On Shellings and Subdivisions of Convex Polytopes

by

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Abstract

Chapter 1 deals with shellings of convex polytopes and polyhedral complexes. R. Stanley first defined the generalized toric h-vector, a fundamental combinatorial invariant of polyhedral complexes (and more general objects). When the complex is simplicial, it is known that this invariant can sometimes be computed by shelling, or taking apart the complex in a certain order. We show how any shellable complex with cubical facets can be dealt with analogously. Based on a result of L. Shapiro we formulate the h-vector of any shellable cubical complex in terms of certain classes of plane trees.

In Chapter 2 we study subdivisions of simplicial complexes. Stanley defined local h-vectors, called l-vectors, to investigate the behavior of h-vectors of simplicial complexes under subdivision. The l-vectors of a large class of simplex subdivisions have certain useful properties. We show that these properties characterize such l-vectors. We also define refined l-vectors (for vertex-colored subdivisions) and show that they have analogous properties.

Chapter 3 is an application of polytope geometry to the enumeration of graphs and degree sequences. To give a combinatorial proof of Stanley's enumeration of quasiforests in terms of degree sequences, we derive a correspondence between quasiforests on n vertices and degree sequences of graphs on n vertices by constructing a canonical decomposition of the appropriate convex polytope into half-open cubes. Using the same method we formulate a bijective map from the set of forests on n vertices to the set of score vectors of tournaments on n vertices.

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Chapter 1

Plane Trees and H-vectors of Shellable Cubical Complexes

1.1 Introduction

R. Stanley first defined the generalized toric h-vector, a fundamental combinatorial invariant of polyhedral complexes (and more general objects). In the case where the complex is simplicial, this invariant can be computed by shelling, or taking apart the complex in a certain order. In this chapter we show how any shellable complex with cubical facets can be dealt with analogously. Based on a result of L. Shapiro we formulate the h-vector of any shellable cubical complex in terms of certain classes of plane trees.

1.2 Background

Let Δ be a *pure* $(d-1)$-dimensional simplicial complex, *i.e.*, its maximal faces $F_1, F_2, \ldots, F_k$ all have dimension $(d-1)$. Sometimes we will denote Δ by $\langle F_1, F_2, \ldots, F_k \rangle$. For each i, let $f_i = f_i(\Delta)$ be the number of i-dimensional faces in Δ. By convention, $f_{-1} = 1$ unless $\Delta = \emptyset$, in which case $f_{-1} = 0$. Then the *f-vector* of Δ is $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$. A linear transformation of this vector is the *h-vector*

$h(\Delta) = (h_0, h_1, \ldots, h_d)$, which is defined by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

It is easy to show that

$$h(\Delta, x) = \sum_{i=0}^{d} h_i x^i = \sum_{F \in \Delta} x^{\#F}(1-x)^{d-\#F}. \qquad (1)$$

The h-vector is often easier to handle than the f-vector, so since knowing the h-vector is equivalent to knowing the f-vector, much research has focussed on the h-vector. (See Chapter 2 for algebraic significance of h-vectors.) Some examples of the convenient properties of h-vectors are outlined below.

Proposition 1.2.1 *If Δ and Γ are simplicial complexes then their* simplicial join *$\Delta * \Gamma$ is the simplicial complex with maximal faces $F \cup G$, where F and G are maximal faces of Δ and Γ, respectively. Then $h(\Delta * \Gamma, x) = h(\Delta, x) \cdot h(\Gamma, x)$.*

Proof: This is immediate from (1). □

For *shellable* simplicial complexes, the h-vector has a simple combinatorial interpretation.

Definition A *shelling* of Δ is an ordering of its maximal faces $F_1, F_2, \ldots, F_k$ such that for each i, $\langle F_i \rangle \cap \langle F_1, F_2, \ldots, F_{i-1} \rangle$ is a pure $(d-2)$-dimensional complex. If such an ordering exists, then Δ is called *shellable.*

Example Let Δ be the boundary complex of a tetrahedron with vertices $1, 2, 3, 4$. Then any ordering of its maximal faces $\{123\}, \{124\}, \{134\}, \{234\}$ is a shelling of Δ.

More generally, we have the following result, due to Brugesser and Mani ([BM]):

Theorem 1.2.2 *The boundary complex of any convex simplicial polytope is shellable.*

Now we have McMullen's interpretation of h-vectors of shellable polytopes (see [McM70, p. 182]), which holds for shellable simplicial complexes in general. For the sake of completeness, we include the proof.

Proposition 1.2.3 *Let $F_1, F_2, \ldots, F_k$ be a shelling of Δ. For each i, define s_i to be the number of facets F_j such that $\langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$ has i maximal faces. Then $h_i(\Delta) = s_i$.*

Proof: Suppose $\langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$ has i maximal faces. Then the contribution of F_j to $h(\Delta, x)$ with respect to the given shelling is

$$\sum_{f \in \langle F_j \rangle \setminus \langle F_1, \ldots, F_{j-1} \rangle} x^{\#f}(1-x)^{d-\#f},$$

Since $\langle F_j \rangle \setminus \langle F_1, \ldots, F_{j-1} \rangle$ contains exactly $\binom{d-i}{r}$ faces of dimension $i + r - 1$ for each $0 \leq r \leq d - i$, the contribution is just x^i, by the binomial theorem. Thus $h(\Delta, x) = \sum_{i=0}^{d} s_i x^i$. □

Example For the shelling $\{123\}, \{124\}, \{134\}, \{234\}$ of the boundary of the tetrahedron with vertices $1, 2, 3, 4$ we have

$$\begin{aligned}
\langle F_1 \rangle \cap \emptyset &= \emptyset, \\
\langle F_2 \rangle \cap \langle F_1 \rangle &= \langle \{12\} \rangle, \\
\langle F_3 \rangle \cap \langle F_1, F_2 \rangle &= \langle \{13\}, \{14\} \rangle, \\
\langle F_4 \rangle \cap \langle F_1, F_2, F_3 \rangle &= \langle \{23\}, \{24\}, \{34\} \rangle. \\
\text{Thus} \quad h(\Delta, x) &= 1 + x + x^2 + x^3.
\end{aligned}$$

Note that in this example the h-vector was symmetric. This is true for the h-vector of the boundary of any simplicial polytope, and for a larger class of simplicial complexes called *Euler complexes*, defined below.

Definition: The *reduced Euler characteristic* of Δ is

$$\tilde{\mathcal{X}}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i.$$

This is a topological invariant, so it depends only on the geometric realization of Δ (see [Stan86, Section 3.8]). For each $F \in \Delta$, the *link* of F with respect to Δ is the simplicial complex $\langle G : G \cup F \in \Delta,\ G \cap F = \emptyset \rangle$. A pure simplicial complex Δ is an *Euler complex* if for all $F \in \Delta$,

$$\tilde{\mathcal{X}}(\mathrm{lk}_\Delta F) = (-1)^{\dim(\mathrm{lk}_\Delta F)}.$$

Example If Δ can be geometrically realized as the $(d-1)$-sphere $\mathbf{S}^{d-1}$, then Δ is an Euler complex.

Theorem 1.2.4 *Let Δ be an Euler complex with h-vector $(h_0, h_1, \ldots, h_d)$. Then we have the* Dehn-Sommerville Equations, *$h_i = h_{d-i}$, for all i.*

Note: The Dehn-Sommerville Equations are the most general linear relations to hold among the components of h-vectors of Euler complexes, and hence translate into the most general linear relations among the components of f-vectors of Euler complexes, but not nearly so elegantly. (See [MS, Section 2.4], [Stan86, p. 151].)

R. Stanley was able to generalize the definition of h-vector to the boundary complex of any convex polytope (not just simplicial) so that the Dehn-Sommerville equations still hold. The ***generalized h-vector***, in fact, is defined for a larger class of objects called ***Eulerian posets***, defined below.

Definitions: A ***poset*** is a partially ordered set. If x is less than any other element of P then x is called $\hat{0}$. If y is greater than any other element of P then y is called $\hat{1}$. A poset P with $\hat{0}$ and $\hat{1}$ is ***graded*** if every maximal chain in P has the same length. If P is graded, the ***rank*** $r(x)$ of $x \in P$ is the maximal length of a chain from 0 to x. The Möbius function μ of a graded poset P is given by

1. $\mu(x, x) = 1$, for any $x \in P$; and
2. $\mu(x, y) = -\sum_{x \le z < y} \mu(x, z)$ for all $x < y$ in P.

See [Stan86, Chapter 3] for background on posets.

Definition Let $\hat{P}$ be a finite graded poset with $\hat{0}$ and $\hat{1}$ such that for all $x, y \in \hat{P}$, we have $\mu(x,y) = (-1)^{r(y)-r(x)}$, where μ is the Möbius function of $\hat{P}$, and r is its rank function. Then $\hat{P}$ is called *Eulerian.*

Example If $\hat{P}$ is the face poset of any convex polytope, then $\hat{P}$ is Eulerian. (See [Stan86, Proposition 3.8.9] for a more general result.)

Definition Let $\hat{P}$ be an Eulerian poset, and $P = \hat{P} - \{\hat{1}\}$. For all $t \in P$, let $P_t = \{s \in P : \hat{0} \le s < t\}$. Let d be the rank of P. Now define $f(P,x)$ and $g(P,x)$ inductively by

1. $f(\phi, x) = g(\phi, x) = 1$
2. $f(P,x) = \sum_{t \in P} g(P_t, x)(x-1)^{d-r(t)}$
3. $g(P,x) = \sum_{i=0}^{\lfloor d/2 \rfloor} (k_i - k_{i-1})x^i$, where k_i is the coefficient of x^i in $f(P,x)$

Then deg $f(P,x) = d$, and the ***generalized h-vector*** of P is $h(P) = (h_0, h_1, \ldots, h_d)$, where $h_i = k_{d-i}$ from above.

Facts (1) If P is a simplicial poset, *i.e.*, P_t is boolean for all $t \in P$, then

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i},$$

where f_{i-1} is the number of elements of rank i in P. In particular, if P is the face poset of a pure simplicial complex Δ, then the generalized h-vector of P is simply the h-vector of Δ.

(2) (For readers familiar with toric varieties) Let $\mathcal{P}$ be a d-dimensional convex polytope in $\mathbf{R}^d$ with rational vertices and the origin in its interior. (Any simplicial convex polytope is combinatorially equivalent to such a $\mathcal{P}$.) Then we can associate with $\mathcal{P}$ a d-dimensional irreducible complex projective toric variety X. The intersection cohomology of X and the generalized h-vector of the boundary of P are related by

$\dim IH^{2i}(X) = h_i$ for $0 \le i \le d$, and $\dim IH^j(X) = 0$ for all other j. Intersection cohomology theory then shows that $h_0 \le h_1 \le \cdots \le h_{\lfloor \frac{d}{2} \rfloor}$ (see [Stan87, Section 3]). Thus, Dehn-Sommerville implies that the h-vector of the boundary of any simplicial convex polytope is unimodal.

(3) For any Eulerian poset $\hat{P}$ we have the *generalized Dehn-Sommerville equations* for the h-vector of P:

$$h_i = h_{d-i}, \qquad \forall 0 \le i \le d.$$

(See [Stan87, Theorem 2.4].) So $g(P, x)$ completely determines $f(P, x)$.

Example The face poset $\hat{L}_d$ of a d-dimensional cube is Eulerian, so its h-vector is determined by its g-vector, which is given by I. Gessel as follows (see [Stan87]).

Proposition 1.2.5 *We have*

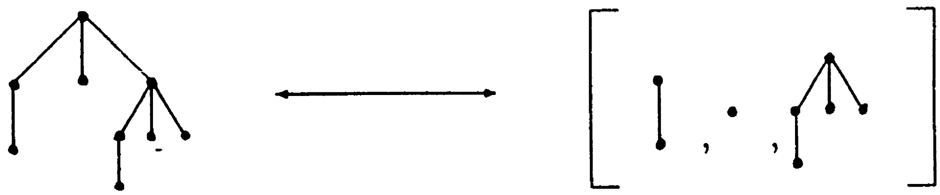
$$g(L_d, x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{1}{d-k+1} \binom{d}{k} \binom{2d-2k}{d} (x-1)^k.$$

Based on this result, L. Shapiro gave the following description of $g(L_d, x)$ in terms of plane trees (see [Stan86], Ex. 3.71g). If two vertices in a plane tree share an edge, we call the lower vertex a *child* of the upper, and write $a_n(i)$ for the number of n-vertex plane trees in which exactly i vertices have more than one child.

Proposition 1.2.6 *We have*

$$g(L_d, x) = \sum_{i=0}^{\lfloor d/2 \rfloor} a_{d+1}(i) x^i.$$

Proof: Since there is only one tree on one vertex, we have $a_1(i) = \delta_{i0}$. By removing the root, we see that a plane tree on $n > 1$ vertices is determined by the ordered set of trees rooted at children of the original root.



Let $\mathbf{N}$ denote the natural numbers, and $\mathbf{P}$ the positive integers. For all $u \in \mathbf{N}$, $v \in \mathbf{P}$, let $[u]_v$ be the set of all $(u_1, \ldots, u_v) \in \mathbf{P}^v$ such that $\sum_{i=1}^{v} u_i = u$, and $[[u]]_v$ the set of all $(u_1, \ldots, u_v) \in \mathbf{N}^v$ such that $\sum_{i=1}^{v} u_i = u$. We have

$$a_n(i) = a_{n-1}(i) + \sum_{j=2}^{n-1} \sum_{b \in [n-1]_j} \sum_{t \in [[i-1]]_j} a_{b_1}(t_1) \cdots a_{b_j}(t_j) \quad \text{for } n > 1.$$

Let $z = \sum_{n \geq 1} \sum_{i \geq 0} a_n(i) y^i x^n$. Then $z = x + xz + xyz^2/(1-z)$, so

$$\begin{aligned}
(1 + xy - x)z &= \frac{1}{2}\left(1 - \sqrt{1 + 4(x^2 - x^2 y - x)}\right) \\
&= -\frac{1}{2} \sum_{k \geq 1} \binom{1/2}{k} 4^k (x^2 - x^2 y - x)^k \\
&= \sum_{k \geq 1} \sum_{s=0}^{k} \frac{1}{k} \binom{2k-2}{k-1} x^{2k} \binom{k}{s} x^{s-k} (y-1)^s \\
&= \sum_{n \geq 1} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{1}{n-s} \binom{2n-2s-2}{n-s-1} \binom{n-s}{s} (y-1)^s x^n \\
&= \sum_{n \geq 1} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{1}{n-s} \left\{ \binom{2n-2s-2}{n-1} \binom{n-1}{s} + \binom{2n-2s-2}{n-2} \binom{n-2}{s-1} \right\} (y-1)^s x^n \\
&= (1 + xy - x) \sum_{n \geq 1} g(L_{n-1}, y) x^n, \quad \text{by Proposition 1.} \quad \square
\end{aligned}$$

Definition A finite graded poset P with $\hat{0}$ is *lower Eulerian* if P_t is Eulerian for all $t \in P$. So for all lower Eulerian P, we can define $f(P, x)$ as in the Eulerian case.

Example Let P be a finite graded poset with $\hat{0}$ such that for all $t \in P$ we have P_t isomorphic to L_r, for some r. In this paper, a *cubical (d-1)-complex* Q is a geometric realization of such a poset P of rank d. Thus, the face poset P_Q of a cubical complex Q is lower Eulerian, and we can define the h-vector of Q by $h(Q) = h(P_Q)$.

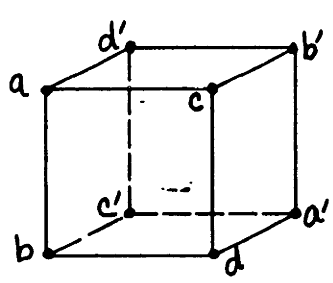
We are interested in h-vectors of shellable cubical complexes, defined analogously to shellable simplicial complexes as follows.

Definition: If Q is a pure cubical $(d-1)$-complex, a *shelling* of Q is an ordering of its maximal faces $F_1, \ldots, F_r$ such that for all $i > 1$ we have that $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ is a $(d-2)$-dimensional cubical complex which is homeomorphic to a ball or sphere. If such an ordering exists, then Q is called *shellable.*

1.3 H-vectors of shellable cubical complexes

Let Q be a shellable cubical $(d-1)$-complex with a given shelling. Our goal is to find a combinatorial interpretation of $h(Q)$ in terms of the given shelling, analogous to Proposition 1.2. The remainder of this chapter appears in [Chan].

A maximal face of Q is called a *facet.* Let F be a facet of Q, and I the intersection of F with previous facets in the shelling. If I is a union of $0 \le i \le d-1$ antipodally unpaired $(d-2)$-faces and $0 \le j \le d-1-i$ pairs of antipodal $(d-2)$-faces, the *h-vector contribution by F* is $\sum_{t \in P_F \setminus P_I} g((P_Q)_t, x)(x-1)^{d-r(t)}$, where r is the rank function of P_Q. We will call F an *(i,j)-facet* (with respect to the given shelling) and denote its h-vector contribution by $f_d(i,j,x) = \sum_{k=0}^{d} b_d(i,j,k)x^k$. (So $f_d(0,d-1,x) = g(L_{d-1},x)$, for example.) Let $b_d(i,j,-1) = 0$ for all i,j.



Let Q be the boundary of the 3-dimensional cube with shelling $(abcd, ab'cd', a'b'cd, abc'd', a'bc'd, a'b'c'd')$.

(0,0)-facet: $abcd$
(1,0)-facet: $ab'cd'$
(2,0)-facets: $a'b'cd, abc'd'$
(1,1)-facet: $a'bc'd$
(0,2)-facet: $a'b'c'd'$

Given a shelling of Q, if we let $s_{i,j}$ be the number of (i,j)-facets in the shelling, then it is clear that $f(P_Q,x) = \sum_{i,j} s_{i,j} f_d(i,j,x)$. Thus $h(Q)$ is given by the sum of the h-vector contributions by facets of Q. We will show that $f_d(i,j,x)$ has nonnegative coefficients by giving them combinatorial interpretations. Consequently we get a combinatorial proof that the h-vector of Q is nonnegative.

Lemma 1.3.1 *For all $0 \le k \le d$ we have that $b_d(0,0,k)$ is the number of d-vertex trees such that exactly k vertices have at most one child.*

Proof: It is clear that $f_d(0,0,x) = (x-1)f(L_{d-1},x) + g(L_{d-1},x)$. We also have $(x-1)f(L_r,x) = x^{r+1}g(L_r,x^{-1}) - g(L_r,x)$ (see [Stan87]). Thus by Proposition 2 we have

$$f_d(0,0,x) = x^d g(L_{d-1},x^{-1}) = \sum_{k=0}^{\lfloor (d-1)/2 \rfloor} a_d(k)x^{d-k},$$

with $a_d(k)$ as defined earlier. The lemma follows. □

Lemma 1.3.2 *For all $1 \le i \le d-1$ and $0 \le k \le d$, we have*

$$b_d(i,0,k) = b_d(i-1,0,k) + b_{d-1}(i-1,0,k) - b_{d-1}(i-1,0,k-1).$$

Proof: The Lemma is true if and only if for all $i \ge 1$ we have

$$f_d(i,0,x) = f_d(i-1,0,x) - (x-1)f_{d-1}(i-1,0,x). \qquad (1)$$

It is easy to see that an $(i,0)$-facet contributes everything that an $(i-1,0)$-facet contributes to the h-vector, except for the contribution by a $(d-2)$-face which intersects the previous facets in $i-1$ antipodally unpaired $(d-3)$-faces. From this we deduce (1). □

Lemma 1.3.3 *For $0 \le k \le d$, $1 \le i \le d-2$, and $1 \le j \le d-1-i$, we have*

$$b_d(i,j,k) = b_d(i,j-1,k) + 2b_{d-1}(i,j-1,k) - 2b_{d-1}(i,j-1,k-1).$$

Proof: Equivalently, we need to show that for all $1 \le i \le d-2$, $1 \le j \le d-1-i$, we have

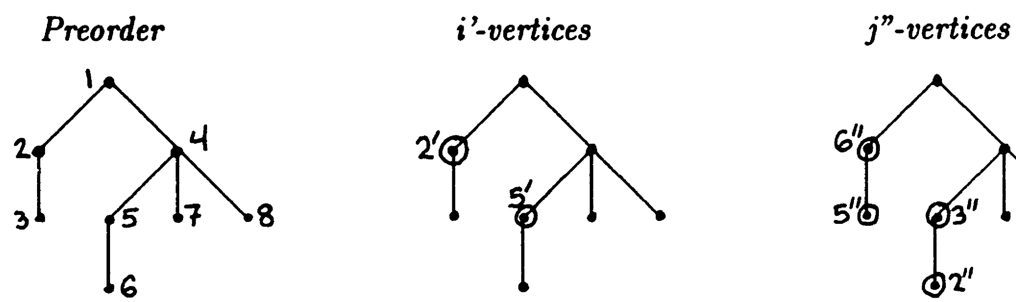
$$f_d(i,j,x) = f_d(i,j-1,x) - 2(x-1)f_{d-1}(i,j-1,x). \qquad (2)$$

Similarly to the proof above, we deduce (2) by comparing the h-vector contribution by an (i,j)-facet with that contributed by an $(i,j-1)$-facet. □

1.4 The Connection to Plane Trees

At this point we introduce some plane tree terminology.

Definitions: An *n-tree* is a plane tree on n vertices. Two children of the same vertex are *siblings.* A vertex is a *fork* if it has more than one child; otherwise it is a *nonfork.* A vertex with no siblings is an *only child.* A child of the root vertex is a *root child.* If a vertex has a sibling to its left and right, it is an *inner child.* In this paper, the vertices of all plane trees are ordered recursively by root first, and then subtrees of the root, from left to right. This is called *preorder.*



If the i-th vertex in an n-tree has exactly one child, we will call this vertex an i'. For $1 \leq j \leq n-2$, if the $(n-j)$-th vertex is followed (in preorder) by an inner, only, or root child, we will call this vertex a j''. For all $j \leq n-2$, let $c_n(i,j,k)$ be the number of n-trees with exactly k nonforks which are not $1', \ldots, i'$ nor $1'', \ldots, j''$. Let $c_n(0, n-1, k) = a_n(k)$ as defined earlier, and $c_n(i,j,-1) = 0$ for all i,j.

We now can state our main result.

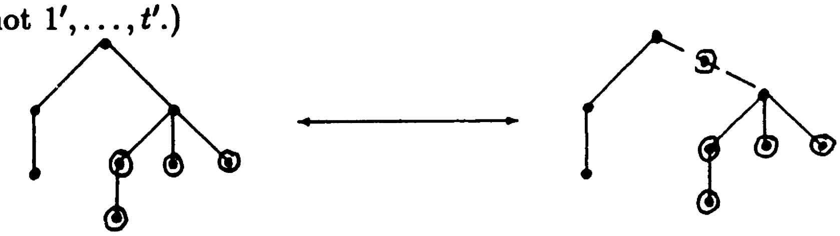
Theorem 1.4.1 *Let F be a facet of a cubical $(d-1)$-complex with given shelling. If F is an (i,j)-facet, then the h-vector contribution by F is $\sum_{k=0}^{d} c_d(i,j,k)x^k$.*

Proof: We need to show that $b_d(i,j,k) = c_d(i,j,k)$ for all $0 \leq k \leq d$. First we consider the case $j = 0$. Since there is no such thing as $0'$ or $0''$, we have

$c_d(0,0,k) = b_d(0,0,k)$ for all $-1 \leq k \leq d$ by Lemma 1. So by Lemma 2, it suffices to show that for all $1 \leq i \leq d-1$, $0 \leq k \leq d$ we have

$$c_d(i,0,k) = c_d(i-1,0,k) + c_{d-1}(i-1,0,k) - c_{d-1}(i-1,0,k-1). \qquad (3)$$

Now given any $(d-1)$-tree with s nonforks which are not $1',\ldots,t'$, we can get a d-tree with $s+1$ nonforks not $1',\ldots,t'$ by inserting a vertex between the $(t+1)$-th vertex and its parent. (In the picture below, circled vertices are nonforks which are not $1',\ldots,t'$.)



$d = 9$

$t = 3$

$s = 4$

This map is injective. From this observation (3) easily follows.

Now consider $j > 0$. As noted in the last section, $f_d(0, d-1, x) = g(L_d, x)$, so the theorem holds for $j = d-1$. By the definition of shelling, $1 \leq j \leq d-2 \Rightarrow 1 \leq i \leq d-2$. Fix such i. Now $c_d(i,0,k) = b_d(i,0,k)$ for all $-1 \leq k \leq d$ from above, so by Lemma 3, it suffices to show that for all $1 \leq j \leq d-1-i$, $0 \leq k \leq d$ we have

$$c_d(i,j,k) = c_d(i,j-1,k) + 2c_{d-1}(i,j-1,k) - 2c_{d-1}(i,j-1,k-1). \qquad (4)$$

Given any $(d-1)$-tree with exactly s nonforks not $1',\ldots,i'$ nor $1'',\ldots,j''$ we can get a d-tree with exactly $s+1$ nonforks not $1',\ldots,i'$ nor $1'',\ldots,j''$ in two ways:
(In the pictures below, circled vertices are nonforks which are not $1',\ldots,i'$ nor $1'',\ldots,j''$.)

1. insert a vertex between the $(d-1-j)$-th vertex and its parent.

$d = 9$

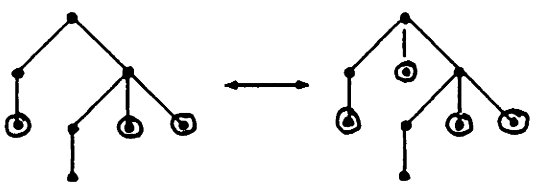
$i = 2$

$j = 4$

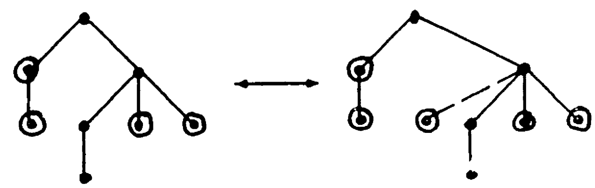
$s = 3$

2. replace the $(d-1-j)$-th vertex and its offspring by a single leaf. If there is a $(d-j)$-th vertex in the cropped tree, call this vertex v, and reinsert the removed subtree so that its root has v as a sibling on its immediate right. If no such v exists, insert the removed subtree so that it is the rightmost subtree directly under the root.

$d = 9, i = 2, j = 4, s = 3$



$d = 9, i = 1, j = 3, s = 4$



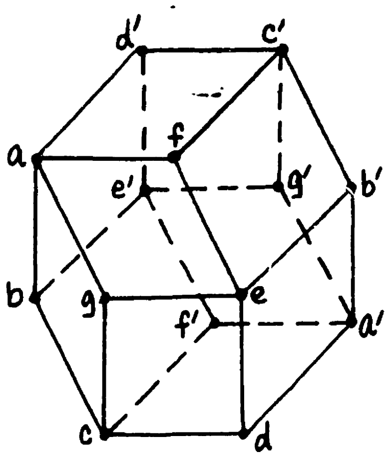
These two maps are injective and have disjoint images. The identity (4) follows.

□

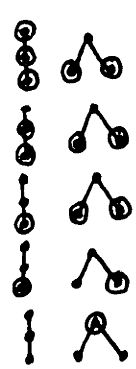
Example Let Q be the rhombic dodecahedron shown below, with shelling

$$(afeg, cged, eda'b', b'c'ef, afc'd', abcg, a'b'c'g, a'f'cd, bce'f', e'd'ab, c'g'e'd', a'f'e'g').$$

We have $s_{0,0} = 1,\ s_{1,0} = 2,\ s_{2,0} = 6,\ s_{1,1} = 2,\ s_{0,2} = 1.$



$$\begin{aligned} f(Q,x) &= (x^2+x^3) \\ &\quad +2(2x^2) \\ &\quad +6(x+x^2) \\ &\quad +2(2x) \\ &\quad +(1+x) \\ &= 1+11x+11x^2+x^3 \end{aligned}$$



Chapter 2

Some Results on Local h-vectors

2.1 Introduction

R. Stanley defined local h-vectors, called l-vectors, to investigate the behavior of h-vectors of simplicial complexes under subdivision. He showed that l-vectors of a large class of simplex subdivisions satisfy certain useful properties. In this chapter we show that these properties characterize such l-vectors. We also show that refined l-vectors (defined for vertex-colored subdivisions) satisfy the same useful properties.

2.2 Background

Let Δ be a finite $(d-1)$-dimensional simplicial complex with f_i i-dimensional faces (i-faces) for each i. Then its *f-vector* $f(\Delta)$ is $(f_0, \ldots, f_{d-1})$. By convention, $f_{-1} = 1$, unless $\Delta = \emptyset$, in which case $f_{-1} = 0$. An algebraically desirable form of the f-vector is the h-vector $h(\Delta) = (h_0, \ldots, h_d)$, defined by $\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}$. We will often work with the h-polynomial $h(\Delta, x) = \sum_{i=0}^{d} h_i x^i$. We will give some background on the algebraic significance of $h(\Delta, x)$ for a certain class of simplicial complexes below. Chapter 1 contains other useful facts about h-vectors, in particular the identities

$$h(\Delta, x) \;=\; \sum_{F \in \Delta} x^{\#F}(1-x)^{d-\#F},$$

$$h(\Delta * \Gamma, x) = h(\Delta, x) \cdot h(\Gamma, x).$$

Definition Let K be an infinite field, and Δ a simplicial complex with vertex set $[n]$. Let $K[x_1, \ldots, x_r]$ denote the ring of polynomials in the formal variables x_i with coefficients in K. Each $F \subseteq [n]$ defines a monomial $x^F = \prod_{i \in F} x_i$. The *face ring* (or *Stanley-Reisner ring*) of Δ is $K[\Delta] = K[x_1, \ldots, x_n]/I(\Delta)$, where $I(\Delta)$ is the ideal $(x^F : F \in 2^{[n]} \setminus \Delta)$.

The standard grading of $K[\Delta]$ is given by $\deg(x_i) = 1$, for each i. With respect to this grading, the *Poincare series* $F(K[\Delta], x)$ is the polynomial $\sum_{i \geq 0} dim_K K[\Delta]_i \cdot x^i$, where $K[\Delta]_i$ denotes the i^{th}-graded piece of $K[\Delta]$.

See [Stan83, Section 2.1] for proof of the following

Proposition 2.2.1 *For any simplicial complex Δ we have*

$$F(K[\Delta], x) = (1 - x)^{-d} h(\Delta, x).$$

Notation For any simplicial complex Δ, let $|\Delta|$ denote the geometric realization of Δ, and for any $F \in \Delta$, let $|F|$ denote the image of F in $|\Delta|$. (see [Stan83, Section 0.3])

Definition For any simplicial complex Δ, let $\tilde{H}_i(\Delta)$ denote the i^{th} reduced simplicial homology group of Δ over a fixed field K. A simplicial complex Δ is *Cohen-Macaulay* over K if for every $F \in \Delta$, $\tilde{H}_i(\mathrm{lk}_\Delta F) = 0$ for all $i < \dim(\mathrm{lk}_\Delta F)$.

Examples (1) Note that this is a topological property, *i.e.*, it depends only on the geometric realization of Δ. For example, all triangulations of balls and spheres are Cohen-Macaulay.

(2)All shellable simplicial complexes are Cohen-Macaulay, as a simple consequence of the Mayer-Vietoris sequence. (See [Stan77, Section 5].)

Definition A set of homogeneous elements $\{\theta_1, \theta_2, \ldots, \theta_d\}$ in $K[\Delta]$ is called a *homogeneous system of parameters* (or *h.s.o.p.*) if it generates an ideal (θ) in $K[\Delta]$ such that $K[\Delta]/(\theta)$ is a finite-dimensional vector space over K. Since $K[\Delta]$ has an

N-grading and K is infinite, existence of an h.s.o.p. of degree one is guaranteed by the Noether Normalization Lemma (see [Stan83, Section 1.5]).

Cohen-Macaulay complexes are characterized algebraically by Reisner's Theorem (see [Reis] for proof):

Theorem 2.2.2 *Δ is Cohen-Macaulay if and only if $K[\Delta]$ is a finitely generated free $K[\theta_1, \theta_2, \ldots, \theta_d]$-module for some $\theta_1, \theta_2, \ldots, \theta_d \in K[\Delta]$, which necessarily form an h.s.o.p. of $K[\Delta]$.*

The following proposition relies on the algebraic characterization of Cohen-Macaulay complexes (see [Stan83, p. 67]).

Proposition 2.2.3 *If Δ is Cohen-Macaulay and $\theta_1, \theta_2, \ldots, \theta_d$ is an h.s.o.p. of degree one for $K[\Delta]$, then*

$$F(K[\Delta]/(\theta), x) = h(\Delta, x).$$

Corollary 2.2.4 *If Δ is Cohen-Macaulay, then $h_i(\Delta) \geq 0$ for all i.*

Corollary 2.2.5 *If Δ is a Cohen-Macaulay simplicial complex on n vertices, then*

$$h_i(\Delta) \leq \binom{n-d+i-1}{i}, \quad 0 \leq i \leq d.$$

Proof: See [Stan85, Theorem 1]. □

A consequence of the last corollary is the *Upper Bound Conjecture for simplicial spheres*:

Definition For every $n \geq d$, a *cyclic polytope* $C(n,d)$ is the convex hull of any n points on the moment curve $\{(t, t^2, \ldots, t^d)\} \subset \mathbf{R}^d$. This is a simplicial d-polytope with $f_i = \binom{n}{i+1}$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor - 1$, so clearly it maximizes these f_i over all simplicial d-polytopes with n vertices. (See [Grün], Section 7.4) for background on cyclic polytopes.)

By Dehn-Sommerville, $f_0, \ldots, f_{\lfloor \frac{d}{2} \rfloor - 1}$ determine the whole f-vector of $C(n,d)$, so the question is, are $f_{\lfloor \frac{d}{2} \rfloor}, \ldots, f_{d-1}$ also maximized by $C(n,d)$? The following theorem gives the answer.

Theorem 2.2.6 *If Δ is a simplicial complex on n vertices and $|\Delta|$ is homeomorphic to $\mathbf{S}^{d-1}$, then*

$$f_i(\Delta) \leq f_i(C(n,d)), \quad 0 \leq i \leq d-1.$$

Proof: Since $|\Delta|$ is homeomorphic to $\mathbf{S}^{d-1}$, the desired inequalities follow from

$$h_i(\Delta) \leq \binom{n-d+i-1}{i}, \quad 0 \leq i \leq d,$$

as proved in [McM70]. Since $|\Delta|$ is homeomorphic to $\mathbf{S}^{d-1}$ implies that Δ is Cohen-Macaulay, Corollary 1.5 holds. □

McMullen's *g-conjecture*, stated below, completely characterizes h-vectors of simplicial d-polytopes on n vertices in algebraic terms. A consequence is a proof of the *Lower Bound Conjecture* for f-vectors.

Theorem 2.2.7 *A polynomial $\sum h_i x^i$ is the h-polynomial of some simplicial d-polytope if and only if $\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor}(h_i - h_{i-1})x^i$ is the Poincare series of some standard graded K-algebra.*

Proof: (1) The "if" direction was proved by Billera and Lee using explicit constructions, in [BL].
(2) The "only if" part was proved by R. Stanley using results from algebraic geometry, in [Stan80]. □

Corollary 2.2.8 *The Lower Bound Conjecture holds for all simplicial d-polytopes Δ on n vertices,* i.e.,

$$\begin{aligned} f_i &\geq \binom{d}{i} f_0 - \binom{d+1}{i+1} i \qquad \forall i \leq d-2, \\ f_{d-1} &\geq (d-1)f_0 - (d+1)(d-2). \end{aligned}$$

Thus ends the background on h-vectors and f-vectors which follows from the algebraic significance of h-vectors. For more background and references, see [Stan85]. Non-algebraic characterizations of h-vectors and f-vectors of simplicial complexes exist, but are not relevant to this paper. (See [GK] and [Stan83, Section 2.2].)

2.3 Subdivisions and local h-vectors

A natural question to ask about $h(\Delta)$ is whether or not it increases, as $f(\Delta)$ does, when some faces of Δ are subdivided. If Δ is Cohen-Macaulay the answer is yes, at least if the subdivision is *quasigeometric* (defined below). Stanley was able to prove this by defining the *local* h-vectors of the subdivision restricted to each face of Δ, and using certain useful properties of these l-vectors.

Definitions (1) A simplicial complex Γ is a *subdivision* of the simplex 2^V if there exists a map $\sigma : \Gamma \to 2^V$ such that for each $W \subseteq V$, the *restriction* $\Gamma_W = \sigma^{-1}(2^W)$ is a subcomplex of Γ with geometric realization homeomorphic to the ball $B^{\#W-1}$ and $\sigma^{-1}(W) = \{$ interior faces of $\Gamma_W\}$. For each $F \in \Gamma$, we call $\sigma(F)$ the *carrier* of F, and say F *lies on* W if $\sigma(F) \subseteq W$.
(2) A simplicial complex Δ' is the *subdivision* of simplicial complex Δ if there exists a map $\sigma : \Delta' \to \Delta$ such that for each $F \in \Delta$, the restriction $\Delta'_F = \sigma^{-1}(F)$ is a subdivision of 2^F.

Definition Let Γ be a subdivision of 2^V. Its *local h-polynomial* is defined by

$$l_V(\Gamma, x) = \sum_{i=0}^{d} l_i x^i = \sum_{W \subseteq V} (-1)^{\#(V \setminus W)} h(\Gamma_W, x),$$

and its *local h-vector* $l_V(\Gamma)$ is $(l_0, \ldots, l_d)$.

Facts (1) $l_\emptyset(2^\emptyset) = 1$, and if $V \neq \emptyset$, $l_V(2^V) = 0$.
(2) $l_V(\Gamma, x) = \sum_{G \in \Gamma} (-1)^{d-\#G} x^{d-e(G)} (x-1)^{e(G)}$, where $e(G) = \#\sigma(G) - \#G$ is called the *excess* of G. ([Stan92, Proposition 2.2])
(3) $l_i = l_{d-i}$ for all i. ([Stan92, Theorem 3.3])
(4) $l_1 \geq 0$. If $d > 0$, then $l_0 = 0$. ([Stan92, Example 2(f)])
(5) $h(\Delta', x) = \sum_{F \in \Delta} l_F(\Delta'_F, x) \cdot h(lk_\Delta F, x)$ when Δ' is any subdivision of a pure simplicial complex Δ. ([Stan92, Theorem 3.2])

Definitions (1) A subdivision Γ of 2^V is *geometric* if it can be realized in $\mathbf{R}^d$ with all convex faces. Γ is *quasigeometric* if for all $F \in \Gamma$, the vertices of F do not all lie on a face of 2^V of dimension less than $\dim(F)$.

(2) If Γ is a geometric subdivision of 2^V, then Γ is called *regular* if it is the projection of a strictly convex polyhedral surface (with boundary) in $\mathbf{R}^{d+1}$. (This is the same idea as Ω_0 being the projection of ∂Z in Chapter 3, Section 2.) Formally, Γ is regular if if there exists a ***height function*** $\omega : |\Gamma| \to \mathbf{R}$ which is *piecewise linear and strictly convex, i.e.,*

(i) ω is *piecewise linear*: For all $F \in \Gamma$, ω restricts to a linear function ω_F on $|F|$.

(ii) ω is *convex*: For all $x, y \in |\Gamma|$, and all $\mu \in [0, 1]$,

$$\omega(\mu x + (1-\mu)y) \geq \mu\omega(x) + (1-\mu)\omega(y).$$

(iii) For all F, G distinct maximal faces of Γ, the functions ω_F, ω_G are distinct (as linear functions).

Examples: (1) Let Γ be the trivial subdivision 2^V. Then $\omega = 0$ satisfies conditions (i)-(iii), so Γ is regular.

(2) Let Γ be the subdivision of 2^V with one interior vertex z and maximal faces of the form $F \cup \{z\}$ where F is any $(d-1)$-dimensional face of 2^V. Then Γ is regular. (see Section 4 for proof.)

Facts (6) (i) All geometric subdivisions of the simplex are quasigeometric, but the converse is not true. For example, you can get a quasigeometric but non-geometric subdivision of the 5-simplex by removing any facet of a non-PL 5-sphere (see [Stan92]). It is easier to see that quasigeometric does not imply regular, since all regular subdivisions are shellable, but not even all geometric subdivisions are.

(ii) Not all subdivisions are quasigeometric. For example: if Δ has maximal faces $\{1,2,3\}, \{1,2,4\}$ and $\sigma : \Delta \to 2^{[3]}$ is given by $\sigma(\{1,4\}) = \sigma(\{2,4\}) = \sigma(\{4\}) = \{1,2\}$, $\sigma(\{1,2\}) = \sigma(\{1,2,4\}) = \{1,2,3\}$, and $\sigma(F) = F$ for all other $F \in \Delta$, then

Δ is a non-quasigeometric subdivision of $2^{[3]}$ with subdivision map σ. See Figure 1.

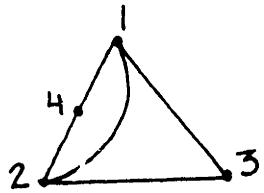


Figure 1

(iii) Let A be a finite affine point configuration, and $\mathcal{P}$ the polytope conv(A), whose vertices are contained in A. Then the regular subdivisions of $\mathcal{P}$ are in one-to-one correspondence with the vertices of the *secondary polytope* $\Sigma(A)$. (See [BFS].)

(7) If Γ is quasigeometric, then $l_V(\Gamma)$ is nonnegative, *i.e.*, $l_i \geq 0$, for all i. (The proof, from [Stan92], uses homological algebra to show that l_i is the dimension of L_i, a vector space associated with Γ.) Consequently, using Fact(5), $h(\Delta') \geq h(\Delta)$ if Δ' is a quasigeometric subdivision of a *Cohen-Macaulay* simplicial complex Δ, since Δ Cohen-Macaulay implies that $\forall F \in \Delta, h(lk_\Delta F) \geq 0$. See Section 5 for an analogous result.

(8) If Γ is regular, then $l_V(\Gamma)$ is unimodal, *i.e.*,

$$l_0 \leq l_1 \leq \ldots \leq l_{\lfloor d/2 \rfloor} \geq l_{\lfloor d/2 \rfloor + 1} \geq \ldots \geq l_{n-1} \geq l_n.$$

(The proof, in [Stan92], uses intersection homology theory.)

2.4 Characterization of Local h-vectors

Let $V = [d]$. We now show that l-vectors of arbitrary subdivisions of 2^V can be completely characterized as follows:

Theorem 2.4.1 *Let $l = (l_0, \ldots, l_d) \in \mathbf{Z}^{d+1}$. Then $l = l_V(\Gamma)$ for some subdivison Γ of 2^V if and only if l is symmetric and $l_0 = 0, l_1 \geq 0$.*

The "only if" direction was proved by Stanley (see [Stan92]). To prove the "if" direction, we will construct a subdivision Γ, given any l satisfying the above requirements, so that $l_V(\Gamma) = l$. We can do so with the help of three lemmas:

Let Γ be a subdivision of 2^V. See Figures 2-4 for illustrations of the subdivisions described in the lemmas below.

Lemma 2.4.2 *Let F be a maximal face of Γ, and Γ' the subdivision of Γ with one new vertex z in $Int(F)$. So the face F of Γ is replaced by the faces $f \cup \{z\}$ in Γ'. Then $h(\Gamma', x) = h(\Gamma, x) + x + x^2 + \cdots + x^{d-1}$, and $l_V(\Gamma', x) = l_V(\Gamma, x) + x + x^2 + \cdots + x^{d-1}$.*

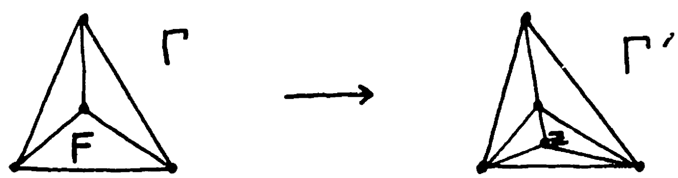


Figure 2

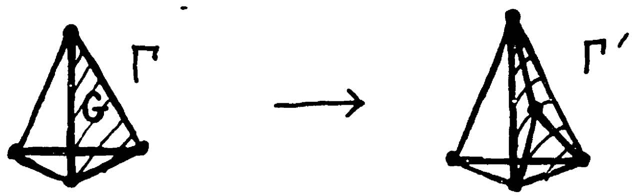
Proof: We first show that $h(\Gamma', x) = h(\Gamma, x) + x + x^2 + \cdots + x^{d-1}$.

$$\begin{aligned}
h(\Gamma', x) &= \sum_{G \in \Gamma'} x^{\#G}(1-x)^{d-\#G} \\
&= \sum_{G \in \Gamma, G \neq F} x^{\#G}(1-x)^{d-\#G} + \sum_{G \subset F} x^{\#(G \cup \{z\})}(1-x)^{d-\#(G \cup \{z\})} \\
&= h(\Gamma, x) - x^{\#F}(1-x)d - \#F + \frac{x}{1-x} \sum_{G \subset F} x^{\#G}(1-x)^{d-\#G} \\
&= h(\Gamma, x) - x^{\#F}(1-x)d - \#F + \frac{x}{1-x}(h(2^F, x) - x^{\#F}(1-x)^{d-\#F}) \\
&= h(\Gamma, x) - x^d + \frac{x}{1-x}(1-x^d) \\
&= h(\Gamma, x) + x + x^2 + \cdots + x^{d-1}.
\end{aligned}$$

Now we show that $l_V(\Gamma', x) = l_V(\Gamma, x) + x + x^2 + \cdots + x^{d-1}$.
Since $\Gamma'_W = \Gamma_W$ for all $W \neq V$,

$$\begin{aligned}
l_V(\Gamma', x) &= \sum_{W \subseteq V} (-1)^{d-\#W} h(\Gamma'_W, x) \\
&= h(\Gamma', x) + \sum_{W \subset V} (-1)^{d-\#W} h(\Gamma_W, x) \\
&= h(\Gamma', x) + l_V(\Gamma, x) - h(\Gamma, x) \\
&= l_V(\Gamma, x) + x + x^2 + \cdots + x^{d-1}. \quad \square
\end{aligned}$$

Lemma 2.4.3 *Let $d \geq 4$. Let G be a $(d-2)$-dimensional face of Γ with $(d-2)$-dimensional carrier W, and Γ' the subdivision of Γ with one new vertex w (with* *carrier W) and one new maximal face* $G \cup \{w\}$*. So in* Γ'*,* G *has carrier* V*. Then* $h(\Gamma', x) = h(\Gamma, x) + x$*, and* $l_V(\Gamma', x) = l_V(\Gamma, x) - x^2 - \cdots - x^{d-2}$*.*



A copy of G is pushed into the interior of Γ′ as in the 2-dim. case of Figure 1.

Figure 3

Proof: We first show that $h(\Gamma', x) = h(\Gamma, x) + x$.

$$\begin{aligned}
h(\Gamma', x) &= \sum_{F \in \Gamma'} x^{\#F}(1-x)^{d-\#F} \\
&= \sum_{F \in \Gamma} x^{\#F}(1-x)^{d-\#F} + \sum_{F \subseteq G} x^{\#F \cup \{w\}}(1-x)^{d-\#F\cup\{w\}} \\
&= h(\Gamma, x) + x \cdot \sum_{F \in 2^G} x^{\#F}(1-x)^{d-1-\#F} \\
&= h(\Gamma, x) + x \cdot h(2^G, x) \\
&= h(\Gamma, x) + x.
\end{aligned}$$

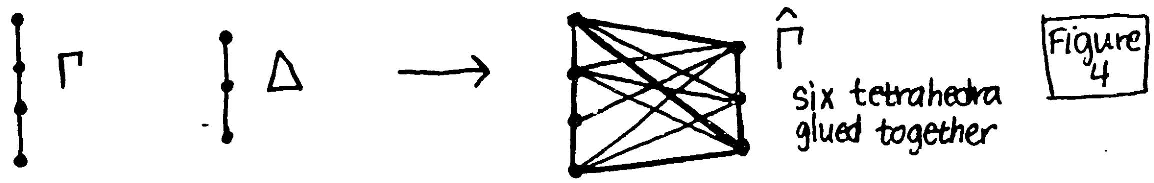
Now we show that $l_V(\Gamma', x) = l_V(\Gamma, x) - x^2 - \cdots - x^{d-2}$.
Since $\Gamma'_U = \Gamma_U$ if $U \not\supseteq W$,

$$\begin{aligned}
l_V(\Gamma', x) &= \sum_{U \subseteq V} (-1)^{d-\#U} h(\Gamma'_U, x) \\
&= h(\Gamma', x) - h(\Gamma'_W, x) + \sum_{U \not\supseteq W} (-1)^{d-\#U} h(\Gamma_U, x)
\end{aligned}$$

(applying Lemma 3.2 to Γ'_W)

$$\begin{aligned}
&= h(\Gamma', x) - h(\Gamma_W, x) - (x + \cdots + x^{d-2}) + l_V(\Gamma, x) - h(\Gamma, x) + h(\Gamma_W, x) \\
&= h(\Gamma', x) - (x + x^2 + \cdots + x^{d-2}) + l_V(\Gamma, x) - h(\Gamma, x) \\
&= l_V(\Gamma, x) - x^2 - \cdots - x^{d-2}. \quad \square
\end{aligned}$$

Lemma 2.4.4 *Let* Δ *be the subdivision of* $2^{\{d+1,d+2\}}$ *with one interior vertex* y*,* $\hat{V} = [d+2]$*, and* $\hat{\Gamma} = \Gamma * \Delta$*. Then* $l_{\hat{V}}(\hat{\Gamma}, x) = x \cdot l_V(\Gamma, x)$*.*



Proof:

$$
\begin{aligned}
l_{\hat{V}}(\hat{\Gamma},x) &= \sum_{\hat{W}\subseteq\hat{V}} (-1)^{d+2-\#\hat{W}} h(\hat{\Gamma}_{\hat{W}},x) \\
&= (-1)^d \sum_{\hat{W}\subseteq\hat{V}} (-1)^{\#\hat{W}} h(\hat{\Gamma}_{\hat{W}},x) \\
&= (-1)^d \sum_{W\subseteq V} \sum_{U\subseteq\{d+1,d+2\}} (-1)^{\#W\cup U} h(\hat{\Gamma}_{W\cup U},x).
\end{aligned}
$$

For each $W \subseteq V$ and $U \subseteq \{d+1, d+2\}$, we have

$$h(\hat{\Gamma}_{W\cup U},x) = h(\Gamma_W * \Delta_U), x) = h(\Gamma_W,x)\cdot h(\Delta_U,x).$$

It is easy to compute

$$h(\Delta_\emptyset,x) = h(\Delta_{\{d+1\}},x) = h(\Delta_{\{d+2\}},x) = 1, \quad h(\Delta_{\{d+1,d+2\}},x) = 1+x.$$

So

$$
\begin{aligned}
l_{\hat{V}}(\hat{\Gamma},x) &= (-1)^d \sum_{W\subseteq V} (-1)^{\#W}\{h(\Gamma_W,x) - h(\Gamma_W,x) - h(\Gamma_W,x) + (1+x)\cdot h(\Gamma_W,x)\} \\
&= x\cdot \sum_{W\subseteq V} (-1)^{d-\#W} h(\Gamma_W,x) \\
&= x\cdot l_V(\Gamma,x). \quad \square
\end{aligned}
$$

Proof of Theorem: Given $l_0 = 0, l_1 \geq 0$, and $l_{d-i} = l_i$, we want to find a subdivision Γ of 2^V, with

$$l_V(\Gamma,x) = l(x) = \sum_{i=0}^{d} l_i x^i.$$

If $d \leq 3$, we can construct Γ directly from 2^V by l_1 applications of Lemma 3.2. For $d \geq 4$, we first find a subdivision Γ_0 of 2^V with

$$l_V(\Gamma_0, x) = l(x) - l_1 \cdot (x + x^2 + \cdots + x^{d-1}).$$

From Γ_0 we get Γ by l_1 applications of Lemma 3.2.

Let $s = l_2 - l_1$.

For $d = 4$: If $s \leq 0$, get Γ_0 from $2^{[4]}$ by $|s|$ applications of Lemma 3.3. If $s > 0$, start with $2^{[2]}$, apply Lemma 3.2 s times, and then apply Lemma 3.4 to get Γ_0.

For $d \geq 5$: Let $W = [d-2]$. We can assume by induction that there exists a subdivision Γ' of 2^W with

$$l_W(\Gamma', x) = x^{-1} \cdot \left(l(x) - l_1 \cdot (x + x^2 + \cdots + x^{d-1}) - s \cdot (x^2 + \cdots + x^{d-2})\right).$$

If $s \leq 0$, apply Lemma 3.4 to get a subdivision of 2^V from Γ', and then get Γ_0 by $|s|$ applications of Lemma 3.3. If $s > 0$, start with Γ', apply Lemma 3.2 s times, and then apply Lemma 3.4 to the resulting subdivision to get Γ_0. □

2.5 Local h-vectors of Regular Subdivisions

Using the same construction as in the previous section, we now show that l-vectors of regular subdivisions of 2^V can be characterized as follows:

Theorem 2.5.1 *If $l = (l_0, l_1, \ldots, l_d) \in \mathbf{Z}^n$ then $l = l_V(\Gamma)$ for some regular subdivision Γ of 2^V if and only if $l_0 = 0$ and l is unimodal and symmetric.*

Proof: Stanley proved the "only if" direction in [Stan92]. To prove the "if" direction, it suffices to show that the subdivision Γ constructed in the proof of Theorem 3.1 is regular when $l_0 = 0$ and l is unimodal and symmetric. For $V = [d]$, let Γ be a regular subdivision of 2^V and let $\omega : |\Gamma| \to \mathbf{R}$ be piecewise linear and strictly convex. See Figures 5-6 for illustrations of the proofs of the following lemmas:

Lemma 2.5.2 $\Gamma' = \Gamma * \{x\}$ *is a regular subdivision of* $2^{V \cup \{x\}}$.

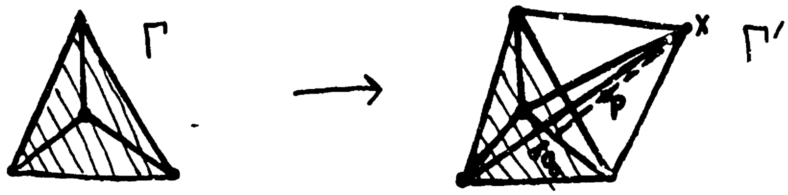


Figure 5

Proof: For every $p \in |\Gamma'|$, there exists a unique $q \in |\Gamma|$ and $\lambda \in [0,1]$ such that $p = \lambda q + (1-\lambda)x$. Let $\omega'(p) = \lambda \cdot \omega(q)$.
If $p_1 = \lambda_1 q_1 + (1-\lambda_1)$, and $p_2 = \lambda_2 q_2 + (1-\lambda_2)$, then for all $\mu \in [0,1]$,

$$\begin{aligned}
\mu p_1 + (1-\mu)p_2 &= \mu(\lambda_1 q_1 + (1-\lambda_1)x) + (1-\mu)(\lambda_2 q_2 + (1-\lambda_2)x) \\
&= \mu\lambda_1 q_1 + (1-\mu)\lambda_2 q_2 + (\mu(1-\lambda_1) + (1-\mu)(1-\lambda_2))x \\
&= (\mu\lambda_1 + (1-\mu)\lambda_2)(rq_1 + (1-r)q_2) + (1 - \mu\lambda_1 - (1-\mu)\lambda_2)x \\
&= \tau(rq_1 + (1-r)q_2) + (1-\tau)x,
\end{aligned}$$

So

$$\begin{aligned}
\omega'(\mu p_1 + (1-\mu)p_2) &= \tau \cdot \omega(rq_1 + (1-r)q_2) \\
&\geq \tau r\omega(q_1) + \tau(1-r)\omega(q_2) \\
&= \mu\lambda_1 w(q_1) + (1-\mu)\lambda_2 w(q_2) \\
&= \mu\omega'(p_1) + (1-\mu)\omega'(p_2),
\end{aligned}$$

with equality if q_1, q_2 are in the same maximal face of $|\Gamma|$, which occurs exactly when p_1, p_2 are in the same maximal face of $|\Gamma|'$.
Also, since $\omega'|_\Gamma = \omega$ is different on each facet of Γ, ω' is different on each facet of Γ'. Thus $\omega' : |\Gamma'| \to \mathbf{R}$ is piecewise linear and strictly convex, and Γ' is regular. □

Lemma 2.5.3 *Let G be an i-face of Γ with $i \geq 2$, and Γ' the subdivision of Γ which results from putting a new vertex v in the interior of G and joining v with all faces of the form $G' - \{z\}$, where $G' \supseteq G$ and $z \in G$. Then Γ' is regular.*

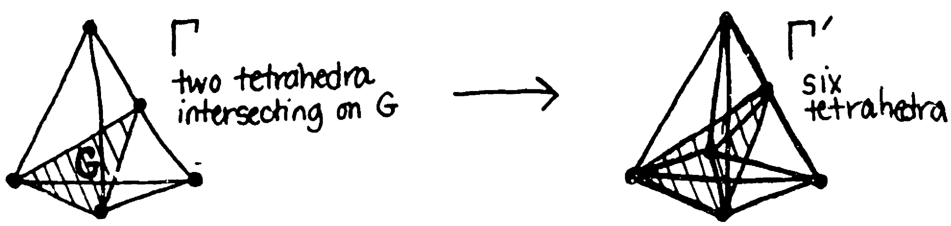


Figure 6

Proof: For every maximal $F \in \Gamma$ such that $F \not\supseteq G$, $\omega_F(v) > \omega(v)$ since ω is strictly convex and piecewise linear. So we can choose $\epsilon > \omega(v)$ such that $\epsilon < \omega_F(v)$ for every such F. Now if G' is any maximal face of Γ' such that $v \in |G'|$, then for all $p \in |G'|$ there exists a unique $q \in |G' - \{v\}|$ and $\lambda \in [0,1]$ such that $p = \lambda q + (1-\lambda)v$. Let $\omega'(p) = \lambda\omega(q) + (1-\lambda)\epsilon$.

For all other $p \in |\Gamma'|$, let $\omega'(p) = \omega(p)$. Then $\omega' : |\Gamma'| \rightarrow \mathbf{R}$ is piecewise linear and strictly convex, by simple geometric arguments. □

Proof of Theorem: As mentioned earlier, we only need to show the "if" direction. Given $l \in \mathbf{Z}^{d+1}$ symmetric and unimodal with $l_0 = 0$, let Γ be the subdivision of 2^V with $l_V(\Gamma) = l$, constructed as in the proof of Theorem 3.1. Since l is unimodal, $s \geq 0$ at every step of the construction. So Γ is built up only from subdivisions described in Lemmas 4.2 and 4.3. Since the construction begins with the trivially regular subdivision 2^W for some W, it remains regular at each step, so Γ is regular. □

2.6 Refined l-vectors

Notation: If S is a finite set and $r(x)$ is any rational expression in the variable x, then $r(\lambda)^S$ denotes the expression $\Pi_{i\in S} r(\lambda_i)$, and $r(\lambda)^{-S}$ denotes the expression $\Pi_{i\in S} r(\lambda_i)^{-1}$.

Definitions (Δ, π) is a *completely balanced* simplicial complex if Δ is a $(d-1)$-dimensional simplicial complex and π is a *coloring* of the vertices of Δ by a d-element *color set* $\pi(\Delta)$ such that every maximal face $F \in \Delta$ has a vertex of each color. For each $F \in \Delta$, let $\pi(F)$ denote the set of colors of vertices in F. Define the *fine f-vector* of (Δ, π) by $f(\Delta, \pi, \lambda) = \sum_{F\in\Delta} \lambda^{\pi(F)} = \sum_{S\subseteq\pi(\Delta)} f_S(\Delta, \pi)\lambda^S$, and the *fine h-vector* by $h(\Delta, \pi, \lambda) = \sum_{F \in \Delta} \lambda^{\pi(F)}(1-\lambda)^{\pi(\Delta)-\pi(F)} = \sum_{S \subseteq \pi(\Delta)} h_S(\Delta, \pi)\lambda^S$. (See [Stan79] for general background.)

The following definition was suggested by R. Stanley and G. Kalai.

Definition Let $V = [d]$, and let $(2^V, id)$ denote the completely balanced simplex with color set V and the identity map as its coloring. Then (Γ, π) is a balanced subdivision of $(2^V, id)$ if Γ is a subdivision of 2^V and π is a coloring of the vertices of Γ such that for every $W \subseteq V$, if π_W is the restriction of π to the vertices of Γ_W, then (Γ_W, π_W) is completely balanced and $\pi_W(\Gamma_W) = W$. (Note that Γ must be quasigeometric.) Then define the *fine l-vector* of (Γ, π) with respect to V by

$$l_V(\Gamma, \pi, \lambda) = \sum_{W \subseteq V} (-1)^{d-\#W} h(\Gamma_W, \pi_W, \lambda).$$

The proofs below are all modifications of analogous results in [Stan92].

The following is an analogue of Fact(2) of Section 2:

Proposition 2.6.1 *Let* (Γ, π) *be a balanced subdivision of* $(2^V, id)$. *Then*

$$l_V(\Gamma, \pi, \lambda) = \sum_{F \in \Gamma} (-1)^{d-\#F} \lambda^{V \setminus \sigma(F)} \lambda^{\pi(F)} (\lambda-1)^{\sigma(F)} (\lambda-1)^{-\pi(F)}.$$

Proof:

$$\begin{aligned}
l_V(\Gamma, \pi, \lambda) &= \sum_{W \subseteq V} (-1)^{d-\#W} h(\Gamma_W, \pi_W, \lambda) \\
&= \sum_{W \subseteq V} (-1)^{d-\#W} \sum_{F \in \Gamma_W} \lambda^{\pi(F)} (1-\lambda)^{W \setminus \pi(F)} \\
&= \sum_{F \in \Gamma} \lambda^{\pi(F)} \sum_{\sigma(F) \subseteq W \subseteq V} (1-\lambda)^{W \setminus \pi(F)} (-1)^{\#W-\#F} (-1)^{d-\#F} \\
&= \sum_{F \in \Gamma} \lambda^{\pi(F)} (\lambda-1)^{-\pi(F)} (\lambda-1)^{\sigma(F)} \sum_{\sigma(F) \subseteq W \subseteq V} (\lambda-1)^{W \setminus \sigma(F)} (-1)^{d-\#F} \\
&= \sum_{F \in \Gamma} \lambda^{\pi(F)} \lambda^{V \setminus \sigma(F)} (\lambda-1)^{\sigma(F)} (\lambda-1)^{-\pi(F)} (-1)^{d-\#F}. \quad \square
\end{aligned}$$

Definition The *reduced Euler characteristic* $\tilde{\mathcal{X}}(\Delta)$ of a simplicial complex Δ is the alternating sum $-1 + f_0 - f_1 + f_2 - \ldots$, where f_i is the number of i-faces of Δ, for each i.

Lemma 2.6.2 *If (Γ, π) is a completely balanced $(d-1)$-dimensional simplicial complex, then*

$$\sum_{F\in\Gamma}(-1)^{d-\#F}h(\mathrm{lk}_\Gamma F, \pi, \lambda) = -(\lambda-1)^{\pi(\Gamma)}\tilde{\mathcal{X}}(\Gamma).$$

Proof:

$$\begin{aligned}
\sum_{F\in\Gamma}(-1)^{d-\#F}h(\mathrm{lk}_\Gamma F,\pi,\lambda) &= \sum_{F\in\Gamma}(-1)^{d-\#F}\sum_{\Gamma\ni G\supseteq F}\lambda^{\pi(G\setminus F)}(1-\lambda)^{\pi(\mathrm{lk}_\Gamma F)\setminus\pi(G\setminus F)}\\
&= \sum_{G\in\Gamma}(-1)^{d}\lambda^{\pi(G)}(1-\lambda)^{\pi(\Gamma)\setminus\pi(G)}\sum_{F\subseteq G}(-\lambda)^{-\pi(F)}\\
&= \sum_{G\in\Gamma}(-1)^{d}\lambda^{\pi(G)}(1-\lambda)^{\pi(\Gamma)\setminus\pi(G)}(1-\lambda^{-1})^{\pi(G)}\\
&= \sum_{G\in\Gamma}(-1)^{d}(1-\lambda)^{\pi(\Gamma)\setminus\pi(G)}(\lambda-1)^{\pi(G)}\\
&= (\lambda-1)^{\pi(\Gamma)}\sum_{G\in\Gamma}(-1)^{\#G}\\
&= -(\lambda-1)^{\pi(\Gamma)}\tilde{\mathcal{X}}(\Gamma). \quad \square
\end{aligned}$$

Theorem 2.6.3 *Let (Δ', π') be a balanced subdivision of a completely balanced $(d-1)$-dimensional simplicial complex (Δ, π). Then*

$$h(\Delta', \pi', \lambda) = \sum_{F\in\Delta} l_F(\Delta'_F, \pi'_F, \lambda)h(\mathrm{lk}_\Delta F, \pi, \lambda).$$

Proof:

$$\begin{aligned}
\sum_{F\in\Delta} l_F(\Delta'_F,\pi'_F,\lambda)h(\mathrm{lk}_\Delta F,\pi,\lambda) &= \sum_{F\in\Delta} h(\mathrm{lk}_\Delta F,\pi,\lambda)\sum_{G\subseteq F}(-1)^{\#(F\setminus G)}h(\Delta'_G,\pi'_G,\lambda)\\
&= \sum_{G\in\Delta} h(\Delta'_G,\pi'_G,\lambda)\sum_{F\supseteq G}(-1)^{\#(F\setminus G)}h(\mathrm{lk}_\Delta F,\pi,\lambda)\\
&= \sum_{G\in\Delta} h(\Delta'_G,\pi'_G,\lambda)\sum_{H\in\mathrm{lk}_\Delta G}(-1)^{\#H}h(\mathrm{lk}_{\mathrm{lk}_\Delta G}H,\pi,\lambda)
\end{aligned}$$

(by Lemma 7)

$$
\begin{aligned}
&= -\sum_{G\in\Delta} h(\Delta'_G,\pi'_G,\lambda)(1-\lambda)^{\pi(\Delta)\setminus\pi(G)}\tilde{\mathcal{X}}(\mathrm{lk}_\Delta G)\\
&= -\sum_{G\in\Delta}\Big(\sum_{K\in\Delta'_G}\lambda^{\pi(K)}(1-\lambda)^{\pi(G)\setminus\pi(K)}(1-\lambda)^{\pi(\Delta)\setminus\pi(G)}\tilde{\mathcal{X}}(\mathrm{lk}_\Delta G)\\
&= -\sum_{K\in\Delta'}\lambda^{\pi(K)}(1-\lambda)^{\pi(\Delta)\setminus\pi(K)}\sum_{G\supseteq\sigma(K)}\tilde{\mathcal{X}}(\mathrm{lk}_\Delta G).
\end{aligned}
$$

If μ is the Möbius function of $P(\Delta)\cup\hat{1}$ (See [Stan86, Sections 3.7-3.8]), then

$$
\sum_{G\supseteq\sigma(K)}\tilde{\mathcal{X}}(\mathrm{lk}_\Delta G)=\sum_{G\supseteq\sigma(K)}\mu(G,\hat{1})=-\mu(\hat{1},\hat{1})=-1,
$$

so

$$
\begin{aligned}
\sum_{F\in\Delta} l_F(\Delta'_F,\pi'_F,\lambda)h(\mathrm{lk}_\Delta F,\pi,\lambda) &= \sum_{K\in\Delta'}\lambda^{\pi(K)}(1-\lambda)^{\pi(\Delta)\setminus\pi(K)}\\
&= h(\Delta',\pi',\lambda).\quad\square
\end{aligned}
$$

Theorem 2.6.4 *If (Γ,π) is a balanced subdivision of $(2^V,id)$, then $l_V(\Gamma,\pi,\lambda)\geq 0$.*

Proof: We may assume that $[n]$ is the vertex set of Γ. We can give $K[\Gamma]$ an $\mathbf{N}^d$-grading by $\deg(x_i)=e_{\pi(i)}\in\mathbf{N}^d$. Let $\theta_i=\sum_{\pi(j)=i}x_j$ for each $i\in[d]$. Then $\{\theta_1,\theta_2,\ldots,\theta_d\}$ is a h.s.o.p. for $K[\Gamma]$ (see [Stan79, Corollary 4.2]). Since $|\Gamma|$ is homeomorphic to a ball, Γ is Cohen-Macaulay, so $F(K[\Gamma]/(\theta))=h(\Gamma,\pi,\lambda)$ (see [Stan79] for proof). Now define the *local face-module* $L_V(\Gamma)$ as follows:

Let $(\mathrm{int}\Gamma)$ be the *interior ideal* $(x^F:\sigma(F)=V)\subset K[\Gamma]$, and $L_V(\Gamma)$ the image of $(\mathrm{int}\Gamma)$ in $K[\Gamma]/(\theta)$. So $L_V(\Gamma)$ inherits the above $\mathbf{N}^d$-grading. We will show that for each $S\subseteq V$, $\dim_K L_S=l_S(\Gamma,V,\pi)$, where L_S is the e^S-th graded piece of $L_V(\Gamma)$.

First we need the following technical lemma:

Lemma 2.6.5 *Let $\mathcal{K}$ be the complex of $K[\Gamma]$-modules*

$$
K[\Gamma]\to\coprod_i\frac{K[\Gamma]}{N_i}\to\coprod_{i<j}\frac{K[\Gamma]}{N_i+N_j}\to\ldots\to\frac{K[\Gamma]}{N_1+\ldots+N_d}\to 0,
$$

where $N_i = (x^F : x_i \in \sigma(F))$ *for each* i*. Then the complex* $\mathcal{K}/\theta\mathcal{K}$ *is exact.*

Proof: See [Stan92, Lemma 4.9].

Note that $\theta_i \in N_i$ for each i, and $K[\Gamma]/(\sum_{i\in S} N_i) \simeq K[\Gamma_{V\setminus S}]$ for each $S \subseteq V$.

Now let $\delta^0, \delta^1, \ldots$ denote the maps in $\mathcal{K}/\theta\mathcal{K}$. Then $\ker(\delta^0)$ is the image of $\cap_i N_i = (\mathrm{int}(\Gamma))$ in $K[\Gamma]/(\theta)$, so $\ker(\delta^0) = L_V(\Gamma)$. Since $\mathcal{K}/\theta\mathcal{K}$ is exact,

$$F(L_V(\Gamma),\lambda) = \sum_{S\subseteq V} (-1)^{\#(V\setminus S)} F(K[\Gamma_S]/(\theta),\lambda).$$

Since Γ is Cohen-Macaulay and $\{\theta_i\}$ is a homogeneous system of parameters for $K[\Gamma]$, for each $S \subseteq V$ we have

$$F(K[\Gamma_S]/(\theta),\lambda) = h(\Gamma_S,\pi,\lambda),$$

so $F(L_V(\Gamma),\lambda) = l_V(\Gamma,\pi,\lambda)$; equivalently $l_S(\Gamma,V,\pi) = \dim_K L_S \geq 0$ for each $S \subseteq V$.
□

Corollary 2.6.6 *If* (Δ',π') *is a balanced subdivision of a completely balanced Cohen-Macaulay complex* (Δ,π)*, then* $h(\Delta',\pi',\lambda) \geq h(\Delta,\pi,\lambda)$.

Proof: For any $F \in \Delta$, since Δ is Cohen-Macaulay, so is $\mathrm{lk}_\Delta F$. So for any $F \in \Delta$, $h(\mathrm{lk}_\Delta F,\pi,\lambda) \geq 0$ (see [Stan79, Theorem 4.4]). By Theorem 4, we also know that $l_F(\Delta'_F,\pi',\lambda) \geq 0$ for each $F \in \Gamma$. So by Theorem 3,

$$\begin{aligned} h(\Delta',\pi',\lambda) &= \sum_{F\in\Delta} l_F(\Delta'_F,\pi'_F,\lambda)h(\mathrm{lk}_\Delta F,\pi,\lambda) \\ &\geq h(\mathrm{lk}_\Delta \emptyset,\pi,\lambda) \\ &= h(\Delta,\pi,\lambda). \quad \Box \end{aligned}$$

Chapter 3

Quasiforests and Degree Sequences By Zonotope Theory

3.1 Introduction

In this chapter we derive a correspondence between the set of quasiforests on n vertices and the set of degree sequences of graphs on n vertices by considering a canonical decomposition of the appropriate zonotope into half-open cubes associated with the quasiforests. Using the same method we formulate a bijective map from the set of forests on n vertices to the set of score vectors of tournaments on n vertices. This work was suggested by R. Stanley's paper, "A Zonotope Associated With Graphical Degree Sequences" ([Stan91]).

3.2 Background

Definitions: For any simple graph G (*i.e.*, no loops or multiple edges) on vertex set $[n]$, the *degree* of vertex i is the number of edges of G containing i, and the *degree sequence of* G is the ordered n-tuple $d(G) = (d_1, d_2, \ldots, d_n)$ where d_i is the degree of vertex i. A *degree sequence of length* n is the degree sequence of any simple graph on $[n]$. The *polytope of degree sequences of length* n, denoted D_n, is the convex hull of all degree sequences of length n, considered as points in $\mathbf{R}^n$.

Fact: The set of points $(d_1, d_2, \ldots, d_n) \in D_n \cap \mathbf{Z}^n$ with even component sum $\sum_{i=1}^{n} d_i$ is exactly the set of degree sequences of length n. (See [Stan91] for details)

D_n is a special kind of polytope, called a *zonotope.*

Definition: A *zonotope* Z is the Minkowski sum of finitely many closed line segments $L_1, L_2, \ldots, L_k$ in $\mathbf{R}^n$, *i.e.*,

$$Z = \{x = \sum_{i=1}^{k} \alpha_i : \alpha_i \in L_i\}.$$

This is a convex polytope with many special properties. (See [McM71]).

Examples: (1) Let $L_1 = [(0,0),(1,0)], L_2 = [(1,0),(2,1)]$ in $\mathbf{R}^2$. Then the zonotope generated by L_1, L_2 is the set of all points of the form $(1,0)+r_1\cdot(1,0)+r_2\cdot(1,1)$ where $0 \leq r_i \leq 1$. This is a parallelogram at the point $(1,0)$, with *basis* $(1,0),(1,1)$. In general, if Z is generated by $\{[0,a_i]\}$, where $\{a_i\}$ is any set of n linearly independent vectors in R^n, then Z is linearly equivalent to a geometric n-dimensional cube. We call Z (and any translation of Z) an *n-cube, e.g.*, the zonotope generated by L_1, L_2 is a 2-cube at the point $(1,0)$. See Figure 1(a).

(2) Let $L_3 = [(0,0),(0,-1)]$ in $\mathbf{R}^2$, and L_1, L_2 as in (1). Then the zonotope generated by L_1, L_2, L_3 is the set of all points of the form $(1,0)+r_1\cdot(1,0)+r_2\cdot(1,1)+r_3\cdot(0,-1)$, where $0 \leq r_i \leq 1$ for each i. This is a hexagon in $\mathbf{R}^2$. See Figure 1(b).

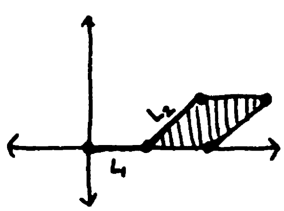


Figure 1(a)

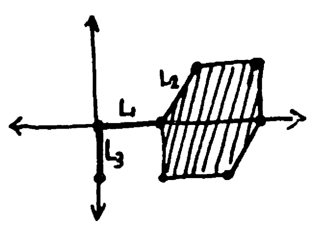


Figure 1(b)

Note: By translation, we can assume that the line segments L_i, called *generators* of Z, all have one endpoint at 0. For convenience, we will usually denote the closed line segment $L_i = [0, a_i]$ simply by its nonzero endpoint a_i. In this paper, Z will denote an n-dimensional zonotope in $\mathbf{R}^n$ and $a_1, a_2, \ldots, a_k$ its generators.

The following proposition is proved in [Stan91].

Proposition 3.2.1 *D_n is a zonotope with generators $\{e_{ij} : 1 \leq i < j \leq n\}$, where $e_{ij} = e_i + e_j$ where e_i, e_j are the standard unit coordinate vectors in $\mathbf{R}^n$.*

Note: We will often let e_{ij} stand for the vector from 0 to e_{ij} as well as for the edge $\{i,j\}$. So we can talk about linearly independent vectors e_{ij} and also about graphs with edges e_{ij}. We will be particularly interested in sets of linearly independent e_{ij}'s, and the corresponding graphs, which are called *quasiforests.*

Definitions: In this paper, all graphs have the vertex set $[n]$. In any graph G, a *cycle of length* s is a set of s edges $\{i,j_1\},\{j_1,j_2\},\ldots,\{j_{s-1},i\}$ in G, such that $i,j_1,j_2,\ldots,j_{s-1}$ are distinct. G is *connected* if there is a path of edges in G from any vertex to any other. A connected graph containing no cycles is called a *tree.* A connected graph with one cycle is called a *quasitree* if the cycle has odd length. A graph made up of trees is called a *forest.* A graph made up of trees and quasitrees is called a *quasiforest.* Quasiforests will be denoted by the letter Q.

It is easy to see that

Proposition 3.2.2 *The graph* $G = \{e_{i_1j_1},\ldots,e_{i_rj_r}\}$ *is a quasiforest if and only if* $\{e_{i_1j_1},\ldots,e_{i_rj_r}\}$ *is linearly independent as a set of vectors in* $\mathbf{R}^n$.

We will use the following notation:

1) If $A \subset \mathbf{R}^n$ is an ordered set $a_1 < \cdots < a_k$ then $A - a_j$ is the ordered set $a_1 < \cdots < a_{j-1} < a_{j+1} < \cdots < a_k$, and for any $c \in \mathbf{R}^n$, $A[c \to a_j]$ denotes the ordered set $a_1 < \cdots < a_{j-1} < c < a_{j+1} < \cdots < a_k$.

2) If $A, B \subset \mathbf{R}^n$ are the ordered sets $a_1 < \cdots < a_k$, $b_1 < \cdots < b_{n-k}$, then (A) denotes the $(k \times n)$-matrix with row vectors $a_1,\ldots,a_k$ in that order, and (A,B) denotes the $(n \times n)$-matrix with row vectors $a_1,\ldots,a_k,b_1,\ldots,b_{n-k}$ in that order.

3) Let M be a square matrix. Then $M_{r,s}$ denotes the matrix M without its r^{th} row and s^{th} column, $\sigma(M)$ denotes the sign of $\det(M)$, and $|M|$ denotes the absolute value of $\det(M)$.

3.3 Cubical dissection of Zonotopes

finitions: (1) If C is an affine n-cube and H is an affine hyperplane in $\mathbf{R}^n$, then H is a *supporting hyperplane* of C if the interior of C lies entirely on one side of H. A *face* of C is either C itself, or the intersection of C with any supporting hyperplane.

A *facet* of C is an $(n-1)$-dimensional face of C.

(2) An *n-dimensional cubical cell complex* Ω is a set of n- dimensional affine cubes such that

(i) If C is in Ω, then any face of C is also in Ω.

(ii) If C_1, C_2 are in Ω, then $C_1 \cap C_2$ is a face of C_1.

A *facet* of Ω is any facet of an n-cube $C \in \Omega$.

(3) A *cubical dissection* of a set $S \subset \mathbf{R}^n$ is a cubical cell complex Ω such that the set union of all $C \in \Omega$ is S. Then we say that S is the *underlying space* of Ω. Clearly, most sets S do not have a cubical dissection.

Any zonotope Z with generators $a_1, \dots, a_k$ can be cubically dissected as follows (due to Bernd Sturmfels [BLSWZ, Section 2.2]):

Let Z, $\{a_i\}$ be as above. Consider $\mathbf{R}^n \subset \mathbf{R}^{n+1}$ by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$.

(1) Choose *heights* $\mu_i \geq 0$ so that if $\tilde{a}_i = a_i + \mu_i \cdot e_{n+1} \in \mathbf{R}^{n+1}$ for each i, then for any $(n+1)$-set $A = \{z_{i_1}, \dots, z_{i_{n+1}}\}$ spanning $\mathbf{R}^n$, the set $\tilde{A} = \{\tilde{a}_i : a_i \in A\}$ spans $\mathbf{R}^{n+1}$.

(2) Let $\tilde{Z}$ be the $(n+1)$-dimensional zonotope generated by $\tilde{a}_1, \dots, \tilde{a}_k, e_{n+1}$. The *lower boundary* of $\tilde{Z}$ with respect to e_{n+1} is

$$\partial Z = \{x \in \tilde{Z} : \forall \epsilon > 0, x - \epsilon \cdot e_{n+1} \notin \tilde{Z}\}.$$

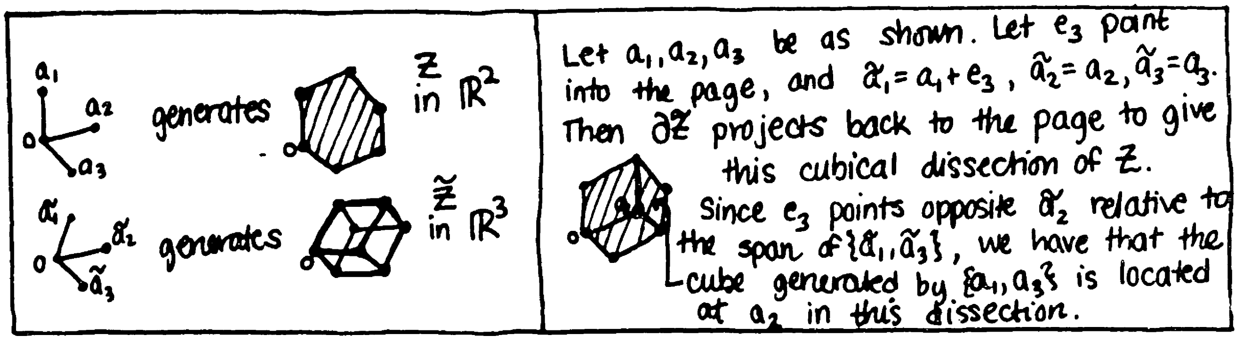
This is a union of n-dimensional faces which projects down to Z by $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$, yielding a cubical dissection Ω_0 of Z.

(3) For each linearly independent $A = \{a_{i_1}, \dots, a_{i_n}\}$, $\tilde{A}$ generates a facet in ∂Z at the point $\sum_{\mathcal{L}(A)} \tilde{a}_j$, where

$\mathcal{L}(A) = \{j : \tilde{a}_j$ points opposite e_{n+1} relative to the hyperplane spanned by $\tilde{A}\}$

$= \{j : \sigma(\tilde{A}, \tilde{a}_j) = -\sigma(\tilde{A}, e_{n+1})\}$.

Thus A generates a cube in Ω_0 at the point $\sum_{\mathcal{L}(A)} a_j$. (See Figure 2.)

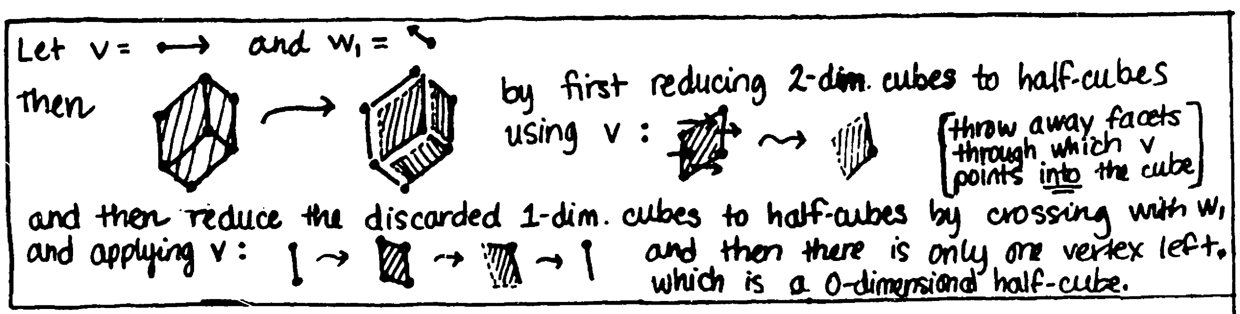


(4) Finding the sets $\mathcal{L}(A)$ is relatively simple if we fix an order on the a_i's and choose the μ_i's so that whenever $a_i < a_j$, μ_i is sufficiently larger than μ_j so that for any $A = \{a_1, \ldots, a_{n+1}\}$ spanning $\mathbf{R}^n$, if a_{n+1} =min$\{a_j \in A\}$, we have $\mu_{n+1}|A - a_{n+1}| > \sum_{i \leq n} \mu_i |A - a_i|$, so $\sigma(\tilde{A}) = \sigma(A - a_{n+1})$.

3.4 Half-cube Decomposition of Zonotopes

Definition: A *half-open cube*, or *half-cube*, is the Minkowski sum of half-open intervals. A half-cube is said to be *generated* by the linearly independent vectors $b_1, \ldots, b_r \in \mathbf{R}^n$ if it is the Minkowski sum of the half-open intervals $[0, b_1), \ldots, [0, b_r)$. An n-dimensional half-cube will be called a *half-n-cube*.

We can decompose Z into half-cubes generated by linearly independent sets of a_i's. Starting with the cubical dissection from Section 2, we will use a *direction vector* v to reduce n-cubes to half-n-cubes, and then *cross-up vectors* $w_1, w_2, ..., w_{n-1}$ to generate n-cubes from the lower-dimensional cubes which remain at each step. (See Figure 3.)



Choose $v, w_1, w_2, \ldots, w_{n-1} \in \mathbf{R}^n$ satisfying:

(*) If $j \leq i$, then w_j points opposite v on each hyperplane generated by $\{w_1, \ldots, w_i, a_{r_1}, \ldots, a_{r_{n-i}}\} \setminus \{w_j\}$, where $\{a_{r_1}, \ldots, a_{r_{n-i}}\}$ is a linearly independent set of generators of Z. This property is crucial for the half-cube decomposition method below.

Notation: If Ω is an $(n-1)$-dimensional cell complex in $\mathbf{R}^n$ and $w \in \mathbf{R}^n$, then $\Omega \times w$ denotes the Minkowski sum of Ω and $[0,w]$, *i.e.*,

$$\Omega \times w = \{x + y : x \in \Omega, y \in [0,w]\}.$$

A *cell* of $\Omega \times w$ is the Minkowski sum of a cell of Ω and $[0,w]$.

Let $W_0 = Z$ and let Ω_0 be the cubical dissection of Z from Sec. 2. Let Ω_0 denote the set of $(n-1)$-faces of Ω_0 through which v points into W_0. For $1 \le i \le n-1$, let Ω_i be the set of n-cubes in $\Omega_{i-1} \cup (\Omega'_{i-1} \times w_i)$, W_i the zonotope generated by $a_1, a_2, \ldots, a_k, w_1, w_2, \ldots, w_i$, and Ω'_i the set of $(n-1)$-faces of Ω_i through which v points into W_i.

Lemma 3.4.1 *Let* $1 \le i \le n-1$. *Then* Ω_i *is a cubical complex with underlying space* W_i.

Proof: We already have that Ω_0 is a cubical complex with underlying space W_0. Fix i. We can assume by induction that Ω_{i-1} is a cubical complex, so Ω'_{i-1} is a cubical complex as well. We now show that $\Omega'_{i-1} \times w_i$ is a cubical complex. For all $(n-1)$-dimensional faces C_1 and C_2 in Ω'_{i-1}, if there exists a point x in $(C_1 \times w_i) \cap (C_2 \times w_i)$, then $x = a + \lambda \cdot w_i = b + \mu \cdot w_i$ for some $a \in C_1, b \in C_2, \lambda, \mu \in [0,1]$. So if $a \neq b$ then the line L through a and b is parallel to w_i. Since W_{i-1} is convex, the part of L between a and b is in W_{i-1}. But then property (*) implies that w_i points in opposite directions (, *i.e.*, out of W_{i-1}) at a and b. This is impossible. Thus $a = b$. So $(C_1 \times w_i) \cap (C_2 \times w_i) \subseteq (C_1 \cap C_2) \times w_i$. So $(C_1 \times w_i) \cap (C_2 \times w_i) = (C_1 \cap C_2) \times w_i = F \times w_i$ for some face F of C_1. So any intersection of faces of $\Omega'_{i-1} \times w_i$ is still a face of $\Omega'_{i-1} \times w_i$. So Ω'_{i-1} is a cubical complex.

Appealing to the property (*) again, we have that the faces of $\Omega'_{i-1} \times w_i$ intersect the faces of Ω_{i-1} only on faces of Ω'_{i-1}. So since Ω_{i-1} and $\Omega'_{i-1} \times w_i$ are both cubical complexes, their union Ω_i itself is also a cubical complex. Since W_{i-1} is the underlying space of Ω_{i-1} (by inductive hypothesis), it is clear that W_i is the underlying space of Ω_i. □

Now let $\Omega = \Omega_{n-1}$. So Ω is a cubical dissection of W_{n-1}. Then we have the following

Lemma 3.4.2 *Ω contains one cube generated by each n-independent set of generators of W_{n-1}.*

Proof: Since Ω is a cubical dissection of W_{n-1}, it is the projection of the lower boundary $\partial\tilde{W}$ of an $(n+1)$-dimensional zonotope $\tilde{W}$ (see [BLSWZ, Section 2.2]). So each n- independent set of generators of W_{n-1} corresponds to a set of generators of a pair of antipodal facets of $\tilde{W}$, exactly one of which belongs to $\partial\tilde{W}$. So each n-independent set of generators of W_{n-1} generates exactly one cube in Ω. □

Now use v to partially decompose W_{n-1} into half-cubes, *i.e.*, for each n-cube C of Ω, throw away all facets through which v points into C. The resulting half-cube is denoted $\dot{C}$.

Lemma 3.4.3 *These half-cubes are disjoint.*

Proof: Suppose $\dot{C}_1$ intersects $\dot{C}_2$. We will show that this forces $C_1 = C_2$. Let x be a point in $\dot{C}_1 \cap \dot{C}_2$.

(1) If x is in the interior of C_1: Since $C_1 \cap C_2$ must be some face of C_1, that face must be C_1 itself, since no proper face of C_1 contains any point in the interior of C_1. Then $C_1 = C_1 \cap C_2$, which must also be a face of C_2. The only face of C_2 of dimension at least n is C_2 itself. So $C_1 = C_2$.

(2) If x not in the interior of C_1: Then x belongs to some facet of C_1. Since x can not be in any facet of C_1 through which v points into C_1, $-v$ points into C_1 through every facet containing x. Thus there exists $\epsilon > 0$ such that $x - \epsilon \cdot v$ is a point of $\text{Int}(C_1)$. But then by argument (1), $C_1 = C_2$. □

Note that, due to the property (*), there is only one discarded $(n-1)$-dimensional face of W_{n-1} which intersects Z. This is generated by $w_1, \ldots, w_{n-1}$, and its intersection with Z is a single point x_0. So the rest of Z belongs to the union of the half-cubes $\dot{C}$ in W_{n-1}. Any such half-cube intersects Z if and only if its generating set is of the form $A \cup \{w_1, w_2, \ldots, w_{n-i}\}$ where A is an i-independent set of a_l's, and $1 \leq i \leq n-1$. Its intersection with Z is then an i-dimensional half-cube generated by A. Since the original half-cubes $\dot{C}$ are disjoint, their intersections with Z are also disjoint.

Thus we get a complete decomposition of Z into half-cubes, one generated by each independent set of a_i's (where the point x_0 is the half-cube generated by $\emptyset$).

3.5 Cubical dissection of D_n

Restricting our attention now to the zonotope D_n, we see that D_n can be dissected into a cell complex of n-cubes generated by maximal quasiforests Q on n vertices. We use the method in Section 2 to cubically dissect D_n,using lexicographic order on the e_{ij}'s, that is, $e_{ij} <_L e_{kl}$ if and only if (1) $i < k$; or (2) $i = k$ and $j < l$. In order to describe the dissection in terms of quasiforests, we need some definitions:

Definitions: If Q has k edges, it is called a *k-quasiforest.* If two vertices belong to the same component in Q, they are *connected* by that component. If V_1, V_2 are two sets of vertices such that some vertex of V_1 is connected to some vertex of V_2 by some component in Q, then V_1, V_2 are *connected* by that component. If $Q \cup e_{ij}$ is not a quasiforest, but $(Q \cup e_{ij}) - e_{kl}$ is, then we say e_{ij} *replaces* e_{kl} in Q. If that is so, then $Q - e_{kl}$ must contain a tree connecting $\{k, l\}$ to $\{i, j\}$, and the number of edges from $\{k, l\}$ to $\{i, j\}$ in such a tree is always even or always odd. (*i.e.*, independent of chosen tree.) The *tree-distance* from e_{ij} to e_{kl} is the smallest such number of edges. For each (i, j), let $\tilde{e}_{ij} = e_{ij} + \mu_{ij} \cdot e_{n+1}$ where the μ_{ij}'s are as in Section2. So if Q is an n-quasiforest and if $Q \cup e_{ij}$ is not a quasiforest, and if $e_{ij} <_L e_{st}$ for all replaceable $e_{st} \in Q$, then $\sigma(\tilde{Q}, \tilde{e}_{ij}) = \sigma(Q)$.
As in Section 2, let $\tilde{D}_n$ be the $(n+1)$-dimensional zonotope generated by the $\tilde{e}_{ij}$'s,

and project its lower boundary onto D_n to get a cubical dissection Ω_0.

Lemma 3.5.1 *In Ω_n, each n- edge quasiforest Q generates a cube C_Q at the point $\sum_{\mathcal{L}(Q)} e_{ij}$, where $\mathcal{L}(Q) = \{e_{ij} \notin Q$: the least replaceable edge of Q is $<_L e_{ij}$ and is even tree-distance from $e_{ij}\}$.*

Before proving the Lemma, we need the following

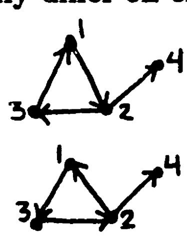
Lemmina 3.5.2 *Let $[q_{ij}] = (Q)$, the incidence matrix of Q with edges in lexicographic order.*

(A) If Q has n edges, then $\sigma(Q)$ =sgn(π), for any $\pi \in S_n$ such that $q_{i,\pi(i)} = 1$ for each i;

(B) If Q has $(n-1)$ edges and $Q \cup e_{ij}$, $Q \cup e_{jk}$ are n-quasiforests, then $\sigma(Q, e_{ij}) = \sigma(Q, e_{jk})$.

Proof: (A) Since all q_{ij} are 0 or 1, $\det(Q) = \sum_{S_k} \mathrm{sgn}(\pi) q_{1\pi(1)} \cdots q_{k\pi(k)} = \sum_T \mathrm{sgn}(\pi)$, where $T = \{\pi \in S_k : \forall i, q_{i,\pi(i)} = 1\}$.

If $\pi \in T$, then for each i, vertex $\pi(i)$ belongs to the i^{th} edge of Q. So $\pi, \tau \in T$ can only differ on entire cycles of Q as shown in Figure 4.



For n=4 and Q = {e12, e13, e23, e34}, π∈T must be one of the two possibilities shown. (Arrows indicate which vertex is assigned to which edge.)

Figure 4

Since all cycles in Q have odd length, π, τ can only differ by an even permutation, so sgn(π) =sgn(τ). Thus $\sigma(Q)$ =sgn(π) for any $\pi \in T$.

(B) Since j is in a tree in Q, we can assign to each edge of Q a vertex $\neq j$ contained in that edge. So we can find $\pi \in S_n$ such that $\pi(n) = j$ and $q_{i,\pi(i)} = 1$ for each $i \leq n-1$. Thus $\sigma(Q, e_{ij})$ =sgn(π) $= \sigma(Q, e_{jk})$, by (A). □

Proof of Lemma: It suffices to show that

$$\mathcal{L}(Q) = \{e_{ij} : \sigma(\tilde{Q}, \tilde{e}_{ij}) = -\sigma(\tilde{Q}, e_{n+1})\}.$$

First we show:

(1)$\sigma(\tilde{Q}, \tilde{e}_{ij}) = \sigma(\tilde{Q}, e_{n+1})$ when $e_{ij} <_L$ all replaceable edges in Q, and

(2)$\sigma(\tilde{Q}, \tilde{e}_{ij}) = -\sigma(\tilde{Q}, e_{n+1})$ when e_{ij} is 0 tree-distance from the least replaceable edge of Q.

(1) This is clear, since in this case μ_{ij} is so relatively large that LHS = $\sigma(Q)$, which equals RHS.

(2) If e_{ij} is 0 tree-distance from the least replaceable edge $e_{i_l j_l}$, we may assume $i = i_l$. Then

$$\begin{aligned}
\sigma(\tilde{Q}, \tilde{e}_{ij}) &= -\sigma(\tilde{Q}[\tilde{e}_{ij} \to \tilde{e}_{i_l j_l}], \tilde{e}_{i_l j_l}) \\
&= -\sigma(Q[e_{ij} \to e_{i_l j_l}]) \\
&= -\sigma(Q) \quad \text{(by the Lemmina)} \\
&= -\sigma(\tilde{Q}, e_{n+1}).
\end{aligned}$$

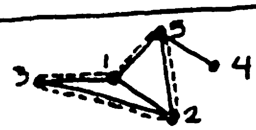
Now assume the hypothesis is true for tree-distance $< t$ and prove for t :

Assume e_{jk} is a tree-edge adjacent to e_{ij} towards $e_{i_l j_l}$. If e_{jk} is odd tree-distance from $e_{i_l j_l}$ then

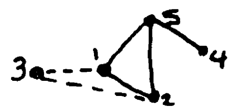
$$\begin{aligned}
\sigma(\tilde{Q}, \tilde{e}_{ij}) &= -\sigma(\tilde{Q}[\tilde{e}_{ij} \to \tilde{e}_{jk}], \tilde{e}_{jk}) \\
&= -\sigma(\tilde{Q}[\tilde{e}_{ij} \to \tilde{e}_{jk}], e_{n+1}) \quad \text{(by induction)} \\
&= -\sigma(\tilde{Q}, e_{n+1}) \quad \text{(as above).}
\end{aligned}$$

Similarly, if e_{jk} is even tree-distance from $e_{i_l j_l}$, then $\sigma(\tilde{Q}, \tilde{e}_{ij}) = \sigma(\tilde{Q}, e_{n+1})$. □

Example: Let $n = 5, Q = \{e_{12}, e_{13}, e_{15}, e_{25}, e_{45}\}, e_{ij} = e_{23}$. Then the least replaceable edge is e_{13}. In $Q - e_{13}$ the tree-distance from e_{13} to e_{23} is 0 since both edges are adjacent to the 0-edged tree $\{3\}$. So by the above proof, $\sigma(\tilde{Q}, \tilde{e}_{23}) = \sigma(\tilde{Q}, \tilde{e}_6)$ and $e_{23} \in \mathcal{L}(Q)$. (see Figure 5.)



$Q \cup \{e_{23}\}$ contains an even cycle (shown with dashed edges), so the least replaceable edge is the least edge in this cycle, e_{13}.



In $Q \setminus \{e_{13}\}$, $\{1,3\}$ and $\{2,3\}$ are connected by the vertex 3, which is a tree, so tree-distance from e_{13} to e_{23} is 0 (even). Thus $e_{23} \in \mathcal{L}(Q)$.

3.6 Half-cube decomposition of D_n

Now we decompose D_n into half-cubes generated by quasiforests. If $v, w_1, \ldots, w_{n-1}$ are carefully chosen, our task is relatively easy. In particular, we want to make it as easy as possible to compute $|A|$, whenever A is a matrix with row vectors $w_{k_1}, \ldots, w_{k_i}, Q$ for Q an $(n-i)$-quasiforest. This is of interest to us since we use such matrices to determine where the n-cubes of Ω lie, as well as which vertices their half-n-cubes contain.

Let $w_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,n})$ for each i, and let $q_{r,s}$ denote the $(r,s)^{th}$ entry of the incidence matrix (Q). Note that $\det(w_{k_1}, \ldots, w_{k_i}, Q)$ is the sum of terms of the form $f_w(\pi) = \mathrm{sgn}(\pi) w_{k_1,\pi(1)} \cdots w_{k_i,\pi(i)} q_{1,\pi(i+1)} \cdots q_{n-i,\pi(n)}$, where $\pi \in S_n$ such that $\pi(1), \ldots, \pi(i)$ is a set of vertices each from a different tree in Q. (Otherwise $q_{k,\pi(i+k)} = 0$ for some $1 \le k \le n-i$, because of the placement of 0's and 1's in the rows of (Q).) Also, if $\pi, \tau \in S_n$ and $\pi(j) = \tau(j)$ for $1 \le j \le i$, then $f_w(\pi) = f_w(\tau)$, by reasoning similar to that in the proof of the Lemmina. We would like there to be some choice of values for $\pi(1), \ldots, \pi(i)$ which would make $f_w(\pi)$ so large (in absolute value) that it determines the sign of the whole determinant.

We first choose $u_0, u_1 \ldots, u_{n-1} \in \mathbf{R}^n_{>0}$ so that $\det(u_{k_1}, \ldots, u_{k_i}, Q)$ is dominated by $f_u(\pi)$ for certain π, as described above. Then we let $w_j = (-1)^j u_j$, so that for the same π, $f_w(\pi)$ dominates $\det(w_{k_1}, \ldots, w_{k_i}, Q)$, and we show that condition (*) from Section 3 is satisfied by $\{w_j\}$.

For each $0 \le j \le n-1$, let $u_j = (u_{j,1}, u_{j,2}, \ldots, u_{j,n})$ where $u_{j,1} > u_{j,2} > \cdots > u_{j,n} > 0$ such that for any $(n-i)$-quasiforest Q, if A is the matrix $(u_{k_1}, \ldots, u_{k_i}, Q)$, where $k_1 > k_2 > \cdots > k_i$, then $u_{k_1,j} \cdot |A_{1,j}| > \sum_{m>j} u_{k_1,m} \cdot |A_{1,m}|$, for all j such that $|A_{1,j}| \neq 0$. Then clearly $\det(u_{k_1}, \ldots, u_{k_i}, Q)$ is dominated by $f_u(\pi)$ where $\pi(1) < \cdots < \pi(i)$ are each the smallest vertex of some tree in Q.

Lemmina 3.6.1 *Let $v = u_0$ and $w_i = (-1)^i u_i$ for each i. Each w_i points opposite v relative to the span of $\{w_1, \ldots, \hat{w}_i, \ldots, w_j\} \cup Q$, where Q is any $(n-j)$-quasiforest.*

Proof:

$$\begin{aligned}\sigma(w_j, \ldots, \hat{w}_i, \ldots, w_1, w_i, Q) &= (-1)^{i-1}\sigma(w_j, \ldots, w_1, Q) \\ &= -\sigma(w_j, \ldots, w_{i+1}, -w_i, -w_{i-1}, \ldots, -w_1, Q) \\ &= -\sigma(w_j, \ldots, w_{i+1}, w_{i-1}, w_{i-2}, \ldots, v, Q). \quad \square\end{aligned}$$

Thus the condition (*) in Section 3 is satisfied, so the half-cube decomposition method applies.

For each $(n-i)$-edge quasiforest Q, let C_Q be the n-cube in Ω generated by $Q \cup \{w_1, \ldots, w_i\}$. The vertex contained in its associated half-cube $\dot{C}_Q$ is the intersection of the facets of C_Q through which v points out of C_Q.

Definitions: If i is the least vertex in some tree of Q, then i is called an *index* of Q. For each e_{jk}, if i is the greatest index connected to e_{jk} in $Q - e_{jk}$, let $r(Q, e_{jk})$ denote the least number of edges from e_{jk} to i in $Q - e_{jk}$, and $s(Q, e_{jk})$ the number of indices larger than i in Q.

Lemma 3.6.2 *C_Q lies at the point $\sum_{\mathcal{L}(Q)} e_{ij}$ and contains the vertex $\sum_{\mathcal{L}(Q)\cup\mathcal{P}(Q)} e_{ij}$, where $\mathcal{L}(Q) = \{e_{ij} \notin Q :$ (1) $e_{ij} \cup Q$ is a quasiforest and $s(Q, e_{ij}) + r(Q, e_{ij})$ is odd, or (2) $e_{ij} \cup Q$ not quasiforest and the least replaceable edge in Q is $<_L e_{ij}$ and an even tree-distance from $e_{ij}\}$, and $\mathcal{P}(Q) = \{e_{ij} \in Q : s(Q, e_{ij}) + r(Q, e_{ij})$ is even $\}$.*

For the proof, we first need the following

Lemmina 3.6.3 *Let $w(k)$ denote the ordered set $\{w_k, \ldots, w_1\}$. If Q is an $(n-i)$-edge quasiforest and $e_{ij} \notin Q$, then*

(A) If $Q \cup e_{ij}$ is a quasiforest, $\sigma(w(i-1), e_{ij}, Q) = (-1)^{r+s}\sigma(w(i-1), v, Q)$, where r, s denote $r(Q, e_{ij}), s(Q, e_{ij})$.

(B) If $Q \cup e_{ij}$ not a quasiforest, then if t is the tree-distance from e_{ij} to the least

replaceable edge $e_{i_t j_t}$ *in* Q, $\sigma(\tilde{Q}', \tilde{e_{ij}}) = (-1)^{t+1}\sigma(\tilde{Q}', e_{n+1})$ *for any* n*-edge quasiforest* Q' *containing* Q.

(If $e_{ij} <_L e_{i_t j_t}$ *then we define* $t = -1$.*)*

Proof: (A) Since Q is an $(n-i)$-edge quasiforest on n vertices it must contain i tree-components. Let $j_1 > \cdots > j_i$ be the indices of the trees in Q. If $r = 0$, then vertex i or j is the index j_{s+1}, so

$$\begin{aligned}
\sigma(w(i-1), e_{ij}, Q) &= \sigma(w(i-1), e_{j_{s+1}}, Q) \\
&= \sigma(w_{i-1}, \ldots, w_{s+1}, (-1)^s e_{j_{s+1}}, w(s), Q) \\
&= \sigma(w_{i-1}, \ldots, w_{s+1}, (-1)^s e_{j_{s+1}}, (-1)^s e_{j_s}, \ldots, -e_{j_1}, Q) \\
&= \sigma(w_{i-1}, \ldots, w_{s+1}, w_s, -w_{s-1}, \ldots, -w_1, -v, Q) \\
&= (-1)^s \sigma(w(i-1), v, Q).
\end{aligned}$$

Now we proceed by induction on r. Let e_{jk} be the first edge of the shortest path in Q from e_{ij} to j_{s+1} (so we assume j is connected to j_{s+1}). Then

$$\begin{aligned}
\sigma(e_{ij}, w(i-1), Q) &= -\sigma(e_{jk}, w(i-1), Q[e_{ij} \to e_{jk}]) \\
&= -(-1)^{r-1+s}\sigma(v, w(i-1), Q[e_{ij} \to e_{jk}]) \quad \text{(by induction)} \\
&= (-1)^{r+s}\sigma(v, w(i-1), Q[e_j \to e_{jk}]) \\
&= (-1)^{r+s}\sigma(v, w(i-1), Q).
\end{aligned}$$

(B) Since the edges in Q' replaceable by e_{ij} are exactly the edges in Q replaceable by e_{ij}, the proof of the previous Lemma applies. □

Proof of Lemma: (1) By the construction of Ω, we know that $C_Q \cap \Omega_{i-1}$ is an $(n-1)$-cube F_Q through which v points into W_{i-1}. F_Q is generated by $Q \cup w(i-1)$ and is located at $\sum_{\mathcal{F}} e_{ij}$, for some set $\mathcal{F}$ of e_{ij}'s not in Q. In particular, if $e_{ij} \notin Q$ and $e_{ij} \cup Q$ is a quasiforest, then $e_{ij} \in \mathcal{F}$ exactly when $\sigma(e_{ij}, w(i-1), Q) = -\sigma(v, w(i-1), Q)$, which is exactly when $e_{ij} \in \mathcal{L}(Q)$, by the Lemmina. If $e_{ij} \notin Q$ and $e_{ij} \cup Q$ is not a quasiforest, then by the Lemmina, if Q' is any n-edge quasiforest containing Q, then $\sigma(\tilde{Q}', \tilde{e_{ij}}) = (-1)^{t+1}\sigma(\tilde{Q}', e_{n+1})$. So $e_{ij} \in \mathcal{F}$ if and only if e_{ij} is even tree-distance from the least replaceable edge in Q, which is exactly when $e_{ij} \in \mathcal{L}(Q)$. Thus C_Q is located at $\mathcal{F} = \mathcal{L}(Q)$.

(2) Since $\dot{C}_Q$ lies at the point $x_L = \mathcal{L}(Q)$, it contains the vertex $x_L + \sum_{\mathcal{P}} e_{ij}$ where $e_{ij} \in \mathcal{P}$ if and only if $e_{ij} \in Q$ and $\sigma(e_{ij}, w(i), Q - e_{ij}) = \sigma(v, w(i), Q - e_{ij})$. By the Lemmina, these conditions are satisfied if and only if $e_{ij} \in \mathcal{P}(Q)$. □

3.7 The correspondence between quasiforests and degree sequences

Now we can formulate the half-cube assignments in terms of the corresponding quasiforests, to get the correspondence between quasiforests and degree sequences. As suggested in [Stan91], we will associate each quasiforest Q with the degree sequences in its corresponding half-cube $\dot{D}_Q = \dot{C}_Q \cap D_n$.

First we want to know which degree sequences lie in $\dot{D}_Q$. We know $\dot{D}_Q$ lies at some point $x = (x_1, \ldots, x_n) \in \mathbf{Z}^n \cap D_n$ with $\sum x_i$ even. Assume $x \in \dot{D}_Q$. (the following is easily modified if a different vertex belongs to $\dot{D}_Q$.)

Then the set of degree sequences in $\dot{D}_Q$ is $\{y = (y_1, \ldots, y_n) \in \mathbf{Z}^n \cap \dot{D}_Q : \sum y_i \text{ is even}\}$ $= \{y \in \mathbf{Z}^n : y = x + \sum_{e_{ij} \in Q} \epsilon_{ij} e_{ij} \text{ where } 0 \le \epsilon_{ij} < 1 \text{ and } \sum y_i \text{ is even}\}$ $\{y = x + \sum_{\mathcal{S}} \frac{1}{2} e_{ij}$ where $\mathcal{S}$ is the set of edges of an even number of cycles in $Q\}$.

(Note: the even number can be 0, *i.e.*, $\mathcal{S} = \emptyset$ is okay.)

So $\#\{$ degree sequences in $\dot{D}_Q\} = \#\{$ choices of even number of cycles in $Q\} = \max\{1, 2^{c(Q)-1}\}$.

The desired rule is:

$Q \mapsto \{d(Q, \mathcal{S}) : \mathcal{S} = \text{set of even number of cycles in } Q\}$

where

$$d(Q, \mathcal{S}) = \sum_{\mathcal{L}(Q)} e_{ij} + \sum_{\mathcal{P}(Q) - \mathcal{S}} e_{ij} + \sum_{\mathcal{S}} \frac{1}{2} e_{ij}.$$

A chart of this correspondence for $n = 4$ appears at the end of this chapter.

Open Problem: Formulate the "inverse" rule $(d_1, \dots, d_n) \mapsto Q$ which assigns to each degree sequence d the quasiforest Q such that $d \in \dot{D}_Q$.

3.8 The correspondence between forests and score vectors

Definitions: A *tournament* on $V = [n]$ is the complete graph on n vertices with a specified direction on each edge. The *score vector* of a tournament T is $s(T) = (s_1, \dots, s_n)$ where s_i is the number of edges directed away from vertex i. For each $1 \leq i < j \leq n$, let $f_{ij} = e_i - e_j \in \mathbf{R}^n$. Then the set of all score vectors of length n is exactly the set of all points $(0, 1, \dots, n-1) + \sum_{\mathcal{F}} f_{ij}$ where $\mathcal{F}$ is any set of f_{ij}'s. We now decompose the zonotope O_n generated by the f_{ij}'s. Since O_n is $(n-1)$-dimensional, we apply the decomposition method above to $O_n \times w_1$, using direction vector v and cross-up vectors $w_2, \dots, w_{n-1}$. As before, we determine the half-cubes generated by $w_1, \dots, w_{n-i}, F$, where F is any i-edge forest, $i \geq 1$. In this way we decompose O_n into half-cubes $\dot{O}_F$ generated by the forests F, and obtain the following one-to-one correspondence between forests and score vectors:

$F \longleftrightarrow (0, 1, 2, \dots, n-1) + \sum_{\mathcal{L}(F)} f_{ij} + \sum_{\mathcal{P}(F)} f_{ij}$

where $\mathcal{L}(F)$ is the set of all $f_{ij} \notin F$ satisfying one of the following conditions:

(1) $f_{ij} \cup F$ is a forest and $s(F, f_{ij}) + r(F, f_{ij})$ is odd, or

(2) $f_{ij} \cup F$ is not a forest and f_{ij} is an odd number of order changes from the least replaceable edge;

$\mathcal{P}(F) = \{f_{ij} \in F : s(F, f_{ij}) + r(F, f_{ij}) \text{ is even }\}$;

$s(F, f_{ij})$ is the number of order changes in the path of vertices in F from $\{i, j\}$ to the greatest connected index i;

and $r(F, f_{ij})$ is the number of indices in F greater than i.

Notes: (A) In the definition of $\mathcal{L}(F)$, f_{ij} satisfies (2) only if it is lexicographically greater than the least replaceable edge, and the order changes are counted on a path in $f_{ij} \cup F$ avoiding the smallest vertex of the illegal cycle.

(B) An *order change* is a change in order of consecutive vertices on a path of edges. For example, in the path of edges $\{1,4\},\{4,3\},\{3,2\}$ the vertices are ordered $1 < 4 > 3 > 2$, so there is just one order change, at the vertex 4.

It is straightforward to reformulate the above rule as follows:

Given a forest F, let i_x denote the index connected to vertex i, for each i, and direct each edge $\{x,y\}$ of a tournament T by

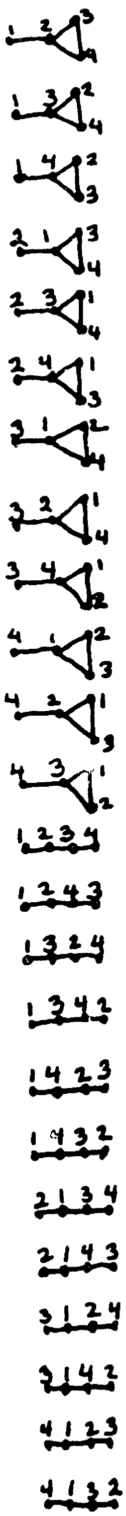
(1) if $i_x < i_y$ and i_y is less than an even number of indices in F, then direct $x \to y$;
if $i_x < i_y$ and i_y is less than an odd number of indices in F, then direct $y \to x$.

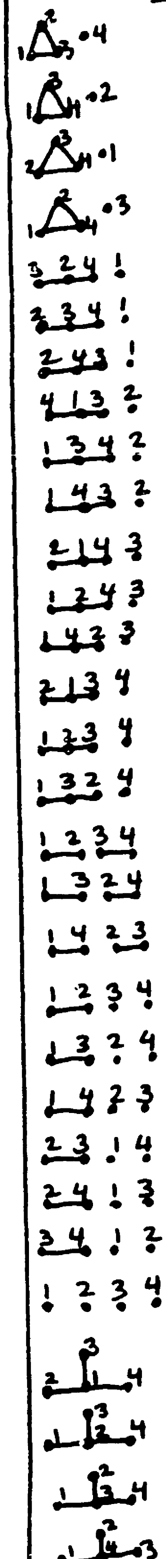
(2) if $\{x,y\} \in F$, let F' denote the forest F with edge $\{x,y\}$ removed, and let i'_x, i'_y denote the indices of x, y in F'. If $i'_x < i'_y$ and i'_y is less than an odd number of indices in F', then direct $x \to y$;
if $i'_x < i'_y$ and i'_y is less than an even number of indices in F', then direct $y \to x$.

(3) if i_{xy} is the smallest index in the path from x to y in F, and if t, u are the vertices adjacent to i_{xy} towards x, y respectively, then direct $x \to y$ if and only if $u < t$.

Then $F \mapsto s(T)$.

Open Problem: Formulate the inverse rule $(s_1, \ldots, s_n) \mapsto F$ which takes each score vector $s(T)$ to the forest F such that $s(T) - (0,1,2,\ldots,n-1) \in \dot{O}_F$.
(Compare to the bijection in [KW].)

Q	$\Sigma_{\mathcal{L}} e_{ij} + (\Sigma_{\mathcal{P}} e_{ij}) = d(Q)$
	$e_{13}+e_{14}+(e_{12}+e_{34}) = (3,1,2,2)$
	$e_{14}+(e_{13}+e_{24}) = (2,1,1,2)$
	$0+(e_{14}+e_{23}) = (1,1,1,1)$
	$e_{23}+e_{24}+(e_{12}+e_{13}+e_{14}) = (3,3,2,2)$
	$0+(e_{13}+e_{14}+e_{23}) = (2,1,2,1)$
	$e_{23}+(e_{13}+e_{14}+e_{24}) = ($[illegible]$,2,2,2)$
	$e_{23}+(e_{12}+e_{13}+e_{14}) = (3,2,2,1)$
	$0+(e_{12}+e_{14}+e_{23}) = (2,2,1,1)$
	$e_{13}+e_{23}+(e_{12}+e_{14}+e_{34}) = (3,2,3,2)$
	$0+(e_{12}+e_{13}+e_{14}) = (3,1,1,1)$
	$e_{14}+(e_{12}+e_{13}+e_{24}) = (3,2,1,2)$
	$e_{14}+e_{24}+(e_{12}+e_{13}+e_{34}) = (3,2,2,3)$
	$e_{14}+e_{24}+(e_{12}+e_{23}+e_{34}) = (2,3,2,3)$
	$e_{13}+e_{23}+(e_{12}+e_{34}) = (2,2,3,1)$
	$e_{14}+e_{34}+(e_{23}+$ [illegible]$) = (1,2,2,3)$
	$e_{23}+(e_{13}+e_{24}) = (1,2,2,1)$
	$e_{34}+(e_{24}+e_{23}) = (0,2,2,2)$
	$e_{24}+(e_{14}+e_{23}) = (1,2,1,2)$
	$e_{23}+e_{24}+(e_{12}+e_{13}+e_{34}) = (2,3,3,2)$
	$e_{24}+e_{23}+(e_{12}+e_{34}) = (1,3,2,2)$
	$e_{23}+(e_{13}+e_{12}+e_{24}) = (2,3,2,1)$
	$e_{34}+e_{23}+(e_{13}+e_{24}) = (1,2,3,2)$
	$e_{24}+(e_{14}+e_{12}+e_{23}) = (2,3,1,2)$
	$e_{34}+(e_{14}+e_{23}) = (1,1,2,2)$

Q	$\Sigma_{\mathcal{L}} e_{ij} + (\Sigma_{\mathcal{P}} e_{ij}) = d(Q)$
	$0+(e_{23}) = (0,1,1,0)$
	$0+(e_{34}) = (0,0,1,1)$
	$0+(e_{23}+e_{24}) = (0,2,1,1)$
	$0+(e_{24}) = (0,1,0,1)$
	$e_{13}+e_{14}+e_{34}+(e_{23}+e_{24}) = (2,2,3,3)$
	$e_{13}+(e_{23}+e_{34}) = (1,1,3,1)$
	$e_{14}+(e_{34}) = (1,0,1,2)$
	$0+(e_{14}+e_{13}) = (2,0,1,1)$
	$e_{14}+(e_{13}+e_{34}) = (2,0,2,2)$
	$e_{13}+(e_{34}) = (1,0,2,1)$
	$0+(e_{14}) = (1,0,0,1)$
	$e_{14}+(e_{24}) = (1,1,0,2)$
	$e_{12}+(e_{14}) = (2,1,0,1)$
	$0+(0) = (0,0,0,0)$
	$e_{13}+(0) = (1,0,1,0)$
	$e_{12}+(e_{13}) = (2,1,1,0)$
	$e_{14}+e_{24}+(e_{34}) = (1,1,1,3)$
	$e_{14}+e_{34}+(e_{13}+e_{24}) = (2,1,2,3)$
	$e_{13}+e_{34}+(e_{14}+e_{23}) = (2,1,3,2)$
	$e_{13}+e_{23}+(e_{12}) = (2,2,2,0)$
	$e_{12}+e_{23}+(0) = (1,2,1,0)$
	$e_{12}+e_{24}+(e_{14}) = (2,2,0,2)$
	$e_{12}+(0) = (1,1,0,0)$
	$e_{12}+(e_{24}) = (1,2,0,1)$
	$e_{14}+e_{24}+e_{12}+(e_{34}) = (2,2,1,3)$
	$e_{13}+e_{23}+(0) = (1,1,2,0)$
	$e_{23}+e_{24}+e_{34}+(e_{12}+e_{13}+e_{14}) = (3,3,3,3)$
	$0+(e_{12}+e_{23}+e_{24}) = (1,3,1,1)$
	$0+(e_{23}+e_{34}) = (0,1,2,1)$
	$0+(e_{24}+e_{34}) = (0,1,1,2)$

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