Simplicial Posets: *f*-vectors and Free Resolutions

by

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B.S., California Institute of Technology (1986)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 1991

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Abstract

Simplicial posets, posets with a $\hat{0}$ element and whose every interval is a Boolean algebra, are a generalization of simplicial complexes, so many results about simplicial complexes can be generalized to simplicial posets.

The *f-vector* of a simplicial complex is widely-studied, and, in particular, the f-vector has been characterized for many classes of simplicial complexes and polytopes. The outstanding problem in the equivalent program for simplicial posets is to characterize the f-vector (equivalently, the *h-vector*) of Gorenstein* simplicial posets. Stanley has almost characterized these h-vectors, but a small infinite segment of potential h-vectors remain in doubt. We show that some of these are not h-vectors of Gorenstein* simplicial posets, primarily by using that the *link* of a Gorenstein* simplicial poset is again a Gorenstein* simplicial poset.

Björner and Kalai completely characterized, for a large class of CW-complexes, including simplicial complexes, which pairs of f-vectors and *Betti sequences* are compatible. We solve the equivalent problem for simplicial posets, showing that some minimal necessary conditions on the f-vectors and Betti sequences are actually sufficient for the pair of sequences to be compatible.

Stanley also defined a ring A_P associated with a simplicial poset P that generalizes the face-ring, $k[\Delta]$ of a simplicial complex Δ. We prove a conjecture of Reiner characterizing when A_P is a *complete intersection ring.* In this case, some minimal sufficient conditions on the poset turn out to be necessary as well.

Hochster, treating $k[\Delta]$ as a $k[V]$-module (for V the vertices of Δ) used free resolutions of $k[\Delta]$ to find the *Betti polynomial* of $k[\Delta]$ and the Hilbert series of the *local cohomology module* of $k[\Delta]$, by splitting the free resolution into subcomplexes. Treating A_P as a $k[V]$-module, we derive the equivalent results for simplicial posets. The proof is similar, but the resulting complex must be split even more finely.

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Acknowledgments

Mathematically, I would like to thank Richard Stanley, who always had ideas and suggestions, even when nothing was working. I also learned a great deal from my fellow graduate students, especially Tom Roby and Vic Reiner, who were always willing to answer questions. Extra thanks also go to Vic for showing me the problem in Chapter 4, and pointing out the flaws in earlier proofs.

I owe a great deal to the late Herb Ryser for introducing me to combinatorics. Rick Wilson became my undergraduate thesis advisor when Herb Ryser died, and helped me learn more than I realized. Kelly still runs that math program where I decided that I would be an engineer (but never a mathematician!) instead of a weatherman.

Financial support was provided by the National Science Foundation and the Mathematics Department at MIT.

A large network of people have helped me to survive and enjoy my stay at MIT. Charles Hershey, Brian Keller, David Nice, and David van Stone were there, at different times, when I needed someone. Tom Roby has had more effect on my life than either of us cares to admit or figure out. Thanks to Brian and the Broadway Bicycle School for cycling, and to Marilyn Richards and a cast of thousands for folk dancing. Marian Wentworth helped me find the courage to try. Michael Lauer and Julia Yang have made our house a home. Mary Helen Michals has made all the difference. My sister and brothers, Julie, Matthew, and Jonathan Duval, keep me in touch with aspects of popular culture that I didn't even know existed.

This thesis is dedicated to my mother, Linda Duval, who, in spite of everything that she herself had to face, has still had the energy left over to be unconditionally supportive.

Contents

List of Notation

$\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{C}$	natural numbers, integers rationals, complex numbers
$\lvert\Delta\rvert$	realization of a simplicial complex Δ
$\mathcal{M}(V), \mathcal{M}(k[V])$	set of monoomials in indeterminates from V
$S \backslash T$	set difference
$\hat{0}$	minimal element of a poset
$\hat{1}$	maximal element of a poset
$\overline{P}$	$P \backslash \{\hat{0}\}$
$\hat{P}$	$P \cup \hat{1}$
∂y	boundary of a face y
$y \lessdot z$	z *covers* y in a poset
$y <_x z$	z covers y in a poset,
	and x is the only atom such that $x \leq z$ and $x \not\leq y$
$A \dot{\cup} B$	disjoint union
2^V	set of all subsetss of V
$\binom{V}{i}$	set of all i-subsets of V
$\mathrm{mub}(x, y)$	set of *m*inimal *u*pper *b*ounds of x and y in a poset
$\emptyset$	the empty set

Chapter 1

Introduction

This thesis is concerned with **simplicial posets**, first studied extensively in their own right in [St7], although [Bj1, GS] mention them first. A poset (partially ordered set) is simplicial if it has a $\hat{0}$ element (*i.e.*, $\hat{0} \leq y$ for all $y \in P$) and (equivalently):

(i) for every $y \in P$, the interval $[\hat{0}, y]$ is a Boolean algebra; or

(ii) every interval $[x, y]$ is a Boolean algebra.

All posets will be assumed to be finite. Simplicial posets are a generalization of **simplicial complexes** ($\Delta \subseteq 2^V$ is a simplicial complex on the set of vertices V if: $V \subseteq \Delta$; and $F \subseteq G \in \Delta \Rightarrow F \in \Delta$) since the face-poset (*i.e.*, poset of elements of Δ, called faces, ordered by inclusion) of a simplicial complex is a simplicial poset. In fact, any simplicial poset that is also a meet-semilattice is the face-poset of a simplicial complex. Simplicial complexes have been studied extensively (references are too numerous to mention; specific references to specific problems will be given below), and most of the motivation for studying simplicial posets comes from generalizing existing results (or questions!) about simplicial complexes. Virtually any statement about simplicial complexes can be phrased for simplicial posets, too; sometimes (see Chapters 2 and 3), it becomes easier to prove, and other times (see Chapters 4 and 5), much more care must be given to extend the simplicial complex proof to simplicial posets. Many times, too, a proof about simplicial complexes carries right over to simplicial posets; but we will not explore such proofs here.

$$\begin{array}{ccc} P & \xleftarrow{\text{face-poset}} & \Gamma \\ \Delta \downarrow & & \downarrow \approx \text{barycentric subdivision} \\ \Delta(\overline{P}) & \xrightarrow{|\cdot|} & |\Delta(\overline{P})| \end{array}$$

Figure 1-1: a commutative diagram

Topological realizations

As with simplicial complexes, we will often think of simplicial posets topologically, rather than as posets. See [Bj1] for an introduction to interpreting certain posets, including simplicial posets, as the face-posets of certain regular CW-complexes. In particular, for any simplicial poset P, there is a well-defined regular CW-complex Γ such that P is the face-poset of Γ. This is a generalization of the realization $|\Delta|$ of a simplicial complex Δ.

Alternatively, the **order complex** $\Delta(\overline{P})$ of $\overline{P} := P \backslash \{\hat{0}\}$, defined to be the simplicial complex consisting of chains of $\overline{P}$, ordered by inclusion, has a realization $|\Delta(\overline{P})|$ that is homeomorphic to Γ. In fact, $\Delta(\overline{P})$ is the face-poset of the barycentric subdivision of Γ. Note, in particular, that the face-poset of $|\Delta(\overline{P})|$ is *not* P again. In short, the diagram in Figure 1-1 commutes. For example, see P, Γ, $\Delta(\overline{P})$, and $|\Delta(\overline{P})|$ in Figure 1-2.

We often abuse notation by writing $|P|$ for Γ, and refer to $|P|$ and P almost interchangeably. In particular, atoms of P (*i.e.* elements that cover $\hat{0}$) are vertices (of $|P|$), elements of P are faces (of $|P|$), and facets of P are the maximal elements of P, corresponding to the maximal faces of $|P|$. If y is a face of P, then

$$\partial y := \{z \in P : z < y\}$$

is the **boundary** of y, corresponding to the boundary of a face of a simplicial complex.

We also define a **topological property** to be one that only depends on (the homeomorphism class of) $|P|$ (equivalently, $|\Delta(\overline{P})|$). Similarly, it is possible, as in [BGS, §3] (among others) to define homology groups (and hence cohomology. too) on a poset P in a natural way so that $\tilde{H}_i(P; k) \cong \tilde{H}_i(|\Delta(\overline{P})|)$. Being topological makes a property more

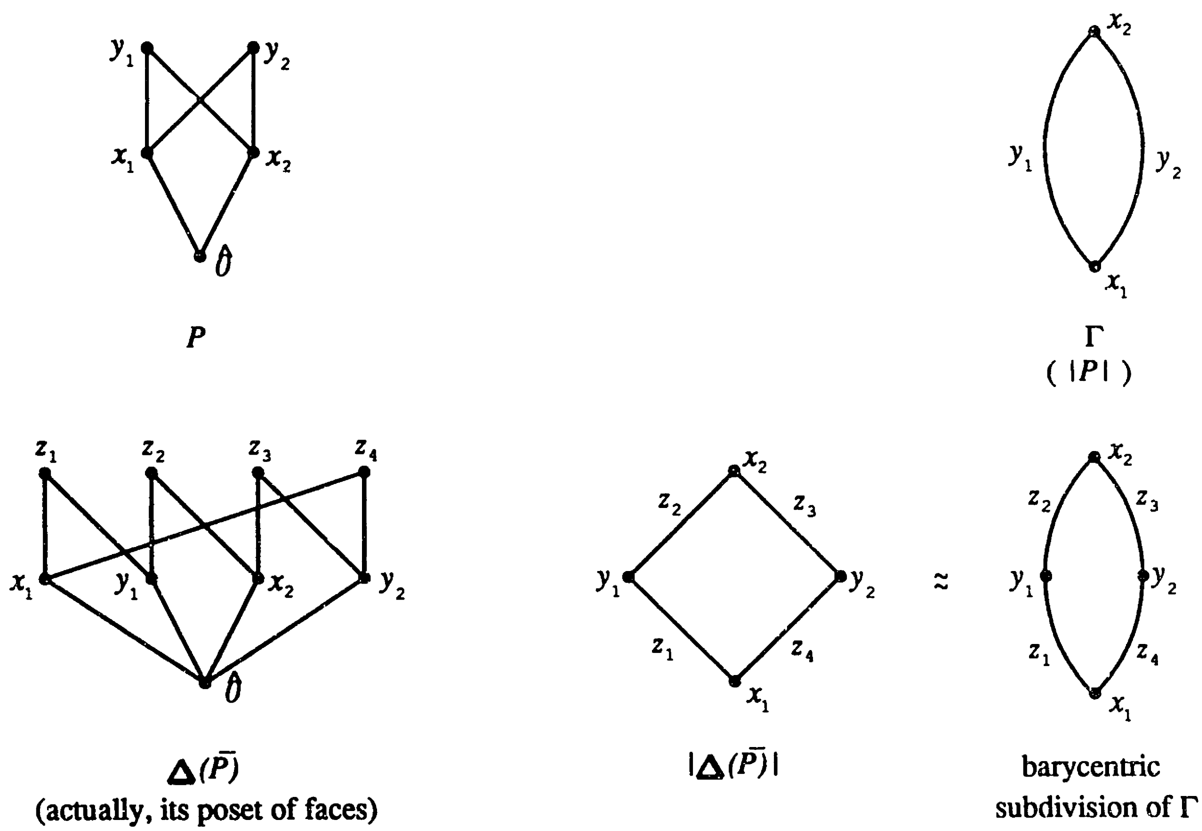


Figure 1-2: an example

interesting; we will deal with several.

f-vectors

One feature of simplicial complexes that has been studied a great deal is the f-vector. In particular, for many classes of simplicial complexes and convex polytopes, the f-vector has been completely characterized (see [Bj2] for an overview and exhaustive bibliography). It is natural then to extend this definition to simplicial posets, and continue this characterization program.

Definitions: Let P be a simplicial poset. It is necessarily graded, that is, for any $y \in P$, every saturated chain from $\hat{0}$ to y contains the same number of elements. Call this number r', and define the **rank** of y to be $\mathrm{r}(y) := r' - 1$. (In particular, $\mathrm{r}(\hat{0}) = 0$, and the rank of an atom is 1.) Then let

$$f_i(P) := |\{y \in P : \mathrm{r}(y) = i + 1\}|.$$

(In particular, $f_{-1}(P) = 1$ for the $\hat{0}$ element and $f_i(P)$ counts the number of i-dimensional faces of $|P|$. The number of vertices of P, $f_0(P)$, will usually just be denoted by n.) Define

$$d := 1 + \max\{i : f_i(P) \neq 0\}.$$

We then say that P has **rank** d and **dimension** $d - 1$ (corresponding to the dimension of $|P|$). The vector $f(P) = (f_0, f_1, \ldots, f_{d-1})$ is the **f-vector** of P.

The characterization of f-vectors of simplicial posets was begun by Stanley in [St7]. The f-vectors of all simplicial posets and of *Cohen-Macaulay* simplicial posets were completely classified. The f-vectors of *Gorenstein** simplicial posets were almost completely classified, leaving a small segment of f-vectors that might or might not be the f-vector of some Gorenstein* simplicial poset. In Chapter 2, we show that some of these vectors cannot be the f-vector of a Gorenstein* simplicial poset, giving more plausibility to the thought that none of the remaining f-vectors show up in Gorenstein* simplicial posets. In general, results about the f-vector of a simplicial poset are easier to state and prove than the corresponding results about the f-vector of a simplicial complex.

Another question about f-vectors of simplicial complexes that has been answered concerns the *Betti sequence*, a sequence of non-negative integers characterizing the homology (over $\mathbf{Q}$) of a simplicial complex. In [BK1], Björner and Kalai characterized pairs of sequences that can be the f-vector and Betti sequence, respectively, of a simplicial complex; such pairs are called **compatible**. The proof relied upon *algebraic shifting* of a ring defined by the simplicial complex. The result was extended to regular CW-complexes that have a certain intersection property in [BK2]. This second proof used completely different techniques, but relied upon an involved induction.

In Chapter 3, we answer the corresponding question for simplicial posets, namely characterizing compatible f-vectors and Betti sequences of simplicial posets. In this case, the characterization is simpler, and nothing more than the Meyer-Vietoris sequence is needed for the proof. In Section 3.2, we explore which Betti sequences are compatible with which f's, and *vice versa.* This is comparable to similar work in [BK1].

The ring A_P

In Chapters 4 and 5 we turn our attention to the ring A_P of a simplicial poset, defined in [St7], which generalizes the *face-ring* of a simplicial complex (defined to be $k[\Delta] := k[x_1, \ldots, x_n]/(\{x_{i_1} \cdots x_{i_m} : \{x_{i_1}, \ldots, x_{i_m}\} \notin \Delta\})$). Fix a field k. (As with most of these matters, cnce the ring k is chosen, it is all but ignored, even though it is technically part of the definition. The idea is that before applying any of these results, one picks the field to be used, often $\mathbf{Q}$ or $\mathbf{C}$, and then simply does not change it again. Often, the resulting combinatorics is independent of the ring chosen.) Let P be a simplicial poset, and let $k[P]$ denote the polynomial ring whose indeterminates are all the elements of P. Then $A_P := k[P]/I$, where I is the ideal of $k[P]$ generated by elements of the following form:

- xy, for any x, y that have no common upper bound in P;
- $xy - (x \wedge y) \sum_{z \in \mathrm{mub}(x,y)} z$, for any x, y incomparable in P, where $\mathrm{mub}(x, y)$ is the (non-empty) set of minimal upper bounds of x and y (if x and y have any upper bound z, then $x \wedge y$ is well-defined, since $x, y \in [\hat{0}, z]$, a Boolean algebra); and
- $\hat{0} - 1$.

If P is the face-poset of a simplicial complex Δ, then $A_P = k[\Delta]$, the face-ring of Δ, first defined, independently, in [Re] and [St1, St2].

One of the most essential features of A_P is that it is a graded algebra.

Definitions: (See [St4, §I.2]). Let k be a field and R be a k-algebra. Then R is a **graded k-algebra** if R has the following decomposition as a vector-space direct sum:

$$R \cong \bigoplus_{\alpha \in A} R_\alpha,$$

where A is an additive semigroup, and $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ (and $R_0 = k$). We will only be concerned with two types of grading: $A = \mathbf{Z}$ (or $\mathbf{N}$) and $A = \mathbf{Z}^n$ (or $\mathbf{N}^n$) (rough and fine grading, respectively). The **homogeneous** elements of R are

$$H(R) := \cup_{\alpha \in A} R_\alpha.$$

We say that $r \in R_\alpha$ is homogeneous of degree α ($\deg(r) = \alpha$). An ideal I is homogeneous if it can be generated by homogeneous elements, and if I is homogeneous, then R/I is still graded. One such ideal, in $\mathbf{N}$- and $\mathbf{N}^n$-grading, is

$$R_+ := \bigoplus_{\substack{\alpha \in A \\ \alpha \neq 0}} R_\alpha.$$

Similarly, a module M is A-graded if it can be written as the vector space direct sum

$$M \cong \bigoplus_{\alpha \in A} M_\alpha,$$

where $R_\alpha M_\beta \subseteq M_{\alpha+\beta}$. If $A = \mathbf{Z}^n$ (or $\mathbf{N}^n$), the **Hilbert series** of M in indeterminates $\lambda_1, \ldots, \lambda_n$ is defined as:

$$F(M, \lambda) := \sum_{\alpha \in \mathbf{Z}^n} \lambda^\alpha \dim_k M_\alpha,$$

where, for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\lambda^\alpha := {\lambda_1}^{\alpha_1} \cdots {\lambda_n}^{\alpha_n}$. It is the obvious generating function for a finely graded module.

When $A = \mathbf{Z}^n$, we make the following definitions on A: If $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $\alpha_+ := (\alpha'_1, \ldots, \alpha'_n)$ where

$$\alpha'_i = \begin{cases} \alpha_i & \text{if } \alpha_i > 0 \\ 0 & \text{if } \alpha_i \leq 0 \end{cases},$$

and α_- is defined similarly. We also use the following partial order on $\mathbf{Z}^n$: If $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, then $\alpha \leq \beta$ iff $\alpha_i \leq \beta_i$ for all i.

There is an obvious fine grading for A_P which we use. Arbitrarily label the vertices of P by $x_1, \ldots, x_n$. Let $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{Z}^n$ be the ith unit vector. Then, for any $y \in P$, let

$$\deg(y) := \sum_{i : x_i \leq y} \epsilon_i.$$

It is easy to see that each generator of I is homogeneous under this grading, so A_P is a graded algebra. We often abuse notation and write

$$\deg(T) := \sum_{i : x_i \in T} \epsilon_i$$

for T a subset of vertices of P. Similarly, define the **type** of $y \in P$ as

$$\text{type}(y) := \{x_i : x_i \leq y\}.$$

More generally, define the **support** of a monomial in A_P as follows: If $m = \prod_{y \in P} y^{a_y}$, then let

$$\text{supp}(m) := \cup_{y : a_y > 0} \{x_i : x_i \leq y\}.$$

Notice that for $y \in P$, $\text{supp}(y) = \text{type}(y)$. We again sometimes abuse notation and write

$$\text{supp}(\epsilon) := \{i : \epsilon_i \neq 0\}$$

for $\epsilon \in \mathbf{Z}^n$.

A_P is an *algebra with straightening laws* (see [St7, §3]), the main consequence of which is, for us, that the monomials corresponding to multichains (*i.e.*, $m = \prod_{i=1}^{j} y_i \in A_P$, for $y_1 \leq \ldots \leq y_j$) form a k-vector space basis of A_P. It is a corollary (Theorem 5.2.1) to this that the set $B_0(P) := \{y\mu : y \in P, \text{ supp}(\mu) = \text{supp}(y)\}$ is also a k-vector space basis of A_P. We will use both bases.

In Chapter 4, we prove a conjecture of V. Reiner describing for which posets P the ring A_P is a *complete intersection ring.* It is comparable to the simplicial complex version (which it generalizes), but the proof is much more involved.

In Chapter 5, we solve two problems involving *minimal free resolutions* (MFR's) of A_P over the subring $k[V]$ (so A_P is treated as a $k[V]$-module), for P a simplicial poset with vertices V. First we generalize a result of Hochster [Ho] describing the *Betti polynomial*, an invariant of any free resolution, of the minimal free resolution of $k[\Delta]$ over $k[V]$, for a simplicial complex with vertices V. Then we generalize another result of Hochster (see [St4, §II.4]) calculating the *Hilbert series* of the *local cohomology modules* of $k[\Delta]$ as a $k[V]$-module for Δ a simplicial complex with vertices V.

In both results from Chapter 5, the outline of the proof follows Hochster's original proofs (which were, in turn, "obtained by a routine elaboration of Reisner's [Re] methods"), but there is an added wrinkle in going from the simplicial complex case to the simplicial poset case. In each case, both problems involve *splitting* a certain *chain complex* (*i.e.*, writing it as a direct sum of subcomplexes that the boundary operator acts upon) as finely as possible; in the simplicial complex case, this means splitting by the usual fine grading of $k[\Delta]$, but in the simplicial poset case, an even finer grading is called for. In this case, it is not enough just to know the fine degree of a basis element $y\mu \in B$, but also which element of a certain degree lies under y. For simplicial complexes, this always reduces to the normal fine grading.

A final warning about necessary background: This thesis assumes on the part of the reader a certain amount of basic knowledge of combinatorics, commutative algebra, and algebraic topology. Definitions of some basic concepts are included, but usually not explained in depth. Any term that is not defined or explained can probably be found in [St6], [AM], or [Mu1].

Chapter 2

f-vectors and h-vectors of Gorenstein* simplicial posets

The characterization of the f-vector (equivalently, h-vector) of various classes of simplicial complexes has been the subject of a great deal of research. It is natural, then, to extend this endeavor to simplicial posets, a program begun by Stanley in [St7]. We start by outlining and summarizing the results from [St7], which completely characterizes f-vectors of all simplicial posets, then of Cohen-Macaulay simplicial posets, and finally gives an incomplete characterization of h-vectors of Gorenstein* simplicial posets (we eventually define all these terms).

The rest of this chapter continues the program of characterization of h-vectors of Gorenstein* simplicial posets, by demonstrating that certain vectors cannot be the h-vectors of a Gorenstein* simplicial poset.

2.1 Background

The background material in this section primarily summarizes [St7], and, unless otherwise stated, all results and definitions are from there. We also introduce the definitions and well-known results that we will need later on.

For completeness, we start with the complete characterization of f-vectors of all simplicial posets.

Theorem 2.1.1 ([St7, Theorem 2.1]) *Let $f = (f_0, \ldots, f_{d-1}) \in \mathbf{Z}^d$. The following are equivalent:*

- *There exists a simplicial poset P of dimension $d-1$ with f-vector $f(P) = f$.*
- *$f_i \geq \binom{d}{i+1}$ for $0 \leq i \leq d-1$.*

Cohen-Macaulay Simplicial Posets

The other class of simplicial posets whose f-vectors have been completely characterized are Cohen-Macaulay simplicial posets. Cohen-Macaulay posets were first studied in [Ba] and [St3], independently, and have been examined extensively since; see [BGS] for an overview and [Ga] for combinatorial interpretations of the Cohen-Macaulayness of $k[\Delta]$. The following definitions are from [BGS]; the equivalence of definitions (i) and (ii) is the celebrated theorem of Reisner [Re], and (ii) $\Leftrightarrow$ (iii) is from [Mu2]. The equivalence of the two definitions of a Cohen-Macaulay simplicial poset is just a result of the definition of order complex.

Definitions: Fix a ring k. A simplicial complex Δ is **Cohen-Macaulay** (over k) if (equivalently):

(i) (algebra) The face ring $k[\Delta]$ is a Cohen-Macaulay ring (see [Ga, St4] for more details).

(ii) (simplicial topology) For all $F \in \Delta$ (including $F = \emptyset$), $\tilde{H}_i(\mathrm{lk}F; k) = 0$ if $i \neq \dim(\mathrm{lk}F)$.

(iii) (topology) If $X = |\Delta|$, then $\tilde{H}_i(X; k) = H_i(X, X - p; k) = 0$ for all $p \in X$ and $i \neq \dim X$.

A finite poset P is **Cohen-Macaulay** (over k) if (equivalently):

(i) Its order complex $\Delta(P)$ (equivalently, if P has a $\hat{0}$, the order complex $\Delta(\overline{P})$ of $\overline{P}$) is.

(ii) For every $x < y$ in $\hat{P}$, the order complex $\Delta(x, y)$ of the open interval (x, y) satisfies

$$\tilde{H}_i(\Delta(x, y); k) = 0 \tag{2.1}$$

if $i < \dim(x, y)$.

Remark: Definition (i) of Cohen-Macaulay posets and definition (iii) of Cohen-Macaulay simplicial complexes show that a simplicial poset being Cohen-Macaulay is a topological property. Also, a Cohen-Macaulay poset is **pure** (see [BGS, remark following Theorem 3.3]), *i.e.*, every facet contains the same number of vertices.

h-vectors

Before we can give the characterization of f-vectors of Cohen-Macaulay simplicial posets, we need to introduce the following definition (see [St5] or [St7] for more details):

Definition: Given $f = (f_0, \ldots, f_{d-1})$ (and $f_{-1} = 1$), define the h-**vector** $h = (h_0, \ldots, h_d)$ by (equivalently):

(i)

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=1}^{d} h_i x^{d-i}; \tag{2.2}$$

or

(ii)

$$h_i = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1}, \tag{2.3}$$

which can be inverted to give

$$f_i = \sum_{j=0}^{d} \binom{d-j}{d-i-1} h_j; \tag{2.4}$$

or

(iii) the following "difference-table," which is computationally the easiest way to compute the h-vector for a particular example. We illustrate how it works with the following example from [St5]:

If P is (the face poset of) the four-dimensional cross-polytope, then $f(P) = (8, 24, 32, 16)$. Write this f-vector along a diagonal, with a 1 to the left of f_0:

$$\begin{array}{ccccc}
1 & 8 & & & \\
 & & 24 & & \\
 & & & 32 & \\
 & & & & 16
\end{array}$$

Then, finish the table by recursively placing the difference of adjacent entries underneath those entries (and placing 1's on the left diagonal):

$$\begin{array}{ccccccccc}
 & & & 1 & & 8 & & & \\
 & & 1 & & 7 & & 24 & & \\
 & 1 & & 6 & & 17 & & 32 & \\
1 & & 5 & & 11 & & 15 & & 16 \\
\hline
1 & 4 & & 6 & & 4 & & 1 &
\end{array}$$

The final row (below f_{d-1}) is h. In this case, $h(P) = (1, 4, 6, 4, 1)$.

Remark: Using any of the definitions of h, $(f_0, \ldots, f_i)$ determines $(h_0, \ldots, h_{i+1})$ and *vice versa*, for all i. In particular, f is recoverable from h. There are also the following simple numerical observations about the h-vector of a $(d-1)$-dimensional simplicial poset P:

$h_0 = 1$; $h_1 = n - d$ (where $n = f_0(P)$, the number of vertices of P); and

$$\sum_{i=0}^{d} h_i = f_{d-1}, \tag{2.5}$$

which is just the number of facets of the poset.

As with the characterization of f-vectors of Cohen-Macaulay simplicial complexes, it is easier not to characterize the f-vectors of Cohen-Macaulay simplicial posets directly, but rather to characterize their h-vectors:

Theorem 2.1.2 ([St7, Theorem 3.10]) *Let $h = (h_0, \ldots, h_d) \in \mathbf{Z}^{d+1}$. The following are equivalent:*

- *There exists a Cohen-Macaulay simplicial poset P of rank d with h-vector $h(P) = h$.*
- *$h_0 = 1$, and $h_i \geq 0$ for all i.*

Gorenstein* simplicial posets

The next step in the f-vector characterization program of [St7] is the class of Gorenstein* simplicial posets. But first we give the definition of a strictly weaker condition.

Definitions: (see [St6, §3.14]) Let μ denote the Möbius function (see [St6, §3.7], for instance) of a poset. A finite graded poset with $\hat{1}$ and $\hat{0}$ is **Eulerian** if

$$\mu(x, y) = (-1)^{\mathrm{r}(y)-\mathrm{r}(x)} \tag{2.6}$$

for any $x <_P y$. If P is simplicial, we will say that P is **Eulerian*** if P is pure and $\hat{P} := P \cup \{\hat{1}\}$ is Eulerian.

Remark: A Boolean algebra is Eulerian, so if P is simplicial, then most instances of equation (2.6) are already true, and to check if P is Eulerian*, we need only check that equation (2.6) holds for $y = \hat{1}$.

Definitions: A graded algebra is said to be **Gorenstein** if its free resolution satisfies a certain duality condition, which we do not actually use here; see [St4, §I.12] for details. We then define a simplicial complex to be **Gorenstein** if $k[\Delta]$ is a Gorenstein ring (see [St4, §II.5]), and a poset P to be **Gorenstein** if its order complex $\Delta(P)$ is Gorenstein (equivalently, if $\Delta(\overline{P})$ is Gorenstein). Following [St7, §4], a poset P is Gorenstein if and only if the subposet $Q := P \backslash \{y \in P : y \text{ is related to all other elements of } P\}$ is non-acyclic Gorenstein (*i.e.*, Gorenstein, and not all reduced homology groups $\tilde{H}_i(\Delta(Q); k)$ vanish). We call non-acyclic Gorenstein posets **Gorenstein***. See [St3, §8], [BGS, §6e], [St4, §II.5], and [Ho, §5,6] for more details. Thus, a simplicial poset may be defined to be Gorenstein* if $\overline{P} = P \backslash \{\hat{0}\}$ is non-acyclic Gorenstein.

We record the following observations about Gorenstein* simplicial posets:

- Boolean algebras are the only Gorenstein simplicial posets which are not Gorenstein*.
- Being Gorenstein* is a topological property.
- Spheres are Gorenstein*.
- A simplicial poset P is Gorenstein* if and only if P is Cohen-Macaulay and Eulerian*.

We can take this last condition as a definition of Gorenstein* simplicial posets. Further, it is almost enough to characterize h-vectors (and hence f-vectors) of Gorenstein* simplicial posets. Eulerian simplicial posets satisfy the **Dehn-Sommerville** equations (see [St6, §3.14]),

$$h_i = h_{d-i} \tag{2.7}$$

for all i. Combining the Dehn-Sommerville equations with Theorem 2.1.2, we get

Proposition 2.1.3 ([St7, §4]) *If P is a Gorenstein* simplicial poset of rank d and $h = h(P)$, then*

- $h_1 = 1$;
- $h_i \geq 0$, *for all* i; *and*
- $h_i = h_{d-i}$.

Further, by construction, we have the following sufficient conditions:

Theorem 2.1.4 ([St7, §4.3]) *Let $h = (h_0, \ldots, h_d) \in \mathbf{N}^{d+1}$, with $h_i = h_{d-i}$ and $h_0 = 1$. Any of the following (mutually exclusive) conditions are sufficient for the existence of a Gorenstein* simplicial poset P of rank d and h-vector $h(P) = h$:*

- d *is odd;*
- d *is even and* $h_{d/2}$ *is even; or*
- d *is even,* $h_{d/2}$ *is odd, and* $h_i > 0$ *for* $0 \leq i \leq d$.

Since $\sum h_i$ counts facets and h is symmetric, these sufficient conditions may be restated as

P has an even number of facets or $h_i > 0$ for $0 \leq i \leq d$.

Completing the characterization of h-vectors of Gorenstein* simplicial posets thus consists of closing the gap between Proposition 2.1.3 and Theorem 2.1.4, *i.e.*, finding which h-vectors (if any) that contain 0's can occur as $h(P)$ for P Gorenstein* simplicial with an odd number of facets. Stanley [St7, §4.5] has already shown that if P is an Eulerian* simplicial poset such that $h_1(P) = 0$, then P cannot have an odd number of facets (since being Gorenstein* implies being Eulerian*, the result may be specialized to Gorenstein* simplicial posets).

In this chapter, we eliminate more such h-vectors. Although not all h-vectors containing 0's are eliminated, none have been found (*i.e.*, no Gorenstein* simplicial poset P with an odd number of facets and $h_i(P) = 0$ has been found yet), lending some credibility to the thought that the conditions of Theorem 2.1.4 are necessary as well as sufficient. This is especially true given that actually very little of the properties of a Gorenstein* simplicial poset are used to eliminate these h-vectors: we use Cohen-Macaulayness only for the 0th homology of certain links, and we use Eulerianness only for the Dehn-Sommerville equations. Perhaps similar techniques using more power will finish the characterization.

The methods used to eliminate these h-vectors, however, are *ad hoc*, and are only effective, so far, on small examples, so there is no particular reason to believe that a counterexample does not exist just past the bounds of the h-vectors that have been eliminated. In particular, if there are unexpected counterexamples, possibly some of the techniques here can be used to pick them out from among the set of all vectors.

2.2 Links of simplicial posets

The link of a face in a simplicial complex is a well-known and useful concept. The definition is usually given as:

Definition: Let Δ be a simplicial complex, and let $F \in \Delta$ be a face. Then

$$\mathrm{lk}_\Delta F := \{G \in \Delta : F \cup G \in \Delta,\ F \cap G = \emptyset\}.$$

In this way, $\mathrm{lk}_\Delta F$ is just a subcomplex of faces in Δ.

What is the "correct" generalization of link to simplicial posets? One possibility is, thinking of our poset P topologically, to define $\mathrm{lk}_P y$ (for $y \in P$) to be the set of all z that have a join with y in P, and whose meet with y is $\hat{0}$ (see Figure 2-1). This has the advantage of being easily expressed and visualized topologically, as well as having a precise non-topological poset definition. This may, in fact, be the definition to use in many applications, but we do not use it at all here.

Instead, we use a definition that, despite appearing to ignore topology, just takes a

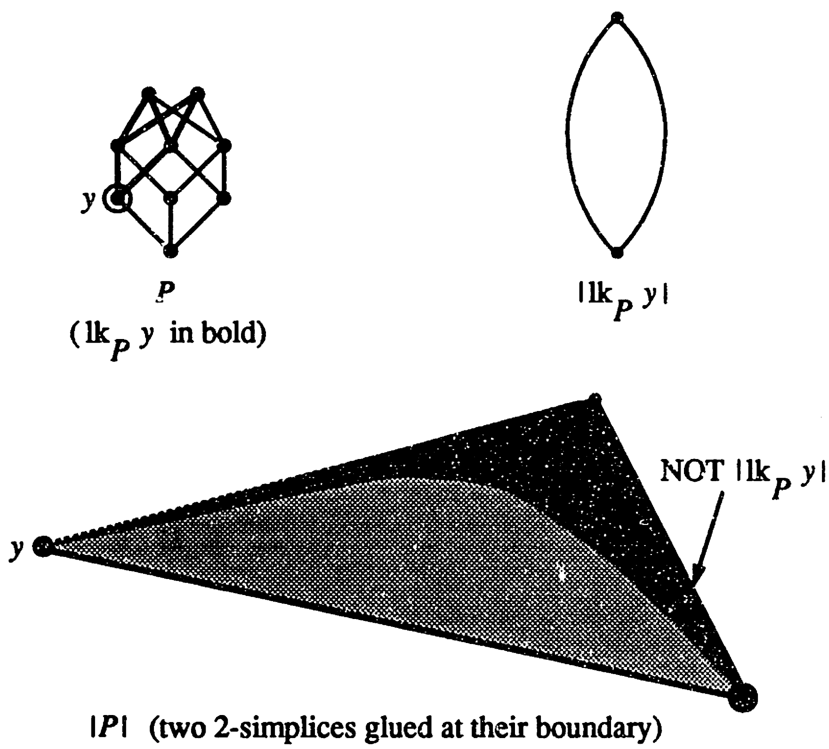


Figure 2-1: what the link is, and what it is not

different approach to the topological interpretation of the link.

Definition: Let P be a simplicial poset, and let $y \in P$. Then

$$\mathrm{lk}_P y := \{z \in P : z \geq y\}$$

(see Figure 2-1).

It is easy to verify that $\mathrm{lk}_P y$ is simplicial, with its $\hat{0}$ element being y. This is a generalization of the simplicial complex link because the face-poset of $\mathrm{lk}_\Delta F$ is isomorphic to $\{H \in \Delta : F \subseteq H\}$ (the isomorphism is given by $\phi : G \mapsto F \cup G$, for any $G \in \mathrm{lk}_\Delta F$).

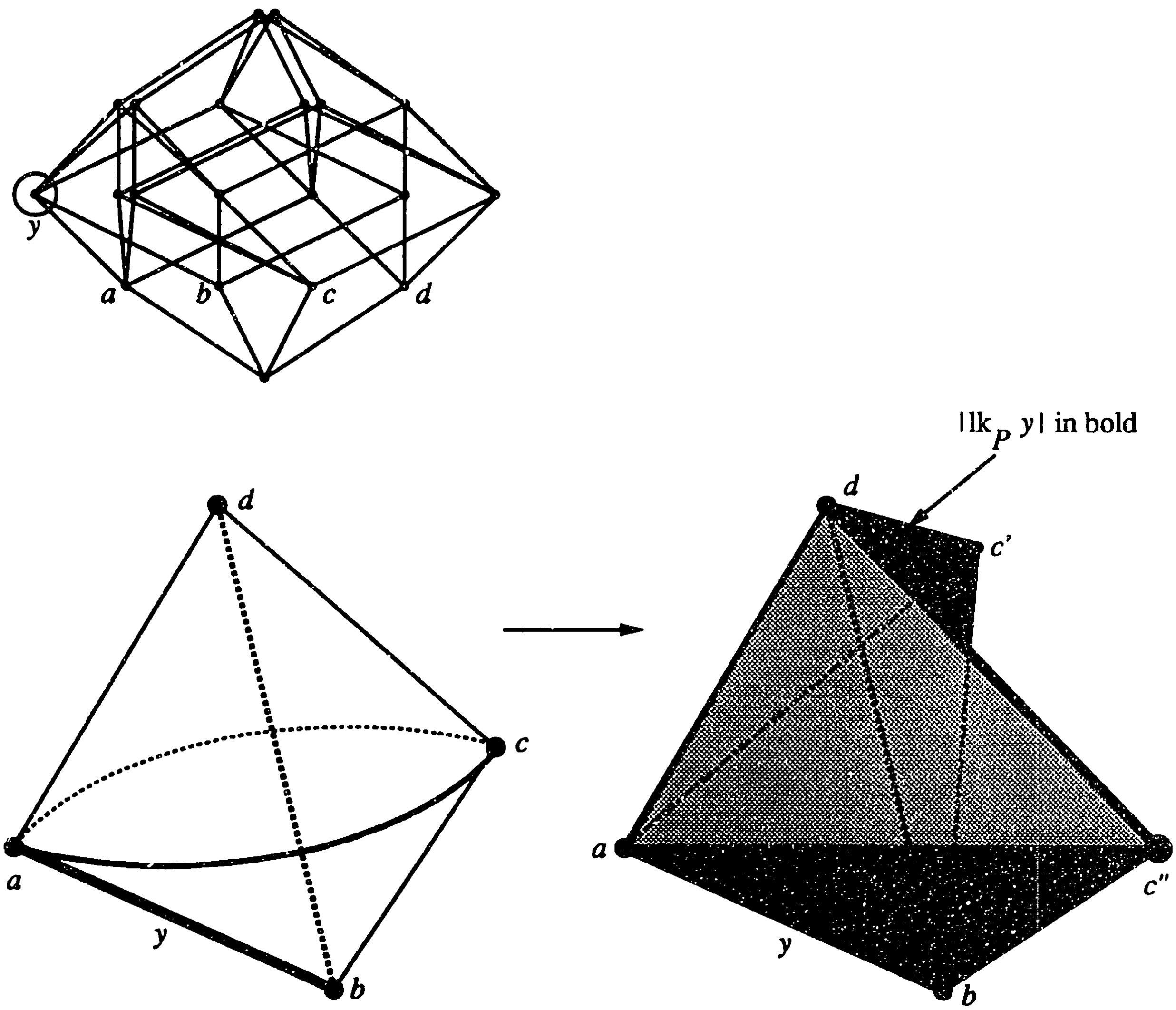
But this is also a generalization topologically if we look more closely at F, and less at all of Δ. The link of F, *via* our isomorphism ϕ, corresponds to all the faces of Δ that contain F. If we look at the subcomplex of just these faces, then we get some idea of what Δ looks like "near" F; it is homeomorphic to $F * \mathrm{lk}_\Delta F$ (the join of F and $\mathrm{lk}_\Delta F$), so $\mathrm{lk}_\Delta F$ carries all the information of this subcomplex. Similarly, looking at the realization of P, if we look at just the faces containing y, ignoring identification of subfaces not containing y (which are, after all, not as "near" to y), we get $y * \mathrm{lk}_P y$, so $\mathrm{lk}_P y$ carries all the information of what P looks like "near" y (see Figure 2-2).

As a final argument for this definition of link, consider the case when y is a vertex. In that case, we can say, using this definition,

$$\Delta(\mathrm{lk}_P y) = \mathrm{lk}_{\Delta(\overline{P})} y$$

(since both sides are the complex of chains of P whose minimal element is y, ordered by inclusion). In other words, the link is topologically unchanged by taking barycentric subdivision (see Figure 2-3). Barycentric subdivision effectively "localizes" P, so this reinforces our idea of the link of a face showing what P looks like near that face.

Of course, the reason for using this definition, ultimately, is simply that it works in each of the applications that calls for it (especially in Chapter 5). However, this explanation will perhaps serve to show the reader that this is not purely a poset-oriented



(two 3-simplices glued along two of the 2-dimensional faces on their boundary, *abd* and *bcd*)

ignore identification of subfaces not containing *y* (The two copies of *abd* are still identified since *abd* is a face of *y*. But *bcd* is no longer identified.)

Figure 2-2: some links near y

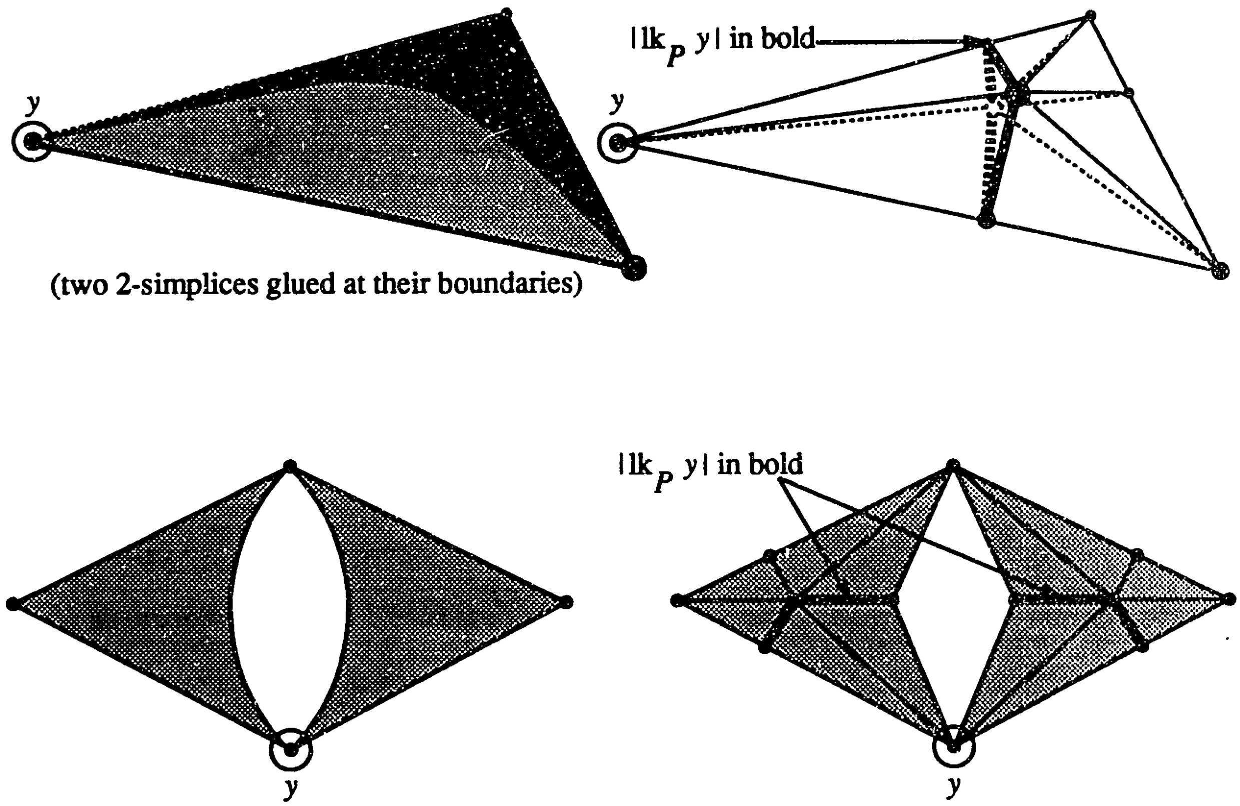


Figure 2-3: barycentric subdivisions

definition, but a topological one as well, that is merely stated most conveniently in terms of the poset.

The first clue that this is a good generalization comes from the following

Proposition 2.2.1 *A simplicial poset P is Cohen-Macaulay if and only if*

$$\tilde{H}_i(\mathrm{lk}\, x) = 0 \tag{2.8}$$

for every $x \in P$ and $i <$ $\dim(\mathrm{lk}\, x)$.

Proof: A Boolean algebra has only top-dimensional homology ($|(x, y)|$ is an $(n-1)$-sphere if $[x, y] \cong B_n$), so, using definition (ii) to test P for Cohen-Macaulayness, we only have to check equation (2.1) for intervals $(x, \hat{1})$. But $(x, \hat{1})$ is just $(\mathrm{lk}_P x) \backslash \{x\}$, so $\tilde{H}_i(\Delta(x, \hat{1}); k) \cong \tilde{H}_i(\mathrm{lk}_P x; k)$. ∎

A simple but useful observation about the link in a simplicial poset is that many properties are *hereditary.*

Definition: A property Q of posets is **hereditary** if

P satisfying $Q \Rightarrow \mathrm{lk}_P y$ satisfies Q (for all $y \in P$).

This concept was mentioned, if not formally defined, in [Ho], following [Re].

Proposition 2.2.2 *Being simplicial is a hereditary property,* i.e., *if P is simplicial, and $y \in P$, then* $\mathrm{lk}_P y$ *is also simplicial.*

Proof: Immediate from definition (ii) of simplicial poset, since every closed interval of $\mathrm{lk}_P y$ is also a closed interval of P. ∎

Proposition 2.2.3 *The following properties are hereditary on simplicial posets,* i.e., *if P is simplicial and has the given property, and $y \in P$, then* $\mathrm{lk}_P y$ *also has the given property (and, of course, is also simplicial):*

*(i) Eulerian**

(ii) Cohen-Macaulay

*(iii) Gorenstein**

Proof: (i) follows from equation (2.6) since the definition is entirely in terms of intervals; (ii) is immediate from equation (2.8), and (iii) then follows from (i) and (ii). ∎

2.3 Eulerian* simplicial posets

In some cases, we need only assume P is Eulerian* simplicial to show that $h(P) \neq h$ for a given h. We first state a number of facts about Eulerian* simplicial posets, leading up to eliminating certain h-vectors as the h-vector of an Eulerian* (and hence Gorenstein*) simplicial poset.

Remark: Eulerian*ness, Cohen-Macaulayness, and Gorenstein*ness all imply purity, so we do not need to explicitly mention purity in most of our assumptions, even though we use purity in the following way: If P is pure of rank d, then every facet has d vertices, and, more importantly (to us), every facet excludes $h_1 = n - d$ vertices.

Lemma 2.3.1 *If P is an Eulerian* simplicial poset of odd dimension d, then P has an even number of facets.*

Proof: By the Dehn-Sommerville equations, $h(P)$ is symmetric, with an even number $(d+1)$ of components, so the sum of its components is even. But this sum is just the number of facets of P. ∎

Lemma 2.3.2 *Let P be an Eulerian* simplicial poset with an odd number of facets. Then every vertex lies in an even number of facets, and, in particular, no vertex lies in every facet.*

Proof: By Lemma 2.3.1, P has even dimension. Now let x be a vertex of P; by Lemma 2.2.3, lk$x$ is also Eulerian* simplicial, of odd dimension. Lemma 2.3.1 now gives us that lk$x$ has an even number of facets. But the number of facets of lk$x$ is just the number of facets of P that x lies in. ■

Corollary 2.3.3 ([St7, §4.5], different proof) *There is no Eulerian* simplicial poset P with an odd number of facets and $h_1(P) = 0$.*

Proof: Let P be Eulerian* simplicial of rank d with $h_1(P) = 0$. Then $n - d = h_1 = 0$, so $n = d$, *i.e.*, every vertex is in every facet. The result now follows from Lemma 2.3.2. ■

This demonstrates the surprising power of Lemma 2.3.2, at least for small h_1. It unfortunately loses its usefulness as h_1 gets larger, but is still quite effective at $h_1 = 1$ and even $h_1 = 2$ or 3.

Proposition 2.3.4 *Let P be an Eulerian* simplicial poset of dimension $d - 1$ with an odd number of facets and $n - d = h_1(P) = 1$. Then*

$$f_i(P) \geq \binom{d+1}{i+1}$$

for $-1 \leq i \leq d - 1$.

Proof: Let V be the set of vertices of P. By Lemma 2.3.2, every vertex is excluded from some facet, and, since $n = d + 1$, each facet excludes just one vertex, so there is a facet of type S for every $S \in \binom{V}{n-1}$. Now let $T \subseteq V$, $T \neq V$. Then there is some $S \in \binom{V}{n-1}$ such that $T \subseteq S$, and hence there is a face of type T (in the facet of type S). Since there are $\binom{n}{i+1} = \binom{d+1}{i+1}$ i-dimensional face-types (*i.e.*, subsets of V of cardinality $i + 1$), the result follows. ■

Corollary 2.3.5 *There is no Eulerian* simplicial poset P with an odd number of facets and*

$$h_0(P) = h_1(P) = \cdots = h_i(P) = 1, h_{i+1}(P) = 0$$

(for $i \geq 1$).

Proof: This is immediate from Proposition 2.3.4 and the following two claims:

Claim 1: The simplicial poset Q that is the face-poset of the boundary of the d-simplex satisfies $h_i(Q) = 1$ and $f_{i-1}(Q) = \binom{d+1}{i}$ for $0 \leq i \leq d$.

This is most easily seen by definition (iii) of h-vector (in this case the difference-table is Pascal's triangle turned sideways!).

Claim 2: If P and Q satisfy $h_j(P) = h_j(Q)$ for $0 \leq j < m$ (equivalently, $f_{j-1}(P) = f_{j-1}(Q)$ for $0 \leq j < m$), then $f_{m-1}(P) - f_{m-1}(Q) = h_m(P) - h_m(Q)$.

This follows by induction on m, using definition (ii) or (iii) of h-vector. ■

Corollary 2.3.6 *There is no Eulerian* simplicial poset P with an odd number of facets, $h_1(P) = 1$, and $\sum_{i=0}^{d} h_i(P) < n$.*

Proof: Immediate from Proposition 2.3.4 and equation (2.5). ■

2.4 The missing-edge graph

Corollary 2.3.6 shows that it can be profitable to look at the faces of a simplicial poset that are *not* there (in that case, the missing i-face), if there are few enough of them. We codify that idea in dimension 1 to help us with the $h_1 = 2$ case.

Definition: Let P be a simplicial poset. The **missing-edge graph** of P, $me(P)$ is the (undirected) graph whose vertices are the vetices of P, and whose edges are pairs $\{x, y\}$ such that x and y do not have a join in P, *i.e.*, such that the potential edge between x and y is "missing." If P is pure, then x and y not having a join is equivalent to x and y not lying in a common facet.

How many edges are in the missing-edge graph? The lower bound is $\binom{n}{2} - f_1$; there could be more if some of the f_1 edges in P are duplicates of one another, *i.e.*, if two or more edges cover the same pair of vertices. In our applications, we will only use the $\binom{n}{2} - f_1$ edges that we are guaranteed.

We use the missing-edge graph by finding "obstructions," graphs H such that if H is a subgraph of $me(P)$, then P cannot be Gorenstein* (or even Eulerian*, sometimes) with an odd number of faces and a given h-vector.

Lemma 2.4.1 *Fix a non-negative integer k, and let $H = \dot\bigcup_{j=1}^{m} K_{i_j}$, a disjoint union of m complete graphs of cardinalities i_1 through i_m, respectively, such that $\sum_{j=1}^{m}(i_j - 1) = k$. Also let P be an Eulerian* simplicial poset with an odd number of facets such that $h_1(P) = k$ and $h_2(P) = 0$. Then H is not a subgraph of $me(P)$, unless every vertex of $me(P)$ is in H (*i.e., *H and P have the same number of vertices).*

Proof: Assume P and H form a counterexample. Each component of H (*i.e.*, each K_{i_j}) disallows $i_j - 1$ vertices from each facet, so, altogether, H disallows k vertices from each facet. Since $n - d = h_1(P) = k$, every vertex of $me(P)$ (or rather, the corresponding vertex in P) that is not in H must be in every facet of P, contradicting Lemma 2.3.2. ∎

Lemma 2.4.2 *Let P be a pure simplicial poset such that $h_1(P) = k$. Then the degree of $me(P)$ is less than $k + 1$.*

Proof: Assume otherwise; then there is some vertex $x \in P$ that does not lie in a facet with (at least) $k + 1$ other vertices, so any facet containing x excludes these $k + 1$ other vertices. But $n - d = h_1(P) = k$, so, by purity of P, only k vertices are excluded from any facet of P. $\Rightarrow\Leftarrow$. ∎

Lemma 2.4.3 *Let $H = K_{1,k} \dot\cup K_2$, the disjoint union of: the complete bipartite graph on parts of size 1 and k; and an edge. Let P be a pure simplicial poset such that $h_1(P) = k$ and $h_2(P) = 0$. Then H is not a subgraph of $me(P)$.*

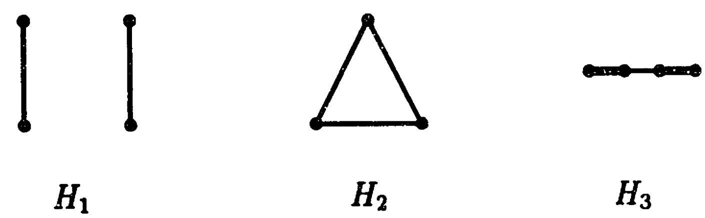


Figure 2-4: Proposition 2.4.4

Proof: Label the vertex in the first part of $K_{1,k}$ by a, and the k vertices in the second part by $b_1, \ldots, b_k$; label the vertices in K_2 by c_1 and c_2. Assume P is a counterexample. Some facet contains a, and any facet that does cannot contain any b_i. But $n - d = h_1(P) = k$, so, by purity of P, any such facet must contain every other vertex of P, including both c_1 and c_2, contradicting the existence of the edge between them in $me(P)$. ∎

Now we use these obstructions to eliminate some h-vectors.

Proposition 2.4.4 *There is no Eulerian* simplicial poset P with an odd number of facets and $h_1(P) = 2, h_2(P) = 0$.*

Proof: Assume otherwise; let P be a counterexample and let $G = me(P)$. The number of edges in G is at least

$$\begin{aligned} \binom{n}{2} - f_1 &= \binom{d+2}{2} - \sum_{j=0}^{2} \binom{d-j}{d-2} h_j = \binom{d+2}{2} - [\binom{d}{d-2} + (d-1)2] \\ &= [1 + 2d + \binom{d}{2}] - [\binom{d}{2} + 2(d-1)] = 3 \end{aligned}$$

(using definition (ii) of h-vector and some binomial identities). Pick 3 of the edges of G; call them E. Let G' be the edge-induced subgraph of G on edges in E.

By the Dehn-Sommerville equations, the smallest d can be is 6 ($h(P) = (1, 2, 0, r, 0, 2, 1)$ for odd r), so $n = d + 2 \geq 8$; thus Lemma 2.4.1 dictates that neither of the graphs H_1 or H_2 in Figure 2-4 are a subgraph of G'. Now G' is a graph of 3 edges, and, by Lemma 2.4.2, has degree no more than 2. Further, since H_1 is an obstruction, G' must have only one connected component, and since H_2 is an obstruction, it cannot be a 3-cycle. This only leaves the graph H_3 in Figure 2-4, which has H_1 (marked in bold) as an obstruction subgraph. $\Rightarrow\Leftarrow$. ∎

2.5 The $h_1 = 3$ case

The missing-edge graph will also be of use for the $h_1 = 3$ case, but now we need to add the Cohen-Macaulay condition.

Lemma 2.5.1 *Let P be a pure $(d-1)$-dimensional simplicial poset with vertices x and y such that each facet that contains x or y has the same type S. Then P is not Cohen-Macaulay.*

Proof: Let $T \subseteq S$ be maximal among subsets of S such that there is a facet containing both T and some vertex not in S. T could be $\emptyset$, but, by assummption on x and y, $|T| \leq d-2$. Let z be a face of type T. Then, by maximality of T, $\mathrm{lk}z$ has (at least) two connected components: one among facets (containing x or y) of type $S \backslash T$, and one among those facets (containing neither x nor y) whose type is disjoint from S.

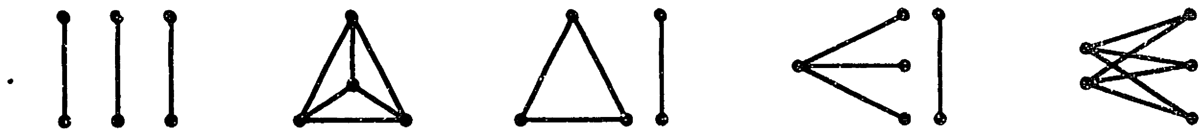
Therefore, $\tilde{H}_0(\mathrm{lk}z) \neq 0$, even though

$$\dim(\mathrm{lk}z) = ((d-1) - (\dim z)) - 1 = (d-1) - |T| \geq 1,$$

so P is not Cohen-Macaulay by Proposition 2.2.1. ∎

Once again, we use this result by showing what it means for the missing-edge graph. Then, in conjunction with our earlier results on the missing-edge graph, we eliminate some more h-vectors.

Corollary 2.5.2 *Let $H = K_{2,k}$, the complete bipartite graph with parts of size 2 and k. Let P be a Cohen-Macaulay simplicial poset such that $h_1(P) = k$, and $h_2(P) = 0$. Then H is not a subgraph of $me(P)$.*



H_4 H_5 H_6 H_7 H_8

Figure 2-5: some obstructions

Proof: Label the 2 vertices in the first part of H by a_1 and a_2, and the k vertices in the second part of H by $b_1, \ldots, b_k$. Assume P is a counterexample; let V be the set of vertices of P and let $S = V \backslash \{b_1, \ldots, b_k\}$. Then any facet that contains a_1 or a_2 cannot contain any b_i. But $n - d = h_1(P) = k$, so any such facet has type S, contradicting Lemma 2.5.1. ∎

Proposition 2.5.3 *There is no Gorenstein* simplicial poset P with an odd number of facets such that $h_1(P) = 3$, and $h_2(P) = 0$.*

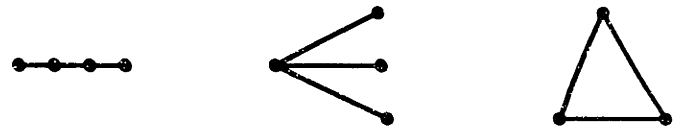
Proof: As with the $h_1 = 2$ case, assume otherwise; let P be a counterexample and let $G = me(P)$. The number of edges in G is at least

$$\begin{aligned}\binom{n}{2} - f_1 &= \binom{d+3}{2} - \sum_{j=0}^{2} \binom{d-j}{d-2} h_j = \binom{d+3}{2} - [\binom{d}{d-2} + (d-1)3] \\ &= [3 + 3d + \binom{d}{2}] - [\binom{d}{2} + 3(d-1)] = 6\end{aligned}$$

(using definition (ii) of h-vector and some binomial identities), so we may pick 6 of the edges of G; call them E. Let G' be the edge-induced subgraph of G on edges in E.

By the Dehn-Sommerville equations, the smallest d can be is 6 (again, $h(P) = (1, 2, 0, r, 0, 2, 1)$ for odd r), so $n = d + 3 \geq 9$; thus, Lemma 2.4.1 dictates that none of the graphs H_4, H_5, nor H_6 in Figure 2-5 is a subgraph of G'. Further, Lemmas 2.4.3 and 2.5.2 show that neither H_7 nor H_8, respectively, in Figure 2-5 is a subgraph of G'.

Now it is just a matter of showing that one of these obstructions appears as a subgraph of G'. This will break down into several cases, none of which is hard. G' has 6 edges and,



H_9 H_{10} H_{11}

Figure 2-6: Case 2

H_{12} H_{13}

Figure 2-7: Case 3.1

by Lemma 2.4.2, degree at most 3. As a result, the possibilities are:

Case 1: G' has at least 3 connected components.

Then H_4 is a subgraph of G'.

Case 2: G' has exactly 2 connected components (see Figure 2-6).

Then at least one of the components must have at least 3 edges; this component must therefore have one of H_9, H_{10}, or H_{11} as a subgraph (as in Proposition 2.4.4), and G' then contains H_4, H_7, or H_6, respectively.

Case 3: G' is connected.

We break this case down by the degree of G'; there are only two possibilities:

Case 3.1: G' has degree 2 (see Figure 2-7).

Then G' is either H_{12} or H_{13} and thus, in either case, contains H_4 (indicated in bold) as a subgraph.

Case 3.2: G' has degree 3.

We break this case down by the size of the smallest cycle of G'.

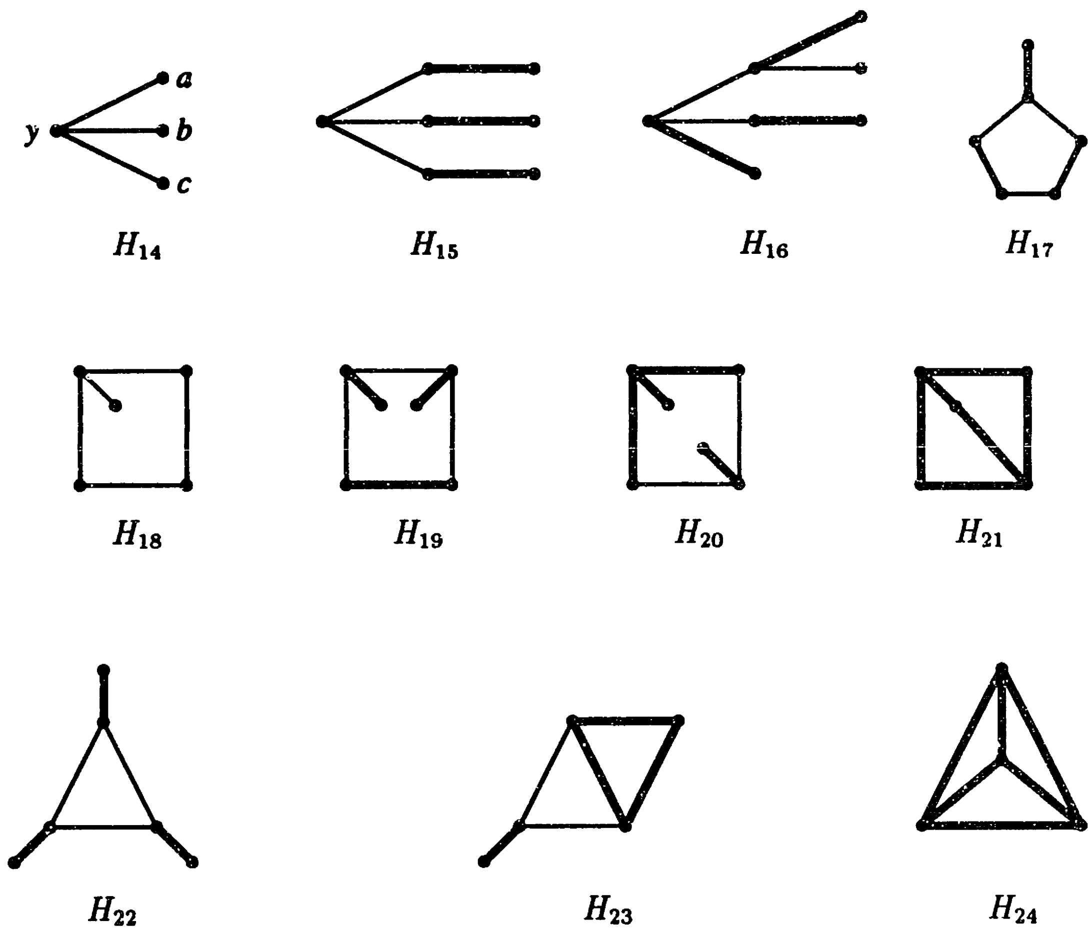


Figure 2-8: Case 3.2

Case 3.2.1: G′ has no cycle (G′ is a tree) (see Figure 2-8).

Then G' starts with H_{14} and, since H_7 is an obstruction (and G' has degree 3), every remaining edge in G' must include one of the vertices a, b, or c. Then, since G' is a tree, G' depends, up to isomorphism, only on the number of edges attached to a, b, and c. So, up to isomorphism, G' is either H_{15} or H_{16}, and, in either case, contains H_4 (indicated in bold) as a subgraph.

Case 3.2.2: The smallest cycle in G′ contains 5 or more edges (see Figure 2-8).

The only possibility for G' in this case is H_{17}, which contains H_4 (indicated in bold) as a subgraph.

Case 3.2.3: The smallest cycle in G′ contains 4 edges (see Figure 2-8).

In this case, G' starts with H_{18}, and the sixth edge makes G' into one of H_{19}, H_{20}, or H_{21}, which contain H_4, H_7, or H_8 (indicated in bold) respectively.

Case 3.2.4: The smallest cycle in G′ contains 3 edges (see Figure 2-8).

Since H_6 is an obstruction, every edge must contain one of the vertices in the 3-cycle. But the degree of G' is 3, so each vertex of the cycle has precisely one extra edge attached to it. Therefore, up to isomorphism, G' is one of H_{22}, H_{23}, or H_{24}, which contain H_4, H_6, or H_5 (indicated in bold) respectively.

■

Chapter 3

f-vectors and Betti sequences

In Chapter 2, we explored possible f-vectors of simplicial posets. Now, we add a twist to this problem by asking, additionally, what homology is possible.

Definition: Fix a field k. Let P be a simplicial poset, and let

$$\tilde{\beta}_i := \dim_k \tilde{H}_i(|P|; k) = \dim_k \tilde{H}_i(\Delta(\overline{P}); k).$$

Then $\beta := (\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \ldots)$ is the **Betti sequence** of P. If P has rank d, then $\tilde{\beta}_i = 0$ for $i > d-1$, so in that case, we will write the Betti sequence as $\beta = (\tilde{\beta}_0, \ldots, \tilde{\beta}_{d-1})$.

In contrast to the situation with f-vectors, *any* sequence of non-negative integers can be the Betti sequence of a simplicial poset, for instance, simply by taking a bouquet of spheres (*i.e.*, a collection of spheres, all intersecting at a unique point) of the appropriate multiset of dimensions. A more interesting question, then, is what *combinations* of f-vectors and Betti sequences of simplicial posets there are. We therefore make the following definition:

Definition: An ordered pair of sequences (f, β) is **compatible** if there is some simplicial poset P such that f is the f-vector of P and β is the Betti sequence of P.

We will be interested in finding compatible sequences.

This problem, and the definition of compatible sequences were examined by Björner and Kalai for the case of simplicial complexes (*i.e.*, when P is the face-poset of a simplicial complex) in [BK1], and then for a larger class of CW-complexes in [BK2].

The characterization of compatible sequences for both simplicial complexes and simplicial posets is purely combinatorial, in that it does not depend on the field k we initially chose. Therefore, it is possible to explore what the range of compatible sequences covers. This is done in [BK1, §5], for simplicial complexes; we do the same for simplicial posets in Section 3.2.

3.1 Necessary and sufficient conditions

In this section, we show that some obvious necessary conditions on the compatibility of f-vectors and Betti sequences of simplicial posets are actually sufficient. This result has a strong resemblance to, but is not identical to, the equivalent question for cell complexes, answered in [BK1, §6].

Theorem 3.1.1 *Suppose that* $f = (f_0, \ldots, f_{d-1})$ *and* $\beta = (\tilde{\beta}_0, \ldots, \tilde{\beta}_{d-1})$ *are two given sequences of non-negative integers,* $f_{d-1} > 0$*, and* k *is a field. Define* $f_i' = f_i - \binom{d}{i+1}$ *and* $\phi_i = f_i' - \tilde{\beta}_i$*. Then the following are equivalent:*

(i) f *is the* f*-vector and* β *the Betti sequence over* k *of some simplicial poset;*

(ii) For all $0 \leq m \leq d-1$,

$$\chi_m := \sum_{i=0}^{m} (-1)^{m-i} \phi_i \geq 0; \tag{3.1}$$

and

$$\chi_{d-1} = 0. \tag{3.2}$$

(iii) f *is the* f*-vector of a simplicial poset homotopic to a* β*-wedge of spheres (* i.e. *a wedge of* $\tilde{\beta}_i$ i*-spheres for* $0 \leq i \leq d-1$*).*

Proof: (iii) ⇒ (i) is trivial.

(i) $\Rightarrow$ (ii): We use induction and the Mayer-Vietoris sequence. Let P be the given simplicial poset. Since $f_{d-1} > 0$, there is a $(d-1)$-dimensional face z, and an entire $(d-1)$-simplex contained in it. Let K_0 be the subposet $\{z' \in P : z' \leq z\}$; then $f_i(K_0) = \binom{d}{i+1}$, so $f_i' = 0$, for all i. Also K_0 is a simplex, and hence contractible, so $\tilde{\beta}_i(K_0) = 0$, for all i, and equations (3.1) and (3.2) are trivially true.

Now, we can build up P by starting with K_0, and then adding the remaining 0-faces, and then the remaining 1-faces, and so on. If we construct P in this way, then every time a face is added, its boundary has already been added. So assume that K_1 is a subposet of P satisfying equations (3.1) and (3.2) that contains every $(i-1)$-face of P, and that K_2 is K_1 with an i-face y added, so $\partial y \subseteq K_1$ (if $i = 0$, then $\partial y = \hat{0} \in K_1$ trivially), and

$$f_p'(K_2) = \begin{cases} f_p'(K_1) & \text{if } p \neq i \\ f_p'(K_1) + 1 & \text{if } p = i \end{cases} .$$

If $i = 0$, then it is easy to see that

$$\tilde{\beta}_p(K_2) = \begin{cases} \tilde{\beta}_p(K_1) & \text{if } p \neq 0 \\ \tilde{\beta}_p(K_1) + 1 & \text{if } p = 0 \end{cases}$$

so $\phi_p(K_2) = \phi_p(K_1)$ for all p, and equations (3.1) and (3.2) continue to hold.

If $i > 0$, first note that the only chain group that is changed is C_i, changing only ∂_i among boundary maps, hence changing only $\tilde{\beta}_i$ and $\tilde{\beta}_{i-1}$. Further, since C_i is increased, *i.e.*, $C_i(K_1) \hookrightarrow C_i(K_2)$,

$$\tilde{\beta}_i(K_2) \geq \tilde{\beta}_i(K_1) \tag{3.3}$$

and

$$\tilde{\beta}_{i-1}(K_2) \leq \tilde{\beta}_{i-1}(K_1). \tag{3.4}$$

To determine more precisely what happens to $\tilde{\beta}_i$ and $\tilde{\beta}_{i-1}$, first note that since $i > 0$, $\partial y \neq \emptyset$, so we obtain the reduced Mayer-Vietoris exact sequence:

$$\begin{aligned} \cdots \quad &\to \tilde{H}_i(\partial y) \to \tilde{H}_i(y) \oplus \tilde{H}_i(K_1) \to \tilde{H}_i(K_2) \\ &\to \tilde{H}_{i-1}(\partial y) \to \tilde{H}_{i-1}(y) \oplus \tilde{H}_{i-1}(K_1) \to \tilde{H}_{i-1}(K_2) \to \tilde{H}_{i-2}(\partial y) \to \cdots \end{aligned} \tag{3.5}$$

(all homology is over k). Now, topologically, ∂y is an $(i-1)$-sphere, so

$$\tilde{H}_p(\partial y) = \begin{cases} k & \text{if } p = i-1 \\ 0 & \text{otherwise} \end{cases},$$

and y is a ball, so $\tilde{H}_p(y) = 0$ for all p; thus, equation (3.5) becomes

$$0 \to \tilde{H}_i(K_1) \to \tilde{H}_i(K_2) \to k \to \tilde{H}_{i-1}(K_1) \to \tilde{H}_{i-1}(K_2) \to 0.$$

Since dimension is additive on exact sequences,

$$0 \;=\; \tilde{\beta}_i(K_1) - \tilde{\beta}_i(K_2) + 1 - \tilde{\beta}_{i-1}(K_1) + \tilde{\beta}_i(K_2),$$

so

$$1 \;=\; (\tilde{\beta}_i(K_2) - \tilde{\beta}_i(K_1)) + (\tilde{\beta}_{i-1}(K_1) - \tilde{\beta}_{i-1}(K_2)),$$

which, with equations (3.3) and (3.4), gives either

$$\tilde{\beta}_p(K_2) \;=\; \begin{cases} \tilde{\beta}_p(K_1) + 1 & \text{if } p = i \\ \tilde{\beta}_p(K_1) & \text{otherwise} \end{cases},$$

or

$$\tilde{\beta}_p(K_2) \;=\; \begin{cases} \tilde{\beta}_p(K_1) - 1 & \text{if } p = i-1 \\ \tilde{\beta}_p(K_1) & \text{otherwise} \end{cases}.$$

Either way, $\phi_p(K_2) = \phi_p(K_1)$ for $0 \leq p \leq d-1, p \neq i, i-1$. Also, in the first case, $\phi_{i-1}(K_2) = \phi_{i-1}(K_1)$ and $\phi_i(K_2) = \phi_i(K_1)$; and in the second case, $\phi_{i-1}(K_2) =$

$\phi_{i-1}(K_1)+1$ and $\phi_i(K_2) = \phi_i(K_1)+1$. Thus, in both cases, $\chi_p(K_2) = \chi_p(K_1)$ if $p \neq i-1$, and $\chi_{i-1}(K_2) \geq \chi_{i-1}(K_1)$, preserving equations (3.1) and (3.2) for K_2, completing the induction.

(ii) $\Rightarrow$ (iii): Proof by construction (similar to [BK1]). For completeness, define $\chi_{-1} = 0$. Then, for $0 \leq i \leq d-1$,

$$f'_i = \chi_{i-1} + \tilde{\beta}_i + \chi_i,$$

so let $E_i = A_i \cup B_i \cup C_i$ be a set of f'_i i-simplices such that $|A_i| = \chi_{i-1}$, $|B_i| = \tilde{\beta}_i$, and $|C_i| = \chi_i$. We now construct a simplicial poset P as follows. Start with a $(d-1)$-simplex Σ with vertices numbered $1,\ldots,d$, and for $1 \leq j \leq d$, let σ_{j-1} denote the face $\{1,\ldots,j\}$. (The vertices and σ_j's are numbered this way only to be definitive; we need only pick σ_j's in Σ such that each σ_j has dimension j and, for $1 \leq j \leq d-1$, σ_{j-1} is a face of σ_j.) Then we add the faces E_i, for every i, by induction on i.

First add the vertices of E_0. Next, for each $i > 0$, add the i-simplices E_i $(i > 0)$ assuming, by induction, that all $(i-1)$-simplices in E_{i-1} have been added such that each $c^j_{i-1} \in C_{i-1}$, $1 \leq j \leq \chi_{i-1}$, has the same boundary as σ_{i-1} (if $i = 1$, this is just the trivial observation that every vertex has $\emptyset$ as its boundary). Then, for each $a^j_i \in A_i$, $1 \leq j \leq \chi_{i-1}$, attach a^j_i by letting

$$\partial a^j_i = ((\partial\sigma_i)\backslash\sigma_{i-1}) \cup c^j_{i-1}, \tag{3.6}$$

which is possible since $\partial c^j_{i-1} = \partial\sigma_{i-1}$. Then attach the i-simplices of B_i and C_i by giving each one the same boundary as σ_i.

Clearly, $f(P) = f$, so it remains to show that P is homotopic to a β-wedge of spheres. We demonstrate the homotopy. For every i, $1 \leq i \leq d-2$, there are χ_i pairs (c^j_{i-1}, a^j_i), satisfying equation (3.6), so apply the homotopy that collapses a^j_i and c^j_{i-1} through a^j_i to $(\partial\sigma_i)\backslash\sigma_{i-1} \subseteq \Sigma$. This can be done for each pair (c^j_{i-1}, a^j_i) independently. What remains is Σ and, for each i, $\tilde{\beta}_i$ i-simplices b^j_i such that $\partial b^j_i \subseteq \Sigma$. So, if we then apply the homotopy that shrinks Σ to a point, the resulting object is a β-wedge of spheres. ■

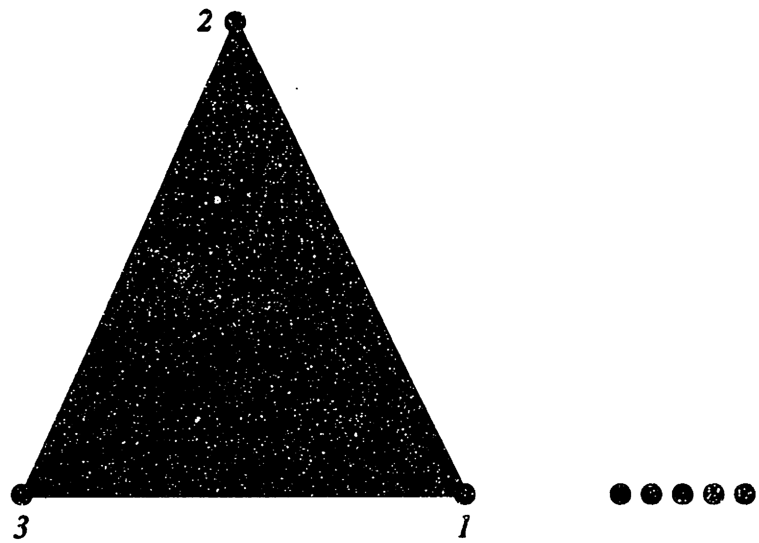


Figure 3-1: first step

Example: Consider the compatible pair of sequences $f = (8, 8, 4)$ and $\beta = (3, 2, 2)$ ($f' = (5, 5, 3)$; $\phi_0 = 2$, $\phi_1 = 3$, and $\phi_2 = 1$; and $\chi_0 = 2$, $\chi_1 = 1$, and $\chi_2 = 0$). To construct a simplicial poset P that has f-vector f and Betti sequence β, start with a 2-simplex (since $d - 1 = 2$). Now, the rest of the steps of the construction are as follows:

(i) $f'_0 = 5$, so add 5 extra vertices (all of which have the same boundary as σ_0, trivially). (See Figure 3-1.)

(ii) $|A_1| = \chi_0 = 2$, $|B_1| = \tilde{\beta}_1 = 2$, and $|C_1| = \chi_1 = 1$, so first add $\chi_0 = 2$ edges (a_1^1, and a_1^2), each attached to vertex 2 of Σ and one of the new vertices added in (i); then add $\tilde{\beta}_1 + \chi_1 = 3$ new edges (b_1^1, b_1^2, and c_1^1), each attached to vertices 1 and 2 (same boundary as σ_1). (See Figure 3-2.)

(iii) $|A_2| = \chi_1 = 1$, $|B_2| = \tilde{\beta}_2 = 2$, and $|C_2| = \chi_2 = 0$, so first add $\chi_1 = 1$ new 2-face (a_2^1), attached to edges 13, 23, and c_1^1; then add $\tilde{\beta}_2 + \chi_2 = 2$ new 2-faces (b_2^1 and b_2^2), each attached to edges 12, 13, and 23 (same boundary as σ_2). (See Figure 3-3.)

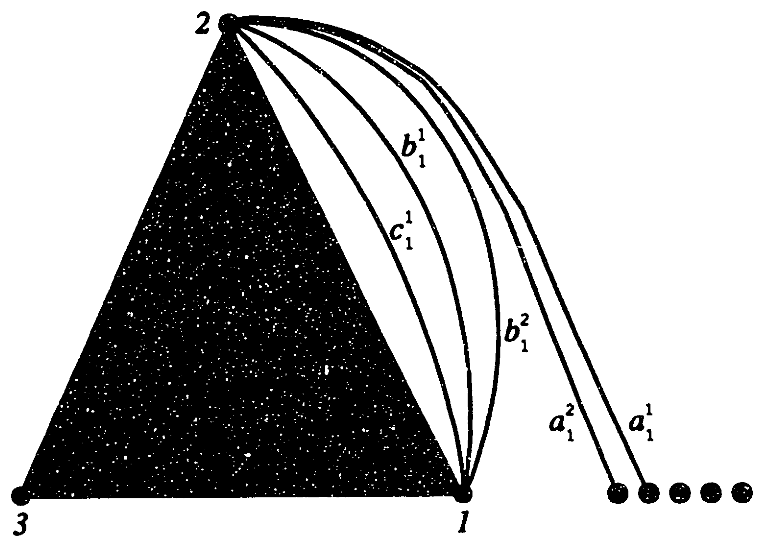


Figure 3-2: second step

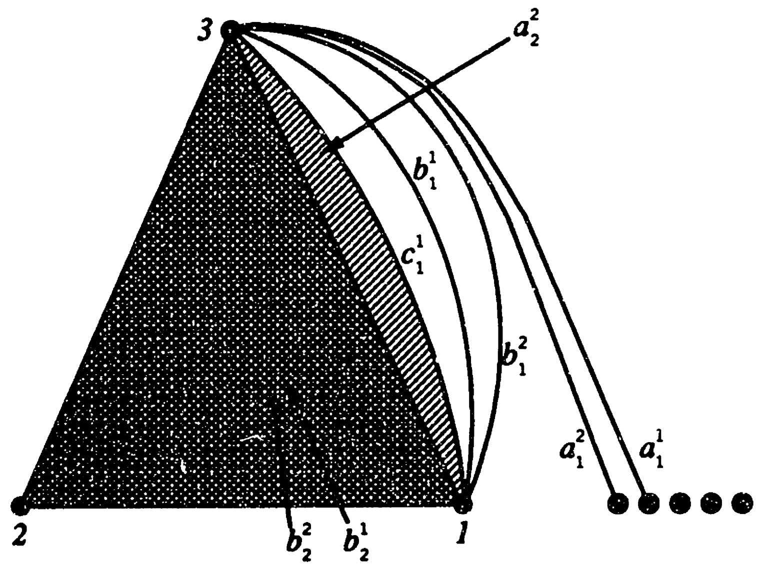


Figure 3-3: third step

3.2 Combinatorics of (f, β) pairs

We use the combinatorial result of the previous section to further examine the set of compatible (f, β) pairs. In particular, we are interested in the set of f's that are compatible with a given β and the set of β's that are compatible with a given f. As in [BK1], we address this problem by looking for the "minimal" or "maximal" f [or β] corresponding to β [or f, respectively]. In our case, three of the four problems are quite easy (we often work with f' instead of f, but this should cause no confusion) :

- There is no "maximal" f for a given β, since $(f + \epsilon_{i,i-1}, \beta)$ is compatible if (f, β) is, where $\epsilon_{i,i-1}$ is the vector with 0's except for 1's in the $(i-1)$st and ith positions $(1 \leq i \leq d-1)$. Toplogically, this corresponds to adding another simplex to each of C_{i-1} and A_i in our construction, which does not change the homology.

- To find the minimal f for a given β or a maximal β for a given f, the answer is the same in both cases: let $f'_i = \tilde{\beta}_i$. This sets $\phi_i = 0$, for all i, which satisfies equations (3.1) and (3.2); and it is easy to see, by adding consecutive equations of 3.1, that each ϕ_i must be at least 0, establishing a lower bound for each f'_i or an upper bound for each $\tilde{\beta}_i$.

On the other hand, finding the minimal β for a given f is less trivial. At first glance, it might seem entirely dual to the maximal f problem; and indeed, decreasing $\tilde{\beta}_i$ and $\tilde{\beta}_{i-1}$ produces a clearly smaller β that still satisfies equations (3.1) and (3.2). However, unlike the f's, there is a limit to how far this can go: every component of β must be non-negative (there was no corresponding upper limit to f). In fact, for some fixed f, there is no compatible β that is componentwise no greater than any other compatible β, as the following example shows:

Example: Let $f' = (2,2,2,0)$. Two compatible β's are $\beta^1 = (2,0,0,0)$ and $\beta^2 = (0,0,2,0)$. But the only non-negative β that is componentwise no greater than both β^1 and β^2 is $(0,0,0,0)$, which is not compatible with f'.

So there is not necessarily a minimal compatible β in the componentwise partial order. We settle for identifying the minimal β in the following total order, $\leq_T$: $\beta^1 \leq_T \beta^2$ if

- $|\beta^1| < |\beta^2|$ (where $|(v_0, \ldots, v_{d-1})| := v_0 + \cdots + v_{d-1}$, the **weight** of v)

  or

- $|\beta^1| = |\beta^2|$ and $\beta^1 \leq_L \beta^2$, where $\leq_L$ is lexicographic order (*i.e.*, for some j, $(\beta^1)_i = (\beta^2)_i$ for $0 \leq i < j$, and $(\beta^1)_j < (\beta^2)_j$).

We also show that a more or less greedy algorithm finds such minimal β.

Proposition 3.2.1 *For a fixed f', the compatible β that is minimal in the above total order $\leq_T$ is the lexicographically minimal compatible β. In other words, the lexicographically minimal compatible β is also a minimal weight compatible β.*

Proof: Let β^0 be the lexicographically minimal compatible β, and let β^1 be any other compatible β, so

$$(\beta^0)_i = (\beta^1)_i \tag{3.7}$$

for $0 \leq i < j$; and

$$(\beta^0)_j < (\beta^1)_j. \tag{3.8}$$

We will find $\beta^2 \leq_T \beta^1$, so that β^1 is not the minimal ($\leq_T$) compatible β. Since $\leq_T$ is a total order, this will show that β^0 is minimal in $\leq_T$. First note that $j \neq d-1$ since, in that case, equations (3.2) and (3.7) force $(\beta^2)_{d-1} = (\beta^1)_{d-1}$. Two cases remain:

Case 1: $j = d-2$.

Let $\beta^2 = \beta^0$. Since $\beta^0 <_L \beta^1$ by assumption, it remains to show $|\beta^0| \leq |\beta^1|$. By equations (3.2), (3.7), and (3.8),

$$(\beta^0)_{d-1} - (\beta^0)_{d-2} = (\beta^1)_{d-1} - (\beta^1)_{d-2}$$

and

$$(\beta^0)_{d-2} < (\beta^1)_{d-2},$$

so

$$(\beta^0)_{d-1} = (\beta^1)_{d-1} + ((\beta^0)_{d-2} - (\beta^1)_{d-2}) < (\beta^1)_{d-1},$$

which, along with equations (3.7) and (3.8), establishes $|\beta^0| < |\beta^1|$.

Case 2: $j < d-2$.

Let

$$(\beta^2)_i = \begin{cases} (\beta^1)_i - 1 & \text{if } i = j \\ (\beta^1)_i + 1 & \text{if } i = j+2 \\ (\beta^1)_i & \text{otherwise} \end{cases}.$$

Clearly $\beta^2 \leq_T \beta^1$; and β^2 is non-negative since

$$(\beta^2)_j = (\beta^1)_j - 1 \geq (\beta^0)_j \tag{3.9}$$

by equation (3.8). It remains to show that β^2 is compatible with f'. But

$$\chi_i(f', \beta^2) = \chi_i(f', \beta^1)$$

if $i \neq j, j+1$ (note that $d - 1 \neq j, j+1$);

$$\chi_j(f', \beta^2) > \chi_j(f', \beta^0) \geq 0$$

(by equation (3.9)); and

$$\chi_{j+1}(f', \beta^2) = \chi_{j+1}(f', \beta^1) + 1 \geq 0.$$

■

The algorithm

Now we describe an efficient method for finding this minimal β. By Proposition 3.2.1, it suffices to show that the β we find is lexicographically minimal among β's compatible with our fixed f'.

We start by setting each $\tilde{\beta}_i = 0$, one at a time, starting with $\tilde{\beta}_0$, then $\tilde{\beta}_1$, and so on, until this forces $\chi_i < 0$ for some i (note that $i > 0$ since χ_0 starts at $f'_0 - \tilde{\beta}_0 = f'_0 \geq 0$). At this point, we change β_{i-1} (not β_i!) to $-\chi_i$, so that now $\chi_i = 0$ and

$$\chi_{i-1} = f'_i - \tilde{\beta}_i - \chi_i = f'_i \geq 0.$$

Also at this point,

$$\chi_{i+1} = f'_{i+1} - \tilde{\beta}_{i+1} - \chi_i = f'_{i+1} \geq 0, \tag{3.10}$$

so $\tilde{\beta}_i$ will not have to be changed.

Now, repeat this process, up to χ_{d-2} (equation (3.10) guarantees that none of our operations will interfere with one another), so $\tilde{\beta}_{d-2}$ and $\tilde{\beta}_{d-1}$ are still 0. Then, to ensure $\chi_{d-1} = 0$:

- If $\chi_{d-1} < 0$, change $\tilde{\beta}_{d-2}$ to $-\chi_{d-1}$, and leave $\tilde{\beta}_{d-1} = 0$, as with all the other χ_i's that were negative;
- If $\chi_{d-1} > 0$, leave $\tilde{\beta}_{d-2} = 0$, change $\tilde{\beta}_{d-1}$ to χ_{d-1}, and then χ_{d-1} will be 0, and in this case there are no other equations or inequalities to check.

At each step of the construction (*i.e.*, possibly fixing $\tilde{\beta}_{i-1}$ to ensure $\chi_i \geq 0$) we consider all previous $\tilde{\beta}_j$'s (*i.e.*, for $j < i - 1$) to be fixed, and set $\tilde{\beta}_{i-1}$ to be the minimal value possible while still setting $\chi_i \geq 0$. Thus, the resulting β is lexicographically minimal since every non-zero $\tilde{\beta}_i$ cannot be made smaller without changing $\tilde{\beta}_j$ for $j < i$.

Example: Say $f' = (3, 7, 2, 5, 10, 2, 1, 5)$. To find the lexicographically minimal compatible β, start with $\beta = (0, \ldots, 0)$. Then $\chi_0 = 3$ and $\chi_1 = 4$, but $\chi_2 = -2$. So change $\tilde{\beta}_1$

to be 2. Then $\chi_1 = 2$, $\chi_2 = 0$, $\chi_3 = 5$ and $\chi_4 = 5$, but $\chi_5 = -3$. So change $\tilde{\beta}_4$ to be 3. Then $\chi_4 = 2$, $\chi_5 = 0$, $\chi_6 = 1$, and $\chi_7 = 4$. Then, to get $\chi_7 = 0$, change $\tilde{\beta}_7$ to be 4. The resulting β is $(0, 2, 0, 0, 3, 0, 0, 4)$.

Chapter 4

Complete intersection rings

In this chapter, we classify all simplicial posets P whose ring A_P is a complete intersection ring; the result was conjectured by V. Reiner.

Definition: Let A be a commutative ring. Then θ is a **non-zero divisor (NZD)** if $u\theta = 0$ implies that $u = 0$.

Definition: Let A be a graded commutative ring. A sequence $\theta_1, \ldots, \theta_r$ of homogeneous elements of A is an **A-sequence** or **regular sequence** if θ_1 is an NZD of A and, for every $1 < i \leq r$, θ_i is an NZD of $A/(\theta_1, \ldots, \theta_{i-1})$.

Remark: Any permutation of a regular sequence is again a regular sequence (implicit from [St4, §I.6]).

Definition: A $\mathbf{Z}^m$-graded algebra R is a **complete intersection** if $R = A/(\theta_1, \ldots, \theta_r)$ for some polynomial ring A, and A-sequence, $\theta_1, \ldots, \theta_r$.

Complete intersections are studied extensively by algebraic geometers, usually in very non-combinatorial terms. Combinatorially, they are of interest because their nice presentation allows certain invariants (generating functions, for instance) to be easily computed.

For instance, it is easy to show that complete intersection rings are Cohen-Macaulay, and even Gorenstein.

4.1 Complete intersections of simplicial complexes

There is a simple classification of all simplicial complexes whose face-ring is a complete intersection:

Theorem 4.1.1 *Let Δ be a simplicial complex. Then its face ring $k[\Delta]$ is a complete intersection ring if and only if Δ is (topologically) the join of simplicial complexes of the form*

- B_n
- $B_n \backslash \hat{1}$

Proof:(sketch) The main point is that $k[\Delta]$ is just the polynomial ring on its vertices, modulo the minimal non-faces. Once enough commutative algebra machinery is brought into play, we can assume that the minimal non-faces form a regular sequence. So say $F = x_{i_1} \cdots x_{i_n}$ and $G = x_{j_1} \cdots x_{j_m}$ are two minimal non-faces, but intersect; say, $x_{i_1} = x = x_{j_1}$, so we can write $F = F'x$ and $G = G'x$. Then modulo F, F' is not zero (since F is a minimal non-face), but $GF' = G'xF' = G'F = 0$, contradicting G being an NZD modulo F. ■

This is a simple proof (modulo the commutative algebra machinery), but it is the heart of what we will do for the more general simplicial poset: find the simplest obstacles to A_P being a complete intersection, and show that they come up except in the simplest of circumstances.

4.2 Definitions and main theorem

First we state the generalization of Theorem 4.1.1 that we will prove in this chapter. As with the simplicial complex case, the sufficiency of the conditions is almost trivial, and we jump right into setting up the machinery to prove the necessity.

Theorem 4.2.1 (conjectured by V. Reiner) *Let P be a simplicial poset. Then A_P is a complete intersection ring if and only if P is the direct product of posets of the form*

- B_n
- $B_n \backslash \hat{1}$
- $B_n \dot{\cup} \hat{1}'$ *(i.e., B_n with $\hat{1}'$ that is greater than everything else in B_n, except for $\hat{1}$, with which it is incomparable.)*

The rest of this chapter is devoted to the proof of this theorem.

First note that some of the simplest generators of I (where $A_P = k[P]/I$ as usual) arise in one of the following two ways:

- **duplication**: If y and y' cover the same elements, then they cannot have an upper bound, so $yy' = 0$ in A_P.
- **elimination**: If there is a set of elements D that form the boundary of a simplex F, but that simplex is not in P, then we could think of A_P as being $A_{P'}/F$, where P' is P with F adjoined in the obvious manner. In this way, F corresponds to a relation.

Elimination may seem somewhat contrived, until we consider the special case of a simplicial complex. We can think of any simplicial complex as a Boolean algebra with some faces modded out. And that is, in fact, the way to think about complete intersection simplicial complexes; the answer in that case is to consider the set of minimal faces that are modded out, and the simplicial complex is a complete intersection if and only if these faces do not intersect.

Similarly, one way to think about Theorem 4.2.1 is that none of the eliminations or duplications overlap. It is actually somewhat more complicated than that, in that there is no systematic way of presenting A_P as a polynomial ring modulo "minimal" relations.

We make the following assumptions throughout the rest of this chapter. P is a simplicial poset with n vertices, $x_1, \ldots, x_n$. The ideal I is defined by $A_P = k[P]/I$ to be the set of all relations among elements of P, and in fact, we often think of A_P this way, even if it is not the most efficient way of presenting A_P. Recall that $\partial y := \{x \in P : x \lessdot y\}$, *i.e.*, the set of elements covered by y, corresponding to the usual definition of boundary in simplicial complexes.

Definitions: Define the equivalence relation $\sim$ by

$$y \sim y' \Leftrightarrow \partial y = \partial y'.$$

Note that, inductively, if $y \sim y'$, then $\deg(y) = \deg(y')$. Further, y and y' are a **duplication**, $y \parallel y'$, iff $y \sim y'$, $y \neq y'$, and $\mathrm{r}(y) = \mathrm{r}(y') > 1$ (note that if x_i and x_j are atoms, then $x_i \sim x_j$, trivially). A **minimal duplication**, $y \parallel_0 y'$ occurs iff $y \parallel y'$ and, for all $z < y$, there is no z' such that $z \parallel z'$. In other words, $y \parallel y'$ and the subposet of $\{z \in P : z < y\}$ is isomorphic to a Boolean algebra with its top element missing. Although duplication and minimal duplication are not equivalence relations, since $y \not\parallel y'$, they are transitive on sets of distinct elements: if x, y, z are three distinct elements of P, $x \parallel y$, and $y \parallel z$, then $x \parallel z$; and similarly for minimal duplications.

An **elimination** is an order ideal $D \subseteq P$ such that D is isomorphic to a Boolean algebra with its top element removed. We often identify D with this removed element by extending certain definitions to D:

$$\deg(D) := \sum_{x_i \in D} \epsilon_i,$$

and

$$\partial(D) := \{z \in D : \nexists z' \in D,\ z < z'\}.$$

Example: (See Figure 4-1.) The edges y_3, y_4, y_5 are all minimal duplications of one another ($y_3 \parallel_0 y_4$, $y_4 \parallel_0 y_5$, and $y_5 \parallel_0 y_3$). The 2-face z_3 does not have any duplications (z_1 and z_2 contain the same vertices x_1, x_2, x_3, but z_3 covers y_4, while the others do not). The 2-faces z_1 and z_2 are a duplication, but not a minimal duplication, since $y_3 \parallel y_4$ and $y_3 < z_3$ (and z_2); therefore $z_1 \parallel z_2$, but $z_1 \nparallel_0 z_2$.

The missing element (D) covers an elimination. Technically, the elimination is $\{\hat{0}, x_1, x_2, x_3, y_1, y_2, y_5\}$, but it is easier to think of (D) as having been eliminated.

Lemma 4.2.2 *If $y \parallel_0 y'$, then $yy' \in I$.*

Proof: Assume otherwise; then there is a $z \geq y, y'$. Therefore, y and y' are distinct elements of the same degree in $[\hat{0}, z]$, which is isomorphic to a Boolean algebra. But every element of a Boolean algebra has distinct degree, $\Rightarrow\Leftarrow$. ∎

4.3 Duplications in regular sequences

We bring in some results that connect the combinatorics of the poset (in particular, a duplication) to the possibility of the ring being a complete intersection.

Lemma 4.3.1 *Let R be a graded k-algebra with ideal I containing homogeneous elements $\theta_1, \ldots, \theta_i$ generated by a regular sequence, S. Then there is another regular sequence S' generating I containing $\theta_1, \ldots, \theta_i$ if $\bar{\theta}_1, \ldots, \bar{\theta}_i$ are linearly independent in the k-vector space I/R_+I.*

Proof: By induction on i. The case $i = 0$ is trivial. So assume that there is another regular sequence $S'' = \{\theta_1, \ldots, \theta_{i-1}, \Psi_i, \ldots, \Psi_r\}$ generating I. Now, $\theta_i \in I$, so we may

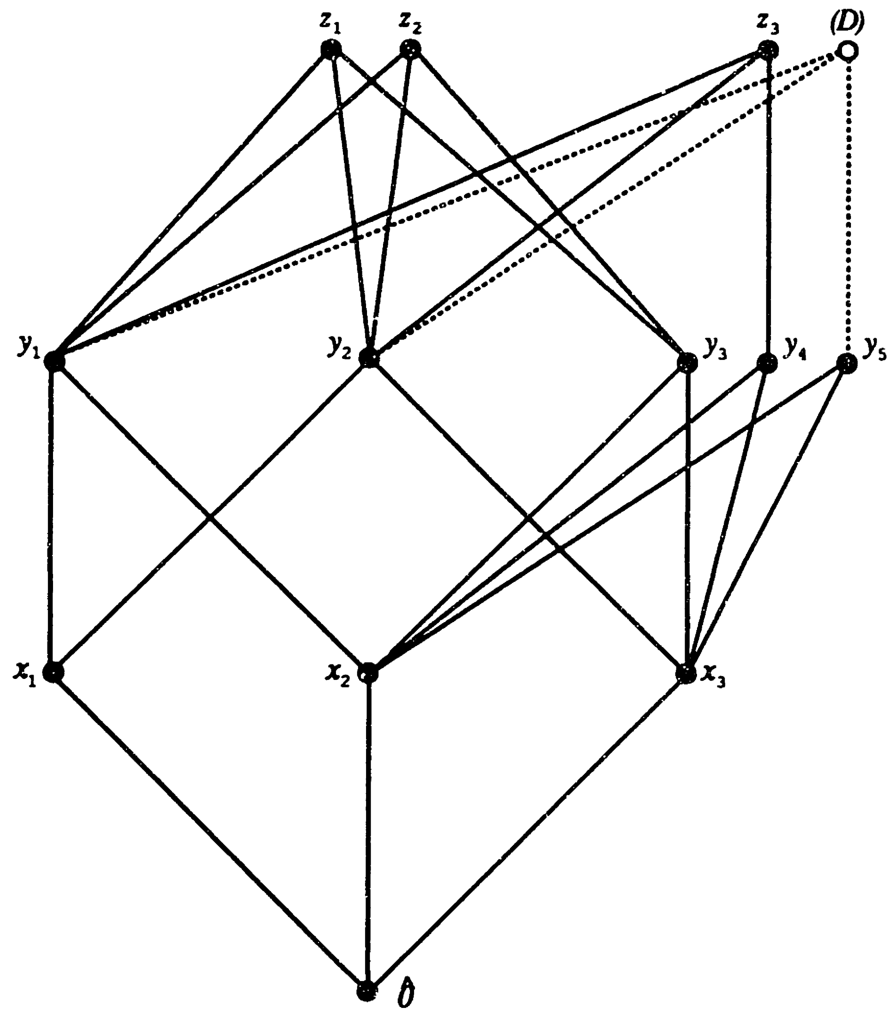


Figure 4-1: eliminations and duplications

write

$$\theta_i = \sum_{j=1}^{i-1} a_j\theta_j + \sum_{j=i}^{r} b_j\Psi_j \tag{4.1}$$

for $b_j \in H(R)$. Rewrite equation (4.1) as

$$\theta_i - \sum_{j=1}^{i-1} a_j\theta_j = \sum_{j=i}^{r} b_j\Psi_j, \tag{4.2}$$

and now consider equation (4.2) modulo R_+I. The left-hand side becomes a non-zero k-linear combination of $\bar{\theta}$'s; the right-hand side, since the b_j's are homogeneous, can be written

$$\sum_{j:b_j\in k} b_j\bar{\Psi}_j.$$

This sum cannot be 0, since the $\bar{\theta}_j$'s are linearly independent, so there is some $b_j \in k\backslash\{0\}$; without loss of generality, since any permutation of a regular sequence is again a regular sequence, $b_r \in k\backslash\{0\}$

Then our new sequence is $S' = (\theta_1, \ldots, \theta_{i-1}, \Psi_i, \ldots, \Psi_{r-1}, \theta_i)$. Now, S' generates I, since

$$\Psi_r = b_r^{-1}(\theta_i - \sum_{j=1}^{i-1} a_j\theta_j - \sum_{j=i}^{r-1} b_j\Psi_j).$$

To show that S' is regular, it suffices to show that θ_i is an NZD $(\bmod\, \theta_1, \ldots, \theta_{i-1}, \Psi_i, \ldots, \Psi_{r-1})$. So assume

$$u\theta_i = 0 \ (\bmod\, \theta_1, \ldots, \theta_{i-1}, \Psi_i, \ldots, \Psi_{r-1}).$$

Then, plugging in equation (4.1),

$$0 = u(\sum_{j=1}^{i-1} a_j\theta_j + \sum_{j=1}^{r} b_j\Psi_j) = ua_r\Psi_r \ (\bmod\, \theta_1, \ldots, \theta_{i-1}, \Psi_i, \ldots, \Psi_{r-1})$$

so $u = 0 \ (\bmod\, \theta_1, \ldots, \theta_{i-1}, \Psi_i, \ldots, \Psi_{r-1})$, since Ψ_r is an NZD $(\bmod\, \theta_1, \ldots, \theta_{i-1}, \Psi_i, \ldots, \Psi_{r-1})$. Thus, θ_i is an NZD. ∎

Lemma 4.3.2 *Let $y \parallel_0 y'$; assume $S = \{\Psi_1, \ldots, \Psi_n\}$ is a regular sequence in $k[P]$ generating I. Then there is another regular sequence S' generating I, containing $\theta = yy'$.*

Proof: First note that $\theta \in I$ by Lemma 4.2.2. Then, by Lemma 4.3.1 it suffices to show, with all calculations done in $k[P]$, not A_P, that $(k[P])_+ I$ does not contain θ. Assume otherwise:

$$\theta = \sum_{i \in I} a_i i, \tag{4.3}$$

where each a_i is homogeneous of non-zero degree, and only finitely many a_i are non-zero, say $a_i \neq 0$ for $i \in I_0$. We may as well assume that every element of I_0 is one of the canonical generators of I, by rewriting and expanding equation (4.3), if necessary. Since $\deg(\theta) = 2\deg(y)$, $\deg(i) < \deg(y)$ for all $i \in I_0$.

But this means that for each relation uv or $uv - (u \wedge v)\sum z$ in I_0, u and v must cover only atoms in $\mathrm{supp}(y)$, *i.e.*, $\deg(u), \deg(v) < \deg(y)$, and at least one of u and v, say v, must have degree $< \deg(y)$. Hence, $\deg(v) < \deg(y)$, so $v < y$, since $y \parallel_0 y'$, and then $\deg(u) < \deg(y)$ (for otherwise $\deg(u) = \deg(y)$ and then $v < u$, since then $y \parallel_0 u$). So all $i \in I_0$ are of the form:

$$i = uv - (u \wedge v)(u \vee v), \tag{4.4}$$

where u and v have a unique join, and $(u \wedge v), (u \vee v) < y$; or

$$i = uv - (u \wedge v)(\sum_{x \sim y} x) \tag{4.5}$$

If i is of the form in equation (4.4), then, by degree considerations, no $a_i i$ will have yy', y^2, or $(y')^2$ as a summand.

For any u, v complements in $[\hat{0}, y]$, define

$$g_{uv} := uv - \sum_{x \sim y} x$$

(this is one of the canonical generators of I). Denote the set of all such pairs $\{u, v\}$ by G. If i is of the form in equation (4.5), $a_i i$ will have a summand yy', y^2, or $(y')^2$ only if

$i = g_{uv}$ and $a_i = cy$ or cy' (for some $c \in k$), by degree considerations again. We may rewrite equation (4.3) as

$$yy' = \sum_{\{u,v\}\in G} c_{uv} y g_{uv} + c'_{uv} y' g_{uv} + \sum_{i \in I_0 \setminus G} a_i i,$$

where $c_{uv}, c'_{uv} \in k$. Since this equation takes place in $k[P]$, and not A_P, we may equate coefficients, first on yy', then on y^2 and $(y')^2$. By the above considerations, this yields

$$\begin{aligned} 1 &= \sum_{\{u,v\}\in G} c_{uv} + c'_{uv}, \\ 0 &= \sum_{\{u,v\}\in G} c_{uv}, \end{aligned}$$

and

$$0 = \sum_{\{u,v\}\in G} c'_{uv},$$

a contradiction, so $\theta \notin (k[P]_+)I$. ∎

Lemma 4.3.3 *If R is a complete intersection ring and $R = B/J$ for some polynomial ring B, then J is generated by a B-sequence.*

Proof: See [Ku, p. 190] ∎

This next lemma is now what this section has been leading up to.

Lemma 4.3.4 *If A_P is a complete intersection ring and $y \parallel_0 y'$, then there is a minimal generating set of I that is a regular sequence and contains $\theta = yy'$.*

Proof: Since $A_P = k[P]/I$ is a complete intersection ring, I is generated by a regular sequence in $k[P]$, by Lemma 4.3.3. But then there is another regular sequence generating I that contains θ, by Lemma 4.3.2. ∎

4.4 Technical lemmas

These just turn out to be precisely what we will need later on.

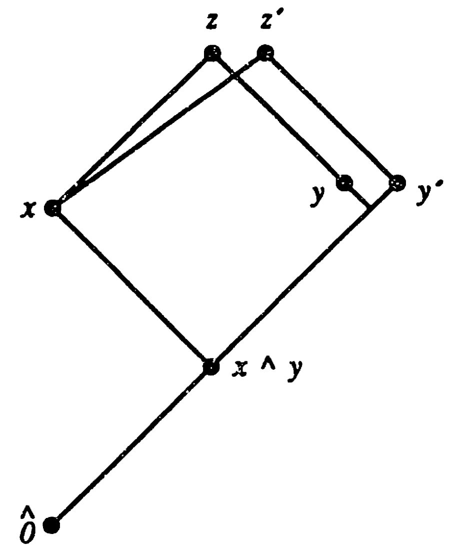


Figure 4-2: Lemma 4.4.1

Lemma 4.4.1 *If*

- $x \wedge y \neq \hat{0}$, $x \neq y$, $y \parallel y'$;
- $z \in \mathrm{mub}(x, y)$ *exists; and*
- $z' \in \mathrm{mub}(x, y')$ *exists (note that* z' *is not a duplication of* z*);*

then z *is not a multiple of* x *in* A_P.

Proof: (See Figure 4-2.) In other words, we must show that in $k[P]$, there is no p such that $xp = z \pmod I$ (p could be a polynomial, it does not need to correspond to an element of P). Assume such a p exists. Since x, z, and I are homogeneous, we need only consider the homogeneous part of p of degree $\deg(z) - \deg(x)$. So we may as well assume that $\deg(p) = \deg(z) - \deg(x) < \deg(y)$ (since $\deg(z) = \deg(y) + \deg(x) - \deg(x \wedge y)$). Modulo I, p is a sum of chains, c_i, with $\deg(c_i) < \deg(y)$, hence each c_i is a single element in P, and has rank $\mathrm{r}(c_i) < \mathrm{r}(y) = \mathrm{r}(y')$.

Thus, for each i, $c_i < y$ iff $c_i < y'$; hence $c_i < z$ iff $c_i < z'$. So, for each i, using relations in I to express xc_i as a sum of chains of $\deg(z)$, *i.e.*, elements of $\deg(z)$,

$$xc_i = a_i(z + z') + \sum_{\substack{\deg(z'')=\deg(z) \\ z'' \neq z, z'}} a_{iz''} z''$$

and

$$xp = a(z + z') + \sum_{\substack{\deg(z'')=\deg(z) \\ z'' \neq z, z'}} a_{z''} z'',$$

where $a_i, a_{iz''}, a, a_{z''} \in k$. But $xp = z \pmod I$, so $a = 1$, and

$$0 = z' + \sum_{\substack{\deg(z'')=\deg(z) \\ z'' \neq z, z'}} a_{z''} z'' \pmod I,$$

so a non-zero sum of elements of $\deg(z)$ is in I, $\Rightarrow\Leftarrow$. ∎

Corollary 4.4.2 *If $y \parallel y'$ and $y > x$, then y is not a multiple of x in A_P.*

The following lemma is what puts everything together, although it is just some elementary arithmetic. It is for duplications what the simple proof for the simplicial complex case is for eliminations. Unfortunately, with duplications, there are more exceptions, which is why we need all the other lemmas.

Lemma 4.4.3 *If A_P is a complete intersection ring, $y \parallel_0 y'$, $z \neq y, y'$, and $y'z \in I$, then z is a multiple of $y \pmod I$.*

Proof: By Lemma 4.3.4, generate I by a regular sequence, S, containing $\theta = yy'$, and let $\{\Psi_1, \ldots, \Psi_n\}$ be the rest of the regular sequence. Since $y'z \in I$,

$$y'z = a_0 yy' + \sum_{i=1}^{m} a_i \Psi_i$$

(all $a_i \in k[P]$). Multiplying through by y,

$$yy'z = a_0 yy'y + \sum_{i=1}^{m} (a_i y) \Psi_i,$$

$$
\begin{aligned}
yy'(z - a_0 y) &= \sum_{i=1}^{m} (a_i y)\Psi_i \\
&= 0 \pmod{\Psi_1, \ldots, \Psi_m}.
\end{aligned}
$$

Since S is a regular sequence, yy' is an NZD $\pmod{\Psi_1, \ldots, \Psi_m}$, and

$$
(z - a_0 y) = 0 \pmod{\Psi_1, \ldots, \Psi_m},
$$

so, since $\Psi_1, \ldots, \Psi_m \in I$,

$$
(z - a_0 y) = 0 \pmod{I}.
$$

■

Corollary 4.4.4 *If A_P is a complete intersection ring, and $y \parallel_0 y'$, then there is no $z \in P$ such that $z \neq y, y'$, and $\deg(z) = \deg(y)$.*

Proof: Assume otherwise; then, by minimality, $z \parallel_0 y$. By Lemma 4.2.2, $y'z \in I$, so, by Lemma 4.4.3, z is a multiple of y' in A_P, contradicting $z \parallel_0 y'$. ■

4.5 Interactions of duplications and elminations

Finally, we start showing that in a complete intersection ring, duplications and eliminations cannot share vertices.

Duplications

Proposition 4.5.1 *If A_P is a complete intersection ring, $x \parallel_0 x'$, and $y \parallel y'$ (with x, x', y, y' all distinct), then $x \wedge y = \hat{0}$.*

Proof: Assume not: $x \wedge y \neq \hat{0}$.

Case 1: Both $\mathrm{mub}(x, y)$ *and* $\mathrm{mub}(x, y')$ *are non-empty, say $z \in$* $\mathrm{mub}(x, y)$*, and $z' \in$* $\mathrm{mub}(x, y')$ *(See Figure 4-3).*

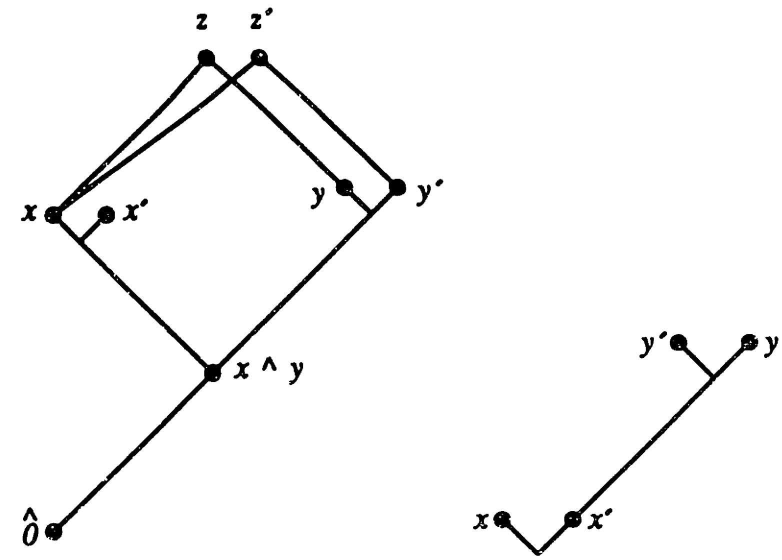


Figure 4-3: Proposition 4.5.1, case 1 and case 2

Since $x' \parallel_0 x < z$, it follows that x' and z have no upper bound, and $x'z \in I$. By Lemma 4.4.1, z is not a multiple of x in A_P, contradicting Lemma 4.4.3.

Case 2: One of $\text{mub}(x, y)$ *and* $\text{mub}(x, y')$ *is empty, say* $\text{mub}(x, y) = \emptyset$ *(See Figure 4-3).*

Then $xy \in I$, so, by Lemma 4.4.3, y is a multiple of x', and therefore $\deg(y) > \deg(x')$. As a result, y must be greater than *some* element whose degree is $\deg(x')$. Then, by Corollary 4.4.4, since $x \parallel_0 x'$, y must be greater than one of x or x'. But $y \not> x$, since $\text{mub}(x, y) = \emptyset$. So $y > x'$, but then, by Corollary 4.4.2, y is not a multiple of x', contradicting our earlier assertion.

■

Corollary 4.5.2 *If A_P is a complete intersection ring, then any duplication is minimal.*

Corollary 4.5.3 *If A_P is a complete intersection ring, then any two duplications have no common vertices, i.e., if $x \parallel x'$ and $y \parallel y'$, then* $\text{supp}(x) \cap \text{supp}(y) = \emptyset$ *unless $y = x$ or $y = x'$.*

Corollary 4.5.4 *Given $y \parallel_0 y'$, vertex set S, $S \cap \deg(y) \neq \emptyset$, there is at most one $z \in P$ such that*

- $\deg(z) = S$, *and*
- *if* $\deg(y) < S$, *then* $y < z$.

Proof: Consider a minimal set S that is a counterexample. If $\deg(y) \not< S$, then this means that there are two elements z, z' with degree $= S$. By minimality of S, there is, for each $x \in \partial z$, one element of P with degree $= \deg(x)$, so $z \parallel_0 z'$, but since $S \cap \deg(y) \neq \emptyset$, this contradicts Lemma 4.5.1.

If $\deg(y) < S$, then $z, z' < y$, and $\deg(z) = \deg(z')$. By minimality of S, there is, for each $x \in \partial z$, only one element of P with degree $= \deg(x)$, unless $\deg(y) > \deg(x)$. But in that case, $y > x$ (since $y, x \in [\hat{0}, z]$), and there is only one element, x' of P with $\deg(x') = \deg(x)$ and $y > x'$. So, again, $z \parallel_0 z'$, contradicting Lemma 4.5.1. ■

Eliminations

Now we show that duplications cannot share vertices with eliminations.

Proposition 4.5.5 *If A_P is a complete intersection ring, $y \parallel y'$, and D is an elimination, then* $\deg(y) \cap \deg(D) = \emptyset$.

Proof: (See Figure 4-4.) Assume not. By Corollary 4.5.2, $y \parallel_0 y'$, so $\deg(D) \not\leq \deg(y)$, and $(\deg(D)) \setminus (\deg(y)) \neq \emptyset$. Let z_0 be the unique element in D with $\deg(z_0) = (\deg(D)) \setminus (\deg(y))$; note that $z_0 \neq \hat{0}$. Let y_0 be the unique element in D with $\deg(y) = (\deg(y)) \cap (\deg(D))$; note that $y_0 \neq \hat{0}$.

Claim: There is no upper bound in P for y_0 and z_0. Assume otherwise: let w be a minimal upper bound of y_0 and z_0. Since $\deg(y_0) \cap \deg(z_0) = \emptyset$, $\deg(w) = (\deg(y_0)) \cup (\deg(z_0)) = \deg(D)$. It will therefore suffice to show that $\partial w = \partial D$, since D is an elimination. By Corollary 4.5.4, for every $u \in \partial D$, except possibly z_0 if $z_0 \in \partial D$, there is at most one element $v \in P$ with $\deg(v) = \deg(u)$, and $v > y > y_0$ if $\deg(u) > \deg(y) >$

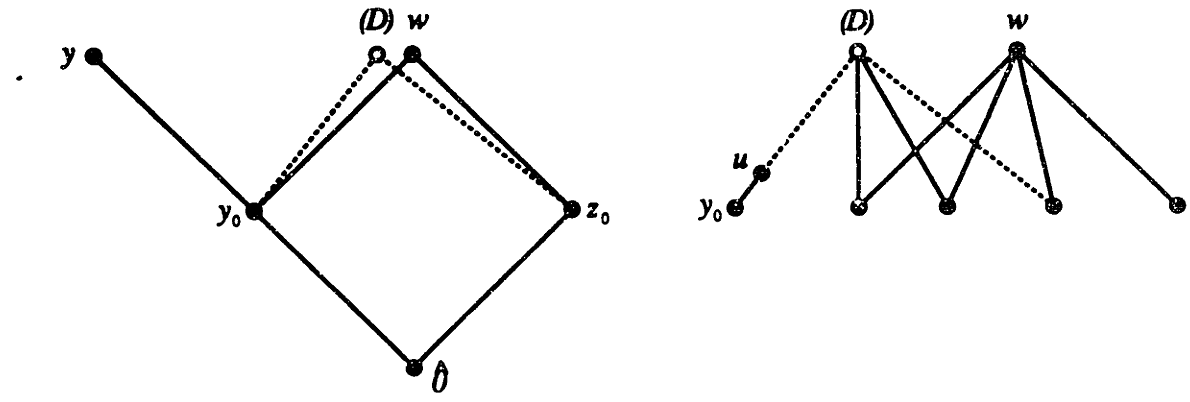


Figure 4-4: Proposition 4.5.5

$\deg(y_0)$. We can apply Corollary 4.5.4 (with $S = \deg(u)$) because $(\deg(u)) \cap (\deg(y)) \neq \emptyset$ unless $|\ (\deg(D)) \cap (\deg(y))\ | = 1$ and the vertex of D that u excludes is the one vertex in $(\deg(D)) \cap (\deg(y))$, in which case $\deg(u) = \deg(D) \backslash (\deg(y)) = \deg(z_0)$. So, for every $u \in \partial D$, there is only one element of $\deg(u)$ that also covers y_0 and z_0 if $\deg(u)$ warrants it. Thus $\partial w = \partial D$, and the claim is established.

Now, since $y > y_0$, it follows that y and z_0 also have no upper bound, so $yz_0 \in I$ and, by degree considerations, z_0 is not a multiple of y', contradicting Lemma 4.4.3. ∎

Finally we show that eliminations cannot share vertices with other eliminations. This is not much more than an elaboration on the proof of Theorem 4.1.1, the simplicial complex case.

Lemma 4.5.6 *Let D and E be eliminations; assume $S = \{\Psi_1, \ldots, \Psi_n\}$ is a regular sequence in $k[P]$ generating I. Then there is another regular sequence S' generating I, containing $D = \prod_{x \in \text{supp}(D)} x$ and $E = \prod_{x' \in \text{supp}(E)} x'$.*

Proof: By Proposition 4.5.5, there is no duplication involving any of the vertices in $\text{supp}(D)$ or $\text{supp}(E)$, so $D = \prod_{x \in \text{supp}(D)} x$ and $E = \prod_{x' \in \text{supp}(E)} x'$ are in I. Then by Lemma 4.3.1 it suffices to show that $(k[P])_+ I$ does not contain any k-linear combination of D and E. Again, assume otherwise:

$$dD + eE = \sum_{i \in I} a_i i,$$

where $d, e \in k$; each a_i is homogeneous of non-zero degree; and only finitely many a_i are non-zero, say $a_i \neq 0$ for $i \in I_0$. As before, we may as well assume that every element of I_0 is one of the canonical generators of I.

But now, looking at homogeneous components, we can restrict attention to the homogeneous component with the same degree as one of D or E, say D. So each relation in I_0 must only cover atoms in $\text{supp}(D)$, and thus every relation is of the form uv, where u and v are in D (since there are no duplications with any vertices in common with D), but have no upper bound. But then, since duplications are "minimal" (*i.e.*, D is a Boolean algebra with only its top element missing), $\deg(uv) = \deg(D)$, but then, if $i = uv$, it is impossible for a_i to have positive degree. ∎

Proposition 4.5.7 *If A_P is a complete intersection ring, and D and E are distinct eliminations, then* $\text{supp}(D) \cap \text{supp}(E) = \emptyset$.

Proof: By Lemma 4.5.6, D and E are part of a regular sequence, so $\prod_{x \in \text{supp}(D)} x$ is an NZD modulo $\prod_{x' \in \text{supp}(E)} x'$. But then, as in the proof simplicial complex case, Theorem 4.1.1, $\text{supp}(D) \cap \text{supp}(E) = \emptyset$. ∎

Proof of main theorem

Finally, we can prove Theorem 4.2.1

Proof: ($\Leftarrow$): This is easy since each of B_n, $B_n \backslash \hat{1}$, and $B_n \dot{\cup} \hat{1}'$ is a complete intersection ($A_P = k[x_1, \ldots, x_n]$, $k[x_1, \ldots, x_n]/(x_1 \cdots x_n)$, and $k[x_1, \ldots, x_n, \hat{1}']/(\hat{1}' - x_1 \cdots x_n)$, respectively), $A_{P \times Q} = A_P \otimes A_Q$ for any simplicial posets P and Q, and the direct product of two complete intersection rings is easily seen to be a complete intersection again.

($\Rightarrow$): Propositions 4.5.1, 4.5.5, and 4.5.7 show that no eliminations or duplications can share vertices with one another. So partition the vertices by which elimination or duplication each one is part of, and also group together those vertices that are not in any duplication or elimination. It is easy to see that the groupings into duplications, eliminations, and others corresponds to the breakdown of the components of P given in Theorem 4.2.1 (in reverse order). ■

Chapter 5

Free resolutions

In this chapter we solve two problems that do not necessarily seem related at first, but that both involve computing invariants of minimal free resolutions of A_P as a $k[V]$-module, where V is the set of vertices of P. In both cases, this is a generalization of work done by Hochster [Ho], who, using ideas from Reisner [Re], worked with simplicial complexes.

We start by introducing a new basis that is especially useful when thinking about A_P as a $k[V]$-module. Then, after reviewing the necessary background on free resolutions, we find the Betti polynomial of A_P as a $k[V]$-module. After that, we review the background material for local cohomology, and find the Hilbert function of the local cohomology modules of A_P. In both cases, the solution involves splitting a chain complex into subcomplexes even more finely in the simplicial poset case than was required in the simplicial complex case. The resulting subcomplexes are then shown to be isomorphic to cochain complexes of certain simplicial posets; the details of signs and dimensions are left to a technical lemma at the end of the chapter. We conclude with an example of each theorem.

Throughout, we assume that k is a field, fixed from the start; P is a simplicial poset with vertices $V = \{x_1, \ldots, x_n\}$; and $S = k[x_1, \ldots, x_n]$. Note that A_P is an S-module.

5.1 Background

The first problem we solve in this chapter is to find the Betti polynomial of a minimal free resolution of A_P over $k[V]$, generalizing the following theorem:

Theorem 5.1.1 (Hochster [Ho, §5]) *The Betti polynomial of $k[\Delta]$ as a $k[V]$-module is*

$$T_i(k[\Delta],\lambda) = \sum_{W \subseteq V} \Big(\prod_{x_i \in W} \lambda_i\Big) \tilde{\beta}_{|W|-i-1}(\Delta_W),$$

where $\Delta_W = \{F \in \Delta : F \subseteq W\}$.

The second problem is to calculate the Hilbert series of the local cohomology of A_P, generalizing the following theorem:

Theorem 5.1.2 (Hochster [St4, §II.4]) *The Hilbert series of the local cohomology module of $k[\Delta]$ is*

$$F(H^i(k[\Delta]),\lambda) = \sum_{F \in \Delta} \tilde{\beta}_{i-|F|-1}(\mathrm{lk} F; k) \prod_{x_j \in F} \frac{\lambda_j^{-1}}{1-\lambda_j^{-1}}.$$

The proofs of both of these results rely upon the following idea:

Definition: Let $\mathcal{K}$ be a chain [cochain] complex, with boundary [coboundary] operator d [δ]. $\mathcal{K}$ is said to **split** into subcomplexes $\{\mathcal{K}_\alpha : \alpha \in A\}$ (for some index set A) if:

(i) $\mathcal{K} \cong \oplus_{\alpha \in A} \mathcal{K}_\alpha$

(ii) d [δ] acts upon each $\mathcal{K}_\alpha$

In both cases, we split a certain chain complex by the usual fine degree. But the elements of this complex are not just elements of $k[\Delta]$; they are ordered pairs, which, for the purposes of using the fine degree to split the complex, are treated as the numerators and denominators, respectively, of a fraction. The subcomplexes are now identified by the (unique) fraction that has the specified degree of the subcomplex and is in "lowest terms," *i.e.*, the numerator and the denominator have no common support (for Betti polynomials) or minimal common support (for local cohomology).

5.2 A new basis for A_P

Both results in this chapter rely upon the following k-vector space basis of A_P.

Definition: Let P be a simplicial poset with vertices V. Then define

$$B_0(P) := \{y\mu : y \in P,\ \mu \in \mathcal{M}(V),\ \text{supp}(\mu) \subseteq \text{supp}(y)\}.$$

Proposition 5.2.1 *$B_0(P)$ is a k-vector space basis of A_P.*

Proof: Recall (from the Introduction) that the standard basis of A_P is the set of (monomials corresponding to) multichains $c = c_1 \leq \ldots \leq c_l$ in P (so the corresponding monomial is $c_1 \cdots c_l \in A_P$). Let $y = c_l$ and $\mu = \prod_{j=1}^{l-1}(\prod_{i:x_i \leq c_j} x_i)$. It is clear that $\text{supp}(\mu) \subseteq \text{supp}(y)$, and $c = y\mu$ follows from the observation

$$w < z \Rightarrow zw = z \prod_{i:x_i \leq w} x_i. \tag{5.1}$$

So $B_0(P)$ contains the standard basis.

Further, every $y\mu \in B_0(P)$ is equal to (the monomial corresponding to) some multichain c, as follows. Say $\mu = \prod_{i=1}^{n} x_i^{a_i}$; then we may rewrite μ as

$$\mu = \prod_{j=1}^{l-1}(\prod_{x_i \in \mu_j} x_i),$$

where $\mu_j := \{x_i : a_i \geq j\}$ and $l := \min\{j : \mu_j = \emptyset\}$. Let c_j be the unique element in $[\hat{0}, y]$ with $\text{supp}(c_j) = \text{supp}(\mu_j)$ Then it is easy to see, by induction and equation (5.1) again, that $y\mu = \prod_{j=1}^{l} c_j$ where $c_1 \leq \ldots \leq c_l$ is a multichain in P. ∎

5.3 Free resolutions

In this section, we collect the facts we need about free resolutions. Although proofs of most of the statements may be found in various algebra texts such as [La], we mostly follow the treatement of [Ho], which gives us just the results we need to start determining the Betti polynomial of A_P.

Theorem 5.3.1 (Hilbert syzygy) *Let M be a finitely-generated $\mathbf{Z}^m$-graded S-module, $S = k[x_1, \ldots, x_n]$. Then there is an exact sequence, called a* **finite free resolution** *(or* **FFR***),*

$$\mathcal{F} : 0 \to M_h \xrightarrow{\rho_h} M_{h-1} \to \cdots \to M_1 \xrightarrow{\rho_1} M_0 \xrightarrow{\rho_0} M \to 0$$

of $\mathbf{Z}^m$-graded S-modules M_i, with ρ_i preserving degree, and $h \leq n$. We also often write $\mathcal{F}_i := M_i$.

Proof: [La, §XVI.10.15]. ∎

Definition: A finite free resolution is **minimal** if equivalently

(i) for each $i \geq 0$, ρ_i maps the free generators of M_i onto a minimal homogeneous basis for $\mathrm{Im}(\rho_i)$; or

(ii) for each $i \geq 1$, $\mathrm{Im}(\rho_i) \subseteq S_+ M_{i-1}$.

We call such a resolution a **minimal free resolution** (or **MFR**).

Definition: Fix a polynomial ring S. Let $0 \to M_h \to \cdots M_0 \to M \to 0$ be a minimal free resolution of an S-module M over S. Then define the ith **Betti number** of M to be

$$\beta_i(M) := \mathrm{rank}_S(M_i),$$

so M_i has β_i S-module generators, $u_1, \ldots, u_{\beta_i}$. Then also define the ith **Betti polynomial** of M to be

$$T_i(M, \lambda) := \sum_{j=1}^{\beta_i} \lambda^{\deg(u_j)}.$$

Remark. These are invariants of M (*i.e.*, any MFR of M will yield the same Betti polynomial. This will follow from torsion (defined below) being well-defined, and the relation between the Betti polynomial and torsion, also given below.

Remark: This is the obvious generating function invariant of a free resolution. In general, it is very difficult to produce a free resolution of M from scratch, so we often settle for the Betti polynomial. In turn, the Betti polynomial can give a strong hint of what an MFR of M is, by providing the degrees of the generators; often, in a particular case, it is then easy to figure out the actual generators, and hence the resolution.

Definition: (see [La, §XVI.3], for instance) If $\mathcal{F} : 0 \to M_h \to \cdots \to M_1 \to M_0 \to M \to 0$ is a free resolution of M, and N is another S-module, then

$$N \otimes \mathcal{F} : 0 \to N \otimes_S M_h \to \cdots \to N \otimes_S M_1 \to N \otimes_S M_0 \to 0$$

is a complex (*i.e.*, $d^2 = 0$) and

$$\mathrm{Tor}_i^S(N, M) := H_i(N \otimes \mathcal{F}).$$

Proposition 5.3.2 $\mathrm{Tor}_i^S(N, M) \cong \mathrm{Tor}_i^S(M, N)$.

Proof: See [La, §XVI.3], for instance. ∎

Regard k as the S-module S/S_+. Then $\mathrm{Tor}_i^S(k, M) = H_i(k \otimes_S \mathcal{F}) =$

$$H_i(\cdots \xrightarrow{0} k \otimes_S M_i \xrightarrow{0} \cdots \xrightarrow{0} k \otimes_S M_0 \xrightarrow{0} 0)$$

(the maps are 0 by definition (ii) of MFR, since $k = S/S_+$)

$$\cong k \otimes_S M_i,$$

which is a k-vector space with k-basis corresponding bijectively to the S-basis of the free S-module M_i. Further, this bijection preserves degree ($k \otimes_S M_i$ is still $\mathbf{Z}^m$-graded), so $F(\mathrm{Tor}_i^S(k, M), \lambda) = T_i(M, \lambda)$. Thus, we need only calculate $\mathrm{Tor}_i^S(k, M)$.

We start by noting that $\mathrm{Tor}_i^S(k, M) \cong \mathrm{Tor}_i^S(M, k)$, so that we may perform this calculation by first finding a free resolution of k.

Definition: Let $S = k[x_1, \ldots, x_n]$, and let $S^n = Su_1 \oplus \cdots \oplus Su_n$ be the free S-module on n generators. The exterior algebra $\Lambda(S^n)$ becomes a free resolution of k, called the **Koszul complex**:

$$\mathcal{K}(x_1, \ldots, x_n; S) : 0 \to \Lambda^n(S^n) \xrightarrow{d_n} \cdots \to \Lambda^1(S^n) \xrightarrow{d_1} S \xrightarrow{d_0} k \to 0$$

with

$$d_r(u_{i_0} \wedge \cdots \wedge u_{i_j} \wedge \cdots \wedge u_{i_{r-1}}) = \sum_{j=0}^{r-1} (-1)^j x_{i_j} (u_{i_0} \wedge \cdots \wedge \hat{u_{i_j}} \wedge \cdots \wedge u_{i_{r-1}})$$

(where $\hat{}$ denotes omission) and $d_0 : x_i \mapsto 0$. (See [La, §XVI.10].)

A simpler and more useful way of expressing the Koszul complex is as follows: Let Σ be the simplicial complex 2^V, where $V = \{x_1, \ldots, x_n\}$. Then $\mathcal{K}_i = \Lambda^i(S^n)$ is the free S-module on the $(n-i)$-faces of Σ, where, if $V \backslash U = \{l_1 < \cdots < l_i\}$, then U corresponds to $u_{l_1} \wedge \cdots \wedge u_{l_i}$. Then

$$dU = \sum_{j:x_j \in V \setminus U} (-1)^{\alpha(j,U)} x_j (U \cup \{x_j\}),$$

where

$$\alpha(j, U) := \#\{t : t < j,\ x_t \in V \setminus U\}. \tag{5.2}$$

5.4 Betti polynomials of simplicial posets

Now we apply the definitions and techniques of the previous section to simplicial posets, and, following the general plan of [Ho], prove the following

Theorem 5.4.1 *If P is a simplicial poset on vertex set $V = \{x_1, \ldots, x_n\}$ and $S = k[V]$, then*

$$\mathrm{Tor}_i^S(k, A_P) \cong \bigoplus_{T \subseteq V} (\mathrm{Tor}_i^S(k, A_P))_{\epsilon(V \setminus T)},$$

and

$$(\mathrm{Tor}_i^S(k, A_P))_{\epsilon(V \setminus T)} \cong \tilde{H}^{n-|T|-i-1}(P \setminus T; k)$$

where $\epsilon(V \setminus T) := \deg(V \setminus T)$, and $(\mathrm{Tor}_i^S(k, A_P))_\epsilon$ is the ϵ-graded piece in the normal fine grading. Consequently,

$$\mathrm{Tor}_i^S(k, A_P) \cong \bigoplus_{T \subseteq V} \tilde{H}^{n-|T|-i-1}(P \setminus T; k).$$

Corollary 5.4.2 *In the MFR of A_P as a $k[V]$-module,*

$$T_i(A_P, \lambda) = \sum_{W \subseteq V} \Big(\prod_{x_i \in W} \lambda_i \Big) \tilde{\beta}_{|W|-i-1}(P_W),$$

where P_W is the simplicial subposet $\{y \in P : \mathrm{supp}(y) \subseteq W\}$.

Proof: By definition of T_i and Theorem 5.4.1,

$$\begin{aligned} T_i(A_P,\lambda) &= F(\mathrm{Tor}_i^S(k,A_P),\lambda) \\ &= \sum_{\epsilon\in\mathbf{Z}^n} \lambda^\epsilon \dim((\mathrm{Tor}_i^S(k,A_P))_\epsilon) \\ &= \sum_{T\subseteq V} \Big(\prod_{i:x_i\in V\setminus T} \lambda_i\Big)\tilde{\beta}^{n-|T|-i-1}(P\setminus T) \\ &= \sum_{W\subseteq V} \Big(\prod_{i:x_i\in W} \lambda_i\Big)\tilde{\beta}^{|W|-i-1}(P_W), \end{aligned}$$

which yields the result, since homology over a field k just gives vector spaces, and then $H_j(\cdot) \cong H^j(\cdot)$ by the usual vector space duality. ■

Lemma 5.4.3 *A_P is a finitely generated free S-module.*

Proof: Use the basis $B_0(P)$ from Section 5.2; as an S-module, then, A_P is just generated by the elements of P, since any multichain is just the product of one element in P, and any number of vertices, which are in S. ■

Thus, we can apply Hilbert's syzygy theorem and find a minimal free resolution of A_P over S. As noted above in the general case, we do not actually find the resolution itself, but settle for Corollary 5.4.2, using the Koszul complex to calculate $H_i(A_P \otimes_S \mathcal{K})$. The rest of this section is devoted to the proof of Theorem 5.4.1.

So now we are looking at

$$\begin{aligned} \mathrm{Tor}_i^S(A_P,k) &= H_i(A_P\otimes_S \mathcal{K}) \\ &= H_i(0 \to A_P\otimes_S \Lambda^n(S^n) \stackrel{id\otimes d}{\to} \cdots \to A_P\otimes_S \Lambda(S^n) \stackrel{id\otimes d}{\to} A_P\otimes_S S \to 0). \end{aligned}$$

Let $\mathcal{L} := A_P \otimes_S \mathcal{K}$; using $B_0(P)$, a basis for $\mathcal{L}_i$ is

$$B_i := \{(y\mu, U) : y\in P,\ \mu\in\mathcal{M}(V),\ \mathrm{supp}(\mu)\subseteq \mathrm{supp}(y),\ U\in \binom{V}{n-i}\}.$$

Also, let $B := \cup_i B_i$. In this case, $d : \mathcal{L}_i \to \mathcal{L}_{i-1}$ is given by

$$d(y\mu, U) = \sum_{x_j \notin U} (-1)^{\alpha(j,U)}(y\mu x_j, U \cup \{x_j\}),$$

which, in terms of our basis, is

$$= \sum_{\substack{x_j \notin U \\ x_j \leq y}} (-1)^{\alpha(j,U)}(y(\mu x_j), U \cup \{x_j\}) + \sum_{\substack{x_j \notin U \\ x_j \not\leq y}} \sum_{y < x_j z} (-1)^{\alpha(j,U)}(z\mu, U \cup \{x_j\}). \tag{5.3}$$

Note that, for a given j, the second summation is empty if x_j and y have no upper bound in P.

Remark: We also attach to B a grading,

$$\deg_0((y\mu, U)) := \deg(y\mu) - \deg(U).$$

Because of this, we think of $(y\mu, U)$ as a fraction of sorts, with $y\mu$ in the numerator and U in the denominator. This fine grading subtracts $\deg(U)$ instead of adding $\deg(V \backslash U)$ (recall the precise definition of $(y\mu, U)$ in terms of $A_P \otimes_S \Lambda^i(S^n)$), so it differs from the standard fine grading by a constant factor:

$$\deg_0((y\mu, U)) = \deg((y\mu, U)) - (1, \ldots, 1),$$

where $\deg((y\mu, U))$ is the normal fine degree. Since the error term is constant, using $\deg_0(\cdot)$ instead of the normal degree does not change the decomposition of $\mathcal{L}$ into homogeneous subcomplexes, only the naming of these subcomplexes.

Splitting into subcomplexes

We now split the complex $\mathcal{L}$ into subcomplexes. In fact, we want to split $\mathcal{L}$ as finely as we can. It is clear that d must preserve our $\deg_0(\cdot)$, and, in fact, this was all that d has to preserve in the simplicial complex case.

But here we notice that not only does d preserve this degree while increasing the degree of the "numerator", but it also only sends basis elements featuring $y \in P$ to basis elements featuring $z \in \mathrm{lk}y$ (in the simplicial complex case, of course, these two concepts are identical, for in a simplicial complex, the degree of an element determines the element precisely). So now we have a decomposition not just into fine grades of degree, but into "extra-fine grades" of pairs: degree, and $y \in P$. We want to capture that difference.

First, as with the simplicial complex case, we have to find those "irreducible" basis elements that do not arise from applying d to another basis element. These elements will index our subcomplexes. In the simplicial complex case, they could be specified entirely by their degree; here we must also specify y.

Of course, not every y can be matched with every possible degree. In fact, as with the simplicial complex case, the "irreducibles" can be characterized by having no common support in the numerator and denominator (for much the same reasons), so if ϵ_0 is the degree of the subcomplex and y_0 is in the numerator of the "irreducible" indexing the subcomplex, we will need $\mathrm{supp}(y_0) = \mathrm{supp}(\epsilon_{0+})$. The need for the rest of the conditions on y_0 and ϵ_0 should be clear.

We let E denote the set of all possible "extra-fine grades:"

$$E := \{(y_0, \epsilon_0) : y_0 \in P,\ \epsilon_0 \in \mathbf{Z}^n,\ \mathrm{supp}(\epsilon_{0+}) = \mathrm{supp}(y_0),\ \epsilon_0 \geq (-1, \ldots, -1)\}.$$

Next, we find the correspondence between E and basis elements of $\mathcal{L}$, with the following

Definition: We define a map $\mathrm{env} : B \to E$ (and then extend it to $\mathrm{env} : \mathcal{L} \to E$) as follows: for a given $(y\mu, U) \in B$, find the unique $(y_0, \epsilon_0) \in E$ such that

(i) $\epsilon_0 = \deg_0((y\mu, U))$; and

(ii) $y_0 \leq y$.

Proposition 5.4.4 *The map* env : $\mathcal{L} \to E$ *described above is well-defined.*

Proof: This ϵ_0 is clearly determined by part (i) of the definition, which, in turn, determines $\mathrm{supp}(y_0)$. But then y_0 is uniquely determined, since part (ii) of the definition specifies that $y_0 \in [\hat{0}, y]$ and $\mathrm{supp}(y_0) = \mathrm{supp}(\epsilon_{0+}) \subseteq \mathrm{supp}(\epsilon_0) \subseteq \mathrm{supp}(y\mu)\backslash\mathrm{supp}(U) \subseteq \mathrm{supp}(y)$. It is also clear that $(y_0, \epsilon_0) \in E$, since $\epsilon_0 \geq -\deg(U) \geq (-1, \ldots, -1)$. ∎

So now, if we define

$$\mathcal{L}_{(y\mu,U)} := \text{the } k\text{-vector space spanned by } (y\mu, U),$$

and

$$\mathcal{L}_{(y_0,\epsilon_0)} := \bigoplus_{\substack{(y\mu,U)\in B \\ \mathrm{env}(y\mu,U)=(y_0,\epsilon_0)}} \mathcal{L}_{(y\mu,U)},$$

each $\mathcal{L}_{(y_0,\epsilon_0)}$ is in the ϵ_0-graded piece of $\mathcal{L}$.

Lemma 5.4.5

$$\mathrm{Tor}_i^S(A_P, k) = \bigoplus_{(y_0,\epsilon_0)\in E} H_i(\mathcal{L}_{(y_0,\epsilon_0)}).$$

Further, then, the ϵ_0-graded piece of $\mathrm{Tor}_i^S(A_P, k)$ *is* $\bigoplus_{y_0:(y_0,\epsilon_0)\in E} H_i(\mathcal{L}_{(y_0,\epsilon_0)})$.

Proof: We already have $\mathrm{Tor}_i^S(A_P, k) = H_i(\mathcal{L})$, so we must show that $\mathcal{L}$ *splits* into $\bigoplus_{(y_0,\epsilon_0)\in E} \mathcal{L}_{(y_0,\epsilon_0)}$ *i.e.*, that d maps $\mathcal{L}_{(y_0,\epsilon_0)}$ into itself. First note that $\mathcal{L} = \oplus\mathcal{L}_{(y_0,\epsilon_0)}$ as S-modules, since $\mathcal{L} = \oplus\mathcal{L}_{(y\mu,U)}$ and env is well-defined.

To show that d maps $\mathcal{L}_{(y_0,\epsilon_0)}$ into itself; *i.e.*, $\mathrm{env}(d(y\mu, U)) = \mathrm{env}(y\mu, U)$, say that $\mathrm{env}(y\mu, U) = (y_0, \epsilon_0)$. Consider the definition of d, equation (5.3). Clearly, d preserves degree (part (i) of the definition of env); and the first component of every basis element composing $d(y\mu, U)$ is either $y\mu'$ or $z\mu$, where $z > y \geq y_0$ (part (ii) of the definition of env). ∎

Irreducible elements

Definition: For every $(y_0, \epsilon_0) \in E$, there is a unique $(y_0\mu_0, U_0) \in B$ such that $\mathrm{env}(y_0\mu_0, U_0) = (y_0, \epsilon_0)$ and $\mathrm{supp}(y_0) \cap U_0 = \emptyset$. It is defined by $U_0 := \mathrm{supp}(\epsilon_{0-})$ and $\deg(\mu_0) = \epsilon_0 - \deg(y_0) + \deg(U_0)$; this follows almost immediately from the definitions of B and E.

Remark: The basis element $(y_0\mu_0, U_0)$ is an "irreducible;" applying the boundary operator δ to any basis element will not produce $(y_0\mu_0, U_0)$ as a summand. Furthermore, this characterizes the set of all $(y_0\mu_0, U_0)$'s. The following lemma recasts $\mathcal{L}$ in terms of these irreducibles.

Lemma 5.4.6 *The complex $\mathcal{L}_{(y_0,\epsilon_0)}$ is isomorphic to the complex*

$$\mathcal{L}'_{(y_0,\epsilon_0)} : 0 \to M_{U_0} \xrightarrow{\delta} \bigoplus_{\substack{|U|=|U_0|+1 \\ U_0 \subseteq U}} M_U \xrightarrow{\delta} \cdots \xrightarrow{\delta} M_V \to 0,$$

where each M_U has basis

$$B'_U := \{(y, U) \ : \ y \geq y_0, \ \mathrm{supp}(y) \cap T = \emptyset, \ T \cup \mathrm{supp}(y) \backslash \mathrm{supp}(y_0) \subseteq U \subseteq T \cup \mathrm{supp}(y)\}$$

with $T := \mathrm{supp}(\epsilon_{0-})$. The isomorphism maps $(y\mu, U)$ to (y, U), and

$$\delta(y, U) = \sum_{\substack{x_j \notin U \\ x_j \leq y}} (-1)^{\alpha(j,U)}(y, U \cup \{x_j\}) + \sum_{\substack{x_j \notin U \\ x_j \not\leq y}} \sum_{y <_{x_j} z} (-1)^{\alpha(j,U)}(z, U \cup \{x_j\}),$$

(where α is defined in equation (5.2)).

Proof: This is equivalent to showing that, for a fixed $(y_0, \epsilon_0) \in E$, there is a bijection between $\{(y\mu, U) \in B : \mathrm{env}(y\mu, U) = (y_0, \epsilon_0)\}$ and pairs $(y, U) \in B'_U$, that takes $(y\mu, U)$

to (y, U).

Given $(y\mu, U) \in B$, $\text{env}(y\mu, U) = (y_0, \epsilon_0)$ it is immediate from the definition of env that $y \geq y_0$. Further, by part (i) of the definition of env, $\text{supp}(y) \cap T = \text{supp}(\epsilon_{0+}) \cap \text{supp}(\epsilon_{0-}) = \emptyset$.

To show that $T \cup \text{supp}(y)\backslash\text{supp}(y_0) \subseteq U$, first note that

$$T = \text{supp}(\epsilon_{0-}) = \text{supp}((\text{deg}_0(y\mu, U))_-) \subseteq U.$$

Then, comparing $(y\mu, U)$ to (y_0, μ_0, U_0), we see that $x_j \notin \text{supp}(y_0) = \text{supp}(y_0\mu_0)$ implies that the jth component of $\epsilon_0 = \text{deg}_0(y_0\mu_0, U_0)$ is at most 0, so that if $x_j \in \text{supp}(y)$, then $x_j \in U$ (otherwise, the jth component of $\epsilon_0 = \text{deg}_0(y\mu, U)$ would be positive).

Similarly, to show $U \subseteq T \cup \text{supp}(y)$, it is necessary and sufficient to show that $U\backslash T \subseteq \text{supp}(y)$. Note that $T = U_0$, and then $x_j \notin U_0$ implies that the jth component of $\epsilon_0 = \text{deg}_0(y_0\mu_0, U_0)$ is at least 0, so that if $x_j \in U$, then $x_j \in \text{supp}(y)$ (otherwise, the jth component of $\epsilon_0 = \text{deg}_0(y\mu, U)$ would be negative).

Conversely, if $(y, U) \in B'_U$, then let μ be the unique monomial such that $\text{deg}_0(\mu) = \epsilon_0 - \text{deg}(y) + \text{deg}(U)$. Then it is easy to see that $\text{env}(y\mu, U) = (y_0, \epsilon_0)$. It remains to show that μ is properly defined ($\mu \geq 0$) and that $(y\mu, U) \in B$ (*i.e.*, $\text{supp}(\mu) \subseteq \text{supp}(y)$).

Since $0 \leq \text{deg}(\mu_0)$, $\text{supp}(\mu_0) \subseteq \text{supp}(y_0) \subseteq \text{supp}(y)$, and $\text{deg}(\mu/\mu_0) = (\text{deg}(U) - \text{deg}(U_0)) - (\text{deg}(y) - \text{deg}(y_0))$, it suffices to show that $U\backslash U_0 \supseteq \text{supp}(y)\backslash\text{supp}(y_0)$ (so $\text{deg}(\mu) \geq 0$) and $U\backslash U_0 \subseteq \text{supp}(y)$ (so $\text{supp}(\mu) \subseteq \text{supp}(y)$). But this is just the last condition on B'_U, rewritten $\text{supp}(y)\backslash\text{supp}(y_0) \subseteq U\backslash T \subseteq \text{supp}(y)$. ∎

Proof of main theorem

Now we are ready to prove Theorem 5.4.1, by plugging Lemma 5.4.6 into Lemma 5.8.1, which will be stated and proved in Section 5.8.

Proof: Momentarily using ϵ_0-grading, we see from Lemmas 5.4.5 and 5.4.6 that

$$(\mathrm{Tor}_i^S(k, A_P))_{\epsilon_0} = \bigoplus_{y_0:(y_0,\epsilon_0)\in E} H_i(\mathcal{L}_{(y_0,\epsilon_0)}) \cong \bigoplus_{y_0:(y_0,\epsilon_0)\in E} H_i(\mathcal{L}'_{(y_0,\epsilon_0)}),$$

which, by Lemma 5.8.1 with $U_0 = T = \mathrm{supp}(\epsilon_{0-})$ and $W = \mathrm{supp}(y_0)$, is

$$\cong \bigoplus_{\substack{y_0:(y_0,\epsilon_0)\in E \\ W=\mathrm{supp}(y_0)=\emptyset}} \tilde{H}^{n-|T|-i-1}(\mathrm{lk}_{P\setminus T}y_0)$$

$$\cong \begin{cases} \tilde{H}^{n-|T|-i-1}(\mathrm{lk}_{P\setminus T}\hat{0}) & \text{if } (\hat{0},\epsilon_0)\in E \\ 0 & \text{if } (\hat{0},\epsilon_0)\notin E \end{cases}$$

Now note that from the definition of env, $\emptyset = \mathrm{supp}(y_0) = \mathrm{supp}(\epsilon_{0+})$, so $T = \mathrm{supp}(\epsilon_{0-}) = \mathrm{supp}(\epsilon_0)$, and also that $\epsilon_0 \geq (-1,\ldots,-1)$, so there is a bijection $\{\epsilon_0 : (\hat{0},\epsilon_0)\in E\} \leftrightarrow 2^V$ given by $\epsilon_0 \mapsto T = \mathrm{supp}(\epsilon_0)$. Also note that $\mathrm{lk}_{P\setminus T}\hat{0} = P\setminus T$. The result then follows, by switching to normal grading: $\epsilon = \epsilon_0 + (1,\ldots,1)$, so $0 \leq \epsilon \leq (1,\ldots,1)$, and $\mathrm{supp}(\epsilon) = V\setminus T$. ∎

5.5 Local cohomology

Definitions: (See [AM, Chapter 3] for a more detailed explanation). Let A be a commutative ring, and let $U \subseteq A$. Define the multiplicative set generated by U to be the multiplicative closure of U; *i.e.*, the smallest subset of A containing U that is closed under multiplication. Now let M be an A-module. If U is a multiplicatively closed set, let M_U (sometimes written $U^{-1}M$) denote the module of fractions: $M \times U$ modulo the relations $\{(m,u) \equiv (m',u') \text{ if } t(um' - u'm) = 0 \text{ for some } t \in U\}$, with the obvious addition and multiplication. We will suggestively write (m,u) as m/u. We call the map $\phi : M \to M_U$, $\phi : m \mapsto m/1$ **the canonical map.**

Definition: Fix a polynomial ring $S = k[V]$, $V = \{x_1,\ldots,x_n\}$. Then for M an S-

module, define the cochain complex

$$\mathcal{K}(\underline{x}^\infty, M): 0 \xrightarrow{\delta_0} M \xrightarrow{\delta_1} \coprod_{U\in\binom{V}{1}} M_U \xrightarrow{\delta_2} \coprod_{U\in\binom{V}{2}} M_U \to \cdots \to \coprod_{U\in\binom{V}{n-1}} M_U \xrightarrow{\delta_n} M_V \to 0$$

(where $M_U := M_{\overline{U}}$, and $\overline{U}$ is the multiplicative set generated by U). To define the coboundary maps $\delta_{s+1} : M_U \to \coprod_{U'\in\binom{V}{s+1}} M_{U'}$, where $V\backslash U = \{x_{l1} < \ldots < x_{l_{n-s}}\}$, first let $\phi_{l_j} : N \to N_{x_{l_j}}$ be the canonical map, $v \mapsto v/1$. Then

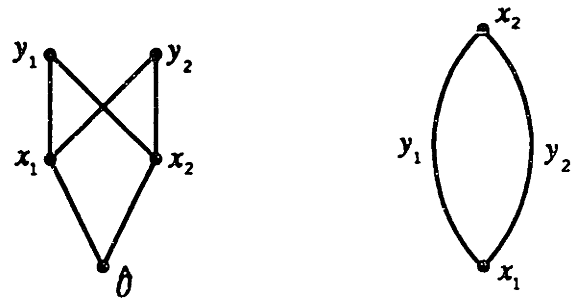
$$\delta_{s+1}(u) := \sum_{j=1}^{n-s}(-1)^{j-1}\phi_{l_j}(u).$$

It is easy to check that this coboundary map turns $\mathcal{K}(\underline{x}^\infty, M)$ into a cochain complex. The ith **local cohomology module** of M is defined by

$$H^i(M) = H^i(\mathcal{K}(\underline{x}^\infty, M)).$$

See [St4, §I.6] for some interseting properties of the local cohomology module.

The goal of the rest of the chapter is to find the Hilbert series of the local cohomology module of A_P, generalizing Theorem 5.1.2. As with the Betti polynomial in section 5.4, we will find a complex to split, and where the simplicial complex case splits just along fine degrees, here we split along pairs of elements in P and degrees. The only real difference, once the language of the two problems is stripped away, is that here, instead of not allowing the "numerator" and "denominator" to overlap, as with the Betti polynomials, we *require* it. This still allows just as much freedom in picking the "numerator" once the "denominator" has been fixed: namely, the complement of the "denominator."



P |P|

Figure 5-1: Why divide by x_U?

5.6 A basis for $(A_P)_U$

Before we do anything else with the local cohomology of $M = A_P$, we must decide on a basis for M_U; say $U = \{x_1, \ldots, x_r\}$, then define

$$B_U := \{\frac{y\mu}{x_U\nu} \; : \; y \in P, \; \mu, \nu \in \mathcal{M}(V),$$
$$\operatorname{supp}(\nu) \subseteq U \subseteq \operatorname{supp}(y), \; \operatorname{supp}(\mu) \subseteq \operatorname{supp}(y), \; \operatorname{supp}(\mu) \cap \operatorname{supp}(\nu) = \emptyset\},$$

where, for any $U \subseteq V$,

$$x_U := \prod_{x_i \in U} x_i.$$

We also define $B := \cup_{U \subseteq V} B_U$.

Part of the motivation of this definition is just to extend the basis $B_0(P)$ from $M = A_P$ to M_U, while expressing the fractions canonically; but x_U is required to be in the denominator to ensure that the proposed basis element cannot be written as the sum of other basis elements, as the following example shows.

Example: (See Figure 5-1.) Let $U = x_2$. Then $x_1 \in M_{x_2}$, but it is not a basis element. Instead, we multiply by x_U in the numerator and denominator, which then shows further decomposition:

$$x_1 = \frac{x_1 x_2}{x_2} = \frac{y_1}{x_2} + \frac{y_2}{x_2},$$

and y_1/x_2 and y_2/x_2 *are* basis elements.

Proposition 5.6.1 *B_U is a basis for M_U.*

Proof: First we check that B_U spans M_U. Since M is spanned by

$$\{y\mu : y \in P,\ \mu \in \mathcal{M}(k[V]),\ \mathrm{supp}(\mu) \subseteq \mathrm{supp}(y)\},$$

M_U is certainly spanned by

$$\begin{aligned}\{\frac{y\mu}{\nu}\ :\ & y \in P,\ \mu, \nu \in \mathcal{M}(V), \\ & \mathrm{supp}(\mu) \subseteq \mathrm{supp}(y),\ \mathrm{supp}(\nu) \subseteq U,\ \mathrm{supp}(\mu) \cap \mathrm{supp}(\nu) = \emptyset\}\end{aligned}$$

(the last condition is not necesary, but it is helpful).

We partition U as follows:

$$U = U_\emptyset \,\dot\cup\, U_y \,\dot\cup\, U_\nu \,\dot\cup\, U_{y\nu}$$

$$\begin{aligned} U_\emptyset &= (U \backslash \mathrm{supp}(y)) \backslash \mathrm{supp}(\nu); \\ U_y &= (U \cap \mathrm{supp}(y)) \backslash \mathrm{supp}(\nu); \\ U_\nu &= (U \backslash \mathrm{supp}(y)) \cap \mathrm{supp}(\nu); \\ U_{y\nu} &= (U \cap \mathrm{supp}(y)) \cap \mathrm{supp}(\nu). \end{aligned}$$

Also let $x_\emptyset = x_{U_\emptyset}$, $x_y = x_{U_y}$, $x_\nu = x_{U_\nu}$, and $x_{y\nu} = x_{U_{y\nu}}$. Note that $x_\nu x_{y\nu}$ divides ν, so we may define ν' such that $\nu = \nu' x_\nu x_{y\nu}$. Then observe that

$$\begin{aligned} \frac{y\mu}{\nu} &= \frac{y\mu x_\emptyset x_y x_\nu}{\nu x_\emptyset x_y x_\nu} = \frac{(y x_\nu x_\emptyset)(\mu x_y)}{(\nu' x_\nu x_{y\nu})(x_\emptyset x_y) x_\nu} \\ &= \frac{(y x_{U\backslash \mathrm{supp}(y)})(\mu x_y)}{x_U(\nu' x_\nu)}. \end{aligned}$$

Now,

$$yx_{U\setminus\operatorname{supp}(\nu)} = \sum_{i=1}^{s} y_i,$$

where $\operatorname{supp}(y_i) = \operatorname{supp}(y) \cup U$, so

$$\frac{y\mu}{\nu} = \sum_{i=1}^{s} \frac{(y_i)(\mu x_y)}{x_U(\nu' x_\nu)}, \tag{5.4}$$

and, for each i,

- $\nu' x_\nu$ divides ν, so $\operatorname{supp}(\nu' x_\nu) \subseteq \operatorname{supp}(\nu) \subseteq U$, and $\operatorname{supp}(\nu' x_\nu) \subseteq \operatorname{supp}(\nu) \subseteq \operatorname{supp}(y) \subseteq \operatorname{supp}(y_i)$;
- $\operatorname{supp}(\mu x_y) \subseteq \operatorname{supp}(\mu) \cup \operatorname{supp}(y) = \operatorname{supp}(y) \subseteq \operatorname{supp}(y_i)$;
- $U \subseteq \operatorname{supp}(y_i)$; and
-

$$\begin{aligned} \operatorname{supp}(\mu x_y) \cap \operatorname{supp}(\nu' x_\nu) &\subseteq (\operatorname{supp}(\mu) \cup \operatorname{supp}(x_y)) \cap \operatorname{supp}(\nu) \\ &= (\operatorname{supp}(\mu) \cap \operatorname{supp}(\nu)) \cup (\operatorname{supp}(x_y) \cap \operatorname{supp}(\nu)) = \emptyset. \end{aligned}$$

Thus, equation (5.4) expresses every element of a spanning set of M_U as a sum of elements in B_U, so B_U spans M_U.

To show that B_U is linearly independent, assume that

$$\sum_{i=1}^{s} k_i \frac{y_i \mu_i}{x_U \nu_i} = 0,$$

where $k_i \in k$, and $(y_i \mu_i)/(x_U \nu_i) \in B_U$. Let $\hat{\rho}_i := \prod_{j \neq i} \nu_j \in \mathcal{M}(U)$ and $\rho_i' := \hat{\rho}_i \mu_i$. Note that $\operatorname{supp}(\rho_i') \subseteq U \cup \operatorname{supp}(\mu_i) \subseteq \operatorname{supp}(y_i)$. Then

$$0 = \sum_{i=1}^{s} k_i \frac{y_i \mu_i}{x_U \nu_i} = \sum_{i=1}^{s} k_i \frac{y_i \mu_i \hat{\rho}_i}{x_U \nu_i \hat{\rho}_i} = \frac{1}{x_U \prod_{i=1}^{s} \nu_i} \sum_{i=1}^{s} k_i y_i \rho_i',$$

so, in M_U,

$$0 = \sum_{i=1}^{s} k_i y_i \rho_i'.$$

In other words,

$$0 = (\sum_{i=1}^{s} k_i y_i \rho_i')\xi$$

for some $\xi \in \mathcal{M}(U)$. Now let $\rho_i = \rho_i'\xi$, and note that $\text{supp}(\rho_i) \subseteq \text{supp}(\rho_i') \cup \text{supp}(\xi) \subseteq \text{supp}(y_i) \cup U \subseteq \text{supp}(y_i)$, so

$$0 = \sum_{i=1}^{s} k_i (y_i \rho_i)$$

where each $y_i\rho_i$ is a basis element of M, and therefore, each $k_i = 0$. ∎

So now, if $V \backslash U = \{x_{l_1} < \ldots < x_{l_s}\}$,

$$\begin{aligned} \delta_{|U|+1}(\frac{y\mu}{x_U \nu}) &= \sum_{j=1}^{n-s} (-1)^{j-1} \frac{y\mu x_{l_j}}{x_{U \cup \{x_{l_j}\}} \nu} \\ &= \sum_{x_i \notin U} (-1)^{\alpha(i,U)} \frac{y\mu x_i}{x_{U \cup \{x_i\}} \nu}, \end{aligned}$$

where α is defined in equation (5.2). In standard form, that is

$$\delta_{|U|+1}(\frac{y\mu}{x_U \nu}) = \sum_{\substack{x_i \notin U \\ x_i \leq y}} (-1)^{\alpha(i,U)} \frac{y\mu x_i}{x_{U \cup \{x_i\}} \nu} + \sum_{\substack{x_j \notin U \\ x_j \not\leq y}} \sum_{y <_{x_j} z} (-1)^{\alpha(i,U)} \frac{z\mu}{x_{U \cup \{x_j\}} \nu}. \qquad (5.5)$$

It is easy to check that each summand in equation (5.5) is a basis element in $B_{U \cup \{x_i\}}$ or $B_{U \cup \{x_j\}}$.

We conclude this section with an elementary, but useful, observation.

Lemma 5.6.2 *If $y\mu/x_U\nu \in B$ then $(\deg(y\mu/x_U\nu))_j < 0$ iff $x_j \in \text{supp}(\nu)$.*

Proof: If $x_j \in \text{supp}(\nu) \subseteq U$, then $(\deg(x_U\nu))_j > 2$. But $\text{supp}(\nu) \cap \text{supp}(\mu) = \emptyset$ and $(\deg(y))_j \leq 1$ for any i, so $(\deg(y\mu))_j \leq 1$. Conversely, if $(\deg(y\mu/x_U\nu))_j < 0$, then $x_j \in U \subseteq \text{supp}(y)$, so $x_j \in \text{supp}(\nu)$. ∎

5.7 More extra-fine grading

Once we have the correct basis, we can find the Hilbert series of the local cohomology modules, which is the goal of this section. The problem becomes quite similar to the Betti polynomial problem, only some of the technical details become a little messier because the numerator and denominator are forced to overlap and because the denominator corresponds to a multiset, not just a set.

In particular, the set E of all possible extra-fine grades becomes slightly less obvious. The "irreducibles," instead of being in "lowest terms" as in the Betti polynomials problem, satisfy $\operatorname{supp}(\nu) = U$ (as opposed to the more general $\operatorname{supp}(\nu) \subseteq U$ required of all basis elements) because that is equivalent to $y\mu/x_U\nu$ not being produced by δ, since this condition means that no x_i can be removed from U without violating $\operatorname{supp}(\nu) \subseteq U$. By Lemma 5.6.2, essentially, this is also equivalent to there being no cancellation of degree between the numerator and denominator, so $\operatorname{supp}(y_0) = \operatorname{supp}(\epsilon)$

More rigorously, we define

$$E := \{(y_0, \epsilon_0) : y_0 \in P,\ \epsilon_0 \in \mathbf{Z}^n,\ \operatorname{supp}(\epsilon_0) = \operatorname{supp}(y_0)\},$$

and then show that this corresponds to the $\operatorname{supp}(\nu) = U$ condition. We do this by finding the correspondence between E and basis elements of $\mathcal{K}(\underline{x}^\infty, M)$. We again make the following

Definition: We define a map $\operatorname{env} : B \to E$ (and then extend it to $\operatorname{env} : \mathcal{K}(\underline{x}^\infty, M) \to E$) as follows: for a given $y\mu/x_U\nu \in B$, find the unique $(y_0, \epsilon_0) \in E$ such that

(i) $\epsilon_0 = \deg(y\mu/x_U\nu)$, and

(ii) $y_0 \leq y$.

Note that this time, ϵ_0 *is* the normal fine degree.

Proposition 5.7.1 *The map* $\operatorname{env} : \mathcal{K}(\underline{x}^\infty, M) \to E$ *described above is well-defined.*

Proof: This ϵ_0 is clearly determined by part (i) of the definition, which, in turn, determines $\text{supp}(y_0)$. But then y_0 is uniquely determined, since part (ii) of the definition specifies that $y_0 \in [\hat{0}, y]$ and $\text{supp}(y_0) = \text{supp}(\epsilon_0) \subseteq \text{supp}(y\mu)\backslash\text{supp}(x_U\nu) \subseteq \text{supp}(y)$. ■

Let

$$\mathcal{K}(\underline{x}^\infty, M)_{(y_0,\epsilon_0)} := \bigoplus_{\substack{y\mu/x_U\nu \in B \\ \text{env}(y\mu/x_U\nu)=(y_0,\epsilon_0)}} \mathcal{K}(\underline{x}^\infty, M)_{y\mu/x_U\nu}.$$

Note that each $\mathcal{K}(\underline{x}^\infty, M)_{(y_0,\epsilon_0)}$ is in the ϵ_0-graded piece of $\mathcal{K}(\underline{x}^\infty, M)$.

Lemma 5.7.2

$$H^i(\mathcal{K}(\underline{x}^\infty, M)) = \bigoplus_{(y_0,\epsilon_0)\in E} H^i(\mathcal{K}(\underline{x}^\infty, M)_{(y_0,\epsilon_0)}).$$

Proof: We must show that $\mathcal{K}(\underline{x}^\infty, M)$ splits into $\bigoplus_{(y_0,\epsilon_0)\in E} \mathcal{K}(\underline{x}^\infty, M)_{(y_0,\epsilon_0)}$ *i.e.*, that δ maps $\mathcal{K}(\underline{x}^\infty, M)_{(y_0,\epsilon_0)}$ into itself. First note that $\mathcal{K}(\underline{x}^\infty, M) = \oplus\mathcal{K}(\underline{x}^\infty, M)_{(y_0,\epsilon_0)}$ as S-modules, since $\mathcal{K}(\underline{x}^\infty, M) = \oplus\mathcal{K}(\underline{x}^\infty, M)_{y\mu/x_U\nu}$ and env is well-defined.

To show that δ maps $\mathcal{K}(\underline{x}^\infty, M)_{(y_0,\epsilon_0)}$ into itself; *i.e.*, $\text{env}(\delta(y\mu/x_U\nu)) = \text{env}(y\mu/x_U\nu)$, say that $\text{env}(y\mu/x_U\nu) = (y_0, \epsilon_0)$. Consider the definition of δ, equation (5.5). Clearly, δ preserves degree (part (i) of the definition of env); and the first component of every basis element composing $\delta(y\mu/x_U\nu)$ is either $y\mu'$ or $z\mu$, where $z > y \geq y_0$ (part (ii) of the definition of env). ■

Irreducible elements

Once again, we look for the "irreducible elements" that cannot be produced by δ. It takes more effort in this case, though.

Lemma 5.7.3 *For every* $(y_0, \epsilon_0) \in E$*, there is a unique* $y_0\mu_0/x_{U_0}\nu \in B$ *such that* $\text{env}(y_0\mu_0/x_{U_0}\nu) = (y_0, \epsilon_0)$*, and* $\text{supp}(\nu) = U_0$*. Further, if* $\text{env}(y\mu/x_U\nu') = (y_0, \epsilon_0)$ *for some other* $y\mu/x_U\nu' \in B$*, then* $\nu' = \nu$*.*

Proof: Let

$$U_0 \;:=\; \text{supp}(\epsilon_{0-})$$

$$\begin{aligned}\deg(\mu_0) &:= \epsilon_{0+} - \deg(\mathrm{supp}(\epsilon_{0+})) \\ \deg(\nu_0) &:= \epsilon_{0-}\end{aligned}$$

(A monomial is defined by its degree as long as that degree is positive, which clearly holds for the definitions of both μ_0 and ν_0 above.) Then

$$\begin{aligned}\deg(y_0\mu_0/x_{U_0}\nu) &= \deg(y_0) + \deg(\mu_0) - \deg(U_0) - \deg(\nu_0) \\ &= \deg(\mathrm{supp}(\epsilon_0)) + \epsilon_{0+} - \deg(\mathrm{supp}(\epsilon_{0+})) - \deg(\mathrm{supp}(\epsilon_{0-})) - \epsilon_{0-} \\ &= \epsilon_0,\end{aligned}$$

so $\mathrm{env}(y_0\mu_0/x_{U_0}\nu) = (y_0, \epsilon_0)$.

Before showing uniqueness, we prove the second part of the lemma. So assume $\mathrm{env}(y\mu/x_U\nu') = (y_0, \epsilon_0)$ also; then $\deg(y\mu/x_U\nu') = \deg(y_0\mu_0/x_{U_0}\nu)$, so $\mathrm{supp}(\nu') = \mathrm{supp}(\nu)$, by Lemma 5.6.2. Also,

$$\deg(\nu') - \deg(\nu) = \deg(\mathrm{supp}(y)\backslash\mathrm{supp}(y_0)) + (\deg(\mu) - \deg(\mu_0)) - \deg(U\backslash U_0) \quad (5.6)$$

($U_0 \subseteq U$ since $U_0 = \mathrm{supp}(\nu_0) = \mathrm{supp}(\nu) \subseteq U$).

Assume that the jth component of $\deg(\nu') - \deg(\nu)$ is positive. Then $x_j \in \mathrm{supp}(\nu') = \mathrm{supp}(\nu) \subseteq \mathrm{supp}(y_0)$, so the jth component of $\deg(\mathrm{supp}(y)\backslash\mathrm{supp}(y_0)) = 0$, and then by equation (5.6), $x_j \in \mathrm{supp}(\mu)$, contradicting $\mathrm{supp}(\mu) \cap \mathrm{supp}(\nu') = \emptyset$. So the jth component of $\deg(\nu') - \deg(\nu)$ is not positive. Now assume that the jth component *of $\deg(\nu') - \deg(\nu)$ is negative. Then $x_j \in \mathrm{supp}(\nu) \subseteq U_0$, so the jth component of* $\deg(U\backslash U_0) = 0$, and then by equation (5.6), $x_j \in \mathrm{supp}(\mu_0)$, contradicting $\mathrm{supp}(\mu_0) \cap \mathrm{supp}(\nu) = \emptyset$. Thus $\deg(\nu') - \deg(\nu) = 0$, and $\nu' = \nu$.

Finally, assume that $\mathrm{env}(y_0\mu/x_U\nu') = (y_0, \epsilon_0)$, and $U = \mathrm{supp}(\nu')$. Then, by the above paragraph, $\nu' = \nu$ and $U = \mathrm{supp}(\nu') = \mathrm{supp}(\nu) = U_0$. Furthermore,

$$\deg(\mu) = \epsilon_0 - \deg(y_0) + \deg(U) + \deg(\nu')$$

$$= \epsilon_0 - \deg(y_0) + \deg(U_0) + \deg(\nu)$$
$$= \deg(\mu_0)$$

so $\mu = \mu_0$. ∎

We make the following simple observation.

Lemma 5.7.4 *With y_0, ϵ_0, U_0, and ν as above, $U_0 = \mathrm{supp}(y_0)$ iff $\epsilon_0 \leq 0$.*

Proof: If $U_0 = \mathrm{supp}(y_0)$, then $\mathrm{supp}(\nu) = U_0 = \mathrm{supp}(y_0) = \mathrm{supp}(\epsilon_0)$, so by Lemma 5.6.2, $\epsilon_0 \leq 0$. Conversely, if $\epsilon_0 \leq 0$, then by Lemma 5.6.2, $U_0 = \mathrm{supp}(\nu) = \mathrm{supp}(\epsilon_0) = \mathrm{supp}(y_0)$. ∎

Irreducible elements in hand, we can give a new description of the components of $\mathcal{L}$ again.

Lemma 5.7.5 *The complex $\mathcal{K}(\underline{x}^\infty, M)_{(y_0,\epsilon_0)}$ is isomorphic to the complex*

$$\mathcal{L}'_{(y_0,\epsilon_0)} : 0 \to M_{U_0} \xrightarrow{\delta} \bigoplus_{\substack{|U|=|U_0|+1 \\ U_0 \subseteq U}} M_U \xrightarrow{\delta} \cdots \xrightarrow{\delta} M_V \to 0$$

where each M_U has basis

$$B'_U := \{(y, U) : y \geq y_0,\ U_0 \cup \mathrm{supp}(y) \backslash \mathrm{supp}(y_0) \subseteq U \subseteq U_0 \cup \mathrm{supp}(y)\}$$

with $U_0 := \mathrm{supp}(\epsilon_{0-})$. The isomorphism maps $y\mu/x_U\nu$ to (y, U). And

$$\delta(y, U) = \sum_{\substack{x_j \notin U \\ x_j \leq y}} (-1)^{\alpha(j,U)}(y, U \cup \{x_j\}) + \sum_{\substack{x_j \notin U \\ x_j \not\leq y}} \sum_{y <_{x_j} z} (-1)^{\alpha(j,U)}(z, U \cup \{x_j\})$$

(where α is defined in equation (5.2)).

Proof: This is equivalent to showing that for a fixed $(y_0, \epsilon_0) \in E$, there is a bijection between $\{y\mu/x_U\nu \in B : \mathrm{env}(y\mu/x_U\nu) = (y_0, \epsilon_0)\}$ and pairs $(y, U) \in B'_U$, that takes $y\mu/x_U\nu$ to (y, U).

Given $y\mu/x_U\nu \in B$, $\text{env}(y\mu/x_U\nu) = (y_0, \epsilon_0)$ it is immediate from the definition of env that $y \geq y_0$.

To show that $U_0 \cup \text{supp}(y)\backslash\text{supp}(y_0) \subseteq U$, first note that

$$U_0 = \text{supp}(\epsilon_{0-}) = \text{supp}((\deg(y\mu/x_U\nu))_-) \subseteq U.$$

Then, comparing $y\mu/x_U\nu$ to $y_0\mu_0/x_{U_0}\nu$, we see that $x_j \notin \text{supp}(y_0) = \text{supp}(y_0\mu_0)$ implies that the jth component of $\epsilon_0 = \deg(y_0\mu_0/x_{U_0}\nu)$ is at most 0, so that if $x_j \in \text{supp}(y)$, then $x_j \in U$ (otherwise, the jth component of $\epsilon_0 = \deg(y\mu/x_U\nu)$ would be positive, since $\text{supp}(\nu) \subseteq U$).

Similarly, to show $U \subseteq U_0 \cup \text{supp}(y)$, it is necessary and sufficient to show that $U\backslash U_0 \subseteq \text{supp}(y)$. Note that $x_j \notin U_0$ (and $\text{supp}(\nu) \subseteq U_0$) implies that the jth component of $\epsilon_0 = \deg(y_0\mu_0/x_{U_0}\nu)$ is at least 0, so that if $x_j \in U$, then $x_j \in \text{supp}(y)$ (otherwise, the jth component of $\epsilon_0 = \deg(y\mu/x_U\nu)$ would be negative).

Conversely, if $(y, U) \in B'_U$, then let μ be the unique monomial such that $\deg(\mu) = (\epsilon_0 - \deg(y) + \deg(U))_+$, and let ν be the unique monomial such that $\deg(\nu) = (\epsilon_0 - \deg(y) + \deg(U))_-$. Then it is easy to see that $\text{env}(y\mu/x_U\nu) = (y_0, \epsilon_0)$. It remains to show that μ and ν are properly defined ($\mu \geq 0$, $\nu \geq 0$) and that $y\mu/x_U\nu \in B$. It is immediate from their definitions that $\text{supp}(\mu) \cap \text{supp}(\nu) = \emptyset$ and that $U \subseteq \text{supp}(y)$ (since $U_0 = \text{supp}(\epsilon_{0-}) \subseteq \text{supp}(\epsilon_0) = \text{supp}(y_0) \subseteq \text{supp}(y)$). It follows from $y_0\mu_0/x_{U_0}\nu$ being a basis element that $\nu \geq 0$ and $\text{supp}(\nu) \subseteq U_0 \subseteq U$.

Now, $0 \leq \deg(\mu_0)$, $\text{supp}(\mu_0) \subseteq \text{supp}(y_0) \subseteq \text{supp}(y)$, and

$$\begin{aligned}
\deg(\mu/\mu_0) &= (\deg(\mu) - \deg(\nu)) - (\deg(\mu_0) - \deg(\nu)) \\
&= (\epsilon_0 - \deg(y) + \deg(U)) - (\epsilon_0 - \deg(y_0) + \deg(U_0) \\
&= (\deg(U) - \deg(U_0)) - (\deg(y) - \deg(y_0)),
\end{aligned}$$

so it suffices to show that $U\backslash U_0 \supseteq \text{supp}(y)\backslash\text{supp}(y_0)$ (so $\deg(\mu) \geq 0$) and $U\backslash U_0 \subseteq \text{supp}(y)$ (so $\text{supp}(\mu) \subseteq \text{supp}(y)$. But this is just the last condition on B'_U, rewritten

$\mathrm{supp}(y)\backslash\mathrm{supp}(y_0) \subseteq U\backslash U_0 \subseteq \mathrm{supp}(y)$. ∎

Main theorem

Theorem 5.7.6 *If P is a simplicial poset on vertex set $V = \{x_1, \ldots, x_n\}$ and $S = k[V]$, then the Hilbert series of the local cohomology modules of A_P as an S-module is*

$$F(H^i(A_P), \lambda) = \sum_{y_0 \in P} \tilde{\beta}_{i-r(y_0)-1}(\mathrm{lk}\, y_0) \prod_{j : x_j \leq y} \frac{\lambda_j^{-1}}{1 - \lambda_j^{-1}}.$$

Proof: Abstractly, we can consider the cochain complex $\mathcal{L}'_{(y_0, \epsilon_0)}$ to be a chain complex with the same maps, but with the modules numbered in reverse. So then by Lemmas 5.7.2 and 5.7.5,

$$\begin{aligned} F(H^i(A_P), \lambda) &:= \sum_{\epsilon_0 \in \mathbf{Z}^n} \dim((H^i(\mathcal{K}(\underline{x}^\infty, M)))_{\epsilon_0})\lambda^{\epsilon_0} \\ &= \sum_{\epsilon_0 \in \mathbf{Z}^n} \dim((H_{n-i}(\mathcal{K}(\underline{x}^\infty, M)))_{\epsilon_0})\lambda^{\epsilon_0}, \end{aligned}$$

which is, using Lemma 5.8.1 with $T = \emptyset$ (and therefore $W = \mathrm{supp}(y_0)\backslash U_0$),

$$= \sum_{\epsilon_0 \in \mathbf{Z}^n} \lambda^{\epsilon_0} \dim\Big(\bigoplus_{\substack{y_0 : (y_0, \epsilon_0) \in E \\ \mathrm{supp}(y_0) = U_0}} \tilde{H}^{n-|U_0|-(n-i)-1}(\mathrm{lk}\, y_0)\Big),$$

which is, by Lemma 5.7.4,

$$\begin{aligned} &= \sum_{\epsilon_0 \leq 0 \in \mathbf{Z}^n} \lambda^{\epsilon_0} \sum_{y_0 : (y_0, \epsilon_0) \in E} \tilde{\beta}^{n-r(y_0)-(n-i)-1}(\mathrm{lk}\, y_0) \\ &= \sum_{\epsilon_0 \leq 0} \lambda^{\epsilon_0} \sum_{y_0 : \mathrm{supp}(y_0) = \mathrm{supp}(\epsilon_0)} \tilde{\beta}^{n-r(y_0)-(n-i)-1}(\mathrm{lk}\, y_0) \\ &= \sum_{y_0 \in P} \tilde{\beta}^{i-r(y_0)-1}(\mathrm{lk}\, y_0) \sum_{\substack{\epsilon_0 \leq 0 \\ \mathrm{supp}(\epsilon_0) = \mathrm{supp}(y_0)}} \lambda^{\epsilon_0}, \end{aligned}$$

which then proves the result, by a standard elementary generating functions technique and vector space duality again. ∎

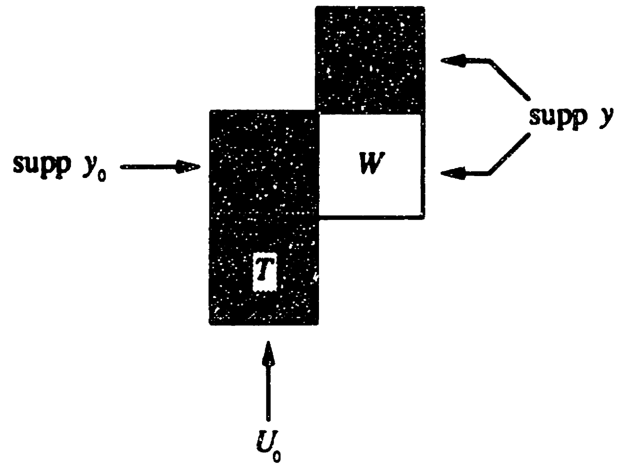


Figure 5-2: Lemma 5.8.1

5.8 Technical lemma

This is the technical lemma that lies at the base of both the results from this chapter. It is almost intuitively clear except for the signs that have to be checked, and also the dimension that everything lies in. It is similar to, but more complicated than (especially in its generality), the equivalent steps in Hochster's proofs of the simplicial complex case.

Once the poset P is fixed, there are three variables to this lemma: T, y_0, and W. When used in Theorem 5.4.1, $W = \text{supp}(y_0)$, and when used in Theorem 5.7.6, $T = \emptyset$.

Lemma 5.8.1 *Given a simplicial poset P on a vertex set $V = \{x_1, \ldots, x_n\}$, $S = k[V]$, $M = A_P$, which is an S-module. Also given a set of vertices $T \subseteq V$, $y_0 \in P \backslash T$ (*i.e., *$y_0 \not\geq x$ for every $x \in T$), and a set of vertices $W \subseteq \text{supp}(y_0)$. Let $U_0 = ((\text{supp}(y_0)) \backslash W) \cup T$. (See the Venn diagram in Figure 5-2.)*

Finally, also assume a complex

$$\mathcal{L} : 0 \to M_{U_0} \xrightarrow{\delta} \bigoplus_{\substack{U_0 \subset U \\ |U| = |U_0|+1}} M_U \xrightarrow{\delta} \bigoplus_{\substack{U_0 \subset U \\ |U| = |U_0|+2}} M_U \xrightarrow{\delta} \cdots \xrightarrow{\delta} M_V \to 0$$

of S-modules defined by:

- *each M_U is a free S-module with basis*

$$B_U = \{b_{(y,U)} : y \in \mathrm{lk}_{P\backslash T} y_0,\ U_0 \cup (\mathrm{supp}(y)\backslash \mathrm{supp}(y_0)) \subseteq U \subseteq U_0 \cup \mathrm{supp}(y)\},$$

  where supp(·) *is in the poset P, not $P\backslash T$, so that U* must *contain the shaded region in Figure 5-2, and is limited to that region and W (note that the second condition on basis elements in B_U restricts y); and*

- *the boundary operator is defined by*

$$\delta(b_{(y,U)}) = \sum_{\substack{x_i \notin U \\ x_i \leq y}} (-1)^{\alpha(i,U)} b_{(y,U\cup\{x_i\})} + \sum_{\substack{x_j \notin U \\ x_j \not\leq y}} \sum_{y <_{x_j} z} (-1)^{\alpha(j,U)} b_{(z,U\cup\{x_j\})},$$

  where $\alpha(i,U) := \#\{t : t < j,\ x_t \in V\backslash U\}$.

Then

$$H_i(\mathcal{L}) = \begin{cases} 0 & \text{if } W \neq \emptyset \\ \tilde{H}^{n-|U_0|-i-1}(\mathrm{lk}_{P\backslash T} y_0) & \text{if } W = \emptyset \end{cases}.$$

Proof: Let P' be the simplicial poset

$$P' := (\mathrm{lk}_{P\backslash T} y_0) \times 2^W$$

and number the vertices as follows:

- number (y_0, x_i) by i; and
- number $(y, \hat{0})$ by j if $y_0 <_{x_j} y$.

Since $W \subseteq \mathrm{supp}(y_0)$, every vertex of P' is numbered distinctly.

Now, it is easy to see that the direct product of simplicial posets corresponds, topologically, to the join, *i.e.*, $|P \times Q| = |P| * |Q|$ for all simplicial posets P and Q. Therefore $|P'|$ is a cone with any vertex of W serving as the apex, unless $W = \emptyset$. It thus remains to show that $H_i(\mathcal{L}) = \tilde{H}^{n-|U_0|-i-1}(P')$. We do this by showing that the diagram

$$\begin{array}{ccc} \tilde{C}^{n-|U_0|-i-1}(P') & \xrightarrow{\delta} & \tilde{C}^{n-|U_0|-i-2}(P') \\ \phi \uparrow & & \uparrow \phi \\ M_i & \xrightarrow{\delta} & M_{i+1} \end{array}$$

Figure 5-3: Relating chains of $\mathcal{L}$ to cochains of P'

in Figure 5-3 commutes, where

$$\phi : M_i \to \tilde{C}^{n-|U_0|-i-1}(P')$$

$$\phi : b_{(y,U)} \mapsto (-1)^{\gamma(y,W\cap U)}(y, W \cap U)^*$$

and

$$\gamma(y, U')) := \sum_{\substack{x_j \in P' \\ x_j \leq (y,U')}} \alpha(j, U_0).$$

So first we calculate

$$\begin{aligned} \delta(\phi(b_{(y,U)})) &= (-1)^{\gamma(y,W\cap U)}\delta(y, W \cap U)^* \\ &= (-1)^{\gamma(y,W\cap U)}[\ \textstyle\sum_{x_i \in W\setminus U}(-1)^{\beta(y,\{x_i\}\cup(W\cap U);i)}(y, \{x_i\} \cup (W \cap U))^* + \\ &\qquad \textstyle\sum_{\substack{z \in P\setminus T \\ y <_{x_j} z}}(-1)^{\beta(z,W\cap U;j)}(z, W \cap U)^*] \\ &= \sum_{\substack{x_i \leq y \\ x_i \notin U}} (-1)^{\gamma(y,W\cap U)+\beta(y,\{x_i\}\cup(W\cap U);i)}(y, \{x_i\} \cup (W \cap U))^* + \\ &\quad \sum_{\substack{x_j \not\leq y \\ x_j \notin U}} \sum_{y <_{x_j} z} (-1)^{\gamma(y,W\cap U)+\beta(z,W\cap U;j)}(z, W \cap U)^* \end{aligned}$$

(where, because of how we numbered the vertices in P',

$$\begin{aligned} \beta(y', U'; i') &= \#\{t : t < i',\ x_t \in \text{atom}(P'),\ x_t \leq (y', U')\} \\ &= \#\{t : t < i',\ x_t \in (\text{supp}(y)\setminus\text{supp}(y_0)) \cup U'\}; \end{aligned}$$

and the change in conditions on the summations is an easy consequence of the definitions of U, U_0, W, and T).

But also,

$$\begin{aligned}
\phi(\delta(b_{(y,U)})) &= \phi(\sum_{\substack{x_i \notin U \\ x_i \leq y}} (-1)^{\alpha(i,U)} b_{(y,U\cup\{x_i\})} + \sum_{\substack{x_j \notin U \\ x_j \not\leq y}} \sum_{y <_{x_j} z} (-1)^{\alpha(j,U)} b_{(z,U\cup\{x_j\})}) \\
&= \sum_{\substack{x_i \notin U \\ x_i \leq y}} (-1)^{\gamma(y,W\cap(U\cup\{x_i\}))-\alpha(i,U)} (y, W\cap(U\cup\{x_i\}))^* + \\
&\quad \sum_{\substack{x_j \notin U \\ x_j \not\leq y}} \sum_{y <_{x_j} z} (-1)^{\gamma(y,W\cap(U\cup\{x_j\}))-\alpha(j,U)} (z, W\cap(U\cup\{x_j\}))^* \\
&= \sum_{\substack{x_i \notin U \\ x_i \leq y}} (-1)^{\gamma(y,\{x_i\}\cup(W\cap U))-\alpha(i,U)} (y, \{x_i\}\cup(W\cap U))^* + \\
&\quad \sum_{\substack{x_j \notin U \\ x_j \not\leq y}} \sum_{y <_{x_j} z} (-1)^{\gamma(y,W\cap U)-\alpha(j,U)} (z, W\cap U)^*
\end{aligned}$$

(since $x_i \in \mathrm{supp}(y)\backslash U \subseteq W$, but $x_j \notin \mathrm{supp}(y) \supseteq W$).

So, to establish the lemma, it remains to show that

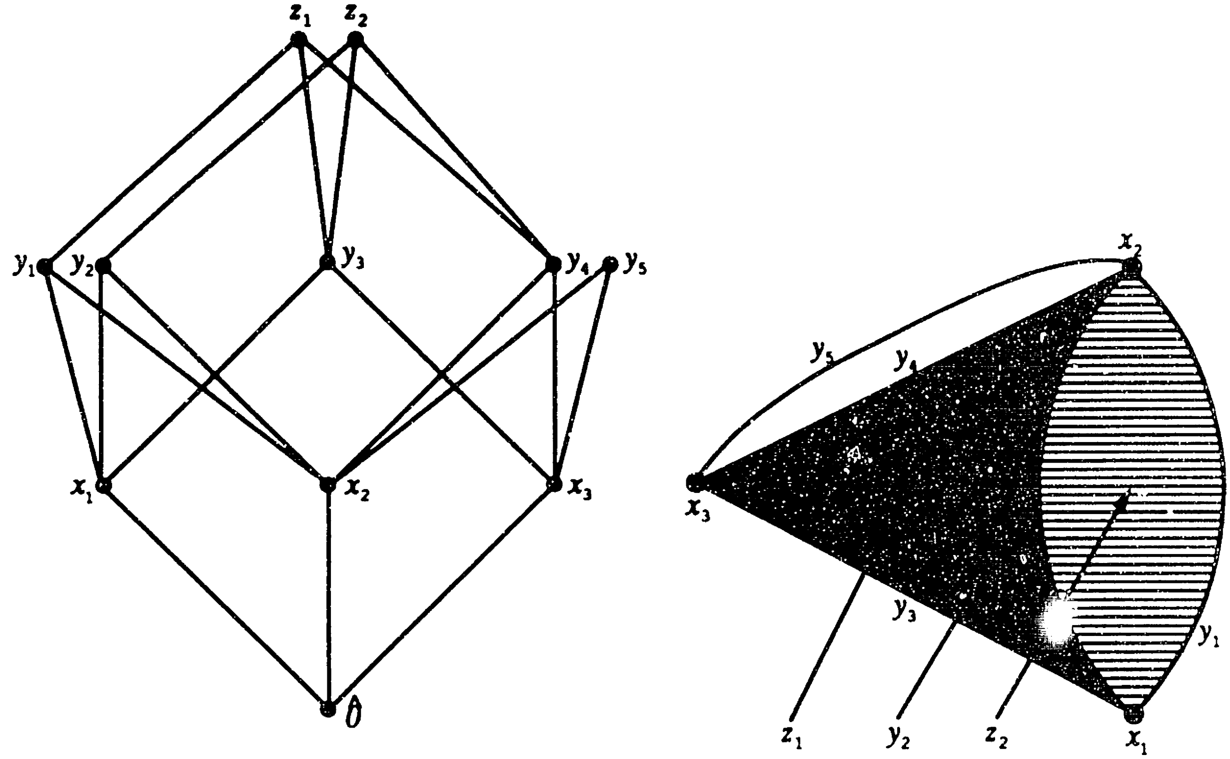
$$\gamma(y, W\cap U) + \beta(y, \{x_i\}\cup(W\cap U); i) = \gamma(y, \{x_i\}\cup(W\cap U)) - \alpha(i,U) \quad (5.7)$$

and

$$\gamma(y, W\cap U) + \beta(z, W\cap U; j) = \gamma(y, W\cap U) - \alpha(j,U). \quad (5.8)$$

We prove equation (5.7); equation (5.8) is similar. By definition of γ, equation (5.7) is equivalent to

$$\begin{aligned}
\beta(y, \{x_i\}\cup(W\cap U); i) &= \alpha(i,U_0) - \alpha(i,U) \\
&= \#\{t : t < i,\ x_t \in (V\backslash U_0)\backslash(V\backslash U)\} \\
&= \#\{t : t < i,\ x_t \in (U\backslash U_0)\}. \quad (5.9)
\end{aligned}$$



P $|P|$

Figure 5-4: An example

But

$$
\begin{aligned}
&\beta(y, \{x_i\} \cup (W \cap U); i) = \\
&\qquad \#\{t : t < i,\ x_t \in (\mathrm{supp}(y) \backslash \mathrm{supp}(y_0)) \cup (\{x_i\} \cup (W \cap U))\}, \qquad (5.10)
\end{aligned}
$$

and equation (5.7) follows, by comparing the right-hand sides of equations (5.9) and (5.10) (see Figure 5-2). ∎

5.9 Examples

Consider the simplicial poset P from Figure 5-4. Its realization is two 2-simplices glued along two of their edges, and an extra edge attached to two of the vertices.

Betti polynomials

Consider first, its Betti polynomials, following the proof of Theorem 5.4.1. Two of its irreducible basis elements are $(y_3x_1, \emptyset)$ and $(x_1, \{x_3\})$ (there are, of course, infinitely many such elements, but we only consider these two). The subcomplexes that they each anchor, of extra-fine grades $(y_3, (2,0,1))$ and $(x_1, (1,0,-1))$ respectively, are shown in Figure 5-5, with edges marking the boundary map. The resulting poset (sitting sideways) is, as described in the proof of Lemma 5.8.1, $\mathrm{lk}_{P\setminus T}y_0 \times 2^{\mathrm{supp}(y_0)}$, where y_0 is y_3 and x_1, respectively, and T is $\emptyset$ and x_3, respectively. The copies of $\mathrm{lk}_{P\setminus T}y_0$ are shown in bold.

Now, all the subcomplexes for which $y_0 \neq \hat{0}$ have no homology, so we only have to consider irreducibles of the form $(\hat{0}, T)$, where $T \subseteq V = \{x_1, x_2, x_3\}$. The result comes from the following table ($W = V\setminus T$):

W	$\tilde{\beta}_l(P_W))$, where $\tilde{\beta} \neq 0$	i ($l = \lvert W\rvert - i - 1$)
$\emptyset$	$\tilde{\beta}_{-1} = 1$	0
$\{x_1\}$	contractible	
$\{x_2\}$	contractible	
$\{x_3\}$	contractible	
$\{x_1, x_2\}$	$\tilde{\beta}_1 = 1$	0
$\{x_1, x_3\}$	contractible	
$\{x_2, x_3\}$	$\tilde{\beta}_1 = 1$	0
$\{x_1, x_2, x_3\}$	$\tilde{\beta}_1 = 1$	1

Thus $T_0(A_P, \lambda) = 1 + \lambda_1\lambda_2 + \lambda_2\lambda_3$, and $T_1(A_P, \lambda) = \lambda_1\lambda_2\lambda_3$. This is confirmed by the free resolution

$$0 \to Sv \to Su_1 \oplus Su_2 \oplus Su_3 \to A_P \to 0,$$

where $u_1 \mapsto 1$, $u_2 \mapsto y_1$, $u_3 \mapsto y_4$, and $v \mapsto x_1u_3 - x_1x_2x_3u_1$.

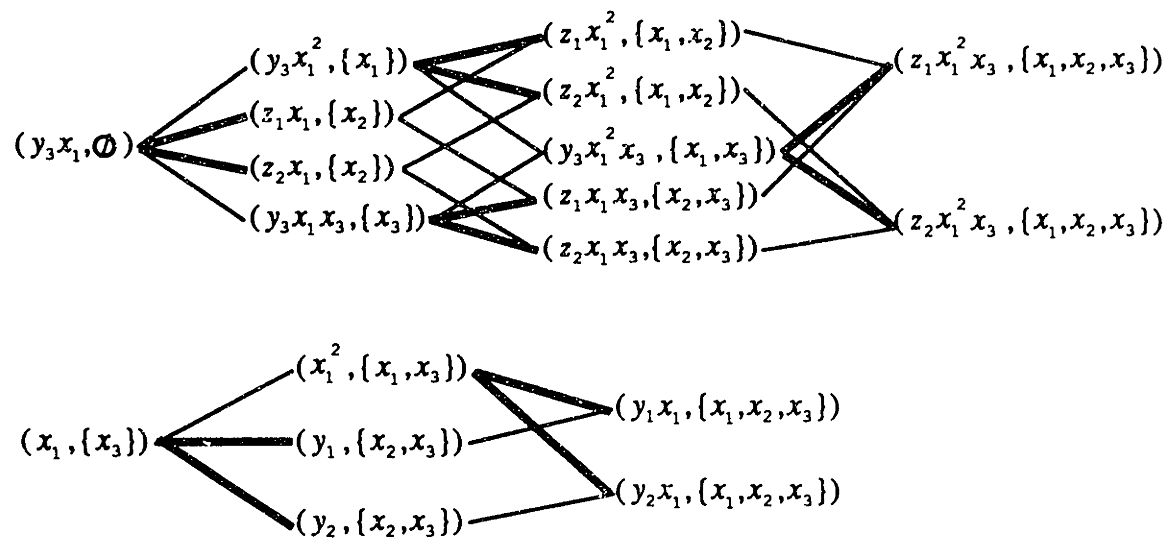


Figure 5-5: subcomplexes for Betti polynomials

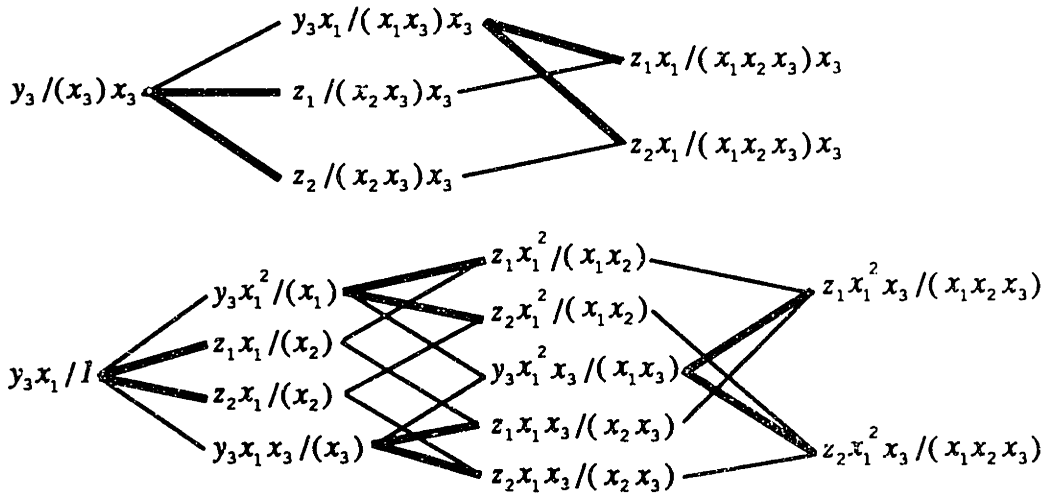


Figure 5-6: subcomplexes for local cohomology

Local cohomology

Now consider the local cohomology modules of A_P, following the proof of Theorem 5.7.6. Again, there are infinitely many irreducible elements, but we only consider two, $y_3x_1/1$ and y_3/x_3^2. The subcomplexes that they each anchor, of extra-fine grades $(y_3, (2, 0, 1))$ and $(y_3, (1, 0, -1))$ respectively, are shown in Figure 5-6, with edges marking the coboundary map. The resulting poset (sitting sideways) is, as described in the proof of Lemma 5.8.1, $\mathrm{lk}_P y_0 \times 2^{\mathrm{supp}(y_0)\setminus U_0}$, where $y_0 = y_3$ in both cases, and $U_0 = \emptyset$ and x_3, respectively. The copies of $\mathrm{lk}_P y_3$ are shown in bold.

Now, in this case, all the non-zero cohomology will arise when $\mathrm{supp}(y_0) = U_0$, which is equivalent, by Lemma 5.7.4, to the normal fine grading being non-positive.

Bibliography

[AM] M. Atiyah and I. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969.

[Ba] K. Baclawski, *Homology and combinatorics of ordered sets*, thesis, Harvard University, 1976.

[Bj1] A. Björner, "Posets, Regular CW Complexes and Bruhat Order," *Europ. J. Combin.* **5** (1984), 7–16.

[Bj2] A. Björner, "Face numbers of complexes and polytopes," *Proc. International Congress of Mathematicians (Berkeley, 1986)*, 1987, 1408–1418.

[BGS] A. Björner, A. Garsia, and R. Stanley, "An introduction to the theory of Cohen-Macaulay partially ordered sets," in *Ordered Sets* (I. Rival, ed.), Reidel, Dordrecht, Holland, 1982, 583–615.

[BK1] A. Björner and G. Kalai, "An extended Euler-Poincaré Theorem," *Acta Math.* **161** (1988), 279–303.

[BK2] ——, "Extended Euler-Poincaré Relations for Cell complexes," preprint.

[Ga] A. Garsia, "Combinatorial methods in the theory of Cohen-Macaulay rings," *Adv. Math.* **38** (1980), 229–266.

[GS] A. Garsia and D. Stanton, "Group actions on Stanley-Reisner rings and invariants of permutation groups," *Adv. Math.* **51** (1984), 107–201.

[Ho] M. Hochster, "Cohen-Macaulay rings, combinatorics, and simplicial complexes," in *Ring Theory II: Proc. of the Second Oklahoma Conference* (B. McDonald and A. Morris, ed.), Dekker, New York, 1977, 171–223.

[Ku] E. Kunz, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser, Boston, 1985.

[La] S. Lang, *Algebra, 2nd ed.*, Addison-Wesley, Menlo Park, CA, 1984.

[Mu1] J. Munkres, *Elements of Algebraic Topology*, Benjamin/Cummings, Menlo Park, CA, 1984.

[Mu2] ——, "Topological Results in Combinatorics," *Mich. Math. J.* **31** (1984), 113-128.

[Re] G. Reisner, "Cohen-Macaulay quotients of polynomial rings,", thesis, University of Minnesota, 1974; *Adv. Math.* **21** (1976), 30–49.

[Sp] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[St1] R. Stanley, "Cohen-Macaulay rings and constructible polytopes," *Bull. Amer. Math. Soc.* **81** (1975), 133–135.

[St2] ——, "The Upper Bound Conjecture and Cohen-Macaulay rings," *Stud. Appl. Math.* **54** (1975), 135–142.

[St3] ——, "Cohen-Macaulay complexes," in *Higher Combinatorics* (M. Aigner, ed.), Reidel, Dordrecht, Holland, 1977, 51–62.

[St4] ——, *Combinatorics and Commutative Algebra*, Progress in Mathematics, vol. 41, Birkhäuser, Boston, 1983.

[St5] ——, "The number of faces of simplicial polytopes and spheres," in *Discrete Geometry and Convexity* (J. Goodman, *et. al.* ed.) *Ann. New York Acad. of Sciences* **440** (1985) 212–223.

[St6] ——, *Enumerative Combinatorics, vol. I*, Wadsworth and Brooks/Cole, Monterey, CA, 1986.

[St7] ——, "f-vectors and h-vectors of simplicial posets," preprint.