

On the classification of Tempered Representations for a Group in the Harish-Chandra class

by

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Abstract

The goal of classifying tempered representations (def. 5.1.2.3) and the goal of decomposing the Langlands subrepresentation for any standard representation (def. 4.1.6) look alike, and in fact they are equivalent.

The main result of this dissertation is theorem 5.4.2. that consists of a formula for the decomposition of the Langlands subrepresentation of a standard representation when the group G is in the Harish-Chandra class (def. 1.2). The classification of tempered representations is consequence of theorem 5.4.2 (corollary 5.4.5).

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TABLE OF CONTENTS

I.- Introduction and Notation	pag. 3.
II.- M -fine representations and K -fine representations	pag. 9.
III.- R -group	pag. 17.
IV.- Real parabolic induction and (\mathfrak{g}, K) -modules	pag. 22.
4.1 Basic properties of real parabolic induction and (\mathfrak{g}, K) -modules	pag. 22.
4.2 Lowest K -types and lambda-lowest K -types	pag. 25.
4.3 Proof of proposition (3.4)	pag. 28.
V.- Classification of tempered representations	pag. 32.
5.1.1 Langlands classification	pag. 32.
5.1.2 Real parabolic induction and regular characters	pag. 33.
5.2 Limit characters	pag. 34.
5.3 Back to fine representations	pag. 39.
5.4 Classification in the general case	pag. 44.
References	pag. 47.

I Introduction and Notation

1.1 Introduction

In order to study spaces of functions on a homogeneous space Harish-Chandra started analyzing the infinite dimensional representations of a semi-simple Lie group G , especially representations that he called “admissible”. His work on admissible representations has been very important source for mathematicians whose main interest is to understand a special type of these representations, the unitary representations of G .

One of the strongest results in representation theory for a semi-simple Lie group G is the Langlands classification, (thm. 5.1.2.2). This consists in the classification of all irreducible admissible representations of G . The Langlands classification was carried out mainly by Langlands, Knapp, Zuckerman and Miličič using real parabolic induction.

For a fixed minimal parabolic subgroup $P = MAN$, the Langlands classification establishes a one-to-one correspondence between equivalence classes of irreducible admissible representations and all triples $(P_1, [\sigma_1], \nu_1)$ such that

- a) $P_1 = M_1 A_1 N_1$ is a parabolic subgroup of G and $P \subseteq P_1$
- b) $[\sigma_1]$ is the class of an irreducible tempered representation σ_1 for M_1
- c) $\exp(\nu_1)$ is a non-unitary representation of A_1 such that $(-\operatorname{Re}(\nu_1))$ is in the open Weyl chamber determined by N_1

The correspondence associates to an irreducible admissible representation X a triple $(P_1, [\sigma_1], \nu_1)$ such that X is the unique irreducible G -subrepresentation of the induced module $\operatorname{Ind}_{P_1}^G(\sigma_1 \otimes \nu_1 \otimes 1)$.

If instead of considering tempered representations in the preceding correspondence we substitute discrete series representations or limit of discrete series representations and allow $\operatorname{Re}(\nu_1)$ to be in the closed Weyl chamber the Langlands subrepresentation (def. 5.1.1.2) could not be irreducible; the correspondence would be finite-to-one.

We have that all irreducible tempered representations for G (def. 4.1.10) are given by all irreducible submodules of the induced representations $\operatorname{Ind}_{P_2}^G(\delta_2 \otimes \nu_2 \otimes 1)$ for any cuspidal parabolic subgroup $P_2 = M_2 A_2 N_2$, for any discrete series representation or limit of discrete series representation δ_2 of M_2 and any unitary representation ν_2 of A_2 .

The goal of classifying tempered representations (def. 5.1.2.3) and the goal of decomposing the Langlands subrepresentation for any standard representation (def. 4.1.6) look alike, and in fact they are equivalent.

The main result of this dissertation is theorem 5.4.2. that consists of a formula for the decomposition of the Langlands subrepresentation of a standard representation when the group G is in the Harish-Chandra class (def. 1.2). The classification of tempered representations is consequence of theorem 5.4.2 (corollary 5.4.4).

The classification of irreducible tempered representations was achieved by Knapp and Zuckerman in [K-Z] for linear reductive Lie groups. One of the tools that they used is the Knapp-Stein theory of the R -group: for a tempered standard representation (def. 4.1.10), there is an R -group acting transitively on the set of its minimal K -types; this group completely determines its reducibility.

Vogan, on the other hand, studies the R -group to decompose the Langlands subrepresentation of a general standard representation (def. 4.1.6). The setting in which he applies the R -group theory in [Green] is narrower than the one in [K-Z]: he uses one of his earlier discoveries; the reduction of the computation of real parabolic induction to “smaller groups” (quasisplit groups def. 2.7) and the analysis of minimal K -types to “smaller K -types” (K -fine representations def. 2.11).

In [Green], Vogan describes explicitly the action of the R -group on the set of minimal K -types of a standard representation $\text{Ind}_P^G(\delta \otimes \nu \otimes 1)$ for a linear reductive group G . Firstly, the description is given by an R -group R_δ when the group G is quasisplit and δ is a fine representation (def. 2.9): the group of characters of R_δ , \widehat{R}_δ , acts on the set of minimal K -types of $\text{Ind}_P^G(\delta \otimes \nu \otimes 1)$, denoted as $A(\delta)$. The action of this group \widehat{R}_δ turns out to be simple and transitive. The extension of this action to the general case is achieved by applying the Zuckerman functors $\mathfrak{R}_q^*()$ (def. 5.1.1.2). This action controls the decomposition of the Langlands subrepresentation of the general standard representation.

To generalize the previous result in [Green] for groups in the Harish-Chandra class (def. 1.2), we have to avoid a main obstacle. The set of characters for R_δ cannot be extended naturally to the maximal compact subgroup K ; as it happens in the linear case (comment previous to proposition (3.1)). The multiplicity one property for minimal K -types disappears in passing from linear groups to our general case. However, it is possible to define an action of the same group \widehat{R}_δ on $A(\delta)$ when δ is a M -fine representation (def. 2.9). \widehat{R}_δ controls not only the reducibility of the Langlands subrepresentation but also the multiplicities of the minimal K -types.

Our main results are theorem 5.4.2 and corollary 5.4.4.

The content of this work is distributed as follows:

In section II, we explain the definition of the R -group R_δ when δ is a M -fine representation (def. 2.9). Next, in sections III and IV, we describe the action \widehat{R}_δ on the set of minimal K -types $A(\delta)$ appearing in $\text{Ind}_P^G(\delta \otimes \nu \otimes 1)$ when δ is a M -fine representation; $A(\delta)$ consists of K -fine representations (def. 2.11 and prop. 4.2.4). First, we prove that $A(\delta)$ is contained in the orbit of the R -group $\widehat{R}_\delta \cdot \mu_\delta$ for any $\mu_\delta \in A(\delta)$. Proposition 3.4 asserts that $A(\delta)$ is equal to this orbit. This implies that in fact we have a transitive action \widehat{R}_δ on $A(\delta)$. The proof of proposition 3.4 is completed by lemma 4.3.2 and lemma 4.3.3.

Section V begins with the formulation of the Langlands classification given in ([Green], Ch. 6) in terms of regular characters and cohomological induction.

The concept of regular characters can be generalized to limit characters ([A-B-V]). As for regular characters, we can induce a (\mathfrak{g}, K) -module $X_G(H, \underline{\gamma})$ from a limit character

$(H, \underline{\gamma})$ (def. 5.2.2). This can be decomposed into subrepresentations each induced from certain limit characters called final (def. 5.2.11). (The module induced from a final character has a unique irreducible submodule.) The idea of limit characters appears earlier in [V-II] and the concept of final in [K-Z]; we discuss this in subsection 5.2.

The decomposition for the Harish-Chandra module $X_G(\delta \otimes \nu)$ when δ is a M -fine representation is given in proposition 5.3.11. We obtain this result with limit characters $(H_F, \underline{\gamma}_F)$ that are not necessarily final but can play their role in the sense of theorem 5.2.12 (lemma 5.3.4).

We lift the action of \widehat{R}_δ on $A(\delta)$ to limit characters related to δ (def. 5.3.6). This action, in turn, can be lifted to the subrepresentations induced from final characters occurring in $X_G(\delta \otimes \nu)$ (def. 5.3.8). In this way we get theorem 5.3.11 which is theorem 5.4.2 for the case of fine representations.

Finally, we get the classification in the general case in subsection 5.4. This is carried out in analogy to the fine case in subsection 5.3. By theorem 5.2.6, we consider a regular character $(H, \underline{\gamma})$ and its corresponding set of θ -stable data $(\mathfrak{q}, H, \underline{\gamma}_1)$. We have that $(H, \underline{\gamma}_1)$ is as in theorem 5.3.11. Since $X_G(H, \underline{\gamma}) = \mathfrak{A}_\mathfrak{q}^S(X_L(H, \underline{\gamma}_1))$ for the Zuckerman functor $\mathfrak{A}_\mathfrak{q}^S(\)$ (notation 5.2.7), we extend theorem 5.3.11 to theorem 5.4.2. One last step is the comparison of the set of lambda-minimal K -types for $X_G(H, \underline{\gamma})$, $A^K(\mathfrak{q}, \delta)$, with the set of minimal $L \cap K$ -types for $X_L(H, \underline{\gamma}_1)$, $A^L(\delta)$; where $\delta = \Gamma_1|_T$. These are in bijection by corollary 4.2.5 and by [Green], thm. 6.5.9. Thus, the action of $\widehat{R}_\delta(L)$ on $A^L(\delta)$ is lifted to an action of $\widehat{R}_\delta(L)$ on $A^K(\mathfrak{q}, \delta)$.

Theorem 5.4.2 asserts that the R -group $\widehat{R}_\delta(L)$ controls the maximal decomposition for the standard representation, how the minimal K -types $A^K(\mathfrak{q}, \delta)$ are distributed among the summands and their multiplicities.

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*“Para mis padres y para mis hermanos;
con mucho cariño”*

1.2 Notation

We will use the following notation [Green]:

Lie groups are denoted by upper case roman letters such as G, H, P, L, \dots , and complex Lie algebras are denoted by german letters such as $\mathfrak{g}, \mathfrak{t}, \mathfrak{u}, \dots$

$Lie(G)$ is used to mean the real Lie algebra of the Lie group G . Real Lie algebras are denoted by German letters with subindex o , and their complexifications are denoted by same letters without the subindex o . Example: $\mathfrak{g}_o = Lie(G)$ and $\mathfrak{g} = \mathfrak{g}_o \otimes \mathbb{C}$

For a Lie group L , L_o will denote the identity component. If $A \subset L$ is a Lie subgroup, $A \cap L_o$ will be denoted by A^o .

If \mathfrak{g} is a reductive Lie algebra with a Cartan subalgebra \mathfrak{h} , $\Delta(\mathfrak{g}, \mathfrak{h})$ will denote the set of roots of \mathfrak{h} in \mathfrak{g} . More generally, if V is the representation of an abelian Lie algebra $\Delta(V) = \Delta(V, \mathfrak{h})$ will denote the set of weights of \mathfrak{h} in V .

For $\Delta(V)$, we write: $\rho(V) = \rho(\Delta(V)) = 1/2(\sum_{\lambda \in \Delta(V)} \lambda)$
(sum with multiplicities)

Now, to describe the type of groups we will be working with, suppose $G_{\mathbb{C}}$ is a connected reductive algebraic group over \mathbb{C} . In this case, denote $\mathfrak{g} = Lie(G_{\mathbb{C}})$.

Definition 1.1 ([A-B-V]) An antiholomorphic involutive automorphism

$$\sigma : G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$$

will be called a real form of $G_{\mathbb{C}}$

Given a real form σ , we get an antiholomorphic involutive automorphism of \mathfrak{g} : $d\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$
Denote,

$$G(\mathbb{R}, \sigma) = \{g \in G_{\mathbb{C}} | \sigma(g) = g\}$$

$$\mathfrak{g}(\mathbb{R}, \sigma) = \{X \in \mathfrak{g} | d\sigma(X) = X\}$$

When $G(\mathbb{R}, \sigma)$ is a real compact Lie group, σ is called a compact real form for $G_{\mathbb{C}}$.

Every connected reductive algebraic group $G_{\mathbb{C}}$ over \mathbb{C} has a compact real form that we denote by σ_c

Definition 1.2 A real Lie group G is said to be in the Harish-Chandra class, if we have

- a) a connected reductive algebraic group $G_{\mathbb{C}}$ over \mathbb{C}
- b) a real form σ of $G_{\mathbb{C}}$ commuting with σ_c
- c) a homomorphism $\varphi : G \rightarrow G(\mathbb{R}, \sigma)$ having finite kernel and finite cokernel
- d) $Ad(G) \subseteq Ad(G_{\mathbb{C}})$

We will identify $\mathfrak{g}_o = Lie(G)$ with $\mathfrak{g}(\mathbb{R}, \sigma)$ and $\mathfrak{g}_o \otimes \mathbb{C}$ with \mathfrak{g} through φ .

Suppose G is in the Harish-Chandra class with data $(G_{\mathbb{C}}, \sigma, \varphi)$ as in (def. 1.2). Consider the Cartan involution $\theta = \sigma \cdot \sigma_c$. We get a maximal compact subgroup K for G , a Cartan

decomposition $G = K \times \mathfrak{p}_\theta$ and an Iwasawa decomposition $G = KAN$ by transferring under the inverse image of φ the corresponding structure from $G(\mathbf{R}, \sigma)$. Every parabolic subgroup P_1 of G is the inverse image under φ of a parabolic subgroup of $G(\mathbf{R}, \sigma)$. So φ^{-1} gives us a Langlands decomposition for P_1 ; $P_1 = M_1 A_1 N_1$.

An important point is that φ^{-1} applied to a Harish-Chandra subgroup of $G(\mathbf{R}, \sigma)$ is automatically a Harish-Chandra subgroup of G ; for instance, $M_1 A_1$ and M_1 are in the Harish-Chandra class.

For a Cartan decomposition $G = K \times \mathfrak{p}_\theta$, $T_1 A_1$ will denote the corresponding Cartan decomposition for a Cartan subgroup H_1 ; ($H_1 \cap K = T_1$). H^s (H^c) will denote a maximal split (fundamental) Cartan subgroup.

Write $G^\# = Ad^{-1}(Ad(G_\theta))$, and for any $S \subseteq G$ $S^\#$ will stand for $S \cap G^\#$

For any compact Lie group C and any Cartan subgroup H , \hat{C} and \hat{H} will denote the corresponding sets of isomorphism classes of irreducible representations.

Suppose $B \subseteq A$, and $\delta \in \hat{B}$ and $\mu \in \hat{A}$. $m(\delta, \mu)$ will denote the multiplicity of δ in $\mu|_B$.

II.- M-fine representations and K-fine representations

Let G be a Lie group in the Harish-Chandra class, (def. 1.2), with a Cartan decomposition $G = K \times \mathfrak{p}_o$. Let \mathfrak{h}_o be a θ -stable Cartan subalgebra of \mathfrak{g}_o .

There is a natural action of θ on $\Delta(\mathfrak{g}, \mathfrak{h})$. This yields different types of roots.

Definition 2.1 A root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is called

- a) *real* if $\theta \cdot \alpha = -\alpha$
- b) *imaginary* if $\theta \cdot \alpha = \alpha$
- c) *complex* if $\theta \cdot \alpha \neq \pm\alpha$

The root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ gives rise to the following root space decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

This can be used to obtain a subdivision among imaginary roots.

Definition 2.2 A root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is called

- a) *compact imaginary* if $\theta \cdot \alpha = \alpha$ and $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$
- b) *noncompact imaginary* if $\theta \cdot \alpha = \alpha$ and $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$

Now we pay special attention to the maximal split Cartan subalgebras.

Definition 2.3 Let \mathfrak{a}_o be a maximal abelian subalgebra of \mathfrak{g}_o contained in \mathfrak{p}_o ($\mathfrak{a}_o = \text{Lie}(A)$)

a) Set,

$$\begin{aligned} M &= \text{centralizer of } \mathfrak{a}_o \text{ in } K \\ M' &= \text{normalizer of } \mathfrak{a}_o \text{ in } K \end{aligned}$$

Consider a Cartan subalgebra \mathfrak{h}_o^* containing \mathfrak{a}_o .

b) Put,

$$\begin{aligned} \overline{\Delta}_r &= \{ \alpha|_{\mathfrak{a}} \mid \alpha|_{\mathfrak{a}} \neq 0 \text{ and } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^*) \} \\ \overline{\Delta} &= \{ \alpha \in \overline{\Delta}_r \mid \frac{1}{2}\alpha \notin \overline{\Delta}_r \} \end{aligned}$$

Here we mention some important results involving $\overline{\Delta}$ that can be found in ([H], pag. 234).

$\overline{\Delta}_r$ and $\overline{\Delta}$ are roots systems. $\overline{\Delta}$ is in fact a reduced root system and its Weyl group $W(\overline{\Delta})$ can be identified with $W = M'/M$.

Iwasawa decompositions of G containing KA are in one-to-one correspondence with minimal parabolic subgroups containing MA and these, in turn, are in one-to-one correspondence with positive systems for $\overline{\Delta}$.

Definition 2.4 A root $\alpha \in \overline{\Delta}$ is called
a) *real* if α is the restriction of real root of Δ
b) *complex* if α is not real

Let G be a group in the Harish-Chandra class (def. 1.2) with data $(G_{\mathbf{C}}, \sigma, \phi)$.
Now we have ([Green], Ch.4), for any real root $\alpha \in \overline{\Delta}$, there is a non-trivial homomorphism
 $SL(2, \mathbf{R}) \xrightarrow{\Phi_{\alpha}} G(\mathbf{R}, \sigma)$ such that $\Phi_{\alpha}(\text{diagonal matrices}) \subseteq H_{\mathbf{R}}^{\alpha}$, and $\phi_{\alpha} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_{\alpha}$.
Here $H_{\mathbf{R}}^{\alpha}$ is the Cartan subgroup of $G(\mathbf{R}, \sigma)$ corresponding to $\mathfrak{h}_{\mathfrak{o}}^{\alpha}$ and $d\Phi_{\alpha} = \phi_{\alpha}$. Moreover
we have the following commutative diagram

$$\begin{array}{ccccc} sl(2, \mathbf{R}) & \xrightarrow{\phi_{\alpha}} & \mathfrak{g}_{\mathfrak{o}} & \xleftarrow{1} & \mathfrak{g}_{\mathfrak{o}} \\ \text{exp} \downarrow & & \text{expL} \downarrow & & \text{exp} \downarrow \\ SL(2, \mathbf{R}) & \xrightarrow{\Phi_{\alpha}} & G(\mathbf{R}, \sigma) & \xleftarrow{\varphi} & G \end{array}$$

Put,

$$\begin{aligned} Z_{\alpha} &= \phi_{\alpha} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \overline{\sigma}'_{\alpha} &= \Phi_{\alpha} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \text{expL} \left(\frac{\pi}{2} Z_{\alpha} \right) \\ m'_{\alpha} &= (\overline{\sigma}'_{\alpha})^2 = \Phi_{\alpha} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Lemma 2.5 ([Green], pag. 172) The map ϕ_{α} is unique up to conjugation
of $sl(2, \mathbf{R})$ by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; so Z_{α} is well defined up to sign, and σ'_{α} up to inverse. Furthermore,
a) $(m'_{\alpha})^2 = 1$ and $m'_{\alpha} \in \varphi(M)$
b) $\sigma'_{\alpha} \in \varphi(M')$, and the image of $\overline{\sigma}'_{\alpha}$ of σ'_{α} in M'/M is the reflexion s_{α} in W .

We are interested in the image of the exponential not in $G(\mathbf{R}, \sigma)$ but rather in G

Definition 2.6 For any real root $\alpha \in \overline{\Delta}$ put

$$\sigma_{\alpha} = \text{exp} \left(\frac{\pi}{2} Z_{\alpha} \right) \text{ and } m_{\alpha} = (\sigma_{\alpha})^2$$

So $\sigma_{\alpha} \in M'$ and $m_{\alpha} \in M$.

Definition 2.7 Let A and M be as in (def. 2.3). Let G be a group in the Harish-Chandra
class and $H^{\sigma} = TA$ the Cartan subgroup corresponding to $\mathfrak{h}_{\mathfrak{o}}^{\sigma} = \mathfrak{t}_{\mathfrak{o}} + \mathfrak{a}_{\mathfrak{o}}$. G is called *quasisplit*
if $M = T$.

Let θ the Cartan involution of G (or \mathfrak{g}). The subgroup of $Aut(\mathfrak{g})$, $\{1, \theta\}$, denoted by Γ , acts naturally on both Δ and \mathfrak{g} . For every root $\alpha \in \Delta$, $\theta \cdot \mathfrak{g}_\alpha = \mathfrak{g}_{\theta\alpha}$.

Suppose G is quasisplit. Thus, $\Delta(\mathfrak{g}, \mathfrak{h}^\theta)$ does not contain imaginary roots, and $\mathfrak{g}_\alpha + \mathfrak{g}_{\theta\alpha}$ is a Γ -irreducible subspace of \mathfrak{g} for any root $\alpha \in \Delta$.

For any root $\beta \in \Delta$, write $[\beta]$ for the set $\{\beta, \theta\beta\}$, and $\Delta/\Gamma = \{[\alpha] \mid \alpha \in \Delta\}$. Therefore, we have the following decompositions:

$$\mathfrak{g} = \mathfrak{h}^\theta + \left(\sum_{[\alpha] \in \Delta/\Gamma} \mathfrak{g}_\alpha + \mathfrak{g}_{\theta\alpha} \right)$$

and

$$\mathfrak{k} = \mathfrak{t}^\theta + \sum_{[\alpha] \in \Delta/\Gamma} \mathbb{C}Z_\alpha + \sum_{[\alpha] \in \Delta/\Gamma} \mathfrak{k}_\alpha$$

where Z_α is as in (def. 2.6) and \mathfrak{k}_β is the one-dimensional space $\mathfrak{k} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\theta\beta})$ given for a complex root β .

Assume from now to the end of this II-part that G is quasisplit. We consider the decomposition $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}$.

Write $\mathfrak{m}^\theta = \mathfrak{m} \cap ([\mathfrak{g}, \mathfrak{g}])$ ($\mathfrak{m} = Lie(M) \otimes \mathbb{C}$). Therefore, we have $\mathfrak{m} = \mathfrak{m}^\theta + (\mathfrak{z} \cap \mathfrak{m})$.

Definition 2.8 An irreducible representation λ of \mathfrak{m} is called *fine* (w.r.t. \mathfrak{g}) if

$$\lambda|_{\mathfrak{m}^\theta} \equiv 0$$

Definition 2.9 A representation $(\delta, V_\delta) \in \widehat{M}$ is called *fine* (w.r.t. G) if under the induced representation of \mathfrak{m} , V_δ is a sum of \mathfrak{m} -fine representations.

Definition 2.10 A representation γ of \mathfrak{k} is called *fine* if

- a) $\forall [\alpha] \in \Delta/\Gamma$, α complex, $\gamma|_{\mathfrak{k}_\alpha} \equiv 0$
- b) $\forall [\alpha] \in \Delta/\Gamma$, α real, the eigenvalues of $\gamma(iZ_\alpha)$ lie in the range $[-1, 1]$.

Definition 2.11 A representation $(\mu, V_\mu) \in \widehat{K}$ is called *fine* if under the induced representation of \mathfrak{k} , V_μ is a sum of \mathfrak{k} -fine representations.

We want to use some results from [V-I] that are proved for $G = G_o$. Therefore, it is important to note:

Remark 2.12 (notation 1.2) For $D = M$ or K , and $\gamma \in \widehat{D}$ the following conditions are equivalent

- a) γ is a D -fine representation
- b) $\gamma|_{D^\circ}$ is a sum of D° -irreducible fine representations
- c) $\gamma|_{D^\circ}$ contains a D° -fine representation

(For $D = M$ is trivial. For $D = K$, we use the fact that $K = MK_o$ and $M=T$).

(2. 13) We will be often in the following situation:

Let A be a compact Lie group and B a normal subgroup of A , and $R = A/B$ a finite group. For these pair of groups, there is a homomorphism of groups $c : A \rightarrow \text{Aut}(B)$ defined by $c(a)(b) = aba^{-1}$ with $a \in A$, $b \in B$.

We can induce an action of A on \hat{B} defining $x \cdot \gamma = \gamma(c(x^{-1}))$ for $x \in A$, $\gamma \in \hat{B}$. Since under this action B acts trivially on \hat{B} , the action of A factors to an action of R on \hat{B} .

If we assume that R is abelian, \hat{R} acts on \hat{A} by tensor product, i.e., given $\gamma \in \hat{A}$ and $\chi \in \hat{R}$, we define $\chi \cdot \gamma = \chi \otimes \gamma$; $\chi \cdot \gamma \in \hat{A}$.

We add to the list of our notation, the following: Given $\delta \in \hat{B}$ and $\mu \in \hat{A}$, put,

$$\begin{aligned} A_\delta &= \text{stabilizer of } \delta \text{ in } A \\ R_\delta &= A_\delta/B = \text{stabilizer of } \delta \text{ in } R \\ \hat{R}_\mu &= \text{stabilizer of } \mu \text{ in } \hat{R} \end{aligned}$$

Suppose δ occurs in $\mu|_B$, define

$$\mu_\delta = \delta\text{-primary subrepresentation of } B \text{ on } \mu|_B$$

The following is a well-known fact (see for example [C-R])

(2.14) Fact.- Suppose $\delta \in \hat{B}$ occurs in $\mu|_B$; $\mu \in \hat{A}$. Then

a) $\mu_\delta \in \hat{A}_\delta$, and A_δ is equal to A_{μ_δ} , the stabilizer of μ_δ in A .

b) $\mu|_{A_\delta} = x_1 \cdot \mu_\delta + x_2 \cdot \mu_\delta + \cdots + x_r \cdot \mu_\delta$

where $\{x_1, \cdots, x_r\} = A/A_\delta$ ($r = |A/A_\delta|$), and δ occurs in $x_j \cdot \mu_\delta|_B$ if and only if $x_j \in A_\delta$

c) $\mu = \text{Ind}_{A_\delta}^A(\mu_\delta)$

(*Ind* is the usual induction functor for compact groups)

From time to time we will write a representation as a pair (γ, V) or (γ, V_γ) to make explicit the underlying complex vector space.

Definition 2.15 Assume $\delta \in \hat{M}$ is a fine representation. The set of good roots for δ , denoted $\overline{\Delta}_\delta$, consists of roots α such that either

a) α is real and $\delta(m_\alpha) \neq -1$, or

b) α is complex

Remark 2.16 Given α real, $(\delta, V_\delta) \in \hat{M}$ a fine representation, $\delta(m_\alpha) \neq -1$ is equivalent to the weaker condition that $\delta(m_\alpha)$ does not have -1 as an eigenvalue.

Proof: Suppose that $\delta(m_\alpha)$ does have -1 as an eigenvalue. Let $M^\circ = G_\circ \cap M$.

Now, by remark (2.12), $\delta|_{M^\circ} = \delta_\circ + m_1 \cdot \delta_\circ + \cdots + m_s \cdot \delta_\circ$ for some elements m_1, \cdots, m_s in M , and for some M° -irreducible fine representations $\delta_\circ, m_1 \cdot \delta_\circ, \cdots, m_s \cdot \delta_\circ$.

Write $V_\delta = V_{\delta_\circ} + V_{m_1, \delta_\circ} + \cdots + V_{m_s, \delta_\circ}$. We may assume that -1 is an eigenvalue for $\delta_\circ(m_\alpha)$. Let $v \in V_{\delta_\circ}$ such that $\delta_\circ(m_\alpha)v = -v$. This implies

$m_j \delta_o(m_\alpha)(\delta(m_j^{-1}))v = \delta_o(m_j m_\alpha m_j^{-1})(\delta(m_j^{-1}))v = \delta(m_j^{-1})(\delta_o(m_\alpha))v = -(\delta(m_j^{-1}))v$
Therefore, for all j , -1 is also an eigenvalue for $m_j \delta_o(m_\alpha)$.

It is enough to prove the observation for $G = G_o$. This is the case proved in [V-I]. \square

Remark 2.17 Taking δ and δ_o from the proof of remark (2.16), we conclude $\overline{\Delta}_\delta = \overline{\Delta}_{\delta_o}$. ($\overline{\Delta}_{\delta_o}$ are the good roots for δ_o). In [V-I], Vogan proves that $\overline{\Delta}_{\delta_o}$ is a root system.

The following property of K -representations will allow us to determine them in terms of the Weyl group $W = M'/M$.

Proposition 2.18 Suppose (μ, V) and (μ', V') are two fine K -representations (def. 2.11). Then,

$$\text{Hom}_K(\mu, \mu') = \text{Hom}_{M'}(\mu, \mu')$$

Proof: \subseteq is obvious. For the other direction, suppose $T : V \rightarrow V'$ is a M' -homomorphism. To prove that T is a K -homomorphism, we use the fact that $K = M'K_o = MK_o$. Thus, it is enough to prove that T is a K_o -homomorphism or, equivalently, \mathfrak{k} -homomorphism.

Recall the decomposition of \mathfrak{k} after definition 2.7 with respect to Δ/Γ . Now, write $\overline{\mu}$ for the induced representation of \mathfrak{k} from μ . For any complex root α , $\overline{\mu}(x)$ and $\overline{\mu}'(x)$ are zero for any $x \in \mathfrak{k}_\alpha$ (def. 2.10).

It follows that for any $v \in V$

$$T(\overline{\mu}(x)v) = T(0) = 0 = \overline{\mu}'(x)T(v)$$

Now let us consider $\alpha \in \overline{\Delta}$ real. We can find a basis $D = \{v_1, v_2, \dots, v_m\}$ (respect. $D' = \{v'_1, v'_2, \dots, v'_{m'}\}$) for V (respect. for V') such that v_j (respect. $v'_{j'}$) is an eigenvector of $\overline{\mu}(iZ_\alpha)$ (respect. of $\overline{\mu}'(iZ_\alpha)$) with eigenvalue s_j (respect. $s'_{j'}$) in $[-1, 1]$ for any $j = 1, 2, \dots, m$ (respect. $j' = 1, 2, \dots, m'$).

Hence $\mu(\sigma_\alpha)v_j = \mu(\exp(\frac{\pi}{2}Z_\alpha))v_j = e^{-\frac{\pi}{2}is_j}v_j$ for any $j = 1, 2, \dots, m$.
(and similarly $\mu'(\sigma_\alpha)v_{j'} = e^{-\frac{\pi}{2}is'_{j'}}v'_{j'}$ for any $j' = 1, 2, \dots, m'$)

So $\mu(\sigma_\alpha)$ and $\overline{\mu}(iZ_\alpha)$ are simultaneously diagonalizable.

Suppose $T(v_j) = a_1 \cdot v'_1 + a_2 \cdot v'_2 + \dots + a_{m'} \cdot v'_{m'}$ where $a_k \in \mathbb{C}$ for any $k = 1, 2, \dots, m'$. Applying $\mu'(\sigma_\alpha)$ to both sides of the equality, we have

$$e^{-\frac{\pi}{2}is_j}T(v_j) = T(\mu(\sigma_\alpha)v_j) = \mu'(\sigma_\alpha)T(v_j) = a_1 e^{-\frac{\pi}{2}is'_1} \cdot v'_1 + \dots + a_{m'} e^{-\frac{\pi}{2}is'_{m'}} \cdot v'_{m'}$$

where the first equality is given by the invariance of T .

This implies that for any $a_k \neq 0$, $s_j = s'_k$; because s_j and s'_k are in $[-1, 1]$. Therefore,

$$\overline{\mu}'(iZ_\alpha)T(v_j) = s_j T(v_j) = T(s_j v_j) = T(\overline{\mu}(iZ_\alpha)v_j)$$

for any $j = 1, 2, \dots, m$. This proves the proposition. \square

Corollary 2.19 Let (μ, V) and (μ', V') two K -representations. Then

a) $\mu|_{M'}$ is irreducible

b) $\mu \cong \mu'$ if and only if $\mu|_{M'} \cong \mu'|_{M'}$ \square

We sometimes omit the symbol $|_{M'}$ when it can be deduced from the context.

(2.20) Example.- We can apply the situation (2.13) to fine representations.

Let $A = M'$ and $B = M$. Suppose $\mu \in \widehat{A}$ is a K -fine irreducible representation (cor. 2.19).

Have

a) $W_{\mu_\delta} = W_\delta$ and $M'_{\mu_\delta} = M'_\delta$

b) $\mu|_{M'_\delta} = w_1 \cdot \mu_\delta + w_2 \cdot \mu_\delta + \cdots + w_r \cdot \mu_\delta$

where $\{w_1, \dots, w_r\} = M'/M'_\delta$ ($r = |M'/M'_\delta|$)

c) $w_i \cdot \mu_\delta \cong w_j \cdot \mu_\delta$ if and only if $w_i = w_j$

d) $\mu = \text{Ind}_{M'_\delta}^{M'}(\mu_\delta)$

e) $m(\delta, \mu) = m(\delta, \mu_\delta) = m(w \cdot \delta, w \cdot \mu_\delta) = m(w \cdot \delta, \mu)$

for every element w in $W = M'/M$.

Definition 2.21 In view of corollary (2.19), for any $\delta \in \widehat{M}$, we define

$$A(\delta) = \{ \mu \in \widehat{M'} \mid \mu \text{ is a } K\text{-fine representation, and } \delta \text{ occurs in } \mu|_M \}$$

(We are omitting the symbol $|_{M'}$)

There is another way of defining $A(\delta)$ that will be applied later on.

Definition 2.22 Suppose $\delta \in \widehat{M}$, we define

$$A(\delta) = \{ \mu_\delta \in \widehat{M'_\delta} \mid \mu_\delta \text{ is the } \delta\text{-primary part of a } K\text{-fine representation } \mu \text{ and is non-zero} \}$$

Corollary (2.19) and example (2.20) imply the equivalence between (def. 2.21) and (def. 2.22).

It turns out that $A(\delta) \neq 0$ (prop. 4.24).

Recall δ , δ_α , and $\overline{\Delta}_\delta = \overline{\Delta}_{\delta_\alpha}$ from remark (2.17). Denote $W(\overline{\Delta}_\delta)$ for the corresponding Weyl group of $\overline{\Delta}_\delta$.

In the proof of (prop. 2.18) we saw that for any real root $\alpha \in \overline{\Delta}$, and for any fine representation $(\mu, V) \in \widehat{K}$, the eigenvalues of $\mu_\delta(\sigma_\alpha)$ are of the form $e^{\frac{s}{2}i\pi}$ with $s \in [-1, 1]$. By definition of $\overline{\Delta}_\delta$, if the real root α is in $\overline{\Delta}_\delta$ then $\delta(m_\alpha) \neq -1$. Then for any $\mu_\delta \in A(\delta)$ (def. 2.16), $\mu_\delta(m_\alpha)$ has eigenvalues of the form $e^{s i \pi}$ with $s \in (-1, 1)$. Therefore, we can compute $\mu_\delta(m_\alpha)^{\frac{1}{2}}$ and it has to be $\mu_\delta(\sigma_\alpha)$. This implies that $\mu_\delta(\sigma_\alpha) = \mu_\delta(m_\alpha)^{\frac{1}{2}}$ leaves invariant each irreducible M -submodule in V .

Let $(\delta, V_\alpha) \in \widehat{M}$ be a subrepresentation of $(\mu, V)|_M$ with $V_\alpha \subseteq V$.

Fix a real root $\alpha \in \overline{\Delta}_\delta$. We have for any $m \in M$ and for any $v \in V_\alpha$

$$\sigma_\alpha \cdot \delta(m)v = \delta(\sigma_\alpha^{-1} m \sigma_\alpha)v = \delta(m_\alpha)^{-\frac{1}{2}} \delta(m) \delta(m_\alpha)^{\frac{1}{2}} v$$

Therefore, $\sigma_\alpha \cdot \delta$ and δ are in the same class in \widehat{M} . Since $\sigma_\alpha \cdot \delta$ is in the class $s_\alpha \cdot \delta$, ($s_\alpha \in W(\overline{\Delta}_\delta)$). We can write $s_\alpha \cdot \delta = \delta$.

On the other hand, for any complex root $\beta \in \overline{\Delta}_\delta$, $s_\beta \in W(\overline{\Delta}_\delta)$ has a representative $k = \exp(x) \in K$ with $x \in \sum_{[\alpha] \in \Delta/\Gamma, \alpha|_\mathfrak{a} = \beta} \mathfrak{k}_\alpha$ (decomposition of \mathfrak{k} after def. 2.7).

Thus, we have that $k \cdot \delta$ is a representative for $s_\beta \cdot \delta$ in \widehat{M} and $\mu_\delta(k)$ is the identity by definition of fine.

For any $v \in V$, and for any $m \in M$

$$k \cdot \delta(m)v = \delta(k^{-1}mk)v = \mu_\delta(k^{-1})\delta(m)\mu_\delta(k)v$$

This implies that $k \cdot \delta$ and δ are representatives of the same class in \widehat{M} . In other words, $s_\beta \cdot \delta = \delta$; identifying δ with its own class.

We conclude that $W(\overline{\Delta}_\delta) \subseteq W_\delta$

In fact, more is true.

Proposition 2.23 Under the Weyl group action, W_δ normalizes $\overline{\Delta}_\delta$. In particular, $W(\overline{\Delta}_\delta)$ is a normal subgroup of W_δ .

Proof: The second statement follows from the first one. To prove the first assertion is enough to assume that $G = G_o$. This case is proved in [V-I]. \square

Definition 2.24 We denote,

$$\begin{aligned} (M'_\delta)^\circ &= \text{inverse image of } W(\overline{\Delta}_\delta) \text{ in } M' \\ W_\delta^\circ &= (M'_\delta)^\circ / M = W(\overline{\Delta}_\delta) \\ R_\delta &= W_\delta / W_\delta^\circ = M'_\delta / (M'_\delta)^\circ \end{aligned}$$

R_δ is called the *R-group associated to δ* .

Now, let us fix a system of positive roots $\overline{\Delta}_\delta^+$ for $\overline{\Delta}_\delta = \overline{\Delta}_{\delta_o}$ and define (notation 1.2):

$$\rho_{\delta_o} = \rho_\delta = \rho(\overline{\Delta}_\delta^+)$$

Choose a positive root system $\overline{\Delta}^+$ for $\overline{\Delta}$ such that ρ_{δ_o} is dominant. Write,

$$\overline{\Delta}_\bullet = \{ \alpha \in \overline{\Delta} \mid \langle \alpha, \rho_{\delta_o} \rangle = 0 \}$$

In [V-I], we find important properties involving $\overline{\Delta}_\bullet$.

Proposition 2.25 In the notation above,

a) $\overline{\Delta}_\bullet$ is a root system, $\overline{\Delta}_\bullet \cap \overline{\Delta}^+$ is a strongly orthogonal set $\{\alpha_1, \dots, \alpha_r\}$ of simple roots of $\overline{\Delta}^+$

b) If $\alpha \in \overline{\Delta}_\bullet$, α is real and $\delta_\alpha(m_\alpha) = -I$

(Using remark (2.12), $\alpha \in \overline{\Delta}_\bullet$ implies $\delta(m_\alpha) = -I$)

Definition 2.26 Write, $R_\delta^c = \{w \in W_\delta \mid w(\overline{\Delta}_\delta^+) = \overline{\Delta}_\delta^+ \}$

By definition, R_δ^c is the stabilizer of $\overline{\Delta}_\delta^+ \subseteq \overline{\Delta}_\delta$ in W_δ (prop. 2.23).

Write $W(\overline{\Delta}_\delta)$ for the Weyl group $\overline{\Delta}_\delta$. It is clear that $W(\overline{\Delta}_\delta) \cong (\mathbf{Z}/2\mathbf{Z})^r$.

Proposition 2.27 (def. 2.24) We have,

a) $W_\delta = R_\delta^c \rtimes W_\delta^\circ$

b) $R_\delta^c \subseteq W(\overline{\Delta}_\delta)$

Proof: The proposition uses only the properties of the group of automorphisms of a root system. For b) we apply Chevalley's theorem [Green]. □

Corollary 2.28 (def. 2.24) R_δ is the sum of copies of $\mathbf{Z}/2\mathbf{Z}$ □

III.- R-group

In [Green] Ch. 4, Vogan constructs an action of \widehat{R}_δ on $A(\delta)$ (def. 2.21) by using certain involutive automorphisms of the maximal compact subgroup K ; they can be thought as extensions for the characters of R_δ . These extensions do not exist when our group G is non-linear. However, we can still build up an action of \widehat{R}_δ on $A(\delta)$ (def. 2.22).

Consider $A(\delta)$ as in (def. 2.22); $A(\delta) \subseteq \widehat{M}'_\delta$.

In the situation (2.13), \widehat{R}_δ acts on \widehat{M}'_δ . Let $\mu_\delta \in A(\delta)$ and write $\widehat{R}_\delta \cdot \mu_\delta$ for the orbit of μ_δ in \widehat{M}'_δ under the action of \widehat{R}_δ .

Proposition 3.1 $A(\delta) \subseteq \widehat{R}_\delta \cdot \mu_\delta$

Proof: Let $\mu_\delta \in A(\delta)$. Define $V_{\mu, \mu'} = \text{Hom}_M(\mu_\delta, \mu'_\delta)$.

a) $V_{\mu, \mu'}$ is a M'_δ -module as follows : $\forall T \in V_{\mu, \mu'}$ and $\forall v \in V_\delta$, define

$$(x \cdot T)(v) = \mu'_\delta(x)T(\mu_\delta(x^{-1})v) = x(T(x^{-1}v))$$

Since M is normal subgroup of M'_δ , the action of M'_δ is well-defined.

b) The action in a) restricted to $(M'_\delta)^\circ$ is trivial:

To see this, we note that $(M'_\delta)^\circ$ is generated by M and $\{\sigma_\alpha\}_{\alpha \in \overline{\Delta}_\delta}$.

For a complex root $\beta \in \overline{\Delta}_\delta^+$, σ_α fixes any element of $V_{\mu, \mu'}$: $\mu_\delta(\sigma_\beta)$ and $\mu'_\delta(\sigma_\beta)$ are the identity. For a real root $\alpha \in \overline{\Delta}_\delta^+$, we observed that $\mu_\delta(\sigma_\alpha)$ and $\mu'_\delta(\sigma_\alpha)$ can be computed as $\mu_\delta(m_\alpha)^{\frac{1}{2}}$ and $\mu'_\delta(m_\alpha)^{\frac{1}{2}}$ respectively.

Replacing the role of σ_α by m_α in the proof of proposition (2.18), our assertion follows.

c) We get that the action of M'_δ on $V_{\mu, \mu'}$ factors to an action of R_δ on $V_{\mu, \mu'}$. Define a homomorphism of M'_δ -modules $\varphi : V_{\mu, \mu'} \otimes \mu_\delta \rightarrow \mu'_\delta$ by $\varphi(F \otimes v) = F(v)$ for $F \in V_{\mu, \mu'}$, and $v \in V_{\mu_\delta}$.

$$\begin{aligned} \text{For all } m \in M'_\delta \quad m(\varphi(F \otimes v)) &= mF(v) = \\ &= (mF)(m^{-1}(mv)) = (mF)(mv) = \varphi(mF \otimes mv) = \varphi(m(F \otimes v)) \end{aligned}$$

On the other hand, $V_{\mu, \mu'} \neq 0$ and $\varphi \neq 0$, for $\mu_\delta, \mu'_\delta \in A(\delta)$

Since R_δ is abelian (cor. 2.24), $V_{\mu, \mu'}$ can be decomposed into a direct sum of one-dimensional representations of R_δ . Therefore, there exists $\chi \in \widehat{R}_\delta$ such that $\varphi| : \chi \otimes \mu_\delta \rightarrow \mu'_\delta$ is a non-zero homomorphism of M'_δ -modules. Thus $\mu'_\delta = \chi \cdot \mu_\delta \in \widehat{R}_\delta \cdot \mu_\delta$. \square

Corollary 3.2 $\forall \mu, \mu' \in A(\delta)$

a) $m(\delta, \mu) = m(\delta, \mu_\delta) = m(\delta, \mu'_\delta) = m(\delta, \mu')$

a) $m(\delta, \mu) = \sum_{\gamma \in (\widehat{M}'_\delta)^\circ} \dim(\text{Hom}_{(M'_\delta)^\circ}(\gamma, \mu_\delta))$

Proof: a) is a direct consequence of proposition (3.1).

b) is consequence of incise b) in the proof of proposition (3.1): $V_{\mu, \mu'} = \text{Hom}_{(M'_\delta)^\circ}(\mu_\delta, \mu'_\delta)$. \square

Definition 3.3 Suppose $\mu_\delta \in A(\delta)$.

Set,

$$\widehat{R}_\delta^0 = \{ \chi \in \widehat{R}_\delta \mid \chi \otimes \mu_\delta = \mu_\delta \}$$

It is clear that our definition (3.3) is independent of μ (prop. 3.1).

Write, $\widehat{R}_\delta^1 = \widehat{R}_\delta / \widehat{R}_\delta^0$

Proposition 3.4 In the notation of definition (3.3), we have:

a) For any $\mu_\delta \in A(\delta)$ $A(\delta) = \widehat{R}_\delta \cdot \mu_\delta = \widehat{R}_\delta^1 \cdot \mu_\delta$

b) For any $\mu_\delta \in A(\delta)$, $m(\delta, \mu_\delta) = |\widehat{R}_\delta^0|^{\frac{1}{2}}$

As consequence of incise a) of proposition (3.4), we have an action of \widehat{R}_δ on $A(\delta)$ that is in fact transitive.

The proof of proposition (3.4) will be given in two parts. Proposition (3.4) a) is lemma (4.3.3) and proposition (3.4) b) is lemma(3.6) b).

As preparation for the proof of proposition (3.4), we formulate three lemmas.

From now to the end of this section III, assume we are in the situation (2.13).

Suppose $R = A/B$ is an abelian group, $\delta \in \widehat{B}$, $\mu \in \widehat{A}$ and δ occurs in $\mu|_B$.

Recall the definitions for A_δ , R_δ , \widehat{R}_μ and μ_δ there.

Write $N = A/A_\delta$.

Lemma 3.5 Denote $(\widehat{R}_\delta)_{\mu_\delta}$ for the stabilizer of μ_δ in \widehat{R}_δ .

We have two short exact sequences

$$a) \quad 1 \rightarrow \widehat{N} \rightarrow \widehat{R} \xrightarrow{\Psi} \widehat{R}_\delta \rightarrow 1$$

$$b) \quad 1 \rightarrow \widehat{N} \rightarrow \widehat{R}_\mu \xrightarrow{\Psi} (\widehat{R}_\delta)_{\mu_\delta} \rightarrow 1$$

where $\Psi(\chi) = \chi|_{A_\delta}$

Proof: $1 \rightarrow R_\delta \rightarrow R \rightarrow N \rightarrow 1$ is short exact sequence of abelian groups.

Hence, a) holds.

b) claims that $\chi \otimes \mu = \mu$ if and only if $\chi|_{A_\delta} \otimes \mu_\delta = \mu_\delta$.

On one hand, $\chi \otimes \mu = \mu$ implies that $\chi|_{A_\delta} \otimes \mu_\delta = x \cdot \mu_\delta$ for some x element of A .

But, $\chi|_{A_\delta} \otimes \mu_\delta$ restricted to B is a sum of $x \cdot \delta$'s. Thus, $x \in A_\delta$ therefore $x \cdot \mu_\delta = \mu_\delta$.

To prove the other direction, we use the fact (2.14) which implies

$$\mu = \text{Ind}_{A_\delta}^A(\mu_\delta) = \text{Ind}_{A_\delta}^A(\chi|_{A_\delta} \otimes \mu_\delta) = \chi \otimes \mu$$

The lemma follows. □

Lemma 3.6 Suppose again that $R = A/B$ is abelian. Then,

- a) $\mu|_B$ is irreducible if and only if $\widehat{R}_\mu = 1$
- b) $m(\delta, \mu) = m(\delta, \mu_\delta) = |(\widehat{R}_\delta)_{\mu_\delta}|^{\frac{1}{2}}$
- c) $A_\delta = 1$ if and only if $\widehat{R}_\mu = \widehat{R}$ and $m(\delta, \mu) = 1$

Proof: Mackey's theory tell us that $Ind_B^A(\mu|_B) = \sum_{\chi \in \widehat{R}} \chi \cdot \mu$. Next, we apply Frobenius Reciprocity theorem

$$Hom_A(\mu, Ind_B^A(\mu|_B)) = Hom_B(\mu|_B, \mu|_B)$$

Therefore, $Hom_B(\mu|_B, \mu|_B) = \sum_{\chi \in \widehat{R}} Hom_A(\mu, \chi \cdot \mu) = \mathbb{C}^m$

where $m = |\widehat{R}_\mu|$.

This implies that $\mu|_B$ is irreducible if and only if $m = 1$. This proves a).

Recall the fact on situation (2.13) incise b).

We get

$$\mathbb{C}^m = Hom_B(\mu|_B, \mu|_B) = \sum_{i=1}^r Hom_B(x_i \cdot \mu_\delta|_B, x_i \cdot \mu_\delta|_B) = \sum_{i=1}^r Hom_B(\mu_\delta|_B, \mu_\delta|_B)$$

where $\{x_1, \dots, x_r\} = (A/A_\delta)$ and $r = |A/A_\delta|$

We conclude that $m(\delta, \mu)^2 = dim(Hom_B(\mu_\delta|_B, \mu_\delta|_B)) = |\widehat{R}_\mu| |A/A_\delta|^{-1}$.

Hence, lemma (3.5) b) implies lemma (3.6) b).

Similarly, c) is consequence of this. The lemma follows. □

From lemma(3.5) and lemma(3.6), we get the following consequence.

Corollary 3.7 We obtain,

- a) $\widehat{R}/\widehat{R}_\mu = \widehat{R}_\delta / (\widehat{R}_\delta)_{\mu_\delta}$
- b) $|\widehat{N}|^{-1} |\widehat{R}_\mu| = m(\delta, \mu)^2$ □

Now, suppose we have a third group C such that $B \subseteq C \subseteq A$. Suppose also that $R = A/B$ is an abelian group.

Let us assume that $\eta \in \widehat{C}$ is such that $\eta|_B$ contains δ as submodule and it occurs in $\mu|_C$.

First note that $C \subseteq A_\eta \subseteq A_\delta C$: since if $x \cdot \eta = \eta$ for some $x \in A$, this implies that $x \cdot \delta = y \cdot \delta$ for some $y \in C$; that is, $y^{-1}x \in A_\delta$ and then $x \in A_\delta C$.

Denote, $N_1 = A/C$ and $S = C/B$.

For η , we write

$$\begin{aligned} Q &= \{ \chi \in \widehat{R} \mid \chi|_C \otimes \eta = x \cdot \eta \text{ for some } x \in A \} \\ Q_o &= \{ \chi \in \widehat{R} \mid \chi|_C \otimes \eta = \eta \} \end{aligned}$$

Lemma 3.8 We have

- a) $Q/Q_o \cong (A_\delta C/A_\eta)$
- b) $m(\delta, \mu) = m(\delta, \eta) m(\eta, \mu) |Q/Q_o|$

Proof: As in lemma 3.5 a), the following is a short exact sequence

$$1 \rightarrow \widehat{N}_1 \rightarrow \widehat{R} \xrightarrow{\Psi} \widehat{S} \rightarrow 1$$

where $\Psi(\chi) = \chi|_C$.

Clearly $Q_o = \Psi^{-1}(\widehat{S}_\eta)$. Thus, $\widehat{R}/Q_o \cong \widehat{S}/(\widehat{S}_\eta)$.

We have natural actions of $\widehat{S} \times (A_\delta C)$ on \widehat{C} and $\widehat{R} \times \widehat{N}_1$ on \widehat{A} .

$\widehat{R} \times \widehat{N}_1$ acts on \widehat{A} by tensor product, as \widehat{R} does.

$\widehat{S} \times (A_\delta C)$ acts on \widehat{C} as follows: given $(\chi, y) \in \widehat{S} \times (A_\delta C)$ and $\omega \in \widehat{C}$, we define,

$$(\chi, y) \cdot \omega = y \cdot (\chi \otimes \omega)$$

(Since $y \cdot \chi = \chi$, this action is well-defined.)

We are interested in the following orbit $\Omega = \widehat{S} \times (A_\delta C) \cdot \eta$ in \widehat{C} .

The action of $\widehat{S} \times (A_\delta C)$ on Ω factors to an action of $\widehat{S} \times (A_\delta C/A_\eta)$.

Write $(\widehat{S} \times (A_\delta C/A_\eta))_\eta$ for the stabilizer of η in $\widehat{S} \times (A_\delta C/A_\eta)$

and $(\widehat{R} \times \widehat{N}_1)_\mu$ for the stabilizer of μ in $\widehat{R} \times \widehat{N}_1$.

We have three short exact sequences

$$\begin{aligned} \text{i)} \quad & 1 \rightarrow \widehat{S}_\eta \rightarrow (\widehat{S} \times (A_\delta C/A_\eta))_\eta \xrightarrow{\pi_1} A_\delta C/A_\eta \rightarrow 1 \\ \text{ii)} \quad & 1 \rightarrow \widehat{N}_1 \rightarrow Q \xrightarrow{\varphi} (\widehat{S} \times (A_\delta C/A_\eta))_\eta \rightarrow 1 \\ \text{iii)} \quad & 1 \rightarrow \widehat{N}_{1\mu} \rightarrow (\widehat{R} \times \widehat{N}_1)_\mu \xrightarrow{\pi_2} Q \rightarrow 1 \end{aligned}$$

where π_i is the projection with respect i -factor, $i = 1, 2$.

φ is defined as follows:

For any $\chi \in Q$, there exists, by definition, an element y in A such that $\chi|_C \cdot \eta = y \cdot \eta$.

Then define $\varphi(\chi) = (\chi|_C, \bar{y}^{-1})$ where \bar{y} is the class of y in A/A_η .

We only need to prove that \bar{y} is an element of $A_\delta C$. But $\chi|_C \cdot \eta|_B$ contains δ as submodule, then $y \cdot \eta|_B$ also contains δ . This implies the existence of an element $c \in C$ such that $\delta = c \cdot y \cdot \delta$. Therefore, $y \cdot c \in A_\delta$ and then $y \in A_\delta C$.

It is clear that Q_o is also equal to $\varphi^{-1}(\widehat{S}_\eta)$.

Hence $Q/Q_o \cong (\widehat{S} \times (A_\delta C/A_\eta))_\eta / \widehat{S}_\eta \cong A_\delta C/A_\eta$. This proves a).

On the other hand, we have the short exact sequence:

$$\text{iv)} \quad 1 \rightarrow \widehat{R}_\mu \rightarrow (\widehat{R} \times \widehat{N}_1)_\mu \xrightarrow{\pi_2} \widehat{N}_1 \rightarrow 1$$

Applying cardinality,

$$|\widehat{R}_\mu| |\widehat{N}_1| = |(\widehat{R} \times \widehat{N}_1)_\mu| = |\widehat{N}_{1\mu}| |Q| = |\widehat{N}_{1\mu}| |\widehat{N}_1| |(\widehat{S} \times (A_\delta C/A_\eta))_\eta| = |\widehat{N}_{1\mu}| |\widehat{N}_1| |\widehat{S}_\eta| |A_\delta C/A_\eta|.$$

In short,

$$|\widehat{R}_\mu| = |\widehat{N}_{1,\mu}| |\widehat{S}_\eta| |A_\delta C / A_\eta|.$$

This, in turn, implies

$$|A/A_\delta|^{-1} |\widehat{R}_\mu| = |A/A_\eta|^{-1} |A_\delta C / A_\eta| |A_\delta C / A_\delta|^{-1} |\widehat{R}_\mu| = |A/A_\eta|^{-1} |\widehat{N}_{1,\mu}| |C/C_\delta|^{-1} |\widehat{S}_\eta| |A_\delta C / A_\eta|^2.$$

In view of corollary (3.7) b), we conclude

$$m(\delta, \mu)^2 = m(\delta, \eta)^2 m(\eta, \mu)^2 |A_\delta C / A_\eta|^2$$

This proves the lemma. □

IV.- Real parabolic induction and (\mathfrak{g}, K) -modules

4.1 Basic properties of real parabolic induction and (\mathfrak{g}, K) -modules

Let G be a group in the Harish-Chandra class with maximal compact subgroup K . Let \mathfrak{g} be the complexification of $\text{Lie}(G)$.

Write $\mathfrak{u}(\mathfrak{g})$ for the universal enveloping algebra associated to \mathfrak{g} .

We cite [Green], [W], [B-W] and [V-I] as reference to understand the following definitions and propositions. Proofs for the results in this subsection can be found there.

Definition 4.1.1 A Harish-Chandra module for G or an admissible (\mathfrak{g}, K) -module is a $\mathfrak{u}(\mathfrak{g})$ -module (Π, V) that is also a K -module such that

- a) any vector v in V satisfies $\dim \langle K \cdot v \rangle < \infty$ (i.e., v is a K -finite vector)
- b) the differential of the representation of K is the representation of \mathfrak{g} restricted to \mathfrak{k}_0
- c) if $X \in \mathfrak{g}$, $k \in K$ and $v \in V$ then $[\text{Ad}(k)X] \cdot v = k(X(k^{-1} \cdot v))$
- d) V can be decompose as $V = \hat{\bigoplus}_{\gamma \in \hat{K}} V(\gamma)$ where $V(\gamma)$ is the γ -isotypic component and $\dim(V(\gamma)) < \infty$

Definition 4.1.2 A Harish-Chandra module for G is called of *finite length* if it is of finite length as a $\mathfrak{u}(\mathfrak{g})$ -module.

Definition 4.1.3 A continuous representation of G , (Π, \mathfrak{H}) , on a complex Hilbert space \mathfrak{H} , is called admissible if the set of all K -finite vectors in \mathfrak{H} is a Harish-Chandra module for G under the natural induced actions of \mathfrak{g} and K .

Now, let $G = KAN$ be an Iwasawa decomposition and $P = MAN$ the corresponding minimal parabolic subgroup.

Let $\bar{\Delta}^+ = \bar{\Delta}(P, A)$ be the corresponding positive root system for $\bar{\Delta}$ (def. 2.3).

Put, $\bar{\Delta}_0 =$ simple roots of $\bar{\Delta}^+$.

For any $F \subseteq \bar{\Delta}_0$, we construct a parabolic subgroup $P_F = M_F A_F N_F$ of G containing P and $A_F \subseteq A$ as follows.

$$\begin{aligned} A_F &= \{a \in A \mid \alpha(a) = 1 \text{ for all } \alpha \in F \} \\ \bar{M}_F &= \text{centralizer of } A_F \text{ in } G \end{aligned}$$

Define, $\Delta_F = \{\alpha \in \bar{\Delta}^+ \mid \alpha(a) \neq 1 \text{ for some } a \in A_F\}$.

Set,

$$N_F = \exp\left(\sum_{\alpha \in \Delta_F} \mathfrak{g}_\alpha\right) \subseteq N$$

We get $P_F = \bar{M}_F N_F$ is a parabolic subgroup with Levi factor \bar{M}_F .

\bar{M}_F can be decomposed as $M_F A_F$ for some group M_F in the Harish-Chandra class of compact center (def. 1.2 and [W]). Thus, $P_F = M_F A_F N_F$ is the Langlands decomposition of P_F .

Definition 4.1.4 A pair (P_F, A_F) as above is called a P -pair for F .
 (The P -pair for $F = \phi$ is (P, A) and in this case $\overline{M}_F = MA$)

We denote $\Delta(P_F, A_F)$ for the set of roots of \mathfrak{a}_F in \mathfrak{n}_F .

Let (P_F, A_F) be a P -pair. Suppose (γ, V_γ) is an admissible representation of M_F on a complex Hilbert space V_γ of finite length. Define $\rho_F = \rho(\Delta_F)$ as in section I, where Δ_F has just been defined.

For any $\nu \in \widehat{A_F}$, define the Hilbert space $\mathfrak{H}_{\gamma \otimes \nu}^{P_F}$ to be the following set

$\{f : G \rightarrow V_\gamma \mid f \text{ is measurable, } f|_K \text{ is square integrable, and for all } m \in M_F, a \in A_F, n \in N_F,$

$$\text{and } g \in G \quad f(gman) = a^{-(\rho_F + \nu)} \gamma(m^{-1}) f(g) \}$$

where the norm is $\|f\|^2 = \int_K \|f(k)\|^2 dk$

It is possible to construct a continuous representation of G on $\mathfrak{H}_{\gamma \otimes \nu}^{P_F}$ by setting
 $[\Pi(\gamma \otimes \nu) \cdot f](x) = f(g^{-1}x)$.

Denote $Ind_{P_F}^G(\gamma \otimes \nu)$ for the representation $(\Pi(\gamma \otimes \nu), \mathfrak{H}_{\gamma \otimes \nu}^{P_F})$.

We call this construction real parabolic induction.

Proposition 4.1.5 ([Green], Ch. 4) We have the following properties for real parabolic induction:

- a) $Ind_{P_F}^G(\gamma \otimes \nu)$ is an admissible representation
- b) if $\gamma \otimes \nu$ is a unitary representation, so is $Ind_{P_F}^G(\gamma \otimes \nu)$
- c) if γ is a discrete series representation of M_F then $Ind_{P_F}^G(\gamma \otimes \nu)$ has finite length
- d) if γ has infinitesimal character λ then $Ind_{P_F}^G(\gamma \otimes \nu)$ has infinitesimal character (λ, ν)

Definition 4.1.6 We define two special types of induced representations

- a) $Ind_{P_F}^G(\gamma \otimes \nu)$ is called a *principal series representation* when $F = \phi$
- b) $Ind_{P_F}^G(\gamma \otimes \nu)$ is called a *standard representation* when γ is a discrete series representation

(4.1.7) There is an important result to obtain a bridge between admissible (\mathfrak{g}, K) -modules and admissible representations for G ; (defs 4.1.1 and 4.1.3). This result (mainly due to Casselman) claims that every Harish-Chandra module of finite length may be realized as the space of K -finite vectors of some admissible representation of G of finite length on a Hilbert space (defs. 4.1.1, 4.1.2 and 4.1.3).

Notation 4.1.8 We will write

$$X(\gamma \otimes \nu) = X_G(\gamma \otimes \nu) = X_G(P_F, \gamma \otimes \nu)$$

for the Harish-Chandra module associated to $Ind_{P_F}^G(\gamma \otimes \nu)$.

Proposition 4.1.9 (Induction by stages) Let P be a minimal parabolic subgroup of G and let $\overline{\Delta}_o$ be as in definition (4.1.4). Consider $F_1 \subseteq F_2 \subseteq \overline{\Delta}_o$ and

(P_{F_1}, A_{F_1}) and (P_{F_2}, A_{F_2}) two corresponding P -pairs (def. 4.1.4) such that $P_{F_1} \subseteq P_{F_2}$ with Langlands decompositions $P_{F_1} = M_{F_1} A_{F_1} N_{F_1}$ and $P_{F_2} = M_{F_2} A_{F_2} N_{F_2}$.

Thus, $P_{F_1} \cap M_{F_2}$ is a parabolic subgroup of M_{F_2} $A_{F_1} = A_{F_2}(A_{F_1} \cap M_{F_2})$. Suppose γ is an admissible representation of M_{F_1} and ν is a representation of A_{F_1} .

Write $\nu_1 = \nu|_{A_{F_1} \cap M_{F_2}}$ and $\nu_2 = \nu|_{A_{F_2}}$. We have,

$$\text{Ind}_{P_1}^G(\gamma \otimes \nu) = \text{Ind}_{P_{F_2}}^G(\text{Ind}_{P_{F_1} \cap M_{F_2}}^{M_{F_2}}(\gamma \otimes \nu_1) \otimes \nu_2)$$

□

Definition 4.1.10 Suppose (Π, V) is an irreducible Harish-Chandra representation of G (def. 4.1.1). (Π, V) is called a *tempered representation* if it may be realized as a subrepresentation of the Harish-Chandra module associated to some standard representation $\text{Ind}_{P_F}^G(\gamma \otimes \nu)$ where $\nu \in \widehat{A_F}$ is a unitary representation (def. 4.1.6).

Such a representation $\text{Ind}_{P_F}^G(\gamma \otimes \nu)$ will be called a *standard tempered representation*.

In case G has compact center, proposition 3.7 in ([B-W], Ch. IV) asserts that an irreducible Harish-Chandra representation (Π, V) of G is tempered if and only if $\forall x, y \in V$, the matrix coefficients $\langle \Pi(g)x, y \rangle: G \rightarrow \mathbb{C}$ are in $L^{2+\epsilon}(G) \forall \epsilon > 0$

In our purpose of decomposing a standard tempered representation, we need to take parabolic subgroups in certain special position.

Definition 4.1.11 ([Green]) Let $P = MAN$ be a minimal parabolic subgroup, $\overline{\Delta}^+$ and $\overline{\Delta}_o$ corresponding to P ; as in definition (4.1.4). $\nu \in \widehat{A}$ is called *negative for P* (or *P negative for ν*) if for all $\alpha \in \overline{\Delta}^+$ $\text{Re}(\langle \alpha, \nu \rangle) \leq 0$.

Put, $F = \{\alpha \in \overline{\Delta}_o \mid \text{Re}(\langle \alpha, \nu \rangle) = 0\}$

If ν is negative for P , the P -pair (P_F, A_F) is said to be defined by $-Re\nu$.

A reason for giving this definition is that for any irreducible tempered representation γ of M_F , $X_G(P_F, \gamma \otimes \nu|_F)$ (notation 4.1.8) has a unique irreducible submodule.

This fact is used in [W] to formulate the classification of all irreducible admissible (\mathfrak{g}, K) -modules, that is the Langlands classification. This will be formulated later on (thm. 5.1.1.3).

4.2 Lowest K -types and lambda-lowest K -types

Suppose G is the Harish-Chandra class with maximal compact subgroup K . Write T^c for a maximal torus of K . Put $\mathfrak{k}_o = \text{Lie}(K)$ and $\mathfrak{t}_o^c = \text{Lie}(T^c)$. Now we fix a positive root system Δ_c^+ for $\Delta(\mathfrak{k}, \mathfrak{t}^c)$, and define $\rho_c = \rho(\Delta_c^+)$.

Any $\pi \in \widehat{K}_o$ is determined by a highest weight $\mu \in \widehat{T}_o^c$. When K is disconnected, we need the normalizer of T^c in K , $N_K(T^c)$. Set,

$$\mathfrak{n}_K = \{n \in N_K(T^c) \mid n\rho_c = \rho_c\}$$

Let $\mu \in \widehat{T}_o^c$. Since T_o^c is a normal subgroup of \mathfrak{n}_K , denote $(\mathfrak{n})_\mu$ for the stabilizer of μ in \mathfrak{n}_K ; as in (2.13).

With respect to Δ_c^+ let us consider a dominant weight $\mu \in \widehat{T}_o^c$, then we write,

$$\widehat{K}(\mu) = \{\pi \in \widehat{K} \mid \pi|_{K_o} \text{ contains the highest weight representation } \pi_\mu \in \widehat{K}_o\}$$

We have a certain generalization of the highest weight representation.

Proposition 4.2.2 With respect to Δ_c^+ ,

- a) For $\pi \in \widehat{K}(\mu)$, the highest weights of $\pi|_{K_o}$ are of the form $x\mu$ for some $x \in \mathfrak{n}_K$
- b) there is a one-to-one correspondence between $\widehat{K}(\mu)$ and the set

$$\{\gamma \in (\mathfrak{n})_\mu \mid \mu \text{ occurs in } \gamma|_{T_o^c}\}$$

□

Now, denote $\bar{\mu}$ for the $d\mu$. For any $\gamma \in (i\mathfrak{t}_o^c)^*$, put $\|\gamma\| = \langle \gamma + 2\rho_c, \gamma + 2\rho_c \rangle$ (which is not a negative number).

Consider $\pi \in \widehat{K}(\mu)$. By proposition (4.2.2), all its highest weights are of the form $x\mu$ for some $x \in \mathfrak{n}_K$. Since $x\rho_c = \rho_c$, we can use any of these to define $\|\pi\| = \|\bar{\mu}\|$.

Definition 4.2.3 Let X be a non-zero Harish-Chandra module for G . The set of lowest K -types of X is defined by

$$\{\pi \in \widehat{K} \mid X(\pi) \neq 0, \text{ and } \|\pi\| \text{ is minimal with this property}\}$$

Since \langle, \rangle is positive definite on the weight lattice in $(i\mathfrak{t}_o^c)^*$ and X is of finite length, the set of lowest K -types of X is non-empty and finite.

Proposition 4.2.4 Suppose G is quasisplit (def. 2.7), $\delta \in \widehat{M}$ a fine representation and $\nu \in \widehat{A}$ unitary. Let X be any irreducible Harish-Chandra module appearing as direct summand in the decomposition of $X_G(\delta \otimes \nu)$ (4.1.9). Then every lowest K -types of X is a K -fine representation (2.11).

Proof: In [V-I], the result is proved for $G = G_o$. Remark (2.12) reduces the general case to this situation. \square

Proposition (4.2.4) implies that $A(\delta)$ (def. 2.21) is non-empty.

On the other hand, for any $\mu \in \widehat{T}_o^c$ Vogan associates to it a unique element $\lambda(\mu) \in (it_o^c)^*$ ([Green], Ch. 5).

For any $\mu \in \widehat{T}_o^c$ define $\|\mu\|_{\text{lambda}} = \langle \lambda(\mu), \lambda(\mu) \rangle$ where \langle , \rangle is as above.

It is possible to prove that for all $x \in \mathfrak{N}_K$, $\lambda(x\mu) = x\lambda(\mu)$. Therefore, in view of proposition (4.2.2) a), there is no ambiguity in defining for any $\pi \in \widehat{K}(\mu')$ $\|\pi\|_{\text{lambda}} = \langle \lambda(\mu'), \lambda(\mu') \rangle$.

Definition 4.2.5 Let X be a Harish-Chandra module for G . The set of lambda-lowest K -types of X is defined by

$$\{ \pi \in \widehat{K} \mid X(\pi) \neq 0, \text{ and } \|\pi\|_{\text{lambda}} \text{ is minimal with this property} \}$$

(Definitions 4.2.3 and 4.2.4 do not coincide for arbitrary Harish-Chandra modules; however, they are equivalent for irreducible representations.)

Definition 4.2.6 ([Green], Ch. 5) Consider the root decomposition for \mathfrak{g} with respect to $\Delta(\mathfrak{g}, \mathfrak{h}^c)$:

$$\mathfrak{g} = \mathfrak{h}^c + \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^c)} \mathfrak{g}_\alpha$$

For any $\lambda \in (it_o^c)^*$ dominant with respect to Δ_c^+ , the θ -stable parabolic subalgebra of \mathfrak{g} given by $\mathfrak{q}(\lambda) = \mathfrak{l} + \mathfrak{u}$, where \mathfrak{l} and \mathfrak{u} are the subalgebras:

$$\mathfrak{l} = \mathfrak{h}^c + \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^c), \langle \alpha, \lambda \rangle = 0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{u} = \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^c), \langle \alpha, \lambda \rangle > 0} \mathfrak{g}_\alpha$$

is called the θ -stable parabolic subalgebra of \mathfrak{g} associated to λ .

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the θ -stable parabolic subalgebra associated to $\lambda(\mu)$ and let L be the normalizer of \mathfrak{q} in G .

$L \cap K$ is a maximal compact subgroup of L and T^c is a maximal torus for $L \cap K$ too.

Write $\Delta_c^+(\mathfrak{l}) = \Delta(\mathfrak{l} \cap \mathfrak{t}, \mathfrak{t}^c) \cap \Delta_c^+$

As in proposition (4.2.4), we can define $(L \cap K)(\mu)$ and $\mathfrak{N}_{L \cap K}$.

Note that $(\mathfrak{N}_K)_\mu \subseteq (\mathfrak{N}_{L \cap K})_\mu$:

If $x \in N_K(T^c)$ then

$$\rho_c = x\rho_c = x(\rho(\Delta_c^+(\mathfrak{l})) + \rho(\mathfrak{u} \cap \mathfrak{t}))$$

Since $x\rho(\mathfrak{u} \cap \mathfrak{t}) = \rho(\mathfrak{u} \cap \mathfrak{t})$, then $x\rho(\Delta_c^+(\mathfrak{l})) = \rho(\Delta_c^+(\mathfrak{l}))$.

This implies $\mathfrak{n}_K \subseteq \mathfrak{n}_{L \cap K}$. Hence, $(\mathfrak{n}_K)_\mu \subseteq (\mathfrak{n}_{L \cap K})_\mu$.

In fact, more is true:

CLAIM: $(\mathfrak{n}_K)_\mu = (\mathfrak{n}_{L \cap K})_\mu$

In view of the property of $\lambda(\mu)$ mentioned previously to definition (4.2.5), we have:

Every $x \in (\mathfrak{n}_K)_\mu$ satisfies $x\lambda(\mu) = \lambda(x\mu) = \lambda(\mu)$. This implies that $(\mathfrak{n}_K)_\mu \subseteq L$. Hence, the claim holds.

Corollary 4.2.7 For any $\mu \in \widehat{T}_\sigma^c$, let \mathfrak{q} be θ -stable parabolic subalgebra of \mathfrak{g} associated to $\lambda(\mu)$ (def. 4.2.5). Then there is a bijection between $\widehat{K}(\mu)$ and $L \cap \widehat{K}(\mu)$.

Proof: This result is consequence of proposition (4.2.4) and the claim above. \square

4.3 Proof of proposition (3.4)

Example 4.3.1 Let G be a group in the Harish-Chandra class satisfying the following conditions:

a) the center of G is compact

b) $\text{Lie}(G) = \mathfrak{g}_0 = \mathfrak{g}_0^1 + \mathfrak{g}_0^2 + \cdots + \mathfrak{g}_0^n + \mathfrak{c}_0$

where \mathfrak{c}_0 is the center of \mathfrak{g}_0 and $\mathfrak{g}_0^i \cong \mathfrak{sl}(2, \mathbb{R})$ $i = 1, 2, \dots, n$.

Therefore, $\mathfrak{h}^s = \mathfrak{a}_0^s + \mathfrak{c}_0$ is a split Cartan subalgebra of \mathfrak{g}_0^i such that $\mathfrak{a}_0^s \cap \mathfrak{g}_0^i$ is a split Cartan subalgebra of \mathfrak{g}_0^i .

We have $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha_1, \dots, \alpha_n\}$ is a strongly orthogonal set of real roots.

$W = M'/M = (\mathbb{Z}/2\mathbb{Z})^n$

Recall the definition of σ_{α_i} and m_{α_i} , $i = 1, 2, \dots, n$ (def. 2.6).

Suppose $\delta \in \widehat{M}$ is a fine representation such that $\delta(m_{\alpha_i}) = -I$ for all $i = 1, \dots, n$.

So, in the notation of proposition (3.4), $\overline{\Delta}_\delta = \emptyset$ and $(M'_\delta)^o = M$. $R_\delta = M'_\delta/M = W_\delta \subseteq W$.

Lemma 4.3.2 Under the conditions in example (4.3.1) (notation in proposition (3.1) and proposition (3.4))

a) \widehat{R}_δ acts on $A(\delta)$ transitively

b) $\text{Ind}_{\widehat{P}}^{\widehat{G}}(\delta \otimes 1) = m(\delta, \mu) \sum_{\mu \in A(\delta)} I_\mu$

where I_μ is an irreducible admissible G -module with unique lowest K -type μ ; (4.2). Moreover, $I_\mu \cong I_{\mu'}$ implies that $\mu \cong \mu'$. I_μ appears in the sum $m(\delta, \mu)$ -times.

Proof: To prove a), it suffices to show that $|\widehat{R}_\delta^1| = |A(\delta)|$ (prop. 3.1).

Case 1.- Let us prove lemma 4.3.2 a) for G^* (notation I).

In this case, $K^* = T^c$. By proposition (4.2.4), let a non-zero element μ in $A(\delta)$.

Fix a system of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{t}^c) = \{\beta_1, \dots, \beta_n\}$ making $-\mu$ dominant.

We identify $\{\beta_1, \dots, \beta_n\}$ with their corresponding elements in \widehat{T}^c . These elements are the extensions of the characters in \widehat{W} .

Now define,

$$A = \{ \beta_{i_1} \otimes \cdots \otimes \beta_{i_r} \otimes \mu \mid i_1 < i_2 < \cdots < i_r ; \text{ where } i_j \in \{1, 2, \dots, n\} \}$$

A contains 2^n T -fine representations.

Proposition (3.1) implies that $A(\delta) \subseteq A$. But $\beta_{i_1} \otimes \cdots \otimes \beta_{i_r} \otimes \mu|_M = \mu|_M$ contains δ as submodule. Therefore, $2^n = |A(\delta)|$, and then a) holds in this case.

We get also that $\widehat{R}_\delta^1 = \widehat{R}_\delta = \widehat{W}$, i.e., $M'_\delta = M'$ and $|\widehat{R}_\delta^o| = 1$

Case 2.- We want to reduce the general case to case 1.

Denote $M_T = M \cap T^c$ and $M'_T = M' \cap T^c$.

Use proposition (4.2.4) to give an element μ in $A(\delta)$.

Let $\delta_o \in \widehat{M}_T$ and $\mu_o \in \widehat{T}^c$ such that δ_o occurs in $\delta|_{M_T}$, $\mu_o \in A(\delta_o)$ and μ_o occurs in $\mu|_{T^c}$. Write $W_c = K/T^c$. This group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$ for some m .

By case 1, $\mathbf{A} = A(\delta_o)$ (using μ_o instead of μ there).

W_c acts on \mathbf{A} . Clearly the stabilizer of μ_o in K is exactly T^c ; equivalently $(\widehat{W}_c)_\mu = \widehat{W}_c$ and $m(\mu_o, \mu) = 1$ (lemma (3.6) c)).

Define, $R = M'/M$, $V = M'_T/M_T$ and $S = M'/M'_T$, $U = M/M_T$; thus $R \cong V$ and $S \cong U$.

We regard μ as element of both \widehat{K} and \widehat{M}' and μ_o as element of both \widehat{T}^c and \widehat{M}'_T (corollary (2.19)).

Thus, we have $\widehat{S}_\mu = \widehat{S}$ and $m(\mu_o, \mu) = 1$, and from case 1, $\widehat{V}_{\mu_o} = 1$ and $(M'_T)_{\delta_o} = M'_T$.

Claim: $m(\delta_o, \delta) = m(\delta, \mu)$

SUBLEMMA: There are two exact sequences:

$$i) \quad 1 \rightarrow \widehat{R}_\mu \rightarrow M'_{\delta_o}/M'_T \rightarrow 1$$

$$ii) \quad 1 \rightarrow \widehat{U}_\delta \rightarrow \widehat{U} \rightarrow M'/M'_\delta \rightarrow 1$$

Assume for a moment this lemma is true. Since $(M'_T)_{\delta_o} = M'_T$, $M'_{\delta_o}/M'_T = M_{\delta_o}/M_T$. Thus, we have

$$|M'/M'_\delta| |\widehat{U}_\delta| = |\widehat{U}| = |M/M_{\delta_o}| |M_{\delta_o}/M_T| = |M/M_{\delta_o}| |\widehat{R}_\mu|$$

Last equality is an application of sublemma a). Hence, $|M/M_{\delta_o}|^{-1} |\widehat{U}_\delta| = |M'/M'_\delta|^{-1} |\widehat{R}_\mu|$. By corollary (3.7) b), we obtain $m(\delta_o, \delta) = m(\delta, \mu)$

On the other hand, since $G/P = G^*/P^*$, we can write

$$\text{Ind}_P^G(\delta \otimes 1)|_{G^*} = m(\delta_o, \delta) \sum_{x \in M/M_{\delta_o}} \text{Ind}_{P^*}^{G^*}(x \cdot \delta_o \otimes 1)$$

The number of T^c -fine representations on the left is equal to $|A(\delta)| m(\delta, \mu) |S|$ ($|S|$ is the number of T^c -fine representations appearing in $\mu|_T$ for $\mu \in A(\delta)$; recall $K_{\mu_o} = T^c$ and $m(\mu_o, \mu) = 1$.

Meanwhile, there are $m(\delta_o, \delta) |M/M_{\delta_o}| |R|$ T^c -fine representations on the right. This implies

$$|M/M_{\delta_o}| |R| = |A(\delta)| |S| = |A(\delta)| |M/M_{\delta_o}| |M_{\delta_o}/M_T| = |A(\delta)| |\widehat{R}_\mu| |M/M_{\delta_o}|$$

where the last equality is an application of sublemma a). We conclude that $|\widehat{R}_\delta| = |A(\delta)|$.

To finish the proof of lemma a) , we prove the sublemma:

i) $\widehat{R} \cong \widehat{V}$. We also have \widehat{V} acts on \widehat{M}'_T and \widehat{R} acts on \widehat{M}' .

In this incise i), for any $\chi \in \widehat{R}$ we write $\chi|$ for the corresponding character in \widehat{V} . First, note that if we take $\chi \in \widehat{R}$ then

$\chi| \otimes \mu_o$ occurs in the restriction of μ to M'_T if and only if there is an $s \in M'$ such that $\chi| \otimes \mu_o = s \cdot \mu_o$ if and only if there is a character ξ of S such that $\chi \otimes \mu = \xi \otimes \mu$.

But $\widehat{S} = \widehat{S}_\mu$.

Therefore, there is an $s \in M'$ such that $\chi| \otimes \mu_o = s \cdot \mu_o$ if and only if $\chi \in \widehat{R}_\mu$.

On the other hand, $\chi| \otimes \mu_o = s \cdot \mu_o$ with $\chi|$ in \widehat{V} implies $s \in M'_{\delta_o}$. Thus,

$\chi \in \widehat{R}_\mu$ if and only if there is an element s in M'_{δ_o} such that $\chi| \otimes \mu_o = s \cdot \mu_o$.

Define $\varphi : \widehat{R}_\mu \rightarrow M'_{\delta_o}/M'_T$ by $\chi \mapsto \bar{s}$.

$M'_T = M'_{\mu_o}$ and $\widehat{V}_{\mu_o} = 1$ imply φ is well-defined and an isomorphism. This proves *i*).

ii) $\widehat{S} \cong \widehat{U}$. We also have \widehat{S} acts on \widehat{M}' and \widehat{U} acts on \widehat{M} .

In this incise *ii*), for any $\chi \in \widehat{S}$ we write $\chi|$ for the corresponding character in \widehat{U} .

Since $\widehat{S} = \widehat{S}_\mu$, for all $\chi| \in \widehat{U}$, there is $s \in M'$ such that $\chi| \otimes \delta = s \cdot \delta$.

As in *i*), we construct a group homomorphism $\psi : \widehat{U} \rightarrow M'/M'_\delta$ by $\chi \mapsto \bar{s}$.

$\ker(\psi) = \widehat{U}_\delta$. We want to proof that ψ is surjective.

We note that for any element x in M' , $x = ym$ for some y in M'_T and m in M ;

for $M' = M'_T M$. Therefore, $x \cdot \delta = y \cdot \delta$. But, $(M'_T)_{\delta_o} = M'_T$ implies that $y \cdot \delta|_{M_T}$ contains δ_o . Thus, there exists $\chi \in \widehat{U}$ such that $x \cdot \delta = y \cdot \delta = \chi \otimes \delta$. The sublemma holds.

Proof of lemma (4.3.2) *b*).

From *a*), consider again the restriction to $G^\#$

$$X_G(\delta \otimes 1)|_{G^\#} = m(\delta_o, \delta) \sum_{\nu \in M/M_{\delta_o}} X_{G^\#}(\nu \cdot \delta_o \otimes 1)$$

In ([Green], Ch. 2), $X_{G^\#}(\delta_o \otimes 1)$ is decomposed as $\sum_{\chi \in \widehat{V}} I_{\chi \cdot \mu_o}$, where $I_{\chi \cdot \mu_o}$ is the unique irreducible (\mathfrak{g}, T^c) -module appearing in the sum whose unique T^c -fine representation is $\chi \cdot \mu_o$. Regarding $\widehat{R}_\mu \subseteq \widehat{V}$, sublemma *i*) implies that

$$\sum_{\nu \in M/M_T} I_{\nu \cdot \mu_o} = \sum_{\nu \in M/M_{\delta_o}} \sum_{\chi \in \widehat{R}_\mu} I_{\nu \cdot \chi \cdot \mu_o}$$

is a (\mathfrak{g}, K) -irreducible submodule of $X_G(\delta \otimes 1)$ containing the unique K -fine representation μ . It appears with multiplicity $m(\delta_o, \delta) = m(\delta, \mu)$. The lemma follows. \square

The next step is a generalization of the previous lemma.

Lemma 4.3.3 Suppose G is quasisplit and $\delta \in \widehat{M}$ is fine. Then we get:

a) \widehat{R}_δ acts on $A(\delta)$ transitively

b) $\text{Ind}_P^G(\delta \otimes 1) = m(\delta, \mu) \sum_{\mu \in \Lambda(\delta)} I_\mu$

where I_μ is an irreducible admissible G -module with unique lowest K -type μ ; (4.2). Moreover, $I_\mu \cong I_{\mu'}$ implies that $\mu \cong \mu'$. I_μ appears in the sum $m(\delta, \mu)$ times.

Proof: Recall $\overline{\Delta}_o$ (prop. 2.25).

Since the nilpotent part of the minimal parabolic subgroup P is not important in this case, we can assume that the positive system $\overline{\Delta}_+$ corresponding to P is as in proposition (2.25).

Let $\overline{\Delta}_o$ be the set of simple roots of $\overline{\Delta}_+$.

We apply induction by stages (prop. 4.1.9).

With $E = \overline{\Delta}_o \cap \overline{\Delta}_s$, consider the associated p -pair (P_E, A_E) and the decomposition $P_E = M_E A_E N_E$. Now, (M_E, δ) satisfy the assumptions in example (4.3.1). We obtain (*):

$$\text{Ind}_P^G(\delta \otimes 1) = \text{Ind}_{P_E}^G(\text{Ind}_{P \cap M_E}^{M_E}(\delta \otimes 1) \otimes 1) = m(\delta, \eta) \sum_{\eta \in A^E(\delta)} \text{Ind}_{P_E}^G(I_\eta \otimes 1)$$

Second equality is an application of lemma (4.3.2), where

$$A^E(\delta) = \{\eta \in \widehat{K}_E \mid \delta \text{ occurs in } \eta|_M\} \text{ and } K_E = K \cap M_E$$

Write $M'_E = M' \cap M_E$, $M'_{E,\delta} = M'_\delta \cap M_E$ and $R_{E,\delta} = M'_{E,\delta}/M$.

By proposition (2.27) and definition (2.24), $M'_\delta = M'_{E,\delta}(M'_\delta)^\circ$ and $M = M'_{E,\delta} \cap (M'_\delta)^\circ$. Hence:

$$R_\delta = M'_\delta / (M'_\delta)^\circ \cong M'_{E,\delta} / M = R_{E,\delta} \cong (\mathbf{Z}/2\mathbf{Z})^n$$

Using corollary (3.2), $\text{Hom}_M(\mu_\delta, \mu'_\delta) = \text{Hom}_{(M'_\delta)^\circ}(\mu_\delta, \mu'_\delta)$, we have

$$\text{Hom}_{M'_{E,\delta}}(\mu_\delta, \mu'_\delta) = \text{Hom}_{M'_\delta}(\mu_\delta, \mu'_\delta)$$

For all $\mu \in A(\delta)$, $\eta = \mu|_{K_E}$ is irreducible and $\eta \in A^E(\delta)$.

By identifying $\widehat{R}_{E,\delta}$ with \widehat{R}_δ , we build up a function

$$\psi : \widehat{R}_\delta^1 \cdot \mu_\delta \rightarrow A^E(\delta) \text{ by } \chi \otimes \mu_\delta \mapsto \chi \otimes \eta_\delta$$

Note $\chi \otimes \mu_\delta = \mu_\delta$ if and only if $\chi \otimes \eta_\delta = \eta_\delta$. Thus, ψ is a bijection. We conclude that $|A^E(\delta)| = |\widehat{R}_\delta^1|$ and $m(\delta, \eta) = |\widehat{R}_\delta^1|^{\frac{1}{2}} = m(\delta, \mu)$ (lemma (3.6) and def. 3.3).

On the right hand side of (*) above we have at least $|\widehat{R}_\delta^1| m(\delta, \mu)$ K -fine representations while on the left we have exactly $m(\delta, \mu) |A(\delta)|$ K -fine representations.

By proposition (3.1), $|A(\delta)| \leq |\widehat{R}_\delta^1|$. Thus, $|A(\delta)| = |\widehat{R}_\delta^1|$ and incise a) holds.

To prove lemma 4.3.3 b), we note that for all η in (*), $\text{Ind}_{P_E}^G(I_\eta \otimes 1)$ is irreducible. With the bijection ψ , $I_\mu = \text{Ind}_{P_E}^G(I_\eta \otimes 1)$ is the unique irreducible submodule that appears in $\text{Ind}_P^G(\delta \otimes 1)$ containing the K -fine representation $\mu = \psi^{-1}(\eta)$. The lemma follows. \square

V.- Classification of tempered representations

5.1 Langlands Classification

The Langlands classification is stated in [Green] in two different ways: Using real parabolic induction (section IV) on one hand, and by Vogan-Zuckerman theory of cohomological induction on the other hand. Although the groups considered there are linear, the result and procedure can be extended to groups in the Harish-Chandra class without any change.

5.1.1 ([Green], Ch. 6) Cohomological induction

Given an irreducible (\mathfrak{g}, K) -representation X , (4.1.1), we can associate to it the following data ([Green], Ch. 5) $(\mathfrak{q}, H, \delta, \nu)$, where

- a) $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ -stable parabolic subalgebra \mathfrak{g}
- b) $H = TA$ is a maximally split Cartan subgroup of L ($L =$ normalizer of \mathfrak{q} in G)
- c) $\delta \in \widehat{T}$ is a fine representation with respect to L (def. 2.9), and $\nu \in \widehat{A}$
- d) \mathfrak{q} is associated to $\rho(\mathfrak{u}) + d\delta$ (def. 4.2.5)

Definition 5.1.1.1 $(\mathfrak{q}, H, \delta, \nu)$ satisfying a), b), c) and d) above is called a set of θ -stable data for G

For $(\mathfrak{q}, H, \delta, \nu)$, we can build up a (\mathfrak{g}, K) -module. First, let P_1 be a parabolic subgroup of L containing a minimal parabolic subgroup of L for which ν is negative (def. 4.1.11). Then let $X_L(P_1, \delta \otimes \nu)$ be the $(\mathfrak{l}, L \cap K)$ -module associated to $Ind_{P_1}^L(\delta \otimes \nu)$.

Now, we consider the Zuckerman functors

$$\mathfrak{R}_\mathfrak{q}^i : \mathfrak{F}(\mathfrak{l}, L \cap K) \rightarrow \mathfrak{F}(\mathfrak{g}, K)$$

where $\mathfrak{F}(\mathfrak{l}, L \cap K)$ and $\mathfrak{F}(\mathfrak{g}, K)$ are the categories of admissible representations of finite length for the pairs $(\mathfrak{l}, L \cap K)$ and (\mathfrak{g}, K) respectively.

It turns out that $\mathfrak{R}_\mathfrak{q}^i(X_L(P_1, \delta \otimes \nu)) \neq 0$ if and only if $i = S = \dim(\mathfrak{u} \cap \mathfrak{k})$.

$\mathfrak{R}_\mathfrak{q}^S(X_L(P_1, \delta \otimes \nu))$ has a maximal completely reducible subrepresentation that we denote by $J(\mathfrak{q}, H, \delta, \nu)$

Definition 5.1.1.2 $J(\mathfrak{q}, H, \delta, \nu)$ is called the *Langlands subrepresentation* associated to $\mathfrak{R}_\mathfrak{q}^S(X_L(P_1, \delta \otimes \nu))$

We have the following classification:

Theorem 5.1.1.3 ([Green], 6.5.12)(Langlands classification)

For every irreducible admissible (\mathfrak{g}, K) -module X

- a) there exists a set of θ -stable data $(\mathfrak{q}, H, \delta, \nu)$ such that X is a submodule of $J(\mathfrak{q}, H, \delta, \nu)$
- b) if $(\mathfrak{q}', H', \delta', \nu')$ is another set of θ -stable data and X is a submodule of $J(\mathfrak{q}', H', \delta', \nu')$ then $(\mathfrak{q}', H', \delta', \nu')$ is conjugate to $(\mathfrak{q}, H, \delta, \nu)$ under K . \square

5.1.2 Real parabolic induction and regular characters

Let $H = TA$ be a θ -stable Cartan subgroup of G , and let $MA = G^A$ be the Langlands decomposition of the centralizer of A in G .

Write $\mathfrak{m} = \text{Lie}(M)$ and $\mathfrak{t} = \text{Lie}(T)$.

Definition 5.1.2.1 A regular character γ is a pair $(\Gamma, \bar{\gamma})$ (w. r. t. H) such that

a) Γ is an irreducible representation of H

b) $\bar{\gamma} \in \mathfrak{h}^*$ is such that for all $\alpha \in \Delta(\mathfrak{m}, \mathfrak{t})$, $\langle \alpha, \bar{\gamma} \rangle$ is a non-zero real number

c) $d\Gamma = \bar{\gamma} + \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m} \cap \mathfrak{t}}$

where $\Delta^+(\mathfrak{m}, \mathfrak{t}) = \{ \alpha \in \Delta(\mathfrak{m}, \mathfrak{t}) \mid \langle \alpha, \bar{\gamma} \rangle > 0 \}$, $\rho_{\mathfrak{m}} = \rho(\Delta^+(\mathfrak{m}, \mathfrak{t}))$ and $\rho_{\mathfrak{m} \cap \mathfrak{t}} = \rho(\Delta^+(\mathfrak{m} \cap \mathfrak{t}, \mathfrak{t}))$.

We will write more often a regular character with the Cartan subgroup that is implicit in its definition, like (H, γ) or like $(H, \Gamma, \bar{\gamma})$.

Now, given a set of character data (H, γ) for G , we construct a (\mathfrak{g}, K) -module.

Let $P = MAN$ be a parabolic subgroup of G containing a minimal parabolic subgroup for which $\Gamma|_A$ is negative (def. 4.1.11). Let,

ω discrete series representation of M based in the parameter $\Gamma|_T$

$\nu = \Gamma|_A$

Thus, we get $X_G(P, \delta \otimes \nu)$; the (\mathfrak{g}, K) -module associated to $\text{Ind}_P^G(\omega \otimes \nu)$.

Write $J(\gamma)$ for the Langlands subrepresentation of $X_G(P, \delta \otimes \nu)$ (def. 5.1.1.2).

Theorem 5.1.2.2 ([Green], 6.6.2)

a) there is a bijection between sets of θ -stable data for G and regular characters for G , which preserves conjugacy under K . The bijection is as follows: for a set of θ -stable data $(\mathfrak{q}, H, \delta, \nu)$, the corresponding set of character data (H, γ) is given by

i) $\bar{\gamma} = (\lambda^G, \nu)$

ii) $\Gamma|_A = \nu$

iii) $\Gamma|_T = \delta \otimes (\Lambda^{\dim(\mathfrak{u} \cap \mathfrak{p})} \mathfrak{u} \cap \mathfrak{p})|_T$ (where $\lambda^G = d\delta + \rho(\mathfrak{u})$)

b) (Langlands classification second version)

$$X_G(P, \delta \otimes \nu) \cong \mathfrak{X}_{\mathfrak{q}}^S(X_L(P_1, \delta \otimes \nu))$$

$$J(\gamma) \cong J(\mathfrak{q}, H, \delta, \nu)$$

□

It is our purpose to describe explicitly the decomposition of the Langlands subrepresentation $J(\gamma) \cong J(\mathfrak{q}, H, \delta, \nu)$; particularly when $\nu \in \hat{A}$ is a unitary representation.

Definition 5.1.2.3 the classification of tempered representations is the explicit decomposition of tempered standard representations (def. 4.1.10) into a direct sum of irreducible representations.

5.2 Limit characters

For linear groups Knapp and Zuckerman gave a complete classification of tempered representations (def. 5.1.2.3) in [K-Z]. Motivated by the technique they used there, we should allow certain generalization of regular characters.

Following [V-II] and [A-B-V], we have:

Definition 5.2.1 Let $H = TA$ be a θ -stable Cartan subgroup. Consider M , \mathfrak{m} and \mathfrak{t} as in definition (5.1.2.1). A set $(H, \underline{\gamma}) = (H, \Gamma, \bar{\gamma}, \Delta_{im}^+)$ is called *pseudo-character* for G , if it satisfies

- a) Γ is an irreducible representation of H and Δ_{im}^+ is a positive root system for $\Delta(\mathfrak{m}, \mathfrak{t})$
- b) $\bar{\gamma} \in \mathfrak{h}^*$ is such that for all $\alpha \in \Delta_{im}^+$, $\langle \alpha, \bar{\gamma} \rangle$ is a non-negative real number
- c) $d\Gamma = \bar{\gamma} + \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m} \cap \mathfrak{t}}$ (where $\rho_{\mathfrak{m}} = \rho(\Delta_{im}^+)$ and $\rho_{\mathfrak{m} \cap \mathfrak{t}} = \rho(\Delta_{im, compact}^+)$)

Δ_{im}^+ is not superfluous in definition (5.2.1). However, it is determined by requiring in condition b) strictly positive instead of non-negative; and in this case this definition coincides with definition (5.1.2.1). Therefore, we can give, without ambiguity, the following definition:

Definition 5.2.2 A pseudo-character $(H, \Gamma, \bar{\gamma}, \Delta_{im}^+)$ for G is called

- a) *regular* character if we require that for all $\alpha \in \Delta_{im}^+$, $\langle \alpha, \bar{\gamma} \rangle$ is a strictly positive real number
- b) *limit* character if we require that for any compact root $\alpha \in \Delta_{im}^+$, $\langle \alpha, \bar{\gamma} \rangle$ is a strictly positive real number

(5.2.3).- In analogy to theorem (5.1.2.2), we construct the following data

$(\mathfrak{q}, H, \Gamma_1, \bar{\gamma}_1, \Delta^+(\mathfrak{m} \cap \mathfrak{l}))$ from a set of pseudo-character character $(H, \underline{\gamma})$ for G as follows:

a) $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ -stable parabolic subalgebra of \mathfrak{g} associated to $\bar{\gamma}|_{\mathfrak{l}}$ (def. 4.2.5)

(H is a θ -stable Cartan subalgebra of L)

b) Γ_1 is the irreducible representation determined by

$$\Gamma_1|_{\mathfrak{A}} = \Gamma|_{\mathfrak{A}} \text{ and } \Gamma_1|_{\mathfrak{T}} = \Gamma|_{\mathfrak{T}} \otimes (\Lambda^{\dim(\mathfrak{u} \cap \mathfrak{p})} \mathfrak{u} \cap \mathfrak{p})^*|_{\mathfrak{T}}$$

c) $\bar{\gamma}_1 = \bar{\gamma} - \rho(\mathfrak{u})$

d) $\Delta^+(\mathfrak{m} \cap \mathfrak{l})$ is the intersection $\Delta_{im}^+ \cap \Delta(\mathfrak{m} \cap \mathfrak{l}, \mathfrak{t})$

Remark 5.2.4 Fix a system of positive roots Δ^+ for $\Delta(\mathfrak{g}, \mathfrak{h})$ such that

$(\Delta_{im}^+ \cup \Delta(\mathfrak{u})) \subseteq \Delta^+$.

Consider a positive system $\Delta^+(\mathfrak{l}, \mathfrak{h})$ determined by $\Delta^+ \cap \Delta^+(\mathfrak{g}, \mathfrak{h})$.

Write (notation 1.2), $\rho = \rho(\Delta^+)$, $\rho_c = \rho(\Delta^+ \cap \Delta(\mathfrak{k}, \mathfrak{t}))$, $\rho_l = \rho(\Delta^+(\mathfrak{l}, \mathfrak{h}))$ and $\rho_{\mathfrak{l} \cap \mathfrak{t}} = \rho(\Delta^+ \cap \Delta(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{h}))$.

a) We have the following equalities:

$$\begin{aligned} d\Gamma_1 &= \bar{\gamma} - 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m} \cap \mathfrak{t}} = (\bar{\gamma} - \rho(\mathfrak{u})) + (\rho(\mathfrak{u}) - 2\rho(\mathfrak{u} \cap \mathfrak{p})) + \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m} \cap \mathfrak{t}} \\ &= (\bar{\gamma} - \rho(\mathfrak{u})) - (\rho - 2\rho_c) + \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m} \cap \mathfrak{t}} + (\rho_l - 2\rho_{\mathfrak{l} \cap \mathfrak{t}}) \end{aligned}$$

Using ([Green], lemma 5.3.29), we have

$$(\rho - 2\rho_c)|_{\mathfrak{t}} = \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m} \cap \mathfrak{t}} \quad \text{and} \quad (\rho_{\mathfrak{t}} - 2\rho_{\mathfrak{t} \cap \mathfrak{t}})|_{\mathfrak{t}} = \rho_{\mathfrak{t} \cap \mathfrak{m}} - 2\rho_{\mathfrak{t} \cap \mathfrak{m} \cap \mathfrak{t}}$$

We conclude

$$d\Gamma_1|_{\mathfrak{t}} = (\bar{\gamma}|_{\mathfrak{t}} - \rho(u)) + \rho_{\mathfrak{t} \cap \mathfrak{m}} - 2\rho_{\mathfrak{t} \cap \mathfrak{m} \cap \mathfrak{t}} = \bar{\gamma}_1|_{\mathfrak{t}} + \rho_{\mathfrak{t} \cap \mathfrak{m}} - 2\rho_{\mathfrak{t} \cap \mathfrak{m} \cap \mathfrak{t}}$$

b) We have for any root α in $\Delta^+(\mathfrak{t}, \mathfrak{h})$, $\langle \alpha, \rho(u) \rangle = 0$ (Hence, this happens for every root in $\Delta(\mathfrak{t}, \mathfrak{h})$): since for each simple root α in $\Delta^+(\mathfrak{t}, \mathfrak{h})$,

$$1 = \langle \alpha, \rho \rangle = \langle \alpha, \rho_{\mathfrak{t}} + \rho(u) \rangle = \langle \alpha, \rho_{\mathfrak{t}} \rangle + \langle \alpha, \rho(u) \rangle = 1 + \langle \alpha, \rho(u) \rangle$$

c) In view of a), $(H, \Gamma_1, \bar{\gamma}_1, \Delta^+(\mathfrak{m} \cap \mathfrak{t}))$ is a pseudo-character for L . We will write it as $(H, \underline{\gamma}_1)$.

From b) we can deduce that for a root α in $\Delta_{\mathfrak{m}}^+$

$$\langle \alpha, \bar{\gamma} \rangle = 0 \text{ if and only if } \langle \alpha, \bar{\gamma}_1 \rangle = 0 \text{ and } \alpha \in \Delta^+(\mathfrak{m} \cap \mathfrak{t})$$

Therefore, by definition,

$(H, \underline{\gamma})$ is a regular character for G if and only if $(H, \underline{\gamma}_1)$ is a regular character for L

$(H, \underline{\gamma})$ is a limit character for G if and only if $(H, \underline{\gamma}_1)$ is a limit character for L .

Definition 5.2.5 (In view of remark (5.2.4)) A set $(\mathfrak{q}, H, \Gamma_1, \bar{\gamma}_1, \Delta^+(\mathfrak{m} \cap \mathfrak{t})) = (\mathfrak{q}, H, \underline{\gamma}_1)$ is called a set of θ -stable pseudo - data for G if the following three conditions are satisfied

a) $\mathfrak{q} = \mathfrak{t} + \mathfrak{u}$ is a θ -stable parabolic subalgebra of \mathfrak{g}

b) $(H, \Gamma_1, \bar{\gamma}_1, \Delta^+(\mathfrak{m} \cap \mathfrak{t}))$ is a pseudo-character data for L

c) \mathfrak{q} is associated to $\bar{\gamma}_1|_{\mathfrak{t}}$

If we require $(H, \underline{\gamma}_1)$ to be a limit character (respect. regular character) in b) for L , we will call $(\mathfrak{q}, H, \underline{\gamma}_1)$ a set of θ -stable limit data (respect. a set of θ -stable data) for G .

In analogy to theorem (5.1.2.2), we have:

Theorem 5.2.6 (Remark 5.2.4 and definition 5.2.5) There is a bijection between pseudo-characters for G and sets of θ -stable pseudo-data for G , which preserves conjugacy under K . This correspondence is given by (5.2.3); it associates sets of θ -stable limit data (respect. sets of θ -stable data) to limit characters (respect. regular characters). \square

For corresponding sets $(\mathfrak{q}, H, \Gamma_1, \bar{\gamma}_1, \Delta^+(\mathfrak{m} \cap \mathfrak{t})) = (\mathfrak{q}, H, \underline{\gamma}_1)$ and $(H, \underline{\gamma})$ for G under the bijection of theorem (5.2.5), we can build up a (\mathfrak{g}, K) -module in the same way as in (5.1).

Write $(H, \underline{\gamma}) = (H, \Gamma, \bar{\gamma}, \Delta_{\mathfrak{m}}^+)$ as in definition (5.2.1). Consider,

Let ω (respect. ω_1) be the discrete series or limit of discrete series representation of M (respect. $L \cap M$) associated to the parameter $\Gamma|_T$ (respect. $\Gamma_1|_T$). Let $P = MAN$ be a parabolic subgroup of G containing a minimal parabolic subgroup for which $\Gamma|_A$ is negative. Thus, $P_1 = P \cap L$ is a parabolic subgroup of L containing a minimal parabolic subgroup for which $\Gamma_1|_A$ is negative.

We have once more

$$\mathfrak{X}_q^i(X_L(P_1, \omega_1 \otimes \nu)) = \begin{cases} X_G(P, \omega \otimes \nu) & \text{if } i = S = \dim(\mathfrak{u} \cap \mathfrak{k}) \\ 0 & \text{otherwise} \end{cases}$$

Definition-Notation 5.2.7 We will call $X_G(P, \omega \otimes \nu)$ a representation *induced from a limit character*. To make explicit the limit character, we will write:

$$X_G(P, \omega \otimes \nu) = X_G(H, \underline{\gamma}) = \mathfrak{X}_q^S(X_L(P_1, \omega_1 \otimes \nu)) = \mathfrak{X}_q^S(X_L(H, \underline{\gamma}_1))$$

We will write

$$\Theta^G(H, \underline{\gamma}) = \Theta^G(H, \underline{\gamma}, \Delta_{im}^+) = \Theta^G(H, \Gamma, \bar{\gamma}, \Delta_{im}^+)$$

for the global character attached to $X_G(H, \underline{\gamma})$.

([V-II], thm. 4.4) Formally we can induce a global character for G (possibly zero) from a pseudo-character $(H, \underline{\gamma})$. Consider again a parabolic subgroup $P = MAN$ containing a minimal parabolic subgroup negative for $\Gamma_1|_A$. We associate to $\Gamma_1|_T$ a character Θ_M for M in the sense of Hecht and Schmid.

$Ind_P^G(\Theta_M \otimes \Gamma_1|_A \otimes 1)$ is a character for G .

This correspondence coincides with the previous one for limit characters. Thus, we write also $\Theta^G(H, \underline{\gamma})$ for this character.

Remark 5.2.8 What we are calling a pseudo-character (def. 5.2.1) agrees with the definition of pseudo-character in [V-II], but this is called a limit character in [A-B-V]. The representation induced from a pseudo-character (def. 5.2.1) is non-zero if and only if it is a limit character (def. 5.2.5); this can be found in [A-B-V], prop. 11.9.

The reason to include pseudo-characters in this subsection is to avoid handling cases, as in [V-II], in the formulation of the Schmid identities.

Let $(H, \Gamma, \bar{\gamma}, \Delta_{im}^+)$ be a pseudo-character for G (def. 5.2.1).

For a simple non-compact root $\beta \in \Delta_{im}^+$, denote $\tilde{\beta}$ for a real root corresponding to β under the Cayley transform $C_\beta()$.

Let $H^\beta = T^\beta A^\beta$ be the θ -stable Cartan subgroup obtained by the application of $C_\beta()$ to H . Recall $Z_{\tilde{\beta}}$ from definition (2.5). Then write $Z_\beta = C_\beta(Z_{\tilde{\beta}})$.

Hence if $\mathfrak{h}_\circ = \mathfrak{t}_\circ + \mathfrak{a}_\circ$ and $\mathfrak{h}_\circ^\beta = \mathfrak{t}_\circ^\beta + \mathfrak{a}_\circ^\beta$, we have

$$\mathfrak{t}_\circ = \mathfrak{t}_\circ^\beta + \mathbf{R}Z_{\tilde{\beta}} \text{ and } \mathfrak{a}_\circ^\beta = \mathfrak{a}_\circ + \mathbf{R}Z_\beta$$

Define $\bar{\gamma}^\beta \in (\mathfrak{h}^\beta)^*$ so that

$$\bar{\gamma}^\beta|_{\mathfrak{m}^\beta} = \bar{\gamma}|_{\mathfrak{m}^\beta}, \quad \bar{\gamma}^\beta|_{\mathfrak{a}} = \bar{\gamma}|_{\mathfrak{a}} \text{ and } \langle \bar{\gamma}^\beta, Z_\beta \rangle = \langle \bar{\gamma}, Z_{\tilde{\beta}} \rangle$$

Put, $T_1^\beta = T^\beta \cap T$.

From [Green], lemma 8.3.5, $T_1^\beta T_\circ = T$ and either $T^\beta/T_1^\beta \cong \mathbf{Z}/2\mathbf{Z}$ or $T_1^\beta = T^\beta$.

Thus, $\Gamma|_{T_1^\beta}$ is an irreducible character of T_1^β .

Similarly, the character Γ_1^β of $H_1^\beta = T_1^\beta A^\beta$, defined by:

$$\Gamma_1^\beta|_{T_1^\beta} = \Gamma|_{T_1^\beta} \quad \text{and} \quad \Gamma_1^\beta|_{A^\beta} = \exp(\bar{\gamma}^\beta|_{\mathfrak{a}_\beta})$$

is irreducible. Therefore, since $H^\beta/H_1^\beta \cong 1$ or $\mathbf{Z}/2\mathbf{Z}$, we have the following decomposition of irreducible H^β -modules (lemma 3.6):

$$\text{Ind}_{H_1^\beta}^{H^\beta}(\Gamma_1^\beta) = \Gamma^\beta \quad \text{or} \quad \Gamma_+^\beta + \Gamma_-^\beta$$

($\text{Ind}_{H_1^\beta}^{H^\beta}(\Gamma_1^\beta)$) is reducible if and only if $H^\beta/H_1^\beta \cong \mathbf{Z}/2\mathbf{Z}$ and for any x in H^β , $x \cdot \Gamma_1^\beta = \Gamma_1^\beta$.)

Next, write

$$\Delta_{im,\beta}^+ = \{ \alpha|_{\mathfrak{t}_\beta} \mid \alpha \in \Delta_{im}^+ \text{ and } \langle \alpha, \beta \rangle = 0 \}$$

Now, set, $(H^\beta, \underline{\gamma}^\beta) = (H^\beta, \Gamma^\beta, \bar{\gamma}^\beta, \Delta_{im,\beta}^+)$ or $(H^\beta, \underline{\gamma}_\pm^\beta) = (H^\beta, \Gamma_\pm^\beta, \bar{\gamma}^\beta, \Delta_{im,\beta}^+)$ accordingly to the decomposition above.

Then, we have obtained (a) new set(s) of pseudo-character data; either $(H^\beta, \underline{\gamma}^\beta)$ or $(H^\beta, \underline{\gamma}_\pm^\beta)$.

It is clear that if $(H, \underline{\gamma})$ is a limit character, the character (s) produced is (are) limit character (s).

Theorem 5.2.9 ([V-II], thm 4.4 Schmid identities) Suppose $(H, \Gamma, \bar{\gamma}, \Delta_{im}^+)$ is a pseudo-character for G . Let β is a simple non-compact root in Δ_{im}^+ with corresponding reflexion s_β . Consider the pseudo-character $(H, \underline{\gamma}) = (H, \Gamma - \beta, \bar{\gamma}, s_\beta(\Delta_{im}^+))$, then we have

$$\Theta^G(H, \underline{\gamma}) + \Theta^G(H, \tilde{\gamma}) = \Theta^G(H^\beta, \underline{\gamma}^\beta) \quad \text{or} \quad \Theta^G(H^\beta, \underline{\gamma}_+^\beta) + \Theta^G(H^\beta, \underline{\gamma}_-^\beta)$$

according to the decompositions above. □

Remark 5.2.10 We can ask when a global character $\Theta^G(H', \underline{\gamma}')$ attached to the limit character $(H', \underline{\gamma}')$ appears in the right hand side of a Schmid identity. In other words, we are asking when there is a global character $\Theta^G(H, \underline{\gamma})$ attached to a limit character $(H, \Gamma, \bar{\gamma}, \Delta_{im}^+)$ appearing on the left of a Schmid identity while $\Theta^G(H', \underline{\gamma}')$ appears on the right.

The first condition we need is $\underline{\gamma}' = \underline{\gamma}_+^\beta, \underline{\gamma}_-^\beta$ or $\underline{\gamma}^\beta$; for certain non-compact root $\beta \in \Delta_{im}^+$.

Suppose that $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is real and $\langle \alpha, \bar{\gamma}' \rangle = 0$. To find $(H, \Gamma, \bar{\gamma}, \Delta_{im}^+)$, we need the following compatibility condition ([S-V], Ch.V), called "*parity condition*" for $(H', \underline{\gamma}')$,

$$\Gamma(m_\alpha) = (-1)^n I \quad \text{where} \quad n = \langle \beta, \rho_m - 2\rho_{m \cap \mathfrak{t}} \rangle$$

(with notation previous to theorem (5.2.6), $\alpha = \tilde{\beta}$)

This, in fact, gives us all we need to describe irreducibility.

Definition 5.2.11 A limit character $(H, \underline{\gamma})$ (def. 5.2.1) is called *final* if whenever there is a real root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ orthogonal to $\bar{\gamma}$, it does not satisfy the parity condition (remark 5.2.10).

Theorem 5.2.12 ([A-B-V], prop. 11.18) If $(H, \underline{\gamma})$ is a final character (def. 5.2.11), the corresponding representation $X_G(H, \underline{\gamma})$ (def. 5.2.7) has a unique non-zero irreducible subrepresentation.

In particular, if $X_G(H, \underline{\gamma})$ is a tempered representation induced from a final character, it is irreducible. \square

Remark 5.2.13 Suppose $(H^\beta, \underline{\gamma}_1)$ is a pseudo -character for L and there is a simple non-compact root β such $\langle \beta, \bar{\gamma}_1 \rangle = 0$.

Then we can find pseudo-characters $(H^\beta, \underline{\gamma}_1^\beta)$ or $(H^\beta, \underline{\gamma}_1^\pm)$ for L as in theorem (5.2.11).

Then if $(\mathfrak{q}, H, \underline{\gamma}_1)$ is a set of θ -stable pseudo-data corresponding to $(H, \underline{\gamma})$, $(\mathfrak{q}, H^\beta, \underline{\gamma}_1^\beta)$ (respect. $(\mathfrak{q}, H^\beta, \underline{\gamma}_1^\pm)$) is (are) a set (s) of θ -stable pseudo-data corresponding to $(H^\beta, \underline{\gamma}^\beta)$ (respect. $(H^\beta, \underline{\gamma}^\pm)$).

Proposition 5.2.14 Consider a limit character $(H', \underline{\gamma}')$ and the corresponding set of θ -stable limit data $(\mathfrak{q}, H', \underline{\gamma}_1')$ under the bijection of theorem (5.2.6).

We have $(H', \underline{\gamma}')$ is a final character for G if and only if $(H', \underline{\gamma}_1')$ is a final character for L .

Proof By remark (5.2.4) c) there is a root $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}')$ such that

$$\langle \alpha, \bar{\gamma} \rangle = 0 \text{ if and only if } \langle \alpha, \bar{\gamma}'_1 \rangle = 0 \text{ and } \alpha \in \Delta^+(\mathfrak{l}, \mathfrak{h}').$$

Suppose $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}')$ is a real root such that $\langle \alpha, \bar{\gamma} \rangle = 0$.

Let $(H, \underline{\gamma}) = (H, \Gamma, \bar{\gamma}, \Delta^+)$ a limit character for G with corresponding set of θ -stable limit data $(\mathfrak{q}, H, \underline{\gamma}_1) = (\mathfrak{q}, H, \Gamma_1, \bar{\gamma}_1, \Delta^+(\mathfrak{m} \cap \mathfrak{l}))$. Remark (5.2.13) implies that $(H, \underline{\gamma})$ appears on the left hand side of a Schmid identity while $(H', \underline{\gamma}')$ appears on the right if and only if $(H, \underline{\gamma}_1)$ appears on the left hand side of a Schmid identity while $(H', \underline{\gamma}'_1)$ appears on the right.

Recall that (by thm. 5.2.6) $\Gamma|_{\mathcal{T}} = \Gamma_1|_{\mathcal{T}} \otimes (\Lambda^{\dim(\mathfrak{u} \cap \mathfrak{p})} \mathfrak{u} \cap \mathfrak{p})|_{\mathcal{T}}$.

We have noted in remark (5.2.4) a) the following identity

$$\rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m} \cap \mathfrak{l}}|_{\mathfrak{l}} = \rho_{\mathfrak{l} \cap \mathfrak{m}} - 2\rho_{\mathfrak{l} \cap \mathfrak{m} \cap \mathfrak{l}} + 2\rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u})|_{\mathfrak{l}}$$

Let β as in theorem (5.2.9) such that $\alpha = \tilde{\beta}$. Since $\langle \beta, \rho(\mathfrak{u}) \rangle = 0$, write

$$n = \langle \beta, \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m} \cap \mathfrak{l}}|_{\mathfrak{l}} \rangle = \langle \beta, \rho_{\mathfrak{l} \cap \mathfrak{m}} - 2\rho_{\mathfrak{l} \cap \mathfrak{m} \cap \mathfrak{l}} \rangle + \langle \beta, 2\rho(\mathfrak{u} \cap \mathfrak{p}) \rangle$$

Therefore,

$$\Gamma(\mathfrak{m}_\alpha) = (-1)^n I \text{ if and only if } \Gamma_1(\mathfrak{m}_\alpha) = (-1)^{n - \langle \beta, 2\rho(\mathfrak{u} \cap \mathfrak{p}) \rangle} I$$

Remark (5.2.10) implies that α satisfies the parity condition for $(H', \underline{\gamma}')$ if and only if it does for $(H', \underline{\gamma}'_1)$. This proves the proposition. \square

5.3 Back to fine representations

Assume in this subsection that $G = KAN$ is a quasisplit group (def. 2.7), $\delta \in \widehat{M}$ is a fine representation, $H = MA$ is a maximally split Cartan subgroup of G and ν is a character of A .

We have a set of limit character data $(H, \underline{\gamma}) = (H, \Gamma, \bar{\gamma}, \Delta_{im}^+)$ for G by putting $\Gamma = \delta \otimes \nu$, $\bar{\gamma} = (d\delta, \nu)$ and $\Delta_{im}^+ = \phi$.

Consider the root systems $\bar{\Delta}$ from definition (2.3) and $\bar{\Delta}_\delta$ from definition (2.15). This time we are more restrictive in choosing a positive system $\bar{\Delta}^+$ for $\bar{\Delta}$.

First, choose a positive system $\bar{\Delta}_\delta^+$ for $\bar{\Delta}_\delta$ such that it contains the intersection of $\bar{\Delta}_\delta$ with the following set

$$Y = \{ \alpha \in \bar{\Delta} \mid \text{either } \langle \alpha, Re(\nu) \rangle < 0 \text{ or } (\langle \alpha, Re(\nu) \rangle = 0 \text{ and } \langle \alpha, Im(\nu) \rangle < 0) \}$$

Hence, consider $\rho_\delta = \rho(\bar{\Delta}_\delta^+)$ (notation 1.2).

Choose $\bar{\Delta}^+$ containing the set

$$Y \cup \{ \alpha \in \bar{\Delta} \mid \langle \alpha, \rho_\delta \rangle > 0 \text{ and } \langle \alpha, \nu \rangle = 0 \}$$

In view of the comment made before definition (2.4), we get a minimal parabolic subgroup P corresponding to $\bar{\Delta}^+$. We note that P is in particular negative for ν (def. 4.1.11).

Notation 5.3.1 Consider $\bar{\Delta}^+$ satisfying the restriction we have just made. Let $\bar{\Delta}_1$ be the set of simple roots of $\bar{\Delta}^+$.

Recall the set $\bar{\Delta}_\delta$ from proposition (2.25). It might happen that ρ_δ is not dominant for the set $\bar{\Delta}^+$; hence $\bar{\Delta}^+ \cap \bar{\Delta}_\delta$ might not be contained in $\bar{\Delta}_1$.

Denote,

$$D = \{ \alpha \in \bar{\Delta}_1 \mid \langle \alpha, \bar{\gamma} \rangle = 0 \}$$

$$F = \{ \alpha \in \bar{\Delta}_1 \cap \bar{\Delta}_\delta \mid \langle \alpha, \bar{\gamma} \rangle = 0 \}$$

Consider the two p -pairs (P_D, A_D) and (P_F, A_F) such that $P_D \supseteq P_F \supseteq P$ with Langlands decompositions $P_D = M_D A_D N_D$ and $P_F = M_F A_F N_F$.

We have (δ, M_F) satisfies the assumptions in example (4.3.1) and (δ, M_D) is as in lemma (4.3.3).

Notation 5.3.2 To study K -fine representations we will need lemma (4.3.3) and the set E consider there. We can assume that $F \subseteq E$ by changing the positive system in lemma (4.3.3) by reflexions with respect to F if necessary (the resulting positive system still makes ρ_δ dominant).

Let (P_E, A_E) be the p -pair and let $P_E = M_E A_E N_E$ be the Langlands decomposition for P_E as in lemma (4.3.3). Then we have $M_F A_F \subseteq M_E A_E$.

Given $X \in \{E, F\}$, define

$H_X = T_X A_X$ a θ -stable Cartan subgroup arising by application of Cayley transforms to H with respect to X (assume $(T_F)_\theta \subseteq T_E$ and $A_E \subseteq A_F$).

We write, $K_X = K \cap M_X$, $M'_X = M' \cap M_X$, $M'_{T_X} = M' \cap T_X$, $R_X = M'_X/M$ and $R_{X,\delta} = M'_{X,\delta}/M$.

Denote,

$$A^X(\delta) = \{ \eta \in \widehat{M}_X \mid \eta \text{ is a } K_X\text{-fine representation and } \eta|_M \text{ contains } \delta \text{ as submodule} \}$$

(defs. 2.21 and 2.22)

For $\eta \in A^F(\delta)$, we write

$$A^E(\eta) = \{ \mu \in A^E(\delta) \mid \mu|_{M_F} \text{ contains } \eta \text{ as submodule} \}$$

Now we apply induction by stages once more (prop. 4.1.9). Thus consider (*):

$$\text{Ind}_P^G(\delta \otimes \nu) = \text{Ind}_{P_D}^G(\text{Ind}_{P_F \cap M_D}^{M_D}(\text{Ind}_{P \cap M_F}^{M_F}(\delta \otimes 1) \otimes 1) \otimes \nu_1) = \text{Ind}_{P_F}^G(\text{Ind}_{P \cap M_F}^{M_F}(\delta \otimes 1) \otimes \nu_F)$$

where $\nu_1 = \nu|_{A_D}$ and $\nu_F = \nu|_{A_F}$.

Substituting the set (δ, F, M_D) instead of the set (δ, E, G) in lemma (4.3.3), we get the following series of equalities (**):

$$\text{Ind}_P^G(\delta \otimes \nu) = m(\delta, \eta) \sum_{\eta \in A^F(\delta)} \text{Ind}_{P_D}^G(\text{Ind}_{P_F \cap M_D}^{M_D}(I_\eta \otimes 1) \otimes \nu_1) = m(\delta, \eta) \sum_{\eta \in A^F(\delta)} \text{Ind}_{P_F}^G(I_\eta \otimes \nu_F)$$

from (*) (and notation 5.3.2).

(5.3.3).- Fix $\eta \in A^F(\delta)$.

Fix any irreducible T_F -module η_o occurring in $\eta|_{T_F}$.

Suppose $(H, \underline{\gamma})$ is the limit character given at the beginning of this subsection.

We define a limit character $(H_F, \underline{\gamma}_F)$ for G as follows:

Let $(H_F, \underline{\gamma}_F) = (H_F, \Gamma_F, \overline{\gamma}_F, \Delta_{F,im}^+)$ given by

i) Γ_F is the irreducible H_F -module determined by

$$\Gamma_F|_{T_F} = \eta_o \text{ and } \Gamma_F|_{A_F} = \Gamma|_{A_F} = \nu_F$$

ii) $\overline{\gamma}_F \in \mathfrak{h}_F^*$ is given by $\overline{\gamma}_F|_{\mathfrak{t}} = d\delta$, $\overline{\gamma}_F|_{\mathfrak{t}^\perp} \equiv 0$ and $\overline{\gamma}_F|_{\mathfrak{a}_F} = \nu_F$ where $\mathfrak{t}_F = \mathfrak{t} + \mathfrak{t}^\perp$

iii) $\Delta_{F,im}^+$ is the unique positive system of $\Delta(\mathfrak{m}_F, \mathfrak{t}_F)$ such that $d\eta_o$ is dominant.

From (**) we first note that $X_G(P_F, I_\eta \otimes \nu_F)$, the Harish-Chandra module associated to $\text{Ind}_{P_F}^G(I_\eta \otimes \nu_F)$, is the representation induced from the limit character $(H_F, \underline{\gamma}_F)$ (notation 4.1.9).

Lemma 5.3.4 In the notation above $X_G(P_F, I_\eta \otimes \nu_F) = X_G(H_F, \underline{\gamma}_F)$ has a unique irreducible (\mathfrak{g}, K) -submodule.

Proof: We note that in (**) $\text{Ind}_{P_D}^G(\text{Ind}_{P \cap M_D}^{M_D}(I_\eta \otimes 1) \otimes \nu_1) = \text{Ind}_{P_F}^G(I_\eta \otimes \nu_F)$.

Denote $Y_D = X_{M_D}(P \cap M_D, I_\eta \otimes 1)$. It is irreducible by the last part in the proof of lemma (4.3.3). Hence, $X_G(P_D, Y_D \otimes \nu_1) = X_G(P_F, I_\eta \otimes \nu_F)$. By the Langlands classification, $X_G(P_D, Y_D \otimes \nu_1)$ contains a unique irreducible submodule.

(To explain this more carefully we apply induction by stages as follows.

Let $D_1 = \{ \alpha \in \overline{\Delta}_1 \mid \langle \alpha, \text{Re}(\nu) \rangle = 0 \}$ and let (P_{D_1}, A_{D_1}) be the corresponding p -pair with Langlands decomposition $P_{D_1} = M_{D_1} A_{D_1} N_{D_1}$ such that $P_{D_1} \supseteq P_D$.

Then (***):

$$\text{Ind}_{P_D}^G(\text{Ind}_{P_F \cap M_D}^{M_D}(I_\eta \otimes 1) \otimes \nu_I) = \text{Ind}_{P_{D_1}}^G(\text{Ind}_{P_D \cap M_{D_1}}^{M_{D_1}}(\text{Ind}_{P_F \cap M_D}^{M_D}(I_\eta \otimes 1) \otimes \nu_I) \otimes \nu_R)$$

where $\nu_I = \nu|_{A_D \cap M_{D_1}}$, $\nu_R = \nu|_{A_{D_1}}$ and $\nu_F = \nu|_{A_F}$.

Since for any root $\alpha \in \Delta(P_D \cap M_{D_1}, A_D \cap M_{D_1})$ (def. 4.1.4) α is not orthogonal to ν_I , $\text{Ind}_{P_D \cap M_{D_1}}^{M_{D_1}}(\text{Ind}_{P_F \cap M_D}^{M_D}(I_\eta \otimes 1) \otimes \nu_I)$ is an irreducible tempered representation by a result of Harish-Chandra in [H-Ch] as in [Green] (4.2.2). So the right hand side of (***) contains a unique irreducible submodule by comment after definition (4.1.11). The lemma follows. \square

Remark 5.3.5 a) The limit character $(H_F, \underline{\gamma}_F)$ in notation 5.3.3 is not necessarily final (def. 5.2.11). However, $(H_F, \underline{\gamma}_F)$ behaves as final character in the sense of theorem (5.2.12); lemma (5.3.4). Exhausting all the real roots orthogonal to $\overline{\gamma}_F$ that satisfy the parity condition, we can get to a final character $(H_{\mathfrak{F}}, \underline{\gamma}_{\mathfrak{F}})$ using the Schmid identities (thm. 5.2.9), but we cannot get more reducibility; that is $X_G(H_{\mathfrak{F}}, \underline{\gamma}_{\mathfrak{F}}) = X_G(H_F, \underline{\gamma}_F)$.

b) By Frobenius reciprocity theorem, the set of minimal K -types for $X_G(H_F, \underline{\gamma}_F)$ is

$$A(\eta) = \{ \mu \in A(\delta) \mid \mu|_{M_F} \text{ contains } \eta \text{ as submodule} \}$$

with $\eta \in A^F(\delta)$. Each appears with multiplicity $m(\eta, \mu)$ for any $\mu \in A(\eta)$.

For any $\eta \in A^F(\delta)$ let η_o be as in notation (5.3.3).

Substituting (δ, η_o, η) for (δ, μ_o, μ) in the proof of lemma (4.3.2) we note that the R -group \widehat{R}_F acts on $A^F(\delta)$ and by restriction to T_E on the set

$$\{ \eta_o \in M'_{T_F} \mid \eta_o \text{ occurs in } \eta|_{M'_{T_F}} \text{ for some } \eta \in A^F(\delta) \}$$

By the same lemma (4.3.2), for any $\eta \in A^F(\delta)$ and any irreducible T_F -module η_o of $\eta|_{T_F}$ $A^F(\delta) = \widehat{R}_F \cdot \eta = \{ \text{Ind}_{T_F}^{K_F}(\chi \cdot \eta_o) \mid \chi \in \widehat{R}_F \}$.

Next, we want to extend the action of the R -group to sets of limit characters.

Definition 5.3.6 Let Ω denote the following set

$$\{ (H_F, \underline{\gamma}_F) \mid (H_F, \underline{\gamma}_F) \text{ is constructed as in (5.3.3) for some } \eta \in A^F(\delta) \}$$

Define an action of \widehat{R}_F on Ω as follows: for any $(H_F, \underline{\gamma}_F) \in \Omega$ and $\chi \in \widehat{R}_F$

$$\chi \cdot (H_F, \underline{\gamma}_F) = \chi \cdot (H_F, \Gamma_F, \overline{\gamma}_F, \Delta_{F,im}^+) = (H_F, \Gamma'_F, \overline{\gamma}_F, \chi \cdot \Delta_{F,im}^+)$$

where $\Gamma'_F|_{T_F} = \chi \cdot \Gamma_F|_{T_F}$ and $\Gamma'_F|_{A_F} = \nu_F$ and $\chi \cdot \Delta_{F,im}^+$ is the unique positive system such that $d(\chi \cdot \Gamma_F)$ is dominant.

Since the construction of $X_G(H'_F, \underline{\gamma}'_F)$ depends only on $(H'_F, \underline{\gamma}'_F)$ up to conjugation by K , given $(H_F, \underline{\gamma}_F) \in \Omega$ and $\chi \in \widehat{R}_F$

$$X_G(\chi \cdot (H_F, \underline{\gamma}_F)) = X_G(H_F, \underline{\gamma}_F) \text{ if and only if } \chi \cdot \Gamma_F|_{T_F} = x \cdot \Gamma_F|_{T_F} \text{ for some } x \in M'_F.$$

That is the case if and only if $\chi \cdot \eta = \eta$; where $\eta = \text{Ind}_{T_F}^{K_F}(\Gamma_F|_{T_F})$.
 (Apply sublemma i) in the proof of lemma (4.3.2) to (δ, η_o, η) for (δ, μ_o, μ) .)

Definition 5.3.7 For any $\bar{\chi} \in \widehat{R}_F/\widehat{R}_{F,\eta}$, define $\bar{\chi} \cdot X_G(H_F, \underline{\gamma}_F) = X_G(\chi \cdot (H_F, \underline{\gamma}_F))$
 (this does not depend on the representative χ of $\bar{\chi}$ in \widehat{R}_F).

Let $\bar{X}_G(H_F, \underline{\gamma}_F)$ be the unique irreducible (\mathfrak{g}, K) -subrepresentation of $X_G(H_F, \underline{\gamma}_F)$. We define $\bar{\chi} \cdot \bar{X}_G(H_F, \underline{\gamma}_F)$ as the unique irreducible (\mathfrak{g}, K) -subrepresentation of $\bar{\chi} \cdot X_G(H_F, \underline{\gamma}_F)$.

We have just proved the following proposition (decomposition (**)):

Proposition 5.3.8 Consider (H, γ) as in the beginning of the subsection. For fixed limit character $(H_F, \underline{\gamma}_F) \in \Omega$ (def. 5.3.6). We have the following decomposition (def. 5.3.7):

$$X_G(\delta \otimes \nu) = m(\delta, \eta) \sum_{\bar{\chi} \in \widehat{R}_F/\widehat{R}_{F,\eta}} X_G(\chi \cdot (H_F, \underline{\gamma}_F))$$

where χ can be any representative of $\bar{\chi}$ in \widehat{R}_F . □

We want to give the decomposition in proposition (5.3.8) in terms of the R -group \widehat{R}_δ (def. 2.24) and the set of K -fine representations $A(\delta)$ (defs. 2.21 and 2.22). To this end, recall in the proof of lemma (4.3.2) that there is a bijection ψ between $A(\delta)$ and $A^E(\delta)$ and that the restriction from \widehat{R}_δ to $\widehat{R}_{E,\delta}$ is an isomorphism of groups.

In particular we note $A^E(\eta) = \psi(A(\eta))$ (notation 5.3.2).

The groups Q and Q_o from lemma (3.8) are going to be necessary too.

Notation 5.3.9 In the setting of notation (5.3.2) and in view of the isomorphism between \widehat{R}_δ to $\widehat{R}_{E,\delta}$, set

$$\begin{aligned} Q(\delta) &= \{ \chi \in \widehat{R}_\delta \mid \chi|_{M'_{F,\delta}} \cdot \eta = x \cdot \eta \text{ for some } x \in M'_{E,\delta} \} \\ Q(\delta)_o &= \{ \chi \in \widehat{R}_\delta \mid \chi|_{M'_{F,\delta}} \cdot \eta = \eta \} \end{aligned}$$

for any $\eta \in A^F(\delta)$.

We also denote, $Q(\delta)^E = \{ \chi|_{M'_{E,\delta}} \mid \chi \in Q(\delta) \}$ and $Q(\delta)_o^E = \{ \chi|_{M'_{E,\delta}} \mid \chi \in Q(\delta)_o \}$

Now choose $\mu \in A^E(\delta)$ and $\eta \in A^F(\delta)$ such that $(\delta, \eta_\delta, \mu_\delta, M, M'_{F,\delta}, M'_{E,\delta})$ can substitute the role of $(\delta, \eta, \mu, B, C, A)$ in lemma (3.8).

In the proof of lemma (3.8), $R_{E,\delta} = R$, $R_{F,\delta} = S$, $Q(\delta)^E = Q$ and $Q(\delta)_o^E = Q_o$.

For any $\chi \in \widehat{R}$ write $\bar{\chi}$ for its class in $\widehat{R}_{E,\delta}/Q(\delta)_o^E$ and $\bar{\bar{\chi}}$ for its class in $\widehat{R}_{E,\delta}/Q(\delta)^E$.

Thus, we have obtained:

Proposition 5.3.10 Recall the definition of $A^E(\eta)$ (notation (5.3.2) and def. 2.22).

a) For any $\chi \in \widehat{R_{E,\delta}}$ denote $\chi| = \chi|_{M'_{F,\delta}}$. Then the following conditions are equivalent

- i) $A^E(\chi| \cdot \eta) \cap A^E(\eta) \neq \phi$
- ii) $A^E(\chi| \cdot \eta) = A^E(\eta)$
- iii) $\chi \in Q(\delta)^E$

b) For any $\bar{\chi} \in \widehat{R_{E,\delta}/Q(\delta)^E}$, define $\bar{\chi} \cdot A^E(\eta) = A^E(\chi| \cdot \eta)$.

In view of a), there is not ambiguity in the definition. Moreover, we have the following disjoint union

$$A^E(\delta) = \bigsqcup_{\bar{\chi} \in \widehat{R_{E,\delta}/Q(\delta)^E}} \bar{\chi} \cdot A^E(\eta)$$

□

In the same way we get in proposition (5.3.9) the decomposition of $A^E(\delta)$, we get

$$A(\delta) = \bigsqcup_{\bar{\chi} \in \widehat{R_\delta/Q(\delta)}} \bar{\chi} \cdot A(\eta)$$

using the bijection between $A(\delta)$ and $A^E(\delta)$ and the isomorphism between $\widehat{R_\delta}$ and $\widehat{R_{E,\delta}}$.

Now, from proof of lemma (3.8), $\widehat{R_\delta/Q(\delta)}_o \cong \widehat{R_{F,\delta}/(\widehat{R_{F,\delta}})_\eta}$. Corollary (3.7) implies that $\widehat{R_\delta/Q(\delta)}_o \cong \widehat{R_F/R_{F,\eta}}$ (we are using the equivalence between (def. 2.21) and (def. 2.22)).

Thus we get the following decomposition of $X_G(H, \underline{\gamma})$, with $(H, \underline{\gamma})$ as in notation (5.3.3), and its Langlands subrepresentation $J(H, \underline{\gamma})$ (def. 5.1.1.2).

Theorem 5.3.11 Consider $(H, \underline{\gamma})$ as in the beginning of the subsection. For fixed limit character $(H_F, \underline{\gamma}_F) \in \Omega$ (def. 5.3.6) and in the notation (5.3.8), we have

a)

$$X_G(H, \underline{\gamma}) = m(\delta, \eta) \sum_{\bar{\chi} \in \widehat{R_\delta/Q(\delta)}_o} \bar{\chi} \cdot X_G(H_F, \underline{\gamma}_F)$$

$$J(H, \underline{\gamma}) = m(\delta, \eta) \sum_{\bar{\chi} \in \widehat{R_\delta/Q(\delta)}_o} \bar{\chi} \cdot \bar{X}_G(H_F, \underline{\gamma}_F)$$

b) the set of minimal K -types of $\bar{\chi} \cdot X_G(H_F, \underline{\gamma}_F)$ (respect. of $\bar{\chi} \cdot \bar{X}_G(H_F, \underline{\gamma}_F)$) is the set $\bar{\chi} \cdot A(\eta)$

c) $\bar{\chi} \cdot A(\eta) = A(\eta)$ if and only if $\bar{\chi} = 1$

d) $\bar{\chi} \cdot X_G(H_F, \underline{\gamma}_F) \cong X_G(H_F, \underline{\gamma}_F)$ if and only if $\bar{\chi} = 1$

e) The multiplicity of $\bar{\chi} \cdot X_G(H_F, \underline{\gamma}_F)$ in $X_G(\delta \otimes \nu)$ is $m(\delta, \eta)$

f) $m(\delta, \eta) = 2^m$ for some $m \in \mathbf{N}$

Proof: a) is proposition (5.3.8) and lemma (5.3.4). Remark (5.3.5) b) implies b).

c) is proposition (5.3.10) and d) is the comment previous to definition (5.3.8). e) is obvious. Corollary (2.28) and lemma (3.6) imply f). The theorem follows. \square

5.4 Classification in the general case

To begin with, let $(H, \underline{\gamma})$ be a regular character with corresponding set of θ -stable limit character $(\mathfrak{q}, H, \underline{\gamma}_1)$ under the bijection in theorem (5.2.3). By remark (5.2.4) $(H, \underline{\gamma}_1)$ is a regular character for L .

As in subsection 5.3, we can find a strongly orthogonal set $F \subset \Delta(\mathfrak{t}, \mathfrak{h}) (\subseteq \Delta(\mathfrak{g}, \mathfrak{h}))$ of real roots. With respect to $(H, \underline{\gamma}_1)$, we construct limit characters $(H_F, \underline{\gamma}_{1F})$ for L as in notation (5.3.3).

Apply again theorem (5.2.6) to get the limit characters $(H_F, \underline{\gamma}_F)$ for G corresponding to all the sets of θ -stable limit data $(\mathfrak{q}, H_F, \underline{\gamma}_{1F})$.

Therefore, we have

$$\mathfrak{R}_{\mathfrak{q}}^S(X_L(H_F, \underline{\gamma}_{1F})) = X_G(H_F, \underline{\gamma}_F)$$

for any two corresponding elements $(H_F, \underline{\gamma}_F)$ and $(\mathfrak{q}, H_F, \underline{\gamma}_{1F})$ ($S = \dim(\mathfrak{u} \cap \mathfrak{p})$, def. 5.2.7).

Suppose $\delta = \Gamma_1|_T$ and $\nu = \Gamma_1|_A$

We write $A^L(\delta)$ for the set of minimal $L \cap K$ -types occurring in $X_L(H, \underline{\gamma}_1)$.

As an application of corollary (4.2.7), write $A^K(\mathfrak{q}, \delta)$ for the K -irreducible representations arisen from the set $\{\pi \otimes \Lambda^{\dim \mathfrak{u} \cap \mathfrak{p}} \mathfrak{u} \cap \mathfrak{p} \mid \pi \in A^L(\delta)\}$ consisting of $L \cap K$ -irreducible representations.

Now, for $(H_F, \underline{\gamma}_{1F}) = (H_F, \Gamma_{1F}, \bar{\gamma}_{1F}, \Delta_{F,im}^{\dagger})$ denote $\eta_o = \Gamma_{1F}|_T$ and $\eta = \text{Ind}_{T_F}^{K_F}(\eta_o)$.

Since we want to apply everything has been said about fine representations to the quasi-split group L and the fine representation δ , particularly the results from subsection (5.3), we complicate the notation a little bit. Denote $\widehat{R}_{\delta}(L)$, $Q(\delta, L)$ and $Q(\delta, L)_o$ for the corresponding groups \widehat{R}_{δ} , $Q(\delta)$ and $Q(\delta)_o$ in (5.3).

Proposition (5.3.3) implies again

$$A^L(\delta) = \bigsqcup_{\bar{\chi} \in \widehat{R}_{\delta}/Q(\delta, L)} \bar{\chi} \cdot A^L(\eta)$$

Write $A^K(\eta)$ for the subset of $A^K(\mathfrak{q}, \delta)$ that arises from elements of $A^L(\delta)$ multiplied by $\Lambda^{\dim(\mathfrak{u} \cap \mathfrak{p})} \mathfrak{u} \cap \mathfrak{p}$, as above.

$A^K(\mathfrak{q}, \delta)$ happens to be the set of lambda-minimal K -types of $X_G(H, \underline{\gamma})$ (def. 4.2.5). This is proved in [Green], theorem 6.5.9. Moreover, we can deduce from there that $A^K(\eta)$ is the set of minimal K -types of $X_G(H_F, \underline{\gamma}_F)$

We know from lemma (5.3.4) that $X_L(H_F, \underline{\gamma}_{1F})$ has a unique irreducible $(\mathfrak{t}, L \cap K)$ -submodule.

By a similar argument for the proof of lemma (5.3.4) (or by [Green],theorem 6.5.10 d)) $X_G(H_F, \underline{\gamma}_F)$ has a unique irreducible (\mathfrak{g}, K) -submodule that we denote by $\overline{X}_G(H_F, \underline{\gamma}_F)$.

As in proposition (5.3.10), we adopt the following notation : For every element χ in $\widehat{R}_\delta(L)$, we write

$\overline{\chi}$ for its corresponding class in $\widehat{R}_\delta(L)/Q(\delta, L)_o$ and $\overline{\overline{\chi}}$ for its corresponding class in $\widehat{R}_\delta(L)/Q(\delta, L)$.

Notation 5.4.1 Now we extend the action of $\widehat{R}_\delta(L)$ in theorem (5.3.11) for the quasisplit group L to the general group G .

$\forall \overline{\chi} \in \widehat{R}_\delta(L)/Q(\delta, L)_o$, define

$$\overline{\chi} \cdot X_G(H_F, \underline{\gamma}_F) = \mathfrak{A}_q^S(\overline{\chi} \cdot X_L(H_F, \underline{\gamma}_{1F}))$$

and $\overline{\overline{\chi}} \cdot \overline{X}_G(H_F, \underline{\gamma}_F)$ to be the unique irreducible (\mathfrak{g}, K) -submodule of $\overline{\chi} \cdot X_G(H_F, \underline{\gamma}_F)$.

$\forall \overline{\overline{\chi}} \in \widehat{R}_\delta(L)/Q(\delta, L)$, define

$\overline{\overline{\chi}} \cdot A^K(\eta)$ to be the subset of $A^K(\mathfrak{q}, \delta)$ corresponding to $\overline{\overline{\chi}} \cdot A^L(\eta)$ under the bijection between $A^K(\mathfrak{q}, \delta)$ and $A^L(\delta)$ established above.

Therefore, we have once more a disjoint union

$$A^K(\mathfrak{q}, \delta) = \bigsqcup_{\overline{\overline{\chi}} \in \widehat{R}_\delta/Q(\delta, L)} \overline{\overline{\chi}} \cdot A^K(\eta)$$

from proposition (5.3.10).

The following result describes the decomposition of the Langlands subrepresentations (def. 5.1.1.2).

Theorem 5.4.2 Assume $(H, \bar{\gamma})$ is a regular for G . Let $(H_F, \underline{\gamma}_F)$ be any limit character constructed in the way mentioned at the beginning of this subsection with respect to $(H, \bar{\gamma})$ and some η .

a)

$$X_G(H, \underline{\gamma}) = m(\delta, \eta) \sum_{\bar{\chi} \in \widehat{R}_\delta(L)/Q(\delta, L)_\circ} \bar{\chi} \cdot X_G(H_F, \underline{\gamma}_F)$$

$$J(H, \underline{\gamma}) = m(\delta, \eta) \sum_{\bar{\chi} \in \widehat{R}_\delta(L)/Q(\delta, L)_\circ} \bar{\chi} \cdot \bar{X}_G(H_F, \underline{\gamma}_F)$$

b) The set of minimal K -types of $\bar{\chi} \cdot X_G(H_F, \underline{\gamma}_F)$ (respect. $\bar{\chi} \cdot \bar{X}_G(H_F, \underline{\gamma}_F)$) is $\bar{\chi} \cdot A^K(\eta)$

c) $\bar{\chi} \cdot A^K(\eta) = A^K(\eta)$ if and only if $\bar{\chi} = 1$

d) $\bar{\chi} \cdot X_G(H_F, \underline{\gamma}_F) \cong X_G(H_F, \underline{\gamma}_F)$ if and only if $\bar{\chi} = 1$

e) the multiplicity of $\bar{\chi} \cdot X_G(H_F, \underline{\gamma}_F)$ in $X_G(H, \underline{\gamma})$ is $m(\delta, \eta)$

f) $m(\delta, \eta) = 2^m$ for some $m \in \mathbf{N}$ □

By proposition (5.2.13) and remark (5.3.5) a) $(H_F, \underline{\gamma}_F)$ in theorem (5.4.2) might not be a final character, however it behaves like such in the sense of theorem (5.2.12).

Corollary 5.4.3 We have obtained the classification of tempered representations (5.1.2.3)

Proof: Assume that the regular character $(H, \bar{\gamma})$ in theorem (5.4.2) is such that $X_G(H, \underline{\gamma})$ is a standard tempered representation. In this case, $X_G(H, \underline{\gamma}) = J(H, \underline{\gamma})$. Theorem (5.4.2) gives a decomposition for it. This proves our assertion. □

REFERENCES

- [A-B-V]- Adams, J., Barbasch, D. and Vogan, D. A. The Langlands classification and irreducible characters for real reductive groups.
Progress in Math. Birkhäuser, Boston - to appear.
- [B-W]- Borel, A. and Wallach, N. Continuous cohomology, discrete subgroups, and representations of reductive groups. Princeton University Press, Princeton, 1980.
- [C-R]- Curtis, W. C. and Reiner, I. Representation theory of finite groups and associative algebras. Interscience (Wiley) New York, 1962.
- [Green]- Vogan, D. A. Representations of real reductive Lie groups,
Birkhäuser, Boston 1981.
- [H]- Helgason, S. Differential Geometry and Symmetric Spaces, Academic Press,
New York, 1962.
- [H-Ch]- Harish-Chandra, Representations of semi-simple Lie groups III,
Trans. Amer. Math. Soc. 76 (1954), 234-253.
- [K]- Knapp, A. W. Representation theory of semi-simple groups "An overview
based on examples". Princeton University Press, Princeton 1986.
- [K-Z]- Knapp, A. W. and Zuckerman, G.J. Classification of irreducible tempered representa-
tions of
semi-simple Lie groups. Ann. of Math. 116 (1982), 389-501.
- [S-V]- Speh, B. and Vogan, D. A. Reducibility of generalized principal series
representations, Acta Math. 145 (1980), 227-299.
- [V-I]- Vogan, D. A. The algebraic structure of the representations of semi-simple Lie groups
I,
Ann. of Math. 109 (1976a) 1-60.
- [V-II]- Vogan, D. A. Irreducible characters of semi- simple Lie groups I.
Duke Math. J 46 (1979), 61-108
- [W]- Wallach, N. Real reductive groups I, Academic Press, Inc. (1988).