof follow.

Lemma A2.1 *The optimal required cost function, $K_i^*(\cdot)$, is of the form*

$$
K_i^*(t) = \frac{1}{n}\,k_i^*(t_i) + \frac{1}{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} t_j .
$$

Proof : This is proven by contradiction. Suppose that $K_i^*(t)$ is not of the previous type. Observe that in the optimal auction $p_i(\cdot)$ will be either zero or one. Choose $k_i(t_i)$ as

$$
k_i(t_i) = \mathrm{E}\left\{ n\,C_i(T) - \sum_{j=1,\,j\neq i}^{n} T_j \,\middle|\, T_i = t_i,\, p_i(T) = 1\right\} . \tag{A2.1}
$$

Moreover,

$$
\mathrm{E}\left\{ p_i(T)\,\psi\left(\frac{1}{n}\sum_{j=1}^{n} T_j - C_i(T)\right)\middle|\, T_i = t_i\right\}
$$
$$
= Q_i(p,t_i)\,\mathrm{E}\left\{\psi\left(\frac{1}{n}\sum_{j=1}^{n} T_j - C_i(T)\right)\middle|\, T_i = t_i,\, p_i(T) = 1\right\} .
$$

Since $\psi(\cdot)$ is concave, by Jensen's inequality I conclude that

$$
\mathrm{E}\left\{\psi\left(\frac{1}{n}\sum_{j=1}^{n} T_j - C_i(T)\right)\middle|\, T_i = t_i,\, p_i(T) = 1\right\} \geq \psi\left(\frac{1}{n}\,t_i - \frac{1}{n}\,k_i(t_i)\right)
$$

so that, replacing $K_i^*(t)$ by $\frac{1}{n}\left(k_i(t_i) + \sum_{j=1,\,j\neq i}^{n} t_j\right)$ the problem's objective function is increased. More-over, because $\psi'(\cdot)$ is concave, by a similar argument,

$$
\mathrm{E}\left\{\psi'\left(\frac{1}{n}\sum_{j=1}^{n} T_j - C_i(T)\right)\middle|\, T_i = t_i,\, p_i(T) = 1\right\} \geq \psi'\left(\frac{1}{n}\,t_i - \frac{1}{n}\,k_i(t_i)\right)
$$

and the firms' utilities associated with the new required cost function will decrease, which also benefits the regulator. Hence, $K_i^*(\cdot)$ cannot be optimal. ∎

Because of this result, given a fixed choice of probabilities, the optimal cost can be determined

solving the following problem, whose solution is independent of the firm i considered,

$$\max_{k(\cdot)} \quad \mathrm{E}\left\{-\lambda U_i(T_i) - (1+\lambda) Q_i(p, T_i) \left(\frac{1}{n} k(T_i) + \psi\left(\frac{1}{n} T_i - \frac{1}{n} k(T_i)\right)\right)\right\}$$

$$\text{s.t.} \quad \begin{cases} \dfrac{\partial U_i(t_i)}{\partial t_i} = -\dfrac{1}{n} Q_i(p, t_i) \psi'\left(\dfrac{1}{n} t_i - \dfrac{1}{n} k(t_i)\right) \\ U_i(\bar{t}) = 0 \,. \end{cases}$$

The first order necessary conditions for a maximum of the problem are

$$\begin{cases} \dfrac{d\mu(t_i)}{dt_i} = \lambda f(t_i) \\ (1+\lambda)\left(1 - \psi'\left(\dfrac{1}{n} t_i - \dfrac{1}{n} k(t_i)\right)\right) f(t_i) = \dfrac{1}{n} \psi''\left(\dfrac{1}{n} t_i - \dfrac{1}{n} k(t_i)\right) \mu(t_i) \\ \mu(t) = 0 \,, \end{cases}$$

for some multiplier $\mu(\cdot)$. These conditions can be solved to give

$$\mu(t_i) = \lambda F(t_i)$$

and

$$\psi'\left(\frac{1}{n} t_i - \frac{1}{n} k^*(t_i)\right) = 1 - \frac{1}{n} \frac{\lambda}{1+\lambda} \frac{F(t_i)}{f(t_i)} \psi''\left(\frac{1}{n} t_i - \frac{1}{n} k^*(t_i)\right) \,;$$

so, the optimal effort is the same that would be required from the winner if there were no other firms.[42] From the previous expression it can be seen that $k^*(\cdot)$ is a nondecreasing function, thus the second order conditions of the problem are satisfied.

To finally get Problem $(\mathcal{P}1.3)$ I just manipulate the objective function of $(\mathcal{P}1.2)$, and the characterization of the solution follows directly from the analysis in Section 1.4 of this chapter.

I now study in detail the implementation of the optimal auction with the mechanisms described in Subsection 1.6.3.

In the average losing cost auction I claim that firms will bid $C_i^{AL}(t_i) = t_i$. To obtain this result, suppose that all firms but firm i bid in this way; the winner decides his bid to solve

$$\max_{C_i^{AL}} \quad \mathrm{E}\left\{\tau_i^{AL}(C_i^{AL}, T_{-i}) - \psi\left(\frac{1}{n} t_i - \frac{1}{n} k^*(C_i^{AL})\right) \,\middle|\, T_{i(1)} > C_i^{AL}, T_i = t_i\right\} \,.$$

Using the expression for $\tau_i^{AL}(\cdot)$, the first order necessary condition is then,

$$\psi'\left(\frac{1}{n} C_i^{AL} - \frac{1}{n} k^*(C_i^{AL})\right) (1 - k^{*\prime}(C_i^{AL})) - \psi'\left(\frac{1}{n} C_i^{AL} - \frac{1}{n} k^*(C_i^{AL})\right)$$

$$+ \psi'\left(\frac{1}{n} t_i - \frac{1}{n} k^*(C_i^{AL})\right) k^{*\prime}(C_i^{AL}) = 0 \,,$$

which can be simplified to

$$\psi'\left(\frac{1}{n} C_i^{AL} - \frac{1}{n} k^*(C_i^{AL})\right) - \psi'\left(\frac{1}{n} t_i - \frac{1}{n} k^*(C_i^{AL})\right) = 0$$

---

[42]To consider a comparable single firm model, I assume that the private information of the other firms in the multiple firm model is common knowledge in the single firm case.

and $C_i^{AL}(t_i) = t_i$ is the solution. Moreover, the second order condition is satisfied, because $k^*(\cdot)$ is a nondecreasing function. Hence, the average losing cost auction selects the winner optimally and provides him with the correct incentives. To prove that the auction is optimal, it remains to be shown that the expected net transfer is optimal; let $p_i^{AL}(T)$ be the optimal probability that firm i will be the winner when private signals are T (i.e., $p_i^{AL}(T) = 1$) if and only if $T_i > T_{i(1)}$, then

$$
\begin{aligned}
&E\{ \tau_i^{AL}(T)\, p_i^{AL}(T) \big| T_i = t_i \} \\
&= E\left\{ \left(\psi\left(\frac{1}{n}T_i - \frac{1}{n}k^*(T_i)\right) + \frac{1}{n}\int_{T_i}^{T_{i(1)}} \psi'\left(\frac{1}{n}x - \frac{1}{n}k^*(x)\right)\, dx \right) p_i^{AL}(T) \bigg| T_i = t_i \right\} \\
&= Q_i(p,t_i)\, \psi\left(\frac{1}{n}t_i - \frac{1}{n}k^*(t_i)\right) + \frac{1}{n}\int_{t_i}^{\bar t}\!\!\int_x^{\bar t} \psi'\left(\frac{1}{n}x - \frac{1}{n}k^*(x)\right) dF_{T_{i(1)}}(t_{i(1)})\, dx \\
&= Q_i(p,t_i)\, \psi\left(\frac{1}{n}t_i - \frac{1}{n}k^*(t_i)\right) + \frac{1}{n}\int_{t_i}^{\bar t} Q_i(p,x)\, \psi'\left(\frac{1}{n}x - \frac{1}{n}k^*(x)\right) dx \\
&= \tau_i^*(t_i),
\end{aligned}
$$

where the third equality follows from the fact that $Q_i(p,x) = 1 - F_{T_{i(1)}}(x)$. Thus, the average losing cost auction generates an optimal expected net transfer, and it is an optimal auction.

The proof that the scoring rule described in Subsection 1.6.3 also implements the optimal auction has three steps. In the first step I show that a firm with private signal t_i will want to bid a target cost equal to $k^*(t_i)$. In the second step I prove the firm will bid a net transfer equal to its optimal disutility of effort. Finally, it is proven that the expected net transfer is optimal.

Suppose that all firms but i use the bidding strategy just described. Given the rules of the auction, if firm i is the winner, its manager *ex-post* utility will be

$$
s_{i(1)}^{SR} - \left(\tilde C_i(C^{SR}) + \psi\left(\frac{1}{n}t_i - \frac{1}{n}C_i^{SR}\right) \right).
$$

Given that the first term is independent of firm i's bid, firm i bids the target cost that minimizes its generalized cost,

$$
\min_{C_i^{SR}}\ \ \tilde C_i^{SR}(C^{SR}) + \psi\left(\frac{1}{n}t_i - \frac{1}{n}C_i^{SR}\right).
$$

Considering the expression of the modified cost, and the other firms' bids, the first order condition is

$$
\psi'\left(\frac{1}{n}k^{*-1}(C_i^{SR}) - \frac{1}{n}C_i^{SR}\right) - \psi'\left(\frac{1}{n}t_i - \frac{1}{n}C_i^{SR}\right) = 0,
$$

whose solution is $C_i^{SR} = k^*(t_i)$, proving that the regulator will require the optimal cost, i.e.,

$$
K_i^{SR}(C_1^{SR}(T_1),\ldots,C_n^{SR}(T_n)) = K_i^*(T).
$$

Having decided the bid for the target cost, firm i must decide its bid for the net transfer, which is equivalent to a choice of its score. This is decided to maximize the manager's utility, which is

$$
\max_{s_i^{SR}}\ \ E\left\{ s_{i(1)}^{SR} - \left(\tilde C_i(C_1^{SR}(T_1),\ldots,C_n^{SR}(T_n)) + \psi\left(\frac{1}{n}T_i - \frac{1}{n}C_i^{SR}(T_i)\right) \right) 1_{\{s_i^{SR} < s_{i(1)}^{SR}\}} \bigg| T_i = t_i \right\},
$$

where $1_{\{s_i^{SR} < s_{i(1)}^{SR}\}}$ is equal to 1 if the condition is satisfied and 0 if it is not. It is immediate that the

previous expectation is maximized by choosing the score s_i^{SR} such that

$$s_i^{SR} < s_{i(1)}^{SR} \implies \tilde{C}_i(C^{SR}) + \psi\left(\frac{1}{n}T_i - \frac{1}{n}C_i^{SR}(T_i)\right) < s_{i(1)}^{SR}$$

which is ensured by bidding

$$c_i^{SR}(t_i) = \psi\left(\frac{1}{n}t_i - \frac{1}{n}k^*(t_i)\right),$$

the cost of the optimal effort.

With these equilibrium strategies, the winner's net transfer will be

$$\tau_i^{SR}(t_i, t_{i(1)}) = \psi\left(\frac{1}{n}t_{i(1)} - \frac{1}{n}k^*(t_{i(1)})\right) + \frac{1}{n}\int_{C_i^{SR}(t_i)}^{C_{i(1)}^{SR}(t_{i(1)})} \psi'\left(\frac{1}{n}k^{*-1}(x) - \frac{1}{n}x\right) dx.$$

Then, noting that

$$\frac{1}{n}\int_{C_i^{SR}(t_i)}^{C_{i(1)}^{SR}(t_{i(1)})} \psi'\left(\frac{1}{n}k^{*-1}(x) - \frac{1}{n}x\right) dx$$

$$= \frac{1}{n}\int_{t_i}^{t_{i(1)}} \psi'\left(\frac{1}{n}x - \frac{1}{n}k^*(x)\right) dk^*(x)$$

$$= \psi\left(\frac{1}{n}t_i - \frac{1}{n}k^*(t_i)\right) - \psi\left(\frac{1}{n}t_{i(1)} - \frac{1}{n}k^*(t_{i(1)})\right) + \frac{1}{n}\int_{t_i}^{t_{i(1)}} \psi'\left(\frac{1}{n}x - \frac{1}{n}k^*(x)\right) dx,$$

it follows that

$$\tau_i^{SR}(t_i, t_{i(1)}) = \psi\left(\frac{1}{n}t_i - \frac{1}{n}k^*(t_i)\right) + \frac{1}{n}\int_{t_i}^{t_{i(1)}} \psi'\left(\frac{1}{n}x - \frac{1}{n}k^*(x)\right) dx,$$

and, as I did to prove that the average losing cost auction generates the optimal expected net transfers, it can be proven that the scoring rule also generates the optimal expected net transfers. This finishes the proof that the auction based on scoring rules presented in the text is optimal.

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Chapter 2

Optimal Multiple Unit Auctions of an Indivisible Good

2.1 Introduction

In the previous chapter of this dissertation I studied auctions of a single indivisible good. However, a large number of observed auctions involve the sale of multiple units of a homogeneous good or several similar objects (e.g., auctions of mineral rights on federal land, offshore drilling rights, procurement contracts, estate collections of stamps, coins, or antiques, fish, flowers, wine).

Even though some basic results and intuitions developed from the single unit analysis are useful in understanding multiple unit auctions, some others are just not present or may induce erroneous conclusions. For example, the long lasting debate about the properties of sequential and simultaneous auctions and the comparison between those two types of auctions cannot be fully addressed without a multiple unit model; the trade-offs involved in the structure of the optimal minimum bids in multiple unit auctions or the optimal stopping rules in sequential auctions do not appear in single unit models; from the analysis of single unit auctions it is tempting to conclude that, under standard assumptions, the sequence of prices paid for each unit in a sequential multiple unit auction should be a martingale, but that may not the case in optimal multiple unit auctions.

Besides the empirical relevance of multiple unit auctions, there is also the theoretical mo-

tivation for the identification of optimal mechanisms for a multiple unit seller. Since Myerson (1981), it has been known that, under a regularity condition, auction-like mechanisms maximize the seller's expected revenue from the sale of a single unit of an indivisible good, but no similar general result is known for the case of multiple unit sales, except for the characterization provided in Maskin and Riley (1989) for the case of symmetric bidders and independent private values.

Among the literature on multiple unit auctions most of the analysis has concentrated on the case in which each consumer demands one unit and valuations are private and identically distributed. Ortega-Reichert (1968) analyzes the performance of three common multiple unit auctions (the discriminatory price, the uniform price and the sequential oral auction),[1] in the unit demand model and provides a revenue equivalence theorem, which states that the three auctions yield the same expected revenue to the seller. Based on an example, the properties of a multiple unit auction consisting of simultaneous and independent single unit auctions of each unit are studied in Engelbrecht-Wiggans and Weber (1979). Harris and Raviv (1981) give the first optimality result: they prove that the mechanisms analyzed in Ortega-Reichert (1968) are optimal if the buyers' valuations are uniformly and independently distributed. Because this result was based on a specific distributional assumption, the work towards the characterization of optimal auctions continued. Hausch (1986) compares the seller's expected revenue from sequential sales with those from simultaneous sales; in his model, either type of auction can dominate. Finally, Maskin and Riley (1989) provide a full characterization of optimal auctions, generalizing Harris and Raviv's result to any distribution on the valuations.

Departures from the unit demand independent private values model have been few. Weber (1983) provides some results for a multiple unit model that generalizes Milgrom and Weber (1982), including affiliation in the valuations; Engelbrecht-Wiggans and Weber (1983) characterize the equilibrium in a sequential auction with asymmetrically informed bidders; Bikhchandani and Huang (1989) compare the discriminatory and uniform price auctions in a model of affiliated valuations to conclude that under different conditions either one may be more desirable

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[1]These are, respectively, generalizations of the first price and second price sealed bid auctions and of the English oral auction. In the discriminatory price auction the units are given to the bidders who submitted the highest bids, and these bidders pay their own bids; in the uniform price auction the same bidders get the units but pay the highest rejected bid; in the sequential oral auction the units are sequentially auctioned in single unit auctions that follow the rules of the English oral auction.

for the seller; Gale (1990) describes optimal auctions in a more general model, showing that if valuations are supperadditive (i.e., the sum of the bidders' valuations from giving m units to one and none to the other is larger than that from a split of the units by both bidders) it is optimal for the seller to bundle all the units and organize an auction for the entire lot.

In this chapter I study the properties of optimal auctions when the seller has multiple indivisible units of a homogeneous good. In the general model I allow bidders to be asymmetric in all their characteristics: support and distribution of private signals, and valuation functions. Moreover, I do not restrict attention to the independent private values case; instead, a continuum of cases that go from the independent private values to common value models are accommodated.

The chapter is organized as follows: in Section 2.2, I introduce the general model of multiple units, which is a multiple unit generalization of the single unit model studied in Chapter 1. Some examples of applications of the model are presented in Section 2.3. Optimal auctions are characterized in Section 2.4. Under regularity conditions on the private signals and the valuation functions, it is shown that the optimal auction is deterministic; i.e., given the bidders' announcements the seller will want to implement a specific allocation with probability one. The optimal auction is a generalization of the single unit optimal auction. The seller allocates the units to the bidders that announce the largest private signals,[2] requires some minimum announcements, which are determined endogenously, and payments that correspond to the winners' expected valuations, had they announced their lowest winning signals, for each unit received.

In Section 2.5, I study the symmetric unit demand model and show that some generalizations of standard auctions allow the seller to implement the optimal mechanism, but some others do not.[3] In the context of an optimal sequential English auction, I show that the sequence of the prices paid for each unit forms a supermartingale. This contrasts with the private values model, because of the way optimal minimum bids are set.

Section 2.6 covers symmetric multiple unit demand models with linear valuations. In a model

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[2] In the model with multiple unit demands the precise statement is that a bidder with a higher private signal will receive at least as many units as a bidder with a lower private signal.

[3] I show that the discriminatory price, the uniform price and the sequential English auction are among those that allow the implementation of the optimal mechanism. Important cases of generalized standard auctions that do not allow it are some generalized oral auctions and the simultaneous and independent auction of each unit.

of constant valuations we show that bundling all the units and selling them as an indivisible lot implements the optimal mechanism; this extends Gale (1990) result on the optimality of bundling, in the sense that supperadditivity is not required. Then, in a model with linear and decreasing valuations, I show that the seller either sells all the units or none, but they are possibly sold to distinct bidders; moreover, the optimal auction can no longer be implemented through generalized standard auctions.

An example of a general private values model, with two bidders and two units, is fully analyzed in the Appendix. The allocation rules in the optimal auction are very complex, which is the main reason why generalized standard auctions cannot be used to implement the optimal mechanism. Instead, the seller can implement the optimal mechanism, through a modification of the discriminatory price auction, that allows the optimal allocation of the units.

Section 2.7 concludes the chapter.

2.2   A Simple Model

I consider a model of a seller who faces a number of potential buyers and is interested in finding a procedure that maximizes his expected revenue from the sale of several units of a homogeneous good. In this section the assumptions of the model, which are analogous to those used in Chapter 1, are presented and their importance for the results is discussed.

Assumptions 2.1 and 2.2 describe the supply side of the model.

Assumption 2.1 *There are m units of a homogeneous indivisible good to be sold.*

Assumption 2.2 *The seller is risk neutral and does not value the good.*

The assumption that the seller does not value the good is made only for the simplification of the analysis, and the results generalize to seller's with positive valuations for some (possibly all) units of the good.

The following set of assumptions describes the demand side and the information structure of the model.

Assumption 2.3 *There are n risk neutral utility maximizing potential buyers.*

Risk neutrality crucially influences the characterization of the optimal auction. A topic for additional research is to generalize the analysis to risk averse buyers.

The structure of information is a key element for the design of optimal auctions. Here, I follow a generalization of the structure of information in a single unit model analyzed in Chapter 1.

Assumption 2.4 *Each potential buyer has his own private information about the value of the good, T_i.*

The fact that the potential buyers are endowed with a unidimensional signal is a limitation of the model. A more general approach would allow multidimensional signals. However, the treatment of multidimensional adverse selection problems still raises many technical difficulties, as pioneer work by Laffont et al. (1987) and McAfee and McMillan (1988) clearly show. Nevertheless, even with a unidimensional signal, the model is sufficiently rich to capture properties of multiple unit auctions which cannot be studied in single unit models.

Given that the potential buyers have private information that is relevant to all of the players, they must have assessments on others' information. In the general model, I allow for heterogeneity among potential buyers; thus, the distributions over types and their supports may vary across potential buyers.

Assumption 2.5 *Other agents do not observe the realization of T_i and treat it as a draw from a cumulative distribution $F_{T_i}(\cdot)$, with support $\mathbf{T}_i = [\underline{t}_i, \bar{t}_i]$.*

The notation to describe vectors of private signals and their realizations is equal to that in the single unit model of Chapter 1.

I assume not only that signals are private, but that a bidder's information is noninformative about the other bidders' information.

Assumption 2.6 *Any two variables T_i and T_j, with $j \neq i$, are independent.*

The mechanism design technique, used in Section 2.4 to identify the optimal auction, would not be possible without the assumption of independent types. This assumption and related statements must be interpreted as conditional on all the (non modelled) available common

information; i.e., the private signals only need to be independent conditional on commonly known information.

The next set of assumptions describes the relationship between valuations and agents' private signals. In particular, the true valuations do not need to be uniquely determined by the agent's private signal, and any agent may want to revise its valuation taking into account others' signals. The assumption restricts the revision process to be additively separable in the bidders' private signals.

Assumption 2.7 *The value of the k-th unit of the good to potential buyer i, V_{ik}, satisfies*

$$E\{V_{ik}(T)|T=t\} = \sum_{j=1}^{n} \nu_{ikj}\bar{V}_{ikj}(t_j)$$

where the parameters ν_{ikj} ($j \neq i$) are nonnegative, ν_{iki} are positive, and the functions $\bar{V}_{ikj}(\cdot)$ are differentiable and increasing.

In a multiple unit model, one needs to specify relationship between the valuations for the several units. I impose a downward sloped demand function condition.

Assumption 2.8 *The expected valuation for additional units is nonincreasing, i.e., for all t,*

$$\sum_{j=1}^{n} \nu_{ikj}\bar{V}_{ikj}(t_j) \geq \sum_{j=1}^{n} \nu_{ikj}\bar{V}_{ik+1j}(t_j), \quad k \in \{1,\ldots,m-1\}.$$

The last two assumptions are technical and are made to guarantee that in the optimal auction the seller will not want to use stochastic allocation rules.

Assumption 2.9 *The expected valuations satisfy the following regularity conditions, for all t_i and $\hat{t}_i$:*

$$\frac{\bar{V}'_{iki}(t_i)}{\bar{V}_{iki}(t_i)} \leq \frac{\bar{V}'_{ik+1i}(t_i)}{\bar{V}_{ik+1i}(t_i)}$$

$$(t_i - \hat{t}_i)\left(\bar{V}'_{iki}(t_i) - \bar{V}'_{iki}(\hat{t}_i)\right) \leq 0.$$

The first condition in Assumption 2.9 is the analog of the nondecreasing price elasticity (Maskin and Riley, 1989) for the case of indivisible goods. It is sufficient to ensure that the

benefit to the seller from allocating the $k + 1$-th unit of the good to bidder i is not great r than the benefit from the allocation of the k-th unit. It will be shown that, under this condition, in the optimal mechanism the seller can simply sequentially look at the contribution to his utility from allocating an additional unit to each bidder; the optimal allocation is implemented by sequentially assigning each unit to the buyer that presents the highest positive marginal contribution, given the previously allocated units, and stopping when all units have been allocated or the highest marginal contribution is nonpositive.

The second condition in the assumption specifies that the uncertainty should be modelled in such a way that the functions $\bar{V}_{ik}(\cdot)$ are concave. It is clear that one may always satisfy this assumption; in fact, the restriction that it imposes is connected with Assumption 2.10, which is affected by the choice of random variables.

I finally introduce the standard monotonic hazard rate condition to ensure that, if the seller wants to allocate k units to a type t_i bidder when others have type t_{-i}, he will want to allocate at least k units to the type $\hat{t}_i$ bidder, with $\hat{t}_i > t_i$, when others have type t_{-i}.

Assumption 2.10 *The hazard rate $h_{T_i}(t_i) = f_{T_i}(t_i)/[1 - F_{T_i}(t_i)]$ is nondecreasing in t_i.*

Assumptions 2.1 to 2.10 define the basic environment in which optimal auctions will be characterized. But, doing that, in the next section I present some examples of applications of the general model.

2.3 Some Examples

The motivation of the model introduced in the previous section is given through several examples. Some of the examples are extensions of examples presented in Chapter 1; others provide new approaches to the model.

2.3.1 Objects of Art

An example of the application of the single unit model to study auctions of objects of art was given in Subsection 1.3.1. Here I show how the example can be adapted to cover multiple unit auctions, describing a model that fits the assumptions of Section 2.2.

There are two paintings to be auctioned between two museums, located in two distinct towns.[4] The value of each object for a museum is given by the expected number of visitors that will go to the museum to see it.

Let n_i be the population of town i. Each person in town i is characterized by two parameters (θ_1, θ_2), uniformly and independently distributed in $[0, \bar{\theta}_i]$, which represent the utility drawn from seeing each object. Let $\theta_{(1)}$ be the maximum between the two parameters, and $\theta_{(2)}$ the minimum. I assume that, because the objects are similar, a person will get utility $\theta_{(1)}$ from the first object seen, independently of which of the objects is seen. Town i's museum knows $\bar{\theta}_i$, but the other only knows that it is distributed in $[0.5, 1]$, and independently distributed across towns.

There is a cost, c_{ij}, for a resident in town i to visit the museum in town j, satisfying $c_{ii} < c_{ij}$, for $j \neq i$. To simplify the algebra, I assume that $c_{ij} < 0.5$, so that a museum displaying an object should expect to receive visitors from both towns. Moreover, I assume that if the same museum is displaying the two objects any person that wants to see both will have to incur in twice the traveling cost.[5]

Let $m_{ikj}(\bar{\theta}_i)$ be the expected number of people from town i that will be willing to visit the museum in town j to see the k-th object. Because it is less expensive to visit the home town museum and utility is not associated with the specific object, if the museum in town i has one painting, a resident in town i will first visit this museum, as along as $\theta_{(1)} > c_{ii}$; therefore,[6]

$$\frac{m_{i1i}}{n_i} = 1 - \left(\frac{c_{ii}}{\bar{\theta}_i}\right)^2 .$$

If the museum also exhibits the second object, the expected proportion of town i's population

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[4]For the simplicity of the example I consider only two objects and two bidders. The main idea generalizes to m objects and n bidders.

[5]With the traveling cost interpretation of the c_{ij}, this amounts to assume that it is not possible to see both objects in the same day.

[6]From the distributional assumption on $(\theta_{(1)}, \theta_{(2)})$ it follows that

$$\text{Prob}\{\theta_{(1)} < c_{11}\} = \left(\frac{\theta_{(1)}}{\bar{\theta}_i}\right)^2 \quad \text{and} \quad \text{Prob}\{\theta_{(2)} < c_{11}\} = 1 - \left(1 - \frac{\theta_{(2)}}{\bar{\theta}_i}\right)^2 .$$

that visit the second object is

$$\frac{m_{i2i}}{n_i} = \left(1 - \frac{c_{ii}}{\theta_i}\right)^2 .$$

Let t_i (the private signal of museum i) be the proportion of people visiting the second painting;[7] then

$$\frac{m_{i1i}}{n_i} = 1 - \left(1 - \sqrt{t_i}\right)^2 .$$

Similar algebra let me obtain the expected number of people from the town j that will visit town i's museum,

$$\frac{m_{j1i}}{n_j} = 1 - \left(\frac{c_{ji}}{c_{jj}}\right)^2 \left(1 - \sqrt{t_j}\right)^2$$

$$\frac{m_{j2i}}{n_j} = \left(1 - \frac{c_{ji}}{c_{jj}}\left(1 - \sqrt{t_j}\right)\right)^2 .$$

It will be shown that in the optimal auction the seller may want to do not sell both objects. However, to simplify the example, I will assume that the parameters are such that it will never be in the seller's interest to hold to any of the objects. In this case, if museum i only buys one object, meaning that the other museum will get the other object, because each person will strictly prefer to see his first object in his home town, museum i will receive visits from the people of town i that go there to see their first object and from the people of town j that will see their second object, i.e.,

$$E\{V_{i1}(T)|T = t\} = n_i \left(1 - \left(1 - \sqrt{t_i}\right)^2\right) + n_j \left(1 - \frac{c_{ji}}{c_{jj}}\left(1 - \sqrt{t_j}\right)\right)^2 .$$

Similarly, if the museum acquires the two objects, it will receive visitors from the home town that will see the second object and visitors from the other town that will see the first object, i.e.,

$$E\{V_{i1}(T)|T = t\} = n_i \, t_i + n_j \left(1 - \left(\frac{c_{ji}}{c_{jj}}\right)^2 \left(1 - \sqrt{t_j}\right)^2\right) .$$

So, by construction, we verify that Assumptions 2.1 to 2.7 of the general Model I are satisfied

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[7]As I mentioned in the previous section, the choice of the random variable that represents the agent's private signal must be done carefully to make it possible to satisfy all the assumptions. For example, defining t_i as the proportion of people that visit the first object, would not work in the example.

in this example. The parameters n_i and c_{ij} can be chosen in a way that makes Assumption 2.8 satisfied. Given that t_i will be in $(0, 1)$, Assumption 2.9 is satisfied. Finally, with an appropriate choice of the distribution on $\bar{\Theta}_i$, the distribution of T_i will satisfy Assumption 2.10.

2.3.2 Consumption Goods with Resale

This example generalizes the example presented in Subsection 1.3.3, assuming that the fisherman has two unit s to sell.[8] I consider the same basic assumptions: the intermediaries have clienteles in the resale market. Each buyer has a unit elastic demand function, and an intermediary has private information on his clientele's expenditure in fish, which the others treat as independent and uniformly distributed in $[1, 2]$. If an intermediary does not have the fish to sell a buyer from his clientele will buy from other seller with probability $\nu > 1/2$, in such a way that the price paid to each seller will be the same. Again, like in the previous example, I will assume that both units will be sold.

Suppose that an intermediary has bought both lots of fish. Then he will be the unique seller in the secondary market and will get the opportunity to sell to his clientele and (with probability ν) to others' clients. Because of the assumptions on the buyers' demands the expected value of the two units is

$$E\{V_{i1}(T) + V_{i2}(T) | T = t\} = t_i + \nu \sum_{\substack{j=1 \\ j \neq i}}^{n} t_j .$$

Now, suppose that he has only bought one unit, and intermediary k has bought the other. Then the consumers' expected expenditure will be

$$E\{V_{i1}(T) + V_{k1}(T) | T = t\} = t_i + t_k + \nu \sum_{\substack{j=1 \\ j \neq i,k}}^{n} t_j$$

half of which will accrue to each of the sellers.

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[8]In this example I consider the market of fish with the following structure. The fisherman captures the fish and sells it to intermediaries; those that have bought the fish then take it to a secondary market and sell it to the final consumers.

Using the previous expressions I obtain

$$\mathrm{E}\{V_{i1}(T)|\, T = t\} = \frac{1}{2}\left(t_i + t_k + \nu \sum_{\substack{j=1 \\ j \neq i,k}}^{n} t_j\right)$$

and

$$\mathrm{E}\{V_{2i}(T)|\, T = t\} = \frac{1}{2}t_i + \left(\nu - \frac{1}{2}\right)t_k + \frac{1}{2}\sum_{\substack{j=1 \\ j \neq i,k}}^{n} t_j .$$

The above expressions depend not only on the intermediaries private information but on who else gets the good, so that the previous valuations do not seem to be well defined. However, it will be seen that the bidders' private information uniquely determines the allocation of the good; thus knowing the vector t is sufficient to know who else has got the good.

It is left to be checked that with this structure of the valuations and distributional assumptions the fisherman will optimally want to sell both units. This result relies on a condition that will be derived in the next section.

In this example Assumptions 2.1 to 2.7 are clearly satisfied. Then note that

$$\mathrm{E}\{V_{i1}(T) - V_{i2}(T)|\, T = t\} = (1 - \nu)\, t_k \geq 0$$

and Assumption 2.8 is satisfied. The valuation functions are linear in the private signals, thus, in particular, $\bar{V}_{i1i}(\cdot)$ and $\bar{V}_{i2i}(\cdot)$ are concave; moreover

$$\frac{\bar{V}_{i1i}'(t_i)}{\bar{V}_{i1i}(t_i)} = \frac{1}{2\,\bar{V}_{i1i}(t_i)} \leq \frac{1}{2\,\bar{V}_{i2i}(t_i)} = \frac{\bar{V}_{i2i}'(t_i)}{\bar{V}_{i2i}(t_i)}$$

and Assumption 2.9 is satisfied.

Finally, the uniform distribution satisfies the hazard rate condition (Assumption 2.10).

2.3.3 Procurement Auctions

In this subsection I consider an example of auctions in procurement. These are usually multidimensional and very complex,[9] which involve not only the price but quality, environmental impact, probability that the project will be realized and date of delivery, among others. Here I will concentrate on auctions with one dimension: the probability that the project will be completed as agreed (characteristics and date).

A public agency is interested in having m projects realized, whose specifications are decided by the agency. The projects can be realized (jointly or independently) by any of n bidders. Let θ_{ij} be the probability that bidder i will be able to complete j projects, during the period agreed and following the specifications given by the agency. From the knowledge accumulated with the realization of other projects and some additional studies each bidder is able to obtain an estimate of the probability that he will be able to complete the m projects, t_i. This depends not only on the firm but also on contingencies specific to the project that are not known by any of the firms, on which the firms' have some prior distribution. Then, let the posterior distribution of the probability that firm i will be able to complete k projects, given the firms' private information, be such that

$$\mathrm{E}\{\Theta_{ik}(T)|\,T = t\} = \sum_{j=1}^{n} \nu_{ikj}\, T_j^{k/m}\,,$$

where the parameters ν_{ikj} are nonnegative and

$$\sum_{j=1}^{n} \nu_{ikj} = 1\,.$$

Let P_1 be the price to be paid by each project and P_2 the penalty to the firm if the project is not completed in the agreed time. Then the valuations are

$$V_{ik}(T) = (P_1 + P_2)\sum_{j=1}^{n} \nu_{ikj}\, T_j^{k/m} - P_2$$

where P_1 and P_2 may be decided in such a way that only firms that have a sufficiently high

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[9]See Chapter 14 of Laffont and Tirole (1991) for a discussion of the implications of multidimensional bids on auction design.

probability of success will have a positive expected payoff.

Then, noting that $T_i \in [0,1]$, one can confirm that the assumptions on the valuations (Assumptions 2.7 to 2.9) are satisfied in this example. The distribution of T_i can be chosen to satisfy Assumption 2.10.

2.3.4 Electricity Spot Markets

In many states of the US, electricity utilities must buy power from the producers through competitive bidding. With privatization, auction like mechanisms have also been used in the UK. Even though these mechanisms usually have multidimensional characteristics, as mentioned in the previous subsection, here I will just consider that the electricity utilities care only about the price that they must pay for the electricity.

In this example consider that electricity is bought in some standard amounts, each normalized as one unit. The electricity utility announces its will to buy m units. There are n power plants that can supply the required electricity at some cost, privately known: let $\bar{V}_{iki}(t_i)$ be the cost to plant i of supplying the k-th unit, where t_i is a privately known efficiency parameter (so this is an example of a private values model).[10] Then the assumptions on the private signals can be chosen to make the assumptions satisfied.

2.4 Optimal Auctions

I now study the optimal mechanisms in the model introduced in the previous section. In general terms, the analysis of multiple unit auctions differs from that of single unit auctions in the greater complexity of the set of feasible allocations. Because agents are risk neutral, a mechanism is simply a mapping from the set of vectors of announcements to a set of probability distributions over feasible allocations and expected payments of each bidder. In the single unit case, there is a *one-to-one* correspondence between winning bidders and feasible allocations. In the multiple unit model, however, no such simple mapping exists, so my analysis of feasible auctions must start with a characterization of feasible allocations.

---

[10]One can introduce correlations between costs.

Definition 2.1 *An* **allocation** *is a vector* $(k_1, \ldots, k_n)$, *such that each element* k_i, *the number of units allocated to agent* i, *is a nonnegative integer and*

$$\sum_{i=1}^{n} k_i \leq m. \tag{2.1}$$

The set of all allocations is denoted by $A(m, n)$.

A particular case of this definition is an allocation in the single unit model: n-dimensional vectors with at most one element equal to 1 and all other elements equal to 0.

Following the Revelation Principle (Myerson, 1979), without loss of generality, I focus on direct revelation mechanisms, in which the potential buyers reveal their private signals and, according to the reports and rules previously specified, the seller decides on the allocation of the goods among bidders and their respective payments.

The formal description of an auction makes clear the complexity that the set of feasible allocations in the multiple unit model brings to the analysis. While in the single unit model the probability that a given bidder will receive the unit is exactly equal to the probability that a given allocation (the one in which the good is allocated to this bidder) will be implemented, in the multiple unit model there is not such an immediate correspondence. Given the assumptions of the model, to compute the expected utilities of the agents it is sufficient to know the probability distribution over the number of units allocated to each bidder and their expected payments. Hence, while a complete analysis of the mechanisms would require the consideration of the probabilities associated to each feasible allocation, a great simplification of the analysis is introduced by the characterization of the mechanisms in terms of the probabilities over the number of units received by each bidder.[11] Therefore, a multiple unit auction is defined as follows.[12]

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[11]Clearly, this will only be a simplification to the extent that one may ignore part of the complex set of restrictions that these probability functions must satisfy.

[12]With this definition I am imposing a constraint on the feasible mechanisms by restricting the seller to mechanisms in which all the payments are made at the time of the auction. More generally, I could consider mechanisms that would require an announcement by the winner, once he learns the true value of the good, which would allow the seller to extract the full surplus.

Definition 2.2 *An* **auction** *is any pair of functions* (p, c), *with*

$$c_i : T_i \to \mathbf{R},$$

and such that there exists a function $\rho(\cdot; \cdot)$ *with*

$$\rho : \mathbf{T} \times \mathcal{A}(m, n) \to [0, 1],$$

satisfying, for all i, k, $(k_1, \ldots, k_n)$ *and* t,

$$p_{ik}(t) = \sum_{\substack{(k_1, \ldots, k_n) \in \mathcal{A}(m, n) \\ k_i \geq k}} \rho(k_1, \ldots, k_n; t) \tag{2.2}$$

$$\sum_{(k_1, \ldots, k_n) \in \mathcal{A}(m, n)} \rho(k_1, \ldots, k_n; t) \leq 1 \tag{2.3}$$

$$\rho(k_1, \ldots, k_n; t) \geq 0. \tag{2.4}$$

In an auction (p, c), given announcements t, a generic element $p_{ik}(t)$ is the probability that bidder i will receive at least k units, and $c_i(t)$ is bidder i's expected payment; the auxiliary element $\rho(k_1, \ldots, k_n; t)$ is the probability that the allocation $(k_1, \ldots, k_n)$ is implemented.

Condition (2.2) says that the probability that bidder i receives at least k units is equal to the sum of the probability over all feasible allocations in which at least k units are given to player i. Conditions (2.3) and (2.4) impose that $\rho(\cdot; t)$ is a probability distribution over the set of feasible allocations.

The analysis of optimal auctions is greatly simplified if the mechanisms are described in terms of (p, c), without explicitly considering the complex set of restrictions that the probabilities ought to satisfy. I replace conditions (2.2) to (2.4) by a set of weaker conditions, which are only written in terms of the probabilities $p_{ik}(\cdot)$, that will be sufficient for the characterization of the optimal mechanism.

Lemma 2.1 *Let (p,c) be an auction. Then the following conditions must be satisfied, for all i, k, and t:*

$$\sum_{i=1}^{n}\sum_{k=1}^{m} p_{ik}(t) \leq m \qquad (2.5)$$

$$p_{ik}(t) \geq p_{i\,k+1}(t) \qquad (2.6)$$

$$p_{ik}(t) \leq 1 \qquad (2.7)$$

$$p_{ik}(t) \geq 0 \,. \qquad (2.8)$$

Proof : See Appendix B1. ∎

Conditions (2.5) to (2.7) are just necessary conditions, and it is possible to construct p's satisfying them for which there are no ρ's satisfying (2.2) to (2.4).[13] However, for a restricted set of auctions these conditions are also sufficient. These auctions are described in the following lemma.

Lemma 2.2 *Let p satisfy conditions (2.5) to (2.7) and $p_{ik}(t) \in \{0,1\}$, for all i, k, and t. Then there exists a function ρ satisfying (2.2) to (2.4).*

Proof : If $p_{i1}(t) = 0$, let $\hat{k}_i = 0$; if $p_{i1}(t) = 1$, let $\hat{k}_i$ be the highest k such that $p_{ik}(t) = 1$. From (2.5) it follows that $\sum_{i=1}^{n} \hat{k}_i \leq m$, so by Definition 2.1 $(\hat{k}_1,\ldots,\hat{k}_n)$ is an allocation. Define the probability distribution over allocations as $\rho(\hat{k}_1,\ldots,\hat{k}_n;t) = 1$ and $\rho(k_1,\ldots,k_n;t) = 0$ for any other allocation. By construction the function $\rho(\cdot;t)$ satisfies (2.2) to (2.4). ∎

If an auction (p,c) satisfies the requirements in Lemma 2.2 the seller is sure about the allocation to be implemented, given the vector of announcements. I will refer to such an auction as a deterministic auction.

In the remainder of the section I will describe auctions in terms of (p,c) without explicitly considering all the requirements in Definition 2.2, but only those in Lemma 2.1; as a conse-

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[13]Consider the following example. Let $m = n = 2$; set $p_{11}(t) = 1$, $p_{12}(t) = 0.2$, $p_{21}(t) = 0.5$ and $p_{22}(t) = 0.3$. Clearly conditions (2.5) to (2.7) are satisfied. However, the fact that $p_{11}(t) = 1$, meaning that bidder 1 gets one unit for sure, implies that, for the ρ's satisfying (2.2) to (2.4) to exist, it must be that $p_{22}(t) = 0$, i.e., bidder 2 cannot get two units for sure. Hence, the p's in the example cannot correspond to an auction.

quence, most of the results are just necessary conditions that are only sufficient for deterministic auctions. This still allows a satisfactory description of the feasible auctions, because in the main result of this section, I will show that the optimal auction is always deterministic.

The remainder of the analysis adapts the technique used in Chapter 1 for single unit auctions to the analysis of multiple unit auctions. The seller's utility from any auction (p, c) is determined by the expected payments of the bidders, and can be written as

$$U_0(p, c) = E\{\sum_{i=1}^n c_i(T_i)\} .$$

On the other hand, the buyers care not only about their expected payments but about the number of units they will receive as well, and buyer i's utility can be written as

$$U_i(p, c, t_i) = E\{\sum_{k=1}^m V_{ik}(T) p_{ik}(T) | T_i = t_i\} - c_i(t_i) .$$

Note that in this expression I am exploiting the fact that $p_{ik}(t)$ is the probability that bidder i receives at least k units, so bidder i will enjoy utility from the k-th unit with probability $p_{ik}(t)$. To simplify the characterization of feasible auctions, generalizing the standard approach to single unit auctions, I will denote bidder i's expected probability of getting at least k units, when his private signal is t_i, by $Q_{ik}(p, t_i)$, i.e.,

$$Q_{ik}(p, t_i) = E\{p_{ik}(T) | T_i = t_i\} . \tag{2.9}$$

Since the analysis is restricted to mechanisms in which bidders report the true private information feasible auctions are defined as in Definition 1.4. The individual rationality condition is that bidders' utilities are nonnegative (Definition 1.3), and so is the same as that for single unit auctions; the incentive compatibility requirement is now written as

$$U_i(p, c, t_i) \geq E\left\{\sum_{k=1}^m V_{ik}(T) p_{ik}(\hat{t}_i, T_{-i}) \middle| T_i = t_i\right\} - c_i(\hat{t}_i) . \tag{2.10}$$

The right hand side of condition (2.10) is the expected utility of a bidder with type t_i who announces type $\hat{t}_i$. Therefore, condition (2.10) states that a truthful report maximizes the utility of any bidder. Like in the case of single unit auctions, the incentive compatibility

requirement can be written in terms of levels of utility associated with two distinct reports, as in the following lemma.

Lemma 2.3 *An auction (p, c) is incentive compatible if and only if, for all i, t_i, and $\hat{t}_i$*

$$U_i(p, c, t_i) \geq U_i(p, c, \hat{t}_i) + \sum_{k=1}^{m} \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}_{iki}(\hat{t}_i) \right) Q_{ik}(p, \hat{t}_i). \tag{2.11}$$

Proof : See Appendix B1. ∎

Condition (2.11) gives a lower bound to the difference in the utility levels associated with any two distinct private signals.

In looking for the optimal auction, attention is restricted to feasible auctions. Proposition 2.1 is a weak generalization of the result for the single unit model (Proposition 1.1), and provides a set of necessary conditions for feasible auctions, which is also sufficient for deterministic auctions that satisfy an additional requirement.

Proposition 2.1 *If an auction (p, c) is feasible then, for all i, k, t, t_i and $\hat{t}_i$,*

$$\sum_{k=1}^{m} \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}_{iki}(\hat{t}_i) \right) Q_{ik}(p, t_i) \geq \sum_{k=1}^{m} \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}_{iki}(\hat{t}_i) \right) Q_{ik}(p, \hat{t}_i) \tag{2.12}$$

$$\frac{\partial U_i(p, c, t_i)}{\partial t_i} = \sum_{k=1}^{m} \nu_{iki} \bar{V}'_{iki}(t_i) Q_{ik}(p, t_i) \tag{2.13}$$

$$U_i(p, c, t_i) \geq 0 \tag{2.14}$$

$$\sum_{i=1}^{n} \sum_{k=1}^{m} p_{ik}(t) \leq m \tag{2.15}$$

$$p_{ik}(t) \geq p_{i\,k+1}(t) \tag{2.16}$$

$$p_{ik}(t) \leq 1 \tag{2.17}$$

$$p_{ik}(t) \geq 0. \tag{2.18}$$

Conversely, a deterministic auction, satisfying conditions (2.13) to (2.18) and

$$(t_i - \hat{t}_i) Q_{ik}(p, t_i) \geq (t_i - 0\hat{t}_i) Q_{ik}(p, \hat{t}_i) \tag{2.19}$$

is feasible.

Proof : I first show necessity. Use condition (2.11) from Lemma 2.3 twice, to show that incentive compatibility implies

$$\sum_{k=1}^{m} \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}_{iki}(\hat{t}_i) \right) Q_{ik}(p, \hat{t}_i) \le U_i(p, c, t_i) - U_i(p, c, \hat{t}_i)$$

$$\le \sum_{k=1}^{m} \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}_{iki}(\hat{t}_i) \right) Q_{ik}(p, t_i), \quad (2.20)$$

which implies (2.12).

Divide the terms in (2.20) by $(t_i - \hat{t}_i)$, and take limits as $\hat{t}_i \to t_i$. The result in the center is $\partial U_i(p, c, t_i)/\partial t_i$, while both bounds converge to $\sum_{k=1}^{m} \nu_{iki} \bar{V}'_{iki}(t_i) Q_{ik}(p, t_i)$, and (2.13) follows.

Expression (2.14) follows directly from individual rationality for the bidder with the lowest private signal. Finally, (2.15) to (2.18) follow from Lemma 2.1.

I now prove that, conditions (2.13) to (2.19) are sufficient for a deterministic auction to be feasible. From Lemma 2.2, conditions (2.15) to (2.18) ensure that (p, c) is in fact an auction. Thus, to prove that (p, c) is feasible it must be proven that (p, c) satisfies incentive compatibility and individual rationality.

I start by noting that $Q_{ik}(p, t_i)$ being nondecreasing in t_i is consistent with (2.12) because $\nu_{iki} > 0$ and $\bar{V}_{iki}(t_i)$ is increasing. Use condition (2.13) to write

$$U_i(p, c, t_i) = U_i(p, c, \underline{t}_i) + \int_{\underline{t}_i}^{t_i} \sum_{k=1}^{m} \nu_{iki} \bar{V}'_{iki}(x) Q_{ik}(p, x) \, dx ; \quad (2.21)$$

then, because $\nu_{iki} > 0$, $\bar{V}'_{iki}(\cdot) > 0$, $Q_{ik}(\cdot) \ge 0$, and condition (2.14) individual rationality is verified.

Now, I prove that (p, c) satisfies incentive compatibility. Use (2.21) to get

$$U_i(p, c, t_i) = U_i(p, c, \hat{t}_i) + \int_{\hat{t}_i}^{t_i} \sum_{k=1}^{m} \nu_{iki} \bar{V}'_{iki}(x) Q_{ik}(p, x) \, dx . \quad (2.22)$$

Again, because $\nu_{iki} > 0$, $\bar{V}'_{iki}(t_i) > 0$ and $Q_{ik}(p, t_i) \ge 0$, it follows that

$$\int_{\hat{t}_i}^{t_i} \sum_{k=1}^{m} \nu_{iki} \bar{V}'_{iki}(x) Q_{ik}(p, x) \, dx \ge \int_{\hat{t}_i}^{t_i} \sum_{k=1}^{m} \nu_{iki} \bar{V}'_{iki}(x) Q_{ik}(p, \hat{t}_i) \, dx$$

$$= \sum_{k=1}^{m} \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}_{iki}(\hat{t}_i) \right) Q_{ik}(p, \hat{t}_i), \qquad (2.23)$$

and incentive compatibility follows from (2.22), (2.23), and Lemma 2.3. ∎

The incentive compatibility constraints imply that the bidders will be given rents that are increasing with the private signal, so the bidders will not want to make a lower announcement. Since the incentive compatibility constraint does not bind on a bidder with the lowest possible signal, there is no reason for the seller to leave such a bidder with a positive rent. This condition helps to characterize the optimal payment schedule.

Proposition 2.2 *Suppose (r^*, c^*) is an optimal auction. Then a bidder reporting the lowest possible private signal has zero utility, i.e.,*

$$U_i(p^*, c^*, \underline{t}_i) = 0 , \qquad (2.24)$$

and, for all i and t_i, the expected payment schedule satisfies

$$c_i^*(t_i) = E\left\{ \sum_{k=1}^{m} \left(V_{ik}(T) p_{ik}^*(T) - \nu_{iki} \int_{\underline{t}_i}^{T_i} \bar{V}_{iki}'(x) p_{ik}^*(x, T_{-i}) \, dx \right) \middle| T_i = t_i \right\} . \qquad (2.25)$$

Proof : Write the seller's utility as

$$U_0(p, c) = E\{\sum_{i=1}^{n} c_i(T_i)\}$$

$$= \sum_{i=1}^{n} E\{\sum_{i=1}^{m} V_{ik}(T) p_{ik}(T)\} - \sum_{i=1}^{n} E\{\sum_{k=1}^{m} V_{ik}(T) p_{ik}(T) - c_i(T_i)\}$$

$$= \sum_{i=1}^{n} E\{\sum_{k=1}^{m} V_{ik}(T) p_{ik}(T)\} - \sum_{i=1}^{n} E\{U_i(p, c, T_i)\}$$

$$= \sum_{i=1}^{n} E\{\sum_{k=1}^{m} V_{ik}(T) p_{ik}(T)\} - \sum_{i=1}^{n} E\left\{ U_i(p, c, \underline{t}_i) + \int_{\underline{t}_i}^{T_i} \sum_{k=1}^{m} \nu_{iki} \bar{V}_{iki}'(x) Q_{ik}(p, x) \, dx \right\}.$$

$$= \sum_{i=1}^{n} E\left\{ \sum_{k=1}^{m} V_{ik}(T) p_{ik}(T) - \sum_{i=1}^{m} \nu_{iki} \bar{V}_{iki}'(x) \frac{1 - F_{T_i}(T_i)}{f_{T_i}(T_i)} p_{ik}(T) \right\} - \sum_{i=1}^{n} U_i(p, c, \underline{t}_i) .$$

$$= \sum_{i=1}^{n} E\left\{ \sum_{k=1}^{m} \left(V_{ik}(T) - \nu_{iki} \bar{V}_{iki}'(x) \frac{1 - F_{T_i}(T_i)}{f_{T_i}(T_i)} \right) p_{ik}(T) \right\} - \sum_{i=1}^{n} U_i(p, c, \underline{t}_i) ,$$

$$(2.26)$$

where the penultimate equality follows from a manipulation similar to that for the single unit auctions. The bidders' expected payments, $c(\cdot)$, appear only in $U_i(p,c,\underline{t}_i)$, so choosing $c(\cdot)$ to maximize the seller's utility is eqrivalent to choosing it to minimize $U_i(p,c,\underline{t}_i)$. From equation (2.21) and the definition of $Q_{ik}(p,t_i)$, $U_i(p,c,\underline{t}_i)$ may be written as

$$
U_i(p,c,\underline{t}_i) = U_i(p,c,t_i) - \int_{\underline{t}_i}^{t_i} \sum_{k=1}^{m} \nu_{iki}\, \bar{V}_{iki}'(x)\, Q_{ik}(p,x)\, dx
$$

$$
= \mathrm{E}\left\{ \sum_{k=1}^{m} \left(V_{ik}(T)\, p_{ik}(T) - \int_{\underline{t}_i}^{t_i} \nu_{iki}\, \bar{V}_{iki}'(x)\, p_{ik}(x,T_{-i})\, dx \right) \middle|\, T_i = t_i \right\} - c_i(t_i)\,.
$$

$$(2.27)$$

By (2.14), $U_i(p,c,\underline{t}_i) \geq 0$, so when minimizing $U_i(p,c,\underline{t}_i)$ the best the seller can do is to make this term equal to zero and (2.24) follows. From (2.27) I observe that this is done by setting $c_i(t_i)$ as in (2.25). ∎

The characterization of optimal auctions is completed with a description of the optimal allocation probabilities, $p(\cdot)$. If the optimal $p(\cdot)$ corresponds to a deterministic auction, it must solve the problem specified in the following proposition.

Proposition 2.3 *Consider an auction* (p^*,c^*) *satisfying (2.25). Let* p^* *solve*

$$
\max_{p(\cdot)} \; E\left\{ \sum_{i=1}^{n} \sum_{k=1}^{m} \left(V_{ik}(T) - \nu_{iki}\, V_{iki}'(T_i) \frac{1 - F_{T_i}(T_i)}{f_{T_i}(T_i)} \right) p_{ik}(T) \right\}
$$

$$
s.t. \;\begin{cases}
\displaystyle\sum_{k=1}^{m} \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}_{iki}(\hat{t}_i) \right) Q_{ik}(p,t_i) \geq \sum_{k=1}^{m} \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}_{iki}(\hat{t}_i) \right) Q_{ik}(p,\hat{t}_i) \\[2mm]
\displaystyle\sum_{i=1}^{n} \sum_{k=1}^{m} p_{ik}(t) \leq m \\[2mm]
p_{ik}(t) \geq p_{i\,k+1}(t) \\[1mm]
p_{ik}(t) \leq 1 \\[1mm]
p_i(t) \geq 0\,.
\end{cases}
$$

$$(\mathcal{P}2.1)$$

Then if (p^*,c^*) *is a deterministic auction, it is an optimal auction.*

Proof: Suppose that the solution to problem ($\mathcal{P}$2.1) is a deterministic auction. Then simple algebra and Propositions 2.1 and 2.2 and condition (2.26) show that it must be optimal. ■

As Proposition 2.3 illustrates, most of the results derived are only useful if the optimal auction is deterministic. To close the analysis, I will show that this is the case . I first introduce some new notation. Denote by $\pi_{ik}(\cdot)$ the contribution to the seller's utility of allocating the k-th unit to bidder i, i.e.,

$$\pi_{ik}(t) \equiv \mathrm{E}\{V_{ik}(T)|\,T=t\} - \nu_{iki}\,\bar{V}'_{iki}(t_i)\,\frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)}. \qquad (2.28)$$

From Proposition 2.3, in the optimal (deterministic) auction the seller will want to allocate the units that are associated with the highest contributions to his utility. Thus, the ordering of the π_{ik}'s will be important for to characterize the optimal auction; I will use $\pi_{(k)}(\cdot)$ to denote the k-th highest contribution.

Lemma B1.1, in Appendix B1, gives some useful properties of the functions $\pi_{ik}(\cdot)$. It is shown that the contribution to the seller's utility from allocating an additional unit to a bidder is nondecreasing in the bidder's private signal and nonincreasing in the number of units already allocated to him.

Define $T^*_{ik}(\cdot)$ as

$$T^*_{ik}(t_{-i}) \equiv \inf\left\{\hat{t}_i \,\middle|\, \pi_{ik}(\hat{t}_i, t_{-i}) \geq \max\{0, \pi_{(m+1)}(\hat{t}_i, t_{-i})\}, \hat{t}_i \in \mathrm{T}_i\right\}. \qquad (2.29)$$

After the characterization of the optimal auction, this will be interpreted as the required minimum announcement from bidder i to receive its k-th unit.

Given the dependence of $T^*_{ik}(\cdot)$ on $\pi_{ik}(\cdot)$, it is possible to show that the minimum announcement associated with an additional unit is not less than the minimum announcement associated with the current unit (see Lemma B1.2 in Appendix B1).

Using the previous definitions I show that the solution to problem ($\mathcal{P}$2.1) is a deterministic auction.

Proposition 2.4 *Let (p^*, c^*) be an optimal auction. Then*

$$p_{ik}^*(t) = \begin{cases} 1 & \text{if } k \leq \arg\sup\left\{\hat{k} \,\middle|\, \pi_{i\hat{k}}(t) > \max\{0, \pi_{(m+1)}(t)\}\right\} \\ 0 & \text{otherwise} \end{cases} \tag{2.30}$$

$$c_i^*(t_i) = E\{\textstyle\sum_{k=1}^{m} V_{ik}(T_{ik}^*, T_i)\, p_{ik}^*(T)|\, T_i = t_i\} \tag{2.31}$$

Proof : The objective function in problem ($\mathcal{P}2.1$) is linear in $p(\cdot)$; so, considering only the restrictions on p, the solution would be $p_{ik}^*(t)$ equal to 1 for the m terms with the largest positive coefficients, and equal to 0 for the remaining terms. From Lemma B1.1 I know that these will correspond to blocks of first units of some bidders, and $p^*(\cdot)$ will be as in (2.30).

It is left to show that the first restriction in problem ($\mathcal{P}2.1$) is satisfied by the auction described by (2.30) and (2.31). This follows from Assumptions 2.9 and 2.10.

Finally, the optimal $c_i^*(\cdot)$, follows from the definition of $p_{ik}^*(\cdot)$ and condition (2.25). ∎

The main result of this analysis is that the optimal auction is a deterministic auction. The important assumptions behind this result are the linearity of the revision process and the (technical) assumptions on the valuation functions and the hazard rate of the distributions of private signals. The characteristics of optimal auctions with a nonlinear revision process is an open question; on the other hand, if the technical assumptions were not satisfied, the optimal mechanism would, in general, be stochastic.

From the characterization it is now clear that the $T_{ik}^*(\cdot)$ are the required minimum announcements for the k-th unit to be allocated to bidder i, which I will refer as the minimum winning signals. In the optimal auction, the seller receives reports of the bidders' private signals. Using the reports, he computes the variables $\pi_{ik}(t)$, orders them and allocates the m units of the good to the bidders that present the m highest positive π's; if less than m π's are positive, the seller will keep the remaining units. The expected payment by each bidder must be equal to the summation of the expected values of the several units had he received a private signal equal to his minimum winning signal.

The analysis in this section is based on revelation mechanisms. Because there are many mechanisms that are equivalent to the same revelation mechanism, in the following sections

I study the implementation of the optimal auction through simple mechanisms. I will consider particular cases of symmetric models, and will look to whether or not generalizations of standard auctions to multiple unit auctions allow the implementation of the optimal auction. In particular, I will contrast sequential single unit auctions and simultaneous multiple unit auctions.

2.5 The Unit-Demand Model

In this section I study mechanisms for the implementation of the optimal auction in a particular case of the general model, where each potential buyer does not value more than one unit of the good, which I will call the unit demand model. The unit demand model has been the most studied case of multiple unit auctions.

Section 2.4 provides the tools to describe optimal auctions in asymmetric models. However, here I will look at a symmetric model that combines features of the single unit model developed in Chapter 1, which is the particular case of this model when there is only a single unit, and the unit demand multiple unit model analyzed in Maskin and Riley (1989), which only considers the private values case.

Several reasons can be given to justify the analysis of such a model. It is very difficult to provide examples of auctions that clearly fit the private values model; elements of a common value are almost always present, but we have a very limited knowledge of common value multiple unit auctions.[14] Most of the analysis of multiple unit auctions has addressed the question of comparing several mechanisms from the seller's point of view: for example, comparing some variants of standard auctions, or comparing sequential versus simultaneous sales; but the seller may also be interested in knowing what are the revenue maximizing mechanisms, either to know how to implement an optimal auction or to know the loss involved with the choice of a suboptimal mechanism. The consideration of a symmetric model is a necessary condition for the optimality of mechanisms based on generalized standard auctions.

In the next subsection I describe the structure of the symmetric unit demand model. Then,

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[14]It seems likely that the justification behind the result that the seller achieves the first best in common value auctions, in models like Crémer and McLean (1985, 1988) and McAfee et al. (1989), will generalize to multiple unit models.

I characterize optimal auctions and show that mechanisms based on generalizations of standard auctions implement the optimal auction.

2.5.1 A Symmetric Model

The symmetric model satisfies the assumptions of the general model plus some additional assumptions.

Assumption 2.11 *The private signals, T_i, are drawn from the same distribution, $F_T(\cdot)$.*

Assumption 2.12 *Each potential buyer is only interested in one unit of the good.*

The potential buyers' valuations for additional units is zero. Moreover, the way the valuation for the first unit depends on the bidders' private signals is restricted to be the same across all bidders, but (possibly) giving a greater weight to the bidder's own signal than to the other bidders' signals.

Assumption 2.13 *The expected valuation for the first unit satisfies the following conditions:*

$$E\{V_{i1}(T)|T = t\} = \alpha_0 W_1(t_i) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_1 W_1(t_j) \tag{2.32}$$

where $W_1(\cdot)$ is a differentiable increasing function, α_0 is positive and α_1 is nonnegative, with $\alpha_0 \geq \alpha_1$.

This set of assumptions defines a symmetric model with unit demands. It spans the private values and common value symmetric models, as well as a continuum of symmetric models in which valuations may be influenced by others' signals but still differ (in a particular way) across bidders.

From Assumptions 2.7 and 2.13 the expected valuation for the first unit can be written as

$$\mathrm{E}\{V_{i1}(T)|T = t\} = (\alpha_0 - \alpha_1) W_1(t_i) + \alpha_1 \sum_{j=1}^{n} W_1(t_j), \tag{2.33}$$

and the interpretation is clear. The first term in the right hand side of (2.33) is the private values component of the valuation, and the second (which is constant across bidders) is the

common value component of the valuation. Observe that if $\alpha_1 = 0$ I obtain a private values model; if $\alpha_0 = \alpha_1$, I have a common value model. Models with $\alpha_0 > \alpha_1 > 0$ give some weight to both private values and common value.

2.5.2 The Optimal Auction

Direct application of Proposition 2.4 yields the optimal auction in the symmetric unit demand model of Subsection 2.5.1. To simplify the description of the optimal auction, I introduce some new notation. Define $T^*_{i1}(t_{-i})$ as the bidder i's private signal that would make the contribution to the seller's utility of allocating one unit to bidder i exactly equal to zero given $T_{-i} = t_{-i}$, i.e.,[15]

$$\pi_{i1}(T^*_{i1}(t_{-i}), t_{-i}) \equiv 0. \tag{2.34}$$

This is the required minimum announcement by player i, above which he has a positive probability of receiving one unit. Announcements below this value will necessarily imply that the seller does not allocate any unit to bidder i.

As seen in Lemma B1.1, the seller's utility of allocating one unit to a bidder with private signal t_i, $\pi_{i1}(t)$, is nondecreasing in t_i. So, in the optimal auction, the seller will want to allocate the units to the bidders that report the highest private signals. I will denote the order statistics of the vector T_{-i} by $T_{i(1)} \geq \cdots \geq T_{i(n-1)}$; so that bidder i's minimum winning private signal is $T^*_{i1} = \max\{T^*_{i1}, T_{i(m)}\}$, i.e., bidder i will receive one unit only if he announces a private signal that is greater than the required minimum announcement (otherwise the seller prefers to keep the unit for himself) and greater than the m-th highest among the other bidders' signals (otherwise there are at least m other bidders to whom the seller would prefer to allocate the units).

Before characterizing the optimal auction, I describe some properties of the required minimum announcements T^*_{i1}. In the first result I show that the required minimum announcements will be higher for a bidder with a higher signal.

Lemma 2.4 *Consider a realization of private signals, t, and the associated required minimum*

---

[15]Equation 2.34 may have no solution; in that case I set $T^*_{i1}(t_{-i}) = \underline{t}_i$. For simplicity of exposition, throughout the text I will assume that an interior solution exists, but none of the results depend on this.

announcements, t^*. Let $((1), \ldots, (n))$ be a permutation of the bidders such that $t_{(1)} \geq \cdots \geq t_{(n)}$; then $t^*_{(1)1} \geq \cdots \geq t^*_{(n)1}$.

Proof : Write the identity (2.34) as

$$\alpha_1 \sum_{\substack{j=1 \\ j \neq i}}^{n} W_1(t_j) + \alpha_0 \left(W_1(t^*_{i1}) - W'_1(t^*_{i1}) \frac{1 - F_T(t^*_{i1})}{f_T(t^*_{i1})} \right) \equiv 0. \tag{2.35}$$

Since the first summation is smaller for a bidder with a higher signal, the second term must be larger for a bidder with a larger signal. The conclusion follows from Assumptions 2.9 and 2.10.

■

The property described in the lemma raises an immediate question. Because a bidder with a larger private signal will have a larger required minimum announcement, is it possible that there is a bidder whose private signal does not exceed his required minimum announcement, while some other bidder has a lower private signal that exceeds his required minimum signal? If such a situation were to exist, the seller would not always want to allocate the units to the bidders with the highest private signals. In Appendix B1 I show that such situations do not arise.

Now I can characterize the optimal auction. Note that, because of the unit demand assumption, in the optimal auction no bidder will receive more than one unit, i.e., for $k > 1$ and any announcements, $p_{ik}(t) = 0$; thus I needed only to specify the probabilities of allocation of the first unit to each bidder.

Proposition 2.5 *Let (p^*, c^*) be an optimal auction in the unit demand symmetric model. Then*

$$p^*_{i1}(t) = \begin{cases} 1 & \text{if } t_i > t^*_{i1} \\ 0 & \text{otherwise} \end{cases} \tag{2.36}$$

$$c^*_i(t_i) = E\{ V_{i1}(T^*_{i1}, T_{-i}) p^*_i(T) | T_i = t_i \} \tag{2.37}$$

Proof : The proof is an application of the result for the optimal auction of the general model (Proposition 2.4) to the symmetric unit demand model. In this case the contribution to the

seller's utility is given by

$$\pi_{i1}(t) = \alpha_1 \sum_{\substack{j=1 \\ j \neq i}}^{n} W_1(t_j) + \alpha_0 \left(W_1(t_i) - W_1'(t_i) \frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)} \right) ,$$

and, by Assumptions 2.9 and 2.10, the highest $\pi_i(t)$'s are associated with the highest private signals. Therefore, by Proposition 2.4 the optimal $p_{i1}(t)$ satisfies (2.36). To get (2.37) I use (2.36) and the assumptions of the symmetric unit demand model into (2.31). ∎

Thus, in the optimal auction the units are allocated to the m bidders that report the highest private signals, provided that they are above the required minimum announcements which are determined endogenously by the seller as a function of the others' bids. The expected payment of each bidder is equal to the expected value of the good, had he received his minimum winning private signal.

The structure of the optimal auction, which allocates the good to the bidders that submit the highest bids, suggests that generalized standard auctions of the type analyzed in Ortega-Reichert (1968) and We,er (1983), with the correct minimum bids, may be implementations of the optimal mechanism. Indeed, in generalized standard auctions there are equilibria in which the bids are increasing in the bidders' private signals; hence, by selecting the bidders that submitted the highest bids as the winners, the seller will be allocating the units optimally. To ensure the implementation of the optimal mechanism, the seller must use an optimal minimum bid requirement. This analysis is done in the next subsection.

2.5.3 Implementing the Optimal Auction with Generalized Standard Auctions

Following the intuition from the previous subsection, here I study some generalized standard auctions that allow the implementation of the optimal auction in the symmetric unit demand model of Subsection 2.5.1.

There are many possible ways to generalize standard auctions to the case of multiple units.[16]

---

[16]See McCabe et al. (1991) for an experimental study of several generalizations of standard auctions.

Here, I will consider only three possible generalizations. The first is a generalization of the second price sealed bid auction: the uniform price auction. The second is a generalization of the first price sealed bid auction: the discriminatory price auction. The third is a generalization of the English auction: the sequential English auction.[17] These correspond to multiple unit auctions that are often used and have been studied in some theoretical works (e.g., Ortega-Reichert, 1968; Weber, 1983).

2.5.3.1 Uniform Price Auction

I start the analysis with the consideration of the uniform price auction.

Definition 2.3 *In an* **uniform** *price sealed bid auction every bidder submits one bid; then the m units of the good are allocated to the bidders that have submitted the m highest bids, provided that they exceed the required minimum bids set by the seller. A winner pays the maximum between the m + 1-th bid and his required minimum bid; others pay nothing.*

In the private values model, Weber (1983) shows that in the unique symmetric equilibrium of the uniform price auction the bidders bid their value for the first unit of the good. In the symmetric model, it is one equilibrium is that each bidder bid the expected value of the good if the m-th highest private signal among the others' signals is equal to the bidder's signal.

Lemma 2.5 *Consider the uniform price auction, without required minimum bids, in the symmetric unit demand model. Let each bidder bid according to*

$$b^U(t_i) = E\left\{V_{i1}(T) \Big| T_{i(m)} = T_i, T_i = t_i\right\}.$$ (2.38)

The profile $(b^U(t_1), \ldots, b^U(t_n))$ is an equilibrium of the uniform price auction.

Proof: See Appendix B1. ∎

Even though in the definition of the equilibrium bidding functions in (2.38) I have used the valuation function of the respective bidder, the symmetry of the model that is being analyzed

---

[17]Other generalizations of standard auctions may allow the implementation of the optimal mechanism in this model, but there are some that clearly will not. This is discussed at the end of the subsection.

makes the bidding function the same for all bidders.[18]

This characterization of the equilibrium bids of the uniform price auction was made for the auction without required minimum bids. It is straightforward to generalize the result to allow minimum bids that can be written in the form of the equilibrium bids. In this case, the required minimum bid implies the existence of a required minimum private signal, so a bidder will bid the expected value of the good if his signal is equal to the maximum between the required minimum private signal and the m highest private signals among the other bidders. This result is used in Proposition 2.6.

Proposition 2.6 *Suppose that the seller sets the required minimum bids*

$$\underline{b}^U(\underline{t}_{i1}^*) = E\{V_{i1}(T)|\, T_{i1}^* = T_i, T_i = \underline{t}_{i1}^*\}.$$ (2.39)

The uniform price auction with minimum bids $\underline{b}^U(\underline{t}_{i1}^*)$ implements the optimal auction in the symmetric unit demand model.

Proof : Let (p^U, c^U) be the allocation probabilities and expected payments generated by the uniform price auction. Suppose that the minimum bid set by the seller is equal to the bid that bidder i would submit if his signal were t_{i1}^*. Then, according to Lemma 2.5, bidder i's equilibrium bid would be

$$b^U(t_i) = \mathrm{E}\{V_{i1}(T)|\, T_{i1}^* = T_i, T_i = t_i\}.$$ (2.40)

Therefore, the required minimum bid, $\underline{b}^U(\cdot)$ is in fact of the form of the bidders' equilibrium bids. Moreover, the equilibrium bids in (2.40) are strictly increasing in the bidder's private signal. Therefore, using the results in Lemma B1.3, I conclude that the uniform price auction with minimum bids $\underline{b}^U(\cdot)$ allocates the units optimally, i.e.,

$$p_{i1}^U(t) = \begin{cases} 1 & \text{if } t_i > t_{i1}^* \\ 0 & \text{otherwise.} \end{cases}$$

---

[18]This is why the equilibrium bids $b^U(\cdot)$ are not indexed by the bidder's index. This will also be the case for the other auctions studied in this subsection.

Bidder *i*'s expected payment is

$$
\begin{aligned}
c_i^U(t_i) &= \mathrm{E}\Big\{ b^U(T_{i1}^*) p_{i1}^U(T) \Big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ \mathrm{E}\Big\{ V_{i1}(T) | T_{i(1)1}^* = T_{i(1)}, T_{i(1)} \Big\} p_{i1}^U(T) \Big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ \mathrm{E}\{ V_{i1}(T) - \alpha_0 \left(W_1'(T_i) - W_1'(T_{i1}^*) \right) | T_{i1}^* < T_i, T_i \} p_{i1}^U(T) \Big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ V_{i1}(T_{i1}^*, T_{-i}) p_{i1}^U(T) \Big| T_i = t_i \Big\} ,
\end{aligned}
$$

where the last equality follows from the definition of $p_{i1}^U(T)$, equal to 1 if $T_{i1}^* < T_i$ and 0 otherwise.

Thus (p^U, c^U) satisfies conditions (2.36) and (2.37), and the uniform price auction implements the optimal auction in the symmetric unit demand model. ∎

The most interesting properties of the optimal uniform price auction relate to the required minimum bids. First, it should be noted that the required minimum bids differ across bidders. In particular, comparing across bidders, from the fact that the required minimum bid increases with the required minimum announcement, Lemma 2.4 implies that the required minimum bid is higher for a bidder with a higher private signal. Second, as a consequence of the first property, the price actually paid in the optimal uniform price auction may be greater for winners with greater private signals (if the required minimum bid is the determinant of their price) than the price paid by the winners with a lower private signal (if the required minimum bid is not binding). Last, since bids are increasing in the bidder's private signal, the highest bids will be submitted by the bidders with the highest signals; hence, as long as the seller sets the appropriate minimum bids, the uniform price auction will allocate the units of the good correctly.

2.5.3.2 Discriminatory Price Auction

Another common generalization of a standard auction is the discriminatory price auction. In this section I design a discriminatory price auction that implements the optimal mechanism in the symmetric unit demand model.

Definition 2.4 *In a discriminatory price sealed bid auction every bidder submits one bid; then the m units of the good are then allocated to the bidders that have submitted the m highest bids, provided that they exceed the minimum bid set by the seller. A winner pays his own bid; others pay nothing.*

In a private values model, Weber (1983) shows that, in the unique symmetric equilibrium of the discriminatory price auction, the bidders bid the good's expected value conditional on having one of the m highest private signals. As a generalization, it is very easy to verify that, in my symmetric model, bidding an amount equal to the expected value of the good if the m-th highest private signal among the others' signals is smaller than the bidder's signal, provides an equilibrium of the discriminatory price auction.

Lemma 2.6 *Consider the discriminatory price auction in the symmetric unit demand model without required minimum bids. Let each bidder bid according to*

$$b^D(t_i) = E\left\{ V_{i1}(T_{i(m)}, T_{-i}) \middle| T_{i(m)} < T_i, T_i = t_i \right\}. \tag{2.41}$$

The profile $(b^D(t_1), \ldots, b^D(t_n))$ is an equilibrium of the discriminatory price auction.

Proof: See Appendix B1. ∎

The lemma characterizes the equilibrium bids of the discriminatory price auction if there are no required minimum bids. The result is easily extended to the case where there are required minimum bids (and corresponding required minimum private signals). Under these circumstances the equilibrium bid is the expected value of the first unit of the good, conditional on having a private signal that is greater than the maximum of the m-th highest signal among the other bidders' private signals and the required minimum private signal. These equilibrium bids are increasing in the bidder's private signal, so a bidder with a larger signal bids a larger value. This suggests that, as long as the seller is able to impose the appropriate required minimum bids, the discriminatory price auction will implement the optimal mechanism. The next proposition presents a discriminatory price auction that implements the optimal mechanism.

Proposition 2.7 *Suppose that the seller sets the required minimum bids*

$$\underline{b}^D(t_{i1}^*) = E\{V_{i1}(T_{i1}^*, T_{-i}) \mid T_{i1}^* < T_i, T_i = t_{i1}^*\}. \tag{2.42}$$

The discriminatory price auction with minimum bids $\underline{b}^D(t_{i1}^*)$ implements the optimal auction in the symmetric unit demand model.

Proof : Let (p^D, c^D) be the allocation probabilities and expected payments generated by the discriminatory price auction. Suppose that the minimum bid set by the seller is equal to the bid that bidder i would submit if his signal were $\underline{t}_{i1}^*$. Then, according with Lemma 2.6, bidder i's equilibrium bid would be

$$b^D(t_i) = \mathrm{E}\{V_{i1}(T_{i1}^*, T_{-i}) \mid T_{i1}^* < T_i, T_i = t_i\}. \tag{2.43}$$

Therefore, the required minimum bid, $\underline{b}^D(\cdot)$, is of the form of the bidders' equilibrium bids. Moreover, the equilibrium bids in (2.43) are strictly increasing in the bidder's private signal. Therefore, using the results in Lemma B1.3, I conclude that the discriminatory price auction with minimum bids $\underline{b}^D(\cdot)$ allocates the units optimally, i.e.,

$$p_{i1}^D(t) = \begin{cases} 1 & \text{if } t_i > t_{i1}^* \\ 0 & \text{otherwise.} \end{cases}$$

Bidder i's expected payment is

$$
\begin{aligned}
c_i^D(t_i) &= \mathrm{E}\Big\{ b^D(T_i) p_{i1}^D(T) \Big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ \mathrm{E}\{V_{i1}(T_{i1}^*, T_{-i}) \mid T_{i1}^* < T_i, T_i\} p_{i1}^D(T) \Big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ \mathrm{E}\{V_{i1}(T) - \alpha_0(W_1'(T_i) - W_1'(T_{i1}^*)) \mid T_{i1}^* < T_i, T_i\} p_{i1}^D(T) \Big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ V_{i1}(T_{i1}^*, T_{-i}) p_{i1}^D(T) \Big| T_i = t_i \Big\},
\end{aligned}
$$

where the last equality follows from the definition of $p_{i1}^D(T)$, equal to 1 if $T_{i1}^* < T_i$ and 0 otherwise. Therefore, (p^D, c^D) satisfies conditions (2.36) and (2.37), and the discriminatory price auction implements the optimal auction in the symmetric unit demand model. ∎

The properties of the optimal discriminatory price auction are similar to those of the uniform price auction. However, in this case, the fact that a bidder with a higher private signal pays more than a bidder with a lower private signal is a direct consequence of the rules of the auction: a winner pays his own bid and bids are strictly increasing in the bidder's private signal.

2.5.3.3 Sequential English Auction

Having analyzed the simultaneous sealed bid auctions, I now look at a sequential oral auction: the seller sequentially auctions each unit of the good following the rules of a single unit auction. I will analyze a case in which the single unit auctions are organized as English auctions.

An important issue in sequential auctions is the information that is released from earlier rounds to later ones. In this model, the later auctions will be nondegenerate only if bidders cannot fully learn others' information from their actions in the current auction. Since private signals are univariate random variables if all bids are seen by every agent and not all private information is released in the first auction, it must be the case that there are some bids that can be chosen by bidders with distinct private signals; consequently such a mechanism cannot be optimal since there is a strictly positive probability that the highest bids will not come from the bidders with the highest private signals. Therefore, I analyze a sequential auction designed in such a way that the information released to the bidders in each auction is limited to whether or not a unit was sold and if a sale occurred, the respective winning bid.[19] This auction is thus strategically equivalent to a sequential first price sealed bid auction, which is analyzed by Weber (1983).[20]

I now formally define the sequential English auction.

Definition 2.5 *In a* **sequential English auction** *the seller sequentially holds an English auction for each unit of the good. In each English auction the bidders only observe the current price, which increases continuously until no bidder wants to bid a higher price. If the stopping price is at least the required minimum bid set by the seller for this bidder, the unit is allocated*

---

[19]This auction is similar to the discriminatory price auction, in the sense that the winner pays his own bid. Note that this is not the standard assumption on theoretical work on English auctions, even though it is what happens in most English auctions.

[20]In the private values model, it is irrelevant in the sequential first price auction whether or not the winning bid is revealed (see Weber, 1983, for details). It is relevant in the non private values model because the previous winning bids convey partial information on the bidders' valuations.

to the last active bidder at the stopping price. Other bidders pay nothing. If the unit is sold, the other bidders observe the stopping price and are allowed to bid in the following auction.

I should note that the stage English auction is not of the type analyzed in Milgrom and Weber (1982) and in Chapter 1, in which bidders could see others' stopping bid and use that information to review their own stopping prices. Here, I consider a thermometer English auction, of the type considered in Graham and Marshall (1987), in which each bidder observes only the current price.

The thermometer English auction can be implemented in the following way. Each bidder has a button that must be pressed if he wants to bid for the unit at the current price, which is observed by every one. If at least one bidder is pressing the button, the price raises continuously. Once the price stops, the winner is the last bidder that was active. He receives the good if he pays the maximum between the current price and a final *take-it-or-leave-it* offer made by the seller. The losers, who have seen the stopping price but not the actual selling price, are allowed to participate in the auction of the following unit.[21]

Because there is no release of information to the bidders during each of the stage auctions and the winner's bid determines the stopping price, the equilibrium in the sequential English auction is essentially equal to the equilibrium in a sequential first price auction, as characterized by Weber (1983), for the private values model. I generalize this result to the symmetric model and show that, if there are k units to be sold, in the unique symmetric equilibrium of the sequential English auction the bidders bid their expected value of the good, given the bids of the previous winners (which, in equilibrium, reveal their types), conditional on having one the k highest private signals among the remaining bidders.

Lemma 2.7 *Consider the sequential English auction in the symmetric unit demand model, without required minimum bids. Consider the k-th auction of the sequence. Let each remaining bidder bid according to*

$$b_k^E(t_i) = E\left\{ V_{i1}(T_{i(m)}, T_{-i}) \,\middle|\, T_{i(m)} < T_i, \, T_{i(k-l)} = t_{i(k-l)} \, (l = 1, \ldots, k-1), \, T_i = t_i \right\}. \quad (2.44)$$

---

[21]The fact that the losers do not observe the price actually paid by the winner is a property of some multiple unit sequential oral auctions in use. After finishing the bidding process, the seller and the winner are allowed to agree on a selling price that diverges from the bidding stopping price.

Let $b^E(t_i)$ be the sequence of such bidding functions, $b^E(t_i) = (b_1^E(t_i), \ldots, b_m^E(t_i))$. The profile $(b^E(t_1), \ldots, b^E(t_n))$ is an equilibrium of the sequential English auction.

Proof : See Appendix B1. ∎

Following the analysis of simultaneous sealed bid auctions, the equilibrium bids generalize to accommodate the case of a sequential English auction with required minimum bids of the same form as the equilibrium bids. This will be the case in the optimal sequential English auction presented in Proposition 2.8.

From the characterization of the equilibrium bids in the sequential English auction, I can conclude that, in each stage auction, the highest bid will be submitted by the bidder with the highest signal; hence, as long as the seller sets the appropriate minimum bids, the units of the good are optimally allocated. The following proposition characterizes a sequential English auction that implements the optimal mechanism.

Proposition 2.8 *Suppose that the seller sets the required minimum bids*

$$\underline{b}_k^E(\underline{t}_{i1}^*) = E\left\{ V_{i1}(T_{i1}^*, T_{-i}) \mid T_{i1}^* < T_i, T_{i(k-l)} = t_{i(k-l)} \ (l = 1, \ldots, k-1), T_i = \underline{t}_{i1}^* \right\} \qquad (2.45)$$

in the stage auction of the k-th unit. Let $\underline{b}^E(\underline{t}_{i1}^*) = (\underline{b}_1^E(\underline{t}_{i1}^*), \ldots, \underline{b}_m^E(\underline{t}_{i1}^*))$. The sequential English auction with minimum bids $(\underline{b}^E(\underline{t}_{11}^*), \ldots, \underline{b}^E(\underline{t}_{n1}^*))$ implements the optimal auction in the symmetric unit demand model.

Proof : The proof follows the steps of the proof of Propositions 2.6 and 2.7. Let (p^E, c^E) be the allocation probabilities and expected payments generated by the optimal sequential English auction. Suppose that the minimum bid set by the seller is equal to the bid that bidder i would submit if his signal were $\underline{t}_{i1}^*$. Then, from Lemma 2.7, bidder i's equilibrium bid in the auction of the k-th unit, if he has received no unit before, will be

$$b_k^E(t_i) = E\left\{ V_{i1}(T_{i1}^*, T_{-i}) \mid T_{i1}^* < T_i, T_{i(k-l)} = t_{i(k-l)} \ (l = 1, \ldots, k-1), T_i = t_i \right\} \qquad (2.46)$$

which is of the form of the required minimum bid, $\underline{b}_k^E(\cdot)$. Moreover, the equilibrium bids in (2.46) are strictly increasing in the bidder's private signal. Therefore, using the results in

Lemma B1.3, I conclude that the sequential English auction with minimum bids $b^E(\cdot)$ allocates the units optimally, i.e.,

$$p_{i1}^E(t) = \begin{cases} 1 & \text{if } t_i > t_{i1}^* \\ 0 & \text{otherwise} \end{cases} .$$

Bidder i's expected payment is

$$
\begin{aligned}
c_i^E(t_i) &= \mathrm{E}\Big\{ \textstyle\sum_{k=1}^m \mathrm{E}\big\{ b_k^E(T_i)\,p_{i1}^E(T) \big| T_{i(k)} < T_i \leq T_{i(k-1)}, T_i \big\} \Big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ \textstyle\sum_{k=1}^m \mathrm{E}\big\{ \mathrm{E}\{ V_{i1}(T_{i1}^*, T_{-i}) | T_{i1}^* < T_i, T_{i(k-l)} \, (l = 1, \ldots, k-1), T_i \big\} \\
&\qquad\qquad\qquad\qquad\qquad p_{i1}^E(T) \big| T_{i(k)} < T_i \leq T_{i(k-1)}, T_i \big\} \Big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ \mathrm{E}\{ V_{i1}(T_{i1}^*, T_{-i}) | T_{i1}^* < T_i, T_i \} \, p_{i1}^E(T) \big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ \mathrm{E}\{ V_{i1}(T_{i1}^*, T_{-i}) - \alpha_0 \, (W_1'(T_i) - W_1'(T_{i1}^*)) | T_{i1}^* < T_i, T_i \} \, p_{i1}^E(T) \big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ V_{i1}(T_{i1}^*, T_{-i}) \, p_{i1}^E(T) \big| T_i = t_i \Big\},
\end{aligned}
$$

where the third and last equalities follow from the definition of $p_{i1}^E(T)$, equal to 1 if $T_{i1}^* < T_i$ and 0 otherwise.

Therefore, (p^E, c^E) satisfies conditions (2.36) and (2.37), and the sequential English auction implements the optimal auction in the symmetric unit demand model. ∎

Two very interesting properties are generated by the equilibrium behavior in the optimal sequential English auction. First, if a stage auction results in no sale, it is optimal for the seller not to auction any of the remaining units. This follows directly from Lemma B1.3; the unit under auction is not sold if the winner's private signal is smaller than the minimum signal required by the seller; but, because the current winner is the bidder with the greatest private signal, this implies that no other bidder will have a private signal greater than his required minimum signal.

Second the expected price paid in any stage auction decreases over the sequence of auctions. It is clear that if there were no minimum bids the expected winning price in any stage auction would be the same. However, since the seller wants to use greater minimum bids for bidders with greater private signals and the winners with high private signals receive a unit before those with low private signals, the expected bid of the winner in the k-th stage auction will be greater than

the expected bid of the winner in the $k + 1$-th stage auction. Moreover, the sequence of prices paid in stage auctions forms a supermartingale.[22] This is mentioned in Ashenfelter (1989), who offered no reasonable explanation for the case of auctions of homogeneous units.[23]

2.5.3.4 Other Generalized Standard Auctions

Having analyzed some generalized standard auctions that implement the optimal mechanism, I now examine some other generalized standard auctions which do not allow the implementation of the optimal mechanism.

The analysis of the single unit model in Chapter 1 suggests that the generalized Dutch auction (simultaneous or sequential) does not allow the implementation of the optimal mechanism, because the seller does not have the information needed to compute the optimal minimum bids.[24]

A similar argument shows that a generalization of the English auction analyzed in Milgrom and Weber (1982) with the bidders observing only the stopping price of the penultimate active bidder, whether simultaneous or sequential, does not allow the implementation of the optimal mechanism. In the simultaneous auction, where the price increases until only m bidders are willing to buy the units at the current price, the seller does not have access to the m highest private signals, and cannot compute the winners optimal minimum bids. In the sequential auction the seller does not have access to the highest private signal,[25] and even though the seller can compute the optimal minimum bid for this first winner he lacks information to compute the other winners' optimal minimum bids.

More generally, simultaneous oral auctions do not allow the implementation of the optimal mechanism,[26] because the seller is unable to learn all the private signals,[27] which are needed

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[22]Even though the winning bid may differ from the price paid, this property also applies to the winning bids.

[23]I am not claiming that I explain Ashenfelter's example; I am solely arguing that there may be auctions among rational agents that would yield such a result.

[24]In a simultaneous generalized Dutch auction the seller would continuously decreasing the price until exactly m bidders wanted to buy at the current price. In a sequential generalized Dutch auction the seller would sequentially organize a single unit Dutch auction to sell each of the units. Bulow and Klemperer (1991) construct and analyze an equilibrium of the sequential auction.

[25]In the first auction the stopping price reveals the second highest signal and the highest signal would never be revealed.

[26]In a simultaneous oral auction the bidding process stops once there are as many active bidders as units.

[27]If the bids are increasing, he cannot learn the $(m-1)$-th highest signals; if the bids are decreasing, he cannot learn the $(n-m)$-th lowest signals.

for the determination of some optimal minimum bids.

In contrast, most of the generalizations of the sealed bid auctions can be used to implement the optimal mechanism. However, a generalization that does not allow it is the case of simultaneous independent auctions; in this case the seller organizes m independent single unit auctions and the highest bidder in each auction is the respective winner; given the unit demand assumption, such a multiple unit auction does not permit the implementation of the optimal mechanism. The conclusion is particularly relevant to procurement problems, in which homogeneous or very close substitute goods are auctioned simultaneously and independently. Weber (1983) cites the use of such auctions by the United States Department of Interior for leasing the mineral rights of off-shore federal properties, and Ramsey (1983) presents an empirical study of these auctions. Porter and Zona (1991) study auctions of highway contracts in the state of New York, which are also auctioned independently.

2.6 The Linear Valuations Model

A special simple case of the general model in which the potential buyers demand more than one unit is the linear valuations model, where the expected valuations are linear in the private signals. There is no particularly interesting case of the linear valuations model that merits full analysis; therefore, I analyze two specific cases in a more restricted manner; for each, I only characterize the optimal auction and construct a mechanism for its implementation. Moreover, because I want to focus the discussion of implementation on generalized standard auctions, I will concentrate on symmetric examples.

2.6.1 A Constant Valuations Model

This is the simplest case that can be analyzed in the linear valuations model. This variant of the general model with multiple unit demands has especially simple optimal mechanisms, mainly because the structure of this model has strong implications on the properties of optimal auctions. This model is characterized by the assumptions of the general model, Assumption 2.11, and the following assumption about the expected valuations.

Assumption 2.14 *The expected valuations satisfy conditions:*

$$E\{V_{ik}(T)|T = t\} = \alpha_0 t_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_1 t_j$$

with $\alpha_0 \geq \alpha_1 \geq 0$.

I first analyze the structure of the optimal required minimum announcements. Let $T^*_{ik}(t_{-i})$ be the generalization of the analogous definition introduced in Subsection 2.5.2, i.e., let it be bidder's i's private signal that would make the contribution to the seller's utility of allocating the k-th unit to bidder i equal to zero. From Assumption 2.14, it is clear that $T^*_{ik} = T^*_i$ for any k, i.e., for each bidder, the required minimum announcement is constant across units. Finally, let $T^*_i = \max\{T^*_i, T_{i(m)}\}$. Using Proposition 2.4, I then construct the optimal mechanism in the constant valuations model.

Proposition 2.9 *Let* (p^*, c^*) *be an optimal auction in the constant valuations model. Then,*

$$p^*_{ik}(t) = \begin{cases} 1 & \text{if } t_i > t^*_i \\ 0 & \text{otherwise} \end{cases} \tag{2.47}$$

$$c^*_i(t_i) = E\{\textstyle\sum_{k=1}^{m} V_{ik}(T^*_i, T_{-i}) p^*_{ik}(T)|T_i = t_i\}. \tag{2.48}$$

Proof : The optimal auction follows directly from Proposition 2.4. ∎

So, as we might expect, the optimal auction requires the seller either to allocate all the m units to the same bidder or to not allocate any of the units. Because valuations are constant, there is no reason to treat the several units separated and the seller may just bundle all the m units in a single lot for auction which may be organized through any of the standard auctions that are optimal for single unit auctions.[28]

A particular case of this result applies to the pure private values model. The present model satisfies the superadditivity condition used by Gale (1990) who proved that, under superadditivity, a policy of bundling is the best for the seller. It is important to note that this requires

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[28]These include the first price and second price sealed bid and the English oral bid auctions. See the analysis in Chapter 1.

symmetry conditions (e.g., if the signals are not drawn from the same distribution the result may no longer be true); but, more important, the analysis in this subsection shows that the optimality of the bundling policy extends to models where valuations are not superadditive; for example, in the common values model, in which any agent knows that, independently of the allocation, the final valuations will be equal across all bidders, it is still optimal to allocate all the units to the bidder that reports the highest signal.

2.6.2 A Proportional Valuations Model

This model is identical to the constant valuations model, except that valuations for an additional unit decrease with the number of units held. I substitute Assumption 2.15 for Assumption 2.14.

Assumption 2.15 *The expected valuations satisfy the following conditions:*

$$E\{V_{ik}(T)|\, T = t\} = \beta_k\, \alpha_0\, t_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_k\, \alpha_1\, t_j$$

with $\alpha_0 \geq \alpha_1 \geq 0$, and $\beta_k > \beta_{k+1} > 0$.

Under this additional assumption, the valuations are decreasing, but the weight given to the bidder's own signal relative to other bidders' signals is constant. Consequently, the contribution to the seller's payoff from allocating the k-th unit to bidder i satisfies the following recursive condition

$$\pi_{i1}(t) = \beta_1 \left(\alpha_1 \sum_{\substack{j=1 \\ j \neq i}}^{n} t_j + \alpha_0 \left(t_i - \frac{1 - F_T(t_i)}{f_T(t_i)} \right) \right)$$

$$\pi_{i\,k+1}(t_i) = \frac{\beta_{k+1}}{\beta_k}\, \pi_{ik}(t_i).$$

It follows that the required minimum announcements are constant across the several units, and must solve

$$\alpha_1 \sum_{\substack{j=1 \\ j \neq i}}^{n} t_j + \alpha_0 \left(t_i - \frac{1 - F_T(t_i)}{f_T(t_i)} \right) = 0\,.$$

However, because the valuations are not constant, in an actual implementation of the optimal auction based on standard auctions, the bids will differ across units.

The description of the optimal auction follows that in Proposition 2.4. One property of the optimal auction emerges. Given that the required minimum announcement is constant for all units, in the optimal auction either all units are allocated (as long as at least one bidder has a private signal that is greater than the required minimum announcement) or no unit is allocated.

The optimal auction in this model cannot be implemented by a generalized standard auction. The analysis in the next section shows that there exists a modification of generalized standard auctions that implements the optimal mechanism, but that such an auction will be very complex (the auction for a two unit and two bidders example can be constructed following the example in Section B2). The main problem is that it is very difficult to write general conditions on the modified bids to ensure that the units are correctly allocated. Given that a more general example is analyzed in Appendix B2, where the main features of the construction of a modified discriminatory price auction that implements the optimal mechanism are captured, I postpone the analysis of these more complex models to the appendix.

2.7 Conclusion

In this chapter I analyze optimal auctions of multiple units of an indivisible good. The model allows for private values as well as common values, and generalizes the model that was developed in Chapter 1.

Some particular symmetric case of the general model is analyzed. In the unit-demand model, I generalize the characterization of optimal auctions done in Maskin and Riley (1989) for the private values case, and show that some generalized standard auctions can be used to implement the optimal auction. In a linear-valuations model I show that generalized standard auctions cannot in general implement the optimal auction, but modifications of them can to implement the optimal auction.

This model is sufficiently general and flexible to allow for the study of many auctions. Even though it does not cover the problem of divisible goods, a similar approach can be followed to generalize the model in Maskin and Riley (1989), allowing for asymmetries and non-private valuations. This will be the topic of Chapter 3.

B1 Appendix: Proofs

In this appendix I present the proofs that were omitted in the main text

Proof of Lemma 2.1: Condition (2.8) follows from (2.2), and (2.4). Then, use (2.2), (2.3) and (2.4) to get (2.7). To prove (2.6) rewrite (2.2) as the following recursive relation (starting with $p_{im}(t) = \rho(k;t)$ with $k_i = m$)

$$p_{ik}(t) = p_{i\,k+1}(t) + \sum_{\substack{(k_1,\ldots,k_n)\in\mathcal{A}(m,n) \\ k_i=k}} \rho(k_1,\ldots,k_n;t) \tag{B1.1}$$

and note that (2.6) follows from (2.4) and (B1.1). To prove (2.5), I construct the p's from the ρ's as follows:

1. Index the allocations with a positive integer, r, lower than $\binom{n+m}{n}$, in such a way that each number indexes one and only one allocation;

2. Start with $r = 1$;

3. Take the allocation indexed by r; let $\rho_r(k_1,\ldots,k_n;t)$ be the probability that it will be implemented when announcements are t. For every i and k, define $\Delta_r p_{ik}(t)$ as

$$\Delta_r p_{ik}(t) = \begin{cases} \rho_r(k_1,\ldots,k_n;t) & \text{if } k \le k_i \\ 0 & \text{if } k > k_i \, ; \end{cases} \tag{B1.2}$$

4. Increase r by 1 and repeat steps 3 and 4 until r exceeds $\binom{n+m}{n}$;

5. Compute $p_{ik}(t)$ according to

$$p_{ik}(t) = \sum_{r=1}^{\binom{n+m}{n}} \Delta_r p_{ik}(t) \, . \tag{B1.3}$$

First, note that, given condition (2.1) in the definition of an allocation and (B1.2) in step 3, for each r, no more than m of the $\Delta_r p$'s will be different than zero, i.e.,[29]

$$\sum_{i=1}^{n}\sum_{k=1}^{m} \Delta_r p_{ik}(t) \le m\,\rho_r(k_1,\ldots,k_n;t) \, . \tag{B1.4}$$

Then, by (B1.3) and (B1.4),

$$\sum_{i=1}^{n}\sum_{k=1}^{m} p_{ik}(t) \le m \left(\sum_{(k_1,\ldots,k_n)\in\mathcal{A}(m,n)} \rho(k_1,\ldots,k_n;t) \right) , \tag{B1.5}$$

and (2.5) follows from (B1.5) and (2.3). ■

Proof of Lemma 2.3: Using Assumption 2.7 the right hand side of (2.10) can be written as

$$\mathrm{E}\Big\{ \textstyle\sum_{k=1}^{m} V_{ik}(T)\,p_i(\hat{t}_i,T_{-i}) \big| T_i = t_i \Big\} - c_i(\hat{t}_i)$$
$$= \mathrm{E}\Big\{ \textstyle\sum_{k=1}^{m} V_{ik}(T_i)\,p_{ik}(T_i) \big| T_i = \hat{t}_i \Big\} - c_i(\hat{t}_i) + \mathrm{E}\Big\{ \textstyle\sum_{k=1}^{m} \nu_{iki}\left(\bar{V}_{iki}(t_i) - \bar{V}_{iki}(\hat{t}_i)\right) p_{ik}(\hat{t}_i,T_{-i}) \Big\}$$

---

[29]The equality is satisfied if and only if the allocation gives all m units to bidders.

$$= U_i(p, c, \hat{t}_i) + \sum_{k=1}^{m} \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}_{iki}(\hat{t}_i) \right) Q_{ik}(p, \hat{t}_i) .$$

Therefore, (2.11) is equivalent to (2.10). ∎

The following lemma presents some properties of the functions $\pi_{ik}(\cdot)$ which were used to characterize the optimal auction.

Lemma B1.1 *The functions $\pi_{ik}(t)$ defined by (2.28) satisfy the following conditions, for all i, k, t_i and $\hat{t}_i$:*

$$(t_i - \hat{t}_i) \left(\pi_{ik}(t) - \pi_{ik}(\hat{t}_i, t_{-i}) \right) \geq 0 \tag{B1.6}$$

and

$$\pi_{ik}(t) \geq \pi_{i\,k+1}(t) . \tag{B1.7}$$

Proof : Write (2.28) as

$$\pi_{ik}(t) \equiv \mathrm{E}\left\{ V_{ik}(T) - \nu_{iki} \, \bar{V}_{iki}(T_i) \big| T = t \right\} + \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}'_{iki}(t_i) \frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)} \right)$$

$$= \sum_{\substack{j=1 \\ j \neq i}}^{n} \nu_{ikj} \, \bar{V}_{ikj}(t_j) + \nu_{iki} \left(\bar{V}_{iki}(t_i) - \bar{V}'_{iki}(t_i) \frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)} \right) . \tag{B1.8}$$

The first term in the right hand side is independent of t_i. So, Assumptions 2.8, 2.9 and 2.10 imply that $\pi_{ik}(\cdot)$ is nondecreasing in t_i and (B1.6) follows. Assumption 2.9 implies (B1.7). ∎

Given the dependence of $T^*_{ik}(\cdot)$ on $\pi_{ik}(\cdot)$, I can order the minimum announcements associated with distinct units.

Lemma B1.2 *The functions $T^*_{ik}(t_{-i})$ defined by (2.29) satisfy the following condition, for all i, k, and t_{-i}:*

$$T^*_{ik}(t_{-i}) \leq T^*_{i\,k+1}(t_{-i}) . \tag{B1.9}$$

Proof : The result is immediate from (2.29) and Lemma B1.1. ∎

In Subsection 2.5.2 it is claimed that the required minimum announcements in the optimal auction of the symmetric model are such that the bidders that meet the requirements are those with the highest private signals. Lemma B1.3 states the precise result.

Lemma B1.3 *Consider a realization of private signals, t, and the associated minimum required announcements, $\underline{t}^*$. Let $((1), \ldots, (n))$ be a permutation of the bidders such that $t_{(1)} \geq \cdots \geq t_{(n)}$; then*

$$t_{(i)} > \underline{t}^*_{(i)1} \Rightarrow t_{(i-1)} > \underline{t}^*_{(i-1)1}$$

and

$$t_{(i)} < \underline{t}^*_{(i)1} \Rightarrow t_{(i+1)} < \underline{t}^*_{(i+1)1} .$$

Proof : Rewrite (2.35) as

$$\pi_{i1}(t) = \alpha_1 \sum_{j=1}^{n} W_1(t_j) + (\alpha_0 - \alpha_1) W_1(t_i) - \alpha_0 \, W'_1(t_i) \frac{1 - F_T(t_i)}{f_T(t_i)} , \tag{B1.10}$$

to see that $\pi_{i1}(t)$ is larger for a bidder with a larger private signal. Now, suppose that $t_{(i)} > t^*_{(i)1}$; this implies $\pi_{(i)1}(t) > 0$ which, from (B1.10), implies $\pi_{(i-1)1}(t) < 0$, which is equivalent to $t_{(i-1)} > t^*_{(i-1)1}$. The second implication can be proven in a similar way. ∎

Proof of Lemma 2.5: Define the following function

$$v(t_i, t_{i(m)}) = \mathrm{E}\{V_{i1}(T) \,|\, T_{i(m)} = t_{i(m)}, T_i = t_i\}.$$

Then, $b^U(t_i) = v(t_i, t_i)$. Suppose that all bidders but i use the strategies prescribed in (2.38). The expected utility of bidder i if he submits bid b_i is

$$U_i(t_i, b_i) = \mathrm{E}\Big\{ (V_{i1}(T) - b^U(T_{i(m)})) \, 1_{\{b^U(T_{i(m)}) < b_i\}} \,\Big|\, T_i = t_i \Big\},$$

where $1_{\{b^U(T_{i(m)}) < b_i\}}$ is a function equal to 1 if $b^U(T_{i(m)}) < b_i$ and 0 otherwise. Using the function $v(\cdot, \cdot)$, the bidder's utility can be rewritten as

$$
\begin{aligned}
\mathrm{E}\Big\{ (V_{i1}(T) - b^U(T_{i(m)})) & \, 1_{\{b^U(T_{i(m)}) < b_i\}} \,\Big|\, T_i = t_i \Big\} = \\
&= \mathrm{E}\Big\{ \mathrm{E}\Big\{ (V_{i1}(T) - v(T_{i(m)}, T_{i(m)})) \, 1_{\{b^U(T_{i(m)}) < b_i\}} \,\Big|\, T_{i(m)}, T_i \Big\} \,\Big|\, T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ (v(T_i, T_{i(m)}) - v(T_{i(m)}, T_{i(m)})) \, 1_{\{b^U(T_{i(m)}) < b_i\}} \,\Big|\, T_i = t_i \Big\} \\
&= \int_{\underline{t}}^{b^{U-1}(b_i)} (v(t_i, x) - v(x, x)) \, f_{T_{i(m)}}(x) \, dx.
\end{aligned}
\tag{B1.11}
$$

The function $v(\cdot, \cdot)$ is increasing in the first argument, so the integrand in (B1.11) is positive if and only if $t_i > x$; therefore, the bid b_i that maximizes bidder i's utility is such that $b^{U-1}(b_i) = t_i$, i.e., $b_i = b^U(t_i)$. ∎

Proof of Lemma 2.6: Suppose that all bidders but i use the strategies prescribed in (2.41). The expected utility of bidder i if he submits bid b_i is then

$$U_i(t_i, b_i) = \mathrm{E}\Big\{ (V_{i1}(T) - b_i) \, 1_{\{b^D(T_{i(m)}) < b_i\}} \,\Big|\, T_i = t_i \Big\}.$$

Using the function $v(\cdot, \cdot)$, defined in the proof of Lemma 2.5 the bidder's utility can be rewritten as

$$
\begin{aligned}
\mathrm{E}\Big\{ (V_{i1}(T) - b_i) & \, 1_{\{b^D(T_{i(m)}) < b_i\}} \,\Big|\, T_i = t_i \Big\} = \\
&= \mathrm{E}\Big\{ \mathrm{E}\Big\{ (V_{i1}(T) - b_i) \, 1_{\{b^D(T_{i(m)}) < b_i\}} \,\Big|\, T_{i(m)}, T_i \Big\} \,\Big|\, T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ (v(T_i, T_{i(m)}) - b_i) \, 1_{\{b^D(T_{i(m)}) < b_i\}} \,\Big|\, T_i = t_i \Big\} \\
&= \int_{\underline{t}}^{b^{D-1}(b_i)} (v(t_i, x) - b_i) \, f_{T_{i(m)}}(x) \, dx.
\end{aligned}
\tag{B1.12}
$$

The bidder chooses his bid to maximize expected utility. Thus from (B1.12) I obtain a necessary first order condition for the optimal bid,

$$\Big(v(t_i, b^{D-1}(b_i)) - b_i\Big) \, f_{T_{i(m)}}(b^{D-1}(b_i)) \, \frac{1}{b^{D'}(b^{D-1}(b_i))} - F_{T_{i(m)}}(b^{D-1}(b_i)) = 0. \tag{B1.13}$$

Suppose that bidder i also wants to use the bidding function $b^D(t_i)$. Then the first order condition can

be written as the following first order differential equation

$$b^{D'}(t_i) = \left(v(t_i, t_i) - b^D(t_i)\right) \frac{f_{T_{i(m)}}(t_i)}{F_{T_{i(m)}}(t_i)}, \tag{B1.14}$$

and $b^D(t_i)$ is really the optimal bid if $b^D(\cdot)$ satisfies the differential equation (B1.14). Let me compute $b^{D'}(\cdot)$; for that write the bid as

$$b^D(t_i) = \int_{\underline{t}}^{t_i} v(x, x) \frac{f_{T_{i(m)}}(x)}{F_{T_{i(m)}}(t_i)} dx$$

and the derivative is

$$b^{D'}(t_i) = v(t_i, t_i) \frac{f_{T_{i(m)}}(t_i)}{F_{T_{i(m)}}(t_i)} - \int_{\underline{t}}^{t_i} v(x, x) \frac{f_{T_{i(m)}}(x) f_{T_{i(m)}}(t_i)}{\left(F_{T_{i(m)}}(t_i)\right)^2} dx$$

$$= \left(v(t_i, t_i) - b^D(t_i)\right) \frac{f_{T_{i(m)}}(t_i)}{F_{T_{i(m)}}(t_i)}$$

so (B1.14) is satisfied and $b_i = b^D(t_i)$. I may also check that, for $b_i = b^D(t_i)$, the second order condition can be written as

$$\frac{\partial}{\partial b_i} \left(\frac{v(t_i, b^{D^{-1}}(b_i)) - b_i}{v(b^{D^{-1}}(b_i), b^{D^{-1}}(b_i)) - b_i} - 1 \right) F_{T_{i(m)}}(b^{D^{-1}}(b_i)) < 0$$

which is in fact satisfied. Thus, given the symmetry of the problem, $(b^D(t_1), \ldots, b^D(t_n))$ is an equilibrium of the discriminatory price auction. ∎

Proof of Lemma 2.7: To show that the entire profile $(b^E(t_1), \ldots, b^E(t_n))$ is a (perfect Bayesian) equilibrium of the sequential English auction I only need to show that the bidding functions in each stage auction form a (Bayesian) equilibrium of the stage auction. For the winner, the k-th stage auction is equivalent to a discriminatory price auction with $n - k + 1$ bidders, and $k - 1$ commonly known signals (the signals that were initially privately known by the winners in the previous auctions). Therefore, Lemma 2.6 shows that the bids in (2.44) form a (Bayesian) equilibrium of the k-th stage auction. ∎

B2 Appendix: An Example of a Two-Unit Auction

In this appendix I analyze an example in which the bidders have multi-unit demands. It illustrates that, even in the case of a symmetric private values model with two units, the optimal auction can be very complex; in particular, the required minimum announcement may vary for distinct units.

To make the example as simple as possible, I consider a symmetric private values model with two units and two bidders; this will permit a simple geometric interpretation of the optimal auction. Conditional on the individual signals, the expected valuations are given by

$$E\{V_{i1}(T)|T = t\} = \sqrt{t_i} \tag{B2.1}$$

$$E\{V_{i2}(T)|T = t\} = t_i, \tag{B2.2}$$

which clearly satisfy Assumption 2.7.

The private signals are independently drawn from an uniform distribution on $[0, 1]$. With this support for the private signals, the valuations in (B2.1) and (B2.2) satisfy the nonincreasing condition in Assumption 2.8. They also satisfy the technical Assumptions 2.9 and 2.10: the hazard rate condition

is satisfied by the uniform distribution, and, from

$$\frac{\bar{V}''_{i1i}(t_i)}{\bar{V}_{i1i}(t_i)} = \frac{1}{2\,t_i} < \frac{1}{t_i} = \frac{\bar{V}''_{i2i}(t_i)}{\bar{V}_{i2i}(t_i)},$$

it is confirmed that Assumption 2.9 is also satisfied.

I define two functions, $\theta_1(\cdot)$ and $\theta_2(\cdot)$, which will be interpreted as the minimum winning announcement given the other bidder's announcement:

$$\theta_1(x) = \max\left\{\frac{1}{3}, \left(\frac{2\,x - 1 + 2\sqrt{x^2 - x + 1}}{3}\right)^2\right\} \tag{B2.3}$$

$$\theta_2(x) = \max\left\{\frac{1}{2}, \frac{3\,x + 2\sqrt{x} - 1}{4\sqrt{x}}\right\}. \tag{B2.4}$$

These expressions are derived as follows. For bidder i to receive at least one unit his private signal must be such that $\pi_{i1}(t) > \max\{0, \pi_{j2}(t)\}$; otherwise the good is not allocated or it is allocated to the other bidder. But, for $\pi_{i1}(t) > 0$ it must be that $t_1 > 1/3$; similarly, for $\pi_{i1}(t) > \pi_{j2}(t)$, it must be that

$$t_i > \left(\frac{2\,x - 1 + 2\sqrt{x^2 - x + 1}}{3}\right)^2.$$

These two conditions yield (B2.3); the expression for $\theta_2(\cdot)$ is derived similarly. The functions $\theta_1(\cdot)$ and $\theta_2(\cdot)$ are used in the following lemma.

Lemma B2.1 *Let (p^*, c^*) be an optimal auction in this example. Then*

$$p^*_{i1}(t) = \begin{cases} 1 & \textit{if } t_i > \theta_1(t_j) \\ 0 & \textit{otherwise} \end{cases} \tag{B2.5}$$

$$p^*_{i2}(t) = \begin{cases} 1 & \textit{if } t_i > \theta_2(t_j) \\ 0 & \textit{otherwise} \end{cases} \tag{B2.6}$$

and

$$c^*_i(t_i) = \begin{cases} 0 & t_i \leq 1/3 \\[2ex] \displaystyle\int_0^{\theta_2(t_i)} \sqrt{\theta_1(x)}\,dx & 1/3 < t_i \leq 1/2 \\[2ex] \displaystyle\int_0^{\theta_2(t_i)} \sqrt{\theta_1(x)}\,dx + \int_0^{\theta_1(t_i)} \theta_2(x)\,dx & t_i > 1/2. \end{cases} \tag{B2.7}$$

Proof : Compute $\pi_{ik}(t_i)$ to get

$$\pi_{i1}(t_i) = \sqrt{t_i} - \frac{1 - t_i}{2\sqrt{t_i}} \tag{B2.8}$$

$$\pi_{i2}(t_i) = 2\,t_i - 1. \tag{B2.9}$$

The optimal auction will allocate one unit to bidder i if $\pi_{i1}(t_i)$ is positive and greater than $\pi_{j2}(t_j)$, which is satisfied if $t_i > \theta_1(t_j)$, proving (B2.5). Similarly, bidder i receives the second unit if $\pi_{i2}(t_i)$ is positive and greater than $\pi_{j1}(t_j)$, which is satisfied for $t_i > \theta_2(t_j)$, proving (B2.6).

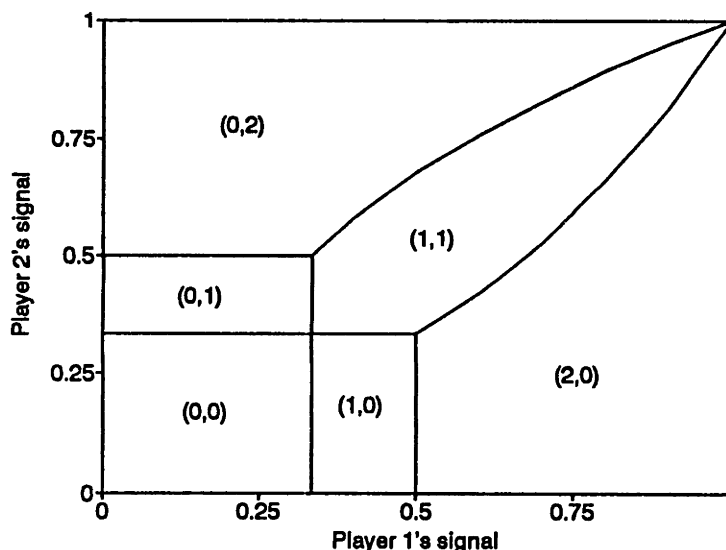To obtain (B2.7), I insert (B2.5) and (B2.6) into (2.31). ∎

Figure B2-1: Optimal allocations in an example of a two unit auction.

Figure B2-1 shows the optimal allocations for the possible vectors of private signals. The ordered pairs in each subset of the signals' space corresponds to the number of units allocated to the bidders: the first number is the number of units allocated to bidder 1, and the second number is the number of units allocated to bidder 2. If bidders have very low private signals they get no unit; with intermediate values they get one unit, and with high values they get two units; however, even though this is a private values model, the boundaries among low, intermediate and high signals, depend on the other bidder's signal.

The optimal mechanism cannot be implemented through a generalized standard auction. However, it is possible to construct a modified discriminatory price auction that implements the optimal mechanism.[30]

To describe the optimal modified discriminatory price auction the seller defines the functions $\vartheta_k(\cdot)$ as

$$\vartheta_1(t_i) = \mathrm{E}\{V_{i1}(\theta_1(T_j'),T_j)|\,\theta_1(T_j) < t_i\}$$
$$\vartheta_2(t_i) = \mathrm{E}\{V_{i2}(\theta_2(T_j),T_j)|\,\theta_2(T_j) < t_i\}\,.$$

Each bidder i submits bids for the two units, b_{i1} and b_{i2}. From the submitted bids the seller computes modified bids $\tilde{b}_{ik}$ as

$$\tilde{b}_{ik} = \pi_{ik}(\vartheta_k^{-1}(b_{ik}));$$

the two units are allocated to the bidders that submitted the two highest modified bids, as long as they are positive; for each unit received a bidder pays the value of the winning original bid; a bidder that receives no unit pays nothing.

To show that this modified discriminatory price auction implements the optimal mechanism, I show that if each bidder bids

$$b_k^{MD}(t_i) = \vartheta_k(t_i) \tag{B2.10}$$

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[30]It is a modified discriminatory price auction because it has the same payment rules as the discriminatory price auction, but the allocation rules are based not on the original bids but on modified bids.

and the pair of bids is denoted $b^{MD}(t_i) = (b_1^{MD}(t_i), b_2^{MD}(t_i))$, then the profile $(b^{MD}(t_1), b^{MD}(t_2))$ is an equilibrium of the auction and generates expected payments as in (B2.7).[31]

Suppose that bidder j uses bids $b^{MD}(t_j)$. Bidder i chooses bids b_1 and b_2 to maximize the objective function

$$\mathrm{E}\big\{ (V_{i1}(T) - b_1)\, 1_{\{\pi_{j2}(T_j) \le \pi_{i1}(\vartheta_1(b_1))\}} \big| T_i = t_i \big\} + \mathrm{E}\big\{ (V_{i2}(T) - b_2)\, 1_{\{\pi_{j1}(T_j) \le \pi_{i2}(\vartheta_2(b_2))\}} \big| T_i = t_i \big\}$$

The first component is the expected utility drawn from receiving one unit: it gives gross benefit $V_{i1}(T)$, and the bidder pays b_1 if bidder j's modified bid for the second unit (which is $\pi_{j2}(T_j)$ by assumption) is not greater than $\tilde{b}_1$. Note that, because $\tilde{b}_{j2}(t_j) \ge 0$ it is not needed to consider the case in which it would be negative. Similarly, the second component corresponds to the expected utility from the second unit.

The objective function is separable in b_1 and b_2, so the optimization can be performed for each bid independently. The optimal bid for the first unit will solve

$$\max_{b_1} \int_0^{\theta_2(\vartheta_1^{-1}(b_1))} \left(\sqrt{t_i} - b_1\right) dx$$

so the first order necessary condition for an optimal bid is

$$\frac{\theta_2'(\vartheta_1^{-1}(b_1))}{b_1'(\vartheta_1^{-1}(b_1))} \left(\sqrt{t_i} - b_1\right) - \theta_2(\vartheta_1^{-1}(b_1)) = 0.$$

Suppose that $b_1 = b_1^{MD}(t_i)$. then the previous condition becomes

$$b_1^{MD'}(t_i) = \frac{\theta_2'(t_i)}{\theta_2(t_i)} \left(\sqrt{t_i} - b_1\right). \tag{B2.11}$$

Compare this expression to

$$b_1^{MD'}(t_i) = \left(\sqrt{\theta_1(\theta_2(t_i))} - b_1^{MD}(t_i)\right) \frac{\theta_2'(t_i)}{\theta_2(t_i)}$$

which follows from (B2.10). For types that get the first unit with positive probability, $\theta_1(\theta_2(t_i)) = t_i$, so b_1 must equal $b_1^{MD}(\cdot)$ in (B2.11) by transitivity. Similarly, it can be shown that $b_2^{MD}(t_i)$ is a bidder i's optimal bid for the second unit. Given the symmetry of the problem I conclude that $(b^{MD}(t_1), b^{MD}(t_2))$ is an equilibrium of the modified discriminatory price auction.

To check that the modified discriminatory price auction implements the optimal mechanism I show that it allocates the goods optimally and that it yields the optimal expected payments. By construction the modified equilibrium bids will be equal to $\pi_k(t_i)$; hence, the units are optimally allocated. The expected payments are

$$c_i(t_i) = \mathrm{E}\big\{ b_1^{MD}(T_i)\, 1_{\{\theta_1(T_j) < T_i\}} + b_2^{MD}(T_i)\, 1_{\{\theta_2(T_j) < T_i\}} \big| T_i = t_i \big\}$$
$$= \mathrm{E}\big\{ V_{i1}(\theta_1(T_j), T_j)\, 1_{\{T_j < \theta_2(T_i)\}} \big| T_i = t_i \big\} + \mathrm{E}\big\{ V_{i2}(\theta_2(T_j), T_j)\, 1_{\{T_j < \theta_1(T_i)\}} \big| T_i = t_i \big\}$$

which is equal to the optimal expected payments given in (B2.7).

The simple private values example of two units and two bidders demonstrates the complexity that the optimal mechanism may have, even in simple frameworks. In particular, the seller may need to modify the standard multiple unit auctions to achieve an optimal mechanism. The generalization to a greater number of units and bidders make it almost impossible to organize an optimal auction without

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[31]I have not tried to identify if there are other equilibria of this game.

a computational procedure, not only for the seller, because the modified bids are very complex, but for the bidders as well, because the optimal bids (particularly in common value auctions) require long calculations.

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Chapter 3

Optimal Auctions of a Divisible Good

3.1 Introduction

Following the common approach to auction theory, the previous chapters of this dissertation analyzed auctions of indivisible goods; in fact, even though there are many situations in which the good to be sold is divisible (e.g., auctions of shares, Treasury bill auctions), we have a very limited knowledge about optimal auctions of divisible goods. In this chapter I adapt the techniques for the analysis of indivisible good auctions to characterize optimal auctions of a divisible good.

The economic relevance of Treasury bill auctions is probably the main justification for this analysis. For over thirty years, there has been a debate about the best procedure for the auctioning of Treasury bills.[1] Bikhchandani and Huang (1989) have recently addressed the question. In a model of affiliated values, where bidders demand at most one unit and a resale market is explicitly considered, they show that, depending on the parameters of the model, either one of the mechanisms may yield a higher revenue to the seller. However, to my

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[1]Most of the discussion has concentrated on the relative merits of the discriminatory price and the uniform price auctions. For a review of the theoretical arguments see Carson (1959), Brimmer (1962), Golstein (1962), Friedman (1963), Smith (1966), Bolten (1973) and Scott and Wolf (1979). Empirical and experimental studies have simultaneously been conducted (see, for example, Smith, 1967, Smith and Williams, 1983, Miller and Plott, 1985, and Cammack, 1991).

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knowledge, there has not been any success in characterizing optimal auctions.

A complete analysis of optimal auctions of a divisible good in a symmetric private values model (hence not applicable to Treasury bill auctions) is given in Maskin and Riley (1989). They show that standard multiple unit mechanisms, such as the discriminatory price and the uniform price auction, are optimal if each bidder demands at most one unit; but this does not generalize to the case of multi-unit demands, instead they construct an optimal procedure, in which the bidders announce demands, and payments and allocations are accordingly decided. This work is the starting point of my analysis, which is aimed at generalizing the characterization of optimal mechanisms to asymmetric models with correlated valuations.

This chapter is composed of two parts. In the first part, which includes Sections 3.2 to 3.5, I analyze auctions when a bidder's valuation does not depend on the other bidders' allocations. The model may apply to several situations, such as those in which the good is used for the bidder's consumption or in which bidders operate in independent resale markets. The analysis generalizes that in Maskin and Riley (1989), because the model in this part allows asymmetric bidders and the valuation functions cover not only the private values model but also common value models, as particular cases.

This first part is organized as follows. In Section 3.2, I introduce the assumptions of the first model, referred as Model I, which are similar to the assumptions in the specification of the multiple unit indivisible good model in Chapter 2. In Model I it is assumed that any bidder's valuation depends only on the quantity that he receives and on the vector of private signals. In Section 3.3, to illustrate the role and meaning of the assumptions, some examples are presented. Section 3.4 is devoted to the characterization of optimal mechanisms; it is shown that mechanisms in which the bidders reveal a demand function are not, in general, optimal.[2] I then consider a symmetric model and four mechanisms that implement it are presented. In one mechanism the bidders place orders for the (expected) quantity desired, and allocations and payments are based on these orders; in the second mechanism, the bidders pay to participate in the auction, and the allocation is decided as a function of the bidders' participation payments. These mechanisms exhibit some possibly undesirable properties, such

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[2]The issue of multiple equilibria is not addressed in this chapter, even though, as Wilson (1979) has shown, these mechanisms may also have multiple equilibria, where sometimes the minimum equilibrium payoff is as low as 50% of the maximum one.

as requiring payments from all bidders, even if some do not receive a positive quantity of the good. The third mechanism have the bidders submitting expected demands, which are used to decide on the allocations and the payments, being guaranteed that a bidder only makes a positive payment if he receives a positive quantity; in the fourth mechanism the bids are payments to be made if the bidder receives a positive quantity. So these two mechanisms will not require a payment from a bidder that receives no good.

The second model (Model II) makes a bidder's valuation contingent on the allocation that is implemented and not only on the quantity received by himself. This assumption seems particularly recommended if there is a resale market, where the bidders that have won in the auction will sell the good. The analysis then proceeds as in Model I. The assumptions are presented and discussed in Section 3.6. Some illustrative examples are described in Section 3.7. Optimal auctions are characterized in Section 3.8; it is shown that, even though the model has similarities with Model I, the optimal auctions may lead to a very different allocation.

Finally, Section 3.9 concludes the chapter.

In an appendix several numerical examples of both models are presented and optimal auctions are explicitly computed.

3.2 Model I: The Assumptions

Model I is very close to the multiple unit model in Chapter 2 and the assumptions are merely adjusted to deal with divisible goods.

The supply side of the model is described by Assumptions 3.1 and 3.2.

Assumption 3.1 *There is a quantity Q of a divisible good to be sold.*

Assumption 3.2 *The seller is risk neutral and does not value the good.*

This is a model of a single seller that wants to sell a fixed quantity of a good with the maximum expected profit. Like in the indivisible good models, the important restriction imposed by Assumption 3.2 is the seller's risk neutrality, which influences the characteristics of the optimal mechanisms. In fact, if the seller had preferences for the good analogous to those of the buyers, all the main results of the paper would be kept, with minor modifications. The assumption of risk neutrality extends to the buyers.

Assumption 3.3 *There are n risk neutral utility maximizers potential buyers.*

I argue that for the main applications that motivated this paper the assumption of agents' risk neutrality is a reasonable one. For example, in the case of Treasury bill auctions, it seems acceptable that the seller does not exhibit risk aversion; similarly, the buyers, mostly banks and financial institutions with other investment possibilities, are reasonably characterized with a neutral attitude towards the risk involved in this particular decision.

The assumptions about the information structure (Assumptions 3.4 to 3.6) are exactly as those in the indivisible good models.

Assumption 3.4 *Each potential buyer has his own private information about the value of the good, T_i.*

The general model allows for asymmetries among bidders: the private signals may be drawn from distinct distributions, with distinct supports. In allowing the private signals to have distinct distributions and supports I am trying to accommodate the possibility that bidders have signals about distinct characteristics of the good, so that T_i and T_j would be signals of two payoff relevant characteristics.

Assumption 3.5 *Other agents do not observe the realization of T_i and treat it as a draw from a cumulative distribution $F_{T_i}(\cdot)$, with support $\mathbf{T}_i = [\underline{t}_i, \bar{t}_i]$.*

The assumption of private signals' independence is very important. Though technically indispensable to drive the results,[3] I sustain that it may be appropriate for a wide range of situations. If the private signals are interpreted as information on distinct characteristics of the product, the independence assumption is reasonable, as long as the characteristics are independently distributed across goods; on the other hand, if the private signals provide information on the same characteristics, the assumption should be interpreted as independence conditional on common information. With this motivation, the assumption is not as strong as it would initially appear.

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[3]In the context of an indivisible good, Crémer and McLean (1985, 1988) show that, provided a technical assumption, even with a small correlation, the seller is able to achieve the first best.

Assumption 3.6 *Any two variables T_i and T_j, with $j \neq i$, are independent.*

Either because the signals convey information about distinct characteristics of the good, or because they are imprecise estimates of the same payoff relevant characteristic that are more informative when considered together, the valuations for the good are allowed to depend on all the available (partially known) information.

Assumption 3.7 *The value for potential buyer i of the quantity q_i of the good satisfies*

$$E\{V_i(q_i,T)|T=t\} = \sum_{j=1}^{n} \nu_{ij} \int_0^{q_i} \bar{V}_{ij}(x,t_j)\,dx$$

where the parameters ν_{ij} ($j \neq i$) are nonnegative, ν_{ii} is positive, and the functions $\bar{V}_{ij}(\cdot,\cdot)$ are twice differentiable, nonincreasing in the first argument and increasing in the second argument.

The function $\bar{V}_{ij}(\cdot,t_j)$ is the expected effect of bidder j's information on bidder i's marginal valuation of the i-th unit. In the next section I present several examples to illustrate possible interpretations for these functions. Note that a private values model is generated if, for $i \neq j$, $\nu_{ij} = 0$; while if, for all i, $\nu_{ij} = \nu_j$ and $\bar{V}_{ij}(\cdot,\cdot) = \bar{V}_j(\cdot,\cdot)$, a common value model is generated.

The specification of the valuations is a very special one: not only the unique relevant element of an allocation is the quantity received by the bidder, but also the information provided by the private signals is aggregated in a separable way. A natural process to generate the additive separable structure is to think about the term ν_{ij} as the probability that t_j will be the realization of the relevant characteristic for potential buyer i. In the next section I will try to justify alternative motivations.

To simplify the characterization of the optimal mechanisms, some regularity conditions are imposed on the private estimates of the demand functions.

Assumption 3.8 *The expected valuations satisfy the following regularity conditions, for all i, q_i and t_i:*

$$\frac{\partial}{\partial t_i}\left(\frac{q_i}{\bar{V}_{ii}(q_i,t_i)}\,\frac{\partial \bar{V}_{ii}(q_i,t_i)}{\partial q_i}\right) \geq 0$$

$$\nu_{ii}\frac{\partial}{\partial t_i}\bar{V}_{ii}(q_i,t_i) \geq \nu_{ji}\frac{\partial}{\partial t_i}\bar{V}_{ji}(q_j,t_i)$$

$$\frac{\partial^2 \bar{V}_{ii}(q_i, t_i)}{\partial t_i^2} \leq 0.$$

The first condition generalizes the nondecreasing price elasticity of the demand used in Maskin and Riley (1989). As illustrated in the multiple unit good model of Chapter 2, this assumption is sufficient to guarantee that the seller's marginal benefit from the quantity allocated to a bidder is nonincreasing in the quantity allocated, which simplifies the structure of the optimal mechanism.

The second condition states that, for any quantities allocated, the effect of an increase in t_i on bidder i's marginal valuation is no lower than the effect on any other bidder's marginal valuation. Thus an increase in i's signal will make the optimal allocation shift towards giving a larger quantity to bidder i.

The third condition imposes the concavity of the marginal valuation functions on the bidder's own signal, which is a standard condition in multiple unit models.

A final regularity condition, the monotone hazard rate condition, is imposed on the distributions of the private signals.

Assumption 3.9 *The hazard rate* $h_{T_i}(t_i) = f_{T_i}(t_i)/[1 - F_{T_i}(t_i)]$ *is nondecreasing in* t_i.

These assumptions specify a model of a divisible good which is the basis for the analysis in the first part of this paper.

3.3 Model I: Examples

In this section I suggest some examples of models that satisfy the assumptions of the previous section, and to which the following analysis may be applied.

3.3.1 Auctions of Shares

The first example applies to auctions of shares. Consider a firm that organizes an initial public offering to raise capital for the realization of a particular project. Let Q be the quantity of shares to be sold (taken as a fraction of the total quantity of shares), which represents a small fraction of the total firm, so that no investor may get control of the firm through the acquisition of these shares.

There are n potential investors, each of whom has private information on a characteristic that affects the value of the firm, t_i. For example t_i may be an estimate of the value of the project that the firm will realize, or an estimate of the probability that the project will be awarded to the firm.

In this framework, interpret $\bar{V}_{ij}(q_i, t_i)$ as the marginal value of q_i shares if the firm's characteristic is t_i, which may also include some nonmonetary benefits of ownership. In particular, let the function satisfy

$$\bar{V}_{ij}(q_i, t_j) = W(t_j)(1 + \phi_i'(q_i)),$$

where $W(t_j)$ is the value of the firm with characteristic t_i, and $\phi_i(q_i)$, is a nondecreasing function that gives investor i's nonmonetary benefits of ownership relative to the value of the firm. Let the probability that the firm's characteristic be equal to any private signal be equal to $1/n$. Then we may write

$$\mathrm{E}\{V_i(T)|\, T = t\} = \sum_{j=1}^{n} \frac{1}{n} \int_0^{q_i} W(t_j)(1 + \phi_i'(x))\, dx.$$

If the uncertainty is modelled in such a way that $W(\cdot)$ is a concave function, and the hazard rate condition is satisfied, all the assumptions of Model I can be satisfied in the example.

3.3.2 Auctioning Pollution Rights

In this example I would like to focus on the possibility of use of auction type mechanisms to control the level of pollution.

There are n plants which generate a pollutant, at levels there are monitored by a public agency. To control the level of pollution the agency regularly auctions *rights to pollute*. To simplify the analysis, let each pollution right entitle a firm to produce one unit and be non-transferable between firms. Moreover, assume that the market price is such that the level of pollution rights really binds the production (in the sense that a firm that has q_i pollution rights will produce q_i units of the good), so that the final market price depends only on the total level of rights that the regulatory agency sells in the auction.

Besides selling the pollution rights, the regulatory agency monitors the level of pollutants. The environmental effect of pollutants created by the plants' operation is influenced by outside

conditions (e.g., atmospheric factors); each plant knows the situation in its region, which constitutes the plant's private information, t_i. As a result of its monitoring, the agency imposes a scheme of payments-reimbursements to each plant which is a function of the pollutant levels in the region; let the effect of such scheme in firm i's cost be

$$\psi_i(t) = \sum_{j=1}^{n} \nu_{ij}\, \phi_j(t_j)$$

where the function $\phi_j(\cdot)$ measures the effect of pollutants in region j, and the weights ν_{ij} take into account the likelihood that pollution generated by plant i will affect region j.

According to the previous description a firm's valuation may satisfy

$$\mathrm{E}\{V_i(q_i,T)|\, T = t\} = \int_0^{q_i} \left(p(Q) - \left(\sum_{j=1}^{n} \nu_{ij}\, \phi_j(t_j) \right) c_i(x) \right)\, dx\,,$$

where $c_i(q_i)$ is the marginal cost of plant i (in the absence of the payment-reimbursement scheme).

Thus, if the cost functions are convex in the level of production and the $\phi_i(\cdot)$ functions are convex (which roughly means that if the state of the atmosphere is deteriorated then the marginal fine imposed on the firms increase), Assumption 3.8 is satisfied.[4]

Finally, the hazard rate condition on the private signals must be satisfied.

3.3.3 Treasury Bill Auctions

With the analysis in this chapter I would like to provide elements for the characterization of optimal Treasury bill auctions. In this example, I suggest reasons to justify that Model I may be used as a description of Treasury bill auctions.

Let me consider the following simplified version of the Treasury bill market. There is a primary market where the bills are sold (in auction) to banks. Then, in a secondary market, the banks that obtained some bills in the auction sell them in a competitive resale market.

The resale market is very simple. I assume that the buyers in the resale market are risk neutral and receive information on the bids submitted in the auction,[5] which are used to

---

[4]The second condition in the assumption imposes some additional restrictions on the functions $\phi_i(\cdot)$.

[5]The Treasury usually releases some statistics about the bids.

compute the expected value of the bills. The bids are informative about the value of the bills because each bank has private information on the effect of its operations on the term structure of interest rates. So, let t_i be bank i's private information; given the vector of private signals, t, the value of each bill may then be written as

$$P(t) = E\left\{\sum_{i=1}^{n} \nu_i \, \phi_i(T_i)\Big| T = t\right\},$$

where ν_i is the effect of bank i's information on the price and the functions $\phi_i(\cdot)$, which account for the effect of bank i' private information, are increasing and concave.

The banks have a cost associated with the participation in the auction, $C_i(q_i)$. This represents the opportunity cost of funds invested in Treasury bills and some administrative costs. Thus, considering the price that will be generated in the secondary market, the valuation of bank i for a quantity q_i of bills satisfies

$$E\{V_i(q_i, T)| T = t\} = P(t)\, q_i(t) - C_i(q_i).$$

If the cost functions are convex the assumptions of Model I may be satisfied.

3.4 Model I: Optimal Auctions

The analysis of optimal auctions in Model I uses a technique similar to the one developed for the indivisible good models. The section is organized in the following way. First, I characterize feasible allocations; then, auctions of divisible goods are defined, and feasible deterministic auctions are characterized; finally, the properties of optimal auctions are studied.

The goal of the seller is to find a mechanism that maximizes his expected revenue from the sale of the good, which, as a first step, requires him to decide the allocation that should be implemented.[6] This is a reason to start the analysis with a characterization of feasible allocations.

Definition 3.1 *An allocation is a vector* $(q_1, \ldots, q_n)$, *such that any element* q_i, *the quantity*

---

[6]I will restrict attention to mechanisms which do not involve payments after the sale of the good.

allocated to bidder i, is nonnegative and

$$\sum_{i=1}^{n} q_i \leq Q \,. \tag{3.1}$$

The set of all allocations is denoted by $\mathcal{A}(Q, n)$.

Given that all agents are risk neutral, a mechanism is fully characterized by a probability distribution over the set of feasible allocations and the bidders' expected payments. Moreover, by the Revelation Principle (Myerson, 1979), any mechanism is equivalent to a revelation mechanism, where buyers are asked to announce their private signals. Therefore, in the context of a divisible good, auctions are defined as follows.

Definition 3.2 *An auction is a pair of functions (q, c), such that*

$$q : \mathbf{T} \to \mathcal{A}(Q, n)$$

and

$$c_i : \mathbf{T_i} \to \mathbf{R} \,.$$

Conditional on the bidders' announcements, t, an auction specifies the allocation that is implemented, $q(t)$, and the bidders' expected payments, $c_i(t_i)$. This, generalizes the definition of deterministic auctions to a divisible good framework.

In general, a mechanism may involve random allocations, i.e., allow for situations in which the seller would randomize among several allocations. However, as the analysis for multiple indivisible units shows,[7] under the regularity assumptions of the model, we expect the optimal mechanism to be deterministic. Thus, the analysis is restricted to deterministic auctions.[8]

The seller's utility from an auction (q, c), $U_0(q, c)$, is equal to the bidders' total expected payments, i.e.,

$$U_0(q, c) = \mathrm{E}\{\textstyle\sum_{i=1}^{n} c_i(T_i)\} \,.$$

---

[7]See Chapter 2 for a treatment of multiple unit auctions of an indivisible good, allowing for stochastic allocation rules.

[8]It can be verified that stochastic mechanisms do not improve the seller's payoff.

On the other hand, the buyers' utilities are also influenced by the quantities allocated and buyer i's utility is given by

$$U_i(q, c, t_i) = \text{E}\{V_i(q_i(T), T) | T_i = t_i\} - c_i(t_i).$$

Given that attention is restricted to revelation mechanisms, a feasible auction must be incentive compatible and individually rational. Let $\mathcal{U}_i(q, c, t_i, \hat{t}_i)$ be the expected utility of bidder i, with type t_i, if he announces to be of type $\hat{t}_i$, i.e.,

$$\mathcal{U}_i(q, c, t_i, \hat{t}_i) = \text{E}\left\{V_i(q_i(\hat{t}_i, T_{-i}), T) \Big| T_i = t_i\right\} - c_i(\hat{t}_i).$$

Then, incentive compatibility can be stated as:

$$U_i(q, c, t_i) = \max_{\hat{t}_i \in T_i} \mathcal{U}_i(q, c, t_i, \hat{t}_i). \tag{3.2}$$

As a direct application of the envelope theorem, Lemma 3.1 provides a necessary condition for incentive compatibility.

Lemma 3.1 *If an auction (q, c) is incentive compatible, then, for all i and t_i,*

$$U_i(q, c, t_i) = U_i(q, c, \underline{t}_i) + \nu_{ii} \, E\left\{ \int_{\underline{t}_i}^{T_i} \int_0^{q_i(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \, \bigg| \, T_i = t_i \right\}. \tag{3.3}$$

Proof: See Appendix C1. ∎

Using the characterization of the necessary condition for incentive compatibility given in Lemma 3.1, it is possible to provide sufficient conditions for an auction to be feasible.

Proposition 3.1 *Suppose that an auction (q, c) satisfies the following conditions, for all i, t, t_i, and $\hat{t}_i$:*

$$(t_i - \hat{t}_i)\left(q_i(t_i, t_{-i}) - q_i(\hat{t}_i, t_{-i})\right) \geq 0 \tag{3.4}$$

$$\frac{\partial U_i(q, c, t_i)}{\partial t_i} = \nu_{ii} E\left\{ \int_0^{q_i(T)} \frac{\partial \bar{V}_{ii}(x, T_i)}{\partial t_i} \, dx \, \bigg| \, T_i = t_i \right\} \tag{3.5}$$

$$U_i(q, c, t_i) \geq 0 \tag{3.6}$$

$$q(t) \in \mathcal{A}(Q, n). \tag{3.7}$$

Then, the auction (q, c) is feasible.

Proof : By condition (3.5) I may write

$$U_i(q, c, t_i) = U_i(q, c, \underline{t}_i) + \nu_{ii} \mathrm{E}\left\{ \int_{\underline{t}_i}^{T_i} \int_0^{q_i(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \, \Big| \, T_i = t_i \right\}, \tag{3.8}$$

and individual rationality follows from (3.6), (3.8) and Assumption 3.7.

To prove incentive compatibility, use (3.8) to write

$$
\begin{aligned}
U_i(q, c, t_i) &= U_i(q, c, \hat{t}_i) + \nu_{ii} \mathrm{E}\left\{ \int_{\hat{t}_i}^{T_i} \int_0^{q_i(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \, \Big| \, T_i = t_i \right\} \\
&\geq U_i(q, c, \hat{t}_i) + \nu_{ii} \mathrm{E}\left\{ \int_{\hat{t}_i}^{T_i} \int_0^{q_i(\hat{t}_i, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \, \Big| \, T_i = t_i \right\} \\
&= U_i(q, c, \hat{t}_i) + \nu_{ii} \mathrm{E}\left\{ \int_0^{q_i(\hat{t}_i, T_{-i})} \left(\bar{V}_{ii}(y, t_i) - \bar{V}_{ii}(y, \hat{t}_i) \right) \, dy \, \Big| \, T_i = t_i \right\} \\
&= \mathcal{U}_i(q, c, t_i, \hat{t}_i),
\end{aligned}
$$

where the inequality follows from (3.4) and Assumption 3.7. But $U_i(q, c, t_i) = \mathcal{U}_i(q, c, t_i, t_i)$; so incentive compatibility is satisfied. Therefore, the auction (q, c) is feasible. ∎

This result can be interpreted in terms of information rents. Bidders enjoy rents that are decided to create the incentive for a bidder to truthfully announce his type, instead of misrepresenting it, announcing a lower signal. However, like in the case of indivisible goods, the argument suggests that the bidder with the lowest possible private signal will have no rent. This result, combined with the incentive compatibility necessary condition, can then be used to derive the expected payment schedule.

Proposition 3.2 *Suppose that (q^*, c^*) is an optimal auction. Then a bidder reporting the lowest possible private signal has zero utility, i.e.,*

$$U_i(q^*, c^*, \underline{t}_i) = 0, \tag{3.9}$$

and, for all i and t_i, the expected payment schedule satisfies

$$c_i^*(t_i) = \mathrm{E}\left\{ V_i(q_i^*(T), T) - \nu_{ii} \int_{\underline{t}_i}^{T_i} \int_0^{q_i^*(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \,\middle|\, T_i = t_i \right\}. \tag{3.10}$$

Proof : The proof is fairly standard. I rewrite the seller's utility as

$$\begin{aligned}
U_0(q, c) &= \mathrm{E}\{\textstyle\sum_{i=1}^n c_i(T_i)\} \tag{3.11}\\
&= \sum_{i=1}^n \mathrm{E}\{V_i(q_i(T), T)\} - \sum_{i=1}^n \mathrm{E}\{V_i(q_i(T), T) - c_i(T_i)\}\\
&= \sum_{i=1}^n \mathrm{E}\{V_i(q_i(T), T)\} - \sum_{i=1}^n \mathrm{E}\{U_i(q, c, T_i)\}\\
&= \sum_{i=1}^n \mathrm{E}\{V_i(q_i(T), T)\} - \sum_{i=1}^n \mathrm{E}\left\{ U_i(q, c, \underline{t}_i) + \nu_{ii} \int_{\underline{t}_i}^{T_i} \int_0^{q_i(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \right\}\\
&= \mathrm{E}\left\{ \sum_{i=1}^n V_i(q_i(T), T) - \nu_{ii} \int_{\underline{t}_i}^{T_i} \int_0^{q_i(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \right\} - \sum_{i=1}^n U_i(q, c, \underline{t}_i). \tag{3.12}
\end{aligned}$$

The bidders' expected payments appear only in $U_i(q, c, \underline{t}_i)$, which has a negative effect in the seller's utility; hence, choosing the expected payments that maximize the seller's payoff is equivalent to choose them to minimize the utilities $U_i(q, c, \underline{t}_i)$. From (3.8), $U_i(q, c, \underline{t}_i)$ can be expressed as

$$\begin{aligned}
U_i(q, c, \underline{t}_i) &= U_i(q, c, t_i) - \nu_{ii} \, \mathrm{E}\left\{ \int_{\underline{t}_i}^{T_i} \int_0^{q_i(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \,\middle|\, T_i = t_i \right\}\\
&= \mathrm{E}\left\{ V_i(q_i(T), T) - \nu_{ii} \int_{\underline{t}_i}^{T_i} \int_0^{q_i(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \,\middle|\, T_i = t_i \right\} - c_i(t_i). \tag{3.13}
\end{aligned}$$

Individual rationality requires that $U_i(q, c, \underline{t}_i) \geq 0$, so the best that the seller can achieve is $U_i(q^*, c^*, \underline{t}_i) = 0$, and, in an optimal auction, condition (3.9) is satisfied; moreover, using (3.9) in (3.13), the expected payment, $c_i^*(t_i)$, must satisfy (3.10). Thus (q^*, c^*) satisfies (3.9) and (3.10). ∎

Proposition 3.2 specifies one of the elements of an optimal auction, the bidders' expected payments, as a function of the optimal allocation rules. Using this result I can construct the problem that the allocation rules of an optimal auction must solve.

Proposition 3.3 *Consider an auction (q^*, c^*) satisfying (3.10). Let $q^*(\cdot)$ solve the following problem*

$$\max_{q(\cdot)} \quad \mathrm{E}\left\{ \sum_{i=1}^{n} \left(V_i(q_i(T), T) - \nu_{ii} \left(\int_0^{q_i(T)} \frac{\partial \bar{V}_{ii}(y, T_i)}{\partial t_i} \, dy \right) \frac{1 - F_{T_i}(T_i)}{f_{T_i}(T_i)} \right) \right\}$$

$$(\mathcal{P}3.1)$$

$$\text{s.t.} \quad \left\{ \begin{array}{l} (t_i - \hat{t}_i)(q_i(t_i, t_{-i}) - q_i(\hat{t}_i, t_{-i})) \geq 0 \\ q(t) \in \mathcal{A}(Q, n). \end{array} \right.$$

Then, (q^*, c^*) is an optimal auction.

Proof : From (3.9) and (3.12), the seller's payoff from an optimal auction is

$$U_0(q^*, c^*) = \mathrm{E}\left\{ V_i(q_i^*(T), T) - \nu_{ii} \int_{\underline{t}_i}^{T_i} \int_0^{q_i^*(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \right\} ; \qquad (3.14)$$

through simple manipulation of the integral, the right hand side of (3.14) can be written as the objective function of Problem ($\mathcal{P}3.1$). Moreover, to ensure the auction's feasibility, the restrictions must be imposed. Therefore, if $q^*(\cdot)$ solves the problem, (q^*, c^*) is an optimal auction. ∎

Using the last proposition, it is possible to present a set of necessary and sufficient conditions that the allocation rule of an optimal auction, $q^*(\cdot)$, must satisfy. The objective function of Problem ($\mathcal{P}3.1$) is a summation, where the i-th term can be interpreted as the expected contribution to the seller's payoff from allocating the quantity q_i to bidder i. The expected marginal contributions play an important role in the determination of optimal allocation rules. Let $\pi_i(q_i, t)$ be the expected marginal contribution of bidder i, if bidders' have types t, i.e.,

$$\pi_i(q_i, t) = \sum_{j=1}^{n} \nu_{ij} \bar{V}_{ij}(q_i, t_j) - \nu_{ii} \frac{\partial \bar{V}_{ii}(q_i, t_i)}{\partial t_i} \frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)} . \qquad (3.15)$$

The functions $\pi_i(\cdot, \cdot)$ satisfy the properties described in the following lemma, which are useful for the characterization of the optimal auction.

Lemma 3.2 *The functions $\pi_i(\cdot, \cdot)$, defined in (3.15), satisfy the following conditions, for all i,*

q_i and t:

$$\frac{\partial \pi_i(q_i, t)}{\partial q_i} \leq 0 \qquad (3.16)$$

and

$$\frac{\partial \pi_i(q_i, t)}{\partial t_i} > 0. \qquad (3.17)$$

Proof : See Appendix C1. ∎

Given that the marginal contributions are nonincreasing, the seller would like to allocate quantities that would make the marginal contributions of bidders that receive a positive quantity equal to zero, while bidders with a negative marginal contribution would receive no quantity. If such allocation is feasible, it is also optimal; otherwise, the allocations would have to be scaled down, while keeping the equality among the marginal contributions of the bidders that receive a positive quantity, until the total quantity allocated equals the quantity available. The statement and proof of the result are given in the following proposition.

Proposition 3.4 *The auction (q^*, c^*) is an optimal auction if and only if there exists a non-negative function $\lambda^*(\cdot)$ such that*

$$\begin{cases} q_i^*(t) \, (\pi_i(q_i^*(t), t) - \lambda^*(t)) = 0 \\[2mm] q_i^*(t) = 0 \implies \pi_i(0, t) \leq \lambda^*(t) \\[2mm] \sum_{i=1}^{n} q_i^*(t) < Q \implies \lambda^*(t) = 0 \\[2mm] q_i^*(t) \in \mathcal{A}(Q, n), \end{cases} \qquad (3.18)$$

and

$$c_i^*(t_i) = E\left\{ V_i(q_i^*(T), T) - \nu_{ii} \int_{\underline{t}_i}^{T_i} \int_0^{q_i^*(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \, \middle| \, T_i = t_i \right\}. \qquad (3.19)$$

Proof : The optimal allocation rule must solve Problem ($\mathcal{P}3.1$). Let me first ignore the allocation's monotonicity constraint and consider the Lagrangean of the resulting problem,

$$\mathcal{L} = E\left\{ \sum_{i=1}^{n} \int_0^{q_i} \pi_i(x, T) \, dx + \lambda(T)(Q - \sum_{i=1}^{n} q_i) \right\}.$$

The first order conditions for a maximum, which are not only necessary but, because of (3.16),

are also sufficient, are

$$
\begin{cases}
q_i(t)\left(\pi_i(q_i(t),t) - \lambda(t)\right) = 0 \\[2mm]
\lambda(t)\left(Q - \sum_{i=1}^{n} q_i(t)\right) = 0 \\[2mm]
q_i(t) = 0 \implies \pi_i(0,t) \leq \lambda(t) \\[2mm]
\lambda(t) = 0 \implies \sum_{i=1}^{n} q_i(t) \leq Q \\[2mm]
q_i(t) \geq 0 \\[2mm]
\lambda(t) \geq 0,
\end{cases}
$$

thus an optimal auction must satisfy conditions (3.18).

For $q^*(\cdot)$ to be a solution to the complete problem, it must then be checked that $q_i^*(\cdot)$ is nondecreasing in t_i. Without loss of generality, let $\hat{t}_i > t_i$. If $q_i^*(t) = 0$, the monotonicity condition is trivially satisfied because of the nonnegativity constraint on a feasible allocation. So suppose that $q_i^*(t) > 0$. From the first order conditions

$$
\pi_i(q_i^*(t),t) = \lambda^*(t); \tag{3.20}
$$

then, by (3.17),

$$
\pi_i(q_i^*(t_i),\hat{t}_i,t_{-i}) > \pi_i(q_i^*(t_i),t). \tag{3.21}
$$

Suppose that $\lambda^*(\hat{t}_i,t_{-i}) \leq \lambda^*(t)$. Then, by (3.20) and (3.21),

$$
\pi_i(q_i^*(t_i),\hat{t}_i,t_{-i}) > \lambda^*(\hat{t}_i,t_{-i}). \tag{3.22}
$$

But, from (3.18),

$$
\pi_i(q_i^*(\hat{t}_i,t_{-i}),\hat{t}_i,t_{-i}) \leq \lambda^*(\hat{t}_i,t_{-i})
$$

which, by (3.22), implies that

$$
\pi_i(q_i^*(\hat{t}_i,t_{-i}),\hat{t}_i,t_{-i}) < \pi_i(q_i^*(t),\hat{t}_i,t_{-i})
$$

and, by (3.16), this requires

$$
q_i^*(\hat{t}_i,t_{-i}) > q_i^*(t)
$$

proving the monotonicity of the quantity allocated to bidder i.

Let me now suppose that $\lambda^*(\hat{t}_i, t_{-i}) > \lambda^*(t)$; then $\lambda^*(\hat{t}_i, t_{-i}) > 0$ and the supply constraint is binding. Suppose that for all $j \neq i$, $q_j^*(\hat{t}, t_{-i}) = 0$; because the supply constraint is binding, $q_i^*(\hat{t}_i, t_{-i}) = Q$, and $q_i^*(\hat{t}_i, t_{-i}) > q_i^*(t)$. So, suppose that, for some $j \neq i$, $q_j^*(\hat{t}_i, t_{-i}) > 0$, implying that

$$\pi_j(q_j^*(\hat{t}_i, t_{-i}), \hat{t}_i, t_{-i}) = \lambda^*(\hat{t}_i, t_{-i}).$$

Note that

$$\pi_j(q_j^*(t), t) \leq \pi_i(q_i^*(t), t),$$

and, by the second condition in Assumption 3.8 and Assumption 3.9,

$$\pi_j(q_j^*(t), \hat{t}_i, t_{-i}) \leq \pi_i(q_i^*(t), \hat{t}_i, t_{-i}).$$

Then, either, for at least one j,

$$\pi_j(q_j^*(\hat{t}_i, t_{-i}), \hat{t}_i, t_{-i}) \leq \pi_j(q_j^*(t), \hat{t}_i, t_{-i})$$

which implies that

$$\pi_i(q_i^*(\hat{t}_i, t_{-i})\hat{t}_i, t_{-i}) \leq \pi_i(q_i^*(t), \hat{t}_i, t_{-i})$$

requiring $q_i^*(\hat{t}_i, t_{-i}) \geq q_i^*(t)$, or, for all $j \neq i$ that get a positive quantity when signals are $(\hat{t}_i, t_{-i})$,

$$\pi_j(q_j^*(\hat{t}_i, t_{-i}), \hat{t}_i, t_{-i}) > \pi_j(q_j^*(t), \hat{t}_i, t_{-i})$$

and $q_j^*(\hat{t}_i, t_{-i}) < q_j^*(t)$, which, because the supply constraint is binding, implies that $q_i^*(\hat{t}_i, t_{-i}) > q_i^*(t)$.

So, in any possible situation, $q_i^*(t)$ is nondecreasing in t_i.

Finally, by Proposition 3.2, $c_i^*(t_i)$ is the optimal expected payment of a bidder with private signal t_i. ∎

The optimal auction can be interpreted in a way similar to the optimal auctions of indivisible goods, which simultaneously specifies a procedure for the identification of the bidders that will

receive a positive quantity of the good. Knowing the bidders private signals, the seller may compute a required minimum announcement for bidder i, T_i^*, which is the cutoff private signal for bidder i, above which there will be a positive probability that he will receive a positive quantity. The required minimum announcements satisfy,[9]

$$\pi_i(0, T_i^*(t_{-i}), t_{-i}) = 0.$$

But, in the optimal auction, the supply constraint may be binding, hence to determine whether or not bidder i receives a positive quantity of the good, the seller must compute bidder i's minimum winning signal, T_i^*, which is such that

$$\pi_i(0, T_i^*(t_{-i}), t_{-i}) = \max \Big\{ 0, \max \big\{ \pi_j(q_j, T_i^*(t_{-i}), t_{-i}) \big|$$
$$(0, q_{-i}) \in \mathcal{A}(Q, n), \sum_{\substack{k=1 \\ k \neq i}}^{n} q_k = Q,$$
$$\forall k, l \neq i \text{ s.t. } q_k, q_l > 0$$
$$\pi_k(q_k, T_i^*(t_{-i}), t_{-i}) = \pi_l(q_l, T_i^*(t_{-i}), t_{-i}) \big\} \Big\}.$$

Then in the optimal auction a bidder receives a positive quantity if $t_i > t_i^*$, as in the optimal auctions of indivisible goods.

In the context of divisible goods, the optimal auction is not similar to any of the usually considered multiple unit auctions.[10] In fact, from the characterization of the optimal auction given in Proposition 3.4, it can be seen that each bidder only needs to submit a bid, which in equilibrium reveals his private information and can be used by the seller to decide on the allocation and payments to be made. In general, a bid may be anything that signals the bidder's type; commonly, the bidders submit a quantity demanded, or a willingness to pay for some (fixed) quantity.

In the next section I present four simple mechanisms for the implementation of the optimal auction in a symmetric model.

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[9]The equation may have no solution, i.e., $\pi_i(0, T_i^*(t_{-i}), t_{-i}) > 0$. In these cases the seller sets $T_i^*(t_{-i}) = t_i$. To simplify the description I will assume that there always exists a solution.

[10]See Chapter 2 for an analysis of standard multiple unit auctions.

3.5   Model I: Implementation in a Symmetric Model

The construction of the optimal auction is the starting point to solve the seller's problem. However, direct revelation mechanisms are seldom used in practice. Instead, in this type of environment, the seller may want to construct mechanisms to implement the optimal auction where the bids are demands or amounts to be paid by the bidders. In this section I look at mechanisms for the implementation of the optimal auction in a symmetric case of Model I.

The additional assumptions of the symmetric model are the following:

Assumption 3.10 *The private signals, T_i, are drawn from the same distribution, $F_{T_i}(\cdot) = F_{T_j}(\cdot)$, for all i and j.*

Assumption 3.11 *The valuation functions satisfy the conditions*

$$\bar{V}_{ij}(\cdot, \cdot) = \bar{V}(\cdot, \cdot)$$

$$\nu_{ij} = \nu .$$

So, the bidders independently get their private signals from the same distribution and the marginal valuation functions will be the same for all bidders. A consequence of the symmetry assumptions is that the marginal contributions to the seller's payoff will satisfy

$$\pi_i(\cdot, \cdot) = \pi_j(\cdot, \cdot) .$$

In the remaining of the section I describe four mechanisms, some of which resemble standard auctions, that implement the optimal auction in the symmetric model.

In the first mechanism the bidders submit orders for quantities, b_i. The bid b_i is treated by the seller as the quantity that would be allocated to bidder i, with private signal t_i, if all other bidders had private signal t_i, in an optimal (unrestricted) auction, i.e., using the bid b_i,

the seller computes the type $t^B(b_i)$ from the condition,[11]

$$\pi_i(b_i, t_i^B, \ldots, t_i^B) = 0.$$

If the true types are equal to t^B, an optimal auction would allocate the quantities $q^*(t^B)$ and require payments such that the expected payment of bidder i would be $c_i^*(t_i^B)$. So, suppose that the allocation rules of the mechanism under consideration are $q^B(\cdot)$ satisfying

$$q_i^B(b_1, \ldots, b_n) = q_i^*(t^B(b_1), \ldots, t^B(b_n)),$$

and a bidder that submits a bid b_i is required to pay c_i^B, which satisfies

$$c_i^B(b_i) = c_i^*(t^B(b_i)).$$

Then, if the bidders truthfully reveal their private signals, by construction, the mechanism will implement the optimal allocation and require the optimal expected payments. I can use the analysis of the optimal auctions to show that in fact the bidders will want to truthfully reveal their private signals.

There are several nonstandard characteristics in this mechanism. Because the identification of all signals is indispensable for the seller to decide on the optimal allocations, bidders must be allowed to place negative orders, which a bidder with a very low signal will, in general, want to do. Also, the payment only depends on the order placed, and not on the final allocation; thus, every bidder, even those that will receive a zero quantity, will be required to pay.[12]

In an alternative mechanism, which is closer to the design of standard auctions than the first one,[13] the bidders pay to participate in the auction (i.e., to be considered as buyers). Let $t^E(\cdot)$ be the inverse of the optimal expected payment function (i.e., $t^E(c)$ is the type of the bidder whose optimal expected payment is equal to c). Using the participation payments, b_i,

---

[11]If the revealed private signal, $t^B(b_i)$ is not a feasible signal the seller will impose a penalty on bidder i. On the other hand, because each bidder must perfectly reveal his signal, the seller may have to allow bids that are negative or larger than the total supply.

[12]This is the case in the example, analyzed in the appendix, because any bidder has a positive probability of getting a positive quantity of the good. In a more general situation, this type of mechanism requires a payment from any bidder that has a positive probability of getting a positive quantity of the good.

[13]This mechanism resembles a discriminatory price auction.

the seller computes the quantities

$$q_i^E(b_1,\dots,b_n) = q_i^*(t^E(b_1),\dots,t^E(b_n)),$$

bidder i receives $q_i^E(b)$ and pays his bid, b_i. By construction, it is clear that, because the seller interprets the bids as correctly revealing the bidders' type, so that the mechanism maps into a revelation mechanism, it will be an equilibrium for each bidder to reveal his type truthfully and submit a bid equal to the expected payment of his type in an optimal auction. Therefore, the mechanism implements the optimal auction.

This mechanism is similar to the discriminatory price auction in the sense that bids are amounts to be paid and each bidder pays his own bid; however, it still allows for the possibility that a bidder that does not receive any good still has to pay. A similar mechanism that would only require payments by the winners is constructed at the end of this section.

General mechanisms that ensure bidders that payments will only be required from bidders that get a positive quantity can also be constructed. In the remaining of this section I present two of such mechanisms. In the first of these examples, the bidders submit an expected demand, b_i. Let $t^D(b_i)$ be the type of a bidder that submits the expected demand b_i. Having received the bids $(b_1,\dots,b_n)$ the seller allocates the quantities

$$q_i^D(b_1,\dots,b_n) = q_i^*(t^D(b_1),\dots,t^D(b_n))$$

and requires payments

$$c_i^D(q_i,b_i) = \mathrm{E}\Big\{ V(q_i^D(b_i,b_{-i}^D(T_{-i})),t^D(b_i),T_{-i}) - \nu \int_{\underline{t}_i}^{t^D(b_i)} \int_0^{q_i^*(x,T_{-i})} \frac{\partial \bar{V}(y,x)}{\partial t_i} \, dy \, dx \Big|$$
$$q_i^D(b_i,b_{-i}^D(T_{-i})) = q_i, \, b_i \Big\}.$$

In this mechanism the seller computes the allocation as if the bid of a bidder with type t_i were equal to the expected quantity that will be allocated to this type of bidder. So, if the bids are equal to the expected quantity that should be allocated to the bidder, the mechanism implements the optimal allocation. To confirm that the bidders really want to submit these

bids, I note that, by doing so, the expected payment of a bidder with type t_i will be

$$
\begin{aligned}
&\mathrm{E}\Big\{ c_i^D(q_i, b_i^D(T_i)) \Big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ \mathrm{E}\Big\{ V(q_i^D(b_i, b_{-i}^D(T_{-i})), t^D(b_i), T_{-i}) - \nu \int_{\underline{t}_i}^{t^D(b_i)} \int_0^{q_i^*(x, T_{-i})} \frac{\partial \bar{V}(y, x)}{\partial t_i} \, dy \, dx \Big| \\
&\hspace{6cm} q_i^D(b_i, b_{-i}^D(T_{-i})) = q_i, \, b_i \Big\} \Big| T_i = t_i \Big\} \\
&= \mathrm{E}\Big\{ V(q_i^*(T), T) - \nu \int_{\underline{t}_i}^{T_i} \int_0^{q_i^*(x, T_{-i})} \frac{\partial \bar{V}(y, x)}{\partial t_i} \, dy \, dx \Big| T_i = t_i \Big\} \\
&= c_i^*(t_i)
\end{aligned}
$$

which is the optimal payment. Therefore, from the fact that the optimal auction is incentive compatible and individual rational, I conclude that it is an equilibrium for the bidders to truthfully reveal their expected demands, and the mechanism implements the optimal auction.

This mechanism is possibly more attractive than the two previous mechanisms, because the payment is contingent not only on the announcements but also on the quantity allocated to the bidder; as a result, a bidder that does not receive any quantity will not be required to pay.

I finally provide a fourth mechanism that can be used to implement the optimal mechanism. Let the bidders submit the payments that they are willing to make if they receive a positive quantity, b_i. Let $t^Q(b_i)$ be the type of bidder whose expected payment, conditional on receiving a positive quantity, in an optimal auction, is b_i. The seller uses these revealed types as if they were the true types to compute the allocation that is implemented, i.e.,

$$
q_i^Q(b_1, \ldots, b_n) = q_i^*\Big(t^Q(b_1), \ldots, t^Q(b_n) \Big) \; ;
$$

a bidder that receives a positive amount pays his bid, and pays zero if no good is received. To check that the mechanism implements the optimal auction I just need to confirm that bidders have an incentive to truthfully reveal their willingness to pay, which, once again, follows from the analysis of optimal auctions.

This mechanism is very close to a generalization of discriminatory price auctions to divisible goods: the bids are amounts to be paid in the event that the bidder receives some quantity of the good.

3.6 Model II: The Assumptions

Having analyzed the optimal auction and mechanisms for its implementation in Model I, I now analyze a second model. Model II makes a different assumption about the valuation functions. Assumptions 3.1 to 3.6, and 3.9 from Model I are kept, but the valuation of any bidder will now not only depend on the quantity of the good that he receives but also on the quantities allocated to others. Because the good being sold is homogeneous, the natural assumption to make about the demand function is that it depends on the bidders' private signals and on the total quantity allocated in the auction, so that bidder i's valuation would satisfy

$$E\{V_i(q,T)|\, T = t\} = \bar{V}\left(\textstyle\sum_{j=1}^n q_j, \hat{r}(t)\right)\, q_i\, ,$$

for some function $\hat{r}(\cdot)$ that aggregates the private signals. Unfortunately, with such assumption the optimal auction is trivially characterized. The intuition is the following. Look at the expected valuation as a function of q_i and $\sum_{j=1}^n q_j$. Fixing the quantity sold, the expected valuation is linear in q_i. In the objective function of the problem that the optimal allocation rules ought to solve, there will be two terms: the first is the bidders' value of the quantity received, which depends on the total quantity sold but not on its allocation among bidders; the second is the total rent given to the bidders, which is linear and, under the standard assumptions, decreasing in the bidders' signals. Therefore, in the optimal auction, the seller should decide the total quantity to be sold, and allocate it to the bidder that announces the largest private signal.[14]

To avoid this trivial solution I assume that the marginal valuation functions are not just functions of the total quantity allocated, and the following assumption is made.[15]

Assumption 3.12 *The value for potential buyer i of the quantity q_i of the good satisfies*

$$E\{V_i(q,T)|\, T = t\} = \nu_{ii}\int_0^{q_i} \bar{V}_{ii}(x,t_i)\, dx + \sum_{\substack{j=1 \\ j \neq i}}^n \nu_{ij}\, \bar{V}_{ij}(q_j,t_j)\, q_i\, ,$$

---

[14]This result is a generalization of the optimality of bundling, for special cases of the multiple indivisible units auctions (see Chapter 2).

[15]If the marginal valuation functions correspond to a demand function in the secondary market, this is equivalent to the assumption that the goods presented for sale in this market will be perceived by the final buyers as differentiated.

where the parameters ν_{ij} ($j \neq i$) are nonnegative, ν_{ii} is positive, and the functions $\bar{V}_{ij}(\cdot, \cdot)$ are differentiable, nonincreasing in the first argument and increasing in the second argument.

The assumption is similar to that of Model I except for the cross effects on marginal valuations. In this case, bidder j's effect on bidder i's valuation depends on j's quantity, not only on his signal.

The structure of the cross effects among valuations greatly complicates the problem that the seller must solve. In particular, it is not impose simple regularity conditions on the valuation functions that ensure that the problem can be solved ignoring the monotonicity condition on the allocation rules. Because of this fact, I make no further regularity assumptions.

These assumptions specify Model II which is analyzed in the second part of the paper.

3.7   Model II: Examples

In this section I present two simple examples of Model II, corresponding to applications that motivated this work: privatizations of state owned firms and electricity spot markets.

3.7.1   Privatization of a State-Owned Firm

The first example of Model II applies to the privatization of a state owned firm.

Let $Q \in (0, 1]$ be the fraction of the firm's capital that is to be privatized. There are n investors interested in bidding for the firm. These are large investors who intend to actively influence the managerial decisions of the firm.[16]

The goal of the investors is to maximize their benefits from ownership, which I assume are just related to the value of the firm.[17] Each investor has certain private information about the profitability of the firm, t_i (this may be information about firm's characteristics, about the market in which the firm operates, or about the investor's skills to influence the firm's decisions). As a result of his private information and the amount of the firm that he owns, an investor will make some effort to influence the management of the firm, which will affect the

---

[16]If they were small investors who had no possibility of influencing the firm's decisions, a model similar to the one presented in Subsection 3.3.1 for auctions of shares could be used.

[17]As I did for the example of auctions of shares, I could also consider some nonmonetary benefits of ownership.

value of the firm and, therefore, the utility of any shareholder. So let $\bar{V}_{ij}(q_j, t_j)$ be the effect on the firm's value due to investor j's actions, which depend on j's ownership share and private information.

In this example investor i's valuation would satisfy

$$\mathrm{E}\{V_i(q, T)| T = t\} = \sum_{j=1}^{n} \nu_j \, \bar{V}_{ij}(q_j, t_j) \, q_i \,,$$

which can be written as in Assumption 3.7.

The appropriate assumptions on the functions $\bar{V}_{ij}(\cdot, \cdot)$ and on the distribution of the random variables will guarantee that the additional assumptions of Model II are also satisfied.

3.7.2   Electricity Spot Markets

In this subsection I generalize the example presented in Chapter 2, relaxing the indivisibility assumption. An electricity utility needs to buy a quantity Q of power, which can be supplied by any of n power plants. The plants' valuations correspond to their cost functions. Assuming that the production function of plant i is homothetic, and affected by an efficiency parameter, t_i, which is privately known, the cost function (denoted by $C_i(q, t)$) may satisfy

$$\mathrm{E}\{C_i(q, T)| T = t\} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \nu_{ij} \, \bar{C}_{ij}(q_j, t_j) \, q_i + \nu_{ii} \int_0^{q_i} \bar{C}_{ii}(x, t_i) \, dx$$

where $\bar{C}_{ij}(q_j, t_j)$ captures the effect of firm j's factor demands on the factor prices.

The firms' valuations will just correspond to the symmetric of the firms' cost. Appropriate conditions on the several functions and random variables will allow the assumptions of Model II to be satisfied.

3.8   Model II: Optimal Auctions

The analysis in this section borrows heavily from that done in Section 3.4. In fact the main modifications are revisions of previous results due to the distinct specification of the expected valuations. The reader is, therefore, referred to the more detailed analysis and comments of

Section 3.4, and in this section I will only state the new version of the results, which go without proof, given the similarity to those for the analogous results in Section 3.4.

I will be consistent with the notation, and whenever a function is used without being explicitly defined, it is implicit that the definition previously given is assumed.

Given that the assumption about the expected valuations was written in such a way that the effect of a bidder's information in his own valuation is equal to that of the first model, the analysis of feasible auctions is equal to that done in Section 3.4. The optimal expected rents are also the same, so the optimal expected payments, $c^*(\cdot)$, are given by

$$c_i^*(t_i) = \mathrm{E}\left\{ V_i(q^*(T), T) - \nu_{ii} \int_{t_i}^{T_i} \int_0^{q_i^*(x, T_{-i})} \frac{\partial \bar{V}_{ii}(y, x)}{\partial t_i} \, dy \, dx \, \middle| \, T_i = t_i \right\}. \tag{3.23}$$

The optimal allocation rules can then be characterized as the solution to an optimization problem presented in the following proposition.

Proposition 3.5 *Consider an auction* (q^*, c^*) *satisfying (3.23). Let* $q^*(\cdot)$ *solve the following problem*

$$\max_{q(\cdot)} \quad E\left\{ \sum_{i=1}^n \left(V_i(q(T), T) - \nu_{ii} \left(\int_0^{q_i} \frac{\partial \bar{V}_{ii}(y, T_i)}{\partial t_i} \, dx \right) \frac{1 - F_{T_i}(T_i)}{f_{T_i}(T_i)} \right) \right\}$$

$$\tag{$\mathcal{P}3.2$}$$

$$s.t. \quad \begin{cases} (t_i - \hat{t}_i)(q_i(t_i, t_{-i}) - q_i(\hat{t}_i, t_{-i})) \geq 0 \\ q(t) \in \mathcal{A}(Q, n). \end{cases}$$

Then (q^*, c^*) *is an optimal auction.*

Proof : The proof is identical to the proof of Proposition 3.3. ∎

Even though the problem that defines the optimal allocation rule is similar to that in Model I, because of the cross effects through quantities, the solution may be fairly different.

The procedure given to solve for the optimal allocation rules in Model I can be used as a first step for the determination of the solution in this model. However, such a procedure may produce an allocation function which has decreasing segments. If this is the case, the monotonicity condition must be explicitly considered and a procedure similar to the one presented in Maskin

and Riley (1989) for the nonregular case must be used.

The mechanisms presented in Section 3.5 can be adapted to implement the optimal auction in Model II.

3.9 Conclusion

Optimal auctions of a divisible good in models that allow for private values and common values as special cases were characterized in this chapter. The main characteristics of the optimal auctions are similar to those for the pure private values case, already analyzed in Maskin and Riley (1989).

These results can be applied to the design of Treasury bill auctions. Moreover, they may also help the design of auctions of divisible goods that are often auctioned as indivisible units, which, in general, is suboptimal. For example, in privatizations of state owned firms, where sometimes indivisible lots are constituted, from the point of view of revenue maximization it is unclear why such a restriction should be considered.

This chapter closes my work on optimal auctions in models of indivisible and divisible goods, under assumptions of risk neutrality and signal independence, but correlated valuations, which shed new light on the design of optimal auctions and offer justifications for a wide range of phenomena observed in actual auctions, suggesting that some empirical work aimed at testing the predictions of the models would be desirable.

Analysis of the effect of departures from the basic model (such as agents' risk aversion or more general revision processes) are among the interesting theoretical topics in an agenda for future research.

C1 Appendix: Proofs

In this Appendix I provide proofs for some of the results presented in the chapter.

Proof of Lemma 3.1: It can be verified that the function $U_i(q, c, t_i)$ is continuous and nondecreasing in t_i, hence differentiable with respect to t_i, almost everywhere; also, from Assumption 3.7, $\mathcal{U}_i(q, c, t_i, \hat{t}_i)$ is differentiable with respect to t_i. Therefore, from (3.2), using the envelope theorem, I conclude that

$$\frac{\partial U_i(q, c, t_i)}{\partial t_i} = \left. \frac{\partial \mathcal{U}_i(q, c, t_i, \hat{t}_i)}{\partial t_i} \right|_{\hat{t}_i = t_i}.$$

Moreover, from the definition of $\mathcal{U}(q, c, t_i, \hat{t}_i)$,

$$\frac{\partial \mathcal{U}_i(q, c, t_i, \hat{t}_i)}{\partial t_i}\bigg|_{\hat{t}_i = t_i} = \nu_{ii}\, \mathrm{E}\left\{ \int_0^{q_i(T)} \frac{\partial \bar{V}_{ii}(x, T_i)}{\partial t_i}\, dx \,\bigg|\, T_i = t_i \right\};$$

thus, a necessary condition for incentive compatibility is

$$\frac{\partial U_i(q, c, t_i)}{\partial t_i} = \nu_{ii}\, \mathrm{E}\left\{ \int_0^{q_i(T)} \frac{\partial \bar{V}_{ii}(x, T_i)}{\partial t_i}\, dx \,\bigg|\, T_i = t_i \right\}$$

which can be written as condition (3.3). ∎

Proof of Lemma 3.2: Rewrite $\pi_i(q_i, t)$ as

$$\pi_i(q_i, t) = \sum_{\substack{j=1 \\ j \neq i}}^n \nu_{ij}\, \bar{V}_{ij}(q_i, t_j) + \nu_{ii}\, \bar{V}_{ii}(q_i, t_i)\left(1 - \frac{\partial \bar{V}_{ii}(q_i, t_i)/\partial t_i}{\bar{V}_{ii}(q_i, t_i)} \frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)} \right).$$

Using the assumption that $\bar{V}_{ij}(q_i, t_j)$ is nonincreasing in q_i, so to prove (3.16) it sufficient to show that

$$\frac{\partial}{\partial q_i}\left(\frac{\partial \bar{V}_{ii}(q_i, t_i)/\partial t_i}{\bar{V}_{ii}(q_i, t_i)} \right) \geq 0.$$

Note that

$$\frac{\partial}{\partial q_i}\left(\frac{\partial \bar{V}_{ii}(q_i, t_i)/\partial t_i}{\bar{V}_{ii}(q_i, t_i)} \right) = \frac{1}{\left(\bar{V}_{ii}(q_i, t_i)\right)^2}\left(\frac{\partial^2 \bar{V}_{ii}}{\partial t_i \partial q_i}\, \bar{V}_{ii} - \frac{\partial \bar{V}_{ii}}{\partial t_i}\, \frac{\partial \bar{V}_{ii}}{\partial q_i} \right)$$

$$= \frac{\partial}{\partial t_i}\left(\frac{\partial \bar{V}_{ii}(q_i, t_i)/\partial q_i}{\bar{V}_{ii}(q_i, t_i)} \right)$$

$$= \frac{1}{q_i}\frac{\partial}{\partial t_i}\left(\frac{q_i}{\bar{V}_{ii}(q_i, t_i)}\, \frac{\partial \bar{V}_{ii}(q_i, t_i)}{\partial q_i} \right)$$

which is nonnegative by the first condition in Assumption 3.8; then (3.16) follows.
    On the other hand,

$$\frac{\partial \pi_i(q_i, t)}{\partial t_i} = \nu_{ii}\left(\frac{\partial \bar{V}_{ii}(q_i, t_i)}{\partial t_i} - \frac{\partial^2 \bar{V}_{ii}(q_i, t_i)}{\partial t_i^2}\, \frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)} - \frac{\partial \bar{V}_{ii}(q_i, t_i)}{\partial t_i}\, \frac{\partial}{\partial t_i}\left(\frac{1 - F_{T_i}(t_i)}{f_{T_i}(t_i)} \right) \right)$$

which, by Assumptions 3.7 and 3.9, is positive; hence, condition (3.17) is satisfied. ∎

C2   Appendix: An Example of Model I

To help understanding the functioning of the optimal auction I now consider a particular example, with the following characteristics:

- the total quantity is $Q = 1$;

- the number of bidders is $n = 2$;

- the private signals are uniformly distributed in $[0, 1]$;

- the functions are $\bar{V}_{ij}(q_i, t_j) = t_j - q_i$;
- the parameters are $\nu_{ij} = 1/2$.

So, this is a two bidder model, with expected valuation

$$E\{V_i(q_i, T) | T_i = t_i\} = \int_0^{q_i} (t_m - x) \, dx,$$

where t_m is the average signal.

The example can correspond to a description of the Treasury bill market, where $(t_1 + t_2)/2$ is the expected value of each bill and $q_i^2/2$ is the opportunity cost of getting q_i bills in the auction.

In this example, the expected marginal contribution of bidder i's allocation to the seller's payoff is

$$\pi_i(q_i, t) = t_m + \frac{1}{2}(t_i - 1) - q_i.$$

I may invoke Proposition 3.4, to show that the quantities allocated in the optimal auction, $q_i^*(t)$, must satisfy the conditions:

$$
\begin{cases}
q_i^*(t) \left(t_m + \frac{1}{2}(t_i - 1) - q_i^*(t) - \lambda^*(t) \right) = 0 \\
\lambda^*(t) \left(1 - \sum_{i=1}^2 q_i^*(t) \right) = 0 \\
q_i^*(t) = 0 \implies t_m + \frac{1}{2}(t_i - 1) \leq \lambda^*(t) \\
\lambda^*(t) = 0 \implies \sum_{i=1}^2 q_i^*(t) \leq 1 \\
q_i^*(t) \geq 0 \\
\lambda^*(t) \geq 0
\end{cases}
$$

The solution of the system is apparently very complex. However, the following procedure can be used to determine the optimal allocation rules. Suppose that the supply constraint is not binding; then $\lambda^*(t) = 0$, and I can compute the quantities allocated to each bidder to be

$$\bar{q}_i(t) = \max \left(0, \frac{t_i + 2t_m - 1}{2} \right).$$

If these quantities do not exhaust the total supply, i.e., $\bar{q}_1(t) + \bar{q}_2(t) \leq Q$, they are the optimal allocations, $q_i^*(t) = \bar{q}_i(t)$. If the total supply is exhausted, i.e., $\bar{q}_1(t) + \bar{q}_2(t) > Q$ the optimal allocations are reduced according to

$$q_i^*(t) = \max(0, \bar{q}_i(t) - \lambda^*(t)),$$

where $\lambda^*(t) > 0$ and

$$\lambda^*(t) = t_m + \frac{\hat{t}_m - 1}{2} - q_m^*(t)$$

where $\hat{t}_m$ and $q_m^*(t)$ are, respectively, the average signal and the average quantity received, among the bidders that received a quantity. Using this procedure, the optimal allocation quantities of the example
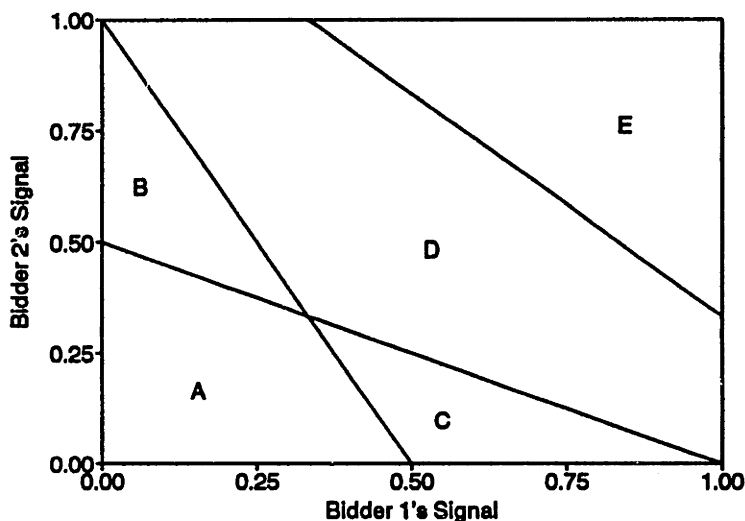
Figure C2-1: Model I: optimal allocations in the example.

were computed to be

$$
q_i^*(t) = \begin{cases} 0 & t_i + 2t_m - 1 \leq 0 \\[2mm] \dfrac{t_i + 2t_m - 1}{2} & t_i + 2t_m - 1 > 0 \text{ and } t_m \leq 2/3 \\[2mm] \dfrac{t_i - t_m + 1}{2} & t_m > 2/3, \end{cases}
$$

which can be checked to satisfy the previous optimality conditions.

Figure C2-1 presents the partition of the types space that is induced by the optimal allocation rules. In subset A no bidder receives any quantity, in B and C only one bidder (respectively, bidder 1 and bidder 2) receive a positive quantity; in D both bidders receive positive quantities and the supply constraint is not binding; finally, in E both bidders receive positive amounts but the quantity constraint is binding.[18]

To finish the description of the optimal auction the expected payments should be computed, using (3.10). After simple but tedious calculations, I obtained

$$
c_i^*(t_i) = \begin{cases} \dfrac{t_i^2\,(3 - 2t_i)}{6} & t_i + 2t_m - 1 \leq 0 \\[3mm] \dfrac{-153t_i^3 + 189t_i^2 + 117t_i - 17}{864} & t_i + 2t_m - 1 > 0 \text{ and } t_m \leq 2/3 \\[3mm] \dfrac{135t_i^3 - 459t_i^2 + 549t_i - 107}{864} & t_m > 2/3 \end{cases}
$$

---

[18]The fact that the total supply is never binding when only one bidder receives a positive quantity is a consequence of the assumption on the total supply. If, in this same example, the total supply is no more 'aan 1/2, then a single bidder may receive the total amount.

C3 Appendix: Two Examples of Model II

In this appendix I describe the optimal auction in two examples that meet the assumptions of Model II. The first example shows that, even in this model, it is possible that the optimal auction will involve "bundling", i.e., the seller will never want to allocate a positive quantity to more than one bidder. The second example, is similar to the one used for Model I, and illustrates a more general situation.

C3.1 Example 1

Consider the following example of Model II:

- the total quantity is $Q = 1/2$;

- the number of bidders is $n = 2$;

- the private signals are uniformly distributed in $[0, 1]$;

- the functions are

$$\bar{V}_{ij}(q_j, t_j) = \begin{cases} t_j - 2q_j & \text{if } i = j \\ t_j - q_j & \text{otherwise;} \end{cases}$$

- the parameters are $\nu_{ij} = 1/2$.

This is a linear duopoly symmetric common value model, in which, following the intuition presented in Section 3.6, the seller should not want to allocate a positive quantity to two distinct bidders. The optimal allocation rules must solve the following problem

$$\max_{q(\cdot)} \quad (t_1 + t_2 - q_1 - q_2)(q_1 + q_2) - \frac{1}{2}\sum_{i=1}^{2}(1 - t_i)q_i$$

$$\text{s.t.} \quad \begin{cases} (t_i - \hat{t}_i)(q_i(t_i, t_{-i}) - q_i(\hat{t}_i, t_{-i})) \geq 0 \\ q(t) \in \mathcal{A}(1/2, 2). \end{cases}$$

The problem is solved using the procedure described in Section 3.5. The first order conditions are

$$\begin{cases} q_i^*(t)\left(t_i + 2t_m - q_1^*(t) - q_2^*(t) - 1 - \lambda^*(t)\right) = 0 \\ \lambda^*(t)\left(\frac{1}{2} - \sum_{i=1}^{2} q_i^*(t)\right) = 0 \\ q_i^*(t) = 0 \implies t_i + 2t_m - 1 \leq \lambda^*(t) \\ \lambda^*(t) = 0 \implies \sum_{i=1}^{2} q_i^*(t) \leq \frac{1}{2} \\ q_i^*(t) \geq 0 \\ \lambda^*(t) \geq 0 \end{cases}$$

From the first condition, we see that the marginal contribution of allocating a positive quantity to the bidder with the largest signal is always greater than that of allocating any positive quantity to the other bidder. Thus, at most one bidder will receive a positive quantity.

I first compute the quantity to be allocated to the bidder with the largest private signal, say i, if the supply constraint is not binding,

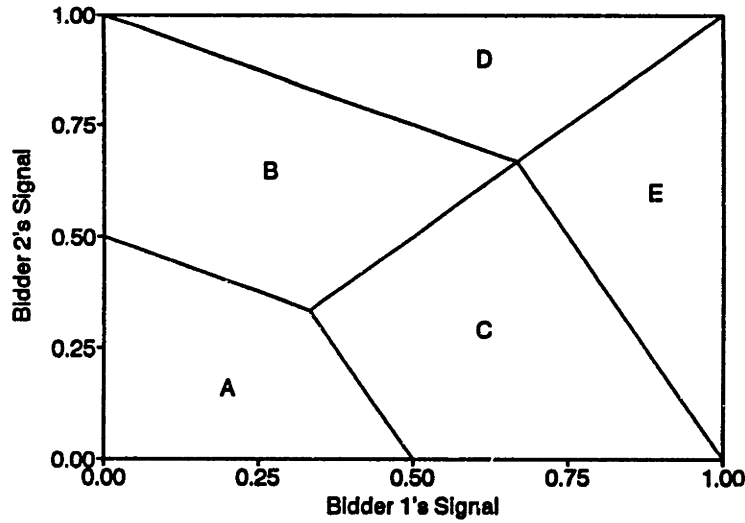$$\bar{q}_i(t) = \max\left\{0, \frac{t_i + 2t_m - 1}{2}\right\}.$$

Figure C3-1: Model II: optimal allocations in example 1.

This allocation is feasible if $t_i + 2t_m - 1 \leq 1$; otherwise, the supply constraint binds and $q_i^*(t) = 1/2$. Hence, the optimal allocation rules are

$$
q_i^*(t) = \begin{cases} 0 & t_i + 2t_m - 1 \leq 0 \text{ or } t_i \leq t_m \\[2mm] \dfrac{t_i + 2t_m - 1}{2} & t_i + 2t_m - 1 \in (0,1] \text{ and } t_i > t_m \\[2mm] \dfrac{1}{2} & t_i + 2t_m - 1 > 1 \text{ and } t_i > t_m \,. \end{cases}
$$

which satisfy the first order conditions written above and are nondecreasing in t_i.

The allocation rules induce the partition of the types' space presented in Figure C3-1. In area A no bidder receives any unit, in areas C and E only bidder 1 receives a positive quantity, which is the total supply in E. Similarly, in areas B and D only bidder 2 receives it.

C3.2 Example 2

I now consider the previous example with the following asymmetry:

$$
\nu_{ij} = \begin{cases} \dfrac{2}{3} & \text{if } i = j \\[2mm] \dfrac{1}{3} & \text{otherwise,} \end{cases}
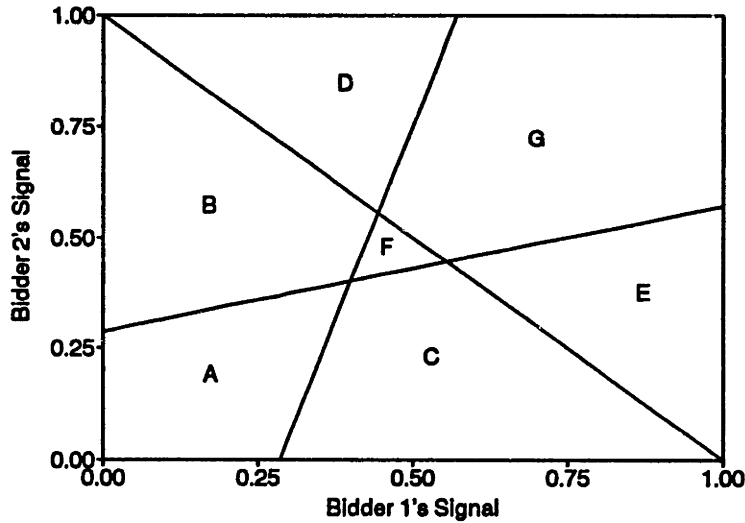$$

Figure C3-2: Model II: optimal allocations in example 2.

so that a bidder's own signal is more important for the determination of his own marginal valuation than the other's signal. The problem that determines the optimal allocation in this example is

$$\max_{q(\cdot)} \ \frac{1}{3}\sum_{i=1}^{2}\left\{(2\,t_m - q_1 - q_2)(q_1 + q_2) + (3\,t_i - q_i - 2)\,q_i\right\}$$

$$\text{s.t.} \quad \begin{cases} (t_i - \hat{t}_i)\left(q_i(t_i,t_{-i}) - q_i(\hat{t}_i,t_{-i})\right) \geq 0 \\ q(t) \in \mathcal{A}(1/2,2). \end{cases}$$

Ignoring the monotonicity condition, the first order conditions are

$$\begin{cases} q_i^*(t)\left(3\,t_i + 2\,t_m - 2\,q_i^*(t) - 2\,(q_1^*(t) + q_2^*(t)) - 2 - \lambda^*(t)\right) = 0 \\ \lambda^*(t)\left(\frac{1}{2} - \sum_{i=1}^{2} q_i^*(t)\right) = 0 \\ q_i^*(t) = 0 \implies 3\,t_i + 2\,t_m - 2\,(q_1^*(t) + q_2^*(t)) - 2 \leq \lambda^*(t) \\ \lambda^*(t) = 0 \implies \sum_{i=1}^{2} q_i^*(t) \leq 1/2 \\ q_i^*(t) \geq 0 \\ \lambda^*(t) \geq 0 \end{cases}$$

From the conditions I can compute the optimal allocation. If the supply constraint is not binding,

$$\bar{q}_i(t) = \max\left\{0, \frac{9\,t_i - 4\,t_m - 2}{6}\right\};$$

from these functions and considering the supply constraint, I compute the optimal allocation rules, which

are

$$q_i^*(t) = \begin{cases} 0 & \text{if } 9t_i - 4t_m - 2 \leq 0 \\[2mm] \dfrac{9t_i - 4t_m - 2}{2} & \text{if } 9t_i - 4t_m - 2 > 0 \text{ and } t_m \leq 1/2 \\[2mm] \dfrac{6(t_i - t_m) + 1}{4} & \text{if } 9t_i - 4t_m - 2 > 0 \text{ and } 9t_i - 14t_m + 2 < 0 \text{ and } t_m > 1/2 \\[2mm] 1/2 & \text{if } 9t_i - 14t_m + 2 \geq 0 \text{ and } t_m > 1/2 . \end{cases}$$

which can be checked that satisfy the first order conditions. Moreover, $q_i^*(t)$ is nondecreasing in t_i.

Figure C3-2 shows the partition of the types' space, according to the allocation that is implemented at each point. Area A corresponds to the region where no good is sold; in area B and C, only bidders 2 and 1, respectively, receive a positive quantity; as well as in D and E, where the supply constraint is binding; regions F and G are those in which both bidders receive positive amounts, with the supply being exhausted in G.

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