Convexity Theorems for Diametral Families of Sets

by

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Abstract

We develop a theory of convexity for arbitrary metric spaces that is based on standard convexity in Euclidean space. Versions of the classical combinatorial theorems of Radon, Helly, and Carathéodory are formulated in the general setting, and numerical invariants associated with these theorems are calculated for two specific classes of metric spaces: the surfaces of Euclidean n-spheres and the *grid*, integer n-tuples with the L^1 metric.

A family $\mathcal{F}$ of subsets of a metric space M is *diametral* provided that $\mathcal{F}$ is closed under intersections and every subset S of M is contained in some member T of $\mathcal{F}$ such that S and T have the same diameter. We show that every metric space has a unique smallest diametral family of subsets. In Euclidean space, the unique smallest diametral family is the family of compact convex sets. Using this minimality, we show that the theorems of Radon and Helly cannot be improved in the sense that no diametral family can have lower Radon and Helly numbers than those assigned by Radon's and Helly's Theorems.

The inductive property is satisfied by a family $\mathcal{F}$ of sets for which the supremum of every directed subfamily of $\mathcal{F}$ is a member of $\mathcal{F}$. Not only does every metric space have a unique smallest inductive diametral family of subsets, but this unique smallest family in the Euclidean metric space is the family of all Euclidean convex sets. By treating the unique smallest diametral and the unique smallest inductive diametral families in a general metric space as analogues of the Euclidean families of compact convex sets and convex sets respectively, we obtain a theory of metric space convexity that allows us to formulate general versions of classical theorems concerning convexity.

Much of our attention is focused on the grid. In particular we introduce *diagonal boxes* which generate the "convex" sets in the grid. In the final chapter we apply our theory to the surfaces of Euclidean n-spheres. Here the results are negative in the sense that any possible notion of spherical convexity which satisfies the basic diametral and inductive properties of Euclidean convexity must necessarily have a family of "convex" sets that is so large as to be non-interesting.

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Contents

Chapter 1

Introduction

This introduction is meant as an overview of the following chapters and as such is merely a broad outline of the many details and proofs that will follow. For unfamiliar terms, see the paper itself.

There are two main goals for this paper. The first goal is to develop a theory of convexity for general metric spaces. That is, given a set of points whose only structure is a known distance between pairs of points, we want to be able to say which subsets are convex and why are they called convex. The second goal is to look at combinatorial results from the literature of convexity theory in Euclidean space and apply these results in the general setting. Along the way, we hope to learn more about the classical results.

The starting point for a theory of convexity is naturally n-dimensional Euclidean space. Standard convexity in Euclidean space can be formulated solely in terms of the standard Euclidean distance: a set is convex provided that it contains every point lying on the shortest path between any two of its elements. This characterization of convexity motivates the usual method of generalizing convexity to metric spaces, a form of convexity called *geodesic* convexity in which a (geodesically) convex set is one which contains every point lying on any shortest path between any two of its elements. Our approach to convexity relies on an alternate characterization of standard convexity that does not utilize geodesics.

Let C_n^* denote the family of all Euclidean convex sets, and let C_n denote the subfamily of compact convex sets. Both of these families are intersecting families, meaning that each family contains all intersections of its members. We begin with a characterization of C_n. Note that every subset of Euclidean space has the same diameter as its convex hull. If $\mathcal{F}$ is an intersecting family of sets, then we can generalize convex hulls by saying that the $\mathcal{F}$-hull of a set S is the intersection of all members of $\mathcal{F}$ which contain S. Let us call an intersecting family $\mathcal{F}$ of subsets of a metric space M *diametral* provided that any subset of M has the same diameter as its $\mathcal{F}$-hull. We show that the unique smallest diametral family in Euclidean space is the family C_n of compact Euclidean convex sets.

The diametral property allows us to characterize the family C_n of Euclidean space. But just as Euclidean space has a unique smallest diametral family, so also we show that every metric space has a unique smallest diametral family which we denote by

$\Omega(M)$. We treat the family $\Omega(M)$ as the analogue in M of the family C_n in Euclidean space. Before proceeding to characterize the family C_n^* of all Euclidean convex sets and determining the appropriate generalization of C_n^* in a general metric space, let us first consider two theorems concerning C_n.

Two of the cornerstones of combinatorial geometry, the theorems of Radon and Helly, present us with a means of assigning positive integers to families of sets, numbers which say something about the "complexity" of the structure of families. Let $\mathcal{F}$ be a family of subsets of U. Informally the Radon number $r(U, \mathcal{F})$ is the least integer such that any subset of U having $r(U, \mathcal{F})$ elements can be partitioned into two blocks with intersecting $\mathcal{F}$-hulls, while the Helly number $h(U, \mathcal{F})$ is the least integer for which we are guaranteed that if any $h(U, \mathcal{F})$ members of a subfamily of $\mathcal{F}$ share a common point of U then all members of that subfamily share some common point of U. In their classical instances, the theorems of Radon and Helly amount to telling us that $r(\mathbf{R}^n, C_n) = n + 2$ and $h(\mathbf{R}^n, C_n) = n + 1$. As a family grows larger, its "complexity" as measured by Radon and Helly numbers increases. Because every diametral family in Euclidean space contains C_n, we deduce that no diametral family in Euclidean space can have smaller Radon and Helly numbers than the family C_n. It is in this sense that we show that Radon's and Helly's Theorems cannot be improved. By substituting the family $\Omega(M)$ of a general metric space for C_n in the classical instances of Radon's and Helly's Theorems, we obtain generalizations of these theorems. And because $\Omega(M)$ is the smallest diametral family in M, the generalized theorems that we obtain cannot be improved as in the sense above.

To pass from the family C_n to the family of all Euclidean convex sets, we consider the *inductive property* which is satisfied by C_n^* but not by C_n. A family of sets has the inductive property provided that the family contains the supremum of any of its chains. (In practice we find it more convenient to work with directed subfamilies which are very similar to chains.) We prove that C_n^* is a subfamily of every inductive diametral family in Euclidean space. In other words, C_n^* is the unique smallest inductive diametral family. As we might expect, there is also a unique smallest inductive diametral family $\Omega(M)^*$ in any metric space M, and it is the family $\Omega(M)^*$ that we treat as the analogue in M of the family C_n^* in $\mathbf{R}^n$.

We close chapter 2 by considering Carathéodory's Theorem which is closely related to the inductive property. Informally, the Carathéodory number of the family $\mathcal{F}$ of subsets of U is the least integer $c(U, \mathcal{F})$ such that any element x taken from the $\mathcal{F}$-hull of any subset $S \subseteq U$ is an element of the $\mathcal{F}$-hull of some subset of S which has no more than $c(U, \mathcal{F})$ elements. Carathéodory's Theorem says that $c(\mathbf{R}^n, C_n^*) = n + 1$. By calculating the number $c(M, \Omega(M)^*)$ we obtain a generalization of Carathéodory's Theorem in M.

It is not difficult to see that the C_n-hull and the C_n^*-hull of any finite Euclidean subset are the same. It is this fact that allows us to say that $r(\mathbf{R}^n, C_n) = r(\mathbf{R}^n, C_n^*)$ and transform Radon's Theorem, which is really a statement about C_n^*, into a statement about C_n. What is more interesting is that C_n^* is the unique inductive family containing C_n for which hulls coincide on all finite sets. We will show that this unique relationship between C_n and C_n^* also holds between their analogues $\Omega(M)$ and $\Omega(M)^*$ in M.

In chapter 3 we leave Euclidean space and consider a discrete metric space called

the grid, which is $\mathbf{Z}^n$ with the L^1 metric. It is known that compact convex sets in Euclidean space can be expressed as intersections of half-spaces. Using the geometry of the grid, we introduce a family of sets which can be expressed as intersections of special half-spaces in the grid. We call this the family of *diagonal boxes*.

The family of diagonal boxes is $\Omega(\mathbf{Z}^n)$, the unique smallest diametral family in $\mathbf{Z}^n$ and hence the analogue of C_n. Therefore the Radon and Helly numbers of the family of diagonal boxes are minimum among all diametral families in the grid. These numbers are calculated and shown to be $2^n + 1$ and 2^n respectively, giving us versions of Radon's and Helly's Theorems for the grid.

In the same way that we pass from C_n to C_n^* in Euclidean space by removing the compactness requirement, we pass from the family of diagonal boxes to a family of *extended diagonal boxes* in the grid. The family of extended diagonal boxes is $\Omega(\mathbf{Z}^n)^*$, the unique smallest inductive diametral family in the grid and as such is the family of "convex" sets in the grid. We show that $\Omega(\mathbf{Z}^n)^*$ deserves the status as the grid's analogue of C_n^* by showing that $\Omega(\mathbf{Z}^n)^*$ is diametral and inductive, has the same Radon number as the family $\Omega(\mathbf{Z}^n)$, and is the unique inductive family containing $\Omega(\mathbf{Z}^n)$ for which hulls agree on finite sets. In other words $\Omega(\mathbf{Z}^n)^*$ bears the same relationship to $\Omega(\mathbf{Z}^n)$ that C_n^* bears to C_n. We close chapter 3 with a version of Carathéodory's Theorem for the grid, showing that $c(\mathbf{Z}^n, \Omega(\mathbf{Z}^n)^*) = 2^{n-1}$.

Though we have been freely considering general metric spaces in this introduction, the first place that we concentrate exclusively on the setting of an arbitrary metric space is in chapter 4. There we give a characterization of $\Omega(M)$ by giving an intersection basis. And by a process which we call *inductive completion*, a means of beginning with one family $\mathcal{F}$ and enlarging it to obtain an inductive family $\mathcal{F}^*$, we characterize $\Omega(M)^*$. We show that $\Omega(M)$ and $\Omega(M)^*$ have the desired relationship, meaning that $\Omega(M)^*$ is diametral and inductive, has the same Radon number as the family $\Omega(M)$, and is the unique inductive family containing $\Omega(M)$ for which hulls agree on finite sets. Thus we can formulate versions of Radon's, Helly's, and Carathéodory's Theorems for M.

Finally in chapter 5 we consider the n-dimensional Euclidean sphere S^n and spherical convexity. We characterize the families $\Omega(S^n)$ and $\Omega(S^n)^*$ and calculate their Radon, Helly, and Carathéodory numbers. It turns out that the every possible subset of S^n which does not contain antipodal points of the sphere is a member of $\Omega(S^n)^*$. Because $\Omega(S^n)^*$ is the smallest inductive diametral family in S^n, any possible notion of spherical convexity which satisfies the basic diametral and inductive properties of Euclidean convexity must necessarily have a family of "convex" sets that is so large as to be non-interesting.

8

Chapter 2

Euclidean Convexity

2.1   Preliminaries

In this section introduce a class of families of sets which we call the *diametral* families and lay the groundwork for an investigation into the relationship between this class and the notion of convexity in Euclidean space.

Capitol letters $A, B, C, \ldots, Z$ will denote sets, while script letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots, \mathcal{Z}$ will denote families of sets. The letter M is reserved for metric spaces. If d is the metric of M, then we denote the diameter of a subset $T \subseteq M$ by

$$\mu(T) = \sup\{d(x,y) : x,y \in T\}.$$

If T is an unbounded set then $\mu(T) = \infty$.

Every member of a given family will be a subset of some fixed set. Often that fixed set will be the Euclidean space $\mathbf{R}^n$. References to distances in $\mathbf{R}^n$ will assume the standard Euclidean metric which we denote by δ.

One special family of sets in the Euclidean space $\mathbf{R}^n$ is the family of all *convex* sets where we say that a subset $T \subseteq \mathbf{R}^n$ is convex provided that given any two elements $x, y \in T$, the entire line segment between x and y lies in T. A Euclidean set that is both compact and convex is called a *convex body*. For any subset T of $\mathbf{R}^n$, there is a unique smallest convex set containing T; this convex set is called the *convex hull* of T and is denoted by $\mathrm{conv}(T)$.

Soon we will define *diametral families*. The definition is based on two important properties of the family of all convex subsets of Euclidean space. The first property is that the intersection of arbitrarily many convex sets is also a convex set. And the second property is based on the fact that though every subset T of Euclidean space is contained in its convex hull $\mathrm{conv}(T)$, both T and $\mathrm{conv}(T)$ always have the same diameter. In order to formulate the definition of diametrality for families in general metric spaces, we now turn to consider the appropriate generalizations of these two properties in a broader setting.

A family of sets is an *intersecting* family when it is closed under arbitrary intersections of its members. The notion of convex hulls can be generalized to any intersecting family as follows. Let U be a fixed set and let $\mathcal{F}$ be an intersecting family of subsets

of U. We define the $\mathcal{F}$-hull of a subset $T \subseteq U$ as the intersection of all members of $\mathcal{F}$ which contain the set T, that is

$$hull_{\mathcal{F}}(T) = \bigcap_{T \subseteq A \in \mathcal{F}} A. \qquad (2.1)$$

Note that $U \in \mathcal{F}$ because intersecting families are closed under the empty intersection. As a consequence for (2.1), there will always be some member A in $\mathcal{F}$ which contains T. The convex hull in Euclidean space is merely the $\mathcal{F}$-hull when the family $\mathcal{F}$ consists of all the convex sets.

Definition: *A family $\mathcal{F}$ of subsets of a metric space M is* **diametral** *if and only if*

>   *1. $\mathcal{F}$ is an intersecting family.*
>
>   *2. For every bounded subset $T \subseteq M$ we have $\mu(hull_{\mathcal{F}}(T)) = \mu(T)$.*

There is an equivalent way of defining diametrality which will be useful to us, a way that does not refer to hulls. Consider an intersecting family $\mathcal{F}$ of subsets of M and a given bounded subset $T \subseteq M$. If $\mathcal{F}$ is a diametral family, then T is contained in some member of $\mathcal{F}$ having the same diameter that T has, namely $T \subseteq hull_{\mathcal{F}}(T)$. Conversely if we know that T is contained in some member $T' \in \mathcal{F}$ having the same diameter that T has, then we know that $\mathcal{F}$ is diametral because

$$T \subseteq hull_{\mathcal{F}}(T) \subseteq T'.$$

Therefore we have shown the following result which is an assertion that diametral families are "sufficiently large".

LEMMA 1 *A family $\mathcal{F}$ of subsets of a metric space is* diametral *if and only if*

>   *1. $\mathcal{F}$ is an intersecting family.*
>
>   *2. Every bounded subset T is contained in a member of $\mathcal{F}$ having the same diameter that T has.*

∎

Beginning in the next section we examine the relationship between diametral families and Euclidean convexity. The definition of diametrality captures much, but not all, of what it means to be convex. But for now we turn to a discussion of hulls, letting $\mathcal{F}$ be an intersecting family of subsets of U for the remainder of this section.

An important property of hulls is that if $\mathcal{F}$ and $\mathcal{H}$ are both intersecting families of subsets of the same set U, where $\mathcal{F} \subseteq \mathcal{H}$ (ie $\mathcal{F}$ is a subfamily of $\mathcal{H}$), then the containment relation of hulls is the reverse of the containment for the families. In other words for any set $T \subseteq U$,

$$hull_{\mathcal{H}}(T) \subseteq hull_{\mathcal{F}}(T). \qquad (2.2)$$

One immediate consequence of (2.2) is the following.

LEMMA 2 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two intersecting families of subsets of a metric space where $\mathcal{F}_1$ is a subfamily of $\mathcal{F}_2$. If $\mathcal{F}_1$ is diametral, then so is $\mathcal{F}_2$.*

PROOF For any bounded subset T we know from (2.1) and (2.2) that

$$T \subseteq hull_{\mathcal{F}_2}(T) \subseteq hull_{\mathcal{F}_1}(T).$$

Together with the fact that $\mathcal{F}_1$ is diametral, this implies that

$$\mu(T) \leq \mu(hull_{\mathcal{F}_2}(T)) \leq \mu(hull_{\mathcal{F}_1}(T)) = \mu(T).$$

Therefore $\mu(T) = \mu(hull_{\mathcal{F}_2}(T))$, and $\mathcal{F}_2$ is by definition diametral. ∎

Note $hull_{\mathcal{F}}$ is a set operator that maps subsets of U to subsets of U. In fact because of the intersecting property of $\mathcal{F}$, the $\mathcal{F}$-hull of any subset of U is a member of $\mathcal{F}$. This gives us a one-to-one correspondence between the collection of $\mathcal{F}$-hulls and the family $\mathcal{F}$ itself, namely

$$\mathcal{F} = \{hull_{\mathcal{F}}(S) : S \subseteq U\}.$$

Equivalently, and more useful for our purposes, we have that for any subset $T \subseteq U$

$$T \in \mathcal{F} \iff T = hull_{\mathcal{F}}(T). \tag{2.3}$$

The hull operator $hull_{\mathcal{F}}$ is a *closure relation* on U, meaning that $hull_{\mathcal{F}}(T)$ is a map from subsets of U to subsets of U which satisfies the following three properties:

$$T \subseteq hull_{\mathcal{F}}(T) \tag{2.4}$$

$$\text{If } S \subseteq T \text{ then } hull_{\mathcal{F}}(S) \subseteq hull_{\mathcal{F}}(T) \tag{2.5}$$

$$hull_{\mathcal{F}}(hull_{\mathcal{F}}(T)) = hull_{\mathcal{F}}(T)$$

On the other hand given any closure relation on U, the image of the closure relation is an intersecting family of subsets of U. What this means is that we can equivalently take either point of view: closure relations or intersecting families. We will concentrate on intersecting families, but everything that we do could be cast in terms of closure relations instead.

2.2 Convex Bodies

As seen in the previous section, the definition of diametrality is motivated by two properties of Euclidean convexity. In this section we will see that these two properties of convexity permit a characterization of the family of convex bodies in Euclidean space. In particular we show that the family of convex bodies is a very special diametral family in that the family of convex bodies is a subfamily of any other Euclidean diametral family. In other words, the family of convex bodies is the

unique smallest diametral family in $\mathbf{R}^n$. As an immediate corollary we will conclude that the class of Euclidean diametral families is closed under intersection.[1]

In the previous paragraph we overlook a technical consideration. The family consisting of the convex bodies alone cannot be a diametral family because it is not even an intersecting family due to the fact that the family of convex bodies is not closed under the empty intersection, ie due to the fact that the set $\mathbf{R}^n$ itself is not a convex body. In order to have an intersecting family, we make the following definition.

Definition: *We let C_n denote the family whose membership includes every convex body in $\mathbf{R}^n$ as well as the set $\mathbf{R}^n$ itself.*

It is C_n which we call the *family of convex bodies*. We claim that C_n is not only an intersecting family (clearly), but also a diametral family. Given any bounded set $T \subseteq \mathbf{R}^n$, the set $hull_{C_n}(T)$ is the topological closure of the convex hull of T. It is well-known that a set, its convex hull, and its closure all have the same diameter in $\mathbf{R}^n$. From this fact we deduce that the family C_n is a diametral family in Euclidean space. Theorem 3 shows that C_n is the unique smallest diametral family in Euclidean space.

THEOREM 3 *If $\mathcal{F}$ is an intersecting family of subsets of $\mathbf{R}^n$, then*

$$\mathcal{F} \text{ is diametral} \iff C_n \subseteq \mathcal{F}.$$

PROOF Let $\mathcal{F}$ be an intersecting family of subsets of $\mathbf{R}^n$.

It is an immediate consequence of lemma 2 and the fact that C_n is a diametral family that if $C_n \subseteq \mathcal{F}$ then $\mathcal{F}$ is also a diametral family. So suppose that $\mathcal{F}$ is a diametral family and let us argue that C_n is a subfamily of $\mathcal{F}$.

Because the set $\mathbf{R}^n$ is a member of both the families C_n and $\mathcal{F}$, it suffices to show that an arbitrary convex body in $\mathbf{R}^n$ is a member of $\mathcal{F}$. Let T be a convex body, that is $T \in C_n - \{\mathbf{R}^n\}$.

The proof that $T \in \mathcal{F}$ will consist of establishing two facts.

1. Every closed ball in $\mathbf{R}^n$ is a member of $\mathcal{F}$.

2. T can be expressed as an intersection of closed balls in $\mathbf{R}^n$.

Because $\mathcal{F}$ contains the intersections of all of its members, the above two facts are sufficient to establish that $T \in \mathcal{F}$.

(Proof of 1): Let B be an arbitrary closed ball[2] in Euclidean space. We know that B is a subset of $hull_{\mathcal{F}}(B)$ by (2.4). Furthermore because B is a bounded set and $\mathcal{F}$ is a diametral family, we have by the definition of diametrality that

$$\mu(B) = \mu(hull_{\mathcal{F}}(B)).$$

---

[1] By the intersection of families, we mean the new family whose members are members of each of the old families.

[2] B is a closed ball provided that there exists a real number $m \geq 0$ and an element $z \in \mathbf{R}^n$ such that $B = \{w \in \mathbf{R}^n : \delta(z, w) \leq m\}$.

It is well-known that a closed ball in Euclidean space cannot be a proper subset of another set having the same diameter. Therefore $B = hull_{\mathcal{F}}(B)$ and hence by (2.3) we know that $B \in \mathcal{F}$.

(Proof of 2): In order to show that T can be written as an intersection of closed balls, it suffices to show that for each $x \notin T$ there exists a closed ball B_x containing the set T but not containing x. If this were the case then

$$T = \bigcap_{x \notin T} B_x$$

is an expression of T as an intersection of closed balls.

Informally, the approach that we take is to show that given any element $x \notin T$, we can always find a closed ball such that T is on the inside of the closed ball and x is on the outside. We do this by considering closed balls which have centers lying on the "opposite" side of T from x and which have diameters just big enough to contain the set T. No matter how close x is to T, it is still possible to find an appropriate closed ball because we can locate the center of a ball arbitrarily far from T, making the ball's boundary that lies between T and x as "flat" as we like.

Let us make these ideas more precise. Choose $x \notin T$. Because T is closed, we know that

$$\inf\{\delta(x,y) : y \in T\} > 0.$$

In fact, there exists a unique element $a \in T$ of minimum distance from x, ie

$$\delta(x,a) = \inf\{\delta(x,y) : y \in T\} > 0.$$

To see the uniqueness of the point a, suppose that a' is an element of T satisfying $\delta(x,a) = \delta(x,a')$. If $a \neq a'$ then there is some point $p \in \mathbf{R}^n$ on the interior of the line segment joining a to a'. The point p lies in T by convexity and is closer to x than either a or a', contradicting the fact that a was chosen in T to have minimum distance from x.

Let $\vec{v}$ denote the ray which begins at x and proceeds in the direction of a. Let c be the point of $\vec{v}$ lying midway between x and a, and let H denote the hyperplane which passes through c and is orthogonal to $\vec{v}$.

Claim: *H strictly separates the point x and the set T.*

Let us see how the theorem follows from the claim. The hyperplane H determines two open half-spaces of $\mathbf{R}^n$; let Λ denote the open half-space which does not contain x. If the claim is true then $T \subseteq \Lambda$. Every point of Λ lies on the surface of a single n-dimensional sphere which is tangent to H at c. Consider the map $f : \Lambda \to \mathbf{R}$ which assigns to each point in Λ the radius of the unique such n-sphere containing that point. Because the map f is continuous, the range $f(T)$ of the compact set T is a compact subset of $\mathbf{R}$. Thus there exists a real number u defined by

$$u = \sup\{f(t) : t \in T\}.$$

Let z denote the point of Λ which lies on $\vec{v}$ and for which $d(z,c) = u$. We show that

13

a closed ball containing T, but not containing x, is given by

$$\text{Ball}(z, u) = \{ y \in \mathbf{R}^n : \delta(z, y) \leq u \}.$$

To see this, first note that c lies on ray $\vec{v}$ between z and x, so $d(z, x) > d(z, c) = u$. Therefore $x \notin \text{Ball}(z, u)$. Also note that any point y in T lies on the surface of some n-sphere S' which is tangent to H at c and which has a radius no greater than u. Hence the n-sphere S' is a subset of $\text{Ball}(z, u)$. Because y is arbitrary, we conclude that $T \subseteq \text{Ball}(z, u)$. Let $B_x = \text{Ball}(z, u)$ and the theorem follows from the claim.

To finish the proof, it remains to establish the claim that the hyperplane H strictly separates T and x. This involves showing that T is bounded away from H, meaning that elements of T are not arbitrarily close to H. This proof will be a variant of the many separating hyperplane theorems for convex sets. (See [3] or [15].)

Let H' be the hyperplane which passes through the point a and which is orthogonal to $\vec{v}$. The hyperplanes H and H' are distinct and parallel. Therefore there is a minimum positive distance which separates them. If T does not intersect the open region between hyperplanes H and H', then T is bounded away from H. So we are finished when we show that H' is a supporting hyperplane of T, which means that if Λ' denotes the open halfspace of $\mathbf{R}^n$ determined by H' which contains the point x, then T does not intersect Λ'.

For contradiction suppose that there is some point $b \in T \cap \Lambda'$. Because $b \in \Lambda'$, the line segment joining b to a contains some point q which lies closer than $\delta(x, a)$ to a. But by the convexity of T, we have $q \in T$. This contradicts the fact that a is the nearest element of T to x. ∎

Recall that the intersection of two families is the new family whose every member is a member of each of the old families.

COROLLARY 4 *The class of diametral families in $\mathbf{R}^n$ is closed under intersections.*

PROOF Let $\mathcal{F}$ denote a family which is an intersection of diametral families. By theorem 3 we know that each of these diametral families that make up the intersection $\mathcal{F}$ must contain $\mathcal{C}_n$, so $\mathcal{F}$ necessarily contains $\mathcal{C}_n$. The intersection of intersecting families is an intersecting family, so it follows by theorem 3 again that $\mathcal{F}$ is a diametral family. ∎

2.3   Combinatorial Invariants of Convex Bodies

We now turn our attention to two combinatorial invariants associated with families of subsets, namely the Radon and Helly numbers. These numbers provide a convenient quantitative tool for working with families. Informally we say that "nice" families have low Radon and Helly numbers. It is in this respect that we demonstrate that no diametral family of Euclidean sets is "nicer" than the family $\mathcal{C}_n$ of convex bodies.

First we look at the classical theorems of Radon and Helly and then we see how these theorems allow us to assign integers to families. Theorems 5 and 9 show a sense in which Radon's Theorem and Helly's Theorem are best possible for diametral families. The reason that these classical theorems are best possible for diametral families is that both involve the family C_n, which we know by theorem 3 is the unique smallest diametral family in Euclidean space. For a general survey of research relating to these theorems, see [5].

Radon's Theorem: *Each set of $n + 2$ or more elements in $\mathbf{R}^n$ can be expressed as the union of two disjoint nonempty sets whose convex hulls share a common element.*

Helly's Theorem was discovered by Helly in 1913 but first published by Radon in 1921. The hypotheses of Helly's Theorem can be given in more generality than we have stated them here.

Helly's Theorem: *Suppose K is a family of at least $n + 1$ convex bodies in $\mathbf{R}^n$. If each $n + 1$ members of K share a common element, then there is an element common to all members of K.*

Both of the above theorems are optimal in the sense that $n+2$ and $n+1$ respectively are the least integers that make each theorem true. These two theorems motivate the definition of Radon numbers and Helly numbers which we now define in a general way. Let U be a fixed set.

Suppose that $\mathcal{F}$ is an intersecting family of subsets of U. The *Radon number* $r(U, \mathcal{F})$ of the family $\mathcal{F}$ is the least positive integer such that each set of $r(U, \mathcal{F})$ elements of U can be expressed as the union of two disjoint nonempty sets whose $\mathcal{F}$-hulls have nonempty intersection. If no such positive integer exists, then we say $r(U, \mathcal{F}) = \infty$.

Following [6], we say that a *Radon $\mathcal{F}$-partition* of a subset T of U is a partition of T into two disjoint nonempty sets which have intersecting $\mathcal{F}$-hulls. Then the Radon number of $\mathcal{F}$ is the least number for which every subset of U having that cardinality admits a Radon $\mathcal{F}$-partition.

Now let $\mathcal{F}$ be any family of subsets of U, not necessarily an intersecting family. The *Helly number* $h(U, \mathcal{F})$ of the family $\mathcal{F}$ is the least positive integer such that if K is a subfamily of $\mathcal{F}$ satisfying the following two conditions,

1. K has at least $h(U, \mathcal{F})$ members and

2. each $h(U, \mathcal{F})$ members of K have nonempty intersection,

then all members of K share some common element of U. If no such positive integer exists, then we say that $h(U, \mathcal{F}) = \infty$.

In theorem 5 we show that a strict lower bound for the Radon and Helly numbers of diametral families in $\mathbf{R}^n$ is given by Radon's and Helly's Theorems respectively. In other words $n + 2$ and $n + 1$ provide strict lower bounds for the Radon and Helly numbers of all diametral families in $\mathbf{R}^n$.

15

Theorem 9 of this section provides an even stronger version of theorem 5 for the case of Helly numbers. In essence theorem 9 says that we can weaken the definition of diametral families to obtain a larger class of *extra-diametral* families and still have the $n + 1$ lower bound for the Helly numbers of families in this larger class.

THEOREM 5 *Let $\mathcal{F}$ be a diametral family of subsets of $\mathbf{R}^n$.*

1. *The Radon number $r(\mathbf{R}^n, \mathcal{F})$ is at least $n + 2$.*

2. *The Helly number $h(\mathbf{R}^n, \mathcal{F})$ is at least $n + 1$.*

PROOF   The proof of this theorem is accomplished by means of three propositions. Proposition 6 tells us that as families become larger, their Radon and Helly numbers also become larger. Therefore no diametral family can have smaller Radon and Helly numbers than the numbers of the diametral family $\mathcal{C}_n$ because all diametral families in $\mathbf{R}^n$ contain $\mathcal{C}_n$ as a subfamily.

Therefore the theorem is proven by calculating the Radon and Helly numbers of the family $\mathcal{C}_n$, showing that $r(\mathbf{R}^n, \mathcal{C}_n) = n + 2$ and $h(\mathbf{R}^n, \mathcal{C}_n) = n + 1$. We calculate these numbers directly from Radon's and Helly's Theorems after dealing with two technical details in propositions 7 and 8. Radon's Theorem is really a statement that the Radon number of the family of all Euclidean convex sets is $n + 2$. Proposition 7 shows how we can easily convert Radon's Theorem into a statement about the Radon number of the family $\mathcal{C}_n$. Helly's Theorem tells us that the Helly number of the family $\mathcal{C}_n - \{\mathbf{R}^n\}$ is $n + 1$. Proposition 8 deals with the fact that the set $\mathbf{R}^n$ was artificially adjoined to $\mathcal{C}_n$ to give us an intersecting family. With these two propositions we obtain the desired Radon and Helly numbers for $\mathcal{C}_n$ directly from Radon's Theorem and Helly's Theorem respectively.   ∎

PROPOSITION 6 *Let $\mathcal{F}_1 \subseteq \mathcal{F}_2$ be two intersecting families of subsets of U. Then*

1. $r(U, \mathcal{F}_1) \leq r(U, \mathcal{F}_2)$.

2. $h(U, \mathcal{F}_1) \leq h(U, \mathcal{F}_2)$.

PROOF   (Proof of 1): The Radon number of any family of sets is at least two, yet if $r(U, \mathcal{F}_1) = 2$ then $r(U, \mathcal{F}_1) \leq r(U, \mathcal{F}_2)$ trivially. So we consider the case where $r = r(U, \mathcal{F}_1) > 2$. Note that r may not be finite.

Given any positive integer m with $2 \leq m < r$, by the minimality involved in the definition of Radon numbers there is a set A_m consisting of m elements of U which does not admit a Radon $\mathcal{F}_1$-partition. In other words A_m cannot be written as the union of two disjoint nonempty sets whose $\mathcal{F}_1$-hulls intersect. By (2.2) we know that $hull_{\mathcal{F}_2}(T) \subseteq hull_{\mathcal{F}_1}(T)$ for any set $T \subseteq \mathbf{R}^n$, so it follows that each set A_m also will not admit an $\mathcal{F}_2$-partition. Therefore $r(U, \mathcal{F}_2) \geq r$.

(Proof of 2): We consider the nontrivial case in which $h = h(U, \mathcal{F}_1)$ is at least two. Note that h may not be finite. By the definition of a Helly number, given any positive integer m where $1 \leq m < h$, there is a subfamily $\mathcal{K}_m$ of $\mathcal{F}_1$ satisfying

1. $|\mathcal{K}_m| \geq m$.

2. Any m members of $\mathcal{K}_m$ have nonempty intersection.

3. There is no element of U common to every member of $\mathcal{K}_m$.

Because each $\mathcal{K}_m$ is also a subfamily of $\mathcal{F}_2$, we conclude that $h(U, \mathcal{F}_2) \geq h$. ∎

As a result of theorem 3 and proposition 6, we know that the value of the Radon number of any diametral family of subsets of $\mathbf{R}^n$ is at least $r(\mathbf{R}^n, \mathcal{C}_n)$. We use proposition 7 to show that the value of the Radon number $r(\mathbf{R}^n, \mathcal{C}_n)$ is equal to $n + 2$. Proposition 7 tells us that when acting on finite sets, we can freely interchange the standard convex hull operator *conv* and the $\mathcal{C}_n$-hull. So when we consider finite sets as we do in Radon's Theorem, the existence of a Radon $\mathcal{C}_n$-partition is equivalent to the existence of a Radon partition with respect to the family of all convex sets. Therefore the Radon number of the family $\mathcal{C}_n$ is equal to the Radon number of the family of all convex sets which by Radon's Theorem is equal to $n + 2$.

The relationship between families whose hulls agree on finite sets will be explored in great detail in later sections.

PROPOSITION 7 *Let T be a finite subset of $\mathbf{R}^n$. Then the convex hull of T is given by*

$$conv\,(T) = \mathrm{hull}_{\mathcal{C}_n}(T).$$

PROOF The operator *conv* is the hull operator for the family of all convex sets. Because $\mathcal{C}_n$ is a subfamily of the family of all convex sets, we know by (2.2) that

$$conv\,(T) \subseteq hull_{\mathcal{C}_n}(T).$$

The reverse containment follows from the fact that $conv(T)$ is a convex body whenever T is finite. ∎

From theorem 3 and proposition 6 we know that the value of the Helly number of any diametral family of subsets of $\mathbf{R}^n$ is at least $h(\mathbf{R}^n, \mathcal{C}_n)$. That the Helly number $h(\mathbf{R}^n, \mathcal{C}_n)$ equals $n + 1$ is a consequence of proposition 8. Helly's Theorem tells us that the Helly number of the family of convex bodies, without the artificial addition of the member $\mathbf{R}^n$, is $n + 1$. Proposition 8 is the price we pay for the convenience of including the set $\mathbf{R}^n$ as a member of $\mathcal{C}_n$ in order to have an intersecting family.

PROPOSITION 8 *If $\mathcal{F}$ is a family of subsets of U, then*

$$h(U, \mathcal{F}) = h(U, \mathcal{F} \cup \{U\}).$$

PROOF We know that $h(U, \mathcal{F}) \leq h(U, \mathcal{F} \cup \{U\})$ by proposition 6. To show the reverse inequality, it suffices to consider the case where $h = h(U, \mathcal{F})$ is finite. Suppose that $\mathcal{K}$ is a subfamily of $\mathcal{F} \cup \{U\}$ having at least h members such that any h members have nonempty intersection. By showing that all members of $\mathcal{K}$ share a common element of U, we establish that $h(U, \mathcal{F} \cup \{U\}) \leq h$. There are two cases to consider.

Case 1 : **Suppose** $U \notin \mathcal{K}$. Then $\mathcal{K}$ is a subfamily of $\mathcal{F}$, and the fact that all members of $\mathcal{K}$ share a common element of U follows from the definition of the Helly number h.

Case 2 : **Suppose** $U \in \mathcal{K}$. If $|\mathcal{K}| = h$ then by assumption all the members of $\mathcal{K}$ have nonempty intersection. Otherwise $|\mathcal{K}| \geq h+1$. Then $\mathcal{K} - \{U\}$ is a subfamily of $\mathcal{F}$ having at least h members such that any h members have nonempty intersection. By the definition of the Helly number h, there is an element of U common to every member of $\mathcal{K} - \{U\}$. This common element is then clearly common to every member of $\mathcal{K}$.

∎

The goal of the remainder of this section is the establishment of theorem 9 as a strengthening of the second part of theorem 5. We show that the class of Euclidean diametral families is part of a larger class of "extra-diametral" families, none of whose Helly numbers have value less than $n + 1$. More generally, in proposition 10, we show that the least Helly number of a diametral family of subsets of U is never greater than the Helly number of an extra-diametral family in U.

We have defined the notion of diametrality to capture two property of Euclidean convexity. The first property is that diametral families are automatically intersecting families, reflecting the fact that the intersection of convex sets is a convex set. An advantage of having intersecting families is that we have hull operators, a necessity for defining the Radon number of a family. But hull operators are not necessary for defining Helly numbers, so families which are not intersecting families still have well-defined Helly numbers. In order to explore Helly numbers for non-intersecting families, we extend the notion of diametrality to include non-intersecting families.

The second property of Euclidean convexity captured by the definition of diametrality is that a set and its hull have the same diameter. But lemma 1 gives an equivalent form of the definition of diametrality that does not require the concept of hulls or intersecting families. Our plan is to drop the intersecting property part of the definition of diametrality and consider the second property of convexity alone as given by lemma 1.

Definition: $\mathcal{F}$ *is an* **extra-diametral** *family of subsets of a metric space M if and only if*

$$T \subseteq M \text{ is bounded } \Rightarrow \exists A \in \mathcal{F} \text{ such that } T \subseteq A \text{ and } \mu(T) = \mu(A).$$

The question is now whether in the larger class of *extra-diametral* families in $\mathbf{R}^n$ there is some family whose Helly number is smaller than $n + 1$.

THEOREM 9 *If $\mathcal{F}$ is an extra-diametral family of subsets of $\mathbf{R}^n$ then*

$$h(\mathbf{R}^n, \mathcal{F}) \geq n + 1.$$

PROOF Let $\mathcal{F}$ be an extra-diametral family of subsets of $\mathbf{R}^n$. By allowing all possible intersections of members of $\mathcal{F}$, we generate the family $\pi\mathcal{F}$. That is, the members of $\pi\mathcal{F}$ are subsets of $\mathbf{R}^n$ that are intersections of members of $\mathcal{F}$. The family $\pi\mathcal{F}$ is both an intersecting family and an extra-diametral family, so by lemma 1 we know that $\pi\mathcal{F}$ is diametral. Therefore by theorem 5, the family $\pi\mathcal{F}$ has a Helly number whose value is at least $n + 1$. But the first part of proposition 10 tells us that the Helly numbers of the families $\mathcal{F}$ and $\pi\mathcal{F}$ are equal. ∎

PROPOSITION 10 *Let $\mathcal{F}$ be a family of subsets of U and let $\pi\mathcal{F}$ denote the intersecting family generated by $\mathcal{F}$. Then*

$$h(U, \mathcal{F}) = h(U, \pi\mathcal{F}).$$

Therefore the Helly number of an extra-diametral family can never be lower than the least Helly number of a diametral family.

PROOF The proof of the second part of proposition 10 follows from the first part in the same way that theorem 9 does. The proof that we give here establishes the first part of proposition 10.

By proposition 6 we know that $h(U, \mathcal{F}) \le h(U, \pi\mathcal{F})$ and also that if $h(U, \mathcal{F})$ is not finite, then neither is $h(U, \pi\mathcal{F})$. Therefore it suffices to consider $h = h(U, \mathcal{F})$ finite and show that $h(U, \pi\mathcal{F}) \le h$.

Let $\mathcal{K}$ be a subfamily of $\pi\mathcal{F}$ having at least h members such that the intersection of any h members from $\mathcal{K}$ is nonempty. The proposition follows when we show that there is an element common to all members of $\mathcal{K}$ because then the definition of Helly numbers tells us that $h(U, \pi\mathcal{F}) \le h$. Consider the family

$$\mathcal{K}' = \{T \in \mathcal{F} : A \subseteq T \text{ for some } A \in \mathcal{K}\}.$$

We claim that there is a element of U common to every member of $\mathcal{K}'$. Let us see how the proposition follows from this claim. Let x be a point common to every member of $\mathcal{K}'$. Note that because $\mathcal{K}$ is a subfamily of $\pi\mathcal{F}$, any $A \in \mathcal{K}$ can be expressed as an intersection of members from $\mathcal{F}$. But this means that any $A \in \mathcal{K}$ can be expressed as an intersection of members of $\mathcal{K}'$. Consequently $x \in A$, and we have shown that the element x is common to every member of $\mathcal{K}$.

Now to prove the claim. Choose an integer m such that $m \le h$ and $m \le |\mathcal{K}'|$. Consider any members $T_1, \ldots, T_m \in \mathcal{K}'$. For each $T_i \in \mathcal{K}'$ where $1 \le i \le m$, there exists $A_i \in \mathcal{K}$ such that $A_i \subseteq T_i$. Because there are at most h different A_i's, all of which are members of $\mathcal{K}$, we know by assumption that the intersection of the A_i's is nonempty. But $\bigcap_{i=1}^{m} A_i \subseteq \bigcap_{i=1}^{m} T_i$ shows that the sets $T_1, \ldots, T_m$ also have nonempty intersection.

Now if $|\mathcal{K}'| \le h$ then letting $m = |\mathcal{K}'|$ allows us to conclude that there is an element of U common to every member of $\mathcal{K}'$. Otherwise $|\mathcal{K}'| > h$. Note that $\mathcal{K}'$ is a subfamily of $\mathcal{F}$. Therefore having shown that the intersection of any h members of $\mathcal{K}'$ is nonempty (the $m = h$ case above), we can conclude from the definition of the

Helly number h that there is an element common to all members of $\mathcal{K}'$. Thus the claim is established. ∎

Given any extra-diametral family $\mathcal{F}$ of subsets, proposition 10 tells us that the "complexity" of $\mathcal{F}$ as measured by Helly numbers is the same as the complexity of the diametral family $\pi\mathcal{F}$. Nothing is gained from the standpoint of generalizations of Helly's Theorem in considering the larger class of extra-diametral families. For this reason we will limit ourselves to discussions concerning intersecting families for which we have the convenient notion of hull operators.

2.4 Inductive Families

We have seen that the two properties of Euclidean convexity captured by the definition of diametrality allow us to characterize the family C_n of convex bodies as the smallest family satisfying these properties. In this section we look at a third property of convexity called the *inductive* property. The family C_n is not inductive, but the family consisting of all convex sets is. We show that the inductive property plus the two properties from the definition of diametrality allow us to characterize C_n^*. The precise statement of this characterization is theorem 11 which tells us that the family of all Euclidean convex sets is the unique smallest inductive diametral family.

Hereafter we will use C_n^* to denote the family of all Euclidean convex sets. The reason for this choice of notation will be apparent in chapter 4 when we introduce inductive completion. We need several definitions in order to proceed and will follow the notation of [2].

Let $\mathcal{F}$ be a family of subsets of U. An *upper bound* of $\mathcal{F}$ is a subset of U which contains every member of $\mathcal{F}$. The family $\mathcal{F}$ is *directed* provided that $\mathcal{F}$ contains an upper bound for any pair of its members. By induction we see that a directed family contains an upper bound for any finite number of its members. The unique smallest upper bound for the family $\mathcal{F}$, called the *supremum* of $\mathcal{F}$, is denoted by

$$\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A.$$

Definition: *A family $\mathcal{F}$ is* **inductive** *provided that $\mathcal{F}$ contains the supremum of each of its directed subfamilies.*

Note that C_n is not an inductive family. To see this, consider the directed subfamily of C_n consisting of all closed balls centered at some fixed point $z \in \mathbf{R}^n$ and having radius strictly less than 1. The supremum of this subfamily is the open unit ball centered at z. But open balls are not convex bodies, and so the fact that C_n has a directed subfamily whose supremum is not a member of C_n tells us that C_n is not inductive.

On the other hand C_n^* is an inductive family. To see this let $\mathcal{F} = \{A_j\}_{j \in J}$ be a directed subfamily of C_n^*. To see that C_n^* is inductive, we must show that the set $\sup \mathcal{F}$, that is the set $\bigcup_{j \in J} A_j$, is convex. Pick $x, y \in \sup \mathcal{F}$. Then there exists $p, q \in J$

such that $x \in A_p$ and $y \in A_q$. Because $\mathcal{F}$ is directed, there also exists $r \in J$ such that $A_p \cup A_q \subseteq A_r$. Thus A_r is a convex set containing both x and y, so A_r contains the entire line segment joining x to y. The line segment between x and y is consequently contained in $\sup \mathcal{F}$, and so we know that $\sup \mathcal{F}$ is a member of C_n^*.

Theorem 11 shows that C_n^* is the unique smallest diametral family in the class of inductive families in $\mathbf{R}^n$, or equivalently that C_n^* is the unique smallest inductive family in the class of all diametral families. This theorem is an analogue for inductive families of theorem 3.

THEOREM 11 *If $\mathcal{F}$ is an inductive intersecting family of subsets of $\mathbf{R}^n$, then*

$$\mathcal{F} \text{ is diametral } \iff C_n^* \subseteq \mathcal{F}.$$

PROOF Let $\mathcal{F}$ be an inductive intersecting family of subsets of $\mathbf{R}^n$.

Suppose $C_n^* \subseteq \mathcal{F}$. The family C_n of convex bodies is a subfamily of the family C_n^* of all convex sets, so it follows that C_n is a subfamily of $\mathcal{F}$ as well. Because $\mathcal{F}$ is an intersecting family, we can conclude by theorem 3 that $\mathcal{F}$ is diametral.

Suppose now that $\mathcal{F}$ is diametral. By theorem 3 we know that $C_n \subseteq \mathcal{F}$. Because $\mathcal{F}$ is inductive, it contains the supremum of all of its directed subfamilies. Therefore we know that $C_n^* \subseteq \mathcal{F}$ when we prove proposition 13. ∎

Before we state proposition 13 and finish the proof of theorem 11, we prove lemma 12. In essence this lemma says that the members of an inductive intersecting family are precisely those sets which can be expressed as the unions of the hulls of their finite subsets.

LEMMA 12 *Let $\mathcal{F}$ be an intersecting family of subsets of U. If $A \in \mathcal{F}$, then the family*

$$\mathcal{K}_A = \{\text{hull}_{\mathcal{F}}(A_f) : A_f \text{ is a finite subset of } A\}$$

is a directed subfamily of $\mathcal{F}$ such that $A = \sup \mathcal{K}_A$. Furthermore if $\mathcal{F}$ is inductive, then the converse is true.

PROOF Let $\mathcal{F}$ be an intersecting family of subsets of U. Suppose that $A \in \mathcal{F}$. Then $\mathcal{K}_A$ is a subfamily of $\mathcal{F}$ by the definition of $\mathcal{F}$-hulls. To see that $\mathcal{K}_A$ is a directed family, let $hull_{\mathcal{F}}(A_1)$ and $hull_{\mathcal{F}}(A_2)$ be two members of $\mathcal{K}_A$ where A_1 and A_2 are finite subsets of A. Then $A_1 \cup A_2$ is also finite subset of A, so $hull_{\mathcal{F}}(A_1 \cup A_2) \in \mathcal{K}_A$. Note that by (2.5), we have

$$hull_{\mathcal{F}}(A_1) \cup hull_{\mathcal{F}}(A_2) \subseteq hull_{\mathcal{F}}(A_1 \cup A_2),$$

so $\mathcal{K}_A$ is by definition a directed family.

We now show that $A = \sup \mathcal{K}_A$. If $x \in A$ then

$$x \in \{x\} \subseteq hull_{\mathcal{F}}(\{x\}) \in \mathcal{K}_A.$$

Therefore $x \in \sup \mathcal{K}_A$, and we deduce that $A \subseteq \sup \mathcal{K}_A$.

To see the reverse containment, suppose $y \in \sup \mathcal{K}_A$. By the definition of supremums, some member of $\mathcal{K}_A$ contains y, say $y \in hull_{\mathcal{F}}(A_f)$ where A_f is a finite subset of A. Then using (2.5) we see that

$$y \in hull_{\mathcal{F}}(A_f) \subseteq hull_{\mathcal{F}}(A) = A.$$

Therefore we have shown that $\mathcal{K}_A$ is a directed subfamily of $\mathcal{F}$ such that $A = \sup \mathcal{K}_A$.

As for the converse, if $\mathcal{K}_A$ is a directed subfamily of $\mathcal{F}$ such that $A = \sup \mathcal{K}_A$, then $A \in \mathcal{F}$ whenever $\mathcal{F}$ is inductive because inductive families contain the supremums of their directed subfamilies. ∎

PROPOSITION 13 *Every member of C_n^* is the supremum of a directed subfamily of C_n.*

PROOF Let $T \in C_n^*$ be an arbitrary convex set.

From lemma 12, a directed subfamily $\mathcal{K}_T$ of C_n^* such that $T = \sup \mathcal{K}_T$ is given by

$$\mathcal{K}_T = \{ hull_{C_n^*}(A) : A \text{ is a finite subset of } T \}.$$

Using proposition 7 we see that

$$\mathcal{K}_T = \{ hull_{C_n}(A) : A \text{ is a finite subset of } T \}.$$

Therefore $T = \sup \mathcal{K}_T$ is an expression of T as the supremum of a directed subfamily of the family C_n. ∎

In proposition 13 we see a relationship between C_n^* and C_n which will be explored further when we introduce inductive completions.

2.5 The Finite Hull Coincidence

Recall that in proposition 7 we showed that C_n-hulls and C_n^*-hulls coincide on all finite subsets of $\mathbf{R}^n$. We used this fact to conclude that the Radon numbers of the families C_n and C_n^* are equal. But proposition 7 is just a hint of a unique relationship that exists between C_n and C_n^*. There is only one inductive intersecting family containing C_n for which hulls of finite subsets coincide.

THEOREM 14 *Let $\mathcal{H}$ be an inductive diametral family of subsets of $\mathbf{R}^n$ such that*

$$\text{hull}_{\mathcal{H}}(T) = \text{hull}_{C_n}(T)$$

for all finite subsets $T \subseteq \mathbf{R}^n$. Then $\mathcal{H} = C_n^*$.

PROOF Let $\mathcal{H}$ be an inductive diametral family that satisfies the hypotheses of the theorem. From theorem 11 we know that $\mathcal{H}$ contains C_n^*. It remains to show that $\mathcal{H} \subseteq C_n^*$. Let $A \in \mathcal{H}$. The remainder of the proof consists of showing that $A \in C_n^*$.

By lemma 12 we know that a directed subfamily $\mathcal{K}_A$ of $\mathcal{H}$ for which $A = \sup \mathcal{K}_A$ is given by

$$\mathcal{K}_A = \{ hull_{\mathcal{H}}(A_f) : A_f \text{ is a finite subset of A} \}.$$

Therefore

$$
\begin{aligned}
A &= \sup\{ hull_{\mathcal{H}}(A_f) : A_f \text{ is a finite subset of A} \} \\
  &= \sup\{ hull_{\mathcal{C}_n}(A_f) : A_f \text{ is a finite subset of A} \} \\
  &= \sup\{ hull_{\mathcal{C}_n^*}(A_f) : A_f \text{ is a finite subset of A} \}.
\end{aligned}
$$

The last line of this series of equations is an expression of the set A as the supremum of a directed subfamily of the inductive family $\mathcal{C}_n^*$. Hence we know by lemma 12 that $A \in \mathcal{C}_n^*$. ∎

The relationship between families that is exhibited in theorem 14 will be explored more fully in a general setting.

2.6   Carathéodory's Theorem

We used Radon's Theorem and Helly's Theorem as the bases for defining Radon and Helly numbers. In this section we will look at another theorem from the classical literature on convexity on which we can base another type of number called *Carathéodory numbers*. The following theorem was published by Carathéodory in 1907.

Carathéodory's Theorem: *When $T \subseteq \mathbf{R}^n$, each element of the convex hull of T is an element of the convex hull of some subset of T which has at most $n + 1$ elements.*

As with Radon's and Helly's Theorems, Carathéodory's Theorem gives us yet another type of number to associate with families of sets. See [13].

The *Carathéodory number* of an intersecting family $\mathcal{F}$ of subsets of U is the least positive integer $c(U, \mathcal{F})$ such that for all $T \subseteq U$ and all $x \in hull_{\mathcal{F}}(T)$, there exists a subset $T_f \subseteq T$ with $|T_f| \leq c(U, \mathcal{F})$ and $x \in hull_{\mathcal{F}}(T_f)$. If no such integer exists, then we say that $c(U, \mathcal{F}) = \infty$.

The integer $n + 1$ is the least number for which Carathéodory's Theorem is true. Therefore Carathéodory's Theorem tells us that the Carathéodory number $c(\mathbf{R}^n, \mathcal{C}_n^*)$ is equal to $n + 1$.

It is natural to ask for which families is the associated Carathéodory number finite. This question leads us to the definition of the finitary property whose importance in the study of abstract convexity was first noted in [10].

Definition: *Let $\mathcal{F}$ be an intersecting family of subsets of U. We say that $\mathcal{F}$ is finitary (or algebraic) if and only if for all $T \subseteq U$ we have*

$$x \in hull_{\mathcal{F}}(T) \Rightarrow x \in hull_{\mathcal{F}}(T_f) \text{ for some finite subset } T_f \text{ of } T.$$

Note that every member of a finitary family is a union of the hulls of its finite subsets.[3] In lemma 12 we saw that the members of an inductive family are precisely those sets which can be expressed as a union of their finite subsets. It is known that an intersecting family $\mathcal{F}$ is finitary if and only if $\mathcal{F}$ is inductive. (See [4]. From this fact we see that Carathéodory's Theorem provides another proof that $\mathcal{C}_n^*$ is inductive.) So a partial answer to the question of which families have finite Carathéodory numbers is that families which are not inductive will never have finite Carathéodory numbers.

Immediately we see that the non-inductive family $\mathcal{C}_n$ has $c(\mathbf{R}^n, \mathcal{C}_n) = \infty$. We can see that $c(\mathbf{R}^n, \mathcal{C}_n) = \infty$ directly by considering an open unit ball B in Euclidean space; the $\mathcal{C}_n$-hull of B is its closure, a closed unit ball, yet any point on the boundary of $hull_{\mathcal{F}}(B)$ is not in the $\mathcal{C}_n$-hull of any finite subset of B.

We close by showing that the fact that a family is inductive does not necessarily imply that the family has finite Carathéodory number, even in $\mathbf{R}^n$.

PROPOSITION 15 *There exists an inductive family $\mathcal{F}$ of subsets of $\mathbf{R}^n$ which has infinite Carathéodory number.*

PROOF Let n be a fixed positive integer throughout the proof. For each integer $i \geq n + 1$, we let p_i denote the element $(i, 0, \ldots, 0)$ in $\mathbf{R}^n$. Also for each integer $i \geq n + 1$, we use closed balls to define the family $\mathcal{B}_i$ of subsets of $\mathbf{R}^n$ by

$$\mathcal{B}_i = \{T : T \subseteq \text{Ball}(p_i, \frac{1}{3}) \text{ and } |T| \leq i\}.$$

Consider now the family

$$\mathcal{F} = \mathcal{C}_n^* \cup \mathcal{B}_{n+1} \cup \mathcal{B}_{n+2} \cup \ldots$$

We first show that the Carathéodory number of the family $\mathcal{F}$ is not finite. Pick an integer m such that $m \geq n + 1$. Let S_m be the boundary of the closed ball $\text{Ball}(p_m, \frac{1}{3})$, that is

$$S_m = \{x \in \mathbf{R}^n : \delta(x, p_m) = \frac{1}{3}\}.$$

Then $hull_{\mathcal{F}}(S_m) = \text{Ball}(p_m, \frac{1}{3})$, and so $p_m \in hull_{\mathcal{F}}(S_m)$. Yet if S_m' is any finite subset of S_m having fewer that m elements, then

$$p_m \notin hull_{\mathcal{F}}(S_m') = S_m'.$$

Therefore $c(\mathbf{R}^n, \mathcal{F}) \geq m$. Because m can be chosen arbitrarily large, we see that $c(\mathbf{R}^n, \mathcal{F})$ cannot be finite.

We now show that $\mathcal{F}$ is inductive by showing that $\mathcal{F}$ is finitary. Let A be any subset of $\mathbf{R}^n$ and let $x \in hull_{\mathcal{F}}(A)$. In order to show that x is an element of the $\mathcal{F}$-hull of some finite subset of A, there are two cases to consider.

CASE 1 In this case suppose that A is not a subset of some single $\text{Ball}(p_i, \frac{1}{3})$ for

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[3]The hulls of finite subsets are called polytopes.

24

any integer $i \geq n + 1$. Then

$$x \in hull_{\mathcal{F}}(A) = hull_{C_n^*}(A).$$

Therefore Carathéodory's Theorem applies to show that $x \in hull_{C_n^*}(A_f)$ where A_f is a subset of A having at most $n + 1$ elements. If A_f is not a subset of some single Ball$(p_i, \frac{1}{3})$ for some $i \geq n + 1$, then we are finished because

$$x \in hull_{C_n^*}(A_f) = hull_{\mathcal{F}}(A_f).$$

Otherwise there is some integer $j \geq n + 1$ such that $A_f \subseteq$ Ball$(p_j, \frac{1}{3})$. By the fact that $A \not\subseteq$ Ball$(p_j, \frac{1}{3})$, we know that there exists $y \in A -$ Ball$(p_j, \frac{1}{3})$. Then

$$x \in hull_{C_n^*}(A_f) \subseteq hull_{C_n^*}(A_f \cup \{y\}) = hull_{\mathcal{F}}(A_f \cup \{y\}).$$

Thus we have shown that an arbitrary element x from $hull_{\mathcal{F}}(A)$ is an element of the $\mathcal{F}$-hull of some finite subset of A.

CASE 2 Suppose now that A is a subset of Ball$(p_m, \frac{1}{3})$ where m is an integer no less than $n + 1$. If A has finitely many elements then we are done. So suppose that A has infinitely many elements. Because $C_n^* \subseteq \mathcal{F}$, (2.2) tells us that $x \in hull_{C_n^*}(A)$. Therefore by Carathéodory's Theorem, $x \in hull_{C_n^*}(A_f)$ for some subset A_f of A having at most $n + 1$ elements. Form the set A' by adjoining to A_f as many elements of A as needed so that $|A'| = m + 1$. Then

$$x \in hull_{C_n^*}(A_f) \subseteq hull_{C_n^*}(A') = hull_{\mathcal{F}}(A').$$

Thus we have again shown that an arbitrary element x from $hull_{\mathcal{F}}(A)$ is an element of the $\mathcal{F}$-hull of some finite subset of A.

∎

Chapter 3

Convexity in the Grid

3.1 Preliminaries

The primary setting of chapter 2 is the Euclidean space $\mathbf{R}^n$. For chapter 3 the setting is a discrete version of Euclidean space known as the *grid*. Our aim is to develop a theory of convexity for this discrete space based on the convexity results for $\mathbf{R}^n$ that we obtained in chapter 2. In particular we obtain analogues in the grid for the families C_n and C_n^* in $\mathbf{R}^n$, and we use these families to obtain Radon, Helly, and Carathéodory type theorems for the grid.

The *grid* is the metric space consisting of the set $\mathbf{Z}^n$ with the L^1 metric, or the Manhattan metric as it is called in [11]. In symbols, given two elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of $\mathbf{Z}^n$, then their distance apart is given by

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|.$$

In this chapter $\mu(T)$ denotes the diameter, possibly infinite, of a set $T \subseteq \mathbf{Z}^n$.

Recall that finite sets played an important role in chapter 2. In the grid, a set is finite if and only if it is bounded.

We can think of the grid as a graph with vertices $\mathbf{Z}^n$ and with edges connecting any pair of vertices whose distance apart is one, ie any pair of vertices whose coordinates are identical except in the single component where they differ by one. Then the distance between two vertices is the number of edges in a shortest path between them. For other treatments of the notion of convexity in graphs, see [1, 7, 8, 9, 14].

In lemma 16 we introduce another perspective on distances in $\mathbf{Z}^n$ that we will take quite frequently. Throughout this chapter let Ψ^n denote the collection of the 2^n n-tuples having the form $[\epsilon_1, \ldots, \epsilon_n]$ where each $\epsilon_i \in \{-1, 1\}$. If $\epsilon = [\epsilon_1, \ldots, \epsilon_n] \in \Psi^n$, then for each $x \in \mathbf{Z}^n$ we define the integer-valued product $\epsilon \cdot x$ as

$$\epsilon \cdot x = [\epsilon_1, \ldots, \epsilon_n] \cdot (x_1, \ldots, x_n) = \epsilon_1 x_1 + \cdots + \epsilon_n x_n.$$

LEMMA 16 *If $x, y \in \mathbf{Z}^n$, then*

$$d(x,y) = \max\{\epsilon \cdot x - \epsilon \cdot y : \epsilon \in \Psi^n\}.$$

PROOF   Fix $x, y \in \mathbf{Z}^n$. Given any $\epsilon \in \Psi^n$,

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |\epsilon_i(x_i - y_i)| \geq |\sum_{i=1}^{n} \epsilon_i(x_i - y_i)| = |\epsilon \cdot x - \epsilon \cdot y|.$$

Now define the element $\eta = [\eta_1, \ldots, \eta_n] \in \Psi^n$ according to $\eta_i = \begin{cases} 1 & \text{if } x_i \geq y_i \\ -1 & \text{otherwise.} \end{cases}$
Then

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} \eta_i(x_i - y_i) = \eta \cdot x - \eta \cdot y.$$

∎

If $\epsilon = [\epsilon_1, \ldots, \epsilon_n] \in \Psi^n$, then we say $-\epsilon = [-\epsilon_1, \ldots, -\epsilon_n] \in \Psi^n$.

We close this section with a lemma that proves an obvious-seeming fact concerning closed balls.

LEMMA 17 *No closed ball in $\mathbf{Z}^n$ is properly contained in another set of the same diameter.*

PROOF   WLOG consider the closed ball $\mathrm{Ball}(c, m)$ where $c = (0, \ldots, 0) \in \mathbf{Z}^n$ and m is a positive integer. It is easy to see that such a ball has diameter at most $2m$. Suppose that $\mathrm{Ball}(c, m)$ is a proper subset of T. Then there exists an element $y = (y_1, \ldots, y_n) \in \mathbf{Z}^n$ such that $y \in T - \mathrm{Ball}(c, m)$, ie

$$d(c, y) = \sum_{i=1}^{n} |y_i| > m.$$

Choose a shortest path in the graph $\mathbf{Z}^n$ from $-y$ to y which goes through c. The length of this path is more than $2m$. As we began at $-y$ and move along this path toward y, we pass two vertices whose distance from c is exactly m. Call the first one of these x. Now a shortest path from x to y has length greater than $2m$. Because $x \in \mathrm{Ball}(c, m)$, we know that

$$\mu(T) \geq d(x, y) > 2m \geq \mu(\mathrm{Ball}(c, m)).$$

The assumption that $\mathrm{Ball}(c, m))$ is a proper subset of T leads us to the conclusion that $\mu(T) > \mu(\mathrm{Ball}(c, m))$. Therefore the lemma is established.   ∎

3.2   Diagonal Boxes

In this section we characterize the family of diagonal boxes, the analogue in the grid of the family $\mathcal{C}_n$ of convex bodies in $\mathbf{R}^n$. The reason that the family of diagonal boxes deserves this status is that the family of diagonal boxes is the unique smallest diametral family in $\mathbf{Z}^n$ just as $\mathcal{C}_n$ is the unique smallest diametral family in $\mathbf{R}^n$.

We begin by noting that every convex body in R^n is a bounded set that can be expressed as the intersection of halfspaces. We define the diagonal boxes as bounded sets which are intersections of special "halfspaces" in Z^n.

Definition: *We say that a set $D \subseteq Z^n$ is a* diagonal box *if and only if*

    1. D is bounded.[1]

    2.

$$D = \bigcap_{\epsilon \in \Psi^n} \{x \in Z^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in D\}.$$

In defining the family C_n we say that it consists of all convex bodies and also the single set R^n, the latter member being included to guarantee an intersecting family. Similarly we define the family $\Omega(Z^n)$ to be the family of all diagonal boxes plus the single set Z^n.

In theorem 19 we show that the family $\Omega(Z^n)$ is the unique smallest diametral family of subsets of Z^n and as such is the analogue in Z^n of the family C_n in R^n. But first we prove that $\Omega(Z^n)$ is an intersecting family and give a representation of its hull operator for finite (ie bounded) sets. (Note that because diagonal boxes are bounded, the $\Omega(Z^n)$-hull of any unbounded subset of Z^n must be all of Z^n.)

LEMMA 18 *The family $\Omega(Z^n)$ is an intersecting family whose hull operator is given by*

$$\text{hull}_{\Omega(Z^n)}(T) = \bigcap_{\epsilon \in \Psi^n} \{x \in Z^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in T\}$$

when T is any bounded subset of Z^n.

PROOF We begin by showing that the family $\Omega(Z^n)$ is an intersecting family. Let $\mathcal{K}$ be an arbitrary subfamily of $\Omega(Z^n)$. It suffices to show that the intersection of all members of $\mathcal{K}$ is a member of $\Omega(Z^n)$. Nothing is lost in supposing that $Z^n \notin \mathcal{K}$ and $\phi \notin \mathcal{K}$. For convenience let J serve as an index set for the family $\mathcal{K}$, that is suppose that $\mathcal{K} = \{D_j\}_{j \in J}$ where each D_j is a diagonal box. Again nothing is lost in supposing that $J \neq \phi$ and $\bigcap_{j \in J} D_j \neq \phi$.

Clearly the intersection $\bigcap_{j \in J} D_j$ is a bounded set because each diagonal box D_j is bounded. Therefore by the definition of diagonal boxes, we show that $\bigcap_{j \in J} D_j$ is a diagonal box by showing that

$$\bigcap_{j \in J} D_j = \bigcap_{\epsilon \in \Psi^n} \{x \in Z^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in \bigcap_{j \in J} D_j\}. \tag{3.1}$$

The containment in one direction of (3.1) is trivial, so we establish the other direction by showing that the right hand side of (3.1) is a subset of $\bigcap_{j \in J} D_j$.

---

[1] We shall consider the empty set ϕ to be bounded. Hence ϕ is a diagonal box.

Fix $i \in J$. If y is an element of the right hand side of (3.1), then for all $\epsilon \in \Psi^n$, there exists an element $t_\epsilon \in \bigcap_{j \in J} D_j$ such that $\epsilon \cdot y \geq \epsilon \cdot t_\epsilon$. Since each $t_\epsilon \in \bigcap_{j \in J} D_j \subseteq D_i$, we know from the definition of diagonal boxes that $y \in L_i$. Because i is arbitrary, we conclude that $y \in \bigcap_{j \in J} D_j$ as well. Thus $\bigcap_{j \in J} D_j$ is a diagonal box, and so $\Omega(\mathbf{Z}^n)$ is an intersecting family.

Now we establish the representation for hulls as given in the statement of the lemma. Let T be a bounded (ie finite) subset of $\mathbf{Z}^n$. We claim that a unique smallest diagonal box D which contains T is given by

$$D = \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbf{Z}^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in T\}.$$

Establishing this claim involves showing:

1. If D' is another diagonal box containing T, then $D \subseteq D'$.

2. D is a diagonal box, meaning

   (a)
   $$D = \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbf{Z}^n : \epsilon \cdot x \geq \epsilon \cdot a_\epsilon \text{ for some } a_\epsilon \in D\}$$

   (b) and D is a bounded set.

The first point (1) is easy to see because if D' is a diagonal box containing T, then

$$D \subseteq \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbf{Z}^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in D'\} = D'.$$

To see the second point (2a), it suffices to suppose that

$$y \in \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbf{Z}^n : \epsilon \cdot x \geq \epsilon \cdot a_\epsilon \text{ for some } a_\epsilon \in D\}$$

and then show that $y \in D$. For each $\epsilon \in \Psi^n$, there exists $a_\epsilon \in D$ such that $\epsilon \cdot y \geq \epsilon \cdot a_\epsilon$. Because $a_\epsilon \in D$, there exists $t_\epsilon \in T$ such that $\epsilon \cdot a_\epsilon \geq \epsilon \cdot t_\epsilon$. Thus $\epsilon \cdot y \geq \epsilon \cdot t_\epsilon$ shows that $y \in D$.

The third point (2b), that D is bounded, is a technical consideration. We proceed as follows. Given $v, w \in D$, there exists $\eta \in \Psi^n$ such that $d(v, w) = \eta \cdot v - \eta \cdot w$ by lemma 16. By the definition of D we know both of the following:

$$\eta \cdot v, \eta \cdot w \geq \min\{\eta \cdot t : t \in T\}$$

$$(-\eta) \cdot v, (-\eta) \cdot w \geq \min\{(-\eta) \cdot t : t \in T\}.$$

It follows that

$$\min\{\eta \cdot t : t \in T\} \leq \eta \cdot v, \eta \cdot w \leq -\min\{(-\eta) \cdot t : t \in T\}.$$

Letting $f : \Psi^n \to \mathbf{Z}$ be given by

$$f(\epsilon) = -\min\{(-\epsilon) \cdot t : t \in T\} - \min\{\epsilon \cdot t : t \in T\},$$

we have $d(v, w) = \eta \cdot v - \eta \cdot w \leq f(\eta)$. We conclude that D has finite diameter, and consequently is bounded, because

$$\mu(D) \leq \max\{f(\epsilon) : \epsilon \in \Psi^n\}.$$

∎

Now we are set to prove the main theorem of this section, which establishes not only that $\Omega(\mathbf{Z}^n)$ is a diametral family, but also that it is the unique smallest diametral family of subsets of $\mathbf{Z}^n$.

THEOREM 19 *If $\mathcal{F}$ is an intersecting family of subsets of $\mathbf{Z}^n$, then*

$$\mathcal{F} \text{ is diametral} \Longleftrightarrow \Omega(\mathbf{Z}^n) \subseteq \mathcal{F}.$$

PROOF Suppose that $\Omega(\mathbf{Z}^n) \subseteq \mathcal{F}$. To show that $\mathcal{F}$ is diametral it suffices to argue that $\Omega(\mathbf{Z}^n)$ is a diametral family, because any intersecting family containing a diametral family is itself a diametral family by lemma 2.

Because lemma 18 tells us that $\Omega(\mathbf{Z}^n)$ is an intersecting family, to show that $\Omega(\mathbf{Z}^n)$ is a diametral family, it is enough to prove that $\mu(hull_{\Omega(\mathbf{Z}^n)}(T)) \leq \mu(T)$ where $T \subseteq \mathbf{Z}^n$ is a given bounded set. Recall

$$hull_{\Omega(\mathbf{Z}^n)}(T) = \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbf{Z}^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in T\}.$$

Let $v, w \in hull_{\Omega(\mathbf{Z}^n)}(T)$. Lemma 16 tells us that there exists $\eta \in \Psi^n$ such that

$$d(v, w) = \eta \cdot v - \eta \cdot w.$$

Because $w \in hull_{\Omega(\mathbf{Z}^n)}(T)$, we know that there exists $w' \in T$ such that

$$\eta \cdot w \geq \eta \cdot w'.$$

Also $v \in hull_{\Omega(\mathbf{Z}^n)}(T)$ and $(-\eta) \in \Psi^n$ implies that there exists $v' \in T$ such that

$$(-\eta) \cdot v \geq (-\eta) \cdot v'.$$

Adding these two equations gives

$$\eta \cdot w - \eta \cdot v \geq \eta \cdot w' - \eta \cdot v'.$$

Using lemma 16 again we have

$$d(v, w) = \eta \cdot v - \eta \cdot w \leq \eta \cdot v' - \eta \cdot w' \leq d(v', w') \leq \mu(T).$$

Because v and w are arbitrary elements of $hull_{\Omega(\mathbf{Z}^n)}(T))$, we conclude that

$$\mu(hull_{\Omega(\mathbf{Z}^n)}(T)) \le \mu(T).$$

Therefore the family $\Omega(\mathbf{Z}^n)$ is diametral.

We have argued that the supposition $\Omega(\mathbf{Z}^n) \subseteq \mathcal{F}$ implies that $\mathcal{F}$ is a diametral family and now will argue the converse of this statement. Let $\mathcal{F}$ be a diametral family.

Our proof will be based on the method used to prove the similar result for the family $\mathcal{C}_n$ in theorem 3. We claim that all closed balls of $\mathbf{Z}^n$ are members of $\mathcal{F}$. To see this claim, note that the fact that $\mathcal{F}$ is diametral implies that any closed ball in $\mathbf{Z}^n$ is contained in some member of $\mathcal{F}$ having the same diameter. But lemma 17 shows that a closed ball cannot be a proper subset of another set having the same diameter, so we know that every closed ball must be a member of $\mathcal{F}$. Therefore we finish the proof when we show that every diagonal box is an intersection of closed balls, because then we will have shown that all diagonal boxes are members of the intersecting family $\mathcal{F}$.

Let D be a given diagonal box for the remainder of this proof. That is,

$$D = \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbf{Z}^n : \epsilon \cdot x \ge \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in D\}.$$

In order to show that D is an intersection of closed balls, it suffices to show that given $z \notin D$, there exists y_z such that $d(y_z, z) > \mu(D \cup \{y_z\})$. Because then if we let $m_z = \mu(D \cup \{y_z\})$, we have that $D \subseteq \text{Ball}(y_z, m_z)$ yet $z \notin \text{Ball}(y_z, m_z)$. Thus we would have an expression of D as an intersection of closed balls, namely

$$D = \bigcap_{z \notin D} \text{Ball}(y_z, m_z).$$

Suppose $z \notin D$. Then there exists $\lambda = [\lambda_1, \ldots, \lambda_n] \in \Psi^n$ such that $\lambda \cdot z < \lambda \cdot t$ for all $t \in D$. Pick any $a = (a_1, \ldots, a_n) \in D$ and define $y \in \mathbf{Z}^n$ componentwise according to

$$y_i = a_i + \lambda_i \mu(D). \tag{3.2}$$

We wish to show that $d(y, z) > \mu(D \cup \{y\})$. First note that

$$
\begin{aligned}
d(y, z) &\ge \lambda \cdot y - \lambda \cdot z \\
&> \lambda \cdot y - \lambda \cdot a \\
&= \sum_{i=1}^n \lambda_i y_i - \sum_{i=1}^n \lambda_i a_i \\
&= \sum_{i=1}^n \left[\lambda_i a_i + \lambda_i^2 \mu(D)\right] - \sum_{i=1}^n \lambda_i a_i \\
&= n\mu(D) \\
&\ge \mu(D).
\end{aligned}
$$

Given any $c \in D$, we know from lemma 16 that

$$d(y, c) = \max\{\epsilon \cdot y - \epsilon \cdot c : \epsilon \in \Psi^n\}.$$

We claim that a maximum occurs when $\epsilon = \lambda$. To see this, note that we can use (3.2) to rewrite this last formula as

$$d(y, c) = \max\{\epsilon \cdot a - \epsilon \cdot c + \sum_{i=1}^{n} \epsilon_i \lambda_i \mu(D) : \epsilon \in \Psi^n\}.$$

Because $a, c \in D$ implies $\epsilon \cdot a - \epsilon \cdot c \leq d(a, c) \leq \mu(D)$ for any $\epsilon \in \Psi^n$, we see that a maximum occurs when $\epsilon = \lambda$. Therefore

$$d(y, c) = \lambda \cdot y - \lambda \cdot c < \lambda \cdot y - \lambda \cdot z \leq d(y, z).$$

We conclude that $d(y, z) > \mu(D \cup \{y\})$ and are done. ∎

As was argued in the proof of theorem 19 and will prove useful later, we record

COROLLARY 20 *Every diagonal box is an intersection of closed balls in $\mathbb{Z}^n$.*

∎

3.3 Radon and Helly Numbers

Earlier we calculated the Radon and Helly numbers of the family $\mathcal{C}_n$ in Euclidean space and showed that no diametral family in $\mathbb{R}^n$ can have lower numbers. In support of the assertion that $\Omega(\mathbb{Z}^n)$ is the analogue in $\mathbb{Z}^n$ of $\mathcal{C}_n$, we show in this section that no diametral family in $\mathbb{Z}^n$ can have lower Radon and Helly numbers than those of the family $\Omega(\mathbb{Z}^n)$. The main work of this section will be in calculating the Radon and Helly numbers of $\Omega(\mathbb{Z}^n)$. We will see that these values give us versions of Radon's and Helly's Theorems for the grid.

THEOREM 21 *Let $\mathcal{F}$ be a diametral family of subsets of $\mathbb{Z}^n$.*

1. *The Radon number $r(\mathbb{Z}^n, \mathcal{F})$ is at least $2^n + 1$.*

2. *The Helly number $h(\mathbb{Z}^n, \mathcal{F})$ is at least 2^n.*

PROOF Because of theorem 19 and proposition 6, it suffices to establish that $r(\mathbb{Z}^n, \Omega(\mathbb{Z}^n)) = 2^n + 1$ and $h(\mathbb{Z}^n, \Omega(\mathbb{Z}^n)) = 2^n$. Consider the set

$$Q_n = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : a_i \in \{0, 1\}\}.$$

Claim: *Any subset of Q_n is a member of $\Omega(\mathbb{Z}^n)$.*

Let us see what follows from this claim.

If every subset of Q_n is a member of $\Omega(\mathbf{Z}^n)$, then we know by (2.3) that every subset of Q_n is its own $\Omega(\mathbf{Z}^n)$-hull. Thus the set Q_n admits no Radon $\Omega(\mathbf{Z}^n)$-partition. Because $|Q_n| = 2^n$, it follows that $r(\mathbf{Z}^n, \Omega(\mathbf{Z}^n)) > 2^n$, and we have a lower bound of $2^n + 1$ on the Radon number of the family $\Omega(\mathbf{Z}^n)$.

The claim also gives us a lower bound on the Helly number $h(\mathbf{Z}^n, \Omega(\mathbf{Z}^n))$. Consider the family

$$\mathcal{K}_n = \{Q_n - \{a\} : a \in Q_n\}.$$

The family $\mathcal{K}_n$ has exactly 2^n members. The intersection of all members of $\mathcal{K}_n$ is empty. But any collection of $2^n - 1$ members of $\mathcal{K}_n$ will intersect at a single element of the set Q_n. If the claim is true then $\mathcal{K}$ is a subfamily of $\Omega(\mathbf{Z}^n)$. Therefore by the definition of Helly numbers we see that $h(\mathbf{Z}^n, \Omega(\mathbf{Z}^n))$ is strictly greater than $2^n - 1$.

We seek now to establish the claim.

The $n = 1$ case is left to the reader, so suppose that $n \geq 2$. Lemma 17 tells us that a closed ball in $\mathbf{Z}^n$ is never properly contained in another set having the same diameter. From this it follows that any diametral family necessarily contains all closed balls. Because diametral families are closed under intersections, any diametral family also contains all sets which are intersections of closed balls. $\Omega(\mathbf{Z}^n)$ is diametral, so we can show that the set Q_n is a member of $\Omega(\mathbf{Z}^n)$ by expressing Q_n as an intersection of closed balls.

Such an intersection is given by

$$Q_n = \bigcap_{a \in Q_n} \text{Ball}(a, n). \tag{3.3}$$

To see this (note that the "$\subseteq$" direction follows because elements of Q_n lie at most a distance of n apart), suppose $b = (b_1, \ldots, b_n) \notin Q_n$. Then for some j^{th} component we know that $b_j \notin \{0, 1\}$. Define the function $f : \mathbf{Z}^n \to Q_n$ componentwise by

$$(f(x))_i = \begin{cases} 1 & \text{if } x_i \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $d(b, f(b)) > n$. Because $b \notin \text{Ball}(f(b), n)$, we see that b is not a member of the right hand side of (3.3). We therefore have an expression of Q_n as an intersection of closed balls, showing $Q_n \in \Omega(\mathbf{Z}^n)$.

To finish establishing the claim, we consider a proper nonempty subset $S \subset Q_n$ and show that S is also in $\Omega(\mathbf{Z}^n)$. For each $a \in Q_n$, let $T_a = Q_n - \{a\}$. Then

$$T_a = Q_n \cap \text{Ball}(f(a), n - 1)$$

is an expression of T_a as an intersection of members of $\Omega(\mathbf{Z}^n)$, showing that $T_a \in \Omega(\mathbf{Z}^n)$. The fact that S is in $\Omega(\mathbf{Z}^n)$ follows from

$$S = \bigcap_{a \in Q_n - S} T_a.$$

33

This establishes the claim.

We now continue with the proof by showing that 2^n is an upper bound for $h(\mathbf{Z}^n, \Omega(\mathbf{Z}^n))$.

By proposition 8, in calculating the Helly number of the family $\Omega(\mathbf{Z}^n)$, nothing is lost in removing the member $\mathbf{Z}^n$ from $\Omega(\mathbf{Z}^n)$, ie

$$h(\mathbf{Z}^n, \Omega(\mathbf{Z}^n)) = h(\mathbf{Z}^n, \Omega(\mathbf{Z}^n) - \{\mathbf{Z}^n\}).$$

Let $\mathcal{K}$ be a subfamily of $\Omega(\mathbf{Z}^n) - \{\mathbf{Z}^n\}$ having at least 2^n members such that any 2^n members of $\mathcal{K}$ have nonempty intersection. We need to show that all members of $\mathcal{K}$ share some common element z.

Fix elements $\epsilon \in \Psi^n$ and $A_0 \in \mathcal{K}$. All the members of $\mathcal{K}$ are bounded sets and consequently finite. Therefore for any $A \in \mathcal{K}$, there exists an integer

$$g_\epsilon(A) = \min\{\epsilon \cdot x : x \in A\}. \tag{3.4}$$

We are interested in finding some set $A_\epsilon \in \mathcal{K}$ for which the integer value of $g_\epsilon(A_\epsilon)$ is as large as possible. To see that there is indeed a maximum to the possible integer values, note that for any $A \in \mathcal{K}$ the nonempty intersection $A \cap A_0$ guarantees that

$$g_\epsilon(A) = \min\{\epsilon \cdot x : x \in A\} \le \max\{\epsilon \cdot x : x \in A_0\}.$$

Therefore we have an upper bound on the size of the integers in (3.4), and so there exists some set $A_\epsilon \in \mathcal{K}$ which makes the value of (3.4) as large as possible.

There are 2^n elements in Ψ^n. So considered over all of Ψ^n, we obtain a subfamily $\mathcal{K}'$ of $\mathcal{K}$ having at most 2^n members by letting

$$\mathcal{K}' = \{A_\eta : \eta \in \Psi^n\}.$$

The intersection of all the members of the subfamily $\mathcal{K}'$ is nonempty by the assumptions on $\mathcal{K}$, and so there is some element z common to every member of $\mathcal{K}'$. We want to show that z is common to every member of $\mathcal{K}$ as well. We will show that $z \in D$ where D is an arbitrary member of $\mathcal{K}$.

By corollary 20 we know that D is an intersection of closed balls. Therefore it suffices to show that any closed ball containing D must also contain z. Suppose that $D \subseteq \text{Ball}(c, m)$ for some nonnegative integer m and some $c \in \mathbf{Z}^n$. We show that $z \in \text{Ball}(c, m)$ by showing that $d(z, c) \le m$.

By lemma 16 there exists an element $\eta \in \Psi^n$ such that $d(z, c) = \eta \cdot z - \eta \cdot c$. Note that $z \in A_{-\eta}$, so

$$
\begin{aligned}
(-\eta) \cdot z &\ge \min\{(-\eta) \cdot x : x \in A_{-\eta}\} \\
&= g_{-\eta}(A_{-\eta}) \\
&\ge g_{-\eta}(D).
\end{aligned}
$$

For any finite set X of integers, $\max\{x : x \in X\} = -\min\{-x : x \in X\}$. Using

34

this fact we obtain

$$
\begin{aligned}
\eta \cdot z &\leq -g_{-\eta}(D) \\
&= -\min\{(-\eta) \cdot x : x \in D\} \\
&= -\min\{-(\eta \cdot x) : x \in D\} \\
&= \max\{\eta \cdot x : x \in D\}.
\end{aligned}
$$

Let y be an element of D for which $\eta \cdot y = \max\{\eta \cdot x : x \in D\}$. Then because $y \in D \subseteq \mathrm{Ball}(c, m)$, we know that

$$
m \geq d(y, c) \geq \eta \cdot y - \eta \cdot c \geq \eta \cdot z - \eta \cdot c = d(z, c).
$$

From the above we can conclude that z is common to every member of $\mathcal{K}$, and so we have shown that $h(\mathbf{Z}^n, \Omega(\mathbf{Z}^n))$ is at most 2^n.

It now remains to establish that $2^n + 1$ is an upper bound for the Radon number $r(\mathbf{Z}^n, \Omega(\mathbf{Z}^n))$. Given any subset $T \subseteq \mathbf{Z}^n$ which has $2^n + 1$ elements, suppose we know that there exists a nonempty subset T_f of T having at most 2^n elements such that

$$
hull_{\Omega(\mathbf{Z}^n)}(T) = hull_{\Omega(\mathbf{Z}^n)}(T_f).
$$

(The actual proof of this fact is the content of the next proposition.) Then we can partition T into two disjoint nonempty sets T_f and $T - T_f$ such that

$$
\begin{aligned}
hull_{\Omega(\mathbf{Z}^n)}(T - T_f) \cap hull_{\Omega(\mathbf{Z}^n)}(T_f) &= hull_{\Omega(\mathbf{Z}^n)}(T - T_f) \cap hull_{\Omega(\mathbf{Z}^n)}(T) \\
&= hull_{\Omega(\mathbf{Z}^n)}(T - T_f) \neq \phi.
\end{aligned}
$$

We now have the required Radon $\Omega(\mathbf{Z}^n)$-partition, namely the partition of T into T_f and $T - T_f$, to show that $r(\mathbf{Z}^n, \Omega(\mathbf{Z}^n)) \leq 2^n + 1$.

All that remains is to prove proposition 22. ∎

PROPOSITION 22 *If T is any bounded subset of $\mathbf{Z}^n$, then there exists a nonempty subset $T_f \subseteq T$ having at most 2^n elements such that*

$$
\mathrm{hull}_{\Omega(\mathbf{Z}^n)}(T) = \mathrm{hull}_{\Omega(\mathbf{Z}^n)}(T_f).
$$

PROOF Let T be a bounded subset of $\mathbf{Z}^n$. We show that there exists a set T_f satisfying the conditions of the proposition.

For each $\epsilon \in \Psi^n$, choose an element $t_\epsilon \in T$ such that $\epsilon \cdot x \geq \epsilon \cdot t_\epsilon$ for all $x \in T$. (We can do this because T is finite.) Let

$$
T_f = \{t_\epsilon : \epsilon \in \Psi^n\}.
$$

Clearly $hull_{\Omega(\mathbf{Z}^n)}(T_f) \subseteq hull_{\Omega(\mathbf{Z}^n)}(T)$ because $T_f \subseteq T$. Now $hull_{\Omega(\mathbf{Z}^n)}(T_f)$ is a member of $\Omega(\mathbf{Z}^n)$ and consequently by corollary 20 is an intersection of closed balls. So to show .hat $hull_{\Omega(\mathbf{Z}^n)}(T) \subseteq hull_{\Omega(\mathbf{Z}^n)}(T_f)$, it suffices to show that any closed ball containing $hull_{\Omega(\mathbf{Z}^n)}(T_f)$ must necessarily contain $hull_{\Omega(\mathbf{Z}^n)}(T)$ as well.

Suppose $hull_{\Omega(\mathbf{Z}^n)}(T_f) \subseteq \text{Ball}(c,m)$ for some nonnegative integer m and some $c \in \mathbf{Z}^n$. Choose $y \in hull_{\Omega(\mathbf{Z}^n)}(T)$. To show that $y \in \text{Ball}(c,m)$, we must show that $d(c,y) \le m$.

By lemma 16 there exists an element $\eta \in \Psi^n$ such that $d(c,y) = \eta \cdot c - \eta \cdot y$. Then

$$d(c,y) = \eta \cdot c - \eta \cdot y \le \eta \cdot c - \eta \cdot t_\eta \le d(c,t_\eta) \le m.$$

∎

Note that proposition 22 is not true for unbounded sets T. Later we will see that this fact is related to the fact that $\Omega(\mathbf{Z}^n)$ is not inductive. An extension of proposition 22 will be given in lemma 30 that will serve as a starting point for computing Carathéodory numbers in $\mathbf{Z}^n$.

For now we deduce a version of Helly's Theorem for the grid from theorem 21 using $\Omega(\mathbf{Z}^n)$ in place of $\mathcal{C}_n$. Note that Radon's Theorem is a statement about convex hulls and not about $\mathcal{C}_n$-hulls. Therefore a version of Radon's Theorem for the grid must wait until we have looked at grid convexity more completely.

A Helly Theorem for the Grid: *Suppose $\mathcal{K}$ is a family of at least 2^n diagonal boxes in $\mathbf{Z}^n$. If each 2^n members of $\mathcal{K}$ share a common element, then there is an element common to all members of $\mathcal{K}$.*

∎

3.4   The Convex Analogues of the Grid

The family $\Omega(\mathbf{Z}^n)$ of diagonal boxes plays a role in $\mathbf{Z}^n$ that the family $\mathcal{C}_n$ of convex bodies plays in $\mathbf{R}^n$, namely each is the unique smallest diametral family. Therefore we say that the family $\Omega(\mathbf{Z}^n)$ is the grid's analogue of $\mathcal{C}_n$. In this section we want to characterize a family in $\mathbf{Z}^n$ which is the analogue of the family $\mathcal{C}_n^*$ of all convex sets in $\mathbf{R}^n$. Recall that $\mathcal{C}_n^*$ is the unique smallest inductive diametral family of $\mathbf{R}^n$. Therefore the candidate family for being the grid's analogue of $\mathcal{C}_n^*$ is the unique smallest inductive diametral family of $\mathbf{Z}^n$.

We begin with a definition. The sets that are members of the family $\mathcal{C}_n^*$ are the result of beginning with the definition of convex bodies and removing the compactness requirement. The sets of the family $\Omega(\mathbf{Z}^n)^*$ of *extended diagonal boxes* are the result of beginning with the definition of diagonal boxes and removing the boundedness requirement.

Definition: *An* extended diagonal box *is a set $A \in \mathbf{Z}^n$ satisfying*

$$A = \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbf{Z}^n : \epsilon \cdot x \ge \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in A\}.$$

36

LEMMA 23 *The family $\Omega(\mathbb{Z}^n)^*$ of extended diagonal boxes is a diametral family whose hull operator is given by*

$$\text{hull}_{\Omega(\mathbb{Z}^n)^*}(T) = \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbb{Z}^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in T\}$$

for any subset $T \subseteq \mathbb{Z}^n$

PROOF Because $\Omega(\mathbb{Z}^n) \subseteq \Omega(\mathbb{Z}^n)^*$, the fact that the family $\Omega(\mathbb{Z}^n)^*$ is a diametral family is a consequence of lemma 2 once we know that $\Omega(\mathbb{Z}^n)^*$ is an intersecting family. We omit the proof that $\Omega(\mathbb{Z}^n)^*$ is an intersecting family because it proceeds exactly like the proof in lemma 18 that $\Omega(\mathbb{Z})$ is an intersecting family. We continue with a

Claim: *An equivalent condition for defining extended diagonal boxes is:*

> *A is an extended diagonal box if and only if there exists some subset $A' \subseteq \mathbb{Z}^n$ such that*
>
> $$A = \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbb{Z}^n : \epsilon \cdot x \geq \epsilon \cdot a_\epsilon \text{ for some } a_\epsilon \in A'\}.$$

To show that the definition of extended diagonal boxes and the seemingly weaker definition given in the claim are indeed equivalent, it suffices to consider a set $A \subseteq \mathbb{Z}^n$ satisfying the condition of the claim and prove that A is an extended diagonal box.

Suppose there exists a set $A' \subseteq \mathbb{Z}^n$ for which

$$A = \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbb{Z}^n : \epsilon \cdot x \geq \epsilon \cdot a_\epsilon \text{ for some } a_\epsilon \in A'\}.$$

Pick an arbitrary $y \in \mathbb{Z}^n$ such that

$$y \in \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbb{Z}^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in A\}.$$

Then it suffices to show that $y \in A$.

For each $\epsilon \in \Psi^n$ there exists $t_\epsilon \in A$ such that $\epsilon \cdot y \geq \epsilon \cdot t_\epsilon$. Because $t_\epsilon \in A$, there exists $a_\epsilon \in A'$ for which $\epsilon \cdot t_\epsilon \geq \epsilon \cdot a_\epsilon$. The fact that $\epsilon \cdot y \geq \epsilon \cdot a_\epsilon$ shows that $y \in A$. Therefore the claim is established.

Now we finish the proof by establishing the representation for $\Omega(\mathbb{Z}^n)^*$-hulls.

Let $T \subseteq \mathbb{Z}^n$. By the claim, we know that

$$D = \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbb{Z}^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in T\}$$

is one extended diagonal box containing containing T. If $T \subseteq D'$ for some extended diagonal box D', then by the definition of extended diagonal boxes we have

$$D \subseteq \bigcap_{\epsilon \in \Psi^n} \{x \in \mathbb{Z}^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in D'\} = D'.$$

Therefore the unique smallest extended diagonal box containing T is given by

$$\bigcap_{\epsilon \in \Psi^n} \{x \in \mathbf{Z}^n : \epsilon \cdot x \geq \epsilon \cdot t_\epsilon \text{ for some } t_\epsilon \in T\},$$

and we have found an expression for the $\Omega(\mathbf{Z}^n)^*$-hull of T. ∎

Comparing lemmas 18 and 23, we record the following useful observation that parallels a similar relationship between C_n and C_n^* in Euclidean space.

LEMMA 24 *For any finite subset T of $\mathbf{Z}^n$,*

$$\text{hull}_{\Omega(\mathbf{Z}^n)}(T) = \text{hull}_{\Omega(\mathbf{Z}^n)^*}(T).$$

∎

In Euclidean space, one difference between the families C_n and C_n^* is that only the latter is inductive. The situation in the grid is parallel to the Euclidean case.

PROPOSITION 25 *The family $\Omega(\mathbf{Z}^n)^*$ is an inductive family whereas $\Omega(\mathbf{Z}^n)$ is not.*

PROOF We begin by showing that $\Omega(\mathbf{Z}^n)$ is not an inductive family.

Fix $n \geq 2$. It suffices to find a directed subfamily $\mathcal{F}$ of $\Omega(\mathbf{Z}^n)$ such that $\sup \mathcal{F}$ is not a member of $\Omega(\mathbf{Z}^n)$. Fix $\epsilon \in \Psi^n$ and make the following definitions:

$$T_n = \{(x_1, \ldots, x_n) \in \mathbf{Z}^n : \epsilon \cdot x = 0\}$$

$$\mathcal{F} = \{\text{hull}_{\Omega(\mathbf{Z}^n)}(A) : A \text{ is a finite subset of } T_n\}.$$

Now $\mathcal{F}$ is a subfamily of $\Omega(\mathbf{Z}^n)$ by construction and is directed because

$$\text{hull}_{\Omega(\mathbf{Z}^n)}(A_1) \cup \text{hull}_{\Omega(\mathbf{Z}^n)}(A_2) \subseteq \text{hull}_{\Omega(\mathbf{Z}^n)}(A_1 \cup A_2).$$

It is easy to see that $T_n \subseteq \mathcal{F}$, so $\sup \mathcal{F}$ is an unbounded set. Because diagonal boxes are bounded sets, $\sup \mathcal{F}$ is a member of $\Omega(\mathbf{Z}^n)$ only if $\sup \mathcal{F} = \mathbf{Z}^n$. But in fact we will show that $\sup \mathcal{F} = T_n \neq \mathbf{Z}^n$. To do this it suffices to choose $y \in \sup \mathcal{F}$ and show that $y \in T_n$.

By the definition of supremums, we know that y is an element of some member of $\mathcal{F}$, say $y \in \text{hull}_{\Omega(\mathbf{Z}^n)}(A)$ where A is a finite subset of T_n. By lemma 18, there exists an element $t \in A$ such that $\epsilon \cdot y \geq \epsilon \cdot t = 0$. But $(-\epsilon) \in \Psi^n$ also, so by lemma 18 there exists an element $t' \in A$ such that

$$-(\epsilon \cdot y) = (-\epsilon) \cdot y \geq (-\epsilon) \cdot t' = -(\epsilon \cdot t') = 0.$$

We conclude that $\epsilon \cdot y = 0$, meaning that $y \in T_n$.

We have shown that if $n \geq 2$, then $\Omega(\mathbf{Z}^n)$ is not inductive. For the $n = 1$ case, consider $T_1 = \{(x) \in \mathbf{Z}^1 : x \geq 0\}$ and proceed as above.

Now we prove that $\Omega(\mathbf{Z}^n)^*$ is an inductive family. To do this we show that if $\mathcal{K}$ is a directed subfamily of $\Omega(\mathbf{Z}^n)^*$, then $\sup \mathcal{K} \in \Omega(\mathbf{Z}^n)^*$. It suffices to show that $hull_{\Omega(\mathbf{Z}^n)^*}(\sup \mathcal{K}) \subseteq \sup \mathcal{K}$.

Pick $x \in hull_{\Omega(\mathbf{Z})^*}(\sup \mathcal{K})$. By lemma 23, for each $\epsilon \in \Psi^n$ there exists an element $t_\epsilon \in \sup \mathcal{K}$ such that $\epsilon \cdot x \geq \epsilon \cdot t_\epsilon$. For each t_ϵ, there exists a member $A_\epsilon \in \mathcal{K}$ such that $t_\epsilon \in A_\epsilon$. And by the directedness of $\mathcal{K}$, there is a member $A \in \mathcal{K}$ containing $\bigcup_{\epsilon \in \Psi^n} A_\epsilon$. Then by lemma 23 we conclude that

$$x \in hull_{\Omega(\mathbf{Z}^n)^*}(\bigcup_{\epsilon \in \Psi^n} t_\epsilon) \subseteq hull_{\Omega(\mathbf{Z}^n)^*}(A) = A.$$

Therefore $x \in \sup \mathcal{K}$. ∎

As a result of proposition 25 we know that $\Omega(\mathbf{Z}^n)$ is not finitary and hence cannot have a finite Carathéodory number. In theorem 31 we calculate the Carathéodory number of $\Omega(\mathbf{Z}^n)^*$.

Not only is $\Omega(\mathbf{Z}^n)^*$ an inductive family, but we can now show that among all diametral families of $\mathbf{Z}^n$, the unique smallest inductive family is $\Omega(\mathbf{Z}^n)^*$.

THEOREM 26 *If $\mathcal{F}$ is an inductive intersecting family of subsets of $\mathbf{Z}^n$, then*

$$\mathcal{F} \text{ is diametral} \iff \Omega(\mathbf{Z}^n)^* \subseteq \mathcal{F}.$$

PROOF Let $\mathcal{F}$ be an inductive intersecting family of subsets of $\mathbf{Z}^n$.

Suppose $\Omega(\mathbf{Z}^n)^* \subseteq \mathcal{F}$. Then because $\Omega(\mathbf{Z}^n)$ is a subfamily of $\Omega(\mathbf{Z}^n)^*$, we know that $\mathcal{F}$ also contains $\Omega(\mathbf{Z}^n)$. By theorem 19 we conclude that $\mathcal{F}$ is diametral.

On the other hand suppose that $\mathcal{F}$ is diametral. Then by theorem 19, we know that $\Omega(\mathbf{Z}^n) \subseteq \mathcal{F}$. We wish to show that $\Omega(\mathbf{Z}^n)^* \subseteq \mathcal{F}$. Because $\mathcal{F}$ is an inductive family and hence contains the supremum of any directed subfamily, it suffices to show that every member of $\Omega(\mathbf{Z}^n)^*$ is the supremum of a directed subfamily of $\Omega(\mathbf{Z}^n)$. This is the content of proposition 27. ∎

PROPOSITION 27 *Every extended diagonal box is the supremum of a directed subfamily of $\Omega(\mathbf{Z}^n)$.*

PROOF Let $A \in \Omega(\mathbf{Z}^n)^*$ be an arbitrary extended diagonal box. By lemma 12, we know that $A = \sup \mathcal{K}_A$ where $\mathcal{K}_A$ is the directed subfamily of $\mathcal{F}$ given by

$$\mathcal{K}_A = \{hull_{\Omega(\mathbf{Z}^n)^*}(A_f) : A_f \text{ is a finite subset of } A\}.$$

But by lemma 24 we see that

$$\mathcal{K}_A = \{hull_{\Omega(\mathbf{Z}^n)}(A_f) : A_f \text{ is a finite subset of } A\}.$$

Therefore $T = \sup \mathcal{K}_A$ is an expression of A as the supremum of a directed subfamily of the family $\Omega(\mathbf{Z}^n)$. ∎

39

3.5 Radon's Theorem Revisited

In this section we uncover a relationship between the families $\Omega(Z^n)$ and $\Omega(Z^n)^*$ that parallels a similar relationship between C_n and C_n^* in Euclidean space. We show that $\Omega(Z^n)^*$ is the unique inductive intersecting family containing $\Omega(Z^n)$ for which the hulls of these two families agree on all finite sets.

Earlier we stated a version of Helly's Theorem for the grid and said that a version of Radon's Theorem for the grid would have to wait until we studied convexity further. At the end of this section we submit a version of Radon's Theorem for the grid.

THEOREM 28 *If $\mathcal{H}$ is an inductive diametral family of subsets of Z^n such that*

$$\text{hull}_{\mathcal{H}}(T) = \text{hull}_{\Omega(Z^n)}(T)$$

for all finite sets $T \subseteq Z^n$, then $\mathcal{H} = \Omega(Z^n)^*$.

PROOF The proof is exactly the same as the proof of theorem 14 when the following substitutions are made:

1. $\Omega(Z^n)$ for C_n

2. $\Omega(Z^n)^*$ for C_n^*

3. Theorem 26 for theorem 11.

∎

COROLLARY 29 *The following Radon numbers are equal:*

$$r(Z^n, \Omega(Z^n)) = r(Z^n, \Omega(Z^n)^*).$$

PROOF Let $r = r(Z^n, \Omega(Z^n))$. By proposition 6, we know $r \leq r(Z^n, \Omega(Z^n)^*)$.

On the the other hand if T is a set of r elements, then by the definition of Radon numbers we know that T admits an $\Omega(Z^n)$-partition. By theorem 28, this partition is also an $\Omega(Z^n)^*$-partition. Therefore $r(Z^n, \Omega(Z^n)^*) \leq r$. ∎

As a result of theorem 21 and corollary 29 we can now state

A Radon Theorem for the Grid: *Each set of $2^n + 1$ or more elements in Z^n can be expressed as the union of two disjoint sets whose $\Omega(Z^n)^*$-hulls share a common element.*

∎

3.6 Carathéodory's Theorem for the Grid

In this section we will prove an analogue of Carathéodory's Theorem for the grid $\mathbf{Z}^n$. As we saw in a previous section, the classical instance of Carathéodory's Theorem in Euclidean space is essentially a calculation of the Carathéodory number $c(\mathbf{R}^n, C_n^*)$. Therefore our formulation of Carathéodory's Theorem for the grid involves a calculation of the Carathéodory number $c(\mathbf{Z}^n, \Omega(\mathbf{Z}^n)^*)$, that is a calculation of the Carathéodory number of the grid's analogue of C_n^*.

We already know that $\Omega(\mathbf{Z}^n)^*$ is inductive and therefore finitary. But as we saw in proposition 15, the fact that a family is finitary is insufficient evidence to conclude that its Carathéodory number is finite. Lemma 30 tells us that $c(\mathbf{Z}^n, \Omega(\mathbf{Z}^n)^*)$ is at most 2^n. We will use this lemma as the starting point in the proof of theorem 31, and then show that we can cut this initial upper bound in half when $n \geq 2$.

LEMMA 30 *Given $T \subseteq \mathbf{Z}^n$ and an element $y \in \mathrm{hull}_{\Omega(\mathbf{Z}^n)^*}(T)$, there is a subset $T_f \subseteq T$ which has at most 2^n elements such that $y \in \mathrm{hull}_{\Omega(\mathbf{Z}^n)^*}(T_f)$.*

PROOF Let $T \subseteq \mathbf{Z}^n$ with $y \in \mathit{hull}_{\Omega(\mathbf{Z}^n)^*}(T)$. By proposition 23, we know that for each $\epsilon \in \Psi^n$ there exists an element $t_\epsilon \in T$ such that $\epsilon \cdot y \geq \epsilon \cdot t_\epsilon$. But then it follows that

$$y \in \mathit{hull}_{\Omega(\mathbf{Z}^n)^*}\left(\bigcup_{\eta \in \Psi^n} t_n \right).$$

Let $T_f = \{t_\eta : \eta \in \Psi^n\}$. ∎

The fact that $c(\mathbf{Z}^1, \Omega(\mathbf{Z}^1)^*) = 2$ is left to the reader. We now deal with all other cases.

THEOREM 31 (A CARATHÉODORY THEOREM FOR THE GRID)
Let $n \geq 2$. When $T \subseteq \mathbf{Z}^n$, each element of the $\Omega(\mathbf{Z}^n)^*$-hull of T is an element of the $\Omega(\mathbf{Z}^n)^*$-hull of some subset of T having at most 2^{n-1} elements.

PROOF We will prove that 2^{n-1} is the least number for which the statement of the theorem is true. That is, we will prove that

$$c(\mathbf{Z}^n, \Omega(\mathbf{Z}^n)^*) = 2^{n-1} \text{ for } n \geq 2. \tag{3.5}$$

There are two parts to the remainder of the proof.

Upper Bound: We first establish the upper bound for (3.5). Consider $T \subseteq \mathbf{Z}^n$ and choose $x \in \mathit{hull}_{\Omega(\mathbf{Z}^n)^*}(T)$. Our goal is to find a subset T' of T having at most 2^{n-1} elements such that $x \in \mathit{hull}_{\Omega(\mathbf{Z}^n)^*}(T')$.

By lemma 30 there is a subset T_f of T having at most 2^n elements such that $x \in \mathit{hull}_{\Omega(\mathbf{Z}^n)^*}(T_f)$. Consider the function $g_x : \Psi^n \times T_f \to \{-1, 1\}$ given by

$$g_x(\epsilon, t) = \begin{cases} 1 & \text{if } \epsilon \cdot x \geq \epsilon \cdot t \\ -1 & \text{otherwise.} \end{cases}$$

Put an equivalence relation $\sim$ on the set T_f by saying that

$$t_1 \sim t_2 \iff g_x(\epsilon, t_1) = g_x(\epsilon, t_2) \ \forall \epsilon \in \Psi^n.$$

Let T' be a complete set of equivalence class representatives taken from T_f, that is given any element $y \in T_f$ there is exactly one element $y' \in T'$ such that for all $\epsilon \in \Psi^n$, $g_x(\epsilon, y) = g_x(\epsilon, y')$.

Claim: $x \in \text{hull}_{\Omega^\bullet(\mathbf{Z}^n)}(T')$.

To establish the claim we must show that given arbitrary $\eta \in \Psi^n$, there is some element t'_η of T' such that $\eta \cdot x \geq \eta \cdot t'_\eta$. We do this now. From the representation of $\Omega(\mathbf{Z}^n)^\bullet$-hulls given in proposition 23, the fact that $x \in \text{hull}_{\Omega(\mathbf{Z}^n)^\bullet}(T_f)$ tells us that there exists an element $t_\eta \in T_f$ such that $\eta \cdot x \geq \eta \cdot t_\eta$. In other words, $g_x(\eta, t_\eta) = 1$. But the fact that T' is a complete set of equivalence representatives means that T' contains some element t'_η equivalent to t_η. Thus $g_x(\eta, t'_\eta) = 1$, meaning that $\eta \cdot x \geq \eta \cdot t'_\eta$, and we can conclude that $x \in \text{hull}_{\Omega^\bullet(\mathbf{Z}^n)}(T')$. The claim is therefore established.

Let us see what follows from the claim. If $|T'| \leq 2^{n-1}$, then the upper bound that we are seeking is established. So suppose that $|T'| > 2^{n-1}$. Note that there are at most 2^n possible equivalence classes. A simple counting argument shows that it must be the case that there exists a pair of elements $p_1, p_2 \in T'$ such that $g_x(\epsilon, p_1) = -g_x(\epsilon, p_2)$ for all $\epsilon \in \Psi^n$. Then $x \in \text{hull}_{\Omega(\mathbf{Z}^n)^\bullet}(\{p_1, p_2\})$ because for any ϵ either $\epsilon \cdot x \geq \epsilon \cdot p_1$ or $\epsilon \cdot x \geq \epsilon \cdot p_2$. So the supposition that $|T'| > 2^{n-1}$ implies that x is really in the $\Omega(\mathbf{Z}^n)^\bullet$-hull of a two element subset of T.

Lower Bound: Now we demonstrate the lower bound for $c(\mathbf{Z}^n, \Omega(\mathbf{Z}^n)^\bullet)$ as given in (3.5). To do this for the general case $n \geq 2$, we find a set $P_n \subseteq \mathbf{Z}^n$ having exactly 2^{n-1} elements such that there exists some element $c \in \text{hull}_{\Omega(\mathbf{Z}^n)^\bullet}(P_n)$ where c is not contained in $\text{hull}_{\Omega(\mathbf{Z}^n)^\bullet}(P'_n)$ for any proper subset $P'_n \subset P_n$.

First consider the $n = 2$ case. Let P_2 be the two element subset $\{(-2, 2), (2, 2)\}$ of $\mathbf{Z}^2$. It is easy to see that the element $(0, 0)$ of $\mathbf{Z}^2$ is an element of $\text{hull}_{\Omega(\mathbf{Z}^n)^\bullet}(P)$. But because $(0, 0)$ is not in the $\Omega(\mathbf{Z}^2)^\bullet$-hull of any proper subset of P, we conclude that $c(\mathbf{Z}^2, \Omega(\mathbf{Z}^2)^\bullet) \geq 2$.

Now for the general case of $n \geq 3$. Consider the set $P_n \subseteq \mathbf{Z}^n$ whose elements have the form $(p_1, \ldots, p_n)$ where

$$p_1 \in \{-2, 2\},$$
$$p_i \in \{-2^{i-2}, 2^{i-2}\} \text{ for } 2 \leq i \leq n - 1,$$
$$p_n = 2^{n-2}.$$

We shall denote an arbitrary element of P_n by

$$(2a_1, \ldots, 2^{i-2}a_i, \ldots, 2^{n-2}),$$

where i is understood to range from 2 through $n - 1$ and each of the numbers $a_1, \ldots, a_{n-1}$ is either 1 or -1. Clearly $|P_n| = 2^{n-1}$.

Consider now $c = (0, \ldots, 0) \in \mathbf{Z}^n$. Our plan is to show that $c \in \text{hull}_{\Omega(\mathbf{Z}^n)^\bullet}(P_n)$ but that $c \notin \text{hull}_{\Omega(\mathbf{Z}^n)^\bullet}(P')$ for any proper subset $P'_n \subseteq P_n$. Note that this will allow us to

42

conclude that $c(\mathbf{Z}^n, \Omega(\mathbf{Z}^n)^*) \geq 2^{n-1}$.

To proceed, we show that for each $\epsilon \in \Psi^n$, there exists $t_\epsilon \in P_n$ such that $\epsilon \cdot c \geq \epsilon \cdot t_\epsilon$. This will establish that $c \in hull_{\Omega(\mathbf{Z}^n)^*}(P_n)$. Given any $\epsilon = [\epsilon_1, \ldots, \epsilon_n] \in \Psi^n$, let $t_\epsilon \in P_n$ be given by

$$t_\epsilon = (-2\epsilon_1, \ldots, -2^{i-2}\epsilon_i, \ldots, 2^{n-2}).$$

Then

$$
\begin{aligned}
\epsilon \cdot t_\epsilon &= -2\epsilon_1^2 - \left[\sum_{i=2}^{n-1} 2^{i-2}\epsilon_i^2\right] + 2^{n-2}\epsilon_n \\
&= -2 - \left[\sum_{i=0}^{n-3} 2^i\right] + 2^{n-2}\epsilon_n \\
&= -2 - (2^{n-2} - 1) + 2^{n-2}\epsilon_n \\
&= (\epsilon_n - 1)2^{n-2} - 1 \\
&< 0 \\
&= \epsilon \cdot c
\end{aligned}
$$

Thus $c \in hull_{\Omega(\mathbf{Z}^n)^*}(P_n)$.

Finally we must prove that c is not contained in the $\Omega(\mathbf{Z}^n)^*$-hull of any proper subset P_n' of P_n. Suppose that P_n' is a proper subset of P_n such that $c \in hull_{\Omega(\mathbf{Z}^n)^*}(P_n')$. We argue to a contradiction. Since P_n' is a proper subset of P_n, there exists an element $y \in P_n - P_n'$, say

$$y = (2a_1, \ldots, 2^{i-2}a_i, \ldots, 2^{n-2})$$

where each $a_1, \ldots, a_{n-1}$ is either -1 or 1. Consider

$$\epsilon = [-a_1, \ldots, -a_i, \ldots, 1] \in \Psi^n.$$

By proposition 23, there exists an element $t_\epsilon \in P_n'$ such that $\epsilon \cdot c \geq \epsilon \cdot t_\epsilon$. Let us say that the coordinates of t_ϵ are given by

$$t_\epsilon = (2b_1, \ldots, 2^{i-2}b_i, \ldots, 2^{n-2})$$

for some $b_1, \ldots, b_{n-1}$ which are either -1 or 1. Because $y \notin P_n'$ and $t_\epsilon \in P_n'$, we know that $a_j \neq b_j$ for some $j \in \{1, \ldots, n-1\}$. Then

$$2a_1 b_1 + \sum_{i=2}^{n-1} 2^{i-2}a_i b_i \leq \left(2 + \sum_{i=2}^{n-1} 2^{i-2}\right) - 2 = 2^{n-2} - 1.$$

Using this equation we reach a contradiction:

$$0 = \epsilon \cdot c \geq \epsilon \cdot t_\epsilon = -2a_1 b_1 - \left(\sum_{i=2}^{n-1} 2^{i-2}a_i b_i\right) + 2^{n-2} \geq 1.$$

Chapter 4

Convexity in Metric Spaces

4.1 The Intersecting-Maximal Family

In chapter 3 we generalized Euclidean convexity and the classical convexity theorems of Radon, Helly, and Carathéodory from their usual Euclidean setting to another metric space, the grid Z^n having the Manhattan metric. In chapter 4 we provide the full generalization of our previous results to arbitrary metric spaces. Throughout let (M, d) be a metric space with $\mu(T)$ denoting the diameter of a subset $T \subseteq M$, that is $\mu(T) = \sup\{d(x, y) : x, y \in T\}$. Any topological reference connected with a metric space refers to the metric topology of the space.

In this section we introduce a special family of subsets associated with a metric space M, a family which we call the *intersecting-maximal* family $\Omega(M)$. We will see that $\Omega(M)$ is the general metric space analogue of the family C_n of convex bodies in R^n. In fact $\Omega(R^n) = C_n$. Previously we used the notation $\Omega(Z^n)$ to denote the family of diagonal boxes in Z^n. We will see that our notation is justified in that the family of diagonal boxes in Z^n is the intersecting-maximal family for Z^n. We begin with a definition.

Definition: *A subset S of a metric space M is a maximal set if and only if S is not properly contained in another subset of M having the same diameter.*

There are a number of facts about maximal sets that we can deduce immediately. Because a set and its closure have the same diameter, we know that all maximal sets are closed sets. Note that in the metric spaces R^n and Z^n discussed earlier, every closed ball is an example of a maximal set. But in both of these spaces there are maximal sets which are not closed balls.

The set M is itself a maximal set. With the possible exception of the set M though, all maximal sets are bounded.

Let $\Omega(M)$ denote the intersecting family of subsets of M generated by the collection of maximal subsets of M. In other words $\Omega(M)$ is the family whose members are those subsets of M which can be expressed as intersections of maximal sets. As such, $\Omega(M)$ is the smallest intersecting family of subsets of M which contains the entire collection of maximal subsets of M. We call $\Omega(M)$ the *intersecting-maximal* family of M.

PROPOSITION 32 *For any metric space M, the family $\Omega(M)$ is diametral.*

PROOF The family $\Omega(M)$ is an intersecting family by construction. Our proof shows that $\Omega(M)$ meets the second condition of diametrality, meaning that for an arbitrary bounded set $T \subseteq M$ we have $\mu(hull_{\Omega(M)}(T)) = \mu(T)$.

Let T be a bounded subset of M. Consider the family

$$\mathcal{G}_T = \{A : T \subseteq A \text{ and } \mu(A) = \mu(T)\},$$

that is all of supersets of T which have the same diameter that T has. Note that $\mathcal{G}_T$ is nonempty because it contains T.

We claim that arbitrary nonempty directed subfamilies of $\mathcal{G}_T$ have upper bounds in $\mathcal{G}_T$. Let us see what follows from this claim. If the claim is true then by the maximal principle (ie Zorn's Lemma) we know that $\mathcal{G}_T$ contains some member T' which is not properly contained in any member of $\mathcal{G}_T$. Then by definition, T' is a maximal set and therefore a member of $\Omega(M)$. It follows from (2.1) that $hull_{\Omega(M)}(T) \subseteq T'$. Thus

$$\mu(T) \leq \mu(hull_{\Omega(M)}(T)) \leq \mu(T') = \mu(T).$$

So if the claim is true, we know that $\Omega(M)$ is a diametral family of subsets of M.

Now to establish the claim. Let $\mathcal{H}$ be an arbitrary nonempty directed subfamily of $\mathcal{G}_T$. We will show that $\sup \mathcal{H} \in \mathcal{G}_T$. We know that $T \subseteq \sup \mathcal{H}$, and consequently that $\mu(T) \leq \mu(\sup \mathcal{H})$, so it remains to show that $\mu(\sup \mathcal{H}) \leq \mu(T)$. To see this, select $x, y \in \sup \mathcal{H}$. Then there are members A_x and A_y of $\mathcal{H}$ containing x and y respectively. Because $\mathcal{H}$ is directed, there is some member $A \in \mathcal{H}$ containing both A_x and A_y and consequently containing both x and y. Therefore

$$d(x, y) \leq \mu(A) = \mu(T).$$

We conclude that $\mu(\sup \mathcal{H}) = \mu(T)$ and therefore that $\sup \mathcal{H} \in \mathcal{G}_T$. ∎

We have a potential problem with notation. Earlier we used $\Omega(\mathbf{Z}^n)$ to denote the family of diagonal boxes in $\mathbf{Z}^n$. Now we have another definition of $\Omega(\mathbf{Z}^n)$ as the family generated by taking all possible intersections of maximal subsets of $\mathbf{Z}^n$. Theorem 33 shows that both of these usages are consistent. In addition, the theorem 33 also gives us the fact that $\Omega(\mathbf{R}^n) = \mathcal{C}_n$.

THEOREM 33 *If $\mathcal{F}$ is an intersecting family of subsets of M, then*

$$\mathcal{F} \text{ is diametral} \Longleftrightarrow \Omega(M) \subseteq \mathcal{F}.$$

PROOF Let $\mathcal{F}$ be an intersecting family of subsets of M.

Suppose that $\mathcal{F}$ is a diametral family. Given a maximal subset S of M, we know both of the following:

1. $S \subseteq hull_{\mathcal{F}}(S)$.

2. $\mu(S) = \mu(hull_{\mathcal{F}}(S))$.

45

The only way that both of these facts can remain consistent with the definition of maximal sets is if $S = hull_{\mathcal{F}}(S)$; in other words if $S \in \mathcal{F}$. Therefore $\mathcal{F}$ contains all maximal subsets of M. The fact that $\mathcal{F}$ is an intersecting family now guarantees that $\Omega(M) \subseteq \mathcal{F}$.

The converse follows from the fact that $\Omega(M)$ is diametral (proposition 32) and the fact that any intersecting family containing a diametral family is itself a diametral family (lemma 2). ∎

COROLLARY 34 *The class of diametral families is closed under intersections.*

PROOF The proof is exactly the same as the proof of corollary 4 with $\Omega(M)$ and theorem 33 substituting for C_n and theorem 3 respectively. ∎

4.2 Numbers and Relationships

As a result of proposition 6, proposition 32, and theorem 33, we immediately have

THEOREM 35 *Let $\mathcal{F}$ be a diametral family of subsets of M.*

1. *The Radon number $r(M, \mathcal{F})$ is at least $r(M, \Omega(M))$.*

2. *The Helly number $h(M, \mathcal{F})$ is at least $h(M, \Omega(M))$.*

The Radon and Helly numbers of diametral families of subsets in any metric space have the given lower bounds. We are led to the following definitions.

Definitions: Let M be a metric space.

1. The *diametral Radon number $r(M)$* is the least Radon number of any diametral family of subsets of M, ie $r(M) = r(M, \Omega(M))$.

2. The *diametral Helly number $r(M)$* is the least Helly number of any diametral family of subsets of M, ie $h(M) = h(M, \Omega(M))$.

Note that either $r(M)$ or $h(M)$ may be infinite.

We have shown that $r(\mathbf{R}^n) = n+2$, $h(\mathbf{R}^n) = n+1$, $r(\mathbf{Z}^n) = 2^n + 1$, and $h(\mathbf{Z}^n) = 2^n$. It is to be observed that $\mathbf{R}^n$ has a diametral Radon number one greater than its diametral Helly number, and that the same is true for $\mathbf{Z}^n$. Earlier we noted that maximal sets are closed and that with the possible exception of M are also bounded. In $\mathbf{R}^n$ and $\mathbf{Z}^n$, closed and bounded sets are compact. In the case of a general metric space M in which all proper maximal sets are compact, we prove in theorem 36 that the diametral Radon number exceeds the diametral Helly number by one. Actually we prove a stronger result in proposition 37 from which theorem 36 follows.

THEOREM 36 *Let M be a metric space in which every proper subset of M that is maximal is also compact. Then*

$$h(M) = r(M) - 1.$$

PROOF If proper maximal subsets of M are compact, then setting $\mathcal{F} = \Omega(M)$ in proposition 37 yields

$$h(M) = h(M, \Omega(M)) = r(M, \Omega(M)) - 1 = r(M) - 1.$$

∎

PROPOSITION 37 *Let M be a metric space and $\mathcal{F}$ an intersecting family of subsets of M. Then*

1. $h(M, \mathcal{F}) \geq r(M, \mathcal{F}) - 1$.

2. If every member of $\mathcal{F} - \{M\}$ is compact, then

$$h(M, \mathcal{F}) = r(M, \mathcal{F}) - 1.$$

PROOF We use r and h to denote $r(\mathcal{M}, \mathcal{F})$ and $h(\mathcal{M}, \mathcal{F})$ respectively. Note $r \geq 2$.

(1) For any positive integer k such that $k < r$, there exists a k-element subset T_k of M which does not admit a Radon partition. Let

$$\mathcal{G}_k = \{T_k - \{x\} : x \in T_k\}.$$

Then $\mathcal{G}_k$ is a family of k members such that any $k-1$ members of $\mathcal{G}_k$ share a common element of M. But no element of M is common to every member of $\mathcal{G}_k$. We conclude that $h > k - 1$. Because k is arbitrary, we have that $h \geq r - 1$ ···henever r is finite and that h is infinite if r is not finite.

(2) Suppose that every member of $\mathcal{F} - \{M\}$ is compact. It suffices to show that $h \leq r - 1$ when r is finite.

Let $\mathcal{K}$ be a subfamily of $\mathcal{F}$ having at least $r - 1$ members such that any $r - 1$ members of $\mathcal{K}$ have a nonempty intersection. We show that there is an element of M common to every member of $\mathcal{K}$. If $\mathcal{K} = \{M\}$ then this is trivially true, so assume that there is a compact member $A \in \mathcal{K}$.

We claim that $\mathcal{K}$ satisfies the finite intersection condition, meaning that every finite collection of members of $\mathcal{K}$ has nonempty intersection. Let us see how the fact that there is an element common to every member of $\mathcal{K}$ follows from this claim.

A well-known theorem from topology says that if A is compact, then for every family $\mathcal{H}$ of subsets of A satisfying the finite intersection condition, there is an element of A common to every member of $\mathcal{H}$. Consider the following family of subsets of A:

$$\mathcal{H} = \{B \cap A : B \in \mathcal{K}\}.$$

If the claim that $\mathcal{K}$ satisfies the finite intersection condition is true, then $\mathcal{H}$ also satisfies the finite intersection condition. Therefore we conclude that there exists an element common to every member of $\mathcal{H}$, which is necessarily a common element of every member of $\mathcal{K}$ also.

Now to establish the claim that $\mathcal{K}$ satisfies the finite intersection condition. (The following proof technique is found in [5].) Let $\mathcal{L}$ be a finite subfamily of $\mathcal{K}$. We argue by induction on the cardinality j of $\mathcal{L}$ to show that there is an element $z \in M$ common to every member of $\mathcal{L}$.

Because we know that any $r-1$ or fewer members of $\mathcal{K}$ have nonempty intersection, we can proceed to the general case $j \geq r \geq 2$. Given any member $T \in \mathcal{L}$, by inductive hypothesis there is an element p_T of M common to every member of the family $\mathcal{L} - \{T\}$. Let

$$V = \{p_T : T \in \mathcal{L}\}.$$

Note that V is a set of at least r elements and so can be partitioned into two disjoint subsets whose $\mathcal{F}$-hulls have nonempty intersection, say including some element $z \in M$. For any $T \in \mathcal{L}$, consider the block B of the partition of V which does not contain the element p_T. Note $B \subseteq T$ and so

$$z \in hull_{\mathcal{F}}(B) \subseteq hull_{\mathcal{F}}(T) = T.$$

Because T is arbitrary, we conclude that z is contained in every member of $\mathcal{L}$. ∎

See [6, 12] for further relationships between Radon, Helly, and Carathéodory numbers.

4.3   Inductive Completion

In this section we obtain a generalization of the family C_n^* of all Euclidean convex sets for arbitrary metric spaces. This family of generalized convex sets will be both diametral and inductive. We take this family to be the unique smallest inductive diametral family in a metric space.

Let us begin by arguing that there is indeed a unique smallest inductive diametral family for a general metric space M. Let the family $\mathcal{X}$ be the intersection of all inductive diametral families of subsets of M, that is $A \in \mathcal{X}$ if and only if A is a member of every inductive diametral family of subsets of M. Note $\mathcal{X}$ is an intersecting family because it is the intersection of intersecting families. This fact, together with theorem 33, tells us that $\mathcal{X}$ is diametral. It is easy to see that the intersection of inductive families remains an inductive family. So we conclude that $\mathcal{X}$ is the unique smallest inductive diametral family.

In order to characterize the unique smallest inductive diametral family of a general metric space, and to show its relationship with the family $\Omega(M)$, we introduce the notion of *inductive completion*, a method of starting with one family and "completing" it to obtain a larger family. One advantage of the completion is that the family obtained upon completion is necessarily inductive.

Definition: *The inductive completion of the family $\mathcal{F}$ is the family*

$$\mathcal{F}^* = \{\sup \mathcal{K} : \mathcal{K} \text{ is a directed subfamily of } \mathcal{F}\}.$$

Clearly if $\mathcal{F}$ is an inductive family, then $\mathcal{F}^* = \mathcal{F}$. More importantly we will show that $\mathcal{F}^*$ is always inductive (and consequently finitary).

LEMMA 38 *For any intersecting family $\mathcal{F}$, the inductive completion $\mathcal{F}^*$ is both intersecting and inductive.*

PROOF First we suppose that $\mathcal{F}$ in an intersecting family and argue that $\mathcal{F}^*$ is intersecting also.

Let $\{T_j\}_{j \in J}$ be a subfamily of $\mathcal{F}^*$. We wish to show that $\bigcap_{j \in J} T_j \in \mathcal{F}^*$. To do this we must show that there exists a directed subfamily $\mathcal{L}$ of $\mathcal{F}$ such that $\bigcap_{j \in J} T_j = \sup \mathcal{L}$.

Note that for each $j \in J$, there exists a directed subfamily $\mathcal{K}_j$ of $\mathcal{F}$ such that $T_j = \sup \mathcal{K}_j$. We want to consider the family $\mathcal{L}$ whose members are those sets which can be expressed as an intersection of exactly one member from each of the $\mathcal{K}_j$'s, that is

$$\mathcal{L} = \{\bigcap_{j \in J} V_j : V_j \in \mathcal{K}_j\}.$$

Note $\mathcal{L}$ is a subfamily of $\mathcal{F}$ because $\mathcal{F}$ is an intersecting family. We claim that $\mathcal{L}$ is a directed family such that $\bigcap_{j \in J} T_j = \sup \mathcal{L}$. From this claim we can conclude that $\mathcal{F}^*$ is an intersecting family.

To see that $\mathcal{L}$ is directed, let $\bigcap_{j \in J} V_j$ and $\bigcap_{j \in J} V_j'$ be two members of $\mathcal{L}$. Each family $\mathcal{K}_j$ is directed and so has some member U_j which contains both V_j and V_j'. Then $\bigcap_{j \in J} U_j$ is an upper bound in $\mathcal{L}$ for $\bigcap_{j \in J} V_j$ and $\bigcap_{j \in J} V_j'$, showing that $\mathcal{L}$ is directed.

Now we show that $\bigcap_{j \in J} T_j = \sup \mathcal{L}$.

To see that $\sup \mathcal{L} \subseteq \bigcap_{j \in J} T_j$, it suffices to choose an arbitrary $A \in \mathcal{L}$ and show that $A \subseteq \bigcap_{j \in J} T_j$. There exists a collection $\{A_j\}_{j \in J}$ where each $A_j \in \mathcal{K}_j$ such that $A = \bigcap_{j \in J} A_j$. Then

$$A = \bigcap_{j \in J} A_j \subseteq \bigcap_{j \in J} \sup \mathcal{K}_j = \bigcap_{j \in J} T_j.$$

To see that $\bigcap_{j \in J} T_j \subseteq \sup \mathcal{L}$, choose $x \in \bigcap_{j \in J} T_j$. For each $j \in J$, the fact that $T_j = \sup \mathcal{K}_j$ tells us that there is a set $B_j \in \mathcal{K}_j$ such that $x \in B_j$. Then $x \in \bigcap_{j \in J} B_j \in \mathcal{L}$. Thus $x \in \sup \mathcal{L}$.

Now we argue the remaining part of the lemma, ie we show that $\mathcal{F}^*$ is inductive.

Let $\mathcal{K}$ be a directed subfamily of $\mathcal{F}^*$. In order to show that $\mathcal{F}^*$ is inductive, we show that $\sup \mathcal{K} \in \mathcal{F}^*$. By the definition of inductive completion, we must show that there exists a directed subfamily $\mathcal{H}$ of $\mathcal{F}$ such that $\sup \mathcal{K} = \sup \mathcal{H}$. Here we produce $\mathcal{H}$ but we must wait until we prove the first part of theorem 42 to see that $\mathcal{H}$ is truly

a subfamily of $\mathcal{F}$. Consider

$$\mathcal{H} = \{hull_{\mathcal{F}^*}(A) : A \text{ is a finite subset of } \sup \mathcal{K}\}.$$

There are three things to show:

(1) $\sup \mathcal{K} \subseteq \sup \mathcal{H}$: If $x \in \sup \mathcal{K}$, then $x \in hull_{\mathcal{F}^*}(\{x\}) \in \mathcal{H}$. Therefore $x \in \sup \mathcal{H}$.

(2) $\sup \mathcal{H} \subseteq \sup \mathcal{K}$: If $x \in \sup \mathcal{H}$, then $x \in hull_{\mathcal{F}^*}(A)$ for some finite $A \subseteq \sup \mathcal{K}$. Each element p of A is contained in some member $A_p \in \mathcal{K}$. Because $\mathcal{K}$ is directed, there exists a set $B \in \mathcal{K}$ such that

$$\bigcup_{p \in A} A_p \subseteq B.$$

Then we see that $x \in \sup \mathcal{K}$ because

$$x \in hull_{\mathcal{F}^*}(A) \subseteq hull_{\mathcal{F}^*}(\bigcup_{p \in A} A_p) \subseteq hull_{\mathcal{F}^*}(B) = B \in \mathcal{K}.$$

(3) $\mathcal{H}$ is directed: Suppose that $hull_{\mathcal{F}^*}(A_1)$ and $hull_{\mathcal{F}^*}(A_2)$ are two members of $\mathcal{H}$ where A_1 and A_2 are finite subsets of $\sup \mathcal{K}$. Then $A_1 \cup A_2$ is also a finite subset of $\sup \mathcal{K}$ and

$$hull_{\mathcal{F}^*}(A_1) \cup hull_{\mathcal{F}^*}(A_2) \subseteq hull_{\mathcal{F}^*}(A_1 \cup A_2).$$

∎

COROLLARY 39 *Inductive completion is a closure relation. That is for any intersecting family $\mathcal{F}$,*

1. $\mathcal{F} \subseteq \mathcal{F}^*$

2. If $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{F}^* \subseteq \mathcal{G}^*$.

3. $(\mathcal{F}^*)^* = \mathcal{F}^*$.

PROOF The first two points follow immediately from the definition of inductive completion. Consider now the third point. In light of the first point, it suffices to show that $(\mathcal{F}^*)^* \subseteq \mathcal{F}^*$.

Let $A \in (\mathcal{F}^*)^*$. Then A is the supremum of a directed subfamily of $\mathcal{F}^*$. By lemma 38 we know that $\mathcal{F}^*$ is an inductive family and so contains the supremum of all of its directed subfamilies Therefore $\mathcal{F}^*$ contains A. ∎

In the proof of corollary 39, we see that the fact that $(\mathcal{F}^*)^* = \mathcal{F}^*$ follows from lemma 38. But we can show that this fact is actually equivalent to the result of lemma 38 that $\mathcal{F}^*$ is always inductive. Suppose $(\mathcal{F}^*)^* = \mathcal{F}^*$ and let $\mathcal{K}$ be a directed subfamily of $\mathcal{F}^*$. Then by the definition of inductive completion, we have that we conclude that $\mathcal{F}^*$ is inductive because $\sup \mathcal{K} \in (\mathcal{F}^*)^* = \mathcal{F}^*$.

We now justify an earlier choice of notation. Recall that in $\mathbf{R}^n$ we denote the family of all Euclidean convex sets by $\mathcal{C}_n^*$, and that in $\mathbf{Z}^n$ we denote the family of extended diagonal boxes by $\Omega(\mathbf{Z}^n)^*$.

PROPOSITION 40 *The inductive completion of the family C_n of convex bodies is the family of all Euclidean convex sets and the inductive completion of the family $\Omega(Z^n)$ is the family of extended diagonal boxes.*

PROOF Let $(C_n)^*$ denote the inductive completion of C_n and let C_n^* denote the family of all Euclidean convex sets. In proposition 13 we showed that every member of C_n^* is the supremum of a directed subfamily of C_n. This shows that $C_n^* \subseteq (C_n)^*$. The reverse containment involves showing that supremums of directed families of convex bodies are convex sets. (It is not necessarily true that nondirected families of convex bodies are convex sets.) To see the reverse containment, note that by corollary 39 and the fact that C_n^* is inductive, we have

$$C_n \subseteq C_n^* \Rightarrow (C_n)^* \subseteq (C_n^*)^* = C_n^*.$$

Now let $(\Omega(Z^n))^*$ denote the inductive completion of the family $\Omega(Z^n)$ of diagonal boxes and let $\Omega(Z^n)^*$ denote the family of extended diagonal boxes.

In the proof of theorem 26 we showed that any extended diagonal box can be expressed as the supremum of a directed family of diagonal boxes, ie $\Omega(Z^n)^* \subseteq (\Omega(Z^n))^*$.

On the other hand in proposition 25 we showed that $\Omega(Z^n)^*$ is inductive. Therefore

$$\Omega(Z^n) \subseteq \Omega(Z^n)^* \Rightarrow (\Omega(Z^n))^* \subseteq (\Omega(Z^n)^*)^* = \Omega(Z^n)^*.$$

∎

We now characterize the unique smallest inductive diametral family of a general metric space as the inductive completion of the unique smallest diametral family.

THEOREM 41 *If $\mathcal{F}$ is an inductive intersecting family of subsets of M, then*

$$\mathcal{F} \text{ is diametral } \Longleftrightarrow \Omega(M)^* \subseteq \mathcal{F}.$$

PROOF Let $\mathcal{F}$ be an inductive intersecting family of subsets of M.

If $\mathcal{F}$ is diametral, then by theorem 33 we know $\Omega(M) \subseteq \mathcal{F}$. So by corollary 39 we know that $\Omega(M)^* \subseteq \mathcal{F}^* = \mathcal{F}$.

On the other hand $\Omega(M) \subseteq \Omega(M)^* \subseteq \mathcal{F}$ would show that $\mathcal{F}$ is diametral. ∎

4.4 An Agreement for Polytopes

In this section we generalize some of our previous results. Recall that hulls of finite sets are called polytopes. In theorem 42 we show that hulls with respect to either a family $\mathcal{F}$ or its inductive completion $\mathcal{F}^*$ agree on all finite sets. Furthermore, $\mathcal{F}^*$ is the unique inductive intersecting family that contains $\mathcal{F}$ for which this is true. As a corollary we will conclude that Radon numbers are blind to inductive closure.

THEOREM 42 *Let U be a set and let $\mathcal{F}$ be an intersecting family of subsets of U. Then*

$$\text{hull}_{\mathcal{F}^*}(T) = \text{hull}_{\mathcal{F}}(T) \text{ for all finite } T \subseteq U.$$

Moreover, $\mathcal{F}^*$ is the unique inductive intersecting family with this property, meaning that if $\mathcal{G}$ is an inductive intersecting family containing $\mathcal{F}$ such that

$$\text{hull}_{\mathcal{G}}(T) = \text{hull}_{\mathcal{F}}(T) \text{ for all finite } T \subseteq U,$$

then $\mathcal{G} = \mathcal{F}^*$.

PROOF Let T be a finite subset of U. We know that $\text{hull}_{\mathcal{F}^*}(T) \subseteq \text{hull}_{\mathcal{F}}(T)$ from (2.2). To see the reverse containment, we make a

Claim: *If A is a set such that $T \subseteq A \in \mathcal{F}^*$, then $\text{hull}_{\mathcal{F}}(T) \subseteq A$.*

Note that from the claim we immediately have

$$hull_{\mathcal{F}}(T) \subseteq \bigcap_{T \subseteq A \in \mathcal{F}^*} A = hull_{\mathcal{F}^*}(T).$$

To establish the claim suppose that we have a set A such that $T \subseteq A \in \mathcal{F}^*$. Then $A = \sup \mathcal{K}$ where $\mathcal{K}$ is a directed subfamily of $\mathcal{F}$. If we can find a set $B \in \mathcal{F}$ such that $T \subseteq B \subseteq A$, then the claim follows from

$$hull_{\mathcal{F}}(T) \subseteq hull_{\mathcal{F}}(B) = B \subseteq A.$$

For each $x \in T$, the fact that $x \in A = \sup \mathcal{K}$ tells us that $x \in B_x$ for some $B_x \in \mathcal{K}$. Because $\mathcal{K}$ is directed and T is finite, there exists a member $B \in \mathcal{K}$ containing these finitely many B_x's. Then $B \in \mathcal{F}$ and $T \subseteq B \subseteq \sup \mathcal{K} = A$. We have established the claim and consequently the fact that $hull_{\mathcal{F}^*}(T) = hull_{\mathcal{F}}(T)$ for all finite $T \subseteq U$.

Now suppose that $\mathcal{G}$ is an inductive intersecting family containing $\mathcal{F}$ which satisfies the hypotheses of the theorem. By corollary 39 we know that

$$\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{F}^* \subseteq \mathcal{G}^* = \mathcal{G}.$$

To show that $\mathcal{G} \subseteq \mathcal{F}^*$, we apply lemma 12. Given $S \in \mathcal{G}$, we know that

$$\mathcal{K}_S = \{hull_{\mathcal{G}}(S_f) : S_f \text{ is a finite subset of } A\}$$

is a directed subfamily of $\mathcal{G}$ such that $S = \sup \mathcal{K}_S$. By hypotheses

$$S = \sup \mathcal{K}_S = \sup\{hull_{\mathcal{F}}(S_f) : S_f \text{ is a finite subset of } A\}$$

shows that we can express S as the supremum of a directed subfamily of $\mathcal{F}$. Therefore $S \in \mathcal{F}^*$ and consequently $\mathcal{G} \subseteq \mathcal{F}^*$. ∎

We remark that the previous results have given us

COROLLARY 43

$$r(U, \mathcal{F}) = r(U, \mathcal{F}^*)$$

PROOF Clearly $r(U, \mathcal{F}) \leq r(U, \mathcal{F}^*)$ because $\mathcal{F} \subseteq \mathcal{F}^*$. Assume $r = r(U, \mathcal{F})$ is finite. Any r elements of U can be partitioned into two disjoint nonempty sets having intersecting $\mathcal{F}$-hulls. These two sets are finite and so have the same $\mathcal{F}^*$-hulls by theorem 42. Thus $r(U, \mathcal{F}^*) \leq r$. ∎

Chapter 5

Spherical Convexity

5.1 Further Study

We define the *diametral Carathéodory number* $c(M)$ of a metric space M as the Carathéodory number of the unique minimum inductive diametral family of M, a family whose existence is guaranteed by theorem 41. In other words, we have that $c(M) = c(M, \Omega(M)^*)$. So Carathéodory's Theorem tells us that $c(\mathbf{R}^n) = n + 1$. And by theorem 31 we know $c(\mathbf{Z}^n) = 2^{n-1}$.

For any metric space M we have developed the underlying theory for two associated families of sets and for three associated types of integers. The first family is $\Omega(M)$ which can be regarded as an analogue of the family of Euclidean convex bodies. The Helly number of this family is the diametral Helly number of the space. The second family is $\Omega(M)^*$ which can be regarded as an analogue of the family of all Euclidean convex sets. The Carathéodory number of this family is the diametral Carathéodory number of the space. The Radon number of either family is the same and is taken as the diametral Radon number of the space.

So for any given metric space M we have the following problems:

1. Characterize $\Omega(M)$, the analogue of Euclidean convex bodies.

2. Characterize $\Omega(M)^*$, the analogue of Euclidean convex sets.

3. Determine the diametral Radon, Helly, and Carathéodory numbers of M.

Having carried out this three part program for a specific metric space M, we arrive at versions of the following classical theorems just as we did for the grid.

A Radon Theorem for M: *Each set of $r(M)$ or more elements in M can be expressed as the union of two disjoint sets whose $\Omega(M)^*$-hulls (or equivalently $\Omega(M)$-hulls) share a common element.*

A Helly Theorem for M: *Suppose $\mathcal{K}$ is a family of at least $h(M)$ members of $\Omega(M) - \{M\}$ in M. If each $h(M)$ members of $\mathcal{K}$ share a common element, then there is an element common to all members of $\mathcal{K}$.*

A Carathéodory Theorem for M: When $T \subseteq M$, each element of the $\Omega(M)^*$-hull of T is an element of the $\Omega(M)^*$-hull of some subset of T having at most $c(M)$ elements.

In the next section, the three part program outlined above is carried out for the Euclidean spheres. But first we give the following technical lemma which is useful for recognizing members of $\Omega(M)$.

LEMMA 44 *Let T be a subset of the metric space M. Then*

$$T \in \Omega(M) \text{ if and only if } \forall z \notin T, \exists y \text{ such that } d(y,z) > \mu(T \cup \{y\}).$$

PROOF   Suppose $T \in \Omega(M)$ and $z \notin T$. Because T can be written as an intersection of maximal subsets of M, there exists some maximal set V containing $T - \{z\}$. V is maximal and does not include z, so we conclude that there exists $y \in V$ such that

$$d(y,z) > \mu(V) \geq \mu(T \cup \{y\}).$$

For the other direction suppose that the condition is satisfied for T, that is given $z \notin T$, there exists y such that $d(y,z) > \mu(T \cup \{y\})$. By the maximal property, there is a maximal set V_z containing $T \cup \{y\}$ with diameter $\mu(V_z) = \mu(T \cup \{y\})$. Because

$$\mu(V_z \cup \{z\}) \geq d(y,z) > \mu(T \cup \{y\}) = \mu(V_z),$$

we can conclude from the fact that V_z is maximal that $z \notin V_z$.

We have shown that given arbitrary $z \notin T$, there is a maximal set V_z which contains T but not z. Thus T can be expressed as an intersection of maximal sets, namely

$$T = \bigcap_{z \notin T} V_z,$$

and so we conclude that $T \in \Omega(M)$.

∎

5.2   The Diametral Numbers of the Sphere

In this section we consider the surface of the Euclidean n-sphere S^n as a metric space in which distance is measured as the length of the shortest path between points along the surface of the sphere. We will be carrying out the three point program outlined in the last section. That is, we characterize $\Omega(S^n)$ and $\Omega(S^n)^*$, the "convex bodies" and "convex sets" of S^n and then calculate the diametral Radon, Helly, and Carathéodory numbers of S^n.

The results of this section indicate that there is only a trivial notion of spherical convexity which satisfies the three basic properties of Euclidean convexity with which we have been dealing.

We begin with a characterization of the intersecting-maximal family of S^n. For every point x of S^n there is a unique point y situated at a maximum distance away x, the *antipodal point* of x. We call the pair x and y a pair of antipodal points.

THEOREM 45 *The family $\Omega(S^n)$ consists of S^n itself and all closed subsets of S^n which do not contain a pair of antipodal points.*

PROOF By the definition of the intersecting-maximal family, we know that any member $V \in \Omega(S^n)$ is an intersection of maximal sets. Because maximal sets are necessarily closed, we have that V is closed. Note that the only maximal set which contains antipodal points is the set S^n itself. Thus if $V \in \Omega(S^n)$ is a subset of S^n that contains antipodal points, then $V = hull_{\Omega(S^n)}(V) = S^n$. So we conclude that every member of $\Omega(S^n)$ must be either S^n itself or a closed subset of S^n which does not contain antipodal points.

Let T be a closed subset of S^n which contains no antipodal points. We claim that $\mu(T)$ is strictly less than $\mu(S^n)$. Let us see what follows from this claim. Choose $z \notin T$, and let y denote the antipode of z. Now $\mu(T \cup \{y\}) < d(y, z) = \mu(S^n)$ because z is not a limit point of V. Therefore by applying lemma 44 we see that $T \in \Omega(M)$.

We now establish the claim. Suppose for contradiction that $\mu(T) = \mu(S^n)$. Then there exists sequences $\{p_i\}_{i \geq 0}$ and $\{q_i\}_{i \geq 0}$ in T such that $d(p_i, q_i) > \mu(S^n) - \frac{1}{2^i}$. Infinite bounded subsets of the surfaces of Euclidean spheres will have limit points, so our two sequences have limit points, say p and q respectively. Because T is a closed set, both p and q are elements of T. Then given any real number $\delta > 0$, there exists M such that

$$i \geq M \Rightarrow d(p_i, p), d(q_i, q) < \delta.$$

Hence $d(p_i, q_i) \leq d(p, q) + 2\delta$ for $i \geq M$. Therefore

$$d(p, q) \geq d(p_i, q_i) - 2\delta > \mu(S^n) - \frac{1}{2^i} - 2\delta.$$

Because δ is arbitrary and i can be chosen arbitrarily large, $d(p, q) = \mu(S^n)$. Therefore $p, q \in T$ are antipodals. But this contradicts the assumption that T has no antipodal points. Therefore $\mu(T) < \mu(S^n)$. ∎

With the characterization of the intersecting-maximal family of S^n, which is the spherical analogue of the Euclidean family of convex bodies, we can calculate the Radon and Helly numbers of this family. We will see that neither of these numbers is finite, which means that no diametral family in the sphere can have a finite Radon or Helly number.

THEOREM 46 *Neither $r(S^n)$ nor $h(S^n)$ is finite.*

PROOF By the definition of $r(S^n)$ and $h(S^n)$, we must show that neither the Radon number nor the Helly number of the intersecting-maximal family $\Omega(S^n)$ is finite.

Fix a positive integer k and consider a set $Q \in \Omega(S^n)$ consisting of $k + 1$ elements of S^n which all lie in an open half-sphere. Every subset of Q is a closed set without

antipodal points and consequently a member of $\Omega(S^n)$ by theorem 45. Immediately we see that Q does not admit a Radon $\Omega(S^n)$-partition. Therefore $r(S^n, \Omega(S^n)) > k + 1$.

For each element $x \in Q$, define the set $Q_x = Q - \{x\}$. The family

$$\mathcal{F} = \{Q_x : x \in Q\}$$

is a family of $k + 1$ members such that any k members of $\mathcal{F}$ have a nonempty intersection. Yet there is no element of S^n common to all members of $\mathcal{F}$. Therefore $h(S^n, \Omega(S^n)) > k$.

The theorem follows from the fact that k can be arbitrarily large. ∎

Now we characterize the inductive closure of the intersecting-maximal family of S^n, the sphere's analogue of convex sets. We see that the smallest family which satisfies the basic properties of Euclidean convexity is quite large.

THEOREM 47 *The family $\Omega(S^n)^*$ consists of S^n itself together with every subset of S^n which does not contain antipodal points.*

PROOF The family $\Omega(S^n)^*$ is the inductive completion of the intersecting-maximal family $\Omega(S^n)$ and must therefore contain S^n as a member. Let $T \subset S^n$ be a proper subset. There are two cases to consider.

Case 1: Suppose that T contains a pair of antipodal points. We further suppose that $T \in \Omega(S^n)^*$ and argue to a contradiction. By the definition of inductive completion, there is a directed subfamily $\mathcal{K}$ of $\Omega(S^n)$ such that $T = \sup \mathcal{K}$. Let x and y denote a pair of antipodal points in T. Then there are members A_x and A_y of $\mathcal{K}$ which contain x and y respectively. By directedness, there is a member $A \in \mathcal{K}$ containing both A_x and A_y and hence both of x and y. Because $\mathcal{K}$ is a subfamily of $\Omega(S^n)$, we know from theorem 45 that $A = S^n$. Then

$$S^n = A \subseteq \sup \mathcal{K} = T$$

contradicts the assumption that T is a proper subset of S^n. Therefore we can conclude that $T \notin \Omega(S^n)^*$.

Case 2: Suppose that T contains no pair of antipodal points. By theorem 45, any finite subset of T is a member of $\Omega(S^n)$. Then the family $\mathcal{F}$ consisting of all finite subsets of T is a directed subfamily of $\Omega(S^n)$ such that $T = \sup \mathcal{F}$. We conclude from the definition of inductive completion that $T \in \Omega(S^n)^*$.

∎

Finally we calculate the diametral Carathéodory number of S^n. We discover that as measured by Carathéodory numbers, the complexity associated with taking "convex hulls" in S^n (for any positive integer n) is equal to the trivial complexity associated with taking convex hulls in $\mathbf{R}^1$.

THEOREM 48

$$c(S^n) = 2$$

PROOF By definition, the diametral Carathéodory number of the n-sphere is equal to the Carathéodory number of the family $\Omega(S^n)^*$. Let $T \subseteq S^n$ and $x \in hull_{\Omega(S^n)^*}(T)$. If $T \in \Omega(S^n)^*$, then $hull_{\Omega(S^n)^*}(T) = T$ and x is an element of the $\Omega(S^n)^*$-hull of the subset $\{x\} \subseteq T$. On the other hand if $T \notin \Omega(S^n)^*$, then by theorem 47 we know that T is a proper subset of S^n which contains a pair p_1 and p_2 of antipodal points. Then

$$x \in hull_{\Omega(S^n)^*}(\{p_1, p_2\}) = S^n.$$

In either case, x is an element of the $\Omega(S^n)^*$-hull of a subset of T that has at most two elements. Hence $c(S^n, \Omega(S^n)) \leq 2$. To see that $c(S^n, \Omega(S^n)) = 2$, consider the case where T consists of only a pair q_1 and q_2 of antipodal points and where $x \in S^n - \{q_1, q_2\}$. Then $x \in hull_{\Omega(S^n)^*}(T) = S^n$, yet $x \notin hull_{\Omega(S^n)^*}(\{q_i\}) = \{q_i\}$ for $i \in \{1, 2\}$.

■

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