MAXIMAL HILBERT SERIES OF QUADRATIC-RELATOR ALGEBRAS

by

DEBRA KAY BORKOVITZ

B.S. Mathematics, University of Illinois, Urbana (1984)
B.S. Computer Science, University of Illinois, Urbana (1984)

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Signature of Author ______________________
Department of Mathematics
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Certified by ______________________
David Anick, Mathematics Professor
Thesis Supervisor

Accepted by ______________________
Alar Toomre, Chairperson
Mathematics Committee

Accepted by ______________________
Sigurdur Helgason, Chairperson
Departmental Graduate Committee

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Abstract

If $H(z) = \sum_{n=0}^{\infty} h_n z^n$ is a formal power series, with the h_i's non-negative integers, there are simple and elegant conditions for $H(z)$ to be the Hilbert series of a commutative graded algebra with g generators. This thesis represents a first attempt to describe conditions characterizing when $H(z)$ can be the Hilbert series of a non-commutative graded algebra with a prescribed number of generators and relators.

We study the problem of finding Hilbert series which are maximal with respect to each of three different orderings: the coefficient-wise partial ordering, the lexicographic total ordering, and an ordering by growth. We focus on quadratic-relator monomial algebras.

For quadratic-relator algebras with g generators and r linearly independent relators, we give a maximal Hilbert series for the total ordering. To accomplish this, we need to arrange $t = g^2 - r$ ones and r zeroes in a g by g matrix M so that the sum of the entries in M^2 is maximized. This matrix problem has a surprising answer. There are two different strategies, one for $t < g^2/2 - g$ and another for $t > g^2/2 + g$. We define an "L-Matrix" and a "Q-Matrix" and show that one of them is always maximal.

We also examine the other two orderings. For the coefficient-wise ordering, we find a number of differences between the non-commutative and commutative cases. For the growth ordering, our Q-Matrix turns out to coincide with a matrix discovered by Brualdi, Hoffman, and Friedland which maximizes spectral radius. Their results are relevant to ours and vice versa.

Thesis Supervisor: David Anick
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Finally, I dedicate this thesis to the person who finds her or himself reading acknowledgement sections, fearing that s/he will never accomplish anything. Others have come before you.

Contents

Chapter 1

Overview and Preliminaries

1.1 Introduction

This thesis began with a simple, but general question: What do Hilbert series of non-commutative graded algebras look like? The question was intended to continue Anick's work [see An.1, An.2, and An.3] on developing a theory of Hilbert series of non-commutative graded algebras. The development of this theory started with trying to find non-commutative analogs for results about Hilbert series of commutative graded algebras. This thesis began similarly.

There is a particularly simple and elegant description of when a formal power series can be a Hilbert series of a commutative graded algebra with a given number of generators. We review this theory in Chapter 4. In order to explore the possibility of a parallel theory for non-commutative graded algebras, this research began with what seemed to be the simplest kinds of non-commutative graded algebras: quadratic-relator algebras. It quickly became clear that even in with this constraint, there was no exact analogy and no clearcut way to describe all possible Hilbert series.

A subsidiary problem, to determine the *maximal* Hilbert series that can occur for a presentation with g generators and r linearly independent quadratic relators, led to a combinatorial problem about $(0, 1)$-matrices, whose solution forms the bulk of this dissertation. We apply this combinatorial problem and other results toward finding maximal Hilbert series.

One subtlety is that there are three reasonable concepts of *maximal*, depending on whether one orders Hilbert series using a lexicographic total ordering, a coefficient-wise partial ordering, or an ordering by growth. These orderings are defined in this chapter and explored more thoroughly in Chapter 4.

Also in Chapter 1, we explain how the Hilbert series problem leads to the combinatorial matrix problem, which is solved completely in Chapter 2. The problem is as follows: Given g and r, how should we arrange $t = g^2 - r$ ones and r zeroes in a $g \times g$ matrix M so that the sum of the entries in M^2 is maximized (we call this value the ***square-sum*** and call a matrix which maximizes it *maximal*). There are complementary strategies. Asymptotically, for $t < g^2/2$ arranging the ones in an L-shape is maximal, and for $t > g^2/2$ arranging the ones in a square shape is maximal. (Note that if the ones form an L-shape the zeroes form a square and vice versa).

At the beginning of Chapter 2 we define the L-Matrix and the Q-Matrix, discrete analogs to the asymptotic L and square shapes. We show that for $t < g^2/2 - g$, the square sum of the L-Matrix is greater than or equal to that of the Q-Matrix. We use a simple but powerful induction step, derived from matrix multiplication, and a complementarity theorem, to show that for $t < g^2/2 - g$, the L-Matrix is always maximal, for $t > g^2/2 + g$, the Q-Matrix is always maximal, and for $g^2/2 - g \leq t \leq g^2/2 + g$ either the L- or the Q- Matrix is maximal. The proof is quite intricate.

In Chapter 3 we compute the entire Hilbert series for algebras corresponding to L- and Q- Matrices. We also explore further terms in the series and maximal Hilbert series under different conditions.

In Chapter 4 we compare the commutative and non-commutative cases. We also explore the growth ordering. Our Q-Matrix coincides with several different results by Brualdi, Hoffman, and Friedland on maximizing the spectral radius of (0,1)-matrices. Our results yield possible explanations for seeming anomolies in their results.

1.2 Background

Throughout this thesis, we let F be a field, $\mathbf{N}$ the non-negative integers, and $\mathbf{Z}_+$ the positive integers.

An $\mathbf{N}$-*graded* F-*algebra* A is an associative F-algebra with identity and a direct sum decomposition, $A = \oplus_{n=0}^{\infty} A_n$, such that $A_a A_b \subseteq A_{a+b}$ for all $a, b \in \mathbf{N}$. We say A is *connected* iff $A_0 = F$. We call A *locally finite* iff each

A_n is a finite dimesional vector space over F. In this thesis, all algebras are $\mathbf{N}$-graded, connected, and locally finite. A non-zero element $x \in A$ is *homogeneous* iff $x \in A_n$ for some n, and we say that n is the *dimension* of x.

If U is any set, we let $F\langle U\rangle$ denote the free associative algebra over F with multiplicative basis U. As usual, $F[U]$ denotes the (commutative) ring of polynomials in U over F. The free monoid B generated by U is a basis for $F\langle U\rangle$ as an F-module. If $e : U \to \mathbf{Z}_+$ is a function which assigns a positive integer to each member of U, we call (U, e) a *graded set.* For $x \in U$, $e(x)$ is called the *degree* or *dimension* of x. We can extend e uniquely to a map of monoids, $e : B \to \mathbf{N}$, and $F\langle U\rangle \approx \text{Span(B)}$ as a graded algebra.

Let A be a graded-algebra with presentation $A = F\langle U\rangle/\langle I\rangle$, where (U, e) is a graded set and $\langle I\rangle$ is a two-sided ideal in $F\langle U\rangle$ generated by a set I of homogeneous elements. Let A inherit the grading from e. If $e(x) = 1$ for all $x \in U$, we say that A is *degree one generated* (or d.o.g.), and A is graded by word length. If $e(\beta) = 2$ for all $\beta \in I$, we say that A is a *quadratic-relator* algebra. If $I \subseteq B$, we call A a *monomial* algebra. In this thesis, all algebras are d.o.g. and quadratic-relator, and we will primarily be studying monomial algebras.

We define the *Hilbert series* $H_A(z)$ of A to be $H_A(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = \text{rank}_F(A_n)$. We view the Hilbert series as a formal power series, and we define three different orderings on them.

Let $B(z) = \sum_{n=0}^{\infty} b_n z^n$ and $C(z) = \sum_{n=0}^{\infty} c_n z^n$. We write $B(z) \ll C(z)$ iff $b_n \leq c_n$ for all n. We call $\ll$ the *coefficient-wise* partial ordering. Note that $H_B(z) \gg H_C(z)$ whenever C is a (graded) quotient algebra of B.

The lexicographic ordering on power series is given by: $B(z) < C(z)$ if for some n we have $b_i = c_i$ for $i < n$, and $b_n < c_n$. The lexicographic ordering extends the coefficient-wise ordering to a total order, and we will sometimes call it the *total* ordering.

For the formal power series $B(z)$, we define

$$\gamma(B) = \limsup_{n\to\infty} |b_n|^{\frac{1}{n}}$$

We call $\gamma(B)$ the *growth* of $B(z)$. In [An.4 Theorem 2] it is proved that for a non-commutative finitely generated graded algebra A (over a field) with Hilbert series $H_A(z)$, $\gamma(H_A)$ exists and equals the inverse of the radius of convergence of H_A. For further discussion of growth, see [Lor]. We write

$B(z) <_g C(z)$ if $\gamma(B) < \gamma(C)$ and call $<_g$ the *growth* ordering. The growth ordering is an ordering of equivalence classes of power series.

We define two classes of quadratic-relator algebras: $\mathcal{A}_{g,r} = \{A|A$ has a presentation with g degree one generators and r linearly independent quadratic relators$\}$ and $\mathcal{A}^{\mathrm{m}}_{g,r} = \{A|A$ has a presentation with g degree one generators and r distinct quadratic monomial relators$\}$. We let $\mathcal{H}_{g,r}$ and $\mathcal{H}^{\mathrm{m}}_{g,r}$ be the corresponding sets of Hilbert series.

In this thesis, we discuss quadratic-relator algebras whose Hilbert series are maximal with respect to each of the three orderings. We will generally fix the number of generators and relators. Most of our discussion will center on monomial algebras, because of the following.

Theorem 1.1 *(a) Let $A \in \mathcal{A}_{g,r}$. There exists $B \in \mathcal{A}^{\mathrm{m}}_{g,r}$ such that $H_B(z) \gg H_A(z)$.*
(b) Suppose $H(z) \in \mathcal{H}_{g,r}$ is maximal with respect to the total ordering. Then $H(z) = H_A(z)$ for some $A \in \mathcal{A}^{\mathrm{m}}_{g,r}$.

Proof: Let $A = F\langle x_1 \dots x_g\rangle/\langle \beta_1 \dots \beta_r\rangle$. In [An.3 Section 3], the associated monomial ring of A, which we call A', is defined. We have $H_A(z) = H_{A'}(z)$. A' is a d.o.g. monomial algebra. It can be assumed, by using row reduction if necessary, that the high terms of the $\{\beta_j\}$ are all distinct. Let X denote the set of these high terms; it is a set of quadratic monomials. The algebra A' has a presentation as $F\langle x_1 \dots x_g\rangle/\langle X \cup Y\rangle$, where Y is a set of monomials disjoint from X. If we let $B = F\langle x_1 \dots x_g\rangle/\langle X\rangle$, then A' is a quotient of B, and clearly $H_B(z) \geq H_{A'}(z) = H_A(z)$ and we have proven both parts of the theorem.

Theorem 1.2 *If (U,e) is a finite graded set and $A = F\langle U\rangle/\langle I\rangle$ where I is a two-sided ideal generated by a finite number of monomials, then $H_A(z)$ is rational.*

Proof: See [Gov].

For an algebra A with rational Hilbert series $H_A(z) = N(z)/D(z)$, where $N(z)$ and $D(z)$ are relatively prime polynomials in $\mathbf{Z}[z]$, we can use partial fractions to show that $\gamma(H_A)$ is equal to $1/z_0$, where z_0 is the smallest positive root of $D(z)$.

We now define some graphs and matrices which will be used throughout the sequel. These concrete combinatorial definitions arise from specializations of more abstract homological structures in [An.3].

Fix a set I of monomials of degree 2 in $U = \{x_1 \ldots x_g\}$. Let B be the free monoid generated by U. We define an ***n*-chain** to be a word w of length $n+1$ such that if $x_i x_j$ is a subword of w, then $x_i x_j \in I$. We define the only ***0-chain*** to be the element $1 \in B$. We call $w \in B$ a ***normal word*** if no subword belongs to I.

If $A = F\langle x_1 \ldots x_g \rangle / \langle I \rangle$ is a quadratic-relator monomial algebra, we associate two matrices and a graph with A. The ***normal words matrix*** for A, $M_A = [m_{ij}]$, is defined by $m_{ij} = 1$ if $x_i x_j$ is a normal word in A, and $m_{ij} = 0$ otherwise. The ***relator matrix*** for A, $M'_A = [m'_{ij}]$, is defined by $m'_{ij} = 1$ if $x_i x_j \in I$ and $m'_{ij} = 0$ otherwise. The ***graph*** associated with A, G_A, is a directed graph with vertices $v_1 \ldots v_g$ such that there is and edge from v_i to v_j iff $x_i x_j$ is a normal word in A. M_A is the adjacency matrix for G_A.

We let $\mathbf{J}_g$ represent the $g \times g$ matrix of ones, and note that $M_A + M'_A = \mathbf{J}_g$, because for every word w of degree two in the free monoid B, we have either $w \in I$ or w is normal.

We let $S(M)$ denote the sum of the entries in a $g \times g$ matrix M, i.e. $S(M) = v^t M v$ where v is a column vector of g ones.

We let the *length* of a path in a directed graph be the number of edges. If $x_{t_1} x_{t_2} \ldots x_{t_n}$ is a normal word in A, then there are edges from v_{t_i} to $v_{t_{i+1}}$ in G_A for all $i < n$, and there is a path $v_{t_1} \to v_{t_2} \to \cdots \to v_{t_n}$ in G_A. If $x_{s_1} x_{s_2} \ldots x_{s_n}$ is not a normal word in A, then there is some $i < n$ such that $x_{s_i} x_{s_{i+1}} \in I$, and there is no edge from v_{s_i} to $v_{s_{i+1}}$ in G_A. Thus $v_{s_1} \to v_{s_2} \to \cdots \to v_{s_n}$ is not a path in G_A. Hence, the number of normal words of length n equals the number of paths of length $n-1$ in G_A. This number equals $S(M_A^{n-1})$. We have proved the important equation,

$$H_A(z) = 1 + gz + \sum_{n=2}^{\infty} S(M_A^{n-1}) z^n. \tag{1.1}$$

Chapter 2

Maximal Matrices

Using (1.1) we can compute the Hilbert series of quadratic monomial algebras by computing powers of a (0,1)-matrix. In this chapter, we work only in the context of matrices. We look at the squares of (0,1)-matrices, in order to find maximal Hilbert series with respect to the total ordering. Our results on matrices are applicable for both normal word and relator matrices, although the normal words matrix is the one used to find maximal Hilbert series.

In this chapter, a (g,t)-*matrix* is defined to be a $g \times g$ matrix M with t entries equal to one and $g^2 - t$ entries equal to zero. We call $S(M^2)$ the *square-sum* of M. A (g,t)-matrix M is *maximal* if $S(M^2) = max\{S(A^2) | A$ is a (g,t)-matrix $\}$. We will use notation R_i to refer to the sum of the elements in the i^{th} row of a matrix and C_j to refer to the sum of the elements in the j^{th} column of a matrix.

In the first section, we define two (g,t)-matrices, the L-Matrix and the Q-Matrix, and we compare their square-sums. We focus on matrices with fewer ones than zeroes, giving a range where square-sum of the L-Matrix is always greater than the square sum of the Q-Matrix.

In the second section, we define two conditions, compression and almost-symmetry, and show that some maximal matrix must satisfy both of them. We then use a complementarity argument and two kinds of induction to prove our main theorem, that the L-Matrix or the Q-Matrix is always maximal.

2.1 The L-Matrix and the Q-Matrix

2.1.1 Definitions and Evaluation

Let $g \geq 1$ and $1 \leq t \leq g^2$. We define the *L-Matrix* for (g, t), denoted $L_{g,t} = [l_{ij}]$, recursively as follows:

$$L_{3,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} L_{3,2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} L_{3,3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_{3,4} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} L_{3,5} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} L_{3,6} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$L_{3,7} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} L_{3,8} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} L_{3,9} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

For $g < 3$, $L_{g,t} =$ the upper left corner of $L_{3,t}$.

For $g > 3$ and $t \leq 2g - 1$, let $l_{1i} = 1$ for $i \leq \lceil \frac{t+1}{2} \rceil$ and $l_{1i} = 0$ otherwise. Let $l_{j1} = 1$ for $j \leq \lceil \frac{t}{2} \rceil$ and $l_{j1} = 0$ otherwise, where $\lceil y \rceil$ refers to the smallest integer n such that $n \geq y$. For $i > 1$ and $j > 1$, let $l_{ij} = 0$.

For $g > 3$ and $t > 2g - 1$, let $L_{g,t}$ be the $g \times g$ matrix whose first row and column consist of 1's and whose (1,1) minor is $L_{g-1,t-(2g-1)}$. For example, we have:

$$L_{7,36} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} L_{7,37} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition: An L-Matrix $L_{g,t}$ is said to be *exact* if there is a positive integer c such that $t = 2cg - c^2$.

The pattern of 1's formed by an exact L-Matrix is an L-Shape of length g and width c, e.g. $L_{3,5}$ (c=1), $L_{3,8}$ (c=2), and $L_{7,33}$ (c=3). Note that $L_{7,36}$ consists of an exact L-Matrix plus 3 other entries, which are arranged in an L-shape. $L_{7,37}$ consists of an exact L-Matrix plus 4 other entries which are arranged in a square shape. In general, L-Matrices consist of exact L-Matrices, plus entries in an L-shape. The case where there are 4 extra entries is an exception, and the existence of this exception will prove important in Chapter 4.

Definition: For a $g \times g$ matrix $M = [m_{ij}]$, let $M^{\mathrm{R}} = [m'_{ij}]$ be defined by $m'_{ij} = m_{g+1-j,g+1-i}$.

M^{R} is found by reflecting M about the top-right to bottom left diagonal. We now define the Q-Matrix.

The Q-Matrix: Let $Q_{g,t} = (\mathbf{J}_g - L_{g,g^2-t})^{\mathrm{R}}$.

$Q_{g,t}$ is found by exchanging the ones and zeroes in L_{g,g^2-t} and reflecting about the top-right to bottom left diagonal. For example we have:

$$Q_{7,13} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad Q_{7,12} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition: A Q-Matrix $Q_{g,t}$ is *exact* if there is an integer m such that $t = m^2$.

Again the motivation for this term is visual: the pattern of ones forms a perfect square in the upper left corner. Note that $Q_{7,13}$ represents a "general" non-exact Q-Matrix. It consists of ones arranged in a perfect square (in this case 3×3) with extra ones divided in half (if possible) and placed on the right and bottom sides of the square. $Q_{7,12}$ represents the exception corresponding to the exception in the L-Matrices. It consists of a perfect square (in this case 4×4) of ones, except for the four zeroes which comprise the bottom right 2×2 corner.

We now compute $S(M^2)$ when M is an L-Matrix or a Q-Matrix.

Lemma 2.1 *For any matrix, M, $S(M^2) = \sum_{i=1}^{g} R_i C_i$.*

Proof: $\sum_i R_i C_i = \sum_i (\sum_j m_{ij})(\sum_k m_{ki}) = \sum_i \sum_j \sum_k m_{ki} m_{ij} = S(M^2)$

Theorem 2.2 *Given an L-Matrix $L_{g,t}$, let c be the unique positive integer such that*

$$2cg - c^2 \leq t < 2(c+1)g - (c+1)^2$$

Let $d = t - (2cg - c^2)$. Then

$$S(L_{g,t}^2) = c^2 g + cg^2 - c^3 + 2cd + d^2/4 + d - \epsilon \tag{2.1}$$

where

$$\epsilon = \begin{cases} 0 & d = 0 \text{ or } d = 4 \\ 1 & d \text{ even, } d \neq 0, 4 \\ 1/4 & d \text{ odd.} \end{cases}$$

Proof: We think of d as measuring the deviation of $L_{g,t}$ from an exact L. First, suppose d is odd. Then for $t \leq 2g - 1$, $L_{g,t}$ is obviously symmetric. If $t > 2g-1$, then the deviation for $L_{g-1,t-(2g-1)}$ is also d, so by induction, if d is odd, $L_{g,t}$ is symmetric. (See Figure 2.1) Hence, $R_i = C_i$ for all i and we have:

$$R_i = \begin{cases} g & i \leq c \\ c + (d+1)/2 & i = c+1 \\ c+1 & c+1 < i \leq c + (d+1)/2 \\ c & c + (d+1)/2 < i \leq g \end{cases}$$

Using Lemma 2.1, we have $S(L_{g,t}^2) = \sum_{i=1}^{g} (R_i)^2$ which simplifies to (2.1). The cases $d = 0$; $d = 4$; and d even, $d \neq 0$, $d \neq 4$, are each computed similarly.

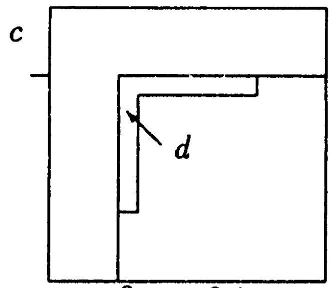

figure 2.1

Corollary: If $L_{g,t}$ is exact, $S(L^2_{g,t}) = 2tg - g^3 + (g^2 - t)^{3/2}$.

Proof: The exact case has $d = 0$ and $c = g - \sqrt{g^2 - t}$.

Definition: $E_L(g,t) = 2tg - g^3 + (g^2 - t)^{3/2}$. $E_L(g,t)$ is called the *Continuous L-approximation.*

The idea behind $E_L(g,t)$ is that it computes what $S(L^2_{g,t})$ would be if the total "mass" of t ones could be distributed into a perfect L-Shape. It agrees with $S(L^2_{g,t})$ in exact cases, and as we shall see provides a convenient upper bound for $S(L^2_{g,t})$ in all cases.

Theorem 2.3 *Given a Q-Matrix, $Q_{g,t}$, let m be the unique positive integer such that $m^2 \leq t < (m+1)^2$, and let $s = t - m^2$. Then*

$$S(Q^2_{g,t}) = m^3 + ms + s^2/4 + s/2 - \epsilon' \tag{2.2}$$

where

$$\epsilon' = \begin{cases} -1/4 & s = 2m - 3 \\ 0 & s \text{ even} \\ 3/4 & s \text{ odd}, s \neq 2m - 3. \end{cases}$$

Proof: See Figure 2.2. The proof of this theorem is similar to the proof of Theorem 2.2.

Corollary: If $Q_{g,t}$ is exact, $S(Q^2_{g,t}) = t^{3/2}$.

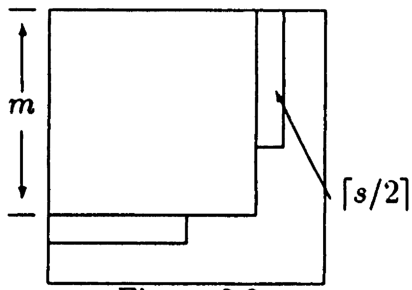

Figure 2.2

Proof: The exact case has $m = \sqrt{t}$ and $s = 0$.

Definition: $E_Q(g,t) = t^{3/2}$. $E_Q(g,t)$ is called the *Continuous Q-approximation.*

The motivation for defining $E_Q(g,t)$ is similar to the motivation for $E_L(g,t)$. Note that $E_Q(g,t)$ doesn't depend on g.

We now prove that for all g and t, $S(L^2_{g,t}) \leq E_L(g,t)$ and $S(Q^2_{g,t}) \leq E_Q(g,t)$. These continuous functions will be used as upper bounds when we compare the L- and Q- Matrices. We first show $S(Q^2_{g,t}) \leq E_Q(g,t)$ and then derive an identity from which the other inequality follows.

Lemma 2.4 *For all Q-Matrices,* $S(Q^2_{g,t}) \leq E_Q(g,t)$.

Proof: When $s = 0$, $S(Q^2_{g,t}) = E_Q(g,t)$. When $s = 1$, we have:

$$S(Q^2_{g,t}) = m(m^2+1) < (m^2+1)^{1/2}(m^2+1) = E_Q(g,t)$$

When $m = 2$ the theorem is easily verified by direct calculation. Thus, for the remainder of the proof, we assume $m \geq 3$ and $2 \leq s \leq 2m$.

For m fixed, the quantity $s/m^2 + 1/s$ is maximized either at $s = 2$ or $s = 2m$. In either case, $s/m^2 + 1/s < 1$ and we have

$$1 + s/m^2 < 2 - 1/s$$

and

$$(s/m^2)(m^2+s) < 2s - 1.$$

Since $s/(2m) \leq 1$, we have

$$(s^2/(2m^3))(m^2+s) \leq (s/m^2)(m^2+s) < 2s - 1$$

and

$$-(s^2/(8m^3))(m^2+s) > (1-2s)/4. \qquad (2.3)$$

We use

$$(2m-s)(m-2) \geq 0$$

to derive the following equivalent inequality:

$$(1-2s)/4 + (s/(2m))(m^2+s) \geq (s+1)^2/4. \qquad (2.4)$$

We add (2.3) and (2.4) to obtain

$$(1-2s)/4 + (s/(2m))(m^2+s) - (s^2/(8m^3))(m^2+s) > (1-2s)/4 + (s+1)^2/4$$

and

$$(m^2+s)((s/(2m)) - s^2/(8m^3)) > (s+1)^2/4.$$

By adding $m(m^2+s)$ to both sides, we obtain:

$$m(m^2+s)(1+s/(2m^2)-s^2/(8m^4)) > m(m^2+s)+(s+1)^2/4 \geq S(Q^2_{g,t}). \qquad (2.5)$$

By Taylor Series expansion, we have

$$(m^2+s)^{1/2} = m(1+s/m^2)^{1/2} \geq m(1+s/(2m^2)-s^2/(8m^4)),$$

which implies

$$E_Q(g,t) = (m^2+s)^{3/2} \geq m(m^2+s)(1+s/(2m^2)-s^3/(8m^4)).$$

Using (2.5), we have

$$S(Q^2_{g,t}) < E_Q(g,t).$$

Lemma 2.5 $E_L(g,t) - S(L^2_{g,t}) = E_Q(g,g^2-t) - S(Q^2_{g,g^2-t})$.

Proof: We observe that

$$E_Q(g,g^2-t) = (g^2-t)^{3/2} = E_L(g,t) - 2gt + g^3. \qquad (2.6)$$

Next, we note that

$$\begin{aligned} S(Q^2_{g,g^2-t}) &= S((\mathbf{J}_g - L_{g,t})^2) \\ &= S(\mathbf{J}_g^2 - \mathbf{J}_g L_{g,t} - L_{g,t}\mathbf{J}_g + L^2_{g,t}) \\ &= g^3 - 2gS(L_{g,t}) + S(L^2_{g,t}) \\ &= g^3 - 2gt + S(L^2_{g,t}) \end{aligned} \qquad (2.7)$$

Combining (2.6) and (2.7), we have

$$S(Q^2_{g,g^2-t}) - S(L^2_{g,t}) = E_Q(g, g^2 - t) - E_L(g,t),$$

and we are done.

Lemma 2.6 *For all L-Matrices,* $S(L^2_{g,t}) \leq E_L(g,t)$.

Proof: By Lemma 2.4 we have for all $0 \leq t < g^2$,

$$E_Q(g, g^2 - t) - S(Q^2_{g,g^2-t}) \geq 0.$$

By Lemma 2.5, we must also have $E_L(g,t) - S(L^2_{g,t}) \geq 0$.

Lemma 2.7 *Suppose* $0 \leq t \leq g^2$ *and* $t = 2cg - c^2 + d$, *with* c *and* d *as in Theorem 2.2. Let* $g' = g - c$. *Then*

$$E_L(g,t) - S(L^2_{g,t}) = E_L(g', d) - S(L^2_{g',d}).$$

Proof: By definition, $d < 2g' - 1$, and by direct calculation using Theorem 2.1

$$S(L^2_{g,t}) - S(L^2_{g',d}) = c^2 g + cg^2 - c^3 + 2cd.$$

Using the definition of $E_L(g,t)$, we also calculate directly

$$E_L(g,t) - E_L(g', d) = c^2 g + cg^2 - c^3 + 2cd,$$

and the theorem follows easily.

We now find a lower bound for $S(L^2_{g,t})$. In the next section, we will compare this bound to $E_Q(g,t)$ to find a range of values of t for which $S(L^2_{g,t}) \geq S(Q^2_{g,t})$ always holds.

Lemma 2.8 *Write $t = 2gc - c^2 + d$ with $0 \leq d < 2(g-c) - 1$ as in Theorem 2.2 for fixed g and c. Then the quantity $E_L(g,t) - S(L^2_{g,t})$ is maximized at $d = g - c$ or $d = g - c - 1$. Putting $k = g^2/2 - t$ we have*

$$S(L^2_{g,t}) \geq (g^2/2 + k + 1/4)^{3/2} - ((1/8)g^2 + 2gk + (1/4)k + 7/8)$$

and

$$\begin{aligned} S(L^2_{g,t}) \geq (g^2/2 + k - 3/4)^{3/2} &+ (3/2)(g^2/2 + k - 3/4)^{1/2} \\ &- ((1/8)g^2 + 2gk + (1/4)k + 5/8). \end{aligned}$$

Proof: Let $g' = g - c$. If $E_L(g', d) - S(L^2_{g',d})$ is maximized on the interval $0 \leq d < 2(g-c) - 1$ at the integer d_0, then by Theorem 2.7, $t = d_0 + 2gc - c^2$ maximizes $E_L(g,t) - S(L^2_{g,t})$. Thus we first consider $E_L(g', d) - S(L^2_{g',d})$. We differentiate

$$E_L(g', d) - S(L^2_{g',d}) = 2dg' - (g')^3 + ((g')^2 - d)^{3/2} - d^2/4 - d + \epsilon \quad (2.8)$$

with respect to d and set equal to zero, to obtain

$$d_1 = 4g' - 13/2 - 3\sqrt{(g'-2)^2 + 1/4}.$$

The point d_1 is a maximum, and the only critical point on the interval $1 \leq d < 2g' - 1$. We note that $g' - 1 \leq d_1 \leq g'$, and determine by computation that for integral values, $E_L(g', d) - S(L^2_{g',d})$ is maximized at $d_0 = g' = g - c$, if g' is even, $g' \neq 4$, or if $g' = 5$. Otherwise, the quantity is maximized when $d_0 = g' - 1 = g - c - 1$.

We finish our proof by using $t = 2cg - c^2 + g - c$ and then $t = 2cg - c^2 + g - c - 1$ to solve for c and then substituting into (2.1).

2.1.2 A Comparison of $L_{g,t}$ and $Q_{g,t}$

When we use $E_L(g,t)$ and $E_Q(g,t)$, we will generally be regarding g as fixed. If we let $x = t/g^2$, we obtain the functions of one variable:

$$\begin{aligned} E_{L1}(x) &= \lim_{g\to\infty} g^{-3} E_L(g,t) = (2x - 1) + (1 - x)^{3/2} \\ E_{Q1}(x) &= \lim_{g\to\infty} g^{-3} E_Q(g,t) = x^{3/2}. \end{aligned}$$

These functions give an asymptotic continuous analog for squaring L- and Q- matrices. If we think of a continuous analog of a matrix as a square with area one, divided into regions which are labelled 0 or 1, then x is the area of the region labelled 1. The function $E_{L1}(x)$ finds a "continuous square-sum" for an L-shaped region of area x; the function $E_{Q1}(x)$ finds a "continuous square-sum" for a square shaped region of area x.

The functions $E_{L1}(x)$ and $E_{Q1}(x)$ are equal at $x = 1/2$; for $x < 1/2$, $E_{L1}(x) > E_{Q1}(x)$; and for $x > 1/2$, $E_{Q1}(x) > E_{L1}(x)$. Thus in our continuous analogy, we have three ranges: for $x < 1/2$, $S(L^2) > S(Q^2)$; for $x = 1/2$, $S(L^2) = S(Q^2)$; and for $x > 1/2$, $S(Q^2) > S(L^2)$;

The discrete case is not as simple. In this section, we show that for $t < g^2/2 - g$, $S(L^2_{g,t}) \geq S(Q^2_{g,t})$. To do so, we first compute small cases explicitly. We then obtain a crude first bound by comparing the lower bounds on $S(L^2_{g,t})$ from Lemma 2.8 with $E_Q(g,t)$. To obtain our desired bound, we determine relationships between variables within the region between the crude bound and the desired bound. These relationships are restrictive enough to reduce symbolic calculations to two cases only.

Lemma 2.9 *If $t < g^2/2 - g$ and $g \leq 6$, then $S(L^2_{g,t}) \geq S(Q^2_{g,t})$.*

Proof: For $t \leq 5$, by definition we have $L_{g,t} = Q_{g,t}$ for all values of g for which these matrices are defined. For $t = 6$ and $t = 7$, the computations when $g = 5$ are identical to the computations when $g = 6$. We explicitly compute $S(L^2_{g,t})$ and $S(Q^2_{g,t})$ for $g = 6$ and $6 \leq t \leq 11$:

t	$S(L^2_{g,t})$	$S(Q^2_{g,t})$
6	14	14
7	19	17
8	23	22
9	29	27
10	34	30
11	41	35

Lemma 2.10 *If $t = g^2/2 - k$, with $g \geq 7$ and $k \geq \frac{3\sqrt{2}+4}{8}g + 4.5$, then $S(L^2_{g,t}) \geq E_Q(g,t)$.*

Proof: We wish to first show that within the given constraints, the lower bounds on $S(L^2_{g,t})$ from Lemma 2.8 are greater than $E_Q(g,t)$. By computing

Taylor approximations, we have:

$$(g^2/2 + k + 1/4)^{3/2} - (g^2/2 - k)^{3/2} > \frac{3kg}{\sqrt{2}} \quad (2.9)$$

and

$$(g^2/2+k-3/4)^{3/2} + (3/2)(g^2/2+k-3/4)^{1/2} - (g^2/2-k)^{3/2} > \frac{3kg}{\sqrt{2}}. \quad (2.10)$$

We wish to show that for $k \geq \frac{3\sqrt{2}+4}{2}g + 4.5$ we have

$$\frac{3kg}{\sqrt{2}} > (1/8)g^2 + 2gk + (1/4)k + 7/8 \quad (2.11)$$

from which the theorem follows easily.

Let $a = \frac{3}{\sqrt{2}} - 2$, $b = 1/8$, $c = 1/4$, $d = 7/8$, and $\phi = b/a = \frac{3\sqrt{2}+4}{8}$. For $g \geq 7$, we have

$$\frac{bc + \frac{ad}{g}}{a(a - \frac{c}{g})} < 4.5.$$

Adding bg/a to each side, we obtain

$$\frac{bg^2 + d}{ag - c} < \phi g + 4.5 \leq k. \quad (2.12)$$

For $g \geq 3$, we have $ag - c > 0$, so (2.12) is equivalent to

$$bg^2 + d < agk - ck$$

or

$$kag > bg^2 + ck + d,$$

which is equivalent to (2.11).

We combine (2.9) and (2.11) to obtain

$$(g^2/2 + k + 1/4)^{3/2} - (g^2/2 - k)^{3/2} > (1/8)g^2 + 2gk + (1/4)k + 7/8$$

and

$$E_Q(g,t) = (g^2/2-k)^{3/2} < (g^2/2+k+1/4)^{3/2} - ((1/8)g^2+2gk+(1/4)k+7/8) \leq S(L^2_{g,t}).$$

Similarly, by changing 7/8 to 5/8 in (2.11) and combining it with (2.10), we obtain

$$\begin{aligned} E_Q(g,t) &= (g^2/2-k)^{3/2} \\ &< (g^2/2+k-3/4)^{3/2}+(3/2)(g^2/2+k-3/4)^{1/2} \\ &\quad -((1/8)g^2+2gk+(1/4)k+5/8) \\ &\leq S(L^2_{g,t}) \end{aligned}$$

and we are done.

Let $\alpha = \frac{3\sqrt{2}+4}{8}g + 4.5$. We have shown that for $t \leq g^2/2-\alpha$, we have $S(L^2_{g,t}) \geq S(Q^2_{g,t})$. We wish to improve this bound to $t < g^2/2 - g$. For the next four lemmas, we assume $g \geq 7$ and let $t = 2cg - c^2 + d = m^2 + s$ with c, d, m, and s non-negative integers and $2cg - c^2 \leq t < 2(c+1)g - (c+1)^2$ and $m^2 \leq t < (m+1)^2$. We also focus exclusively on the range $g^2 - \alpha < t < g^2/2 - g$.

We first show that in this small range, we must have either $c = g - m - 2$ or $c = g - m - 3$. In the $c = g - m - 3$ case, we show that $S(L^2_{g,t}) > S(Q^2_{g,t})$, for all values of g and t within the range. In the $c = g - m - 2$ case, we show that $S(L^2_{g,t}) \leq S(Q^2_{g,t})$, with equality possible only if $t = g^2/2 - g - 1/2$. We also show that for $t = g^2/2 - g$, $S(Q^2_{g,t}) > S(L^2_{g,t})$ is possible.

Lemma 2.11 $c \leq g - m - 2$.

Proof: We first show that in the range $t < x \leq g^2 - t$, there is at least one perfect square. This range includes the range $g^2/2 - g < x < g^2/2 + g$. Observe that whenever $a > b > 0$, we have $\sqrt{a+b} + \sqrt{a-b} < 2\sqrt{a}$, so

$$\sqrt{g^2/2+g} - \sqrt{g^2/2-g} = \frac{2g}{\sqrt{g^2/2+g}+\sqrt{g^2/2-g}} > \frac{2g}{2\sqrt{g^2/2}} = \sqrt{2}$$

Since the interval $\sqrt{g^2/2-g} < x < \sqrt{g^2/2+g}$ has length greater than 1, there must be at least one integer y such that $\sqrt{g^2/2-g} < y < \sqrt{g^2/2+g}$. Thus we have

$$g^2/2 - g < y^2 < g^2/2 + g$$

and

$$m^2 \le t < y^2 \le g^2 - t \le (g-c)^2$$

Thus $g - c \ge m + 2$.

Lemma 2.12 $c \ge g - m - 3$.

Proof: If $c < g - m - 3$, we have

$$g^2-(m+3)^2 = 2g(g-m-3)-(g-m-3)^2 \ge 2g(c+1)-(c+1)^2 > t \ge g^2/2-\alpha,$$

and $m < -3 + \sqrt{g^2/2+\alpha}$. Squaring, we obtain

$$\begin{aligned} m^2 < g^2/2+\alpha-6\sqrt{g^2/2+\alpha}+9 &\le g^2/2+\alpha+9-3\sqrt{2}g \\ &= g^2/2 - \frac{21\sqrt{2}-4}{8}g + 13.5. \end{aligned}$$

Then $m^2 + s > g^2/2 - \alpha$ implies

$$g^2/2 - \frac{21\sqrt{2}-4}{8}g + 13.5 + s > m^2 + s > g^2/2 - \alpha = g^2/2 - \frac{3\sqrt{2}+4}{8}g - 4.5,$$

and

$$18 + s > (\frac{9\sqrt{2}}{4} - 1)g.$$

Since $s \le 2m$ and $m^2 < g^2/2$, we have $-s > -\sqrt{2}g$. Adding, we obtain

$$18 > .768g,$$

and the only cases where $c < g-m-3$ is possible are those with $g \le 23$. The values of c, g, m, and $g - m - c$ for $7 \le g \le 23$ and all t with $g^2/2 - \alpha < t < g^2/2-g$ are listed in the appendix. In every case, $g-m-c = 2$ or $g-m-c = 3$.

The next two lemmas use explicit calculation to determine the quantity $S(L^2_{g,t}) - S(Q^2_{g,t})$ when $c = g - m - 3$ or $c = g - m - 2$. Although the calculations are cumbersome, the conclusion of Lemma 2.14 is particularly elegant.

Lemma 2.13 *Let $t = g^2/2 - g - e = m^2 + s$, with $e > 0$. If $c = g - m - 3$, then $S(L^2_{g,t}) > S(Q^2_{g,t})$.*

Proof: We have

$$\begin{aligned} g &= 1 + \sqrt{1 + 2m^2 + 2e + 2s} \\ c &= -2 - m + \sqrt{1 + 2m^2 + 2e + 2s} \\ d &= 6m + 7 - s - 2e - 2\sqrt{1 + 2m^2 + 2e + 2s} \end{aligned}$$

Using (2.1) and (2.2), we calculate

$$\begin{aligned} S(L^2_{g,t}) - S(Q^2_{g,t}) &= 4m^2 + 4m + es + e^2 + 5/4 + \epsilon' \\ &\quad - \left(2em + 2ms + s + e + \epsilon + ((2m+1) - s)\sqrt{1 + 2m^2 + 2e + 2s}\right). \end{aligned}$$

We let $s = 2m - p$ with $0 \le p \le 2m$ and substitute to obtain

$$S(L^2_{g,t}) - S(Q^2_{g,t}) = (p+1)\left(2m - e - \sqrt{1 + 2m^2 + 4m + 2e - 2p}\right) + \left(e^2 + p + 5/4 + \epsilon' - \epsilon\right).$$

Since $\epsilon < 5/4$, if we show that $2m > e + \sqrt{1 + 2m^2 + 4m + 2e - 2p}$, it will follow that $S(L^2_{g,t}) > S(Q^2_{g,t})$.

For each $7 \le g \le 19$ and $g^2/2 - \alpha < t < g^2/2 - g$, the appendix lists the values of $g - m - c$, $S(L^2_{g,t})$, and $S(Q^2_{g,t})$. In every case where $g - m - c = 3$, we have $S(L^2_{g,t}) > S(Q^2_{g,t})$.

For $g > 19$, we have

$$.052g + 11.81 < \sqrt{g^2/2 - \alpha} < m + 1.$$

Recall that $e \le \alpha - g = \frac{3\sqrt{2}-4}{8}g + 4$. Thus we have

$$m > .052g + 10.81 > e + 1 + (1/\sqrt{2})(e + 3) > e + 1 + (1/\sqrt{2})\sqrt{e^2 + 6e + 3}$$

and

$$\begin{aligned} 2(m - e - 1)^2 &> e^2 + 6e + 3 \\ 2(m^2 - 2m(e+1)) + 2(e+1)^2 &> 1 + 2e - e^2 + 2(e+1)^2 \\ 2m^2 - 4me - 4m &> 1 + 2e - e^2 \\ 4m^2 - 4me + e^2 &> 1 + 2e + 2m^2 + 4m \end{aligned}$$

$$2m - e > \sqrt{1 + 2e + 2m^2 + 4m} > \sqrt{1 + 2e + 2m^2 + 4m - 2p}$$

and we are done.

Lemma 2.14 *Let $t = g^2/2 - g - e = m^2 + s$, with $e > 0$. If $c = g - m - 2$, then $S(L^2_{g,t}) \geq S(Q^2_{g,t})$, with equality iff $e = 1/2$ and $s = 0$.*

Proof: We have

$$\begin{aligned} g &= 1 + \sqrt{1 + 2m^2 + 2e + 2s} \\ c &= -1 - m + \sqrt{1 + 2m^2 + 2e + 2s} \\ d &= 4m + 2 - s - 2e - 2\sqrt{1 + 2m^2 + 2e + 2s} \end{aligned}$$

Using (2.1) and (2.2), we calculate

$$\begin{aligned} S(L^2_{g,t}) &= m^3 + e^2 - \epsilon + (1/4)s^2 + es + s\sqrt{1 + 2m^2 + 2e + 2s} \\ &= m^3 + e^2 - \epsilon + (1/4)s^2 + es + s(g - 1) \\ S(Q^2_{g,t}) &= m^3 + sm + s^2/4 + s/2 - \epsilon' \end{aligned}$$

and we obtain,

$$S(L^2_{g,t}) - S(Q^2_{g,t}) = s\sqrt{1 + 2m^2 + 2e + 2s} + es - ms - s/2 + e^2 - \epsilon'', \quad (2.13)$$

with $\epsilon'' = \epsilon - \epsilon'$ and $-5/4 \leq \epsilon'' \leq 3/4$.

If $s > 0$, (2.13) is positive. Assume $s = 0$. We have

$$S(L^2_{g,t}) - S(Q^2_{g,t}) = e^2 - \epsilon'',$$

with $\epsilon' = 0$ and $\epsilon = \epsilon'' = 0, 1/4$, or 1. We also have $g^2/2 - g - e = m^2$ or

$$(g - 1)^2 - 2m^2 = 2e + 1.$$

This equation has no solutions when $2e + 1 \equiv 3 \pmod 8$, so $e = 1/2$ is the only possible solution to $e^2 - \epsilon \leq 0$. If $e = 1/2$, we must have d odd, hence $\epsilon = 1/4$ and $S(L^2_{g,t}) = S(Q^2_{g,t})$, and we are done.

Although the case $e = 0$ is outside the range of the previous lemma, the calculations are still valid for this value. If $e = 0$, we must have d even, and $\epsilon = 1$, hence $S(L^2_{g,t}) - S(Q^2_{g,t}) = -1$. Thus the bound $t < g^2/2 - g$ is as strict as possible.

We can use powers of the unit $(\sqrt{2}-1)$ in the ring $\mathbf{Z}[\sqrt{2}]$ to find solutions to $(g-1)^2-2m^2=2e+1=1$ or 2, and these solutions will be the borderline cases where $S(L^2_{g,t})-S(Q^2_{g,t})$ equals 0 or -1. If $e=0$, we have $(\sqrt{2}-1)^4=17-12\sqrt{2}$. If we let $g-1=17$ and $m=12$, we have $t=144$, $S(L^2_{18,144})=1727$, and $S(Q^2_{18,144})=1728$. If $e=1/2$, we have $(2-\sqrt{2})(\sqrt{2}-1)^2=(10-7\sqrt{2})$. Proceeding similarly, we let $g=11$, $m=7$, and $t=49$. $S(L^2_{11,49})=S(Q^2_{11,49})=343$.

We remove our assumptions about g and t and compile the last six lemmas to prove this section's main theorem.

Theorem 2.15 *If $t<g^2/2-g$ then $S(L^2_{g,t})\geq S(Q^2_{g,t})$. Equality is possible only when $t\leq 6$ or $t=g^2/2-g-1/2$.*

Proof: By Lemma 2.9, the theorem holds for $g\leq 7$. By Lemma 2.10 and Lemma 2.4, the theorem holds for $g>7$ and $t\leq g^2/2-\alpha$. We thus restrict ourselves to the range $g^2/2-\alpha<t<g^2/2-g$.

By Lemmas 2.11 and 2.12, in this range we have $c=g-m-2$ or $c=g-m-3$. By Lemma 2.13, if $c=g-m-3$, the theorem holds with a strict inequality. By Lemma 2.14, if $c=g-m-2$, the theorem holds with a strict inequality, except when $t=g^2/2-g-1/2$ and $s=0$. We have covered all cases.

For all g and g' such that $Q_{g,t}$ and $Q_{g',t}$ are defined, we easily see that $S(Q^2_{g,t})=S(Q^2_{g',t})$. The value of $S(Q^2_{g,t})$ does not depend on g at all. In the next theorem, we fix t and compare L-Matrices while g is allowed to vary. As a corollary, we find the maximum possible value of $S(L^2_{g,t})$ for a given t.

Theorem 2.16 *If $t\leq g^2/2$, then $S(L^2_{g+1,t})\geq S(L^2_{g,t})$, with equality at $c=0$ or $d=0$.*

Proof: Let $t=2cg-c^2+d=2c'(g+1)-c'^2+d'$. If $c=0$, it is clear that equality holds, so assume $c>0$. We divide the proof into two cases:

<u>Case 1</u>: $d\geq 2c$

In this case, $c'=c$ and $d'=d-2c$. By (2.1) we have

$$S(L^2_{g+1,t})=c(g+1)^2+(g+1)c^2-c^3+2c(d-2c)+\frac{(d-2c)^2}{4}+d-2c-\epsilon$$

and

$$S(L^2_{g+1,t}) - S(L^2_{g,t}) = c(2g - 2c - d - 1) + \omega, \tag{2.14}$$

where

$$\omega = \begin{cases} -1 & \text{if } d = 4 \text{ and } c = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $d = 4$ and $c = 1$, in order to have $t \leq g^2/2$, we must have $g > 5$, and $S(L^2_{g+1,t}) - S(L^2_{g,t}) = 2g - 8$ is positive. Otherwise,

$$2c + d + 1 < 2c + (2(g - c) - 1) + 1 = 2g$$

which implies that (2.14) is positive.

<u>Case 2</u>: $d < 2c$

Since $t \leq g^2/2$, we have $c \leq g - g/\sqrt{2}$. For $g \geq 3$, we have $c \leq g/2 - 1/2$, or equivalently, $2(g - c) - 1 > 2c$. Thus $d + 2(g - c) - 1 > 2c$, and we must have $c' = c - 1$. We compute d' by solving

$$2(c - 1)(g + 1) - (c - 1)^2 + d' = 2cg - c^2 + d$$

to obtain $d' = 2g - 4c + 3 + d$. By (2.1), we have

$$\begin{aligned} S(L^2_{g+1,t}) &= (c-1)(g+1)^2 + (g+1)(c-1)^2 - (c-1)^3 + 2(c-1)(2g - 4c + 3 + d) \\ &\quad + \frac{(2g - 4c + 3 + d)^2}{4} + (2g - 4c + 3 + d) - \epsilon \end{aligned}$$

and

$$S(L^2_{g+1,t}) - S(L^2_{g,t}) = d(g - 2c - 1/2) + 1/4 - \omega,$$

where

$$\omega = \begin{cases} 3/4 & d \text{ odd, and } d' \text{ even, } d' \neq 0, 4 \\ -3/4 & d \text{ even, } d' \neq 0, 4, \text{ and } d \text{ odd} \\ 1/4 & d = 0 \text{ or } d = 4, \text{ and } d' \text{ odd} \\ -1/4 & d \text{ odd, and } d' = 0 \text{ or } d' = 4. \end{cases}$$

If $d = 0$, then $S(L^2_{g+1,t}) = S(L^2_{g,t})$. Otherwise, in order to have $S(L^2_{g+1,t}) \leq S(L^2_{g,t})$, we would have to have, $d(g - 2c - 1/2) \leq 1/2$. Since $(g - 2c - 1/2) = 0$ has no solutions with g and c integers, the only other possible solution is $c = (g-1)/2$ and $d = 1$, which is outside the range $t \leq g^2/2$. We have shown that $S(L^2_{g+1,t}) \geq S(L^2_{g,t})$, with equality iff $d = 0$.

Corollary: If we fix t and let g vary, the maximum possible value of $S(L^2_{g,t})$ is $t^2/4 + t - \epsilon$ where ϵ is as in Theorem 2.2.

Proof: We have shown that as g increases, $S(L^2_{g,t})$ does not decrease. However, for all $g \geq \lfloor \frac{t+1}{2} \rfloor$, we have $S(L^2_{g,t}) = t^2/4 + t - \epsilon$, where $\lfloor y \rfloor$ is the greatest integer n such that $n \leq y$.

This Corollary, together with Theorem 2.15, implies that for a given t, $L_{\lfloor \frac{t+1}{2} \rfloor, t}$ is the matrix with maximal square sum amongst all L- and Q- Matrices.

2.2 Proof That $S(L^2_{g,t})$ or $S(Q^2_{g,t})$ is Always Maximal

In this section, we prove our main theorem about matrices. First, we define two conditions, compression and almost-symmetry, and show that some maximal matrix must satisfy both of them. We then prove two theorems which give important techniques for proving our main theorem. The first is a complementarity condition which allows us to only consider the case $t \leq g^2/2$. The second is an induction technique which allows us to deduce information about square-sums of matrices from inequalities involving smaller or larger matrices. We then prove the thesis's main theorem.

Throughout this section, let $\mathcal{M}_{g,t}$ be the set of maximal (g,t)-matrices.

Definition: A (g,t)-matrix $M = [m_{ij}]$ is *row-compressed* (resp. *column-compressed*) if $m_{ij} = 0 \Rightarrow m_{i,j+1} = 0$ (resp. $m_{ij} = 0 \Rightarrow m_{i+1,j} = 0$) A matrix which is both row and column compressed is *compressed.*

Lemma 2.17 *There is a compressed matrix M such that $M \in \mathcal{M}_{g,t}$.*

Proof: This theorem is a special case of a theorem of Schwartz [Sch Theorem 1]. In [Sch] a compressed matrix is defined, although not named as such, as a square matrix with non-negative elements whose rows and columns are monotonically *non-decreasing.* In our definition, the rows and columns are *non-increasing*, but for maximizing square-sums, the definitions are equivalent.

Definition: A square matrix, $M = [m_{ij}]$, is *almost-symmetric* if $m_{ij} = m_{ji}$ with at most one exception. That is, if $m_{ij} \neq m_{ji}$ and $m_{i'j'} \neq m_{j'i'}$, then the index pair (i', j') equals either (i, j) or (j, i).

Lemma 2.18 *Let $0 < t < g^2$. Let $\mathcal{M}^c_{g,t}$ be the set of compressed maximal (g,t)-matrices. There is an almost-symmetric matrix M such that $M \in \mathcal{M}^c_{g,t}$.*

Proof: Assume the contrary. By Lemma 2.17 let $M \in \mathcal{M}_{g,t}$ be compressed, but suppose there are indices $i \neq l$ and $j \neq k$ with $m_{ij} = 0$, $m_{ji} = 1$ and $m_{kl} = 1$, $m_{lk} = 0$. Without loss of generality, let $i + j \geq k + l$. Since M is compressed, $m_{ji} = 1 \Rightarrow R_j \geq i$ and $C_i \geq j$. Similarly, $m_{lk} = 0 \Rightarrow R_l < k$ and $C_k < l$. Let E be the $g \times g$ matrix with $e_{ij} = 1$ and $e_{kl} = -1$ and all other entries zero. Then $M + E$ is a (g,t)-matrix, and

$$S((M+E)^2) = S(M^2) + S(ME) + S(EM) + S(E^2)$$

Note that $S(E^2) = 0$ or -1, $S(EM) = R_j - R_l \geq i - k + 1$ and $S(ME) = C_i - C_k \geq j - l + 1$. Thus

$$S((M+E)^2) \geq S(M^2) + (i+j) - (k+l) + 1 > S(M^2),$$

contradicting the maximality of M.

We now prove the complementarity theorem, which will allow us to restrict our attention to the case $t \leq g^2/2$.

Theorem 2.19 *If M is a compressed, almost-symmetric, maximal, (g,t)-matrix, then $(\mathbf{J}_g - M)^{\mathrm{R}}$ is a maximal $(g, g^2 - t)$-matrix.*

Since M is compressed, the (i,j) entry of M^2 is $min(R_i, C_j)$. Thus $S(M^2) = \sum_{i=1}^{g} \sum_{j=1}^{g} min(R_i, C_j)$. We first consider the case when M is symmetric, and

$$S(M^2) = \sum_{i=1}^{g} \sum_{j=1}^{g} min(R_i, R_j) = \sum_{i=1}^{g} \sum_{j=1}^{g} min(C_i, C_j) = \sum_{i=1}^{g} \sum_{j=1}^{g} R_{max(i,j)}.$$

There are $2i - 1$ distinct pairs (i, j) with $1 \leq j \leq i$, so

$$S(M^2) = \sum_{i=1}^{g} (2i-1) R_i = 2x^* - t$$

where x^* is the "first moment with respect to the x-axis" of M. Since t is fixed, M maximal implies that x^* is maximal. Let y^* be the "first moment with respect to the x-axis" of $\mathbf{J}_g - M$. Since $x^* + y^* = g^2(g+1)/2$, we have x^* maximal implies that y^* is minimal. If we make the transformation to $(\mathbf{J}_g - M)^{\mathrm{R}}$, y^* is maximal.

If M is not symmetric let $m_{ij} = 1$ and $m_{ji} = 0$. If $i < j$, the terms $min(R_i, C_i)$ and $min(R_i, C_j)$ are the only terms which differ from the symmetric case. We have $min(R_i, C_i) = C_i = R_i - 1$ and $min(R_i, C_j) = C_j = R_j + 1$. Thus $min(R_i, C_i) + min(R_i, C_j) = R_i + R_j$, which was is the same sum as in the symmetric case.

If $i > j$, the terms $min(R_j, C_j)$ and $min(R_i, C_j)$ are the only terms different from $min(C_l, C_m)$. We have $min(R_j, C_i) = R_j = C_j - 1$ and $min(R_i, C_j) = R_i = C_i + 1$, and once again the total agrees with the symmetric case.

The next theorem provides us with the key induction step for proving our main theorem. We first define notation that will be used throughout the sequel. The notation corresponds to forming a $(g+1) \times (g+1)$ matrix from a $g \times g$ matrix by adding a "layer" of ones around the upper left corner or by adding a "layer" of zeroes around the bottom right corner.

Let $M = [m_{ij}]$ be a $g \times g$ matrix. Let $T_1(M)$ denote the $(g+1) \times (g+1)$ matrix $M' = [m'_{ij}]$ given by $m'_{1j} = m'_{j1} = 1$ and $m'_{ij} = m_{i-1,j-1}$ if $i \neq 1$ and $j \neq 1$. For any positive integer c, we define $T_1^c(M)$ recursively in the obvious way, i.e. $T_1^c(M) = T_1(T_1^{c-1}(M))$ and note that if M is a (g,t)-matrix, then $T_1^c(M)$ is a $(g+c, t+2gc+c^2)$-matrix.

Let $T_0(M) = [m''_{ij}]$ denote the $(g+1) \times (g+1)$ matrix given by $m''_{ij} = m_{ij}$ for $i \leq g$ and $j \leq g$, and $m''_{ij} = 0$ if $i = g+1$ or $j = g+1$.

Note that $T_0^{c'} T_1^c(M)$ is a $(g+c+c', t+2gc+c^2)$-matrix defined by adding c "layers" of ones outside the upper left corner of M and c' "layers" of zeroes outside the bottom right corner of M.

Theorem 2.20 *Let $c \geq 1$. Let K and M be $(g-c, t+c^2-2gc)$-matrices. Then $S(K^2) \geq S(M^2)$, if and only if $T_1^c(K)$ and $T_1^c(M)$ are (g,t)-matrices satisfying $S((T_1^c(K))^2) \geq S((T_1^c(M))^2)$.*

Proof: Let O denote the $(g-c) \times (g-c)$ matrix of zeroes, and let $E = T_1^c(O)$

(E is exact). Then

$$S((T_1^c(K) - E)^2) = S(K^2)$$

and

$$S((T_1^c(M) - E)^2) = S(M^2).$$

Let $K' = T_1^c(K) - E$ and $M' = T_1^c(M) - E$, then

$$\begin{aligned} S(T_1^c(K)^2) = S((K' + E)^2) &= S((K')^2) + S(K'E) + S(EK') + S(E^2) \\ &= S(K^2) + 2c(t + c^2 - 2cg) + c^2 g + cg^2 - c^3. \end{aligned}$$

Let $y = 2c(t + c^2 - 2cg) + c^2 g + cg^2 - c^3$. We do a similar computation to show that $S(T_1^c(M)^2) = S(M^2) + y$, and $S(K^2) \geq S(M^2)$ if and only if $S((T_1^c(K))^2) \geq S((T_1^c(M))^2)$.

We now come to the main theorem of this thesis.

Theorem 2.21 *Let $g \geq 1$. For each t, $0 < t < g^2$, either $L_{g,t}$ is a maximal (g,t)-matrix or $Q_{g,t}$ is a maximal (g,t)-matrix. If $t < g^2/2 - g$ then $L_{g,t}$ is maximal, and if $t > g^2/2 + g$ then $Q_{g,t}$ is maximal.*

We begin with a brief overview of the proof. We use induction on g, assuming that $g \geq 4$ and that the theorem holds for $n < g$ (it is trivially checked for small g). We have already seen that there exists a maximal (g,t)-matrix that is both compressed and almost-symmetric. We partition the compressed almost-symmetric (g,t)-matrices into three subsets, knowing that the square-sum maximizing matrix for at least one of the subsets must be a maximal (g,t)-matrix. We examine a variety of candidates for maximality within each subset, frequently invoking the inductive hypothesis to eliminate or compare them. We then compare between subsets, eliminating matrices which are not L- or Q- Matrices, and showing that the ranges where L- or Q-Matrices are maximal correspond with the desired ranges.

Proof: For $g \leq 3$, it is easy to check that $L_{g,t} = Q_{g,t}$ and they are maximal. We henceforth assume $g \geq 4$. We will examine the cases where $t \leq g^2/2$ and use complementarity for $t \geq g^2/2$.

By our inductive hypothesis, for $n < g$, if $t < n^2/2 - n$ then $L_{n,t}$ is maximal; if $n^2/2 - n \leq t \leq n^2/2 + n$, either $L_{n,t}$ or $Q_{n,t}$ is maximal; and if

$t > n^2/2 + n$, $Q_{n,t}$ is maximal. We first wish to prove the theorem for all $t \leq g^2/2$.

Write $\mathcal{C} = \mathcal{C}_{g,t}$ for the set of all compressed almost-symmetric (g,t)-matrices. Let $\mathcal{C}_1 = \{K = [k_{ij}] \in \mathcal{C} | k_{1g} = k_{g1} = 1\}$, $\mathcal{C}_2 = \{K = [k_{ij}] \in \mathcal{C} | k_{1g} = k_{g1} = 0\}$, and $\mathcal{C}_3 = \{K = [k_{ij}] \in \mathcal{C} | k_{1g} \neq k_{g1}\}$. For $i = 1,2,3$, let $C\max(i)$ denote the supremum of $S(K^2)$ as K runs throught the set $\mathcal{C}_i$. For $i = 1,2,3$, we choose matrices M_i such that $S(M_i^2) = C\max(i)$. By Lemma 2.18, the intersection $\mathcal{C} \cap \mathcal{M}_{g,t} = (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3) \cap \mathcal{M}_{g,t}$ is non empty, so at least one of the three matrices M_1, M_2, M_3 is a maximal (g,t)-matrix. Let $S\max(n,q)$ denote the supremum of $S(N^2)$ as N runs through all (n,q)-matrices. Clearly $S\max(g,t) = \max\{C\max(1), C\max(2), C\max(3)\}$.

First consider $M_1 = [m_{ij}]$. Because M_1 is compressed and $M_1 \in \mathcal{C}_1$, we have $m_{1i} = m_{i1} = 1$, for all i, and $t \geq 2g - 1$. Let I_1 denote the $(1,1)$-minor of M_1, so that I_1 is a $(g-1, t-(2g-1))$-matrix and

$$M_1 = T_1(I_1)$$

By Theorem 2.20, I_1 must be maximal. Since $t \leq g^2/2$, we have

$$t - (2g-1) \leq g^2/2 - (2g-1) < (g-1)^2/2 - (g-1).$$

By the inductive assumption,

$$S(I_1^2) = S\max(g-1, t-(2g-1)) = S(L^2_{g-1,t-(2g-1)}).$$

By the definition of $L_{g,t}$ and Theorem 2.20,

$$C\max(1) = S(M_1^2) = S((T_1(I_1))^2) = S((T_1(L_{g-1,t-(2g-1)}))^2) = S(L^2_{g,t}).$$

Now consider $M_2 = [m'_{ij}]$. Because M_2 is compressed and $M_2 \in \mathcal{C}_2$, we have $m'_{1i} = m'_{i1} = 0$, for all i. Thus, $M_2 = T_0(I_2)$ for some $(g-1,t)$-matrix I_2. If I_2 is not maximal, we can choose a $(g-1,t)$-matrix I'_2 such that $S(I'^2_2) > S(I_2^2)$ and clearly $M'_2 = T_0(I'_2)$ has larger square-sum than M_2. Thus, we must have

$$S(I_2^2) = S\max(g-1,t),$$

and by the inductive hypothesis we have

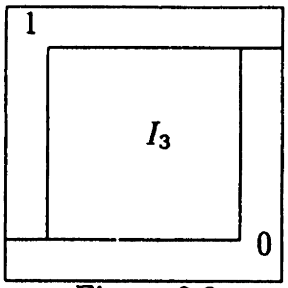

Figure 2.3

$$S(I_2^2) = \begin{cases} S(L_{g-1,t}^2) & t < (g-1)^2/2 - (g-1) \\ S(L_{g-1,t}^2) \text{ or } S(Q_{g-1,t}^2) & (g-1)^2/2 - (g-1) \le t \le g^2/2. \end{cases}$$

Thus we either have

$$C\text{max}(2) = S(M_2^2) = S((T_0(L_{g-1,t}))^2)$$

or

$$C\text{max}(2) = S(M_2^2) = S(T_0(Q_{g-1,t}))^2) = S(Q_{g,t}^2).$$

Finally, consider $M_3 = [m''_{ij}]$. After replacing M_3 by M_3^T if necessary, we can assume $m''_{1g} = 1$ and $m''_{g1} = 0$. By compression and almost-symmetry, $m''_{1j} = 1$ for all j, $m''_{j1} = 1$ for all $j \neq g$, $m''_{gj} = 0$ for all j, and $m''_{jg} = 0$ for all $j \neq 1$. (See Figure 2.3). Note that

$$t - (2g-2) \le g^2/2 - (2g-2) = (g-2)^2/2$$

and we write

$$M_3 = T_0 T_1(I_3) + E$$

where I_3 is a $(g-2, t-(2g-2))$-matrix and E is a $g \times g$ matrix with $e_{1g} = 1$ and all other entries equal to zero. We compute

$$S(M_3^2) = S((T_0 T_1(I_3) + E)^2) = S((T_1(I_3))^2) + (g-1). \qquad (2.15)$$

By Theorem 2.20,

$$S(I_3^2) = S\text{max}(g-2, t-(2g-2)),$$

and by our inductive hypothesis we have either

$$C\text{max}(3) = S(M_3^2) = S((T_0T_1(L_{g-2,t-(2g-2)}) + E)^2)$$

or

$$C\text{max}(3) = S(M_3^2) = S((T_0T_1(Q_{g-2,t-(2g-2)}) + E)^2).$$

We summarize our results so far. At least one of the following matrices must be maximal:

(In $\mathcal{C}_1$): $L_{g,t}$
(In $\mathcal{C}_2$): $T_0(L_{g-1,t})$ or $Q_{g,t}$
(In $\mathcal{C}_3$): $(T_0T_1(L_{g-2,t-(2g-2)}) + E)$ or $(T_0T_1(Q_{g-2,t-(2g-2)}) + E)$.

We now compare the five possibilities, with the goal of showing that $L_{g,t}$ or $Q_{g,t}$ is always maximal. We first show that for all $t \neq 2g-2$, the maximal matrices in $\mathcal{C}_3$ cannot have square-sums greater than both those in $\mathcal{C}_1$ and those in $\mathcal{C}_2$.

If necessary, we rename so that we have either $M_3 = (T_0T_1(L_{g-2,t-(2g-2)}) + E)$ or $M_3 = (T_0T_1(Q_{g-2,t-(2g-2)}) + E)$. For brevity, we also write either $M_3 = T_0T_1(L) + E$ or $M_3 = T_0T_1(Q) + E$, respectively. We note that if $t = 2g-2$, $M_3 = L_{g,t}$. In all other cases, we show that by exchanging either m''_{1g} or m''_{g1} with another entry in M_3, we obtain a matrix $M' \in \mathcal{C}_1 \cup \mathcal{C}_2$ with $S(M'^2) \geq S(M_3^2)$. Thus for $t \neq 2g-2$, we show that there cannot be a maximal matrix whose square sum is unique to those in $\mathcal{C}_3$.

Assume $t > 2g-2$. We first consider the case where $M_3 = T_0T_1(L) + E$. Write $t-1 = 2c(g-1) - c^2 + d$, where $2c(g-1) - c^2 \leq t-1 < 2(c+1)(g-1) - (c+1)^2$. The parameters c and d are those of Theorem 2.2 for the matrix $T_1(L)$.

If $d = 0$, then $T_1(L)$ is an exact $(g-1, t-1)$-matrix. In this case, we use (2.1) and (2.15) to compute

$$S(M_3^2) = S(T_1(L)^2) + (g-1) = c^2(g-1) + c(g-1)^2 - c^3 + g - 1.$$

We let $M' = L_{g,t} \in \mathcal{C}_1$ and compare $S(M_3^2)$ to $S((M')^2)$ by first writing $t = 2c'g - (c')^2 + d'$ where $0 \le d' < 2(g - c') - 1$. By an argument similar to that in Case 2 of Theorem 2.16, we have $c' = c - 1$ and $d' = 2g - 4c + 2$. Using (2.1), we compute that $S((M')^2) = S(L_{g,t}^2) = S(M_3^2)$, and we have $C\max(3) = C\max(1)$.

If $d \neq 0$, we show that $C\max(3) \le C\max(1)$ or $C\max(3) \le C\max(2)$. First, we designate coordinates (l_1, l_2) and (n_1, n_2) as follows:

$$(l_1, l_2) = \begin{cases} (c+2, c+2) & d = 4 \\ (c+1, c+d/2+1) & d \text{ even}, d \neq 4 \\ (c + \frac{d+1}{2}, c+1) & d \text{ odd}; \end{cases}$$

$$(n_1, n_2) = \begin{cases} (c+1, c+3) & d = 4 \\ (c+d/2+1, c+1) & d \text{ even}, d \neq 4 \\ (c+1, c + \frac{d+1}{2} + 1) & d \text{ odd}. \end{cases}$$

Note that in each case, $m''_{l_1,l_2} = 1$ and $m''_{n_1,n_2} = 0$.

Recall that we used R_i and C_j to refer to the sum of the entries of the i^{th} row and the j^{th} column, respectively, of a given matrix, in this case M_3. If $R_{l_2} + C_{l_1} \le g + 1$, we let $K = [k_{ij}]$ be a $g \times g$ matrix with $k_{g1} = 1, k_{l_1,l_2} = -1$, and all other entries equal to 0. Let $M' = M_3 + K$, and note that $M' \in \mathcal{C}_1$. We compute

$$\begin{aligned} S((M')^2) &= S(M_3^2) + S(MK) + S(KM) + S(K^2) \\ &= S(M_3^2) + (1 - C_{l_1}) + (g - R_{l_2}) + \delta(l_1, l_2), \end{aligned}$$

where δ is the Kronecker delta. We have $S((M')^2) \ge S(M_3^2)$ and $C\max(3) \le C\max(1)$.

We now assume $R_{l_2} + C_{l_1} > g + 1$. For $d \neq 4$, with d even, we have $(R_{n_2} + C_{n_1}) > (R_{l_2} + C_{l_1})$. For d odd, we have $(R_{n_2} + C_{n_1}) - (R_{l_2} + C_{l_1}) = -1$. For $d = 4$, we have $(R_{n_2} + C_{n_1}) = (R_{l_2} + C_{l_1})$. In each of these cases we have, $R_{n_2} + C_{n_1} \ge g + 1$. We let $K' = [k'_{ij}]$ be a $g \times g$ matrix with $k'_{1g} = -1, k'_{n_1,n_2} = 1$, and all other entries equal to zero. We let $M' = M_3 + K'$, and note that $M' \in \mathcal{C}_2$. We compute

$$\begin{aligned} S((M')^2) &= S(M_3^2) + S(MK') + S(K'M) + S((K')^2) \\ &= S(M_3^2) + (C_{n_1} - (g-1)) + R_{n_2} + \delta(n_1, n_2). \end{aligned}$$

We have $S((M')^2) \ge S(M_3^2)$ and $C\max(3) \le C\max(2)$.

We have now shown that if $M_3 = T_0T_1(L)+E$ then there is an $M' \in \mathcal{C}_1 \cup \mathcal{C}_2$ with $S((M')^2) \geq C\text{max}(3)$.

We use a similar argument to show that the same conclusion holds when $M_3 = T_0T_1(Q)+E$. Assume $M_3 = T_0T_1(Q)+E$. Let $t-(2g-2) = (m-1)^2+s$, where $0 \leq s < 2(m-1)+1$. The parameters $m-1$ and s are those of Theorem 2.3 for the matrix Q.

If $s = 0$, then Q exact, and we use (2.2) and (2.15) to compute

$$S(M_3^2) = g^2 - 2 + 2(m-1)^2 + (m-1)^3.$$

Let $M' = Q_{g,t} \in \mathcal{C}_2$. We compare $S(M_3^2)$ to $S((M')^2)$ by first writing $t = (m')^2 + s'$ where $0 \leq s' < 2m' + 1$. We have $m' = m$ and $s' = 2g - 2m - 1$. Using (2.2), we compute that for $s' \neq 2m-3$, we have $S(M_3^2) = S((M')^2)$, and for $s' = 2m-3$, we have $S(M_3^2) < S((M')^2))$. In either case, $C\text{max}(3) \leq C\text{max}(2)$.

If $s \neq 0$, we show that $C\text{max}(3) \leq C\text{max}(1)$ or $C\text{max}(3) \leq C\text{max}(2)$. First, we designate coordinates (l_1, l_2) and (n_1, n_2) as follows:

$$(l_1, l_2) = \begin{cases} (m+1, s/2) & s \text{ even} \\ (\frac{s+1}{2}, m+1) & s \text{ odd}, s \neq 2m-3 \\ (m-1, m+1) & s = 2m-3; \end{cases}$$

$$(n_1, n_2) = \begin{cases} (s/2+1, m+1) & s \text{ even} \\ (m+1, \frac{s+1}{2}) & s \text{ odd}, s \neq 2m-3 \\ (m, m) & s = 2m-3. \end{cases}$$

Note that in each case, $m''_{l_1,l_2} = 1$ and $m''_{n_1,n_2} = 0$.

If $R_{l_2} + C_{l_1} \leq g+1$, we let $K = [k_{ij}]$ be a $g \times g$ matrix with $k_{g1} = 1, k_{l_1,l_2} = -1$, and all other entries equal to 0. Let $M' = M_3 + K$, and note that $M' \in \mathcal{C}_1$. We compute

$$\begin{aligned} S((M')^2) &= S(M_3^2) + S(MK) + S(KM) + S(K^2) \\ &= S(M_3^2) + (1 - C_{l_1}) + (g - R_{l_2}) + \delta(l_1, l_2), \end{aligned}$$

where δ is the Kronecker delta. We have $S((M')^2) \geq S(M_3^2)$ and $C\text{max}(3) \leq C\text{max}(1)$.

We now assume $R_{l_2} + C_{l_1} > g+1$. For s odd, $s \neq 2m-3$, we have $(R_{n_2} + C_{n_1}) > (R_{l_2} + C_{l_1})$. For s even, we have $(R_{n_2} + C_{n_1}) - (R_{l_2} + C_{l_1}) = -1$. Thus for $s \neq 2m-3$, $R_{l_2} + C_{l_1} > g+1$ implies $R_{n_2} + C_{n_1} \geq g+1$. If $s = 2m-3$, we have $(R_{n_2}+C_{n_1})-(R_{l_2}+C_{l_1}) = -2$, and $R_{l_2}+C_{l_1} > g+1$ implies $R_{n_2}+C_{n_1} \geq g$. We let $K' = [k'_{ij}]$ be a $g \times g$ matrix with $k'_{1g} = -1, k'_{n_1,n_2} = 1$, and all other entries equal to zero. We let $M' = M_3 + K'$, and note that $M' \in \mathcal{C}_2$. We compute

$$\begin{aligned} S((M')^2) &= S(M_3^2) + S(MK') + S(K'M) + S((K')^2) \\ &= S(M_3^2) + (C_{n_1} - (g-1)) + R_{n_2} + \delta(n_1, n_2). \end{aligned}$$

Note that if $s = 2m-3$, we have $\delta(n_1,n_2) = 1$, and in all cases $S((M')^2) \geq S(M_3^2)$ and $C\max(3) \leq C\max(2)$.

We have now shown that $(\mathcal{C}_1 \cup \mathcal{C}_2) \cap \mathcal{M}_{g,t}$ is non-empty, and at least one of the following must be maximal:

(In $\mathcal{C}_1$): $L_{g,t}$
(In $\mathcal{C}_2$): $T_0(L_{g-1,t})$ or $Q_{g,t}$.

Note that for $t < 2g-2$, we have $T_0(L_{g-1,t}) = L_{g,t}$. For $t \geq 2g-2$, we have $S(T_0(L_{g-1,t})^2) = S(L^2_{g-1,t})$. By Theorem 2.16,

$$S(L^2_{g,t}) \geq S(L^2_{g-1,t}) = S(T_0(L_{g-1,t})^2),$$

and we must have either $L_{g,t} \in \mathcal{M}_{g,t}$ or $Q_{g,t} \in \mathcal{M}_{g,t}$.

By Theorem 2.15, if $t < g^2/2 - g$, $S(L^2_{g,t}) \geq S(Q^2_{g,t})$. We have completed our proof for $t \leq g^2/2$. By Theorem 2.19, the theorem also holds for $t \geq g^2/2$.

Corollary 2.22 *Let $M_{g,t}$ be maximal among all (g,t) matrices, where t is seen as fixed and g is allowed to vary.* $S(M^2_{g,t}) = S(L^2_{\lfloor\frac{t+1}{2}\rfloor,t})$.

Proof: By the Corollary to Theorem 2.16, $L_{\lfloor\frac{t+1}{2}\rfloor,t}$ is maximal amongst the L-Matrices with t fixed. For $t \leq 3$ all cases are trivial, and in all other cases we have $t < (\lfloor\frac{t+1}{2}\rfloor)^2 - (\lfloor\frac{t+1}{2}\rfloor)$ so the square-sum of this maximal L matrix is larger than that of the Q- Matrix, and we are done.

Chapter 3

Further terms in the Series

In this chapter, we compute a general form for Hilbert series of algebras with certain normal word matrices. These matrices include all L- and Q- matrices. The series are for general reference and will also be used in Chapter 4. We also examine the fourth term of the Hilbert series and give examples of maximal Hilbert series.

3.1 Hilbert Series of L- and Q- Matrices

Let $R(j)$ be the j^{th} row of a matrix, and let R_j be the sum of the entries in this row.

We first compute a general form of the Hilbert series for symmetric, compressed matrices with rank ≤ 4. Later we will compute the generating functions for asymmetric L- and Q- matrices. Let $M = [m_{ij}]$ be a compressed, symmetric $(0,1)$-matrix corresponding to the degree two normal words of a quadratic monomial algebra H, with Hilbert series $H(z)$. Let $R = \{R_j\}$ be the set of non-zero row-sums in M. If $|R| \leq 4$, let $\rho_1 > ... > \rho_{|R|} > 0$ be the distinct elements of R. Note that we are excluding rows with sum zero, which correspond to superfluous generators. Later, we will include these generators in the Hilbert series. Let $r = \sum n_i \rho_i$ where n_i is the number of rows of length ρ_i. Since M is symmetric, there are also n_i columns of length ρ_i. See Figure 3.1.

Let G_M be the graph associated with M, i.e. a graph with vertices $v_1, ..., v_g$, with an edge, e_{ij}, connecting v_i to v_j if and only if $m_{ij} = 1$. Using

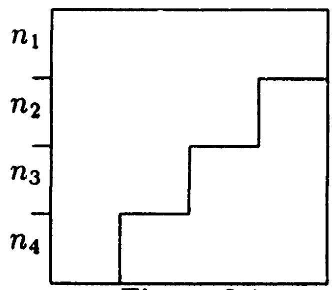

Figure 3.1

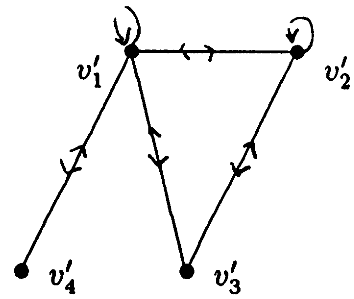

Figure 3.2

a techinique of Ufnarovskii [Uf, Section 5], we "glue" vertices together to define a graph G'_M with four vertices labelled by (possibly empty) sets of points, and with edges, e'_{ij}. We let $v'_i = \{R(j) | R_j = \rho_i\}$ be the vertex labels, and let G'_M be as in Figure 3.2. Note that superfluous generators of H are not represented in the graph G'_M, and that v'_1 and v'_2 correspond to complete graphs on n_1 and n_2 vertices in G_M, and v'_3 and v'_4 correspond to sets of n_3 and n_4 vertices in G_M.

We follow [Uf], slightly changing some of the notation. We let $h_0 = (n_1 z, n_2 z, n_3 z, n_4 z)$ and

$$M' = z \begin{pmatrix} n_1 & n_2 & n_3 & n_4 \\ n_1 & n_2 & n_3 & 0 \\ n_1 & n_2 & 0 & 0 \\ n_1 & 0 & 0 & 0 \end{pmatrix}$$

We define $h_n = h_0(M')^n$ and let H_n be the sum of the coordinates in h_n. The Hilbert series for the algebra corresponding to $G_{M'}$ is then defined by $H_{M'}(z) = 1 + \sum_{i=0}^{\infty} H_i$. Note that $H_{M'}(z)$ and $H(z)$ differ only at the z term.

Let $P(\lambda) = \sum_{i=1}^{4} a_i \lambda^i$ be the characteristic polynomial of M'. By Theorem 5 of [Uf] we have

$$b_0 H_{M'}(z) = b_0 + b_1 H_0 + b_2 H_1 + b_3 H_2 + b_4 H_3$$

where

$$b_i = \sum_{j=i}^{4} a_j$$

Since the H_i's are polynomials in z, b_0 is the denominator of $H_{M'}(z)$. Let $D(z) = b_0 = P(1) = det(I - M')$. We compute

$$D(z) = 1 - (n_1 + n_2)z - (n_1 n_3 + n_1 n_4 + n_2 n_3)z^2 + n_1 n_2 n_4 z^3 + n_1 n_2 n_3 n_4 z^4.$$

We will use initial conditions to solve for the numerator, which must have degree ≤ 4. Note that this numerator will account for all the vertices "glued" together to form G'_M, but it will not account for superfluous generators of H, which were not included in G'_M. These generators affect only the z term of $H(z)$, and we will include them later.

Let

$$H_M(z) = \frac{N(z)}{D(z)} = \sum f(n) z^n,$$

with

$$N(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3.$$

We have (using Lemma 2.1 to compute f(3))

$$\begin{aligned} f(0) &= 1 \\ f(1) &= n_1 + n_2 + n_3 + n_4 \\ f(2) &= (n_1 + n_2)^2 + 2n_1 n_3 + 2n_1 n_4 + 2n_2 n_3 \\ f(3) &= n_1(n_1 + n_2 + n_3 + n_4)^2 + n_2(n_1 + n_2 + n_3)^2 + n_3(n_1 + n_2)^2 + n_4 n_1^2. \end{aligned}$$

We compute

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= n_3 + n_4 \\ \alpha_2 &= -n_2 n_4 \\ \alpha_3 &= -n_2 n_3 n_4 \end{aligned}$$

and

$$H_M(z) = \frac{1 + (n_3 + n_4)z - n_2 n_4 z^2 - n_2 n_3 n_4 z^3}{1 - (n_1 + n_2)z - (n_1 n_3 + n_1 n_4 + n_2 n_3)z^2 + n_1 n_2 n_4 z^3 + n_1 n_2 n_3 n_4 z^4}$$

and

$$H(z) = H_M(z) + (g - (n_1 + n_2 + n_3 + n_4))z.$$

We substitute for $\vec{v} = (n_1, n_2, n_3, n_4)$ to obtain generating functions for all symmetric L and Q matrices:

Exact L-Matrix: $\vec{v} = (c, 0, 0, g - c)$

L-Matrix with d odd: $\vec{v} = (c, 1, (d-1)/2, g - c - (d+1)/2)$

L-Matrix with $d = 4$, $\vec{v} = (c, 2, 0, g - c - 2)$

Exact Q-Matrix, $\vec{v} = (m, 0, 0, 0)$

Q-Matrix with s even, $\vec{v} = (s/2, m - s/2, 0, 1)$

Q-Matrix with $s = 2m - 3$, $\vec{v} = (m-1, 0, 0, 2)$ (Note that this case is also an exact L-Matrix)

We use the same method, with more specific matrices, to calculate series for asymmetric L and Q matrices, i.e. the L-Matrix with d even, $d \neq 4$ and the Q-Matrix with s odd, $s \neq 2m - 3$.

To compute the L-Matrix with d even, $d \neq 4$, we let

$$M' = z \begin{pmatrix} c & 1 & d/2-1 & 1 & g-c-1-d/2 \\ c & 1 & d/2-1 & 1 & 0 \\ c & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We compute $H_{M'}(z) = N(z)/D(z)$, where

$$N(z) = 1 + (g-c-1)z - (g-c-d/2-1)z^2 - (g-c-d/2-1)(d/2-1)z^3$$

and

$$D(z) = 1-(c+1)z-(cg+d/2-c^2-c-1)z^2+(cg-cd/2-c^2-c)z^3+c(d/2-1)(g-c-d/2-1)z^4.$$

To compute the Q-Matrix with s odd, $s \neq 2m-3$, we let

$$M' = z \begin{pmatrix} (s-1)/2 & 1 & m-(s+1)/2 & 1 \\ (s-1)/2 & 1 & m-(s+1)/2 & 1 \\ (s-1)/2 & 1 & m-(s+1)/2 & 0 \\ (s-1)/2 & 0 & 0 & 0 \end{pmatrix}.$$

We compute $H_{M'}(z) = N(z)/D(z)$, where

$$N(z) = 1 + z - (m - (s+1)/2)z^2$$

and

$$D(z) = 1 - mz - ((s-1)/2)z^2 + ((s-1)/2)(m - (s+1)/2)z^3.$$

3.2 The Fourth Term

In this short section we give two easy results relevant to the fourth term of maximal Hilbert series.

Lemma 3.1 *If $H_A(z) = \sum h_n z^n$ is the Hilbert series of graded algebra A, then for all positive integers m and n, $h_m h_n \geq h_{m+n}$.*

Proof: Since $A_m \otimes A_n \to A_{m+n}$ is a surjection of vector spaces, the lemma must hold.

Corollary: If A is a quadratic monomial algebra with g generators and r relators and Hilbert series $H_A = \sum h_n z^n$, then $h_4 \leq t^2$ where $t = g^2 - r$.

Lemma 3.2 *If $H_A(z) = \sum h_n z^n$ is the Hilbert series of quadratic monomial algebra A with $h_4 = h_2^2$, then $h_2 = m^2$ for some integer m, and one possible basis for A_2 consists of all possible quadratic words on a subset of m generators.*

Proof: Write $A = F\langle U\rangle/\langle I\rangle$, where I consists of the quadratic monomial relators on the generating alphabet U. Let W_m consist of all length-m words on U which are non-zero in A_m. Let

$$L = \{x \in U | xy \in W_2 \text{ for some } y \in U\}$$

and

$$R = \{x \in U | yx \in A_2 \text{ for some } y \in U\}.$$

The sets L and R are the sets of generators which can (respectively) begin and end quadratic words in W_2. Let $l \in L$ and $r \in R$ and choose $x_1, x_2 \in U$ so that $lx_1, x_2r \in W_2$. Since $h_4 = h_2^2$, for every $w_1, w_2 \in W_2$ we have $w_1w_2 \in A_4$. Thus $x_2rlx_1 \in A_4$, and its subword $rl \in W_2$. Thus $l \in R$, $r \in L$, and we must have $R = L$. For every $x_1, x_2 \in R$ (and L) we must have $x_1x_2, x_2x_1 \in W_2$.

3.3 Some Maximal Series

In this section we show that $1/(1-mz)$ can be a maximal Hilbert series with respect to the coefficient-wise ordering. Recall that for formal power series $B(z) = \sum b_i z^i$ and $C(z) = \sum c_i z^i$ we have $B(z) \gg A(z)$ iff $b_i \geq a_i$ for all i. We also give the maximal series with respect to the total ordering, for a class of algebras with a fixed number of normal words of degree two and an unrestricted number of generators.

Theorem 3.3 *Let g and r be integers, with $r < g^2/2 - g$. If for some integer m we have $m^2 = g^2 - r$, then $H(z) = 1/(1-mz)$ is maximal in $\mathcal{A}^{\mathrm{m}}_{g,r}$ with respect to the coefficient-wise ordering.*

Proof: Recall that $\mathcal{A}^{\mathrm{m}}_{g,r}$ is the collection of quadratic monomial algebras with g generators and r relators. By Theorem 2.21, the z^3 term of $H(z)$ has the largest possible coefficient, and by the previous two lemmas, the same is true of the z^4 and all higher terms.

Our last result for Chapter 3 gives the Hilbert series which is maximal relative only to fixing the number t of quadratic normal words. For the collection of graded algebras over which this supremum is taken, the number of generators g can vary (and become arbitrarily large), but the number r of linearly independent quadratic relators is always related to g via the expression $r = g^2 - t$. Technically there is no maximal Hilbert series for all algebras in this collection because the first (or z^1) term is gz, and g can be arbitrarily large. However, if this term is excluded from consideration, there is a maximal series. Now the problem is essentially the same as the problem of maximizing the square-sum of a (g,t)-matrix for fixed t and arbitrary g. By Corollary 2.22, these maxima stabilize once g exceeds $\lfloor \frac{t+1}{2} \rfloor$. We use the results of Section 3.1 to compute Hilbert series, and the following theorem makes sense and is true:

Theorem 3.4 *For a graded non-commutative algebra with $t > 2$ normal words of degree two, the following Hilbert series are maximal with respect to the total ordering:*

$$H(z) = \begin{cases} (1 + \frac{t-1}{2} z)/(1 - z - \frac{t-1}{2} z^2) & t \text{ odd} \\ (1 + \frac{t}{2} z)/(1 - z - \frac{t-2}{2} z^2) & t \text{ even, } t \neq 4 \\ 1/(1-2z) & t = 4. \end{cases}$$

Chapter 4

Further Results

In this chapter we apply the results of Chapters 2 and 3 in several different contexts. In Section 1, we compare maximal commutative algebras to maximal quadratic relator algebras. In Section 2, we look at the growth ordering. In Section 3, we look at the geometric structure of Hilbert series of quadratic-relator algebras.

4.1 Comparison with the Commutative Case

We now compare some of the properties of maximal Hilbert series of $\mathbf{N}$-graded commutative algebras with the Hilbert series we have been studying.

We first briefly review the commutative case, essentially following the exposition in [St.1 II.2]. For proofs of the theorems, see [St.2 Thm 2.1] or [Mac].

Given integers $l, i > 0$, we write the unique i-binomial representation

$$l = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}, \quad n_i > n_{i-1} \cdots > n_j \geq j \geq 1.$$

We define

$$l^{<i>} = \binom{n_i + 1}{i+1} + \binom{n_{i-1} + 1}{i} + \cdots + \binom{n_j + 1}{j+1}, \quad 0^{<i>} = 0$$

A *multicomplex*, Γ, on $V = \{x_1, \ldots, x_n\}$ is a set of monomials $x_1^{a_1} \cdots x_n^{a_n}$ such that $u \in \Gamma, v|u$ implies $v \in \Gamma$. A multicomplex is also an order ideal in the partially ordered set of monomials defined by divisibility, and in [An.2] the term *order ideal of monomials* is used instead of multicomplex. Let $\Gamma_i = \{u \in \Gamma | \deg(u) = i\}$. For a multicomplex, Γ, let $h_i = \#\{u \in \Gamma | \deg(u) = i\}$, and define the h-vector, $h(\Gamma) = (h_0, h_1, \ldots)$. A sequence $(h_0, h_1, \ldots)$ which is the h-vector of some non-void multicomplex, Γ, is called an *M-vector.*

We have:

Theorem 4.1 *$(h_0, h_1, \ldots)$ is an M-vector iff $h_0 = 1$ and for all $i \geq 0$, we have $0 \leq h_{i+1} \leq h_i^{<i>}$.*

Theorem 4.2 *If $(h_0, h_1, \ldots)$ is an M-vector, then there exists a commutative graded algebra $A = A_0 \oplus A_1 \oplus \cdots$, generated by A_1, such that $H_A(z) = \sum_{i=0}^{\infty} h_i z^i$.*

If we are given $(h_0, h_1, \ldots)$ with $h_0 = 1$ and $0 \leq h_{i+1} \leq h_i^{<i>}$, we can construct a commutative graded algebra with Hilbert series $\sum_{i=0}^{\infty} h_i z^i$ by listing monomials of degree i in reverse lexicographic order, and then taking Γ_i to be the first h_i monomials of degree i. We then let

$$A = F[x_1, \ldots, x_n]/(x_1^{a_1} \cdots x_n^{a_n} \notin \Gamma)$$

and $H_A(z) = \sum_{i=0}^{\infty} h_i z^i$.

Theorem 4.3 *Let $C_{g,r}$ be the set of $\mathbf{N}$-graded commutative F-algebras with g generators and r relators. $C_{g,r}$ has a maximal element in the coefficient-wise ordering.*

Proof: We let $h_0 = 1, h_1 = g, h_2 = \binom{g+1}{2} - r$, and $h_{i+1} = h_i^{<i>}$ for $i \geq 2$. By the previous theorems, there is an algebra $A \in C_{g,r}$ with $H_A(z) = \sum_{i=0}^{\infty} h_i z^i$, and given any $A' \in C_{g,r}$ with $H_{A'}(z) = \sum_{i=0}^{\infty} h'_i z^i$, for $i \geq 3$ we cannot have $h'_i > h_i$.

This result is in marked contrast with the non-commutative case.

Theorem 4.4 *Let $\mathcal{B}_{g,r}$ be the set of maximal quadratic-relator algebras with g generators and r relators. $\mathcal{B}_{g,r}$ may or may not have a maximal element in the coefficient-wise ordering, depending upon the particular g and r.*

Proof: We show that $\mathcal{B}_{5,16}$ has no maximal element in the coefficient-wise partial ordering, and for $m^2 > g^2/2 + g$, $\mathcal{B}_{g,g^2-m^2}$ has a maximal element in the coeffiecient-wise partial ordering, thus exibiting one $\mathcal{B}_{g,r}$ with a maximal element and one without.

Let $L(z) = \sum l_i z^i$ and $Q(z) = q_i z^i$ be the Hilbert series of algebras in $\mathcal{B}_{5,16}$ whose normal words matrices are the L- and Q- Matrices, respectively. Let s_3 =max(q_3, l_3) By Theorem 2.21, $a_3 \leq s_3$ for all $H_A(z) = \sum a_i z^i \in \mathcal{B}_{5,16}$. By theorem 3.1, $a_4 \leq q_4$ for all $H_A(z) = \sum a_i z^i \in \mathcal{B}_{5,16}$. We have by Chapter 3:

$$L(z) = \frac{1+4z}{1-z-4z^2} = 1 + 5z + 9z^2 + 29z^3 + 65z^4 + \cdots$$

$$Q(z) = \frac{1}{1-3z} + 2z = 1 + 5z + 9z^2 + 27z^3 + 81z^4 + \cdots$$

Thus, neither $L(z)$ nor $Q(z)$ is maximal under the coefficient-wise ordering. Suppose $H_A(z) \gg Q(z)$ and $H_A(z) \gg L(z)$. Then we must have

$$H_A(z) = 1 + 5z + 9z^2 + 29z + 81z^4 + \cdots$$

By Lemma 3.2, we cannot have $A \in \mathcal{B}_{5,16}$, and $\mathcal{B}_{5,16}$ has no maximal element under the coefficient-wise partial order.

In $\mathcal{B}_{g,g^2-m^2}$, by Theorem 3.3, $Q(z)$ is maximal under the coefficient-wise ordering. We have given one example of a $\mathcal{B}_{g,r}$ with a maximal element in the coefficient-wise partial ordering and one without.

In the commutative case, the reverse lexicographic ordering provides us with a sequential way of forming maximal Hilbert Series. If $H_A(z)$ is a maximal Hilbert series for h_1 generators and $h_2 \neq h_1^{<1>}$ normal words of degree two, we can choose one word of degree two to adjoin to a basis for A_2 and use A_1 and the enlarged A_2 to form an algebra which has h_1 generators and $h_2 + 1$ normal words of degree two, and whose Hilbert series is maximal under the total ordering. We cannot, in general, do such a thing in the non-commutative case.

Definition: Let $\mathcal{A}$ be a set of graded algebras, each with generators in a finite set U. Let $\mathcal{A}_1 \cup \mathcal{A}_2 \cdots \cup \mathcal{A}_n = \mathcal{A}$, with $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ be a partition of $\mathcal{A}$. Let $\mathcal{H}_i$ be the set of Hilbert series of algebras in $\mathcal{A}_i$. Let $\mathcal{H}_i^*$ be the set of elements of $\mathcal{H}_i$ whose third-order term is maximal (i.e. the coefficient of

z^3 in the series is as large as it can be within $\mathcal{H}_i$). Let $\mathcal{A}_i^* \subseteq \mathcal{A}$ consist of those algebras whose Hilbert series are in $\mathcal{H}_i^*$. If for each i, $\mathcal{A}_i^* \neq \emptyset$ and we can select $A_i \in \mathcal{A}_i^*$ so that for all $i, (A_i)_2 \subseteq (A_{i+1})_2$ and $(A_{i+1})_2/(A_i)_2$ has rank one as a vector space, we say that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ is a *MacCaulay-type two-ordering* of $\mathcal{A}$.

Theorem 4.5 *Fix n and let $\mathcal{C}$ be the set of commutative graded algebras generated by $x_1, \ldots, x_n$. Let $\mathcal{C}_i \subseteq \mathcal{C}$ consist of those algebras C for which $dim(C_2) = i$. Then $\mathcal{C}_0, \ldots, \mathcal{C}_{\frac{n(n+1)}{2}}$ is a MacCaulay-type two-ordering of $\mathcal{C}$.*

Proof: The proof is a simple consequence of Theorems 4.1 and 4.2.

For the next two proofs, we abuse notation slightly, and let $\mathcal{A}^{\mathrm{m}}_{g,r}$ be the set of quadratic monomial algebras generated by $U = \{x_1, x_2, \ldots, x_g\}$ with r distinct monomial relators. We show that there is no MacCaulay-type two-ordering of $\mathcal{A}^{\mathrm{m}}_{g,r}$ which would be analogous to that of $\mathcal{C}$, but there are such orderings of proper subsets of $\mathcal{A}^{\mathrm{m}}_{g,r}$.

Theorem 4.6 *Fix $g \geq 3$ and let $\mathcal{A}_i = \mathcal{A}^{\mathrm{m}}_{g,g^2-i}$. Then $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{g^2}$ is not a MacCaulay-type two-ordering of $A^{\mathrm{m}}_{g,r}$*

Proof: By Theorem 2.21, for any $B \in \mathcal{A}_4^*$ there is a pair of generators $\{y_1, y_2\} \subseteq U$ for which $\{y_1^2, y_1y_2, y_2y_1, y_2^2\}$ spans B_2. Similarly, associated with any $C \in \mathcal{A}_5^*$ there exists a set of generators $\{z_1, z_2, z_3\} \subseteq U$ for which $\{z_1^2, z_1z_2, z_1z_3, z_2z_1, z_3z_1\}$ spans C_2. There is no way to choose B and C so that $B_2 \subseteq C_2$, hence $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{g^2}$ is not a MacCaulay-type two-ordering of $A^{\mathrm{m}}_{g,r}$.

Theorem 4.7 *Let $g \geq 3$ and $\mathcal{A}_i = \mathcal{A}^{\mathrm{m}}_{g,g^2-i}$. Then $\mathcal{A}_j, \mathcal{A}_{j+1}, \ldots, \mathcal{A}_{j+k}$ is a MacCaulay-type two-ordering of $\cup_{i=j}^{j+k} \mathcal{A}_i$ if one of the following holds:*
(a): $j + k < g^2/2 - g$; and for each integer n, with $j \leq n \leq j + k$, and for non-negative integers c, $n = 2cg - c^2 + 4$ is possible only if $n = j + k$, and $n = 2cg - c^2 + 5$ is possible only if $n = j$.
(b): $j > g^2/2 + g$; and for each integer n, with $j \leq n \leq j + k$, and for non-negative integers m, $n = m^2 + (2m - 4)$ is possible only if $n = j + k$, and $n = m^2 + (2m - 3)$ is possible only if $n = j$.

Proof: If (a) holds, the L-Matrix is maximal. The additional constraints ensure that we can avoid the anomolous $d = 4$ case and choose our basis for $(A_i)_2$, $A_i \in \mathcal{A}_i^*$ by choosing normal words of degree two corresponding to the ones in $L_{g,i}$. In this range, $L_{g,i+1}$ is formed from $L_{g,i}$ by changing a zero to a one. If (b) holds, the Q-Matrix is maximal, and the reasoning is similar.

4.2 The Growth Ordering

For many cases, the Q-Matrix is maximal with respect to the growth ordering. In this section, we summarize results from the literature. All of the results were found in the context of maximizing the spectral radius of a (0,1)-Matrix with a prescribed number of ones. The dimensions of the matrix were not fixed, so these results depend on t but not on g.

Theorem 4.8 *(Friedland, also Brualdi/Hoffman) Let $t = m^2 + s$. If $s = 0, 1, 2m - 3$, or $2m$, the Q-Matrix is maximal with respect to the growth ordering. If we fix s, then for $m > 2s + \sqrt{s}$, the Q-Matrix is maximal with respect to the growth ordering.*

Proof: See [Fr] and [BH].

Hilbert series of quadratic-relator algebras provide another context for interpreting these results on spectral radii, especially in the anomolous case when $s = 2m - 3$. Theorem 2.20 shows that this anomoly is rooted in the fact that

$$S\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 > S\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 .$$

With this justification, the following seems reasonable:

Conjecture 4.9 *The Q-Matrix is maximal with respect to the growth ordering, for any g and $t = g^2 - r$.*

4.3 Algebraic Geometry of Hilbert Series

In [An.1] Anick defines a set of Zariski-closed algebraic sets, which allow us to look at $\mathcal{A}_{g,r}$ in another way. We first define the sets, specializing to the degree one generated quadratic-related case, and then we examine some of our earlier results in this context. The results, while very preliminary, indicate that the algebraic sets can have complex structures.

Let $T = F\langle x_1 \dots x_g \rangle$ be a d.o.g. free algebra on g generators. Let G be any algebra which has a presentation as $G = T/\langle \beta_1 \dots \beta_r \rangle$, where $\langle \beta_1 \dots \beta_r \rangle$ denotes the two-sided ideal of T generated by the set $\{\beta_1 \dots \beta_r\}$ of r (not necessarily linearly independent) quadratic relators. Each β_k can be written uniquely as

$$\beta_k = \sum_{i=1}^{g} \sum_{j=1}^{g} c_{ijk} x_i x_j,$$

where $c_{ijk} \in F$. Viewing the c_{ijk}'s as indeterminates, we see that specifying such a presentation for G is equivalent to picking a point $c = (c_{ijk})$ in the N-dimensional affine space F^N, where $N = g^2 r$.

Theorem 4.10 *Let G^c denote the graded algebra associated with the point $c = (c_{ijk}) \in F^N$. For any formal power series $A(z) = \sum_{n=1}^{\infty} a_n z^n$, the sets $\{c \in F^N | H_{G^c} \gg A(z)\}$ and $\{c \in F^N | H_{G^c} \geq A(z)\}$ are Zariski-closed subsets of F^N.*

Proof: See [An.1, pg 261].

We will write the sets referred to in Theorem 4.10 as $V(H(z) \gg A(z))$ and $V(H(z) \geq A(z))$. These sets are not nested in any obvious way. For example, following the proof of Theorem 4.4 we have for $g = 5$ and $r = 16$:

$$V(H(z) \gg 29z^3 + 81z^4) = V(H(z) \gg 29z^3) \cap V(H(z) \gg 81z^4) \subseteq V(H(z) \gg 10z^2)$$

while

$$V(H(z) \gg 9z^2 + 29z^3) = V(H(z) \gg 29z^3) \cap V(H(z) \gg 9z^2) \not\subseteq V(H(z) \gg 10z^2)$$

and

$$V(H(z) \gg 9z^2 + 81z^4) = V(H(z) \gg 81z^4) \cap V(H(z) \gg 9z^2) \not\subseteq V(H(z) \gg 10z^2).$$

Appendix

Values of g, t, c, d, m, s, $g-m-c$, $S(L^2_{g,t})$, and $S(Q^2_{g,t})$, for $7 \le g \le 23$ and $g^2/2 - \frac{3\sqrt{2}+4}{8}g - 4.5 < t < g^2/2 - g$.

g	t	c	d	m	s	$g-m-c$	$S(L^2_{g,t})$	$S(Q^2_{g,t})$
7	13	1	0	3	4	3	55	45
7	14	1	1	3	5	3	58	50
7	15	1	2	3	6	3	61	57
7	16	1	3	4	0	2	66	64
8	20	1	5	4	4	3	92	86
8	21	1	6	4	5	3	97	93
8	22	1	7	4	6	3	104	100
8	23	1	8	4	7	3	110	107
9	27	1	10	5	2	3	143	137
9	28	1	11	5	3	3	152	143
9	29	1	12	5	4	3	160	151
9	30	1	13	5	5	3	170	158
9	31	1	14	5	6	3	179	167
10	36	2	0	6	0	2	232	216
10	37	2	1	6	1	2	237	222
10	38	2	2	6	2	2	242	230
10	39	2	3	6	3	2	249	237
11	45	2	5	6	9	3	309	295
11	46	2	6	6	10	3	316	306
11	47	2	7	6	11	3	325	317
11	48	2	8	6	12	3	333	330
11	49	2	9	7	0	2	343	343
12	56	2	12	7	7	3	423	407
12	57	2	13	7	8	3	435	419
12	58	2	14	7	9	3	446	430
12	59	2	15	7	10	3	459	443

Values of g, t, c, d, m, s, $g-m-c$, $S(L^2_{g,t})$, and $S(Q^2_{g,t})$, for $7 \le g \le 23$ and $g^2/2 - \frac{3\sqrt{2}+4}{8}g - 4.5 < t < g^2/2 - g$, continued.

g	t	c	d	m	s	$g-m-c$	$S(L^2_{g,t})$	$S(Q^2_{g,t})$
13	67	2	19	8	3	3	567	539
13	68	2	20	8	4	3	581	550
13	69	3	0	8	5	2	597	560
13	70	3	1	8	6	2	604	572
13	71	3	2	8	7	2	611	583
14	80	3	5	8	16	3	728	712
14	81	3	6	9	0	2	737	729
14	82	3	7	9	1	2	748	738
14	83	3	8	9	2	2	758	749
15	93	3	12	9	12	3	902	879
15	94	3	13	9	13	3	916	894
15	95	3	14	9	14	3	929	911
15	96	3	15	9	15	3	944	928
15	97	3	16	9	16	3	958	945
16	108	3	21	10	8	3	1142	1100
16	109	3	22	10	9	3	1159	1114
16	110	3	23	10	10	3	1178	1130
16	111	3	24	10	11	3	1196	1145
17	123	4	3	11	2	2	1393	1355
17	124	4	4	11	3	2	1404	1367
17	125	4	5	11	4	2	1415	1381
17	126	4	6	11	5	2	1426	1394
17	127	4	7	11	6	2	1439	1409
18	139	4	11	11	18	3	1649	1619
18	140	4	12	11	19	3	1663	1640
18	141	4	13	11	20	3	1679	1661
18	142	4	14	11	21	3	1694	1682
18	143	4	15	11	22	3	1711	1705

Values of g, t, c, d, m, s, $g-m-c$, $S(L^2_{g,t})$, and $S(Q^2_{g,t})$, for $7 \le g \le 23$ and $g^2/2 - \frac{3\sqrt{2}+4}{8}g - 4.5 < t < g^2/2 - g$, continued.

g	t	c	d	m	s	$g-m-c$	$S(L^2_{g,t})$	$S(Q^2_{g,t})$
19	157	4	21	12	13	3	1983	1932
19	158	4	22	12	14	3	2002	1952
19	159	4	23	12	15	3	2023	1971
19	160	4	24	12	16	3	2043	1992
19	161	4	25	12	17	3	2065	2012
20	175	5	0	13	6	2	2375	2287
20	176	5	1	13	7	2	2386	2303
20	177	5	2	13	8	2	2397	2321
20	178	5	3	13	9	2	2410	2338
20	179	5	4	13	10	2	2423	2357
21	195	5	10	13	26	3	2739	2717
21	196	5	11	14	0	2	2756	2744
21	197	5	12	14	1	2	2772	2758
21	198	5	13	14	2	2	2790	2774
21	199	5	14	14	3	2	2807	2789
22	215	5	20	14	19	3	3164	3109
22	216	5	21	14	20	3	3186	3134
22	217	5	22	14	21	3	3207	3158
22	218	5	23	14	22	3	3230	3184
22	219	5	24	14	23	3	3252	3209
23	237	5	32	15	12	3	3702	3597
23	238	5	33	15	13	3	3730	3618
23	239	5	34	15	14	3	3757	3641
23	240	6	0	15	15	2	3786	3663
23	241	6	1	15	16	2	3799	3687

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