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GROMOV'S AMENABLE LOCALIZATION AND GEODESIC FLOWS

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ABSTRACT. Let M be a compact smooth Riemannian n-manifold with boundary. We combine Gromov's amenable localization technique with the Poincaré duality to study the traversally generic geodesic flows on SM, the space of the spherical tangent bundle. Such flows generate stratifications of SM, governed by rich universal combinatorics. The stratification reflects the ways in which the flow trajectories are tangent to the boundary $\partial(SM)$. Specifically, we get lower estimates of the numbers of connected components of these flow-generated strata of any given codimension k in terms of the normed homology $H_k(M;\mathbb{R})$ and $H_k(DM;\mathbb{R})$, where $DM = M \cup_{\partial M} M$ denotes the double of M. The norms here are the simplicial semi-norms in homology. The more complex the metric on M is, the more numerous the strata of SM and $S(DM)$ are. It turns out that the normed homology spaces form obstructions to the existence of globally k-convex traversally generic metrics on M. We also prove that knowing the geodesic scattering map on M makes it possible to reconstruct the stratified topological type of the space of geodesics, as well as the amenably localized Poincaré duality operators on SM.

1. INTRODUCTION

This paper is an extension of [AK] and especially of [K3]. As in the latter articles, it draws its inspiration from the papers [Gr], [Gr1] of Gromov, where the machinery of amenable localization has been developed. Here we combine Gromov's amenable localization with the Poincaré duality operators (this combination was introduced in [K3]) to study the traversally generic geodesic flows on compact connected smooth Riemannian n-manifolds M with boundary (see Definition 2.3 and formula (2.5) for the definition of a traversally generic vector field.). For such flows, the metric on M is non-trapping (see Definition 3.1).

The application of the techniques from [K3] to geodesic flows is relatively straightforward; however, the idea that such applications may be fruitful is a novelty. These applications reveal a new phenomenon: the simplicial norms (see Definition 1.1) of homology classes of a Riemannian manifold M impose restrictions on the tangency patterns of geodesic curves to its boundary ∂M. A tangency pattern $\omega =_{\mathsf{def}} (j_1, j_2, \dots, j_i, \dots)$ is just an ordered finite string of natural numbers. They encode the degree of tangency of a geodesic curve $\gamma \subset M$ to the boundary ∂M.

The fundamental groups of the boundary components of ∂M are assumed to be amenable, while the non-trivial results arise only for manifolds M with non-amenable fundamental group.

For a given Riemannian metric g on a compact manifold M, we denote by v^g the geodesic vector field on SM, the space of the g-unitary spherical tangent bundle of M.

In [K5], we have studied metrics g on M such that the geodesic field v^g on SM is of the gradient type; that is, there exists a smooth Lyapunov function $F : SM \to \mathbb{R}$ with the property $dF(v^g) > 0$. By Theorem 2.1 and Corollary 2.3 from [K5], such metrics g coincide with the class of non-trapping metrics on M. They form an open nonempty set $\mathcal{G}(M)$ in the space $\mathcal{M}(M)$ of all Riemannian metrics on M.

In [K5], Definition 2.3, we also introduced the notion of boundary generic metrics (see Definitions 2.2 and 3.1 below). For them, the boundary ∂M is "generically curved" and the geodesics $\gamma \subset M$ do not have high order of tangency to ∂M in comparison to its dimension. We denote by $\mathcal{B}^\dagger(M)$ the space of boundary generic metrics, and by $\mathcal{G}^\dagger(M)$ the space of boundary generic *and* non-trapping metrics. If each component of ∂M is either strictly convex or concave in g, then the metric g is boundary generic.

Boundary generic metrics form an open set in the space $\mathcal{M}(M)$ of all metrics. Conjecturally, $\mathcal{B}^\dagger(M)$ is dense in $\mathcal{M}(M)$, and $\mathcal{G}^\dagger(M)$ is dense in the space $\mathcal{G}(M)$.

Finally, we consider a subspace $\mathcal{G}^\ddagger(M) \subset \mathcal{G}^\dagger(M)$, formed by the traversally generic metrics on M. Their Definition 2.3 is more involved. In short, a metric g is traversally generic if the geodesic vector field v^g on SM is traversally generic with respect to the boundary $\partial(SM)$ in the sense of [K2], Definition 3.2. Conjecturally, $\mathcal{G}^\ddagger(M)$ is also dense in the space $\mathcal{G}(M)$. It is proven to be open in $\mathcal{M}(M)$ (see [K2]).

For a given metric $g \in \mathcal{G}^\dagger(M)$, let $\mathcal{T}(v^g)$ denote the space of trajectories of the v^g-generated flow on SM. We call $\mathcal{T}(v^g)$ the space of geodesics on M. We denote by $\Gamma : SM \to \mathcal{T}(v^g)$ the obvious map that takes each point $x \in SM$ to the v^g-trajectory through x. In general, $\mathcal{T}(v^g)$ is not a manifold, but, for $g \in \mathcal{G}^\ddagger(M)$, a compact CW-complex ([K5]).

For a smooth traversally generic vector field v on a compact $(d+1)$-manifold X with boundary, the trajectory space $\mathcal{T}(v)$ acquires a stratification $\{\mathcal{T}(v,\omega)\}_\omega$, labeled by the combinatorial patterns of tangency ω that belong to an universal poset $\mathbf{\Omega}^\bullet_{'\langle d]}$ (see [K2] and Section 2). This poset depends only on d.

Similarly, for a traversally generic vector field v^g on a compact $(2n-1)$-manifold SM, the $(2n-2)$-dimensional space of geodesics $\mathcal{T}(v^g)$ acquires a stratification $\{\mathcal{T}(v^g,\omega)\}_\omega$ by the combinatorial patterns of tangency ω that belong to the universal poset $\mathbf{\Omega}^\bullet_{'\langle 2n-2]}$.

The more numerous the connected components of these stratifications are, the more *complex* the v^g-flow is, and thus the more complex (in relation to ∂M) the metric g on M is.

So, our goal here is to find *lower bounds* of the numbers such connected components in terms of the topology of SM or M. The key tool, enabling such estimates, has been developed in [K3] for arbitrary traversally generic flows. We call it "Gromov's amenable localization of the Poincaré duality".

The $\mathbf{\Omega}^\bullet_{'\langle 2n-2]}$-stratification of the geodesic space $\mathcal{T}(v^g)$ generates the stratification

$$\big\{SM(v^g,\omega) =_{\mathsf{def}} \Gamma^{-1}(\mathcal{T}(v^g,\omega))\big\}_{\omega \in \mathbf{\Omega}^\bullet_{'\langle 2n-2]}}$$

of SM and the stratification

$$\big\{\partial(SM)(v^g,\omega) =_{\mathsf{def}} SM(v^g,\omega)\cap \partial(SM)\big\}_\omega$$

of its boundary $\partial(SM)$. These stratifications can be refined by considering the connected components of the sets $\{\partial(SM)(v^g,\omega),\ SM^\circ(v^g,\omega)\}_\omega$. Here

$$SM^\circ(v^g,\omega) =_{\mathsf{def}} SM(v^g,\omega)\cap \mathrm{int}(SM).$$

We construct an auxiliary closed manifold, the double $D(SM) =_{\mathsf{def}} SM \cup_{\partial(SM)} SM$ of SM. The double comes equipped with an involution τ so that $(D(SM))^\tau = \partial(SM)$ and the orbit-space $D(SM)/\{\tau\}$ is homeomorphic to SM. We stratify $D(SM)$ by the connected components of the following sets:

$$\big\{\partial(SM)(v^g,\omega),\ \ SM^\circ(v^g,\omega),\ \ \tau(SM^\circ(v^g,\omega))\big\}_{\omega\in \mathbf{\Omega}^{\bullet}_{'\langle 2n-2]}}$$

All these v^g-induced stratifications of $\mathcal{T}(v)$, SM, and $D(SM)$ are the foci of our investigation here.

Let $SM^\circ_{-j}(g)$ denote the union of strata of codimension j in SM°. Similarly, let $D(SM)_{-j}(g)$ denote the union of codimension j strata in $D(SM)$.

Let $\mathcal{D}$ stand for the Poincaré Duality operator (over the coefficient rings $\mathbb{R}$ or $\mathbb{Z}$) on an oriented manifold with boundary. For each tranversally generic metric g on M, we introduce two localized Poincaré Duality linear operators:

$$\mathcal{L}_j(g;\mathbb{R}):\ H_j(D(SM);\mathbb{R}) \stackrel{\approx\mathcal{D}}{\longrightarrow} H^{2n-1-j}(D(SM);\mathbb{R}) \stackrel{i^*_{\mathsf{loc}}}{\longrightarrow} H^{2n-1-j}(D(SM)_{-j}(g);\mathbb{R}),$$

$$\mathcal{M}_j(g;\mathbb{R}):\ H_j(SM;\mathbb{R}) \stackrel{\approx\mathcal{D}}{\longrightarrow} H^{2n-1-j}(SM,\partial(SM);\mathbb{R})$$

$$\stackrel{i^*_{\mathsf{loc}}}{\longrightarrow} H^{2n-1-j}(SM_{-j}(g), SM_{-j}(g)\cap\partial(SM);\mathbb{R}),$$

where i^*_{loc} are the natural homomorphisms in the cohomology, induced by the corresponding embeddings of spaces.

The source spaces of these operators come naturally equipped with the Gromov simplicial semi-norms $\|\sim\|_{\mathbf{\Delta}}$ ([Gr]). Let us recall their definition.

Definition 1.1. *Let X be a topological space, and let $h\in H_j(X;\mathbb{R})$ be a real homology class. We consider singular real cycles $c=\sum_{i=1}^\infty r_i\sigma_i$ that represent h, where the coefficients $r_i\in\mathbb{R}$ and $\sigma_i:\Delta^j\to X$ are singular j-simplexes, such that each compact $K\subset X$ intersects only with the images $\{\sigma_i(\Delta^k)\}_i$ of finitely many singular simplexes σ_i.*

Put $\|c\|_{\ell_1} =_{\mathsf{def}} \sum_i |r_i|$.[1] *Then*

$$\|h\|_{\mathbf{\Delta}} =_{\mathsf{def}} \inf_{\{c\}}\{\|c\|_{\ell_1}\}.$$

A similar definition is available for relative classes $h\in H_j(X,Y;\mathbb{R})$, where Y is a subspace of X. $\diamondsuit$

[1]For a non-compact X, $\|c\|_{\ell_1}$ may be infinite.

We denote by $\tilde{\mathsf{B}}^{\mathbf{\Delta}}_j(X)$ the unit balls in these semi-norms. Thus $\tilde{\mathsf{B}}^{\mathbf{\Delta}}_j(X) \subset H_j(X;\mathbb{R})$ is a convex set, possibly non-compact.

We will see that the g-dependent target spaces of the operators $\mathcal{L}_j(g;\mathbb{R})$ and $\mathcal{M}_j(g;\mathbb{R})$ may be also equipped with some norms $\| \sim \|^{\bullet}_{\mho}$ (see the text that follows the proof of Theorem 3.1). Their definition depends only on the codimension j connected components of the $\mathbf{\Omega}^{\bullet}_{\langle 2n-2]}$-stratifications of the spaces $D(SM)$ and SM, respectively. For each j, the unit balls, $\diamondsuit^{(j)}_{\mho}(SM,g)$ and $\diamondsuit^{(j)}_{\mho}(D(SM),g)$, in these norms $\| \sim \|^{\bullet}_{\mho}$ are compact convex polyhedra. They depend on the geodesic flow, and thus on the metric g. In fact, the balls are linear projections of some perfect polyhedra in the appropriate vector spaces (these perfect polyhedra are the duals of the hypercubes "$\square$"; thus the notation "$\diamondsuit$").

Theorem 3.2 describes the images, under the localized Poincaré Duality operators $\mathcal{L}_j(g;\mathbb{R})$ and $\mathcal{M}_j(g;\mathbb{R})$, of the unitary spheres $\partial\tilde{\mathsf{B}}^{\mathbf{\Delta}}_j(\sim)$ in Gromov's semi-norms.

We stress that all these results require that the fundamental groups $\pi_1(M)$ and $\pi_1(DM)$ are quite big (non-amenable); for the amenable fundamental groups, all our results are vacuous!

For a given homology group $H_j(\sim;\mathbb{R})$, we form its quotient space

$$H^{\mathbf{\Delta}}_j(\sim;\mathbb{R}) =_{\mathsf{def}} \; H_j(\sim;\mathbb{R})\big/ H_j^{\{\|\sim\|_\Delta=0\}}(\sim;\mathbb{R}), \tag{1.1}$$

where the vector subspace $H_j^{\{\|\sim\|_\Delta=0\}}(\sim;\mathbb{R})$ is spanned by all the homology classes whose simplicial semi-norm vanishes. So there is an obvious epimorphism $H_j(\sim;\mathbb{R}) \to H^{\mathbf{\Delta}}_j(\sim;\mathbb{R})$, which converts the semi-norm $\| \sim \|_{\mathbf{\Delta}}$ on $H_j(\sim;\mathbb{R})$ into a *norm* $\| \sim \|^{\bullet}_{\mathbf{\Delta}}$ on $H^{\mathbf{\Delta}}_j(\sim;\mathbb{R})$. We call the *normed* vector space $H^{\mathbf{\Delta}}_j(\sim;\mathbb{R})$ "the reduced j-homology".

Theorem 3.2 implies that the kernels of the localized Poincaré Duality operators $\mathcal{L}_j(g;\mathbb{R})$ and $\mathcal{M}_j(g;\mathbb{R})$ are contained in the spaces $H_j^{\{\|\sim\|_\Delta=0\}}(D(SM);\mathbb{R})$ and $H_j^{\{\|\sim\|_\Delta=0\}}(SM;\mathbb{R})$, respectively.

This observation leads to our main result, Theorem 3.4. It claims that, for $j \in [1,n]$, the ranks of the reduced homology groups $H^{\mathbf{\Delta}}_j(D(SM);\mathbb{R}) \approx H^{\mathbf{\Delta}}_j(DM;\mathbb{R})$ give a *lower bound* for the number of connected components of the codimension j strata

$$\{\partial(SM)(v^g,\omega),\; SM^\circ(v^g,\omega),\; \tau(SM^\circ(v^g,\omega))\}_\omega.$$

Similarly, for $j \in [1,n]$, the ranks of the reduced homology groups $H^{\mathbf{\Delta}}_j(SM;\mathbb{R}) \approx H^{\mathbf{\Delta}}_j(M;\mathbb{R})$ give a lower bound for the number of connected components of the codimension j strata $\{SM^\circ(v^g,\omega) \cap \mathrm{int}(SM)\}_\omega$.

Both claims may be regarded as vague analogues of the Morse inequalities for the spaces of geodesics.

The reader may glance at reveling Examples 3.1 and 3.2 to get a better feel for the claims of our main results, Theorem 3.2 and Theorem 3.4.

We say that a metric $g \in \mathcal{G}^\ddagger(M)$ is globally j-convex if the v^g-induced stratification of $\mathcal{T}(v^g)$ has no strata of codimension $\geq j$ (compare this with Definition 2.3). For example, if ∂M is strictly convex or concave in g, then g is globally 3-convex. By Corollaries 3.2 and 3.3, the non-triviality of the groups $H^{\mathbf{\Delta}}_j(SM;\mathbb{R})$ and $H^{\mathbf{\Delta}}_j(SM;\mathbb{R})$ constitutes an *obstruction* to the existence of a global j-convex metric $g \in \mathcal{G}^\ddagger(M)$.

For $g \in \mathcal{G}^\ddagger(M)$, we are also investigating the connection of the localized Poincaré Duality operators with the inverse geodesic scattering problem.

The scattering map $C_{v^g} : \partial_1^+(SM) \to \partial_1^-(SM)$, generated by the v^g-flow, takes a domain $\partial_1^+(SM)$ in the boundary $\partial(SM)$ to the complementary domain $\partial_1^-(SM) \subset \partial(SM)$, both domains being diffeomorphic and g-independent.

Let us outline the construction of C_{v^g}. For any point $x \in \partial M$ and any unit tangent vector $w \in T_xM$ that points inside of M or is tangent to ∂M, we consider the geodesic $\gamma \subset M$. We take the next along γ point $x' \in \gamma \cap \partial M$ and register the unit tangent to γ vector $w' \in T_{x'}M$. By definition, $C_{v^g}(x, w) = (x', w')$. In fact, $\partial_1^+(SM)$ is diffeomorphic to $\partial_1^-(SM)$, the smooth topological types of both domains being g-independent. With a few exceptions, fundamentally, C_{v^g} is a *discontinuous* map!

It turns out that C_{v^g} allows for a reconstruction of the $\mathbf{\Omega}^\bullet_{'\langle 2n-2]}$-*stratified* topological type of SM (see Theorem 3.3 from [K5]). This reconstruction is an instance of a more general phenomenon, which we call "The Holographic Principle" (see Theorem 3.1 from [K4] and [K6]). By applying the Holographic Principle to the geodesic flows, in Theorem 3.3 below, we prove that it is possible to reconstruct the localized Poincaré Duality operators from the geodesic scattering map C_{v^g}.

2. Basics of Traversally Generic Vector Fields

In an attempt to make this text more self-contained, we start with presenting few basic definitions and facts related to the traversing and traversally generic vector fields, as they appear in [K1] - [K5] and [K7].

Let X be a compact connected smooth $(n+1)$-dimensional manifold with boundary. A vector field v is called traversing if each v-trajectory is ether a closed interval with both ends residing in ∂X, or a singleton also residing in ∂X (see [K1] for the details). In particular, a traversing vector field does not vanish in X. In fact, v is traversing if and only if $v \neq 0$ and v is of the gradient type ([K1], Corollary 4.1).

We denote by $\mathcal{V}_{\mathsf{trav}}(X)$ the space of traversing fields on X.

For traversing fields v, the trajectory space $\mathcal{T}(v)$ is homology equivalent to X [K1].

In this paper, we consider an important subclass of traversing fields which we call traversally generic (see formula (2.5) below and Definition 3.2 from [K2]).

For a traversally generic field v, the trajectory space $\mathcal{T}(v)$ is stratified by closed subspaces, labeled by the elements ω of an *universal* poset $\mathbf{\Omega}^\bullet_{'\langle n]}$. It depends only on $\dim(X) = n+1$. The partial order "$\succ_\bullet$" in $\mathbf{\Omega}^\bullet_{'\langle n]}$ mimics the bifurcations of real roots of real polynomials of degree $2n+2$ (see Section 2, for the definition and properties of $\mathbf{\Omega}^\bullet_{'\langle n]}$). The elements $\omega \in \mathbf{\Omega}^\bullet_{'\langle n]}$ correspond to combinatorial patterns that describe the way in which v-trajectories $\gamma \subset X$ intersect the boundary ∂X. Each intersection point $a \in \gamma \cap \partial X$ acquires a well-defined multiplicity $m(a)$, a natural number that reflects *the order of tangency* of γ

to ∂X at a (see [K1] and Definition 2.1 for the expanded definition of $m(a)$). So $\gamma \cap \partial X$ can be viewed as a *divisor* D_γ on the trajectory γ, an ordered set of points in γ together with their multiplicities. Then ω is just the ordered sequence of multiplicities $\{m(a)\}_{a \in \gamma \cap \partial X}$, the order being prescribed by v.

The support of the divisor D_γ is either a singleton a, in which case $m(a) \equiv 0 \mod 2$, or the minimum and maximum points of $\sup D_\gamma$ have *odd* multiplicities, and the rest of the points have *even* multiplicities.

Let

$$m(\gamma) =_{\mathsf{def}} \sum_{a \in \gamma \cap \partial X} m(a) \quad \text{and} \quad m'(\gamma) =_{\mathsf{def}} \sum_{a \in \gamma \cap \partial X} (m(a) - 1). \tag{2.1}$$

Similarly, for $\omega =_{\mathsf{def}} (j_1, j_2, \dots, j_i, \dots)$, where $j_i \in \mathbb{N}$, we introduce the norm and the reduced norm of ω by the formulas:

$$|\omega| =_{\mathsf{def}} \sum_i j_i \quad \text{and} \quad |\omega|' =_{\mathsf{def}} \sum_i (j_i - 1). \tag{2.2}$$

Let $\partial_j X =_{\mathsf{def}} \partial_j X(v)$ denote the locus of points $a \in \partial X$ such that the multiplicity of the v-trajectory γ_a through a at a is greater than or equal to j.

We may embed the compact manifold X into an open manifold $\hat{X}$ of the same dimension, so that v extends smoothly to a non-vanishing gradient-like vector field $\hat{v}$ in $\hat{X}$. We treat the extension $(\hat{X}, \hat{v})$ as "a germ at (X, v)".

Now, the locus $\partial_j X =_{\mathsf{def}} \partial_j X(v)$ has a description in terms of an auxiliary function $z : \hat{X} \to \mathbb{R}$ that satisfies the following three properties:

(2.3)

- 0 is a regular value of z,
- $z^{-1}(0) = \partial X$, and
- $z^{-1}((-\infty, 0]) = X$.

In terms of z, the locus $\partial_j X =_{\mathsf{def}} \partial_j X(v)$ is defined by the equations:

$$\{z = 0,\ \mathcal{L}_v z = 0,\ \dots,\ \mathcal{L}_v^{(j-1)} z = 0\},$$

where $\mathcal{L}_v^{(k)}$ stands for the k-th iteration of the Lie derivative operator $\mathcal{L}_v$ in the direction of v (see [K2]). The pure stratum $\partial_j X^\circ \subset \partial_j X$ is defined by the additional constraint $\mathcal{L}_v^{(j)} z \neq 0$. The locus $\partial_j X$ is the union of two loci: (1) $\partial_j^+ X$, defined by the constraint $\mathcal{L}_v^{(j)} z \geq 0$, and (2) $\partial_j^- X$, defined by the constraint $\mathcal{L}_v^{(j)} z \leq 0$. The two loci, $\partial_j^+ X$ and $\partial_j^- X$, share a common boundary $\partial_{j+1} X$.

Definition 2.1. *The* multiplicity $m(a)$*, where* $a \in \partial X$*, is the index* j *such that* $a \in \partial_j X^\circ$. ♢

Definition 2.2. *The vector field v on X is called* boundary generic, *if for all j and each point $a \in \partial_j X^\circ$, there exists a neighborhood $V_a \subset \hat{X}$ of a and some local coordinates $(u, \vec{x}, \vec{y}) : V_a \to \mathbb{R} \times \mathbb{R}^{j-1} \times \mathbb{R}^{n-j}$ so that ∂X is given by the polynomial equation*

$$\wp(u, \vec{x}) =_{\mathsf{def}} u^j + \sum_{l=0}^{j-2} x_l\, u^l = 0 \tag{2.4}$$

of degree $j = m(a)$ in u, while X by the polynomial inequality $\pm\wp(u, \vec{x}) \leq 0$. Each v-trajectory in V is obtained by freezing the $\vec{x}, \vec{y}$ coordinates. $\diamondsuit$

The characteristic property of traversally generic fields is that they admit special flow-adjusted coordinate systems, in which the boundary is given by quite special polynomial equations (see formula (2.5)), and the trajectories are parallel to one of the preferred coordinate axis (see [K2], Lemma 3.4). For a traversally generic v on a $(n+1)$-dimensional X, the vicinity $U \subset \hat{X}$ of each v-trajectory γ of the combinatorial type $\omega = (j_1, j_2, \dots, j_i, \dots)$ has a special coordinate system

$$(u, \vec{x}, \vec{y}) : U \to \mathbb{R} \times \mathbb{R}^{|\omega|'} \times \mathbb{R}^{n-|\omega|'}.$$

In these coordinates, by Lemma 3.4 and formula (3.17) from [K2], the boundary ∂X is given by the polynomial equation:

$$\wp(u, \vec{x}) =_{\mathsf{def}} \prod_{i=1}^{|\omega|-|\omega|'} \Big[(u-i)^{j_i} + \sum_{l=0}^{j_i-2} x_{i,l}\,(u-i)^l\Big] = 0 \tag{2.5}$$

of an even degree $|\omega|$ in u. Here $i \in \mathbb{Z}_+$ runs over the distinct roots of $\wp(u, \vec{0})$, and the vector $\vec{x} =_{\mathsf{def}} \{x_{i,l}\}_{i,l}$ is sufficiently small. At the same time, X is given by the polynomial inequality $\{\wp(u, \vec{x}) \leq 0\}$. Each v-trajectory in U is produced by freezing all the coordinates $\vec{x}, \vec{y}$, while letting u to be free.

Here we treat formula (2.5) as *the working definition* of a traversally generic vector field; for a more conceptual definition see [K2].

We denote by $\mathcal{V}^\ddagger(X)$ the space of traversally generic fields on X. In fact, $\mathcal{V}^\ddagger(X)$ is an *open* and *dense* (in the C^∞-topology) subspace of $\mathcal{V}_{\mathsf{trav}}(X)$ (see [K2], Theorem 3.5).

We denote by $X(v, \omega)$ the union of v-trajectories whose divisors are of a given combinatorial type $\omega \in \mathbf{\Omega}^\bullet_{'\langle n]}$. Its closure $\bigcup_{\omega' \preceq_\bullet \omega} X(v, \omega')$ is denoted by $X(v, \omega_{\succeq_\bullet})$.

Each *pure* stratum $\mathcal{T}(v, \omega) \subset \mathcal{T}(v)$ is an open smooth manifold of dimension $n - |\omega|'$ and, as such, has a "conventional" tangent bundle.

Definition 2.3. *We say that a traversing field v on X is* globally k-convex *if $m'(\gamma) < k$ for any v-trajectory γ. In other words, all strata $\mathcal{T}(v, \omega)$ of codimension $\geq k$ are empty.* $\diamondsuit$

3. The localized Poincaré Duality for geodesic flows

We are now in position to apply Gromov's amenable localization to the study of a certain class of Riemmanian metrics (called "traversally generic") on compact connected smooth manifolds M with boundary. We will focus on the tangency patterns, exhibited by the geodesic curves with respect to the boundary ∂M. We follow closely the general arguments in [K3], as they apply to geodesic flows for non-trapping metrics.

Consider the unit spherical fibration $SM \to M$, associated with the tangent bundle $TM \to M$ and a Riemmanian metric g on M.

Definition 3.1. *We say that a Riemmanian metric g on M is:*

- of the gradient type *or* non-trapping*, if the geodesic vector field v^g on SM is traversing,*
- boundary generic*, if the geodesic vector field v^g on SM is boundary generic (in the sense of Definition 2.2) with respect to the boundary $\partial(SM)$,*
- traversally generic*, if the geodesic vector field v^g on SM is traversally generic with respect to $\partial(SM)$.*[2] $\diamondsuit$

Since v^g depends smoothly on g and since the traversally generic fields form an open set in the space of all smooth vector fields on SM ([K5], Theorem 2.2), we conclude that the traversally generic metrics $\mathcal{M}^\ddagger(M)$ form an open set in the space $\mathcal{M}(M)$ of all smooth Riemannian metrics on M. The question whether, for a given M, the space $\mathcal{M}^\ddagger(M)$ is nonempty remains wide open. For example, if ∂M is strictly convex, and g is non-trapping, then g is traversally generic.

We conjecture that $\mathcal{M}^\ddagger(M)$ is actually *dense* in the space of non-trapping metrics.

Now, for a smooth compact manifold M with boundary, we form its double $DM =_{\mathsf{def}} M \cup_{\partial M} M$. We denote by $\rho_M : DM \to DM$ a smooth involution such that the quotient space DM/ρ_M is homeomorphic to M. Let

$$D(SM) =_{\mathsf{def}} SM \cup_{SM|_{\partial M}} SM.$$

Let $\rho_{SM} : D(SM) \to D(SM)$ be an involution such that the quotient $D(SM)/\rho_{SM}$ is homeomorphic to SM and the fixed set $D(SM)^{\rho_{SM}} = SM|_{\partial M}$, the restriction of the bundle $SM \to M$ to $\partial M \subset M$. The construction of $D(SM)$ gives rise to a spherical fibration $q : D(SM) \to DM$ over DM.

Warning: ρ_{SM} is not induced by the differential of ρ_M, and q is not the tangent spherical bundle of DM, i.e., $D(SM) \neq S(DM)$!

For a compact connected smooth Riemannian n-manifold M with boundary, any *traversally generic* metric g, via its geodesic flow v^g, defines a stratification of the space SM by the pure strata $\big\{SM(v^g, \omega)\big\}_{\omega \in \mathbf{\Omega}^\bullet_{'\langle 2n-2]}}$, which organize the v^g-trajectories by their tangency to $\partial(SM)$ patterns $\{\omega\}$. Let

$$SM^\circ(v^g, \omega) =_{\mathsf{def}} SM(v^g, \omega) \cap \mathrm{int}(SM).$$

In turn, these strata generate the filtration

$$\Big\{SM^\circ_{-(k+1)}(g) =_{\mathsf{def}} \bigcup_{\omega \in \mathbf{\Omega}^\bullet \,\big|\, |\omega|' \geq k+1} SM^\circ(v^g, \omega)\Big\}_k \tag{3.1}$$

[2]In particular, traversally generic metrics are boundary generic and of the gradient type.

of $SM \setminus \partial(SM)$ and the filtration

$$\Big\{SM_{-(k+1)}(g) =_{\mathsf{def}} \Big(\bigcup_{\omega\in\mathbf{\Omega}^\bullet \,\big|\, |\omega|'\geq k+1} SM^\circ(v^g,\omega)\Big)$$

$$\bigcup \Big(\bigcup_{\omega\in\mathbf{\Omega}^\bullet \,\big|\, |\omega|'\geq k} \partial(SM)\cap SM(v^g,\omega)\Big)\Big\}_k \tag{3.2}$$

of SM. The filtration $\{SM_{-(k+1)}(g)\}$ induces the ρ_{SM}-equivariant filtration $\{D(SM)_{-(k+1)}(g)\}$ of the double $D(SM) \approx S(DM)$.

For any commutative ring R, we introduce the free R-modules:

$$\mathsf{C}^{2n-1-j}_{\mho}\big(D(SM), g; \mathsf{R}\big) =_{\mathsf{def}}\; H^{2n-1-j}\big(D(SM)_{-j}(g),\; D(SM)_{-(j+1)}(g);\; \mathsf{R}\big)$$

and

$$\mathsf{C}^{2n-1-j}_{\mho}\big(SM^\circ, g; \mathsf{R}\big) =_{\mathsf{def}}\; H^{2n-1-j}\big(SM_{-j}(g),\; SM_{-(j+1)}(g)\cup\big(SM_{-j}(g)\cap\partial(SM)\big);\; \mathsf{R}\big).$$

So each traversally generic metric g on M gives rise to a differential complex

$$\mathbf{C}^*_{\mho}\big(D(SM),\, g;\, \mathsf{R}\big) =_{\mathsf{def}} \Big\{0 \to \mathsf{C}^0_{\mho}\big(D(SM), g; \mathsf{R}\big) \xrightarrow{\delta_0} \mathsf{C}^1_{\mho}\big(D(SM),\, g;\; \mathsf{R}\big) \xrightarrow{\delta_1} \dots$$

$$\dots \xrightarrow{\delta_{2n-2}} \mathsf{C}^{2n-1}_{\mho}\big(D(SM), g; \mathsf{R}\big) \to 0\Big\}, \tag{3.3}$$

where the differentials $\{\delta_j\}$ are the boundary homomorphisms from the long exact cohomology sequences of the triples

$$\big\{D(SM)_{-(j-1)}(g) \supset D(SM)_{-j}(g) \supset D(SM)_{-(j+1)}(g)\big\}_j.$$

Similarly, g produces the differential complex

$$\mathbf{C}^*_{\mho}\big(SM^\circ, g;\, \mathsf{R}\big) =_{\mathsf{def}} \Big\{0 \to \mathsf{C}^0_{\mho}\big(SM^\circ, g; \mathsf{R}\big) \xrightarrow{\delta_0} \mathsf{C}^1_{\mho}\big(SM^\circ, g; \mathsf{R}\big) \xrightarrow{\delta_1} \dots$$

$$\dots \xrightarrow{\delta_{2n-2}} \mathsf{C}^{2n-1}_{\mho}\big(SM^\circ, g; \mathsf{R}\big) \to 0\Big\}, \tag{3.4}$$

where the differentials $\{\delta_j\}$ are the boundary homomorphisms from the long exact homology sequences of the triples

$$\big\{SM_{-(j-1)}(g)\cup\partial(SM) \;\supset\; SM_{-j}(g)\cup\partial(SM) \;\supset\; SM_{-(j+1)}(g)\cup\partial(SM)\big\}_j.$$

We denote by

$$\mathsf{B}^{2n-1-j}_{\mho}\big(SM^\circ, g; \mathsf{R}\big) \subset \mathsf{C}^{2n-1-j}_{\mho}\big(SM^\circ, g; \mathsf{R}\big)$$

the image of the differential δ_{2n-2-j} from (3.4). Similarly, let

$$\mathsf{B}^{2n-1-j}_{\mho}\big(D(SM), g; \mathsf{R}\big) \subset \mathsf{C}^{2n-1-j}_{\mho}\big(D(SM), g; \mathsf{R}\big)$$

stand for the image of the differential δ_{2n-2-j} from (3.3).

Conjecture 3.1. *Let g be a traversally generic Riemmanian metric on a compact connected smooth n-manifold M with boundary. Then, as described above, the metric g generates the differential complexes $\mathbf{C}^\ast_\mho\big(SM^\circ, g; \mathsf{R}\big)$ and $\mathbf{C}^\ast_\mho\big(D(SM), g; \mathsf{R}\big)$ of free R-modules.*

We conjecture that the homology groups of these differential complexes depend only on the connected component of the space of traversally generic metrics on M, to which g belongs. $\diamondsuit$

The next theorem claims that, for a non-trapping metric g, the differential complexes $\mathbf{C}^\ast_\mho\big(SM^\circ, g\big)$ and $\mathbf{C}^\ast_\mho\big(D(SM), g\big)$ can be reconstructed from the scattering map C_{v^g}. We call such reconstructions "holographic" since the objects that are affiliated with the $(2n-1)$-dimensional bulk SM or $S(DM)$ and the geodesic flow are recorded on the pair $\partial_1^+(SM)$, $\partial_1^-(SM)$ of diffeomorphic $(2n-2)$-dimensional screens.

Theorem 3.1. *Let g be a traversally generic Riemmanian metric on a compact connected smooth n-manifold M with boundary.*

The differential complexes $\mathbf{C}^\ast_\mho\big(SM^\circ, g; \mathsf{R}\big)$ and $\mathbf{C}^\ast_\mho\big(D(SM), g; \mathsf{R}\big)$ can be reconstructed from the geodesic scattering map $C_{v^g} : \partial_1^+(SM) \to \partial_1^-(SM)$.

Proof. Let $\mathcal{F}(v^g)$ be the oriented 1-dimensional foliation on SM, produced by the v^g-flow. Any traversally generic geodesic field v^g is automatically boundary generic and of the gradient (non-trapping) type. If g is non-trapping and the geodesic field v^g is boundary generic, then by Theorem 3.3 from [K5], the scattering map $C_{v^g} : \partial_1^+(SM) \to \partial_1^-(SM)$ allows for a reconstruction of the pair $(SM, \mathcal{F}(v^g))$, up to a homeomorphism of SM which is the identity on the boundary $\partial(SM)$. That homeomorphism preserves the stratification $\mathcal{S}^\bullet_{v^g}(SM)$, whose strata are the connected components of the stratification $\mathcal{S}_{v^g}(SM) = \{\partial(SM)(v^g, \omega),\ SM^\circ(v^g, \omega)\}_\omega$. Therefore, the topological types of the stratifications $\mathcal{S}^\bullet_{v^g}(SM^\circ)$ and $\mathcal{S}^\bullet_{v^g}(D(SM))$ are determined by C_{v^g}. As a result, the differential complexes $\mathbf{C}^\ast_\mho\big(SM^\circ, g; \mathsf{R}\big)$ and $\mathbf{C}^\ast_\mho\big(D(SM), g; \mathsf{R}\big)$, whose construction depends only on the *stratified* topological types of the spaces SM and $D(SM)$, can be reconstructed from the geodesic scattering map C_{v^g} along the lines of [K4]. $\square$

Abusing notations, we denote by $C^j_\mho\big(D(SM), g; \mathbb{Z}\big)$ the image of $C^j_\mho\big(D(SM), g; \mathbb{Z}\big)$ in the vector space $C^j_\mho\big(D(SM), g; \mathbb{R}\big)$, viewed as an integral lattice. Similarly, we consider the integral lattice $C^j_\mho\big(SM^\circ, g; \mathbb{Z}\big) \subset C^j_\mho\big(SM^\circ, g; \mathbb{R}\big)$. The integral lattice $C^j_\mho\big(SM, g; \mathbb{Z}\big)$ comes equipped with a *basis* whose vectors correspond to the connected components of the strata

$$\big\{SM(v^g, \omega) \cap \mathrm{int}(SM)\big\}_{\{\omega \in \mathbf{\Omega}^\bullet \,\big|\, |\omega|' = j\}}.$$

Similarly, the integral lattice $C^j_\mho\big(D(SM), g; \mathbb{Z}\big)$ comes equipped with a basis whose vectors correspond to the connected components of the strata

$$\big\{SM(v^g, \omega)\big\}_{\{\omega \in \mathbf{\Omega}^\bullet \,\big|\, |\omega|' = j\}},$$

each stratum being considered twice, together with the connected components of the strata

$$\big\{SM(v^g,\omega)\cap\partial(SM)\big\}_{\{\omega\in\mathbf{\Omega}^\bullet\,\big|\,|\omega|'=j-1\}}.$$

Using these bases, we introduce the l_1-norms $\|\sim\|_{\mho}$ in the vector spaces $C^j_{\mho}\big(SM^\circ, g;\mathbb{R}\big)$ and $C^j_{\mho}\big(D(SM), g;\mathbb{R}\big)$ so that the basic vectors (which belong to the lattices $C^j_{\mho}\big(SM^\circ, g;\mathbb{Z}\big)$ and $C^j_{\mho}\big(D(SM), g;\mathbb{Z}\big)$, respectively) have lengths 1. The unit balls in these norms are the perfect polyhedra, dual to the hypercubes in the corresponding spaces.

The semi-norms $\|\sim\|_{\mho}$ induce "honest" norms $\|\sim\|^\bullet_{\mho}$ in the quotient spaces[3]

$$C^j_{\mho}\big(SM^\circ, g;\mathbb{R}\big)\big/B^j_{\mho}\big(SM^\circ, g;\mathbb{R}\big) \ \text{ and } \ C^j_{\mho}\big(D(SM), g;\mathbb{R}\big)\big/B^j_{\mho}\big(D(SM), g;\mathbb{R}\big),$$

respectively.

We denote by $\diamondsuit^j_{\mho}\big(SM^\circ, g\big)$ and $\diamondsuit^j_{\mho}\big(D(SM), g\big)$ the unit balls in these quotient norms $\|\sim\|^\bullet_{\mho}$. They are convex hulls of the images, under the quotient maps, of the verticies of the perfect polyhedron $\{\|\sim\|_{\mho}=1\}$.

The manifolds SM and $S(DM)$ are orientable. So the Poincaré Duality is available for their homology and cohomology with coefficients in $\mathbb{R}$ or $\mathbb{Z}$. As in [K3] (where we dealt with arbitrary traversally generic flows), for each j, we consider the *localized* Poincaré Duality operators over the coefficient rings $\mathsf{R}=\mathbb{Z},\mathbb{R}$ [4]:

$$\begin{aligned}
\mathcal{L}_j(g): H_j(D(SM)) &\xrightarrow{\approx\mathcal{D}} H^{2n-1-j}(D(SM)) \xrightarrow{i^*_{\mathsf{loc}}} H^{2n-1-j}\big(D(SM)_{-j}(g)\big)\\
&\approx\ C^{2n-1-j}_{\mho}(D(SM), g)\Big/B^{2n-1-j}_{\mho}(D(SM), g),\\
\mathcal{M}_j(g): H_j(SM) &\xrightarrow{\approx\mathcal{D}} H^{2n-1-j}(SM, \partial(SM)) \xrightarrow{i^*_{\mathsf{loc}}} H^{2n-1-j}\big(SM_{-j}(g), SM_{-j}(g)\cap\partial(SM)\big)\\
&\approx\ C^{2n-1-j}_{\mho}(SM^\circ, g)\Big/B^{2n-1-j}_{\mho}(SM^\circ, g).
\end{aligned}$$

(3.5)

Here the natural homomorphisms i^*_{loc} are induced by the inclusions of the strata

$$SM^\circ_{-j}(g)\subset SM \text{ and } D(SM)_{-j}(g)\subset D(SM),$$

thus the term "localized" in the names of the two operators.

The RHS of isomorphisms "$\approx$" in (3.5) can be justified exactly by the same homological argument as in [K3], page 516; just replace an arbitrary traversally generic vector fields v on X with the geodesic traversally generic vector fields v^g on SM.

$\mathcal{L}_j(g;\mathbb{R})$ maps the lattice $H_j(D(SM);\mathbb{Z})$ to the lattice $H^{2n-1-j}\big(D(SM)_{-j}(g);\mathbb{Z}\big)$. Similarly, $\mathcal{M}_j(g;\mathbb{R})$ maps the lattice $H_j(SM;\mathbb{Z})$ to the lattice $H^{2n-1-j}\big(SM^\circ_{-j}(g);\mathbb{Z}\big)$.

Consider a vector space $\tilde{\mathcal{V}}$ over the reals, equipped with a semi-norm $\|\sim\|$. For any pair of vectors $v,w\in\tilde{\mathcal{V}}$ such that $\|w\|=0$, by using twice the triangle inequality, we get

[3] Recall that the "quotient norm" of a given vector $\vec{V}$ in the quotient space is defined to be the infimum of the norms of all the vectors (in the original space) that represent $\vec{V}$.

[4] We have suppressed in (3.5) the dependence of homology and cohomology on the coefficients R.

$\|v+w\| = \|v\|$. Therefore $\| \sim \|$ becomes a *norm* on the quotient $\mathcal{V} =_{\mathsf{def}} \tilde{\mathcal{V}}/\tilde{\mathcal{V}}^{\{\|\sim\|=0\}}$ of the space $\tilde{\mathcal{V}}$ by the linear subspace $\tilde{\mathcal{V}}^{\{\|\sim\|=0\}}$ of vectors whose semi-norms vanish. The unit ball $\tilde{B}$ in $\tilde{\mathcal{V}}$ consists of vectors whose simplicial semi-norms do not exceed 1. It is a product of the unit ball B in the quotient normed space $\mathcal{V}$ times the vector space $\tilde{\mathcal{V}}^{\|\sim\|=0\}}$.

Let us apply this construction to our setting. We denote by $\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(D(SM)) \subset H_j(D(SM);\mathbb{R})$ and $\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(SM) \subset H_j(SM;\mathbb{R})$ the set of vectors (homology classes) whose simplicial semi-norms $| \sim |_{\mathbf{\Delta}}$ do not exceed 1. These are the "unit balls". Let $\partial\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(D(SM))$ and $\partial\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(SM)$ denote the sets of vectors whose semi-norms are equal 1 (these "spheres" may not be compact!).

So we may form the compact convex ball $\mathsf{B}_j^{\mathbf{\Delta}}(D(SM)) \subset H_j^{\mathbf{\Delta}}(D(SM);\mathbb{R})$, the image of $\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(D(SM))$ under the quotient map $H_j(D(SM);\mathbb{R}) \to H_j^{\mathbf{\Delta}}(D(SM);\mathbb{R})$; similarly, we may form the compact convex ball $\mathsf{B}_j^{\mathbf{\Delta}}(SM) \subset H_j^{\mathbf{\Delta}}(SM;\mathbb{R})$, the image of $\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(D(SM))$ under the quotient map $H_j(SM;\mathbb{R}) \to H_j^{\mathbf{\Delta}}(SM;\mathbb{R})$.

We will use the localized Poincaré Duality operators $\mathcal{L}_j(g;\mathbb{R})$ and $\mathcal{M}_j(g;\mathbb{R})$ from (3.5) to project linearly the unit balls $\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(D(SM))$ and $\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(SM)$ on the g-dependent "screens"

$$C_{\mho}^{2n-1-j}(D(SM), g;\mathbb{R})\big/ B_{\mho}^{2n-1-j}(D(SM), g;\mathbb{R})$$

and

$$C_{\mho}^{2n-1-j}(SM^\circ, g;\mathbb{R})\big/ B_{\mho}^{2n-1-j}(SM^\circ, g;\mathbb{R}),$$

respectively. These screens are manufactured with the help various metrics $g \in \mathcal{G}^\ddagger(M)$.

The next theorem, one of our main results, makes several claims about the geometry of these projections.

Theorem 3.2. *Let M be a compact connected smooth n-manifold with boundary, where $n \geq 3$. Let $j \in [0, n]$.*

- *Assume that, for each connected component of the boundary ∂M, the image of its fundamental group in $\pi_1(DM)$ is amenable.*

  Then there is a universal constant $\lambda = \lambda(n, j) \geq 1$ such that, for every M and every traversally generic Riemannian metric g on M, the image of the unit (in the simplicial semi-norm) sphere $\partial\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(D(SM))$ under the localized Poincaré Duality operator $\mathcal{L}_j(g;\mathbb{R})$, is contained in the complement to the radius λ^{-1} ball

  $$\lambda^{-1} \cdot \diamondsuit_{\mho}^{2n-1-j}\big(D(SM), g\big).$$

  For $j \leq n$, the unit sphere $\partial\mathsf{B}_j^{\mathbf{\Delta}}(D(SM))$ is isometric to the unit sphere $\partial\mathsf{B}_j^{\mathbf{\Delta}}(DM) \subset H_j^{\mathbf{\Delta}}(DM;\mathbb{R})$.
- *Assume that, for each connected component of the boundary ∂M, the image of its fundamental group in $\pi_1(M)$ is amenable.*

  Similarly, there is a universal constant $\mu = \mu(n, j) \geq 1$ such that, for every M and every traversally generic Riemannian metric g on M, the image of the *unit sphere* $\partial\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(SM)$ *under the localized Poincaré Duality operator* $\mathcal{M}_j(g;\mathbb{R})$ *is contained in the complement to the radius* μ^{-1} *ball*

$$\mu^{-1}\cdot\diamondsuit_{\mho}^{2n-1-j}\big(SM,g\big).$$

For $j<n$*, the unit sphere* $\partial\mathsf{B}_j^{\mathbf{\Delta}}(SM)$ *is isometric to the unit sphere* $\partial\mathsf{B}_j^{\mathbf{\Delta}}(M)\subset H_j^{\mathbf{\Delta}}(M;\mathbb{R})$.

Proof. Theorem 4.2 from [K3] (see also Theorem 8.8 from [K7]) forms the foundation of our arguments. The latter theorem applies to *any* traversally generic vector field v on a connected compact smooth manifold X with boundary; in particular, we will apply it to any traversally generic geodesic vector field v^g on SM.

Let us first discuss Theorem 4.2 from [K3] at some length, since we will use its reformulation. Let X be a compact connected and smooth $(n+1)$-manifold with boundary, equipped with a traversally generic vector field v. The localized Poincaré Duality operators

$$\mathcal{L}_{k+1}^{v,\mho}: H_{k+1}(DX)\to C_{\mho}^{n-k}(DX,v)/B_{\mho}^{n-k}(DX,v)$$

,

$$\mathcal{M}_{k+1}^{v,\mho}: H_{k+1}(X)\to C_{\mho}^{n-k}(X,v)/B_{\mho}^{n-k}(X,v)$$

in formula (4.9) from [K3] have been already adapted, via formula (3.5), to the environment of geodesic flows v^g on SM.

Theorem 4.2 from [K3] claims the inequality $\|h\|_{\mathbf{\Delta}}\leq\lambda\cdot\|\mathcal{L}_j(h)\|_{\mho}^{\bullet}$ for every class $h\in H_{k+1}(DX;\mathbb{R})$, and of the inequality $\|h\|_{\mathbf{\Delta}}\leq\mu\cdot\|\mathcal{M}_j(h)\|_{\mho}^{\bullet}$ for every class $h\in H_{k+1}(X;\mathbb{R})$, where $\lambda\geq\mu\geq 1$ being some universal constants. These constants depend only on $\dim(X)$ and the index $k+1$.

Adapting these estimates to our setting, we validate of the inequality

$$\|h\|_{\mathbf{\Delta}}\leq\lambda\cdot\|\mathcal{L}_j(g)(h)\|_{\mho}^{\bullet} \tag{3.6}$$

for every class $h\in H_j(D(SM);\mathbb{R})$, and of the inequality

$$\|h\|_{\mathbf{\Delta}}\leq\mu\cdot\|\mathcal{M}_j(g)(h)\|_{\mho}^{\bullet} \tag{3.7}$$

for every class $h\in H_j(SM;\mathbb{R})$. Here $\lambda\geq\mu\geq 1$ are some universal constants. They depend only on $\dim(M)$ and the index $j\in[0,2n-1]$.

We will see soon that, in fact, one gets non-trivial results only for $j\in[0,n]$.

The inequality (3.6) may be interpreted as claiming that the $\mathcal{L}_j(g;\mathbb{R})$-image of the unit sphere $\partial\tilde{\mathsf{B}}_j^{\mathbf{\Delta}}(D(SM))$ (in the simplicial semi-norm) is contained in the complement to the ball $\lambda^{-1}\cdot\diamondsuit_{\mho}^{2n-1-j}\big(D(SM),g\big)$ of radius λ^{-1} in the target space. A similar interpretation is available for $\mathcal{M}_j(g;\mathbb{R})$ from (3.7). So we use this more geometric interpretation of inequalities (3.6) and (3.7) in the formulation of Theorem 3.2.

Let us describe the source of the universal constants λ, μ, participating in these inequalities[5]. Recall that, in Section 2, we have introduced the model space $\mathsf{E}_\omega \subset \mathbb{R} \times \mathbb{R}^{|\omega|'} \times \mathbb{R}^{n-|\omega|'}$, given by the polynomial inequality $\{\wp(u, \vec{x}) \leq 0\}$, where $\wp$ is defined by the LHS of (2.5) and $\vec{x} \in \mathbb{R}^{|\omega|'}$.

Consider the obvious projection $p : \mathbb{R} \times \mathbb{R}^{|\omega|'} \times \mathbb{R}^{n-|\omega|'} \to \mathbb{R}^{|\omega|'} \times \mathbb{R}^{n-|\omega|'}$. The fibers of the projection $p : \mathsf{E}_\omega \to \mathbb{R}^{|\omega|'} \times \mathbb{R}^{n-|\omega|'}$ are unions the ∂_u-trajectories in E_ω. Therefore E_ω acquires a stratification $\big\{\mathsf{E}_\omega(\partial_u, \hat{\omega})\big\}_{\hat{\omega}}$, labeled by the elements $\hat{\omega} \succeq \omega$. They form the sub-poset $\omega_{\preceq} \subset \mathbf{\Omega}^{\bullet}_{'\langle 2n-2]}$. Put

$$\partial\mathsf{E}_\omega(\partial_u, \hat{\omega}) =_{\mathsf{def}} \mathsf{E}_\omega(\partial_u, \hat{\omega}) \cap \partial\mathsf{E}_\omega$$

and

$$\mathsf{E}^\circ_\omega(\partial_u, \hat{\omega}) =_{\mathsf{def}} \mathsf{E}_\omega(\partial_u, \hat{\omega}) \cap \mathrm{int}(\mathsf{E}_\omega).$$

We denote $\mathcal{S}^\bullet(\mathsf{E}^\circ_\omega)$ the stratification of the space E°_ω by the connected components of these strata.

The double $D\mathsf{E}_\omega$ of E_ω is stratified by the connected components of $\big\{\partial\mathsf{E}_\omega(\partial_u, \hat{\omega})\big\}_{\hat{\omega} \in \omega_{\prec}}$, together with the connected components of $\big\{\mathsf{E}^\circ_\omega(\partial_u, \hat{\omega})\big\}_{\hat{\omega} \in \omega_{\preceq}}$ and their images under the involution $\tau : D(\mathsf{E}_\omega) \to D(\mathsf{E}_\omega)$ that is a part of the doubling construction. We denote by $\mathcal{S}^\bullet(D(\mathsf{E}_\omega))$ this stratification.

The universal constant μ is the maximum of the $\mathcal{S}^\bullet(\mathsf{E}^\circ_\omega)$-stratified[6] relative (to their boundaries) simplicial norms of small j-disks, each one being normal to a particular connected component of the $(2n+1-j)$-dimensional strata $\{\mathsf{E}^\circ_\omega(\partial_u, \hat{\omega})\}_{\omega, \hat{\omega} \in \omega_{\preceq}}$. The maximum being taken over all pairs $\omega \preceq \hat{\omega}$, where $\omega \in \mathbf{\Omega}^{\bullet}_{'\langle 2n-2]}$.

The universal constant λ is the maximum of the $\mathcal{S}^\bullet(D(\mathsf{E}_\omega))$-stratified relative (to their boundaries) simplicial norms of small j-disks, each one being normal to a particular connected component of the $(2n+1-j)$-dimensional strata

$$\Big\{\mathsf{E}^\circ_\omega(\partial_u, \hat{\omega}),\ \tau(\mathsf{E}^\circ_\omega(\partial_u, \hat{\omega})),\ \partial\mathsf{E}_\omega(\partial_u, \hat{\omega})\Big\}$$

in the double $D(\mathsf{E}_\omega)$. The maximum is taken over all pairs $\omega \preceq \hat{\omega}$, where $\omega \in \mathbf{\Omega}^{\bullet}_{'\langle 2n-2]}$.

Now we are ready to show that the hypotheses of Theorem 3.2 about the fundamental groups of M, DM, and ∂M imply the validity of hypotheses of Theorem 4.2 from [K3], as it applies to the fundamental groups of SM, $D(SM)$, and $\partial(SM)$.

Consider the tangent $(n-1)$-spherical fibration $p : SM \to M$ and its double, the fibration $q : D(SM) \to DM$. We denote by $\partial_\alpha M$ a typical connected component of ∂M.

Using the long homotopy sequences of the fibration p and of its restriction to $\partial_\alpha M$, we notice that $p_* : \pi_1(p^{-1}(\partial_\alpha M)) \approx \pi_1(\partial_\alpha M)$ and $p_* : \pi_1(SM) \approx \pi_1(M)$ since $\pi_1(S^{n-1}) = 0$ for $n \geq 3$ and $\pi_0(\text{spherical fiber}) \approx \pi_0(\text{total space})$.

Consider the square diagram that is formed by these four fundamental groups, where the two vertical homomorphisms are induced by the inclusions $\partial_\alpha M \hookrightarrow M$ and $p^{-1}(\partial_\alpha M) \hookrightarrow$

[5]This is the only place where the hypotheses that v^g is traversally generic (and not only boundary generic and traversing) seems to be important!

[6]See [Gr1] and [AK] for an accurate definition of the stratified simplicial norm.

SM. Using its commutativity, we conclude that, if the image of $\pi_1(\partial_\alpha M) \to \pi_1(M)$ is an amenable group, so is the image of $\pi_1(p^{-1}(\partial_\alpha M)) \to \pi_1(SM)$. Similarly, if the image of $\pi_1(\partial_\alpha M) \to \pi_1(DM)$ is an amenable group, so is the image of $\pi_1(q^{-1}(\partial_\alpha M)) \to \pi_1(D(SM))$.

For $n \geq 3$, the map $p : SM \to M$ induces an isomorphism of the fundamental groups $\pi_1(SM)$ and $\pi_1(M)$, and the map $q : D(SM) \to DM$ induces an isomorphism of the fundamental groups $\pi_1(D(SM))$ and $\pi_1(DM)$.

We rely on following proposition ([Gr], Corollaries, page 248, [Gr1], Section 3.2).

If a continuous map $f : X \to Y$ of finite CW-complexes induces an isomorphism of their fundamental groups, then the induced homomorphism $f_* : H_*(X;\mathbb{R}) \to H_*(Y;\mathbb{R})$ is an isometry with respect to the simplicial semi-norms $\| \sim \|_{\mathbf{\Delta}}$. In other words, $f_* : H^{\mathbf{\Delta}}_*(X;\mathbb{R}) \to H^{\mathbf{\Delta}}_*(Y;\mathbb{R})$ is an isomorphism of the *normed* vector spaces.

This proposition implies instantly that, for $n \geq 3$, $p_* : H^{\mathbf{\Delta}}_*(SM;\mathbb{R}) \to H^{\mathbf{\Delta}}_*(M;\mathbb{R})$ and $q_* : H^{\mathbf{\Delta}}_*(D(SM);\mathbb{R}) \to H^{\mathbf{\Delta}}_*(DM;\mathbb{R})$ are isometries of normed spaces (see (1.1) for the definition of the reduced homology $H^{\mathbf{\Delta}}_*(\sim;\mathbb{R})$).

Let us spell out how these isometries manifest themselves. We will use this understanding in the proof of Theorem 3.3 and in Examples 3.1 and 3.2.

Since M is of a homotopy type of an $(n-1)$-dimensional CW-complex (recall that $\partial M \neq \emptyset$), the fibration $p : SM \to M$ admits a section $\sigma : M \to SM$. With the help of σ, $H_*(M;\mathbb{R})$ is direct summand of $H_*(SM;\mathbb{R})$. By the Leray-Hirsh Theorem (see Theorem 4D.1 in [Hat]), the existence of σ implies the isomorphism $H_*(SM;\mathbb{R}) \approx H_*(S^{n-1};\mathbb{R}) \otimes H_*(M;\mathbb{R})$. In other words, $H_k(SM;\mathbb{R}) \approx H_k(M;\mathbb{R}) \oplus H_{k-n+1}(M;\mathbb{R})$ for all $k \in [0, 2n-1]$. The first summand is delivered by $\sigma_* : H_*(M;\mathbb{R}) \to H_*(SM;\mathbb{R})$.

By its definition, the simplicial semi-norm of a homology class $h \in H_q(X;\mathbb{R})$ does not increase under continuous maps $f : X \to Y$ of spaces, that is, $\|h\|_\Delta \geq \|f_*(h)\|_\Delta$, where $f_*(h) \in H_q(Y;\mathbb{R})$. Therefore, $\|h\|_\Delta \leq \|\sigma_*(h)\|_\Delta$. On the other hand, $\|\sigma_*(h)\|_\Delta \geq \|p_*(\sigma_*(h))\|_\Delta = \|h\|_\Delta$. So p_* is an isometry on $H_0(S^{n-1};\mathbb{R}) \otimes H_*(M;\mathbb{R})$. At the same time, the simplicial norm of any class of the form $[S^{n-1}] \otimes \rho \in H_{n-1}(S^{n-1};\mathbb{R}) \otimes H_*(M;\mathbb{R})$ vanishes. Indeed, the homomorphism $p_* : H_{n-1}(S^{n-1};\mathbb{R}) \otimes H_*(M;\mathbb{R}) \to H_*(M;\mathbb{R})$ is trivial, since $p_*(H_{n-1}(S^{n-1};\mathbb{R})) = 0$. Therefore by [Gr], $\|[S^{n-1}] \otimes \rho\|_{\mathbf{\Delta}} = 0$. So we get that the inverse of the canonical *isomorphism* $p_* : H^{\mathbf{\Delta}}_*(SM) \approx H^{\mathbf{\Delta}}_*(M)$ is delivered by σ_*. In particular, $H^{\mathbf{\Delta}}_*(SM) = 0$ for all $j \geq n$.

Similar arguments, applied to the spherical fibration $q : D(SM) \to DM$, lead to the isometry $q_* : H^{\mathbf{\Delta}}_k(D(SM)) \approx H^{\mathbf{\Delta}}_k(DM)$ for all $k \in [0, n]$. Indeed, the spherical fibration $q : D(SM) \to DM$ admits a section $D\sigma$, the double of the section σ of p. (Note that the tangent spherical bundle $S(DM) \to DM$ may not have a section!) In particular, $H^{\mathbf{\Delta}}_*(D(SM)) = 0$ for all $j > n$.

Let us consider the unit (in the norm $\| \sim \|_{\mathbf{\Delta}}$) spheres $\partial B^{\mathbf{\Delta}}_j(M) \subset H^{\mathbf{\Delta}}_j(M)$ and $\partial B^{\mathbf{\Delta}}_j(DM) \subset H^{\mathbf{\Delta}}_j(DM)$. We may use the sections σ and $D\sigma$ to map these spheres isometrically onto the spheres $\partial \tilde{B}^{\mathbf{\Delta}}_j(SM) \subset H_j(SM)$ and $\partial \tilde{B}^{\mathbf{\Delta}}_j(D(SM)) \subset H_j(D(SM))$, respectively. Then we compose the isometries $D\sigma_*$ and σ_* with the localized Poincaré

Duality operators $\mathcal{L}_j^{v,\mho}$ and $\mathcal{M}_j^{v,\mho}$ to conclude that the $(\mathcal{L}_j^{v,\mho} \circ D\sigma_*)$-image of the sphere $\partial B_j^{\mathbf{\Delta}}(DM)$ is contained in the compliment to the ball $\lambda^{-1} \cdot \diamondsuit_{\mho}^{2n-1-j}\big(D(SM), g\big)$ of radius λ^{-1} and the $(\mathcal{M}_j^{v,\mho} \circ \sigma_*)$-image of the sphere $\partial B_j^{\mathbf{\Delta}}(M)$ is contained in the compliment to the ball $\mu^{-1} \cdot \diamondsuit_{\mho}^{2n-1-j}\big(SM, g\big)$ of radius μ^{-1}. These facts will be instrumental in the proof of Theorem 3.4.

Thus we completed translation of the notions and constructions from [K3] to the language of traversally generic geodesic flows. We have shown that, for traversally generic metrics g on M, the hypotheses of Theorem 3.2 imply the validity of the hypotheses of Theorem 4.2 from [K3]. This concludes the proof of Theorem 3.2. □

Corollary 3.1. *Let (N, g) be a closed connected smooth Riemannian n-manifold, where $n \geq 3$. Let $U \subset N$ be a codimension zero submanifold with a smooth boundary so that U is contained in a topological n-ball. Put $M =_{\mathsf{def}} N \setminus \mathrm{int}(U)$, and let us assume that the metric $g|_M$ is traversally generic.*

Then the number of connected codimension n components of the $\mathcal{S}^\bullet_{v^g}(D(SM))$-stratification of $D(SM)$ exceeds $\lambda^{-1} \cdot \|[DM]\|_{\mathbf{\Delta}}$, where $\lambda \geq 1$ depends only on n.

In particular, if $\|[DM]\|_{\mathbf{\Delta}} \neq 0$, there exists a $(n-1)$-dimensional family of geodesics γ in M such that their reduced multiplicity $m'(\gamma) = n$ (see formula (2.1)). $\diamondsuit$

Remark 3.1. Note that, by the previous arguments, $\|[D(SM)]\|_{\mathbf{\Delta}} = 0$. Therefore Theorem 3.2 does not tell anything about the number of codimension $2n-1$ connected components of the $\mathcal{S}^\bullet_{v^g}(D(SM))$-stratification of $D(SM)$. These components arise from the finitely many geodesics γ that have the *maximal* reduced multiplicity $m'(\gamma) = 2n-2$ to the boundary ∂M. $\diamondsuit$

Example 3.1. Let (N, g) be a closed hyperbolic n-manifold, $n \geq 3$. Let U be a smooth n-ball in N. Put $M = N \setminus U$. We assume that $g|_M$ is traversally generic.

Then $DM = N \# N$, where "#" stands for the connected sum. For $n \geq 3$, we get $\|[DM]\|_{\mathbf{\Delta}} = 2 \cdot \|[N]\|_{\mathbf{\Delta}}$ ([Gr], Section 3.5). By the hyperbolicity of N,

$$\|[N]\|_{\mathbf{\Delta}} = vol_{\mathsf{hyp}}(N)/vol_{\mathsf{hyp}}(\Delta^n),$$

the normalized hyperbolic volume of N. Here $vol_{\mathsf{hyp}}(\Delta^n)$ stands for the hyperbolic volume of an ideal simplex Δ^n in the hyperbolic space. In such a case, the number of codimension n connected components of the $\mathcal{S}^\bullet_{v^g}(D(SM))$-stratification of $D(SM)$ exceeds

$$2\big(\lambda \cdot vol_{\mathsf{hyp}}(\Delta^n)\big)^{-1} \cdot vol_{\mathsf{hyp}}(N) > 0.$$

In particular, there exists a $(n-1)$-dimensional family of geodesics γ in M such that their reduced multiplicity $m'(\gamma) = n$.

Therefore, for a given family of closed hyperbolic n-manifolds $\{N_k\}_{k\to\infty}$ of increasing volumes and the corresponding compact hyperbolic manifolds $M_k = N_k \setminus U_k$, the number of codimension n connected components of the $\mathcal{S}^\bullet_{v^g}(D(SM))$-stratification of $D(SM_k)$ grows at least as fast as $\{vol_{\mathsf{hyp}}(N_k)\}_{k\to\infty}$. $\diamondsuit$

Theorem 3.3. *For any metric $g \in \mathcal{G}^{\ddagger}(M)$, the localized Poincaré Duality operators $\mathcal{L}_j(g;\mathbb{R})$ and $\mathcal{M}_j(g;\mathbb{R})$ in (3.5), as well as the simplicial semi-norms $\|\sim\|_{\mathbf{\Delta}}$ in their source spaces, and the norms $\|\sim\|^{\bullet}_{\mho}$ in their target spaces, can be reconstructed from the geodesic scattering map $C_{v^g} : \partial_1^+(SM) \to \partial_1^-(SM)$.*

Proof. By Theorem 3.3 from [K5] (see also [K6]), the scattering map C_{v^g} allows for a reconstruction of the pair $(SM, \mathcal{F}(v^g))$, up to a stratification-preserving homeomorphism of SM which is the identity on $\partial(SM)$. Therefore, with the help of the involution $\tau : D(SM) \to D(SM)$, we get also a reconstruction of the stratified topological type of the double $D(SM)$.

The localized Poincaré Duality operators $\mathcal{L}_j(g;\mathbb{R})$ and $\mathcal{M}_j(g;\mathbb{R})$ also depend only on the $\mathcal{S}^{\bullet}_{v^g}(D(SM))$-stratified and $\mathcal{S}^{\bullet}_{v^g}(SM^\circ)$-stratified topological types of $D(SM)$ and SM°, respectively. As a result, $\mathcal{L}_j(g;\mathbb{R})$ and $\mathcal{M}_j(g;\mathbb{R})$ can be recovered from C_{v^g}.

The semi-norm $\|\sim\|_{\mathbf{\Delta}}$ of a given homology class is an invariant of the topological type of the underlying space. Therefore (in accordance with Theorem 3.5 from [K5]), the semi-norms $\|\sim\|_{\mathbf{\Delta}}$ on the source spaces of $\mathcal{L}_j(g;\mathbb{R})$ and $\mathcal{M}_j(g;\mathbb{R})$ can be recovered from C_{v^g}. The norm $\|\sim\|^{\bullet}_{\mho}$ on the target spaces is defined also solely in terms of the stratified topological types of $D(SM)$ and SM°, and therefore, by Theorem 3.3 from [K5], depends on the scattering map C_{v^g} only. □

We are in position to state our main result.

Theorem 3.4. *Let (M, g) be a compact connected smooth n-manifold with boundary, where $n \geq 3$. Let $j \in [0, n]$.*

- *Assume that, for each connected component of the boundary ∂M, the image of its fundamental group in $\pi_1(M)$ is amenable.*

  Then, for any traversally generic Riemannian metric g on M, the number of $(2n-1-j)$-dimensional connected components in the stratification $\{SM^\circ(v^g, \omega)\}_\omega$ is greater than or equal to $\mathrm{rank}\big(H_j^{\mathbf{\Delta}}(M)\big)$.
- *Assume that, for each connected component of the boundary ∂M, the image of its fundamental group in $\pi_1(DM)$ is amenable.*

  Then, for any traversally generic Riemannian metric g on M, the number of $(2n-1-j)$-dimensional connected components in the stratification $\{D(SM)(v^g, \omega)\}_\omega$ is greater than or equal to $\mathrm{rank}\big(H_j^{\mathbf{\Delta}}(DM)\big)$.

Proof. By Theorem 3.2, the kernel $\ker(\mathcal{L}_j(g))$ is contained in the subspace

$$H_j^{\{\|\sim\|=0\}}(D(SM);\mathbb{R}) \subset H_j(D(SM);\mathbb{R}).$$

Therefore, employing (3.5), we get

$$\mathrm{rank}(H_j^{\mathbf{\Delta}}(D(SM);\mathbb{R}) \leq \mathrm{rank}\big(H_j(D(SM);\mathbb{R})\big/\ker(\mathcal{L}_j(g)\big) = \mathrm{rank}(\mathrm{im}\big(\mathcal{L}_j(g)\big)$$

$$\leq \mathrm{rank}\big(C^{2n-1-j}_{\mho}(D(SM), g)\Big/B^{2n-1-j}_{\mho}(D(SM), g)\big) \leq \mathrm{rank}\big(C^{2n-1-j}_{\mho}(D(SM), g)\big),$$

the number of codimension j connected components in the stratification $\{D(SM)(v^g, \omega)\}_\omega$. Similarly,

$$\mathrm{rank}(H^{\mathbf{\Delta}}_j(SM; \mathbb{R}) \leq \mathrm{rank}\big(C^{2n-1-j}_{\mho}(SM, g)\big),$$

the number of codimension j connected components in the stratification $\{SM^\circ(v^g, \omega)\}_\omega$.

Using that, by Theorem 3.2, $p_* : H^{\mathbf{\Delta}}_j(SM) \approx H^{\mathbf{\Delta}}_j(M)$ (for $j \geq n$, both spaces are trivial) and $q_* : H^{\mathbf{\Delta}}_j(D(SM)) \approx H^{\mathbf{\Delta}}_j(DM)$ (for $j > n$, both spaces are trivial) are isometries, the claim follows. □

The next corollary spells out the claims of Theorem 3.4 by applying the formula on the bottom of page 537 in [K3] to the settings of the theorem.

Corollary 3.2. *Let M be a compact connected smooth n-manifold with boundary, and g a traversally generic metric on M. Let $n \geq 3$.*

- *Assume that, for each connected component of the boundary ∂M, the image of its fundamental group in $\pi_1(M)$ is amenable.*

  Then, for each $j \in [0, n-1]$, the space of geodesics $\mathcal{T}(v^g)$ has the property

$$\sum_{\omega \in \mathbf{\Omega}^\bullet_{'\langle 2n-2]} \big| \, |\omega|'=j} \mathrm{sup}(\omega) \cdot \#\Big(\pi_0\big(\mathcal{T}(v^g, \omega)\big)\Big) \geq \mathrm{rank}\big(H^{\mathbf{\Delta}}_j(M)\big),$$

  where $\mathcal{T}(v^g, \omega)$ stands for the family of geodesics $\gamma \subset M$ whose intersections $\gamma \cap \partial M$ generate the combinatorial tangency pattern ω, $\mathrm{sup}(\omega) =_{\mathsf{def}} |\omega| - |\omega|'$, and $\pi_0(\sim)$ denotes the set of path connected components of the appropriate space.

- *Assume that, for each connected component of the boundary ∂M, the image of its fundamental group in $\pi_1(DM)$ is amenable.*

  Then, for each $j \in [0, n]$, the space of geodesics $\mathcal{T}(v^g)$ has the property

$$\sum_{\omega \in \mathbf{\Omega}^\bullet_{'\langle 2n-2]} \big| \, |\omega|'=j} \mathrm{sup}(\omega) \cdot \#\Big(\pi_0\big(\mathcal{T}(v^g, \omega)\big)\Big) +$$

$$+ \, 2 \cdot \sum_{\hat\omega \in \mathbf{\Omega}^\bullet_{'\langle 2n-2]} \big| \, |\hat\omega|'=j+1} (\mathrm{sup}(\hat\omega) - 1) \cdot \#\Big(\pi_0\big(\mathcal{T}(v^g, \omega)\big)\Big) \geq \mathrm{rank}\big(H^{\mathbf{\Delta}}_j(DM)\big). \qquad \diamondsuit$$

Example 3.2. Consider a collection $\{\Sigma'_k\}_{k \in [1,s]}$ of closed surfaces of genera $\mathsf{g}_k \geq 2$. Let $N'_k = \Sigma'_k \times S^2$. We denote by N' the connected sum of all N'_k's. Let M' be the compact 4-dimensional manifold, obtained from N' by removing a number of smooth 4-balls D^4 and solid tori of the form $T^2 \times D^2$ and $S^1 \times D^3$, residing in N'. We assume that these domains do not intersect the surfaces $\{\Sigma'_k \times pt_k\}$, where the points $pt_k \in S^2$. Finally, let M be any smooth compact 4-dimensional manifold which is homotopy equivalent to M' and such that $\partial M = \partial M'$. Let $H : M \to M'$ be this homotopy equivalence. We denote by Σ_k be the H-preimage of Σ'_k. We may assume that H is transversal to $\coprod_{k=1}^s (\Sigma'_k \times pt_k) \subset M'$.

Then the hyperbolicity of the Σ'_k's implies that

$$\mathrm{rank}\big(H^{\mathbf{\Delta}}_2(SM)\big) \geq s \;\text{ and }\; \mathrm{rank}\big(H^{\mathbf{\Delta}}_2(DM)\big) \geq 2s.$$

By Theorem 3.4, for any traversally generic metric g on M, the number of 5-dimensional connected components in the stratification $\{SM^\circ(v^g, \omega)\}_\omega$ of SM° is s at least. These are connected components of the strata, indexed by ω's with the property $|\omega|' = 2$. Such ω's belong to the list $\{(1221), (13), (3,1)\}$.

Let us rephrase these conclusions in terms of M. For any traversally generic metric g on M, there exists at least s distinct four-dimensional families of geodesics γ in M, such that either γ is quadratically tangent to ∂M at a pair of distinct points, or γ has a single tangency to ∂M of order 3. Each family is continuously parametrized by an open smooth 4-manifold. Different families do not share the same geodesics.

Next, let us interpret the claims of Theorem 3.2 in this setting. Take, for example, the 2-cycle $h = \sum_{k=1}^{s}[\Sigma_k]$. Its simplicial norm $\|h\|_\Delta = \sum_{k=1}^{s}(2\mathsf{g}_k - 2)$.

For an universal constant $\mu \geq 1$ and any traversally generic metric g on M, the number of 5-dimensional connected components in the stratification $\{SM^\circ(v^g, \omega)\}_\omega$ of SM, where $\omega = (1221), (13), (3,1)$, is greater than or equal to $\mu^{-1} \cdot \sum_{k=1}^{s}(2\mathsf{g}_k - 2)$. Although estimating μ from above may be challenging, at least we know the rate of growth of the number of 5-dimensional connected components in the $\mathcal{S}_{v^g}(SM^\circ)$-stratification, as $s \to \infty$ or as individual genus $\mathsf{g}_k \to \infty$. $\diamondsuit$

Revisiting Definition 2.3, the formulas from Corollary 3.2 have an instant implication.

Corollary 3.3. *Under the hypotheses of Theorem 3.4, the non-triviality of the groups $H_j^{\Delta}(M)$, where $j \in [1, n-1]$, and/or $H_j^{\Delta}(DM)$ where $j \in [1, n]$, represents an obstruction to the existence of a globally j-convex and traversally generic metric on M.* $\diamondsuit$

The assumption that a metric g on a given manifold M is traversally generic, perhaps, could be relaxed. To extend all the results of this paper to the broader class of boundary generic and traversing geodesic flows, we need to consider only such geodesic flows that match a *finite list* of a priori fixed semi-local models (in the spirit of (2.5)) of the vicinity of every v^g-trajectory $\tilde{\gamma}$. Perhaps, these models could be determined not only by the combinatorics of tangency (like ω) of $\tilde{\gamma}$ to $\partial(SM)$, but also by some continuous parameters.

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References

[AK] Allpert, H., Katz, G., *Using Simplicial Volume to Count Multi-tangent Trajectories of Traversing Vector Fields*, Geometriae Dedicata, DOI 10.1007/s10711-015-0104-6. (arXiv:1503.02583v1 [math.DG] (9 Mar 2015)).

[Gr] Gromov, M., *Volume and bounded cohomology* Publ. Math. I.H.E.S., tome 56 (1982), 5-99.

[Gr1] Gromov, M., *Singularities, Expanders and Topology of Maps. Part I: Homology versus Volume in the Spaces of Cycles*, Geom. Funct. Anal. vol 19 (2009), 743-841.

[Hat] Hatcher, A., *Algebraic Topology*, Cambridge University Press, 2002.

[K1] Katz, G., *Stratified Convexity & Concavity of Gradient Flows on Manifolds with Boundary*, Applied Mathematics, 2014, 5, 2823-2848, http://www.scirp.org/journal/am

[K2] Katz, G., *Traversally Generic & Versal Flows: Semi-algebraic Models of Tangency to the Boundary*, Asian J. of Math., vol. 21, No.1 (2017), 127-168.

[K3] Katz G., *Complexity of shadows and traversing flows in terms of the simplicial volume*, Journal of Topology and Analysis, Vol. 8, No. 3 (2016), 501-543.

[K4] Katz G., *Causal Holography of Traversing Flows*, to appear in Journal of Dynamics and Differential Equations (arXiv:1409.0588v4 [mathGT] 16 Aug 2020).

[K5] Katz G., *Causal Holography in Application to the Inverse Scattering Problem*, Inverse Problems & Imaging, 2019, 13 (3) : 597-633. doi: 10.3934/ipi.2019028 (arXiv: 1703.08874v3 [Math.GT], 7 May 2018).

[K6] Katz G., *The Ball-Based Origami Theorem and a Glimpse of Holography for Traversing Flows*, Qual. Theory Dyn. Syst. 19, 41 (2020). https://doi.org/10.1007/s12346-020-00364-7.

[K7] Katz G., *Morse Theory of Gradient Flows, Concavity and Complexity on Manifolds with Boundary*, World Scientific (2020), ISBN 978-981-4368-75-9.

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