The Intersection Homology With Twisted Coefficients of Toric Varieties

by

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Abstract

A toric variety is a complex algebraic variety on which $(\mathbf{C}^*)^n$ acts with finitely many orbits, one of which is dense. It can be equivalently defined, topologically, as the product of an n-dimensional polytope with an n-dimensional (real) torus modulo certain identifications.These identifications are closely related to the geometry of the underlying polytope. With this definition, a toric variety is a stratified pseudomanifold with a natural stratification in which each stratum corresponds to a skeleton of the polytope.

The main result of this thesis is the construction of a chain complex for the intersection homology with twisted coefficients of a general toric variety. The construction of the chain complex incorporates the combinatorial structure of the polytope as well as the structure of the torus.

A decomposition of the toric variety is described and is used in defining the chain complex. The subspaces in this decomposition are to a large extent transverse to the strata, and are therefore well suited to support intersection homology chains. This decomposition is dual to that used in various previous treatments of toric varieties (*e.g.* [1]) in which the subspaces are unions of strata. The duality can be seen at the level of the polytope, in that the subspaces in treatments such as in [1] correspond to unions of faces of the polytope whereas the subspaces in the decomposition described here correspond to subspaces of the polytope which are transverse to the skeleta.

A related result is the construction of a chain complex for the intersection homology with twisted coefficients of an n-fold product of 2-disks.

Thesis Supervisor: Robert MacPherson
Professor of Mathematics

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1. Introduction

Toric varieties first arose in the context of algebraic geometry. They were defined in 1970 by Demazure [10] and in 1973 by Kempf, Knudesen and Mumford [11]. They are algebraic varieties and are generalizations of both the affine spaces $\mathbb{A}^n$ and the complex projective spaces $\mathbb{P}^n$. Because of their structure, they served as interesting examples on which one could illustrate many concepts of algebraic geometry such as: linear systems, invertible sheaves, cohomology, resolution of singularities, as well as the more recent theory of intersection homology. From the algebro-geometric point of view, their main importance lay in the fact that many algebraic varieties embed more naturally in a suitable toric variety than in $\mathbb{P}^n$, and in that varieties frequently are "locally" *toroidal.*

Since then, toric varieties have shown up in a variety of fields ranging from differential geometry to the combinatorial theory of rational convex polytopes. In fact, many prominent results about convex polytopes have relied almost entirely on the theory of toric varieties for their proof.
In this context, the intersection homology of toric varieties has played a central role. The most noteable example perhaps is the *h-vector*, an important combinatorial invariant of a convex polytope whose i^{th} component is precisely equal to the i^{th} Betti number of the middle perversity intersection homology of the associated toric variety. It was only through this link for example, that the non-negativity of the components of the *h-vector* was proved, despite some 50 years of empirical evidence of this fact.

The middle perversity intersection homology $\mathbf{IH}_*^{\bar{m}}(\mathbf{X};\mathbf{Q})$ for a general toric variety $\mathbf{X}$ was computed in the early 1980's, independently by J.N. Bernstein, A.G. Khovanski and R.D. MacPherson, and was published by Stanley [12] in 1987. These computations were carried out in a seperate fashion for smooth toric varieties, and in the passage to the singular case fairly involved methods of characteristic-p algebraic geometry were used. As a result, the computations were abstract and somewhat impenetrable without a significant background in algebraic geometry. In addition, these computations were done only for constant (rational) coefficients, and only for middle perversity. In fact, the characteristic-p methods used *cannot* be generalized to other perversities, and a full classification of the local coefficient systems for which these methods are valid is not known.

The main result of this thesis is the construction of a chain complex for the intersection homology with twisted coefficients $IH_*^{\bar{p}}(\mathbf{X};\mathcal{L})$ of a toric variety $\mathbf{X}$. The chain complex is constructed is section 8 and is proved (theorem 8.7) to be quasi-isomorphic to the intersection homology chain complex $IC_*^{\bar{p}}(\mathbf{X};\mathcal{L})$.

The complex is infinitely generated and hence, in its present form, falls short of en-

abling direct computation of $IH^{\bar{p}}_{\bullet}(\mathbf{X};\mathcal{L})$. It has however the following advantages :
(i) It relates the intersection homology of a toric variety to a very natural algebraic construction, namely the bar-complex of $\mathbf{Z}^n$.
(ii) It is explicitly constructed in a manner which is closely related to the structure of the toric variety and the chains have an explicit geometric interpretation.
(iii) It is defined with no distinction between singular and non-singular toric varieties.
(iv) It is defined for a very general class of local systems, and
(v) It is defined for *any* perversity $\bar{p}$.

A related result is presented in section 5. A chain complex is constructed which is finitely generated but is nevertheless very similar in nature to the complex described above. This complex is shown to compute the intersection homology with twisted coefficients $IH_{\bullet}(\mathbf{Y},\partial\mathbf{Y};\mathcal{L})$ where $\mathbf{Y}$ is an n-fold product of 2-dimensional disks and is viewed (and stratified) as a "corner" of a smooth toric variety $\mathbf{X}$.

The constructions of both of these complexes as well as the proofs run along similar lines. In each case the toric variety is decomposed into subspaces $\{\mathbf{X}_K\}$ corresponding to the cones K of the underlying cone complex (see section 2.1 for the relevant definitions). A spectral sequence argument is used to show that it suffices to prove the theorem locally for each subspace $\mathbf{X}_K$.
These subspaces are "sufficiently transverse" to the singular strata so as to contain all of the local intersection-homological information. In addition, for each K, $\mathbf{X}_K$ is either the product with a circle of an analogous subspace in a toric variety of one dimension lower, or the topological cone on the union of certain lower dimensional sets $\mathbf{X}_{K'}$. The associated chain complex is shown to posess the algebraic analogues of these properties, and hence the construction lends itself nicely to an inductive proof.

Section 2 lays down the basic definitions and some fundamental facts concerning toric varieties along with some examples.

Section 3 presents some background about local systems, about intersection homology and about the two put together. In addition, a lemma is proved (lemma 3.14) which facilitates the computation of intersection homology in certain settings. Section 3.6 deals with the intersection homology of the product of a space with a circle, and with the intersection homology of a cone.

In section 4 a standard CW-decomposition of the n-torus is used to compute the homology with twisted coefficients of the torus.

Section 6 deals with the bar complex of $\mathbb{Z}^n$. The Eilenberg-Zilber theorem is stated in this context using explicit maps defined by Eilenberg and Maclane in [9], and certain properties concerning allowability are proved for the maps involved.

Section 7 proves that that the bar complex of $\mathbb{Z}^n$, the cellular complex of section 4 and the complex of *P.L.* geometric chains on the n-torus (all with twisted coefficients) are all quasi-isomorphic.

2 Preliminaries

2.1 Basic definitions

Definition 2.1 *Polyhedral cones:* A **rational polyhedral cone** in $\mathbf{R}^n$ is a set $K = \mathbf{R}^{\geq 0}v_1 + \mathbf{R}^{\geq 0}v_2 + \ldots + \mathbf{R}^{\geq 0}v_k$, where $v_i \in \mathbf{Z}^n$ $\forall i$. ($\mathbf{R}^{\geq 0}$ denotes the set of non-negative real numbers.) We say that K *is spanned by* $\{v_1, \ldots, v_k\}$.
K is **proper** if it is not spanned by any proper subset of $\{v_1, \ldots, v_k\}$, and $v_1, \ldots, v_k$ all lie strictly on one side of some hyperplane in $\mathbf{R}^n$.
A cone spanned by a (proper) subset of $\{v_1, \ldots, v_k\}$ which is contained in the topological boundary of K is a (proper) **face** of K.
The **dimension** of K is the dimension of the smallest subspace of $\mathbf{R}^n$ containing K.
A k-dimensional cone K is **simplicial** if it is spanned by precisely k rays. If K is simplicial then all of its faces are too.
A 1-dimensional cone is called a **ray**. An n-dimensional cone is called a **chamber**. For a ray K, the unique $v = (z_1, \ldots, z_n) \in \mathbf{Z}^n$ which spans K and for which the z_i's are coprime will be referred to as the **coordinates** of K. For a k-dimensional cone K, we define the **coordinate matrix** of K to be the $k \times n$ matrix $M(K)$ whose rows are the coordinates of the spanning rays of K.

Definition 2.2 *Cone Complex:* A (rational polyhedral) **cone complex** $\mathcal{K} \subset \mathbf{R}^n$ is a partition of $\mathbf{R}^n$ into a finite number of proper rational polyhedral cones such that the intersection of any two cones K_1 and K_2 is a face of both K_1 and K_2.
The graded poset of cones of $\mathcal{K}$, graded by dimension and ordered by inclusion is called the **face lattice** of $\mathcal{K}$. The apex of all the cones (the origin of $\mathbf{R}^n$) is considered as a 0-dimensional cone and it is a face of all the other cones.
The **i-skeleton** of $\mathcal{K}$ is $\mathcal{K}^i = \{K \in \mathcal{K} : dim(K) \leq i\}$.
A cone complex is often referred to (elsewhere) as a *complete fan.*

Definition 2.3 *The dual polyhedron:* Let $\mathcal{K} \subset \mathbf{R}^n$ be a rational polyhedral cone complex. Reversing the inclusions and the grading in the face lattice of $\mathcal{K}$ we obtain the face lattice of an abstract polyhedron which can be realized in $\mathbf{R}^n$ as a regular (polyhedral) cell complex. Fix one such realization and denote it by $\mathcal{P}(\mathcal{K})$ or, when no confusion can arise, simply $\mathcal{P}$. $\mathcal{P}$ is called **the dual polyhedron** or **the dual polytope.**
Note: $\mathcal{P}$ is homeomorphic to D^n, the closed unit ball in $\mathbf{R}^n$, and its cells (which we will also refer to as faces) are dually paired with the cones of $\mathcal{K}$ in complimentary dimensions.
The 0-dimensional faces of $\mathcal{P}$ are called **vertices**,the 1-dimensional faces are called **edges** and the codimension-1 faces are **facets.**
The **i-skeleton** of $\mathcal{P}$ is $\mathcal{P}^i = \{F : F \text{ is a face of } \mathcal{P},\ dim(F) \leq i\}$.

Remark 2.4 $\mathcal{P}$ can be explicitly constructed in a manner analogous to the "cell - dual cell" construction in the original proof of Poincaré duality as follows :
Let $S^{n-1} \subset \mathbf{R}^n$ be any sphere centered at the origin. Then $\mathcal{K}$ determines a polyhedral cell decomposition of S^{n-1} by simply intersecting the sphere with the positive dimensional cones of $\mathcal{K}$. Denote the cell corresponding to a cone K by σ_K and let $\mathcal{S}$ be this polyhedral sphere. Let $\mathcal{S}' = sd(\mathcal{S})$ be the barycentric subdivision of $\mathcal{S}$.
For each $K \in \mathcal{K}$ $(\dim(K) \geq 1)$, let $D(\sigma_K)$ be the subcomplex of $\mathcal{S}'$ consisting of all simplices whose vertices are barycenters of cells of $\mathcal{S}$ containing σ_K. Then the sets $D(\sigma_K)$ partition $\mathcal{S}'$ into polyhedral cells dual to to the σ_K's and hence to the (positive dimensional) cones of $\mathcal{K}$ and for each K, $\dim(D(\sigma_K)) = n - \dim(K)$. This completes the construction of $\partial\mathcal{P}$. Now simply take the the cone on $\mathcal{S}$ with apex at the origin as the unique n-dimensional cell dual to the 0-dimensional cone.

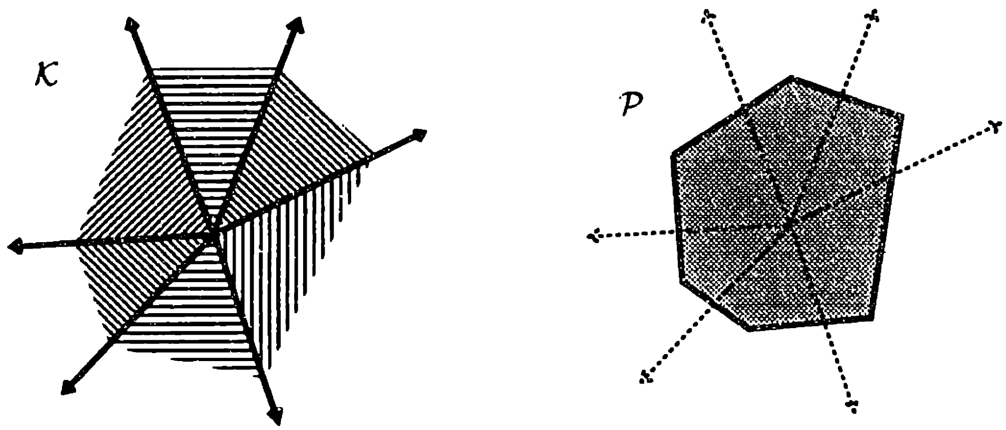

Figure 1: Cone complex $\mathcal{K} \subset \mathbf{R}^2$ an its dual polytope $\mathcal{P}$

In order to define a toric variety it is first necessary to describe a certain procedure for colapsing an n-torus to a lower dimensional torus. Denote by $\mathcal{T}^n$ the n-torus $\mathbf{R}^n/\mathbf{Z}^n \cong \overbrace{S^1 \times \ldots \times S^1}^{n}$ and consider the projection : $\mathbf{R}^n \xrightarrow{proj} \mathbf{R}^n/\mathbf{Z}^n = \mathcal{T}^n$. Let F be a k-dimensional rational subspace of $\mathbf{R}^n$ (*i.e.* F has a basis in $\mathbf{Z}^n$) and set $\hat{F} = proj(F)$. The rationality of F implies that $\hat{F}$ is a compact k-dimensional subtorus of $\mathcal{T}^n$. The class of affine k-planes in $\mathbf{R}^n$ which are parallel to F determines a class of subtori "parallel" to $\hat{F}$ in $\mathcal{T}^n$. Denote by $\mathcal{T}^n/\hat{F}$ the space obtained from $\mathcal{T}^n$ by collapsing each of these parallel subtori to a point. In the language of Lie groups, this amounts to modding out the group $\mathcal{T}^n$ by the subgroup $\hat{F}$, whence it is clear that $\mathcal{T}^n/\hat{F} \cong \mathcal{T}^{n-k}$.
Note: If the subspace F is not rational, $\hat{F}$ is non-compact and the resulting quotient space is not even Hausdorff.

Definition 2.5 *Toric Variety:* Let $\mathcal{K}$ be a cone complex in $\mathbf{R}^n$, and let $\mathcal{P}$ be the dual polyhedron. For each cone $K \in \mathcal{K}$, let F_K denote the linear span of K in $\mathbf{R}^n$. Set

$\hat{\mathbf{X}} = \mathcal{P} \times \mathcal{T}^n$. The **toric variety associated to $\mathcal{K}$** is obtained from $\hat{\mathbf{X}}$ as follows : for each $x \in \mathcal{P}$, $\{x\} \times \mathcal{T}^n$ is collapsed to $\{x\} \times (\mathcal{T}^n / \hat{F}_K)$, where K is the cone dual to the unique open cell of $\mathcal{P}$ containing x.

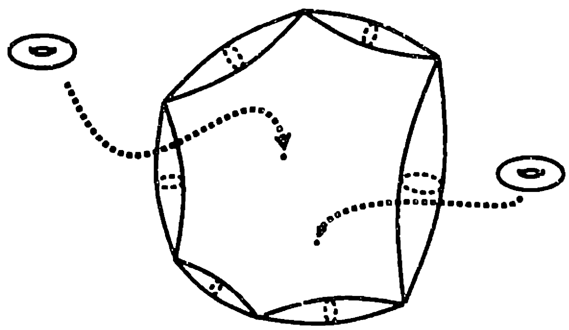

Figure 2: The toric variety associated to $\mathcal{K}$ (and $\mathcal{P}$) from figure 1

Remark 2.6 No (non-trivial) collapsing occurs over the interior of $\mathcal{P}$.

We have a commuting diagram of projections :

$$\begin{array}{ccc} \hat{\mathbf{X}} & \xrightarrow{\hat{p}} & \mathbf{X} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{P} & \end{array}$$

which we will refer to several times throughout this paper.

2.2 First examples

We describe now a certain class of examples of non-singular $2n$-dimensional toric varieties for $n \geq 1$. Despite their simplicity, the local structure of the toric varieties in these exaples will come in handy in following sections. In fact, any non-sigular toric variety is locally homeomorphic (in a P.L. stratum preserving way) to one of the toric varieties in these examples.

Example 2.7 $\mathbf{n = 1}$: There is only one cone complex $\mathcal{K}$ in $\mathbb{R}^1$. It has (0) as a 0-dimensional cone and the positive and negative rays emanating from (0) as 1-dimensional cones.

$\mathcal{P}$ is a closed interval I (with 2 0-cells at the ends and a single 1-cell connecting them).
$\hat{\mathbf{X}}$ is a cylinder $I \times S^1$, and since $\hat{p}$ simply collapses each end of the cylinder to a point, $\mathbf{X} \cong \Sigma S^1 \cong S^2$ (the 2-sphere). (See figure 3).

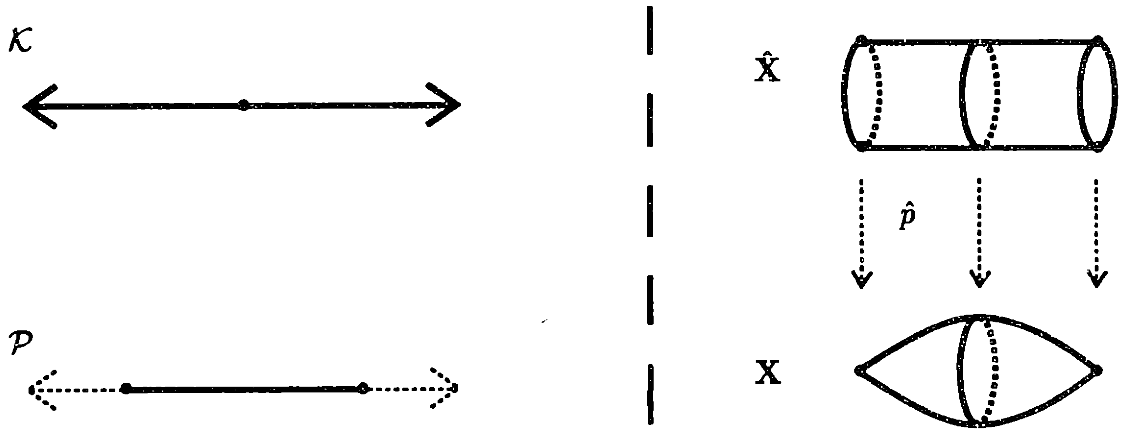

Figure 3: $n = 1$

Example 2.8 Let $\mathcal{K} \subset \mathbf{R}^2$ be the cone complex with 4 rays $R_1, \ldots, R_4$ having respective coordinates $(1,0),(0,1),(-1,0),(0,-1)$, and whose chambers are the 4 quadrants of $\mathbf{R}^2$.
$\mathcal{P}$ is a square $I_1 \times I_2$, and $\hat{\mathbf{X}} = I_1 \times I_2 \times \mathcal{T}^2$.
In this example it is convenient to consider the torus as a product : $\mathcal{T}^2 = c_1 \times c_2$ where $c_1 = proj$(x-axis) and $c_2 = proj$(y-axis). The collapsing map $\hat{p}$ is the canonical projection of $\mathcal{T}^2$ onto c_2 over points in $\partial I_1 \times I_2$ and the canonical projection onto c_1 over points in $I_1 \times \partial I_2$.
In view of this and rewriting $\hat{\mathbf{X}}$ as $(I_1 \times c_1) \times (I_2 \times c_2)$, we see that $\hat{p}$ collapses the c_1 component to a point over the endpoints of I_1 and the c_2 component to a point over the endpoints of I_2 (hence the whole torus gets collapsed to a point over each of the four vertices of the square).
Thus $\mathbf{X} = \Sigma S^1 \times \Sigma S^1 \cong S^2 \times S^2$ (see figure 4).

Example 2.9 We now construct the higher dimensional analogue of the previous examples. Let $\mathcal{K}$ be the cone complex in $\mathbf{R}^n$ whose chambers are those cut out by the coordinate hyperplanes, and whose lower dimensional cones are finite intersections of chambers. The dual polytope $\mathcal{P}$ is an n-dimensional cube $I_1 \times \ldots \times I_n$. As in the previous example, $\hat{X} = \mathcal{P} \times \mathcal{T}^n$ can be written as $\prod_{j=1}^{n}(I_j \times c_j)$. The collapsing map $\hat{p}$

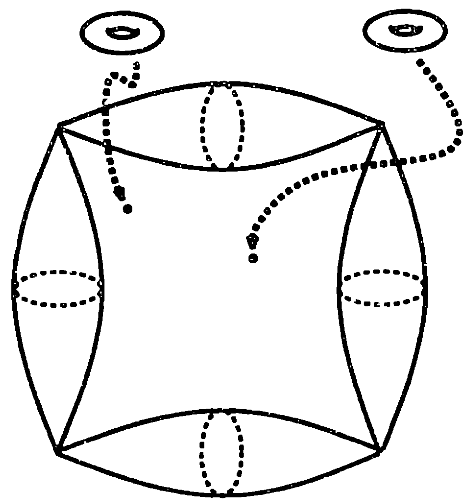

Figure 4: $S^2 \times S^2$

collapses each c_j to a point when the I_j coordinate is 0 or 1 (accordingly, over faces of $\mathcal{P}$ of codimension ≥ 1, when the I_j coordinate is 0 or 1 for more than one j, all of the associated c_j's are collapsed). Thus a seperate collapsing occurs at the ends of each cylinder ($I_j \times c_j$) and the resulting toric variety $\mathbf{X}$ is the n-fold product $S^2 \times \ldots \times S^2$ (See figure 5).

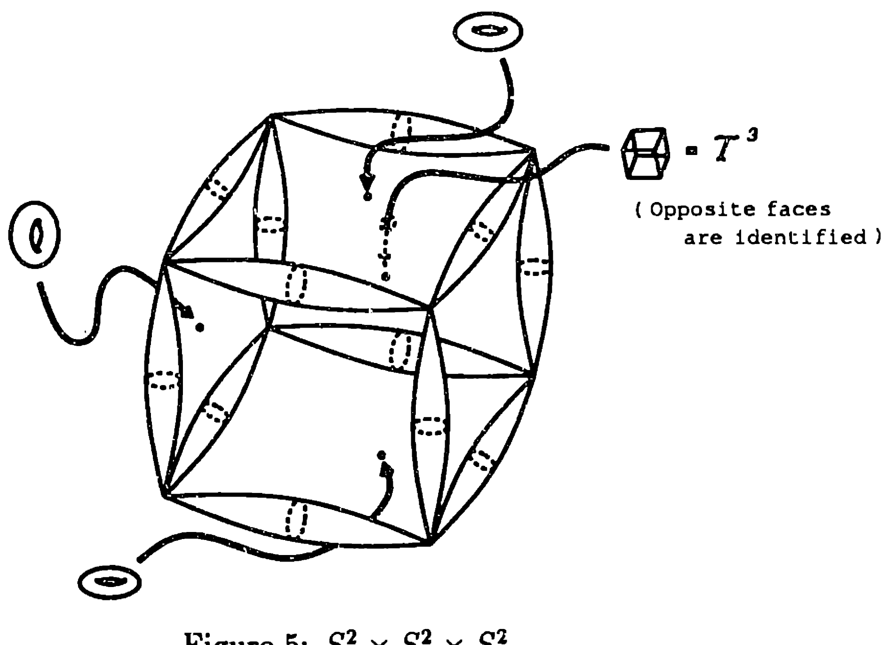

Figure 5: $S^2 \times S^2 \times S^2$

Finally, note that in ΣS^1, if we restrict the t coordinate (coming from the interval) to $0 \leq t \leq \frac{1}{2}$, then we obtain a 2-disk D^2. Consequently, the restriction of the toric varieties in these examples to the subspaces $p^{-1}([0,\frac{1}{2}] \times \ldots \times [0,\frac{1}{2}])$ is a product of 2-disks $D^2 \times \ldots \times D^2$ (see figure 6).

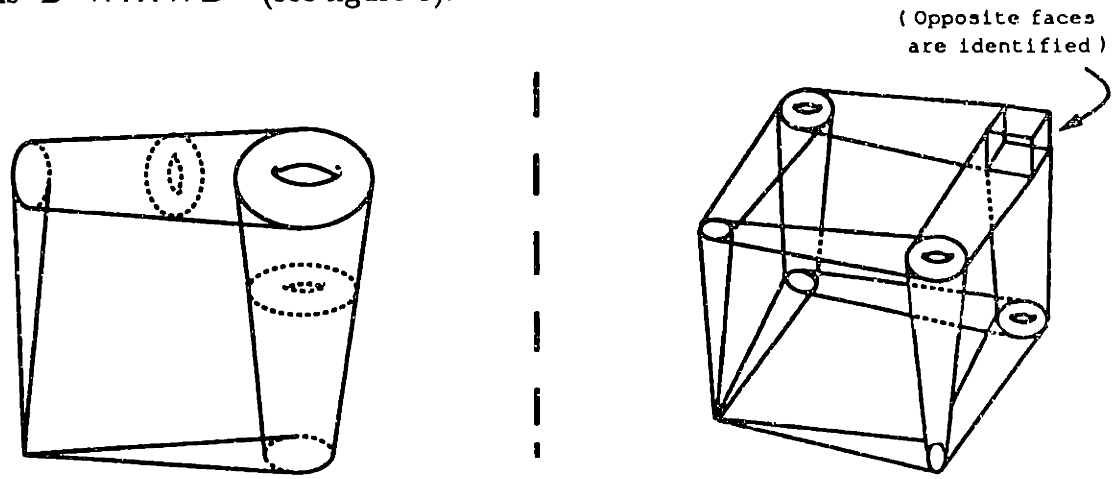

Figure 6: $D^2 \times D^2$ and $D^2 \times D^2 \times D^2$

2.3 Some fundamental facts and observations about toric varieties

Fact 2.10 A toric variety $\mathbf{X}$ is compact.

Proof : $\mathbf{X}$ is a quotient of the compact space $\hat{\mathbf{X}}$ by a closed relation.

Fact 2.11 The action of $(\mathbf{C}^*)^n$: For any toric variety $\mathbf{X}$ there is an action of $(\mathbf{C}^*)^n$ on $\mathbf{X}$ with finitely many orbits, one of which is dense. In fact, the orbits are in one to one correspondence with the cones of $\mathcal{K}$ (equivalently, with the faces of $\mathcal{P}$).

Proof : $p^{-1}(int(\mathcal{P})) \cong int(D^n) \times \mathcal{T}^n \cong (\mathbf{C}^*)^n$. In fact, by the construction it is clear that for any k-face F of $\mathcal{P}$, $p^{-1}(int(F)) \cong int(D^k) \times \mathcal{T}^k \cong (\mathbf{C}^*)^k$. Moreover, this $(\mathbf{C}^*)^k$ is canonically isomorphic to the quotient $(\mathbf{C}^*)^n/(\mathbf{C}^*)^{n-k}$ of the "big" $(\mathbf{C}^*)^n$ by the appropriate subgroup. Thus the action of $(\mathbf{C}^*)^n$ extends to all of $\mathbf{X}$. The orbits of this action are precisely the sets $p^{-1}(int(F))$ where F ranges over all of the faces of $\mathcal{P}$. The orbit over the unique open n-face of $\mathcal{P}$ (namely, over the interior of $\mathcal{P}$) is dense in $\mathbf{X}$.

Remark 2.12 It follows from the above that $\mathbf{X}$ is a compactification of $(\mathbf{C}^*)^n$

Remark 2.13 Let F be a (closed) k-face of $\mathcal{P}$. Upon close examination of how the $(\mathbf{C}^*)^k$ over $int(F)$ gets collapsed to $(\mathbf{C}^*)^i$ over the i-faces in ∂F we see that $p^{-1}(F)$ is a $2k$-dimensional toric variety.

The stratification of X : A toric variety $\mathbf{X}$ has a natural stratification with only even dimensional strata :

$$\mathbf{X}_0 \subset \mathbf{X}_2 \subset \ldots \subset \mathbf{X}_{2(n-1)} \subset \mathbf{X}_{2n}. \tag{1}$$

Where for each k, $\mathbf{X}_{2k} = p^{-1}(\mathcal{P}^k)$, and $dim(\mathbf{X}_{2k}) = 2k$.
With this stratification $\mathbf{X}$ is a ***stratified pseudomanifold*** of (real) dimension $2n$. (See [2] for the precise definitions of stratified pseudomanifolds).

Fact 2.14 Let $\mathcal{K}$ be a cone complex and $\mathbf{X}$ the associated toric variety. Then:
(i) $\mathbf{X}$ is rationally nonsingular if and only if $\mathcal{K}$ is simplicial.
(ii) $\mathbf{X}$ is integrally nonsingular (i.e. a smooth manifold) if and only if $\mathcal{K}$ is simplicial and $|\det(M(K))| = 1$ for every maximal dimensional cone $K \in \mathcal{K}$.

3. Local coefficients and intersection homology

In this section we describe some of the necessary fundamentals of homology with twisted coefficients (a.k.a. local coefficients). Section 3.2 is a brief discussion of ordinary homology with twisted coefficients. The case of intersection homology with twisted coefficients is somewhat more complex. We expand on this in sections 3.3 and 3.4. Finally, in section 3.6, we present two fundamental lemmas - *the combing lemma* and *the coning lemma*. The first deals with the intersection homology with twisted coefficients of the product of a stratified pseudomanifold $\mathbf{X}$ with a circle S^1, and the second with the intersection homology, again with twisted coefficients, of the cone $c(\mathbf{X})$. In both cases, the appropriate homology is proved to be related to that of $\mathbf{X}$ in a way that will be very useful in section 8 in the inductive step of the proof of the main theorem of this thesis.

3.1 Local systems

We will use a definition of a local system which is essentially that of [5] and [6]. The only difference is that we require the fiber to be a vector space over $\mathbf{Q}$.

Definition 3.1 Local system. Let $\mathbf{X}$ be a topological space and $\mathbf{V}$ a vector space over $\mathbf{Q}$. A local system $\mathcal{L}$ on $\mathbf{X}$ with fibre $\mathbf{V}$ is a vector bundle over $\mathbf{X}$ whose fiber over any point $x \in \mathbf{X}$, denoted $\mathbf{V}_x$, is isomorphic to $\mathbf{V}$, along with isomorphisms $\Phi_\omega : \mathbf{V}_{x_1} \to \mathbf{V}_{x_2}$ for every pair of points $x_1, x_2 \in \mathbf{X}$ and for every path ω from x_1 to x_2. These maps are required to satisfy the following conditions :
(i) The map Φ_ω depends only on the path-homotopy class $[\omega]$ of ω.
(ii) If $[\omega] \in \pi_1(\mathbf{X}, x_0)$ is the identitiy, then $\Phi_\omega : \mathbf{V}_{x_0} \to \mathbf{V}_{x_0}$ is the identity.
(iii) If ω is a path from x_1 to x_2 and υ is a path from x_2 to x_3 then $\Phi_{\omega * \upsilon} = \Phi_\upsilon \circ \Phi_\omega : \mathbf{V}_{x_1} \to \mathbf{V}_{x_3}$ (here $\omega * \upsilon$ denotes the path ω followed by υ from x_1 to x_3).

Note that it follows from (ii) and (iii) that if ω^{-1} denotes the path ω traversed in reverse, then $\Phi_{\omega^{-1}} = \Phi_\omega^{-1}$.

Remark 3.2 It is a standard fact that any such local system $\mathcal{L}$ (more precisely - any isomorphism class of local systems) uniquely determines a representation of $\pi_1(\mathbf{X}, x_0)$ on $\mathbf{V}$ and vice-versa.

3.2 Ordinary homology

Detailed descriptions of homology with twisted coefficients in the simplicial category as well as in the category of regular cell complexes can be found in [5] and [6]. We review here some of the essential ideas in the simplicial category.

Let $\mathbf{X}$ be a topological space, triangulated by a simplicial complex K. To simplify notation, we will not distinguish between $\mathbf{X}$ and K, nor will we distinguish between a simplex $\sigma \in K$ and $|\sigma| \subset \mathbf{X}$. We assume all simplices are oriented, determining well defined incidence numbers $[\sigma;\tau] = \pm 1$ whenever τ is a face of σ.

Let $\mathcal{L}$ be a local system on $\mathbf{X}$ with fiber $\mathbf{V}$.
For each simplex σ, fix a designated point $x_\sigma \in \sigma \backslash \partial\sigma$ (*e.g.* the barycenter of σ), and denote by $\mathbf{V}_\sigma$ the fiber $\mathbf{V}_{x_\sigma}$. If τ is a face of σ, then since σ is simply connected, there is a canonical isomorphism $\Phi_{\sigma,\tau} : \mathbf{V}_\sigma \rightarrow \mathbf{V}_\tau$ determined by any path ω in σ from x_σ to x_τ.

Definition 3.3 The simplicial chain complex with twisted coefficients. The chain complex $S_\bullet(\mathbf{X};\mathcal{L})$ of simplicial chains on $\mathbf{X}$ with coefficients in the local system $\mathcal{L}$ is defined by :

$$S_q(\mathbf{X};\mathcal{L}) = \text{ formal finite sums } \sum_{i=1}^{m} \alpha_i \cdot \sigma_i$$

where for each i, σ_i is a q-simplex and $\alpha_i \in \mathbf{V}_{\sigma_i}$. Chains are added formally except that $\alpha \cdot \sigma + \beta \cdot \sigma = (\alpha + \beta) \cdot \sigma$.
The boundary map $\partial_q : S_q(\mathbf{X};\mathcal{L}) \rightarrow S_{q-1}(\mathbf{X};\mathcal{L})$ is defined by

$$\partial_q(\alpha \cdot \sigma) = \sum_{\tau \text{ a face of } \sigma} [\sigma;\tau]\Phi_{\sigma,\tau}(\alpha) \cdot \tau$$

and it is a standard result that $\partial_{q-1} \circ \partial_q = 0$

Remark 3.4 One of the the most notable differences from a geometrical point of view, between constant coefficient chains and twisted coefficient chains, is that twisted coefficient chains have a "non-topological" boundary arising from the existence of the monodromies in addition to the regular topological boundary. For example, in the constant coefficient case, any compact orientable n-dimensional smooth manifold M has a fundamental class, *i.e.* in any triangulation of M the n-simplices may be oriented in such a way so that their sum (each with coefficient 1) is a cycle. In the twisted coefficient case this is not so, as can clearly be seen in the case of a circle S^1 (or any torus for that matter) with non-trivial local coefficient system, in all of the examples

of section 4.
Thus a "topological cycle" (*i.e.* an oriented closed submanifold) might not support *any* cycle in the *P.L.* geometric chain complex. (Note however that the support of a *P.L.* geometric cycle is always without topological boundary).

Definition 3.5 homology with twisted coefficients. The homology of $\mathbf{X}$ with coefficients in the local system $\mathcal{L}$, denoted $H_*(\mathbf{X};\mathcal{L})$ is defined to be the homology of the chain complex $S_*(\mathbf{X};\mathcal{L})$. Analogous definitions are given in [6] using cellular chains and using singular chains, and it is proved that all these methods are equivalent.

Intersection homology is defined using *P.L.* geometric chains. There are some subtleties involved in defining twisted coefficient geometric chains. We address these issues first and then proceed to define intersection homology with twisted coefficients.

3.3 Geometric chains with twisted coefficients

3.3.1 A (brief) review of the constant coefficient case

Let $\mathbf{X}$ be a *P.L.* space. Let T be a triangulation of $\mathbf{X}$ which is compatible with the *P.L.* structure of $\mathbf{X}$. Denote by $C_*^T(\mathbf{X})$ the chain complex of simplicial chains of $\mathbf{X}$ with respect to T. The class of all such complexes on $\mathbf{X}$ is a directed system under chain maps induced by refinements of triangulations. The *P.L.* geometric chain complex $C_*(\mathbf{X})$ is defined to be the direct limit $\varinjlim_T C_*^T(\mathbf{X})$.

One of the advantages of working with $C_*(\mathbf{X})$ is that one is not bound to a specific triangulation of $\mathbf{X}$, and can work with certain "elementary" chains which are more general than simplices. These are oriented, *P.L.* "pieces" of the space, with multiplicities. Each such subspace ξ is $|c|$ (the *support* of c) for some simplicial chain $c \in C_*^T(\mathbf{X})$ (and consequently it is also equal to $|c'|$ whenever c' corresponds to c under some refinement T' of T). Thus for any coefficient a in our fixed coefficient group, we may think of the chain $a \cdot \xi$ as the simplicial chain whose simplices are all of those simplices in c and all have the same coefficient a. Then one can use these chains much the same as one uses simplices in a fixed triangulation. The boundary of such a chain $a \cdot \xi$ is, geometricaly, $\partial|c|$, possibly divided into several pieces, each of which has coefficient $\pm a$. These notions are invariant under refinement of the triangulation, and hence, for any oriented i-dimensional *P.L.* subspace ξ of $\mathbf{X}$ and for any coefficient a, the chains $a \cdot \xi$ and $\partial(a \cdot \xi)$ are well defined chains in $C_i(\mathbf{X})$.

The formal definition of the geometric chain complex as a direct limit might give the appearance that it is a difficult complex to work with. In fact, due to the ability to define these elementary chains, $C_o(\mathbf{X})$ allows for much more flexibility than the simplicial complex associated to a fixed triangulation.

3.3.2 The twisted coefficient case

Let T be a triangulation of $\mathbf{X}$, and let $S^T_*(\mathbf{X};\mathcal{L})$ be the associated chain complex as in definition 3.3. Let T' be a refinement of T. Any simplex $\sigma' \in T'$ is carried by a unique simplex $\sigma \in T$, and since σ is simply connected, there is a canonical identification of the fibres $\mathbf{V}_{x_\sigma}$ and $\mathbf{V}_{x_{\sigma'}}$, mapping $\alpha \in \mathbf{V}_{x_\sigma}$ to some $\alpha' \in \mathbf{V}_{x_{\sigma'}}$. Thus there is a canonical chain map from $S^T_*(\mathbf{X};\mathcal{L})$ to $S^{T'}_*(\mathbf{X};\mathcal{L})$ mapping $\alpha\cdot\sigma$ to $\sum \alpha'\cdot\sigma'$, where the sum runs over all $\sigma' \in T'$ which are carried by σ. Thus we can formally define :

Definition 3.6 . The geometric chain complex on $\mathbf{X}$ with coefficients in the local system $\mathcal{L}$, denoted $C_*(\mathbf{X};\mathcal{L})$ is the direct limit (under chain maps induced by refinement) $\varinjlim_{T} S^T_*(\mathbf{X};\mathcal{L})$.

As in the constant coefficient case, the formal definition of the geometric chain complex is quite inconvenient to work with, and we would like to define certain "elementary chains" similar to those defined above. As it turns out, there are some obstacles one must overcome if one wishes to maintain the freedom of not being bound to any specific triangulation, and to continue to think of elementary chains as "*P.L.* subspaces with a (single) coefficient attached", out of which general chains are formed. The obvious naive approach would be to mimick the costruction of section 3.2 and definition 3.3, namely to take formal finite sums of geometric chains where to each chain is attached a coefficient from the fiber over a designated interior point of the chain.
However, given a *P.L.*-subspace ξ, $x_\xi \in int(\xi)$ and $\alpha \in \mathbf{V}_{x_\xi}$, one encounters two difficulties in attempting to define a chain $\alpha\cdot\xi$ analogous to those above.

The first is that given a triangulation T of which ξ is a subcomplex, and given two simplices σ and $\sigma' \in \xi$ with respective designated interior points x_σ and $x_{\sigma'}$, there is no canonical identification of the fibers $\mathbf{V}_{x_\sigma}$ and $\mathbf{V}_{x_{\sigma'}}$ and therefore there is no way to attach "the same" coefficient α to each of the simplices in ξ.

The second problem, which is in fact a consequence of the first, is that of remark 3.4. Consequently, specifying a *P.L.* subspace and a single coefficient over some point in its interior does not uniquely define the support of $\partial\xi$.

Thus we define :

Definition 3.7 Elementary chains. An *elementary chain* $\alpha \cdot \xi$ in $C_k(\mathbf{X};\mathcal{L})$ consists of :
(i) A k-dimensional *P.L.* subspace ξ (the *support* of the chain).
(ii) A $(k-1)$-dimesional *P.L.* subspace of ξ, denoted $\nabla\xi$, so that $\xi \setminus \nabla\xi$ is simply connected.
(iii) A designated point $x_\xi \in int(\xi \setminus \nabla\xi)$.
(iv) A coefficient $\alpha \in \mathbf{V}_{x_\xi}$.

Let T be any triangulation of $\mathbf{X}$ in which both ξ and $\nabla\xi$ are subcomplexes. Any k-simplex $\sigma \in \xi$ has its interior in $\xi \setminus \nabla\xi$, whence the fibre $\mathbf{V}_\sigma$ over the designated point $x_\sigma \in int(\sigma)$ is *canonically* isomorphic to $\mathbf{V}_{x_\xi}$, (by taking any path in $\xi \setminus \nabla\xi$ connecting x_σ and x_ξ) and it is thus understood that the coefficient of σ is the image of α under this canonical isomorphism. since the same holds for any refinement T' of T, $\alpha \cdot \xi$ is a well defined element of $C_k(\mathbf{X};\mathcal{L})$, and $|\partial(\alpha \cdot \xi)| \subset |\nabla\xi| \cup \partial|\xi|$.

3.4 Intersection homology

Definition 3.8 Perversity. A perversity is a vector

$$\bar{p} = (p_2, p_3, \ldots, p_n)$$

satisfying :
(i) $p_2 = 0$, and
(ii) $p_i \leq p_{i+1} \leq p_i + 1$, for $2 \leq i \leq n$.
For each n there are 4 special perversities with various properties (which we will not list here) which do not hold for general perversities. They are :
0-perversity :

$$\bar{0} = (0, 0, \ldots, 0),$$

lower and upper *middle-perversities*:

$$\bar{m} = (0,0,1,1,2,\ldots) \text{ and } \bar{n} = (0,1,1,2,2,\ldots),$$

and *total-perversity* :

$$\bar{t} = (2, 3, \ldots, n-2).$$

Let $\mathbf{X}$ be an n-dimensional stratified pseudomanifold and Let $\Sigma = \mathbf{X}_{n-2}$ denote the singular set. All triangulations of $\mathbf{X}$ in the following discussion are assumed to be compatible with the *P.L.* structure, and with the stratification.
Fix a perversity $\bar{p}$.

3.4.1 The constant coefficient case

Definition 3.9 Allowability. Let $\xi \in C_i(\mathbf{X})$. ξ is allowable (*w.r.t.* perversity $\bar{p}$) if

$$\dim(|\xi| \cap \mathbf{X}_{n-c}) \leq i - c + p_c, \text{for every } 2 \leq c \leq n. \tag{2}$$

Note that for 0-perversity, condition (2) states that a chain is allowable if and only if it is transverse to all of the strata of $\mathbf{X}$ and for a general perversity $\bar{p}$, each p_i is a measure of the extent to which chains are allowed to deviate from being transverse to $\mathbf{X}_{n-i}$.

Definition 3.10 The intersection homology chain-complex for a given perversity $\bar{p}$ is defined to be the subcomplex

$$IC_*^{\bar{p}}(\mathbf{X}) \subset C_*(\mathbf{X})$$

consisting of all allowable chains with allowable boundaries. The perversity $\bar{p}$ will henceforth be suppressed when the discussion applies to all perversities.

3.4.2 The twisted coefficient case

Given a local system $\mathcal{L}$ on $\mathbf{X}$, the geometric chain complex with twisted coefficients $C_*(\mathbf{X};\mathcal{L})$ is defined as in definition 3.6 and we could restrict to allowable chains with allowable boundaries to define the intersection homology chain complex with twisted coefficients. However, note that if $\xi \in C_i(\mathbf{X};\mathcal{L})$ is allowable then, in particular,

$$\dim(|\xi| \cap \Sigma) \leq i - 2 \tag{3}$$

whence for any triangulation of ξ, all of the i-simplices as well as all of their $(i-1)$-faces have their interiors disjoint from Σ. Thus we need not require that $\mathcal{L}$ be defined over all of $\mathbf{X}$, and in fact for intersection homology with twisted coefficients we require the local system $\mathcal{L}$ to be defined only over the non-singular set $\mathbf{X} \setminus \Sigma$. In so doing we obtain a much larger and more interesting class of local systems. The intersection homology chain complex with twisted coefficients is formally defined as follows :

Let $\mathcal{L}$ be a local system on $\mathbf{X} \setminus \Sigma$.

Definition 3.11 $\tilde{\mathbf{C}}_*(\mathbf{X};\mathcal{L})$. Let T be a triangulation of $\mathbf{X}$. Set

$$\tilde{C}_k^T(\mathbf{X};\mathcal{L}) = \{\text{formal sums} \textstyle\sum_{i=1}^m \alpha_\sigma \cdot \sigma \mid \sigma \in T,\ \dim(\sigma) = k,$$

$$\dim(\sigma \cap \Sigma) \leq k-1 \text{ and } \alpha_\sigma \in \mathbf{V}_\sigma\}.$$

Note: We do not (in fact - cannot) define an appropriate boundary map on $\tilde{C}_*^T(\mathbf{X};\mathcal{L})$ since for a general chain $\xi \in \tilde{C}_*^T(\mathbf{X};\mathcal{L})$, $\partial\xi$ could consist in part of chains ξ' which are contained in Σ. In particular, $\tilde{C}_*^T(\mathbf{X};\mathcal{L})$ is not a chain complex. Nevertheless, if T' is a refinement of T, then there is a canonical map from $\tilde{C}_*^T(\mathbf{X};\mathcal{L})$ to $\tilde{C}_*^{T'}(\mathbf{X};\mathcal{L})$, and thus we can

Define $\tilde{C}_*(\mathbf{X};\mathcal{L})$ to be the direct limit of the $\tilde{C}_*^T(\mathbf{X};\mathcal{L})$.

Note that although $\tilde{C}_*(\mathbf{X};\mathcal{L})$ is not a chain complex, any *allowable* chain (and in particular, any allowable elementary chain) in $\tilde{C}_*(\mathbf{X};\mathcal{L})$ has a well defined boundary (which nevertheless may be non allowable).

Definition 3.12 $\mathbf{IC}_*^{\bar{p}}(\mathbf{X};\mathcal{L})$. Let $\bar{p}$ be a perversity. Define

$$IC_k^{\bar{p}}(\mathbf{X};\mathcal{L}) = \{\xi \in \tilde{C}_k(\mathbf{X};\mathcal{L}) \mid \xi \text{ and } \partial\xi \text{ are } \bar{p}\text{-allowable}\}$$

Definition 3.13 $\mathbf{IH}_*^{\bar{p}}(\mathbf{X};\mathcal{L})$. Define $IH_*^{\bar{p}}(X;\mathcal{L})$ to be the homology of the complex $IC_*^{\bar{p}}(\mathbf{X};\mathcal{L})$.

3.4.3 A useful lemma for computing intersection homology

Lemma 3.14 Let $\tilde{D}_*(\mathbf{X};\mathcal{L}) \subset \tilde{C}(\mathbf{X};\mathcal{L})$ and suppose that for each i, there is a finite set of elementary geometric chains $\xi_1^i, \ldots, \xi_{n_i}^i$ so that

$$\tilde{D}_i(\mathbf{X};\mathcal{L}) = \{\sum_{j=1}^{n_i} \alpha_j \cdot \xi_j^i \mid \alpha_j \in \mathbf{V}_{\xi_j^i}\}$$

(where $\mathbf{V}_{\xi_j^i}$ is the fiber of $\mathcal{L}$ over the designated interior point of ξ_j^i).

Suppose further that $\tilde{D}_*(\mathbf{X};\mathcal{L})$ is closed under ∂_* *i.e.* that $\partial(\alpha \cdot \xi_j^i) \in \tilde{D}_{i-1}(\mathbf{X};\mathcal{L})$ whenever $\partial(\alpha \cdot \xi_j^i)$ is defined. (In fact it is enough to require this only when ξ_j^i is allowable).

For each i, let

$$ID_i(\mathbf{X};\mathcal{L}) = \tilde{D}_*(\mathbf{X};\mathcal{L}) \cap IC_i(\mathbf{X};\mathcal{L}).$$

$\tilde{D}_*(\mathbf{X};\mathcal{L})$ can be decomposed as :

$$\tilde{D}_i(\mathbf{X};\mathcal{L}) = \tilde{D}_i^A(\mathbf{X};\mathcal{L}) \oplus \tilde{D}_i^B(\mathbf{X};\mathcal{L})$$

where $\tilde{D}_i^A(\mathbf{X};\mathcal{L}) =$ chains of the form $\sum_j \alpha_j \cdot \xi_j^i$ with all ξ_j^i's allowable, and $\tilde{D}_i^B(\mathbf{X};\mathcal{L}) =$ chains of the form $\sum_j \alpha_j \cdot \xi_j^i$ with all ξ_j^i's being non-allowable.

Accordingly, the restriction

$$\partial_i|_{\tilde{D}_i^A(\mathbf{X};\mathcal{L})} : \tilde{D}_i^A(\mathbf{X};\mathcal{L}) \to \tilde{D}_{i-1}(\mathbf{X};\mathcal{L})$$

decomposes as $\partial^A \oplus \partial^B$ with $\mathrm{Im}\,(\partial^A) \subset \tilde{D}^A_{i-1}(\mathbf{X};\mathcal{L})$ and $\mathrm{Im}\,(\partial^B) \subset \tilde{D}^B_{i-1}(\mathbf{X};\mathcal{L})$.

Then

$$ID_i(\mathbf{X};\mathcal{L}) = \ker(\partial^B_i).$$

□

<u>**Why the lemma is useful :**</u> The early developers of homology theory in their attempts to compute what we now call the homology of a space tried to construct generating sets of cycles and describe the equivalence relations (homologies) among them. Finding the cycles was no easy task, nor was it easy to prove that one had found *all* of the cycles and all of the homology relations.
The discovery of simplicial homology theory dramatically simplified the problem. Namely, by cutting up the space into relatively simple pieces (simplices), and describing the realtions among them (the simplicial boundary), computing homology is reduced to computing kernels, images and quotients (all finite dimensional if the space is compact), tasks for which easy finite algorithms exist.

In the case of intersection homology one can frequently describe a finitely generated subcomplex $D_*(\mathbf{X}) \subset C_*(\mathbf{X})$ (resp. $D_*(\mathbf{X};\mathcal{L}) \subset \tilde{C}_*(\mathbf{X};\mathcal{L})$) and prove that its restriction to allowable chains with allowable boundaries in fact computes $IH_*(\mathbf{X})$ (resp.$IH_*(\mathbf{X};\mathcal{L})$). For example, in [3] it is shown that if a triangulation of $\mathbf{X}$ is fine enough (*e.g.* the barycentric subdivision of any triangulation), then the associated simplicial complex will do.
In such cases, it may be relatively easy to determine for each individual generator (*e.g.* simplex) whether it is allowable or not, but significantly more difficult to explicitly compute the subcomplex $ID_*(\mathbf{X})$ (resp $ID_*(\mathbf{X};\mathcal{L})$). Lemma 3.14 asserts that the latter task can be "left to elementary linear algebra".
Section 5 deals precisely with such a construction for the computation of the intersection homology of a product of disks (relative to the boundary).

3.5 Local sytems on toric varieties

For a toric variety $\mathbf{X}$,

$$\mathbf{X} \setminus \Sigma = \mathrm{int}(\mathcal{P}) \times \mathcal{T}^n$$

which is homotopy equivalent to $\mathcal{T}^n$. Thus any local system on $\mathbf{X} \setminus \Sigma$ uniquely determines (by restriction) a local system on $\mathcal{T}^n$ and any local system on $\mathcal{T}^n$ determines (by trivial extension) a local system on $\mathbf{X} \setminus \Sigma$ which is unique up to equivalence of local systems.

Equivalent local systems on any stratified pseudomanifold produce (canonically) isomorphic intersection homology groups, and thus we may assume that any local system $\mathcal{L}$ on a toric variety $\mathbf{X}$ is the trivial extension to $\mathbf{X} \setminus \Sigma$ of a local system on $\mathcal{T}^n$ which, for simplicity, we also denote by $\mathcal{L}$.
Thus to specify a local system for intersection homology on a toric variety $\mathbf{X}$ it suffices to specify a local system on $\mathcal{T}^n$ or, in view of remark 3.2, a representation of $\mathbb{Z} = \pi_1(\mathcal{T}^n, t_0)$ on $\mathbf{V}$. Explicitly, we must specify:
(i) The fiber $\mathbf{V}$,
(ii) a base point t_0 of $\mathcal{T}^n$,
(iii) a basis $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ of $\pi_1(\mathcal{T}^n, t_o)$, and
(iv) n commuting monodromies $T_1, \ldots, T_n \in Aut(\mathbf{V})$ corresponding (respectively) to the $\mathcal{C}_i$'s.

3.6 Combing and Coning

3.6.1 Combing

Let $\mathbf{X}$ be a stratified pseudomanifold with strata $\{S_\alpha\}$ and with base point $x \in \mathbf{X} \setminus \Sigma$. Let (S^1, s) denote a circle with base point s. $S^1 \times \mathbf{X}$ is a pseudomanifold (stratified with strata $\{S^1 \times S_\alpha\}$). Let $\mathcal{L}'$ be a local system with fibre $\mathbf{V}$ on $\{s\} \times (\mathbf{X} \setminus \Sigma)$. Let $T \in Aut(\mathbf{V})$ and let $\mathcal{L}''$ be the local system on S^1 with fibre $\mathbf{V}_s = \mathbf{V}$ and monodromy $T : \mathbf{V}_s \to \mathbf{V}_s$ associated to one of the two generators of $\pi_1(S^1, s)$. Also let $s' \in S^1$ be a point "nearby" s and assume that $\mathcal{L}''$ is constant on some connected subset of S^1 containing s and s'. Let $\mathcal{L}$ be the (unique) local system on $S^1 \times \mathbf{X}$ such that $\mathcal{L}|_{\{s\}\times\mathbf{X}} = \mathcal{L}'$ and $\mathcal{L}|_{S^1\times\{x\}} = \mathcal{L}''$.

Identify in the obvious way $\mathbf{X}$ and $\{s\} \times \mathbf{X}$ and let $\alpha \cdot (\xi, \nabla\xi) \in \tilde{C}_i(\mathbf{X}, \mathcal{L}')$ be an elementary chain. Consider the "suspension" of $\alpha \cdot \xi$

$$\overline{susp}(\alpha \cdot \xi) \in \tilde{C}_{i+1}(S^1 \times \mathbf{X}; \mathcal{L})$$

which is defined to be the chain $\alpha \cdot \xi'$ satisfying :
(i) $\xi' = S^1 \times \xi$,
(ii) $\nabla\xi' = (S^1 \times \nabla\xi) \cup \{s\} \times \xi$,
(iii) $x_{\xi'} = (s', x_\xi)$.

(Note that we are identifying the fibers over $\{s\} \times x_\xi$ and $\{s'\} \times x_\xi$).

$\overline{susp}(\alpha \cdot \xi)$ is an elementary chain in $\tilde{C}_{i+1}(S^1 \times \mathbf{X}; \mathcal{L})$.

For any $x \in \mathbf{X}$, denote by $T_x : \mathbf{V}_{(s,x)} \to \mathbf{V}_{(s,x)}$ the monodromy corresponding to the path which goes once around $S^1 \times \{x\}$ (in the direction of the orientation of S^1).

Remark 3.15
With ξ and α as above,

$$\partial(\overline{susp}(\alpha \cdot \xi)) = (T_{x_\xi} - I)(\alpha) \cdot \xi - \overline{susp}(\partial(\alpha \cdot \xi)).$$

A simple dimension check gives :

Lemma 3.16
$\overline{susp}(\alpha \cdot \xi)$ is allowable $\iff$ $\alpha \cdot \xi$ is.

Combining remark 3.15 with lemma 3.16 we have :

Corollary 3.17
a. $\overline{susp}$ maps $IC_*(\mathbf{X}; \mathcal{L}')$ to $IC_{*+1}(S^1 \times \mathbf{X}; \mathcal{L})$, and
b. The composition

$$IC_*(\mathbf{X}; \mathcal{L}') \xrightarrow{\overline{susp}} IC_{*+1}(S^1 \times \mathbf{X}; \mathcal{L}) \xrightarrow{q} IC_{*+1}(S^1 \times \mathbf{X}, \{s\} \times \mathbf{X}; \mathcal{L}) \qquad (4)$$

is a chain map (q is the obvious quotient map).

In [4] it is proved that a map analogous to the suspension map defined above induces an isomorphism

$$IH_*(\mathbf{X}) \xrightarrow{\cong} IH^{BM}_{*+1}(\mathbf{R} \times \mathbf{X}) \qquad (5)$$

for any sratified pseudomanifold $\mathbf{X}$, where IH^{BM}_* denotes the Borel-Moore intersection homology (in which geometric chains are not required to have compact support).

The name **combing** is derived from the method of proof of (5). A cycle ξ_k in $\mathbf{R} \times \mathbf{X}$ is "straightened out" by a process similar to combing a head of hair. The chain is cut transversely with $\{t\} \times \mathbf{X}$ for some suitably chosen $t \in \mathbf{R}$, and the intersection is extended ("combed left and right") giving a chain of the form $\mathbf{R} \times \xi'_{k-1}$.

A close examination of the proof of (5) shows that it carries over in its entirety when the local system $\mathcal{L}'$ on $\mathbf{X}$ is introduced and extended trivially to a local system on $\mathbf{R} \times \mathbf{X}$ which we denote by $\mathbf{R} \times \mathcal{L}'$, and we obtain an isomorphism

$$IH_*(\mathbf{X}; \mathcal{L}') \xrightarrow{\cong} IH^{BM}_{*+1}(\mathbf{R} \times \mathbf{X}; \mathbf{R} \times \mathcal{L}'). \qquad (6)$$

However, the homology groups on the right hand side of (6) are canonically isomorphic to $IH_{*+1}(S^1 \times \mathbf{X}, \{s\} \times \mathbf{X}; \mathcal{L})$ proving

Lemma 3.18 The combing lemma. $q\circ\overline{susp}$ induces an isomorphism

$$IH_*(\mathbf{X};\mathcal{L}') \xrightarrow{\cong} IH_{*+1}(S^1\times\mathbf{X},\{s\}\times\mathbf{X};\mathcal{L}).$$

□

Corollary 3.19
Let $ID_*(\{s\}\times\mathbf{X};\mathcal{L}')$ be a subcomplex of $IC_*(\{s\}\times\mathbf{X};\mathcal{L}')$ and suppose that the inclusion

$$i: ID_*(\{s\}\times\mathbf{X};\mathcal{L}') \hookrightarrow IC_*(\{s\}\times\mathbf{X};\mathcal{L}')$$

induces an isomorphism on homology.
Then the inclusion of the subcomplex $ID_*(\{s\}\times\mathbf{X};\mathcal{L}')\oplus\overline{susp}(ID_*(\{s\}\times\mathbf{X};\mathcal{L}'))$ into $IC_*(S^1\times\mathbf{X};\mathcal{L})$ induces an isomorphism on homology. (Note that by remark 3.15 the direct sum is in fact a subcomplex).

Proof: The proof follows from a simple spectral sequence argument by filtering both complexes as

$$IC_*(\{s\}\times\mathbf{X};\mathcal{L}')\subset IC_*(S^1\times\mathbf{X};\mathcal{L})$$

and

$$ID_*(\{s\}\times\mathbf{X};\mathcal{L}')\subset ID_*(\{s\}\times\mathbf{X};\mathcal{L}')\oplus\overline{susp}(ID_*(\{s\}\times\mathbf{X};\mathcal{L}'))$$

and noting that the hypothesis on $ID_*(\{s\}\times\mathbf{X};\mathcal{L}')$ combined with the combing lemma imply that the inclusion i induces an isomorphism on the E^1 terms of the spectral sequences corresponding to these filtrations. □

Using the monodromies T_x as defined above for every point $x\in\mathbf{X}$, we obtain a chain map

$$T_\#: IC_*(\{s\}\times\mathbf{X};\mathcal{L}')\to IC_*(\{s\}\times\mathbf{X};\mathcal{L}').$$

which maps an elementary chain $\alpha\cdot\xi$ to $T_{x_\xi}(\alpha)\cdot\xi$.
(It is easy to verify that $T_\#$ is in fact a chain map and that it preserves allowability since it does not change the support of a chain).
Denoting the induced map on homology by T_* we have :

Corollary 3.20 For all $k\geq 0$

$$\begin{aligned} IH_k(S^1\times\mathbf{X};\mathcal{L}) &\cong \operatorname{coker}(\,(T_*-I): IH_k(\mathbf{X};\mathcal{L}')\to IH_k(\mathbf{X};\mathcal{L}')\,)\\ &\bigoplus \ker(\,(T_*-I): IH_{k-1}(\mathbf{X};\mathcal{L}')\to IH_{k-1}(\mathbf{X};\mathcal{L}')\,). \end{aligned}$$

□

3.6.2 A Z-action

As a final remark concerning the combing lemma we note that with all the definitions as above, there is a $\mathbf{Z}$-action on each fiber $\mathbf{V}_x$ for any $x \in \mathbf{X}$ given by

$$z \cdot \alpha = (T_x)^z(\alpha). \tag{7}$$

This induces a $\mathbf{Z}$-action on elementary chains and consequently on $IC_*(\{s\} \times \mathbf{X}; \mathcal{L}')$. We shall refer to this $\mathbf{Z}$-action in later sections after addressing certain issues concerning resolutions of $\mathbf{Z}$ over $\mathbf{Z}[\mathbf{Z}^n]$ in section 6.

3.6.3 Coning

Let $\mathbf{Y}$ be an $(m-1)$-dimensional stratified pseudomanifold and let $c(\mathbf{Y})$ denote the topological cone on $\mathbf{Y}$. Define a map

$$coning : IC_*(\mathbf{Y}) \longrightarrow C_{*+1}(c(\mathbf{Y}))$$

which maps a chain $\psi = \sum \alpha_\xi \cdot \xi$ to $\sum \alpha_\xi \cdot c(\xi)$ where $c(\xi)$ denotes the cone on ξ. Let $\bar{p}$ be a perversity. In [4] it is proved that $coning(\psi) \in IC_{*+1}(\mathbf{Y})$ if and only if $\dim(\psi) \geq m - \bar{p}_m$ or ψ is an $(m - \bar{p}_m - 1)$-dimensional cycle. It is then proved that the composition

$$IC_{\tau * \geq r}(\mathbf{Y}) \xrightarrow{coning} IC_{*+1}(c(\mathbf{Y})) \xrightarrow{q} IC_{*+1}(c(\mathbf{Y}), \mathbf{Y}) \tag{8}$$

induces an isomorphism on homology, where q is the obvious quotient map, $r = m - \bar{p}_m - 1$ and $\tau * \geq k$ is the *truncation* operator defined for any chain complex $\mathbf{C}_*$ and non negative integer k by :

$$(\mathbf{C}_{\tau * \geq k})_i = \begin{cases} \mathbf{C}_i & \text{if } i > k \\ \text{the } i\text{-cycles of } \mathbf{C}_* & \text{if } i = k \\ 0 & \text{if } i < k \end{cases} \tag{9}$$

Once again, the assertion (and proof) continue to hold when a local system is introduced. Explicitly -

Lemma 3.21 The coning lemma. Let $\mathbf{X}$ be an $(m-1)$-dimensional stratified pseudomanifold, $\mathcal{L}'$ a local system on $\mathbf{X} \setminus \Sigma$ and $\mathcal{L}$ is its trivial extension to $c(\mathbf{X}) \setminus c(\Sigma)$. Let $\bar{p}$ be a perversity and set $r = m - \bar{p}_m - 1$. then the composition

$$IC_{\tau * \geq r}(\mathbf{X}; \mathcal{L}') \xrightarrow{coning} IC_{*+1}(c(\mathbf{X}); \mathcal{L}) \xrightarrow{q} IC_{*+1}(c(\mathbf{X}), \mathbf{X}; \mathcal{L}) \tag{10}$$

induces an isomorphism on homology.

□

4 The homology of a torus

In this section we look at (literally...) a specific CW-decomposition of the n-torus $\mathcal{T}^n$ for $n = 1, 2$ and 3. We construct in each case the associated chain complex with local coefficients, and from these we are able to deduce the form of the analogous chain complex for arbitrary n which we denote by $C_*^{CW}(\mathcal{T}^n; \mathcal{L})$.
The discussion in this section is not rigorous but rather is meant as a description of an intuitive, geometric approach to a computation of homology with local coefficients. The complex $C_*^{CW}(\mathcal{T}^n; \mathcal{L})$ is precisely the Koszul complex for $H_*(\mathbb{Z}^n, \mathcal{L}) \cong H_*(\mathcal{T}^n, \mathcal{L})$. Section 7.1 describes an explicit chain-homotopy equivalence between $C_*^{CW}(\mathcal{T}^n; \mathcal{L})$ and the bar complex $\bar{W}(\mathbb{Z}^n; \mathcal{L})$.
Fix $n > 0$ and let

$$proj : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n = \mathcal{T}^n$$

be the canonical projection.
For any $1 \le i \le n$ let $e_i = (0, 0, \dots, 1, 0, \dots, 0)$ (1 in the i^{th} position) and let $\tilde{\mathcal{C}}_i$ denote the directed linear path in $\mathbb{R}^n$ from the origin to e_i. Set $\mathcal{C}_i = proj(\tilde{\mathcal{C}}_i)$.
For any set $J = \{j_1, \dots, j_k\} \subseteq \{0, 1, \dots, n\}$ $(k \ge 1)$, denote

$$\mathcal{C}_J = \mathcal{C}_{j_1} \times \dots \times \mathcal{C}_{j_k},$$

and let $\mathcal{C}_\emptyset = proj(\mathbb{Z}^n)$ be the base point of the torus.

We think of the torus $\mathcal{T}^n$ as the unit cube in $\mathbb{R}^n$ with opposite faces identified. The images (under the identification) of these pairs of faces constitute the cells in a standard CW-decomposition of the torus. The closures of these cells are precisely the sets $\mathcal{C}_J$ described above. Consequently, the corresponding chain complex will have one copy of $\mathbf{V}$ corresponding to each $\mathcal{C}_J$. We denote each such copy of $\mathbf{V}$ by $\mathbf{V}_{j_1 \dots j_k}$ to indicate which cell it is associated to.

Note that the $\mathcal{C}_i$'s $(1 \le i \le n)$ form a basis of $\pi_1(\mathcal{T}^n, \mathcal{C}_\emptyset)$.

Let $\mathcal{L}$ be a local system with fiber $\mathbf{V}$ on the torus, and for each $1 \le i \le n$, let $T_i \in Aut(\mathbf{V})$ be the monodromy 'around' $\mathcal{C}_i$.

<u>n = 1; The homology of a circle.</u>
In figure 7 the 1-torus $\mathcal{C}_1$ is shown both as an interval with the endpoints identified and as a circle.

WOLOG, we may assume that the local system is constant away from the checkered area since the space with the checkered area removed is simply-connected (in fact

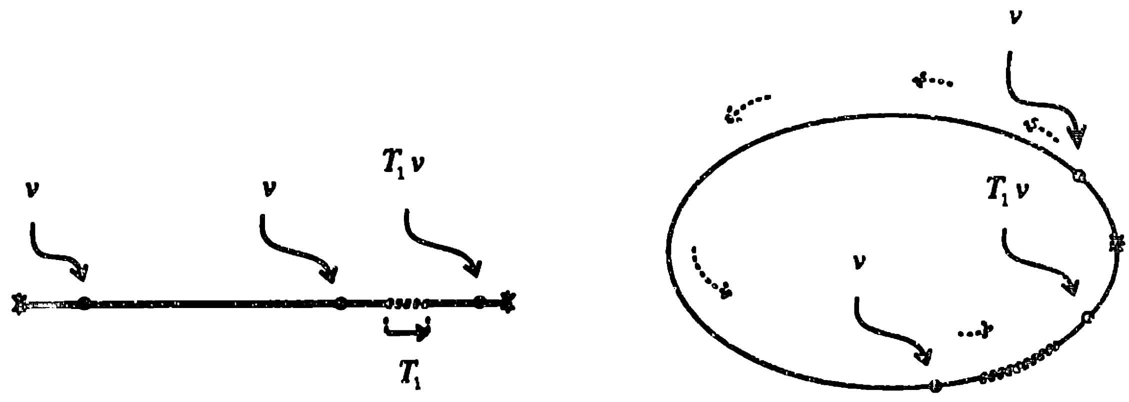

Figure 7: $\mathcal{C}_1$ ($= S^1$)

- contractible). Thus we assume that the monodromy T_1 occurs only when passing through this area.
There are only two cells, and therefore

$$C_i^{CW}(\mathcal{C}_1;\mathcal{L}) = \begin{cases} \mathbf{V} & \text{if } i = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

The only nontrivial boundary map is ∂_1, and it is clear from the picture that

$$\partial_1(v) = T_1 v - v, \qquad \forall v \in \mathbf{V}_1.$$

Thus $C_i^{CW}(\mathcal{C}_1;\mathcal{L})$ has the form

$$\begin{array}{cc} & 0 \\ & \downarrow \\ 2 & 0 \\ & \downarrow \\ 1 & \mathbf{V}_1 \\ & \downarrow {\scriptstyle T_1 - I} \\ 0 & \mathbf{V}_0 \end{array}$$

<u>n = 2.</u>
In figure 8 we represent the 2-torus $T^2 = \mathcal{C}_1 \times \mathcal{C}_2$ by a square with opposite faces identified.

Once again we may assume that $\mathcal{L}$ is constant away from the gray strips, and that the monodromies occur only when passing through the gray strips as in the figure. Note that each of the 1-cells has the same structure as the 1-torus in the previous case (but each with a different monodromy). Again by counting the cells we have :

$$C_i^{CW}(\mathcal{C}_1 \times \mathcal{C}_2;\mathcal{L}) = \begin{cases} \mathbf{V} & \text{if } i = 0,2 \\ \mathbf{V} \oplus \mathbf{V} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

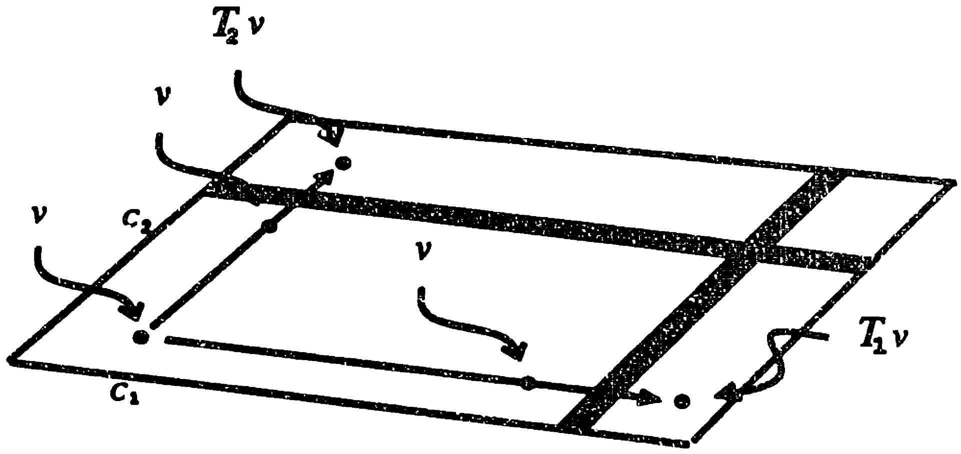

Figure 8: $\mathcal{T}^2 = C_1 \times C_2$

∂_1 is computed seperately for each of the 1-cells precisely as in the 1-dimensional example above, whereas for any $v \in \mathbf{V}_2$,

$$\partial_2(v) = (\,-(T_2 v - v), T_1 v - v\,)$$

where the negative sign in the first component is due to orientations.
Thus $C_*^{CW}(\mathcal{T}^2; \mathcal{L})$ is as shown in figure 9.

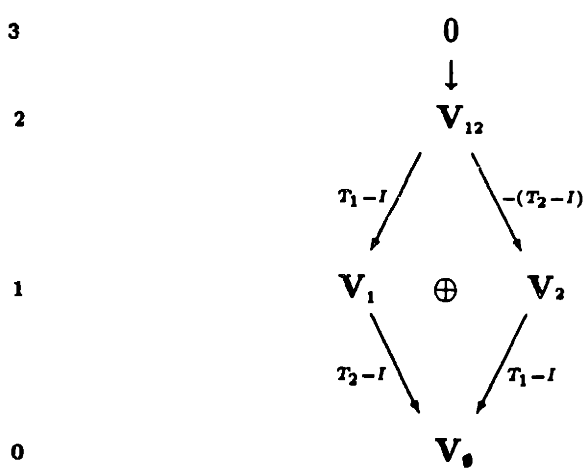

Figure 9: $C_*^{CW}(\mathcal{T}^2; \mathcal{L})$. $\mathbf{V}_J$ denotes the copy of $\mathbf{V}$ associated to the cell $\prod_{j \in J} C_j$ (for all $J \subseteq \{1, 2\}$).

n = 3.
The analogous picture of the 3-torus is shown in figure 10.

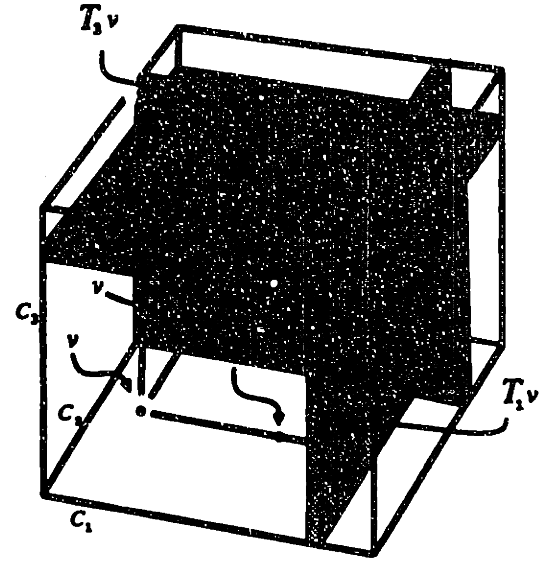

Figure 10: $\mathcal{T}^3 = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$

Note that each of the 2-cells is equivalent (up to a change in the monodromies) to the 2-dimensional example above as seen in figure 11.

Thus, by considerations similar to those in the previous example, we obtain the complex $C_*^{CW}(\mathcal{T}^3;\mathcal{L})$ (figure 12).

Remark 4.1 Note that $\mathcal{T}^3 = \mathcal{T}^2 \times \mathcal{C}_3$. Furthermore, each cell in $\mathcal{T}^3$ is either of the form

$$\mathcal{C}_J \times \mathcal{C}_\emptyset$$

or of the form

$$\mathcal{C}_J \times \mathcal{C}_3$$

for some cell $\mathcal{C}_J$ in $\mathcal{T}^2$. Algebraically this fact can be seen in the chain complex $C_*(\mathcal{T}^3;\mathcal{L})$, as it consists of two copies of $C_*(\mathcal{T}^2;\mathcal{L})$ with one shifted up by 1 in degree. In the language of section 3.6 we can simply state that

$$C_*^{CW}(\mathcal{T}^n;\mathcal{L}) = C_*^{CW}(\mathcal{T}^2;\mathcal{L}|_{\mathcal{T}^2}) \oplus \overline{susp}(C_*^{CW}(\mathcal{T}^2;\mathcal{L}|_{\mathcal{T}^2})).$$

The general case.
From these examples one is led to the construction of $C_*^{CW}(\mathcal{T}^n;\mathcal{L})$ for general n,

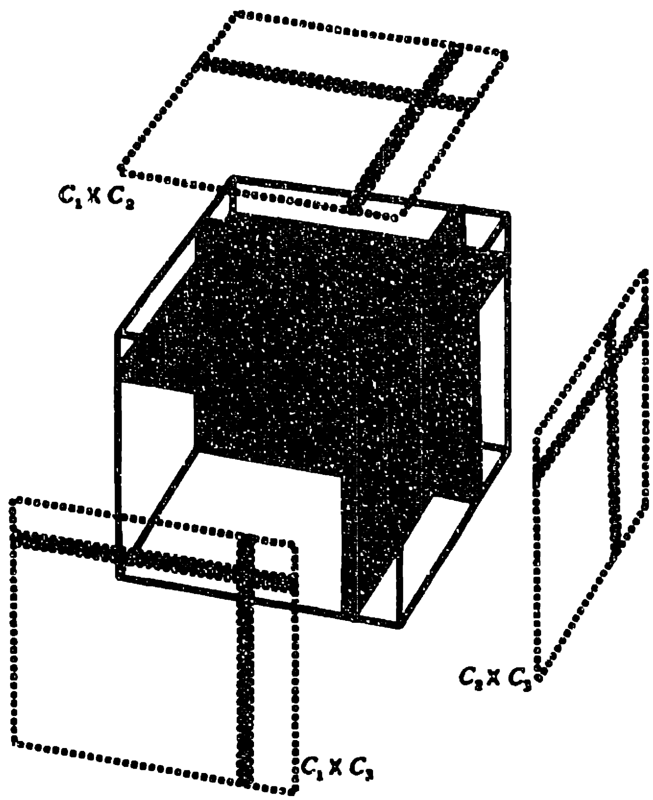

Figure 11: The 2-faces of $\mathcal{T}^3$.

namely - $C_*^{CW}(\mathcal{T}^n;\mathcal{L})$ has the form of an exterior algebra on n generators so that for each $0 \le k \le n$,

$$C_k^{CW}(\mathcal{T}^n;\mathcal{L}) = \bigoplus_{\substack{J\subseteq\{1,\ldots,n\}\\ |J|=k}} \mathbf{V}_K$$

and the boundary maps for each $J = \{j_1,\ldots,j_k\}$ are given by

$$\partial^J = \sum_{i=1}^{k}(-1)^i \partial^J_{j_i}$$

where

$$\partial^J_{j_i} \;=\; T_{j_i} - I \;:\mathbf{V}_J \longrightarrow \mathbf{V}_{\{j_1,\ldots,\hat{j_i},\ldots,j_k\}}.$$

Furthermore,

$$C_*^{CW}(\mathcal{T}^n;\mathcal{L}) = C_*^{CW}(\mathcal{T}^{n-1};\mathcal{L}|_{\mathcal{T}^{n-1}}) \oplus \overline{susp}(C_*^{CW}(\mathcal{T}^{n-1};\mathcal{L}|_{\mathcal{T}^{n-1}}). \qquad (11)$$

4.1 A final note regarding CW-decompositions of the torus

The e_i's from the beginning of the section which determined the basis of $\pi_1(\mathcal{T}^n,\mathcal{C}_\emptyset)$ can be replaced by any other basis $(e'_1,\ldots,e'_n)$ of $\mathbf{Z}^n$, thus determining a different

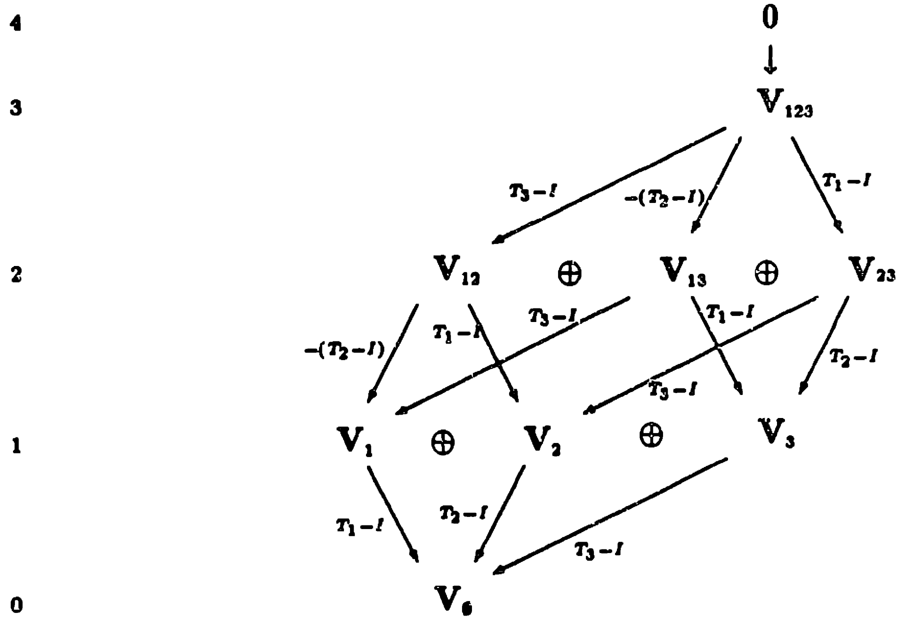

Figure 12: $C_*^{CW}(\mathcal{T}^3;\mathcal{L})$

basis $\{C_1',\ldots,C_n'\}$ of $\pi_1(\mathcal{T}^n,\mathcal{C}_\emptyset)$, a different set of monodromies $T_1',\ldots,T_n'$ and a different cell decomposition of the torus, all of which form a different representation of the same torus with the same local system.

For example - any simplicial n-dimensional cone K in a cone complex $\mathcal{K}\subset\mathbf{R}^n$ for which $|\det(M(K))|=1$, determines such a representation.

5 The intersection homology of a product of disks

In example 2.9 we described a cone complex whose associated toric variety $\mathbf{X}$ is an n-fold product of 2-spheres $\Sigma S^1 \times \ldots \times \Sigma S^1$ (Recall that ΣS^1 denotes $S^1 \times [0,1]/\sim$, where $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$, $\forall x, x' \in S^1$), and for which the underlying polytope $\mathcal{P}$ was an n-cube. We later proved that the inverse image (under p) of a neighborhood of a vertex of $\mathcal{P}$ is homeomorphic to an n-fold product of 2-disks $\mathbf{Y}$, and inherits a natural stratification as a subspace of $\mathbf{X}$. In this section we use this construction, along with the chain complex of section 4 to construct a chain complex which computes the intersection homology with twisted coefficients of a product of disks (relative to the boundary).

We begin with an intuitive description, since the fundamental geometric ideas of this construction are quite simple, whereas when done formally, the beauty of the intuitive forest is somewhat obscured by the denseness of the trees of rigor.

5.1 The intuitive description

Let v be the given vertex of the cube $\mathcal{P}$. Let Q be the closed star of v in the barycentric subdivision of $\mathcal{P}$. The face structure of the cone dual to v induces a cell structure on the n-torus $\mathcal{T}^n$. Corresponding to this cell structure, we obtain, using the methods of section 4, a chain complex which computes the homology of the n-torus. The product of this chain complex with the simplicial complex Q yields a geometric chain complex on $Q \times \mathcal{T}^n$. Each chain ξ in this complex projects to a geometric chain $p(\xi)$ in $\mathbf{Y}$ and thus the whole chain complex projects to a geometric chain complex on $\mathbf{Y}$. The restriction of the projected complex to allowable chains with allowable boundaries is shown to compute the desired intersection homology.

The combinatorial nature of this construction enables us to give a combinatorial description of the entire chain complex, as well as a combinatorial algorithm for determining allowability of each chain.

5.2 The formal construction

Denote $\Sigma^{(-)}S^1 = \{[(x,t)] \in \Sigma S^1 : t \leq \frac{1}{2}\}$. $\Sigma^{(-)}S^1$ is the cone $cS^1 \cong D^2$ (the closed 2-disk). Set $\mathbf{X} \supset \mathbf{Y} = \Sigma^{(-)}S^1 \times \ldots \times \Sigma^{(-)}S^1$ (n terms). $\mathbf{Y}$ is stratified in the obvious way, namely $\mathbf{Y}_{2k} = \mathbf{Y} \cap \mathbf{X}_{2k}$. If each disk is stratified with one 0-stratum (at the apex of the cone) and one 2-stratum, then this stratification of $\mathbf{Y}$ is simply the natural product stratification.

The interior of the "non-singular" part of $\mathbf{Y}$, $\mathbf{Y} \setminus \mathbf{Y}_{2n-2}$ is a complex n-torus. Thus, as is the case for the toric variety $\mathbf{X}$, a local system $\mathcal{L}$ for intersection homology on $\mathbf{Y}$ consists of a local system on the torus, namely of a vector space $\mathbf{V}$ over $\mathbb{Q}$, and n commuting monodromies: $T_1, \ldots, T_n \in Aut(\mathbf{V})$. Each T_i is the monodromy "around" the boundary of the $i^{\underline{th}}$ disk.

Recall that the underlying polytope $\mathcal{P}$ is an n-cube $I_1 \times \ldots \times I_n$, where $\forall j,\ I_j = [0,1]$. Set $I_j^{(-)} = [0, \frac{1}{2}]$, and set $\mathcal{Q} = \prod_{j=1}^{n} I_j^{(-)}$. Then $\mathbf{Y} = p^{-1}(\mathcal{Q})$.
For each j, we denote by D_j the disk $p^{-1}(\{0\} \times \{0\} \times \ldots \times I_j^{(-)} \times \ldots \times \{0\})$ and by C_j its boundary $p^{-1}(0,0,\ldots,\frac{1}{2},0,\ldots,0)$. Thus in fact $\mathbf{Y} = D_1 \times \ldots \times D_n$, and "the torus in the middle" is $C_1 \times \ldots \times C_n$.
We use the following convention for indexing k-fold subproducts of any of the various n-fold products mentioned above : Let [n] denote the set $\{1, \ldots, n\}$ and let $J \subseteq [\mathrm{n}]$. Then I_J denotes the product $\prod_{j \in J} I_j$ and likewise for all other products. We denote by J^c the complement of J, $[\mathrm{n}] \setminus J$.
We index 0-fold products by $\emptyset$. Thus $I_\emptyset = I_\emptyset^{(-)} =$ the point $(0,0,\ldots,0)$ which we denote by v, and $D_\emptyset = p^{-1}(v) = C_\emptyset =$ "the" base point of $C_{[\mathrm{n}]} = \mathcal{T}^n$.
Note that over points in I_J , $C_{[\mathrm{n}]}$ gets collapsed to $C_{[\mathrm{n}]}/C_{J^c}$ which is canonically isomorphic to C_J.

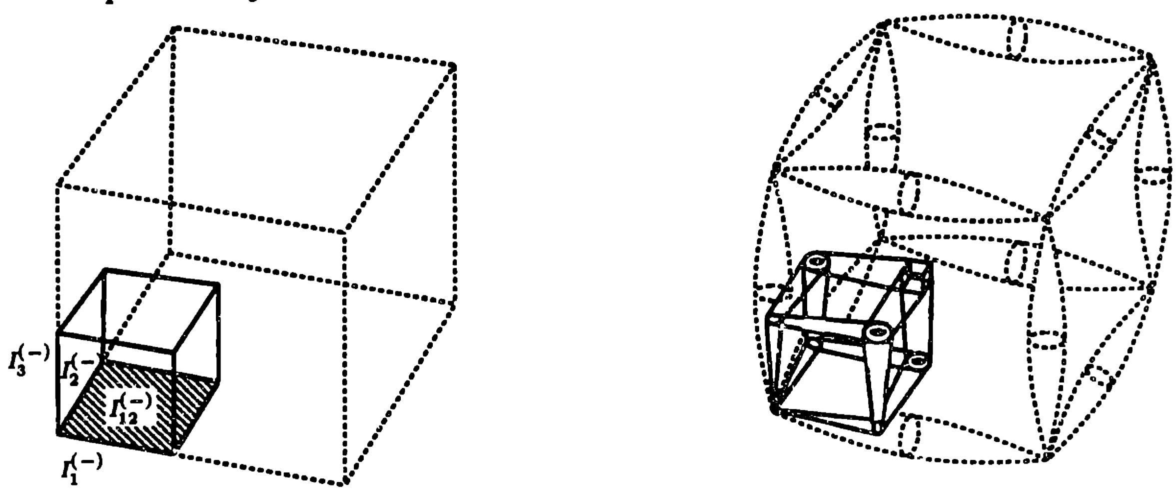

Figure 13: $n = 3$: $\mathcal{Q} \subset \mathcal{P}$ and $\mathbf{Y} = D^2 \times D^2 \times D^2 \subset S^2 \times S^2 \times S^2$

For any $J \subseteq [\mathrm{n}]$, let $\mathcal{L}_J = \mathcal{L}|_{C_J}$ and let $C_\bullet^{CW}(C_J; \mathcal{L}_J)$ be the chain complex for the homology of the torus C_J, as in section 4. Denote the associated boundary maps of this complex by $\partial_\bullet^J$. Recall that the k-chains in this complex have the form $\sum(\alpha_K \cdot C_K)$ where $\alpha_K \in \mathbf{V}$ and K runs over all the subsets of J of order k.

Let $sd(\mathcal{P})$ be the barycentric subdivision of $\mathcal{P}$, suitably oriented, and let $[\sigma, \sigma'] \in \{0, 1, -1\}$ denote the appropriate incidence number for two simplices $\sigma, \sigma' \in sd(\mathcal{P})$. $\mathcal{Q}$ is triangulated by a subcomplex of $sd(\mathcal{P})$, and we shall henceforth identify it with

this subcomplex.
Each m-dimensional simplex $\sigma \in \mathcal{Q}$ corresponds to a flag of faces of $\mathcal{P}$ of the form $I_{J_0} \subset I_{J_1} \subset \ldots \subset I_{J_m}$, where $J_0 \subset J_1 \subset \ldots \subset J_m \subseteq [n]$.
Set $J(\sigma) = J_m$.
Note : If $\sigma \in \mathcal{Q}$ corresponds to the flag $J_0 \subset J_1 \subset \ldots \subset J_m$, then $\partial\sigma$ is made up of the simplices $\sigma^1, \ldots, \sigma^m$ where $\sigma^i = \partial^i\sigma$ corresponds to the flag $J_0 \subset \ldots \subset J_{i-1} \subset J_{i+1} \subset \ldots \subset J_m$. Thus for all $i < m$, $J(\sigma^i) = J(\sigma) = J_m$, and correspondingly, if $J(\sigma) = [n]$, then σ^m is the only face of σ which is contained in $\partial\mathcal{P}$ (see figure 14).

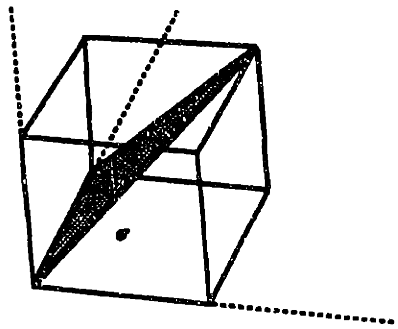

Figure 14: $\sigma \in \mathcal{Q}$ corresponds to the flag $I_\emptyset \subset I_2 \subset I_{\{1,2,3\}} = I_{[3]}$.

Next we define a geometric chain complex (with twisted coefficients) on $\mathcal{Q} \times T^n$, which we will then project down to $\mathbf{Y}$.

Definition 5.1 Denote $\hat{\mathbf{Y}} = \mathcal{Q} \times T^n$.
Define $\hat{D}_\bullet(\hat{\mathbf{Y}};\mathcal{L}) \subset C_\bullet(\hat{\mathbf{Y}};\mathcal{L})$ to be the product of the simplicial complex $\mathcal{Q}$ with the CW-complex $C_\bullet^{CW}(C_{[n]};\mathcal{L})$.
Explicitly - Let σ be a simplex in $\mathcal{Q}$ and $K \subseteq [n]$. Let $\hat{\xi} = \sigma \times C_K$.
By intentional abuse of notation, if $K = \{i_1, \ldots, i_{|K|}\}$ then for $1 \leq j \leq |K|$, denote

$$\partial^j(C_K) = C_1 \times \ldots \times C_{j-1} \times C_{j+1} \times \ldots \times C_{|K|},$$

and denote the $(|K|-1)$-skeleton of C_K by

$$\partial C_K = \bigcup_{j=1}^{|K|} \partial^j(C_K).$$

Set $\hat{\xi}' = (\partial\sigma \times C_K \cup \sigma \times \partial C_K)$.
Note : $\hat{\xi}'$ is a *codimension*-1 subspace of $\hat{\xi}$ and $\hat{\xi} \setminus \hat{\xi}'$ is contractible and hence

is simply connected. Thus if $\mathbf{V}_{\hat\xi} = \mathbf{V}_{x_{\hat\xi}}$ is the fiber of $\mathcal{L}$ over some designated point $x_{\hat\xi} \in \hat\xi \setminus \hat\xi'$ and $\alpha \in V_{\hat\xi}$, then the chain $\alpha \cdot \hat\xi \in C_*(\hat{\mathbf{Y}}, \mathcal{L})$ is unambigously defined.

$\hat{D}_i(\hat{\mathbf{Y}}; \mathcal{L})$ consists of all sums of chains $\alpha \cdot \hat\xi$ where $\hat\xi = \sigma \times C_K$ and $\dim(\sigma) + |K| = i$. $\partial(\alpha \cdot \hat\xi)$ is a local-coefficient geometric chain whose support is in ξ'. In fact, if $\hat\xi = \alpha \cdot C_K \in \hat{D}_i(\hat{\mathbf{Y}}; \mathcal{L})$ then

$$\partial(\hat\xi) = \underbrace{\sum_{i=0}^{m} [\sigma; \sigma^i] \alpha \cdot (\sigma^i \times C_K)}_{\text{"topological" boundary}} + \underbrace{\sum_{j=1}^{|K|} (-1)^{j-1} (T_j \alpha - \alpha) \cdot (\sigma \times \partial^j(C_K))}_{\text{boundary attributed to the local sytem}}. \qquad (12)$$

Thus $\partial(\hat\xi) \in \hat{D}_{i-1}(\hat{\mathbf{Y}}; \mathcal{L})$, so that $\hat{D}_*(\hat{\mathbf{Y}}; \mathcal{L})$ is a chain complex.

Now, with $\hat\xi$ and $\hat\xi'$ as above, set $\xi = \hat{p}(\hat\xi)$ and $\xi' = \hat{p}(\hat\xi')$. ξ' is a *codimension*-1 subspace of ξ and $\xi \setminus \xi'$ is contractible.
If $J(\sigma) \neq [\mathrm{n}]$ (*i.e.* $\sigma \subset \partial\mathcal{P}$) then $\xi \subset \mathbf{Y}_{2n-2}$ and hence is not allowable. Otherwise, $\xi \setminus \xi' \subset \mathbf{Y} \setminus \mathbf{Y}_{2n-2}$, and thus $x_\xi = \hat{p}(x_{\hat\xi}) \in \mathbf{Y} \setminus \mathbf{Y}_{2n-2}$ whence $\mathbf{V}_\xi = \mathbf{V}_{x_\xi}$ is defined, so that for $\alpha \in \mathbf{V}_\xi$ we can define unambiguously the chain $\alpha \cdot \xi$. (Here $\mathbf{V}_\xi$ denotes the fiber of $\mathcal{L}$ over x_ξ). Thus, for any $\xi = \hat{p}(\sigma \times C_K)$, $\alpha \cdot \xi$ is defined whenever $J(\sigma) = [\mathrm{n}]$, even though ξ might not be allowable.
Note that in order for a chain $\xi = \hat{p}(\sigma \times C_K)$ to be allowable we must have that $\dim(\xi \cap \mathbf{Y}_{2n-2}) \leq \dim(\xi) - 2$, which implies in particular :
(*i*) $J(\sigma) = [\mathrm{n}]$, and
(*ii*) $\dim(\hat{p}(\sigma^m \times C_K)) < \dim(\sigma \times C_K)$.
Note that (*ii*) $\Rightarrow \partial(\xi) \cap \hat{p}(int(\sigma^m) \times C_K)) = \emptyset$.

On the other hand, $p : (\hat\xi,\ \hat\xi \setminus (\sigma^m \times C_K)) \to (\xi,\ \xi \setminus \hat{p}(\sigma^m \times C_K))$ is a relative homemorphism. Combining this observation with the above remarks and with (12), we see that if ξ is allowable, then

$$\partial(\xi) = \sum_{i=0}^{m-1} [\sigma; \sigma^i] \alpha \cdot \hat{p}(\sigma^i \times C_K) + \sum_{j=1}^{|K|} (-1)^{j-1} (T_j \alpha - \alpha) \cdot \hat{p}(\sigma \times \partial^j(C_K)), \qquad (13)$$

and since for each of the simplices σ^i in this formula, as well as for σ, $J(\sigma^i) = J(\sigma) = [\mathrm{n}]$, $\partial(\xi)$ is well defined.

Definition 5.2 $\mathbf{ID}_*(\mathbf{Y}; \mathcal{L})$. Define $D_*(\mathbf{Y}; \mathcal{L}) = \hat{p}_\#(\hat{D}_*(\hat{\mathbf{Y}}; \mathcal{L})) \subset \tilde{C}_*(\mathbf{Y}; \mathcal{L})$.
Define $ID_*(\mathbf{Y}; \mathcal{L})$ to be the subcomplex $D_*(\mathbf{Y}; \mathcal{L}) \cap IC_*(\mathbf{Y}; \mathcal{L})$. thus, $ID_*(\mathbf{Y}; \mathcal{L})$ is

the chain complex whose k-chains are of the form

$$\psi = \sum_{\xi} \alpha_{\xi} \cdot \xi$$

where each ξ is a k-chain of the form $\xi = \hat{p}(\sigma \times C_K)$, and $\forall \xi$, $\alpha_{\xi} \in \mathbf{V}_{\xi}$. In particular, $\psi \in IC_k(\mathbf{Y};\mathcal{L}) \Rightarrow \xi$ is allowable $(\forall \xi)$.

The diagrams on the following pages show all of the chains of the form $\hat{p}(\sigma \times C_K)$ for $n = 1$ and $n = 2$, followed by some examples of boundaries. The indicated allowability/non-allowability is with respect to middle perversity.

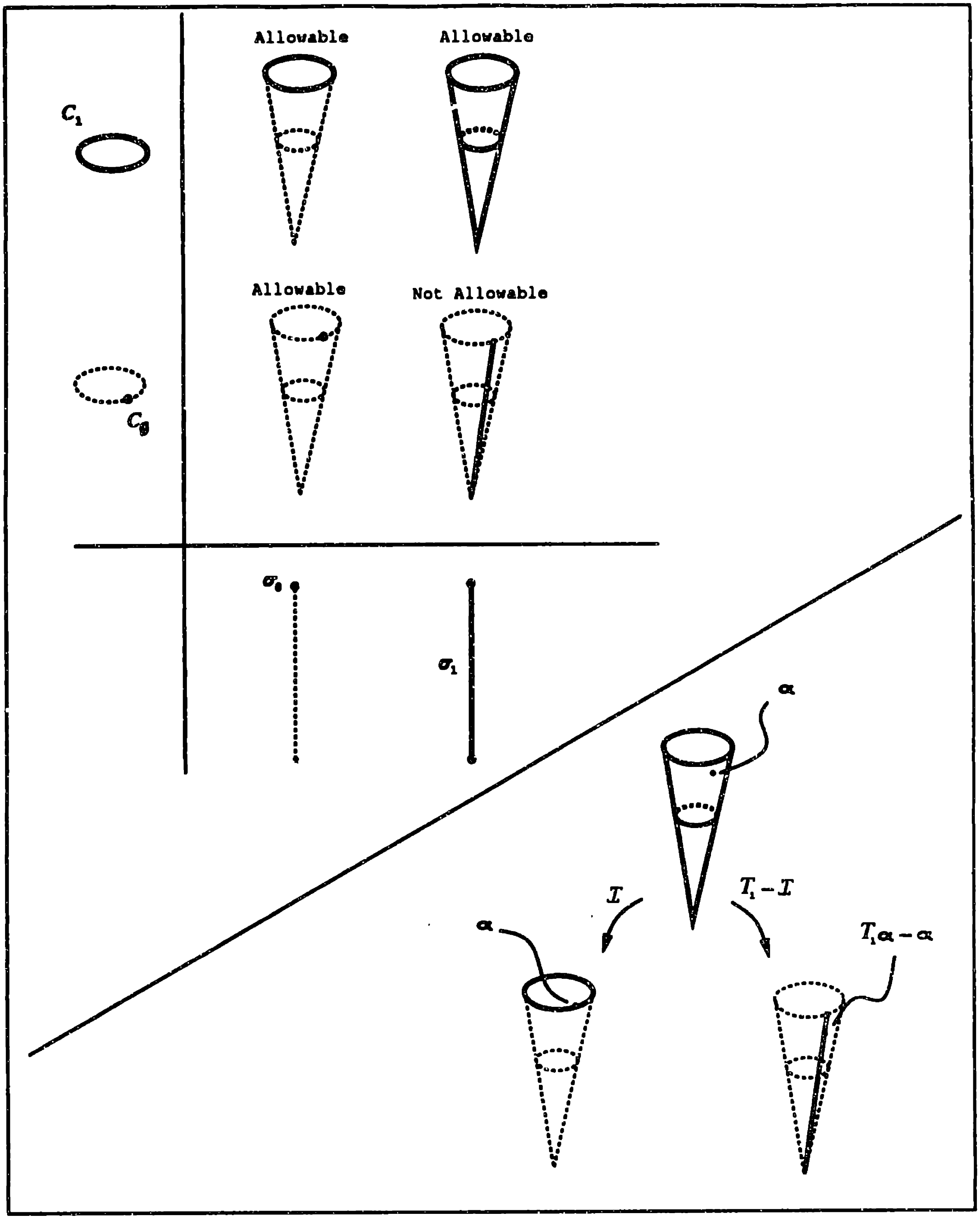

Figure 15: **Top** : The chains in D^2 $(n = 1)$. **Bottom** : Sample boundary computation.

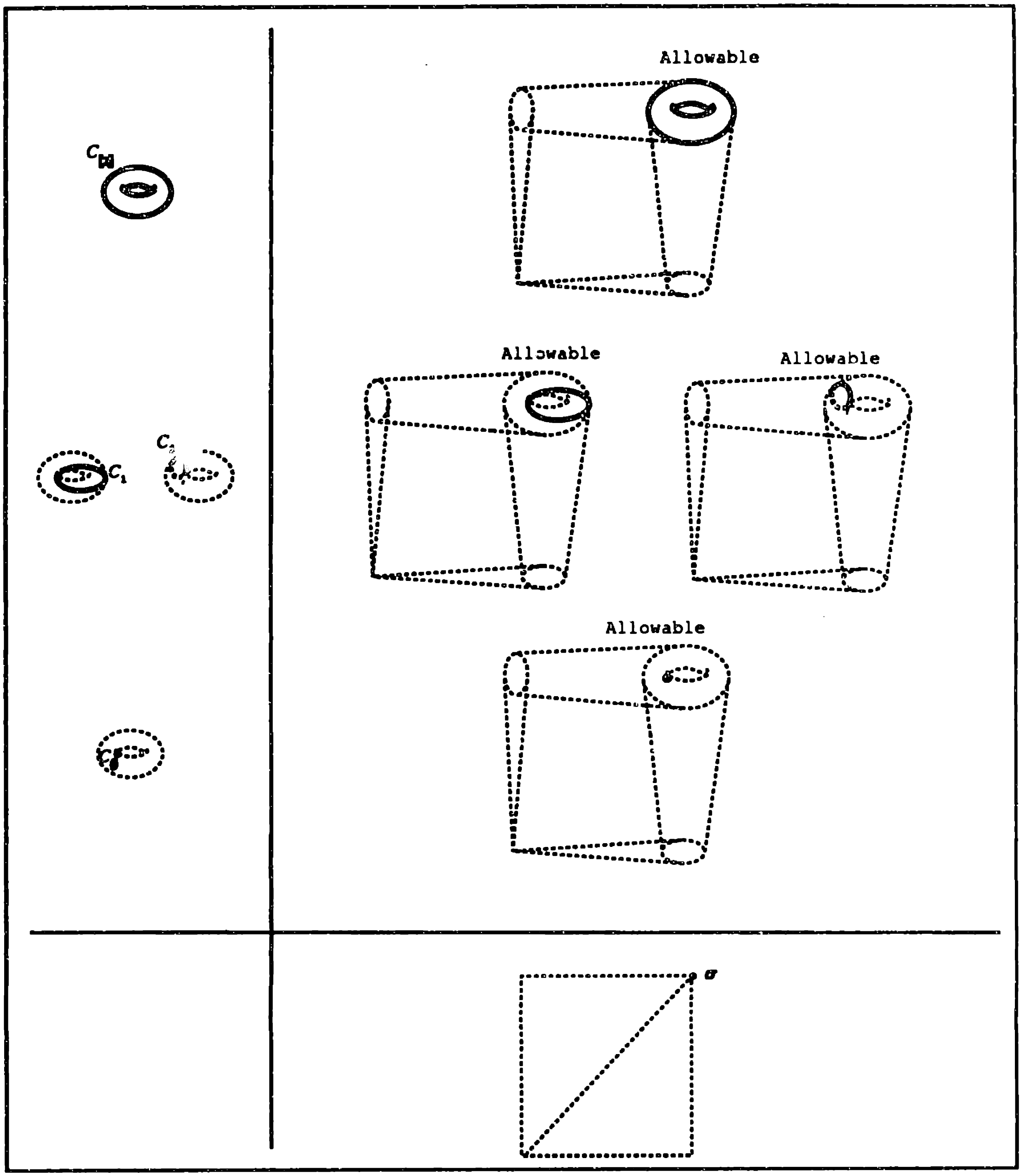

Figure 16: Chains in $D^2 \times D^2$ with $\sigma =$ the barycenter of $\mathcal{P}$.

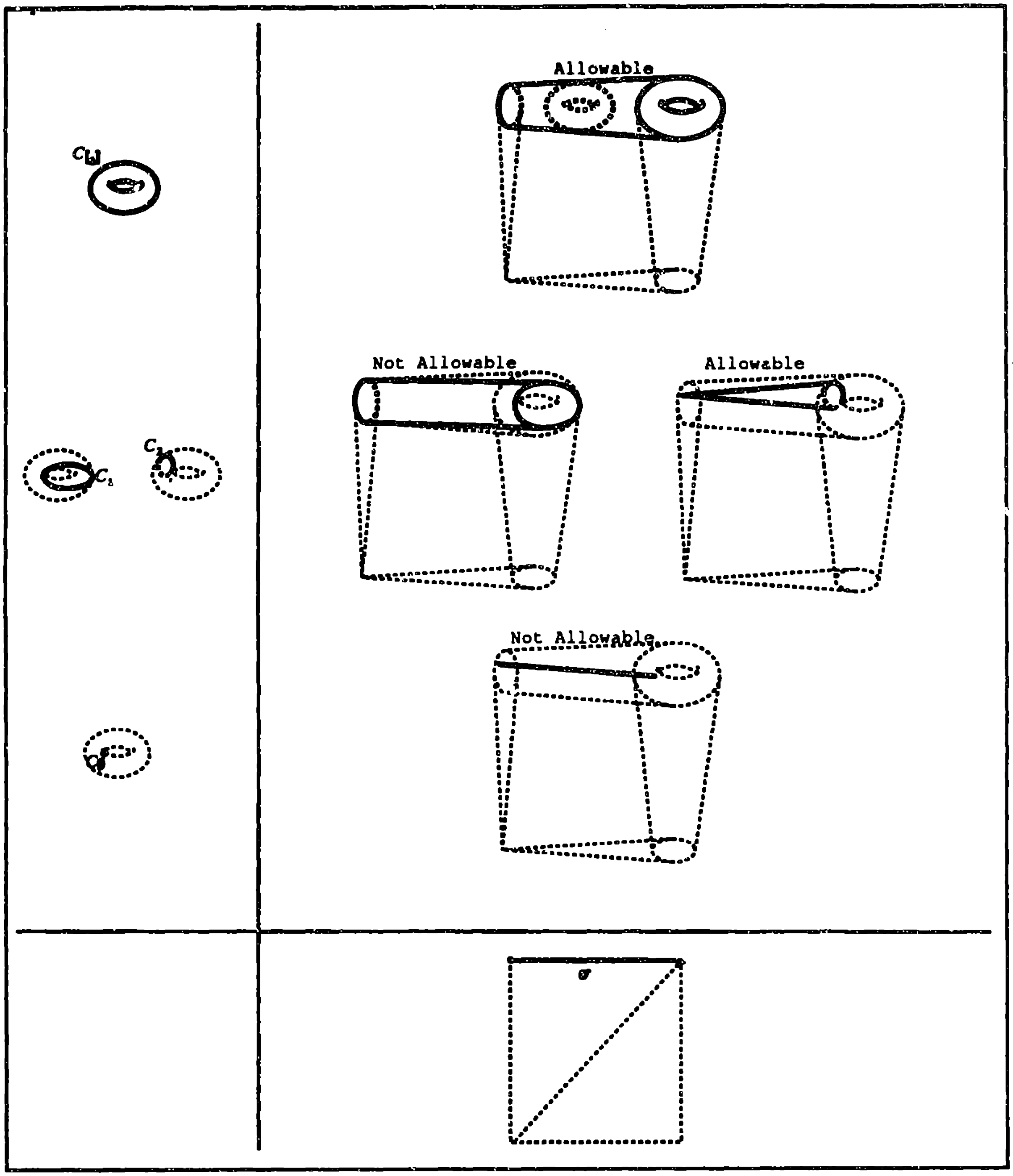

Figure 17: Chains in $D^2 \times D^2$ of the form $\hat{p}(\sigma \times \mathcal{C}_K)$.

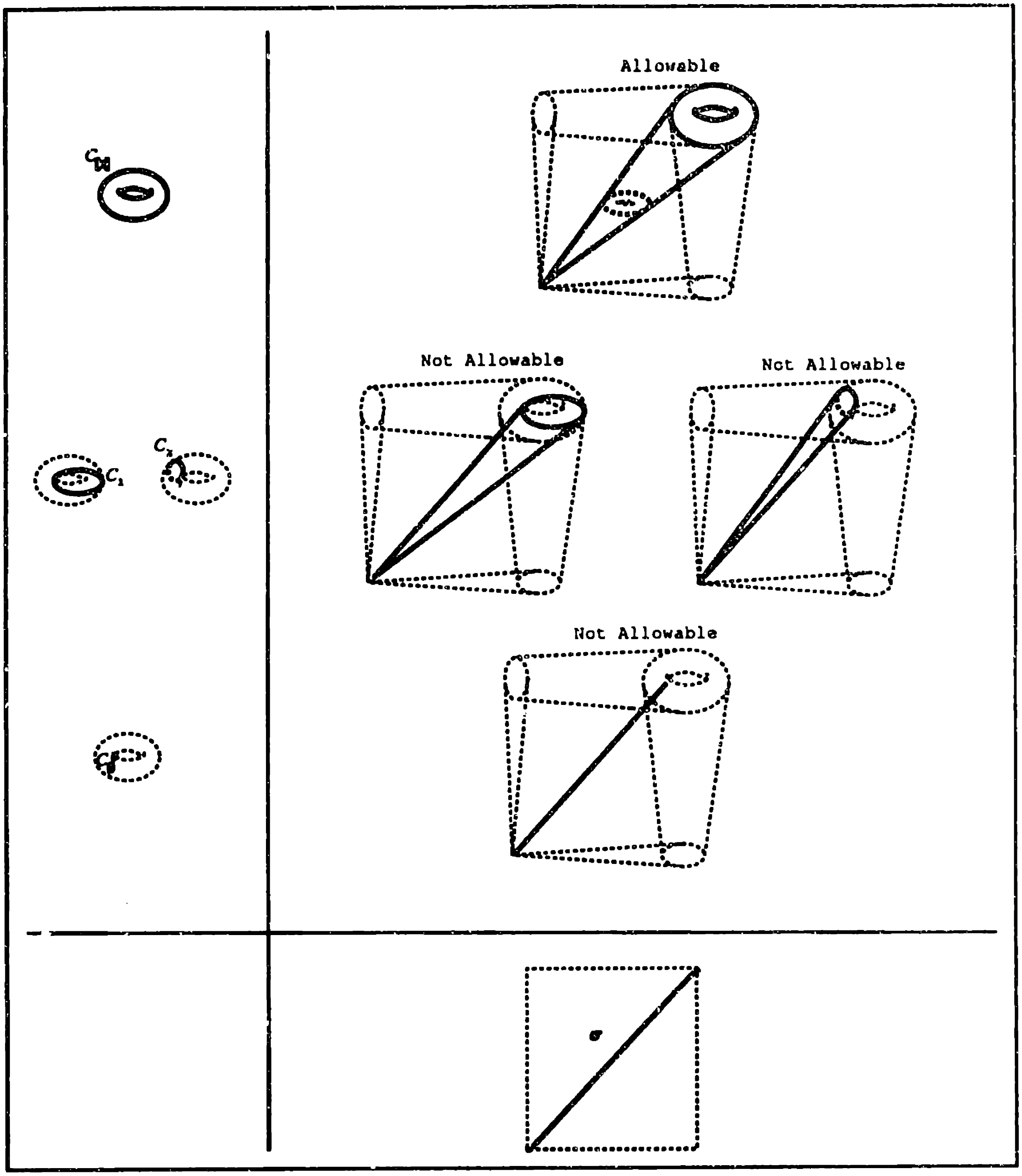

Figure 18: Chains in $D^2 \times D^2$ of the form $\hat{p}(\sigma \times C_K)$.

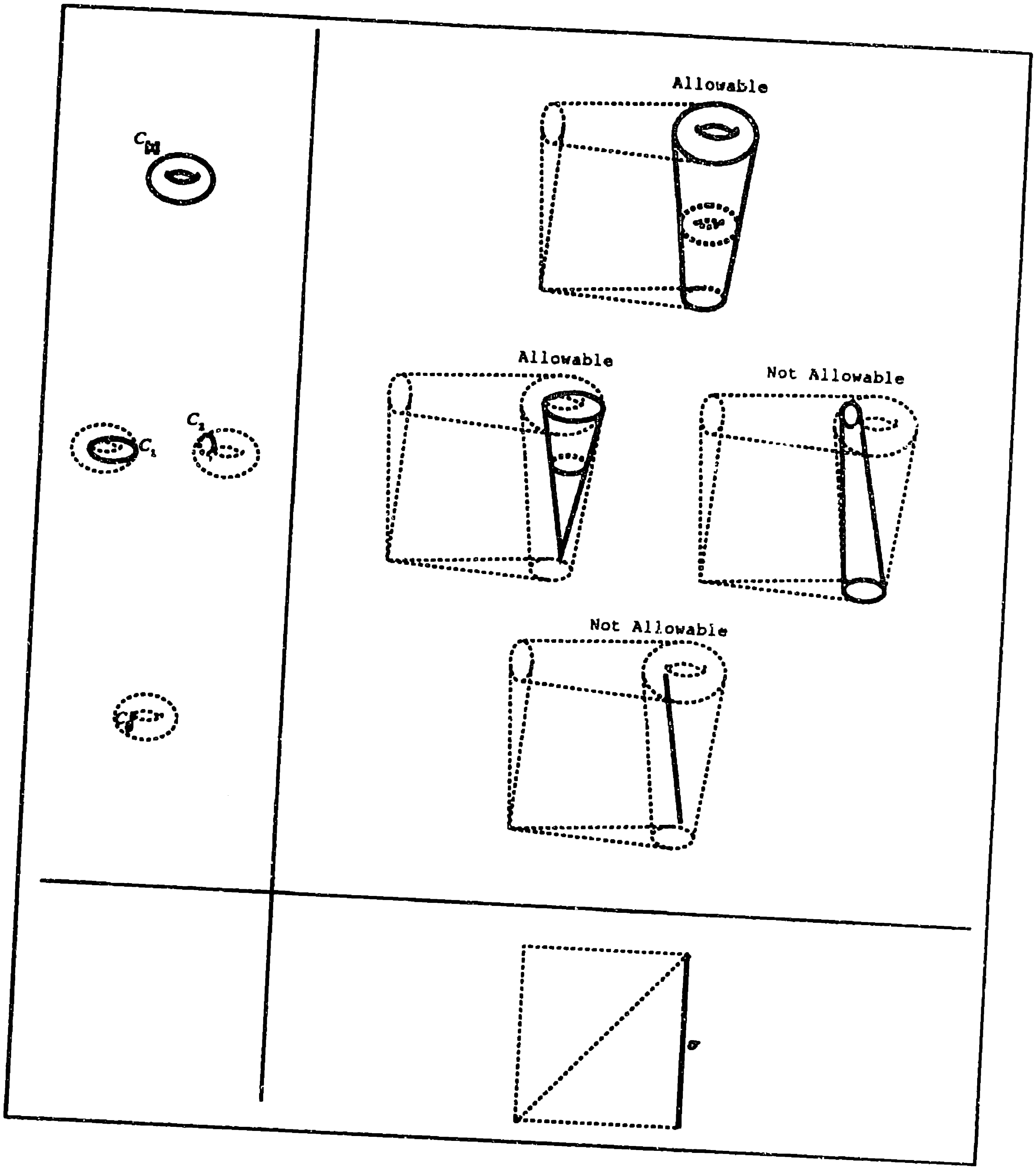

Figure 19: Chains in $D^2 \times D^2$ of the form $\hat{p}(\sigma \times C_K)$.

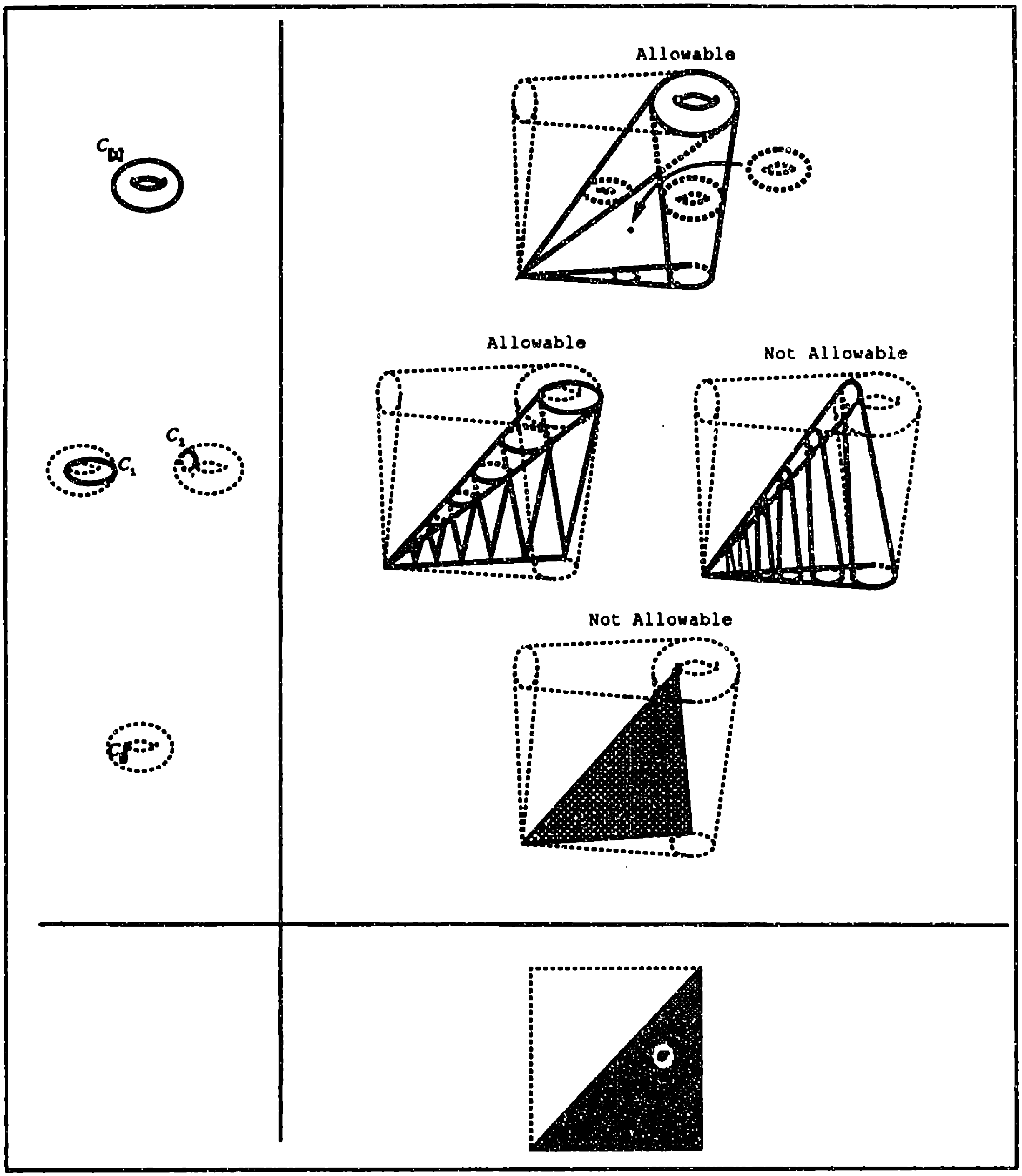

Figure 20: Chains in $D^2 \times D^2$ of the form $\hat{p}(\sigma \times C_K)$.

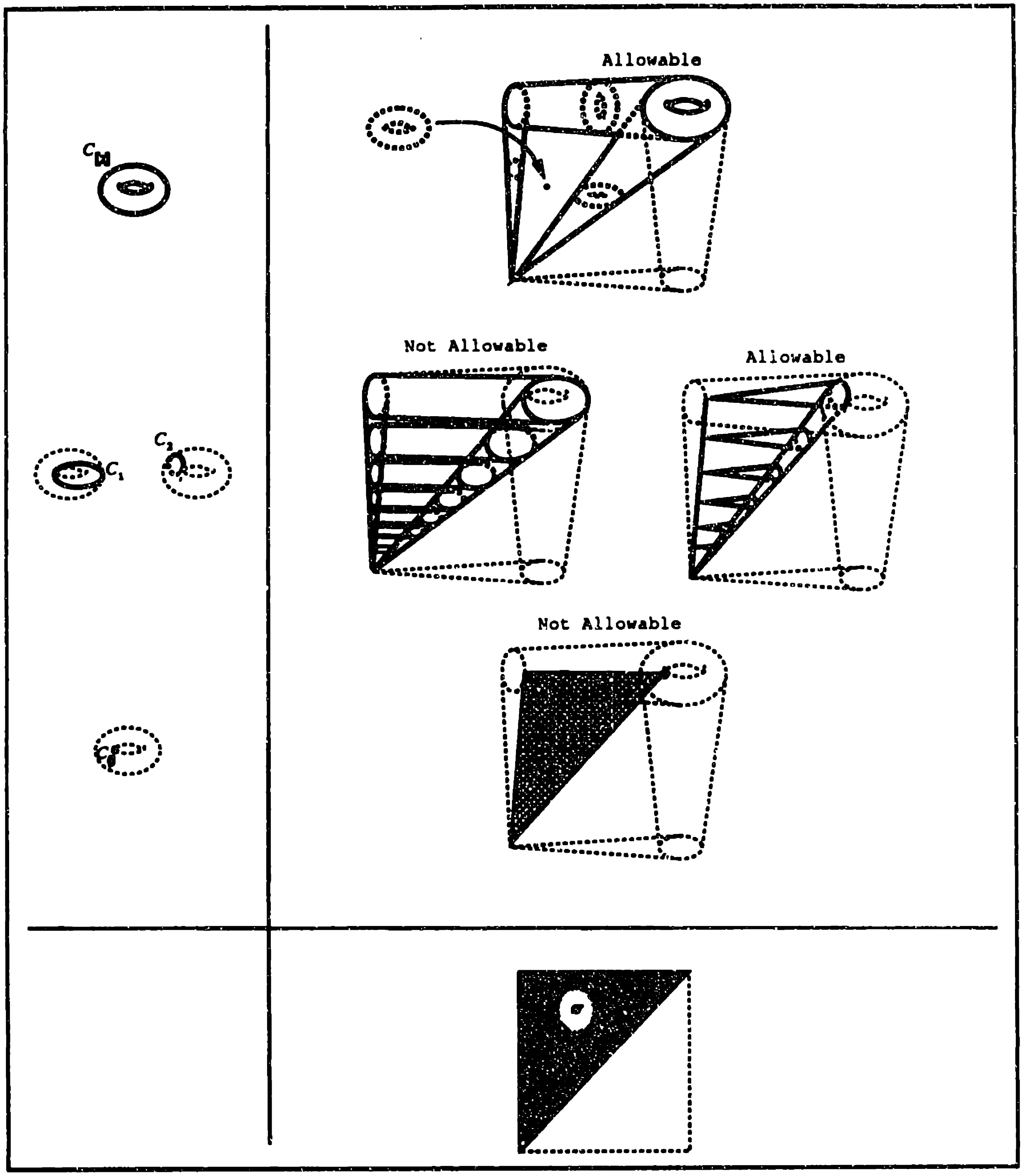

Figure 21: Chains in $D^2 \times D^2$ of the form $\hat{p}(\sigma \times C_K)$.

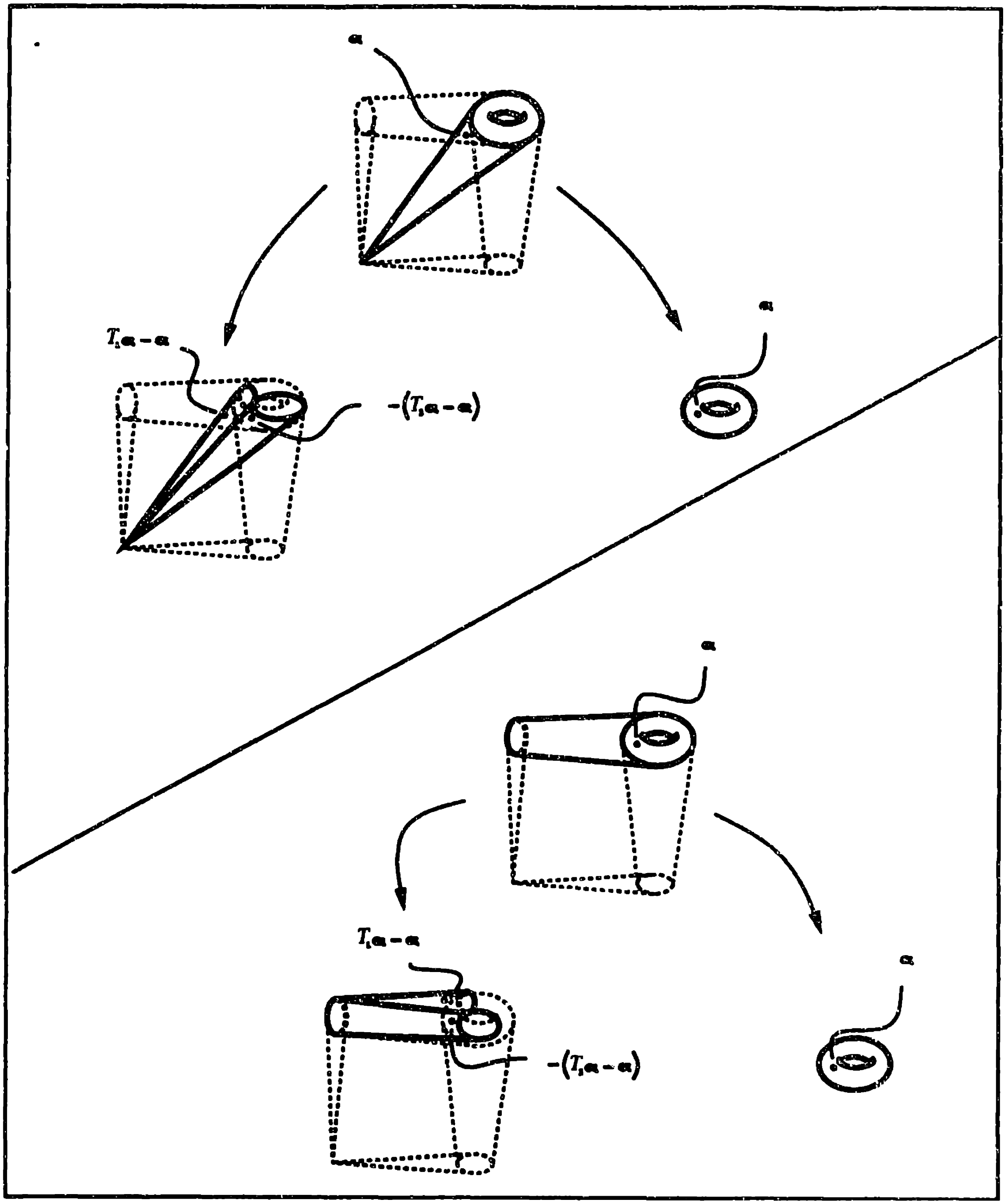

Figure 22: Examples of boundaries.

We can now state the main theorem of this section :

5.3 The main theorem

Theorem 5.3 The inclusion $i\colon ID_\bullet(\mathbf{Y};\mathcal{L}) \hookrightarrow IC_\bullet(\mathbf{Y};\mathcal{L})$ induces an isomorphism $i_*\colon H_*(ID_\bullet(\mathbf{Y},\partial\mathbf{Y})) \xrightarrow{\cong} IH_*(\mathbf{Y},\partial\mathbf{Y};\mathcal{L})$.

Proof: We describe a filtration of $\mathbf{Y}$ by subspaces which are dual in a certain sense to the strata of $\mathbf{Y}$. Corresponding to this filtration we filter the two chain complexes $ID_\bullet(\mathbf{Y};\mathcal{L})$ and $IC_\bullet(\mathbf{Y};\mathcal{L})$ so that i preserves the filtrations.Then we set up the spectral sequences corresponding to these two filtered complexes, and prove that i induces an isomorphism on the E^1 terms, and hence by the spectral sequence comparison theorem, it induces an isomorphism on the homologies of the two complexes.

5.3.1 The filtration of Y

We define a filtration of $\mathbf{Y}$ by closed subspaces $\mathbf{Y}^0 \subset \mathbf{Y}^1 \subset \mathbf{Y}^2 \subset \ldots \subset \mathbf{Y}^n$. Recall first that $Q = \prod_{i=1}^n [0, \frac{1}{2}]$. Let $Q^k \subset Q$ be the set of points of Q for which at least $n-k$ of the coordinates are equal to $\frac{1}{2}$. Now set $\mathbf{Y}^k = p^{-1}(Q^k)$. Thus we have :

$$\begin{aligned}
\mathbf{Y}^0 &= C_1 \times C_2 \times \ldots \times C_n \\
\mathbf{Y}^1 &= \bigcup_{i=1}^n C_1 \times \ldots \times C_{i-1} \times D_i \times C_{i+1} \times \ldots \times C_n \\
\mathbf{Y}^2 &= \bigcup_{0 \le i_1 < i_2 \le n} C_1 \times \ldots \times D_{i_1} \times C_{i_1+1} \times \ldots \times D_{i_2} \times \ldots \times C_n \\
&\vdots \\
\mathbf{Y}^n &= D_1 \times D_2 \times \ldots \times D_n.
\end{aligned}$$

We further decompose each $\mathbf{Y}^k$ as follows : for $K \subseteq [\mathrm{n}]$ and $i \in [\mathrm{n}]$, set

$$B(i,K) = \begin{cases} D_i & \text{if } i \in K \\ C_i & \text{if } i \notin K \end{cases}$$

Set

$$\mathbf{Y}_K = \prod_{i=1}^n B(i,K).$$

Then

$$\mathbf{Y}^k = \bigcup_{K \subseteq [\mathrm{n}],\ |K| = k} \mathbf{Y}_K. \tag{14}$$

Also, for any $K \subseteq [\mathbf{n}]$, set

$$\partial K = \{J \subset K \ : \ |J| = |K| - 1\}. \tag{15}$$

Then

$$\partial \mathbf{Y}_K = \bigcup_{K' \in \partial K} \mathbf{Y}_{K'} \tag{16}$$

and thus we denote

$$\partial \mathbf{Y}_K = \mathbf{Y}_{\partial K}. \tag{17}$$

Remark 5.4 Note that

$$\mathbf{Y}_K \cap \mathbf{Y}^{k-1} = \mathbf{Y}_{\partial K}.$$

Corresponding to this decomposition of $\mathbf{Y}^k$ we define a similar decomposition of $\mathcal{Q}^k$ by setting

$$\mathcal{Q}_K = \{(t_1, t_2, \ldots, t_n) \in \mathcal{Q} : \ t_i = \tfrac{1}{2}, \ \forall i \in K^c\} \tag{18}$$

So that $\forall K$, $\mathbf{Y}_K = p^{-1}(\mathcal{Q}_K)$, and $\mathcal{Q}^k = \bigcup_{K \subseteq [\mathbf{n}],\, |K| = k} \mathcal{Q}_K$. (See figure 23).

The filtration of the chain complex $IC_*(\mathbf{Y}; \mathcal{L})$ is defined by

$$\psi \in IC_*(\mathbf{Y}^k; \mathcal{L}) \iff \psi \in IC_*(\mathbf{Y}; \mathcal{L}) \ \underline{\text{and}} \ supp(\xi) \subseteq \mathbf{Y}^k. \tag{19}$$

$IC_*(\mathbf{Y}_K; \mathcal{L})$ is defined analagously.
$ID_*(\mathbf{Y}^k; \mathcal{L})$ and $ID_*(\mathbf{Y}_K; \mathcal{L})$ are defined to be the respective subcomplexes $ID_*(\mathbf{Y}; \mathcal{L}) \cap IC_*(\mathbf{Y}^k; \mathcal{L})$ and $ID_*(\mathbf{Y}; \mathcal{L}) \cap IC_*(\mathbf{Y}_K; \mathcal{L})$.
It is worth noting that if $\psi = \sum_\xi \alpha_\xi \cdot \xi \in ID_i(\mathbf{Y}; \mathcal{L})$, then

$$\psi \in ID_i(\mathbf{Y}^k; \mathcal{L}) \text{ (resp. } ID_i(\mathbf{Y}_K; \mathcal{L}) \text{) } \Leftrightarrow \forall \xi, \ p(\xi) \subset \mathcal{Q}^k \text{ (resp. } \mathcal{Q}_K).$$

In particular, since each ξ is of the form $\xi = \hat{p}(\sigma \times C_K)$, this is equivalent to saying that $\sigma \in \mathcal{Q}^k$ (resp. $\mathcal{Q}_K$).
Thus we obtain the filtrations

$$IC_*(\mathbf{Y}^0) \subset IC_*(\mathbf{Y}^1) \subset \ldots \subset IC_*(\mathbf{Y}^n). \tag{20}$$

and

$$ID_*(\mathbf{Y}^0) \subset ID_*(\mathbf{Y}^1) \subset \ldots \subset ID_*(\mathbf{Y}^n). \tag{21}$$

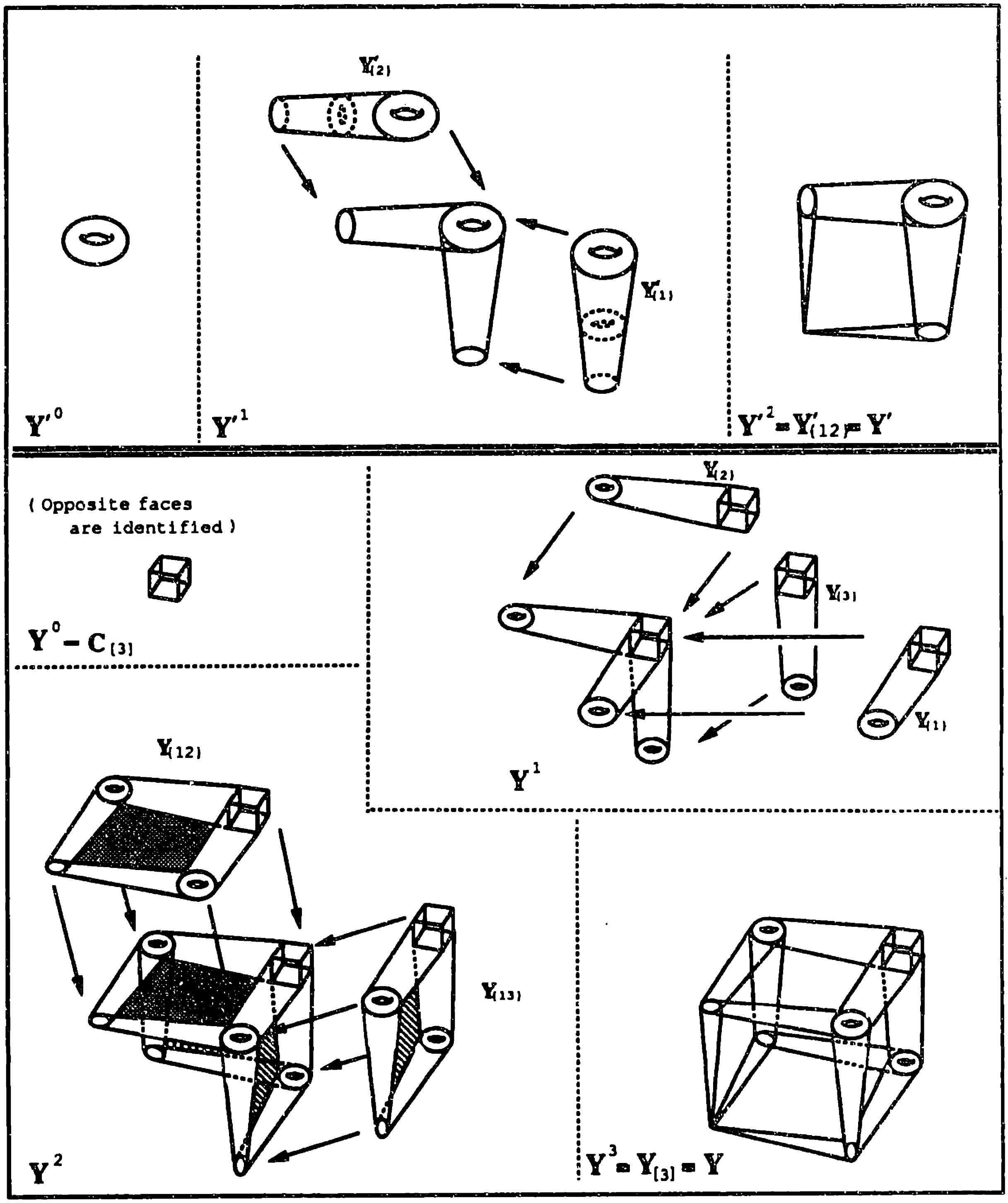

Figure 23: The filtrations of $\mathbf{Y}' = D^2 \times D^2$ (top) and $\mathbf{Y} = D^2 \times D^2 \times D^2$ (bottom). Note that for all $K \subseteq \{1,2\}$, $\mathbf{Y}_K = \mathbf{Y}'_K \times \mathcal{C}_3$.

Now let $\mathbf{Y} = D_1 \times \ldots \times D_n$, with the above filtration. We must prove that for each k, i induces an isomorphism

$$i_*\colon H_*(ID_*(\mathbf{Y}^k, \mathbf{Y}^{k-1}; \mathcal{L})) \xrightarrow{\cong} IH_*(\mathbf{Y}^k, \mathbf{Y}^{k-1}; \mathcal{L}). \tag{22}$$

Remark 5.5 For any two subsets $K_1, K_2 \subset [\mathbf{n}]$,

$$\mathbf{Y}_{K_1} \cap \mathbf{Y}_{K_2} = \mathbf{Y}_{K_1 \cap K_2}.$$

Also, if K_1 and K_2 are two different subsets of $[\mathbf{n}]$ of order $k < n$ then $|K_1 \cap K_2| \leq k - 1$.
It follows that

$$\mathbf{Y}_{K_1} \cap \mathbf{Y}_{K_2} \subseteq \mathbf{Y}^{k-1}.$$

Therefore for any $k < n$

$$IH_*(\mathbf{Y}^k, \mathbf{Y}^{k-1}; \mathcal{L}) = \bigoplus_{K \subset [\mathbf{n}], |K| = k} IH_*(\mathbf{Y}_K, \mathbf{Y}_K \cap \mathbf{Y}^{k-1}; \mathcal{L}),$$

and by remark 5.4

$$IH_*(\mathbf{Y}^k, \mathbf{Y}^{k-1}; \mathcal{L}) = \bigoplus_{K \subset [\mathbf{n}], |K| = k} IH_*(\mathbf{Y}_K, \mathbf{Y}_{\partial K}; \mathcal{L}).$$

Moreover, the analogous decomposition holds on the level of the chain complexes $IC_*(\mathbf{Y}^k, \mathbf{Y}^{k-1}; \mathcal{L})$ and $ID_*(\mathbf{Y}^k, \mathbf{Y}^{k-1}; \mathcal{L})$, and the map i preserves these decompositions. Therefore, to prove (22) it is enough to show that for any $K \subseteq [\mathbf{n}]$, i induces an isomorphism

$$i_*\colon H_*(ID_*(\mathbf{Y}_K, \mathbf{Y}_{\partial K}); \mathcal{L}) \to IH_*(\mathbf{Y}_K, \mathbf{Y}_{\partial K}; \mathcal{L}). \tag{23}$$

The proof of the theorem is by induction on n. The case $n = 1$ is an easy application of lemma 3.18 to the complex $C_*^{CW}(\mathcal{C}_1; \mathcal{L})$ of section 4. There are only two subsets $\emptyset$, $\{1\} \subseteq [\mathbf{1}]$. The assertion holds since
$\mathbf{Y}^0 = \mathbf{Y}_\emptyset = \mathcal{C}_1$, $ID_*(\mathbf{Y}^0; \mathcal{L}) = C_*^{CW}(\mathcal{C}_1; \mathcal{L})$, and
$\mathbf{Y}^1 = \mathbf{Y}_{\{1\}} = D^2 = c(\mathbf{Y}^0)$.

So let $n > 1$ and let $K \subseteq [\mathbf{n}]$ with $|K| = k$.

We distinguish three cases :

<u>Case 1 : k = 0.</u>
$\mathbf{Y}^0 = \mathbf{Y}_\emptyset$ is simply the n-torus $C_1 \times \ldots \times C_n$, which does not meet the singular strata

at all, and $\partial\mathbf{Y}_K = \emptyset$. $ID_*(\mathbf{Y}^0;\mathcal{L})$ is the CW chain complex for the torus $\mathcal{T}^n$, as in section 4 and i is the natural inclusion into $C_*(\mathcal{T}^n;\mathcal{L})$ and hence is an isomorphism on homology.

Case 2 : 0 < k < n.

Let $K \subset [\mathbf{n}]$ be a subset of order k. *WOLOG* we may assume that $K = [\mathrm{k}] = \{1, 2, \ldots, k\} \subseteq [\mathbf{n}-\mathbf{1}] \subset [\mathbf{n}]$, so that $\mathbf{Y}_K = D_1 \times D_2 \times \ldots \times D_k \times C_{k+1} \times \ldots \times C_n$.

Set

$$\mathbf{Y}' = D_1 \times D_2 \times \ldots \times D_{n-1}$$

and let $Q' = \prod_{j=1}^{n-1} I_j$ be the underlying "corner" of the $(n-1)$-polytope over which $\mathbf{Y}'$ is defined. Correspondingly, denote by $\hat{p}'$ the collapsing map from $Q' \times \mathcal{T}^{n-1}$ to $\mathbf{Y}'$.

By identifying in the obvious way $D_1 \times D_2 \times \ldots \times D_{n-1}$ with $D_1 \times D_2 \times \ldots \times D_{n-1} \times C_\emptyset$ we may assume that

$$\mathbf{Y}' \subset \mathbf{Y},$$

and by definition

$$\mathbf{Y} = \mathbf{Y}' \times C_n.$$

Also, by identifying Q' with $Q' \times \{\frac{1}{2}\}$, we may assume that Q' is a subcomplex of Q. Moreover, under this identification and since $K \subseteq [\mathbf{n}-\mathbf{1}]$:

$$Q'_K = Q_K$$

and in fact for every $L \subset K$

$$Q'_L = Q_L.$$

Note also that with these conventions we may view $\hat{p}' : Q' \times \mathcal{T}^{n-1} \to \mathbf{Y}'$ as a restriction of $\hat{p} : Q \times \mathcal{T}^n \to \mathbf{Y}$.
Finally, identify $\mathcal{T}^{n-1}$ with $\mathcal{T}^{n-1} \times C_\emptyset$ and let $\mathcal{L}' = \mathcal{L}|_{\mathcal{T}^{n-1}}$.
Recall that $\mathbf{Y}_K$ is obtained from $Q_K \times \mathcal{T}^n$ by collapsing certain subtori $C_J \subset \mathcal{T}^n$ over certain faces of Q. However for any such C_J which gets collapsed, $J \subset [\mathbf{n}-\mathbf{1}]$, and hence each such C_J is in fact a subtorus of $\mathcal{T}^{n-1}$, and $\mathbf{Y}'_K$ is obtained from $Q' \times \mathcal{T}^{n-1}$ by collapsing the same C_J*'s* over the same faces of Q_K $(= Q'_K)$.

Thus we have shown :

Proposition 5.6 For any $L \subseteq K$, $\mathbf{Y}_L = \mathbf{Y}'_L \times C_n$ and in particular

$$\mathbf{Y}_K = \mathbf{Y}'_K \times C_n \quad \text{and} \quad \mathbf{Y}_{\partial K} = \mathbf{Y}'_{\partial K} \times C_n. \tag{24}$$

Furthermore, the equalities in (24) hold as stratified spaces *i.e.* the strata of $\mathbf{Y}_K$ are precisely the products of the strata of $\mathbf{Y}'_K$ with C_n.

Recall equation (11) of section 4 :

$$C_*^{CW}(\mathcal{T}^n;\mathcal{L}) = C_*^{CW}(\mathcal{T}^{n-1};\mathcal{L}|_{\mathcal{T}^{n-1}}) \oplus \overline{susp}(C_*^{CW}(\mathcal{T}^{n-1};\mathcal{L}|_{\mathcal{T}^{n-1}}).$$

This implies by the definition of $\hat{D}_*(\hat{\mathbf{Y}};\mathcal{L})$ that

$$\hat{D}_*(\hat{\mathbf{Y}}_K;\mathcal{L}) = \hat{D}_*(\hat{\mathbf{Y}}'_K;\mathcal{L}') \oplus \overline{susp}(\hat{D}_*(\hat{\mathbf{Y}}'_K;\mathcal{L}')), \tag{25}$$

(where $\hat{\mathbf{Y}}_K = \hat{p}^{-1}(\mathbf{Y}_K)$ and $\hat{\mathbf{Y}}'_K = \hat{p}^{-1}(\mathbf{Y}'_K)$).

Lemma 5.7
(i) $\mathcal{C}_n$ is not collapsed over any of the faces of $\mathcal{Q}'$.
(ii) For any $J \subseteq [\mathrm{n}]$ set

$$J' = \begin{cases} J & \text{if } n \notin J \\ J \setminus \{n\} & \text{if } n \in J \end{cases}$$

Then for any $\sigma \in \mathcal{Q}'$,and for any $J \subseteq [\mathrm{n}]$,

$$\hat{p}(\sigma \times \mathcal{C}_J) \text{ is allowable} \Longleftrightarrow \hat{p}'(\sigma \times \mathcal{C}_{J'}) \text{ is allowable.}$$

Proof:
(i) $\mathcal{C}_n$ is only collapsed over points $(t_1, \ldots, t_n)$ of $\mathcal{Q}$ in which $t_n = 0$, whereas for any point in $\mathcal{Q}'$, $t_n = \frac{1}{2}$.
(ii) This follows proposition 5.6 and from the fact that $\mathcal{C}_J = \mathcal{C}_{J'} \times \mathcal{C}_n$. □

Corollary 5.8

$$ID_*(\mathbf{Y}_K;\mathcal{L}) = ID_*(\mathbf{Y}'_K;\mathcal{L}') \oplus \overline{susp}(ID_*(\mathbf{Y}'_K;\mathcal{L}')),$$

and in fact, by restriction,

$$ID_*(\mathbf{Y}_{\partial K};\mathcal{L}) = ID_*(\mathbf{Y}'_{\partial K};\mathcal{L}') \oplus \overline{susp}(ID_*(\mathbf{Y}'_{\partial K};\mathcal{L}')),$$

whence

$$ID_*(\mathbf{Y}_K, \mathbf{Y}_{\partial K};\mathcal{L}) = ID_*(\mathbf{Y}'_K, \mathbf{Y}'_{\partial K};\mathcal{L}') \oplus \overline{susp}(ID_*(\mathbf{Y}'_K, \mathbf{Y}'_{\partial K};\mathcal{L}')).$$

□

Now by the inductive hypothesis, the inclusion

$$ID_\bullet(\mathbf{Y}'_K, \mathbf{Y}'_{\partial K}; \mathcal{L}') \hookrightarrow IC_\bullet(\mathbf{Y}'_K, \mathbf{Y}'_{\partial K}; \mathcal{L}')$$

induces an isomorphism on homology, and thus combining corollary 5.8 and the corollary of the combing lemma (cor 3.19) the proof of this case is complete.

Case 3 : k = n.

Remark 5.9 Note that the filtrations and the further decompositions of section 5.3.1 restrict to the subspace $\mathbf{Y}^{n-1} \subset \mathbf{Y}$ and hence the two chain complexes $ID_\bullet$ and $IC_\bullet$ restrict (as *filtered* complexes) to $\mathbf{Y}^{n-1}$, as do the associated spectral sequences. By the previous two cases, (23) holds for any $K \subset [n]$ with $|K| \leq n-1$. It follows that i: $ID_\bullet(\mathbf{Y}^{n-1}; \mathcal{L}|_{\mathbf{Y}^{n-1}}) \longrightarrow IC_\bullet(\mathbf{Y}^{n-1}; \mathcal{L}|_{\mathbf{Y}^{n-1}})$ induces an isomorphism

$$i_*: H_\bullet(ID_\bullet(\mathbf{Y}^{n-1}; \mathcal{L}|_{\mathbf{Y}^{n-1}})) \xrightarrow{\cong} IH_\bullet(\mathbf{Y}^{n-1}; \mathcal{L}|_{\mathbf{Y}^{n-1}}). \quad (26)$$

Now note that $\mathbf{Y} = \mathbf{Y}^n = c(\mathbf{Y}^{n-1})$.
Let $\sigma \in \mathcal{Q}^{n-1}$ correspond to the flag $I_{J_0} \subset \ldots \subset I_{J_m}$. Note that $\sigma \in \mathcal{Q}^{n-1} \Rightarrow J_0 \neq \emptyset$. Let $c(\sigma) \in \mathcal{Q}^n$ $(= \mathcal{Q})$ be the simplex corresponding to the flag $I_\emptyset \subset I_{J_0} \subset \ldots \subset I_{J_m}$. $c(\sigma)$ is in fact the topological cone on σ with apex = the vertex $v = I_\emptyset$. *WOLOG* We may assume that the simplices in $\mathbf{Y}$ are oriented in such a way so that $[c(\sigma); c(\sigma')] = [\sigma; \sigma']$. Note also that $J(c(\sigma)) = J(\sigma) = J_m$.

Consider the map *coning* defined in section 3.21.

Proposition 5.10 .
(i) *coning* maps $ID_{\tau_\bullet \geq n}(\mathbf{Y}^{n-1}; \mathcal{L})$ to $ID_{\bullet+1}(\mathbf{Y}; \mathcal{L})$.
(ii) The composition

$$ID_{\tau_\bullet \geq n}(\mathbf{Y}^{n-1}; \mathcal{L}) \xrightarrow{coning} ID_{\bullet+1}(\mathbf{Y}; \mathcal{L}) \xrightarrow{q} ID_{\bullet+1}(\mathbf{Y}, \mathbf{Y}^{n-1}; \mathcal{L})$$

is a chain isomorphism.
(iii) The following diagram commutes

$$\begin{array}{ccc} ID_{\tau_\bullet \geq n}(\mathbf{Y}^{n-1}; \mathcal{L}) & \xrightarrow{q \circ coning} & ID_{\bullet+1}(\mathbf{Y}, \mathbf{Y}^{n-1}; \mathcal{L}) \\ \downarrow i & & \downarrow i \\ IC_{\tau_\bullet \geq n}(\mathbf{Y}^{n-1}; \mathcal{L}) & \xrightarrow{q \circ coning} & IC_{\bullet+1}(\mathbf{Y}, \mathbf{Y}^{n-1}; \mathcal{L}) \end{array} \quad (27)$$

Proof:
(*i*) By definition, *coning* maps $\xi = \sum \alpha_i \cdot \hat{p}(\sigma_i \times C_{K_i})$ to $\sum \alpha_i \cdot \hat{p}(c(\sigma_i) \times C_{K_i})$, which is in $ID_{*+1}(\mathbf{Y})$ if (and only if) $\xi \in ID_{\tau *\geq n}(\mathbf{Y}^{n-1}; \mathcal{L})$.

(*ii*) $q \circ$*coning* is a chain map and is clearly injective. Thus it suffices to show that it is surjective too.
Any m-simplex $\sigma \in \mathcal{Q}$ which is not contained in $\mathcal{Q}^{n-1}$ (except for the unique 0-simplex corresponding to the flag $I_\emptyset$) has the form $c(\sigma')$ for some $(m-1)$-simplex $\sigma' \in \mathcal{Q}^{n-1}$.
Let $i \geq n$ and let $\bar{\psi} \in ID_{i+1}(\mathbf{Y}, \mathbf{Y}^{n-1}; \mathcal{L})$. $\bar{\psi}$ is represented by a chain $\psi = \sum_j \alpha_j \cdot \xi_j$, where for each j, $\xi_j = \hat{p}(\sigma_j \times C_{K_j})$ and so that for each j:
(*a*) $\xi_j \not\subseteq \mathbf{Y}^{n-1}$, and
(*b*) ξ_j is allowable (whence in particular , $\sigma_j \neq v$).
These two conditions imply that for each j, $\sigma_j = c(\sigma'_j)$ for some $\sigma'_j \in \mathcal{Q}^{n-1}$, and hence that $\xi_j = c(\xi'_j)$ where $\xi'_j = \hat{p}(\sigma'_j \times C_{K_j})$, and since ξ_j is allowable, ξ'_j is too (an elementary fact about coning). •
Thus $\psi \in \mathrm{Im}(coning)$ and $q \circ$*coning* is surjective.

(*iii*) This is clear since the maps i are inclusions and the upper map *coning* is the restriction of the lower map *coning*.

We have shown that both horizontal maps induce isomorphisms on homology.
Now, by remark 5.9, i: $ID_*(\mathbf{Y}^{n-1}; \mathcal{L}) \to IC_*(\mathbf{Y}^{n-1}; \mathcal{L})$ induces an isomorphism on homology, whence its restriction $ID_{\tau *\geq n}(\mathbf{Y}^{n-1}; \mathcal{L}) \xrightarrow{i} IC_{\tau *\geq n}(\mathbf{Y}^{n-1}; \mathcal{L})$, which is the vertical map on the left, also induces an isomorphism on homology.

Thus the vertical map on the right induces an isomorphism on homology, and the case $k = n$ is complete and along with it - the entire proof.

□

5.4 A combinatorial description of allowability in $D_*(\mathbf{Y}; \mathcal{L})$

Let $\xi = \hat{p}(\sigma \times C_K) \in D_*(\mathbf{Y}; \mathcal{L})$ where $\sigma \in \mathcal{Q}$ and $K \subseteq [\mathrm{n}]$.
Suppose that σ is an m-dimensional simplex corresponding to the flag of faces of $\mathcal{P}$:

$$I_{J_0} \subset I_{J_1} \subset \ldots \subset I_{J_{m-1}}$$

with

$$J_0 \subset J_1 \subset \ldots \subset J_{m-1} \subseteq [\mathrm{n}]. \tag{28}$$

Let S_i be the stratum over I_{J_i} (*i.e.* $p^{-1}(I_{J_i})$), and note that

$$\dim(S_i) = 2\dim(I_{J_i}) = 2|J_i|. \tag{29}$$

We describe a combinatorial formula for the allowability of ξ in terms of m, K, and the J_i's.

Recall that $J^c = [\mathbf{n}] \setminus J$, and that $\mathcal{C}_{J^c}$ is the subtorus of $\mathcal{C}_{[\mathbf{n}]} = \mathcal{T}^n$ which gets collapsed over every point of I_J.

For each $0 \leq i \leq m-1$, set $J'_i = J^c_{m-1-i}$.
Dual to the flag (28) is the flag :

$$J'_0 \subset J'_1 \subset \ldots \subset J'_{m-1} \subseteq [\mathbf{n}]. \tag{30}$$

Now, over every point of I_{J_i}, $\mathcal{C}_{J_i^c}$ gets collapsed to a point and hence $\mathcal{C}_K$ gets collapsed to

$$\mathcal{C}_K/\mathcal{C}_{K\cap J_i^c}, \cong \mathcal{C}_{K\setminus(K\cap J_i^c)} = \mathcal{C}_{K\cap J_i}$$

which is a torus of dimension $|K \cap J_i|$.
Also,

$$\dim(\sigma \cap I_{J_i}) = i.$$

It follows that

$$\dim(\xi \cap S_i) = i + |K \cap J_i|.$$

Let $\bar{p}$ be a perversity, and to avoid long indices denote $\bar{p}_i$ by $\bar{p}(i)$.
The allowability condition states that ξ is allowable if and only if for each i,

$$\dim(\xi \cap S_i) \leq \dim(\xi) - \operatorname{codim}(S_i) + \bar{p}(\operatorname{codim}(S_i)). \tag{31}$$

By (29), $\operatorname{codim}(S_i) = 2n - 2|J_i| = 2|J_i^c|$.
Thus, ξ is allowable if and only if for each i,

$$i + |K \cap J_i| \leq (m + |K|) - 2|J_i^c| + \bar{p}(2|J_i^c|)$$

$\Leftrightarrow$

$$|K| - |K \cap J_i| \geq i - m + 2|J_i^c| - \bar{p}(2|J_i^c|)$$

$\Leftrightarrow$

$$|K \cap J_i^c| \geq (2|J_i^c| - \bar{p}(2|J_i^c|)) - (m - i).$$

Now since this must hold for all $0 \leq i \leq m-1$, we may replace all i's by $m-1-i$ to obtain :

ξ is allowable if and only if for all $0 \leq i \leq m-1$,

$$|K \cap J'_i| \geq 2|J'_i| - \bar{p}(2|J'_i|) - i - 1. \tag{32}$$

Finally, note that for $\bar{p}$ = middle-perversity, $\bar{p}(2c) = c - 1$, whence $2|J'_i| - \bar{p}(2|J'_i|) = |J'_i| + 1$ and thus we have :

ξ is allowable *w.r.t.* middle-perversity if and only if for all $0 \leq i \leq m-1$,

$$|K \cap J'_i| \geq |J'_i| - i. \tag{33}$$

6 The bar resolution, The bar complex and the Eilenberg-Zilber theorem

In section 5 we constructed a subcomplex of $IC_*(\mathbf{Y};\mathcal{L})$ which computes the relative intersection homology $IH_*(\mathbf{Y},\partial\mathbf{Y};\mathcal{L})$ where $\mathbf{Y}$ was an n-fold product of 2-disks, and $\mathcal{L}$ was a local system. The space $\mathbf{Y}$ was viewed as a "corner" of a smooth toric variety $\mathbf{X}$ which in this case was taken to be the n-fold product $S^2 \times \ldots \times S^2$. The construction involved tensoring the (relative) simplicial complex of the barycentric subdivision of the underlying corner of the polytope $\mathcal{P}$, with the chain complex obtained from a suitable CW-decomposition of the n-torus. The restriction of this complex to allowable chains with allowable boundaries was shown to compute $IH_*(\mathbf{Y},\partial\mathbf{Y};\mathcal{L})$.

A naive attempt to construct a global chain complex which would compute $IH_*(\mathbf{X};\mathcal{L})$ for a general toric variety $\mathbf{X}$ would be to tensor the relative simplicial complex of the barycentric subdivision of the entire polytope $S_*(sd\mathcal{P},\partial\mathcal{P})$ with such a CW-complex of the torus, and then restrict to the allowable subcomplex. This, unfortunately, does not work since any one such CW-decomposition of the torus, although it carries all the Intersection-homological information of one "corner" of $\mathbf{X}$ is simply not fine enough to contain all of the Intersection-homological information at other corners. Thus, if this approach is to be followed, the CW-complex of a given corner must be replaced by a larger complex on the torus. This complex must be large enough so as to contain in a natural way, as a subcomplex, each of the cells in each of the local CW-complexes . To this end we use the *bar complex* obtained from the bar resolution of $\mathcal{A}$ (with local coefficients), where $\mathcal{A}$ is a free abelian group of rank n.

In this section we discuss certain aspects of free resolutions of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{A}]$ as related to our construction, in particular - the bar resolution, and their relation to the local coefficient homology of a torus. We then describe an explicit chain homotopy equivalence between the bar resolution of $\mathbb{Z}^n$ and the tensor product of of the bar resolutions of $\mathbb{Z}^{n-1}$ and $\mathbb{Z}$ and prove certain properties of the maps involved which will be used later on in proving that certain maps (which will be defined in terms of these maps) preserve allowability.

Since we deal only with free abelian groups, we use additive notation as opposed to the standard multiplicative notation commonly used in the homology theory of general groups. For detailed descriptions and proofs of the standard results see [7].

6.1 The bar resolution

Let $\mathcal{A}$ denote a free abelian group of rank n.

Definition 6.1 $\mathbf{W}_*(\mathcal{A})$. For all $k \geq 0$, let $W_k(\mathcal{A})$ be the free abelian group generated by the ordered $(k+1)$-tuples $(z_0, z_1, \ldots, z_k)$, where $\forall k,\ z_k \in \mathcal{A}$.
Define $\tilde{\partial}_k : W_k(\mathcal{A}) \to W_{k-1}(\mathcal{A})$ by

$$\tilde{\partial}_k(z_0, z_1, \ldots, z_k) = \sum_{i=0}^{k}(-1)^i \tilde{d}_i,$$

where

$$\tilde{d}_i(z_0, z_1, \ldots, z_k) = (z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_k).$$

Define the augmentation $\epsilon : W_0(\mathcal{A}) \to \mathbb{Z}$ by : $\epsilon(z_0) = 1,\ \forall z_0 \in \mathcal{A}$.
$W_k(\mathcal{A})$ is a free module over $\mathbb{Z}[\mathcal{A}]$ where the $\mathcal{A}$ action is given by :

$$z \cdot (z_0, z_1, \ldots, z_k) = (z + z_0, z + z_1, \ldots, z + z_k).$$

So defined, $W_*(\mathcal{A})$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{A}]$.
We refer to the generators of $W_*(\mathcal{A})$ as simplices and call the z_i's vertices.
We say that $(z_0, \ldots, z_k)$ is *(strongly) degenerate* if $z_i = z_j$ for some $i \neq j$.
We say that $(z_0, \ldots, z_k)$ is *weakly degenerate* if the dimension of the affine span of $\{z_0, \ldots, z_k\}$ is less than k.

6.2 The non-homogeneous description

It is often convenient to describe the $\mathbb{Z}$-generators of $W_*(\mathcal{A})$ in the form :

$$z_0[z_1|z_2|\ldots|z_k]$$

where z_0 is the initial vertex and for $i > 0$, z_i is the "vector" from the $(i-1)^{st}$ vertex to the the i^{th} vertex. In other words, $z_0[z_1|z_2|\ldots|z_k]$ denotes the simplex whose vertices are $(z_0, z_0 + z_1, \ldots, z_0 + \ldots + z_k)$. This is known as the *non-homogeneous* description of $W_*(\mathcal{A})$.
For $i > 0$, we call the z_i's *edges*.
In this description, the boundary map $\tilde{\partial}_k : W_k(\mathcal{A}) \to W_{k-1}(\mathcal{A})$ is given by :
$\tilde{\partial}_k = \sum_{i=0}^{k}(-1)^i \tilde{d}_i$, where

$$\tilde{d}_i(z_0[z_1|\ldots|z_k]) = \begin{cases} (z_0 + z_1)[z_2|\ldots|z_k] & i = 0 \\ z_0[z_1|\ldots|z_i + z_{i+1}|\ldots|z_k] & 1 \leq i \leq k-1 \\ z_0[z_1|\ldots|z_{k-1}] & i = k \end{cases}$$

The $\mathcal{A}$-action simply translates the initial vertex namely :

$$z \cdot (z_0[z_1|\dots|z_k]) = (z + z_0)[z_1|\dots|z_k].$$

In this description, a simplex is degenerate if one of the z_i's is equal to 0, and weakly degenerate if $\{z_1, \dots, z_k\}$ are linearly dependent (over $\mathbf{R}$).

6.3 The bar complex

Let $\mathbf{V}$ be a (left) $\mathcal{A}$-module (in our case a rational vector space on which $\mathcal{A}$ acts by automorphisms). Then for each k, $\mathbf{V} \otimes W_k(\mathcal{A})$ is an $\mathcal{A}$-module with $\mathcal{A}$ acting diagonally on the tensor product, namely

$$z \cdot (\alpha \otimes z_0[z_1|z_2|\dots|z_k]) = (z \cdot \alpha) \otimes (z + z_0)[z_1|z_2|\dots|z_k]. \tag{34}$$

Definition 6.2 The bar complex. The bar complex of $\mathcal{A}$ with coefficients in a local system $\mathcal{L}$ (with fiber $\mathbf{V}$), denoted $\bar{W}_*(\mathcal{A};\mathcal{L})$ is the chain complex obtained from $\mathbf{V} \otimes W_*(\mathcal{A})$ by modding out by the $\mathcal{A}$-action.
Equivalently it can be defined as :

$$\bar{W}_k(\mathcal{A};\mathcal{L}) = \mathbf{V} \otimes_{\mathbf{Z}[\mathcal{A}]} W_k(\mathcal{A}),$$

with associated boundary homomorphism :

$$\partial_k = I_{\mathbf{V}} \otimes_{\mathbf{Z}[\mathcal{A}]} \tilde{\partial}_k$$

(I_V denotes the identity on $\mathbf{V}$).

The bar complex can be described more explicitly as follows :
$\bar{W}_k(\mathcal{A};\mathcal{L})$ consists of finite sums of elements of the form

$$\alpha \otimes [z_1|z_2|\dots|z_k]$$

where $\alpha \in \mathbf{V}$ and $z_i \in \mathcal{A}$ for each i.
The boundary homomorphism $\partial : \bar{W}_k(\mathcal{A};\mathcal{L}) \to \bar{W}_{k-1}(\mathcal{A};\mathcal{L})$ is given by

$$\begin{aligned} \partial(\alpha \otimes [z_1|z_2|\dots|z_k]) &= z_1 \cdot \alpha \otimes [z_2|\dots|z_k] \\ &+ \sum_{i=1}^{k-1}(-1)^i \alpha \otimes [z_1|\dots|z_i + z_{i+1}|\dots|z_k] \\ &+ (-1)^k \alpha \otimes [z_1|z_2|\dots|z_{k-1}]. \end{aligned} \tag{35}$$

Fact 6.3 With boundary homomorphism as above, $\bar{W}_*(\mathcal{A};\mathcal{L})$ is a chain complex, and $H_*(\bar{W}_*(\mathcal{A};\mathcal{L})) \cong H_*(T^n;\mathcal{L})$.

6.4 Normalization

In the bar resolution, as well as in the bar complex, the degenerate chains (sums of simplices which are (strongly) degenerate) form a subcomplex called the *degenerate* complex. The quotient of the bar complex by the degenerate subcomplex is called the *normalized bar complex.* The projection onto the normalized complex is a quasi-isomorphism (induces an isomorphism on homology).

6.5 Some general remarks about group-homology with local coefficients

Given a group G, a G-module $\mathbf{V}$ and any free (projective) resolution $F_\bullet$ of $\mathbb{Z}$ over $\mathbb{Z}[G]$, the homology of G with coefficients in $\mathbf{V}$ is defined as in definition 6.2 as the homology of the chain complex obtained by modding out $F_\bullet \otimes \mathbf{V}$ by the diagonal G action. (It is a standard result that the homology is independent of the resolution $F_\bullet$).

The quotient obtained by modding out by the G action is denoted $(F \otimes \mathbf{V})_G$.

If there is a short exact sequence $0 \to G_1 \to G \to G_2 \to 0$ (in particular if G_1 and G_2 are subgroups of G such that $G = G_1 \times G_2$) then $(F \otimes \mathbf{V})_G$ can be computed in two steps, (see [7]) by first dividing out by the action of G_1, and then dividing out by the induced action of G_2 on the quotient :

$$(F \otimes \mathbf{V})_G = ((F \otimes \mathbf{V})_{G_1})_{G_2}. \tag{36}$$

We shall use this fact in section 8.

The operator $(\_)_G$ which takes any G-module to its quotient modulo the G action is called the *co-invariants fuctor.*

6.6 The Eilenberg-Zilber theorem

The following are two fundamental facts from the theory of the homology of groups :

Fact 6.4 Uniqueness of resolutions. Let G be a group. Then any two free (or projective) resolutions of $\mathbb{Z}$ over $\mathbb{Z}[G]$ are chain homotopy equivalent.

Fact 6.5 Let G' and G'' be groups, and let $G = G' \times G''$. Let $F'_\bullet$ and $F''_\bullet$ be free resolutions of $\mathbb{Z}$ over $\mathbb{Z}[G']$ and $\mathbb{Z}[G'']$ (respectively).
Set $F_\bullet = (F' \otimes F'')_\bullet$. Then $F_\bullet$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$.

Corollary 6.6 $W_*(\mathbb{Z}) \otimes W_*(\mathbb{Z}^{n-1})$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}^n]$ and it is chain homotopy equivalent to $W_*(\mathbb{Z}^n)$.

We will rely heavily on this last fact in the proof of the major result of this thesis. However, we will need to prove that the homotopy equivalences (as well as the homotopy itself) preserve certain subcomplexes and for this we will need explicit formulas. Such fomulas were given by Eilenberg and MacLane in [9]. In order to exploit these maps for our needs, we first need to describe the bar resolution as a simplicial abelian group and to recall some background from [8].

Definition 6.7 Let $k \geq 0$. Define degeneracy operators $s_i : W_k(\mathcal{A}) \to W_{k+1}(\mathcal{A})$ for all $0 \leq i \leq k$ by:

$$s_i(z_0[z_1|\ldots|z_k]) = z_0[z_1|\ldots|z_i|\,0\,|z_{i+1}|\ldots|z_k].$$

In the homogeneous descrition this amounts to duplicating the i^{th} vertex.

Fact 6.8 With these degeneracy operators, and the face operators $\tilde{d}_i$ defined in section 6.2, $W_*(\mathcal{A})$ is a simplicial abelian group.

Definition 6.9 In [8] the authors define for any two simplicial abelian groups K and L, a simplicial abelian group $K \times L$ by :

$$(K \times L)_p = K_p \otimes L_p.$$

The generators of $K \times L$ are denoted $k \times l$ (for $k \in K$, $l \in L$).
The face and degeneracy operators are applied componentwise, namely :

$$d_i(k \times l) \stackrel{def}{=} d_i(k) \times d_i(l), \text{ and}$$

$$s_i(k \times l) \stackrel{def}{=} s_i(k) \times s_i(l).$$

Remark 6.10 As for any simpliial abelian group, $K \times L$ is a chain complex with boundary map $\partial = \sum(-1)^i d_i$.

The following definitions and theorems (through section 6.7.1) appear in [8] in the general context of simplicial abelian groups.

Let p and q be non negative integers.

Definition 6.11 Simplicial operators. A simplicial operator $S_{(p,q)}$ in the category of simplicial abelian groups is an operator which for every simplicial abelian group G, maps G_p to G_q homomorphically, and which is natural with respect to simplicial maps (a simplicial map is a degree 0 homomorphism which commutes with all face and degeneracy operators).

Definition 6.12 Monotonic simplicial operators. A monotonic simplicial operator (in the category of simplicial abelian groups) is an operator $M_{(p,q)}$, natural with respect to simplicial maps, which for any simplicial abelian group G maps G_p homomorphically to G_q, and which can be written (uniquely) as

$$M = s_{i_r} \dots s_{i_1} d_{j_t} \dots d_{j_1}, \tag{37}$$

with

$$p > i_r > \dots > i_1 \geq 0, \quad 0 \leq j_t < \dots < j_1 \leq q, \quad \text{and } q - t + r = p. \tag{38}$$

Theorem 6.13 . Any simplicial operator S can be uniquely expressed as a sum of monotonic simplicial operators.

Definition 6.14 Derived operators. Let M be a monotonic simplicial operator say

$$M = s_{i_r} \dots s_{i_1} d_{j_t} \dots d_{j_1}.$$

The derived operator is defined to be

$$M' = s_{i_r+1} \dots s_{i_1+1} d_{j_t+1} \dots d_{j_1+1}. \tag{39}$$

Thus for any simplicial abelian group G, M' maps G_{p+1} to G_{q+1}.
Let S be any simplicial operator. If S is expressed in its canonical form as a sum of monotonic operators:

$$S = \sum M_i,$$

then we define the derived operator :

$$S' = \sum M_i'.$$

Definition 6.15 Frontal operators. A monotonic simplicial operator M is *frontal* if in its canonical form (37), $j_t > 0$. A general simplicial operator is frontal if it is a sum of frontal monotonic operators.

Note: Any derived operator is frontal.

Theorem 6.16 Eilenberg-Zilber. Let K and L be simplicial abelian groups. There are maps

$$K \times L \underset{\nabla}{\overset{f}{\rightleftarrows}} K \otimes L \tag{40}$$

and a degree $+1$ map

$$\Phi : (K \times L)_* \to (K \times L)_{*+1} \tag{41}$$

satisfying (modulo degeneracies):
(i) $f \circ \nabla = I$.
(ii) $\nabla \circ f = \partial\Phi + \Phi\partial$. (Recall that $K \times L$ and $K \otimes L$ are chain complexes).
In addition, each of the maps takes degenerate chains to degenerate chains and hence they induce maps on the normalized resolutions. On the normalized resolutions, (i) and (ii) hold precisely.
We describe the explicit formulas for f, ∇ and Φ as pertaining to our setting in 6.7.1 below.

6.7 The E-Z theorem and $W_*(\mathbb{Z}^n)$

We shall henceforth consider only the normalized bar resolution and bar complex. To avoid overly cumbersome notation we use the same notation for the normalized complexes as we had used up to now for the general complexes

Lemma 6.17 Let $\mathbb{Z}^n$ be decomposed as $\mathbb{Z} \oplus \mathbb{Z}^{n-1}$. Then $W_*(\mathbb{Z}^n)$ is canonically isomorphic with $W_*(\mathbb{Z}) \times W_*(\mathbb{Z}^{n-1})$.

Proof: $\forall z \in \mathbb{Z}^n$, write $z = (z', z'')$, where $z' \in \mathbb{Z}$ and $z'' \in \mathbb{Z}^{n-1}$. Then make the identification:

$$z_0[z_1|\dots|z_k] \longleftrightarrow z'_0[z'_1|\dots|z'_k] \otimes z''_0[z''_1|\dots|z''_k], \tag{42}$$

which is clearly a bijection, and it is trivial to verify that the face and degeneracy operations are preserved under this identification.

Thus we have a chain homotopy equivalence as in theorem 6.16 :

$$W_*(\mathbb{Z}^n) \underset{\nabla}{\overset{f}{\rightleftarrows}} W_*(\mathbb{Z}) \otimes W_*(\mathbb{Z}^{n-1}). \tag{43}$$

We now give an explicit description of the maps involved as in [9].

6.7.1 The maps f, ∇, Φ

<u>f :</u>

The map f is the Alexander Whitney map. It is defined in general on $K \times L$ (where K and L are any simplicial ab-lian groups) by $f(a_k \otimes b_k) =$

$$s_1 \dots s_k\, a_k \otimes b_k + \sum_{i=1}^{k-1} (-1)^i s_{i+1} \dots s_k\, a_k \ \otimes \ s_{i-1} \dots s_0\, b_k + (-1)^k a_k \otimes s_0 \dots s_{k-1}\, b_k.$$

Explicitly in our setting, let $\omega = z_0[z_1|\dots|z_k] \in W_*(\mathbb{Z}^n)$, and for each i, let $z_i = (z_i', z_i'')$, where $z_i' \in \mathbb{Z}$ and $z_i'' \in \mathbb{Z}^{n-1}$. Set

$$\begin{aligned}
f_0(\omega) &= z_0'[\ ] \otimes z_0''[z_1''|\dots|z_k''], \\
f_i(\omega) &= z_0'[z_1'|\dots|z_i'] \otimes (\sum_{j=0}^{i} z_j'')[z_{i+1}''|\dots|z_k''], \quad \text{for } 1 \le i \le k-1, \\
f_k(\omega) &= z_0'[z_1'|\dots|z_k'] \otimes (\sum_{j=0}^{k} z_j)[\ ].
\end{aligned}$$

Define

$$f = f_0 - f_1 + \dots + (-1)^k f_k. \tag{44}$$

<u>∇ :</u>

Denote by $[m]$ the set $\{0, 1, \dots, m\}$. Let p and q be non negative integers s.t. $p + q = m + 1$. A $(p,q) - shuffle$ (μ, ν) of $[m]$ is a partition of $[m]$ into two disjoint sets

$$\mu = \{\mu_1, \dots, \mu_p\} \text{ and } \nu = \{\nu_1, \dots, \nu_q\}$$

so that $\mu_1 < \dots < \mu_p$ and $\nu_1 < \dots < \nu_q$.

$$\nabla : W_p(\mathbb{Z}) \otimes W_q(\mathbb{Z}^{n-1}) \to W_{p+q}(\mathbb{Z}^n)$$

is the "shuffle product" defined by:

$$\nabla(\omega_p' \otimes \omega_q'') = \sum_{(\mu,\nu)} (-1)^{\epsilon(\mu)} (s_{\nu_q} \dots s_{\nu_1}\, \omega_p') \times (s_{\mu_p} \dots s_{\mu_1}\, \omega_q''), \tag{45}$$

where

$$\epsilon(\mu) \stackrel{def}{=} \sum_{i=1}^{p} (\mu_i - (i-1))$$

is the *signature* of the shuffle (μ, ν) and the summation in (45) runs over all (p,q) shuffles.

Remark 6.18
(*i*) In (45) the summand corresponding to a given pair (μ, ν) is the $(p+q)$-simplex whose edges are the edges of ω' and of ω'', "shuffled" so that the edges of ω' are in positions $\nu_1, \ldots, \nu_q$ and the edges of ω'' are in positions $\mu_1, \ldots, \mu_q$.

(*ii*) In the context of (oriented) simplicial complexes, equation (45) gives a (standard) way of triangulating the product $\Delta^p \times \Delta^q$ of a p-simplex with a q-simplex, so that with this triangulation,

$$\partial(\Delta^p \times \Delta^q) = \partial\Delta^p \times \Delta^q + (-1)^p \, \Delta^p \times \partial\Delta^q.$$

Set

$$h = \nabla \circ f$$

and note that $h_0(z_0[\ ]) = z_0[\ ]$.
h is a simplicial operator, whence its derived operator h' is defined.

Φ :
$\Phi_q : W_q(\mathbb{Z}^n) \rightarrow W_{q+1}(\mathbb{Z}^n)$ is defined by induction as a natural homomorphism as follows:

$$\Phi_0 = 0,$$

and assuming $\Phi_{q-1} : W_{q-1}(\mathbb{Z}^n) \rightarrow W_q(\mathbb{Z}^n)$ is defined for $q > 0$, set

$$\Phi_q(\omega_q) = -\Phi'_{q-1}(\omega_q) + h's_0(\omega_q). \tag{46}$$

Note: Since h' and Φ' are frontal, we conclude by induction that Φ is frontal.

6.8 Properties of the maps pertaining to allowability

In this section we prove certain properties of the maps f, ∇ and Φ which will be used later on in proving that allowability is preserved by certain maps based on these maps.

Definition 6.19 Intersection number. Let F be a linear subspace of $\mathbb{R}^n$, and let $\omega = z_0[z_1|\ldots|z_k] \in W_k(\mathbb{Z}^n)$.
Define the intersection number of ω with F :

$$\omega|_F = \#\{\ \{z_1, \ldots, z_k\} \cap F\ \}.$$

i.e. $\omega|_F =$ the number of edges of ω parallel to F.

Theorem 6.20 Let $\mathbb{Z}^n = \mathbb{Z} \oplus \mathbb{Z}^{n-1}$ and correspondingly let $\mathbb{R}^n = \mathbb{R} \oplus \mathbb{R}^{n-1}$. Let $F \subseteq \mathbb{R}^{n-1}$ be a linear subspace.

(*i*) Let $\omega \in W_k(\mathbb{Z}^n)$ be a k-simplex, and let $f(\omega) = \sum(\omega'_i \otimes \omega''_i)$.
Then for each i,

$$\omega'_i \text{ non-degenerate} \Longrightarrow \omega''_i|_F \geq \omega|_F.$$

(*ii*) Let $\omega' \otimes \omega'' \in W_*(\mathbb{Z}) \otimes W_*(\mathbb{Z}^{n-1})$ (where ω' and ω'' are simplices), and let $\nabla(\omega' \otimes \omega'') = \sum \omega_i$.
Then for each i,

$$\omega_i|_F \geq \omega''|_F.$$

Note : as a corollary of (*i*) and (*ii*), $\nabla \circ f$ preserves (or increases) intersection numbers with any subspace $F \subseteq \mathbb{R}^{n-1}$ (modulo degeneracies).

(*iii*) Let $\omega \in W_k(\mathbb{Z}^n)$ be a simplex and let $\Phi(\omega) = \sum a_i \omega_i \quad (a_i \in \mathbb{Z},\ \forall i)$.
Then for each i for which $a_i \neq 0$ and ω_i is non degenerate,

$$\omega_i|_F \geq \omega|_F.$$

Proof:
(*i*) Let $\omega = z_0[z_1|\ldots|z_k] \in W_k(\mathbb{Z}^n)$, and for each j, let $z_j = (z'_j, z''_j)$. Since $F \subseteq \mathbb{R}^{n-1}$,

$$z_j \in F \iff z'_j = 0 \iff z''_j = z_j \quad \text{(for each } j\text{)}.$$

Now for any $0 \leq i \leq k$, $\omega'_i = z'_0[z'_1|\ldots|z'_i]$ and $\omega''_i = z''_0[z''_{i+1}|\ldots z''_k]$, and thus

$$\begin{aligned} \omega'_i \text{ is non-degenerate} &\Leftrightarrow z'_j \neq 0 \ \ for \ \ 1 \leq j \leq i \\ &\Leftrightarrow z_j \notin F \ \ for \ \ 1 \leq j \leq i \\ &\Leftrightarrow \text{ all the edges of } \omega_i \text{ which are in } F \text{ are also edges of } \omega''_i. \end{aligned}$$

(*ii*) This is clear since by the definition of ∇, each of the edges of ω'' is an edge of each of the ω_i's.

The proof of (*iii*):

Definition 6.21 Define *the join* of a vertex with a simplex by :
In the homogeneous description :

$$z_0 * (z_1, z_2, \ldots, z_k) = (z_0, z_1, \ldots, z_k)$$

and in the non-homogeneous description :

$$z_0 * z_1[z_2|\ldots|z_k] = z_0[z_1 - z_0|z_2|\ldots|z_k] \tag{47}$$

or

$$z_0 * (z_0 + z_1)[z_2| \dots |z_k] = z_0[z_1|z_2| \dots |z_k]. \tag{48}$$

Lemma 6.22 Let M be a monotonic operator and let $\omega \in W_*(\mathbb{Z}^n)$ and $z \in \mathbb{Z}$. Then

$$M'(z * \omega) = z * M(\omega).$$

Proof: The proof is immediate from the definition of derived operators.

Now with M, ω, z as above, set $M(z_0[z_1| \dots |z_k]) = \tilde{z_0}[\tilde{z_1}| \dots |\tilde{z_m}]$.

Lemma 6.23 If M is frontal then $\tilde{z_0} = z_0$.

Proof: This follows immediately from the fact that in the canonical form of M, d_0 does not appear.

Corollary 6.24 Let $M : W_p(\mathbb{Z}^n) \to W_q(\mathbb{Z}^n)$ be a <u>frontal</u> monotonic operator and let $F \subseteq \mathbf{R}^n$ be a linear subspace. If M preserves (or increases) the intersection number of every simplex with F then so does $M' : W_{p+1}(\mathbb{Z}^n) \to W_{q+1}(\mathbb{Z}^n)$.

Proof: Let $z_0[z_1| \dots |z_{p+1}] \in W_{p+1}(\mathbb{Z}^n)$. In view of definition 6.21 and lemmas 6.22 and 6.23 we have

$$\begin{aligned} M'(z_0[z_1| \dots |z_{p+1}]) &= M'(z_0 * (z_0 + z_1)[z_2| \dots |z_{p+1}]) \\ &= z_0 * M((z_0 + z_1)[z_2| \dots |z_{p+1}]) \\ &= z_0 * (z_0 + z_1)[\tilde{z_2}| \dots |\tilde{z_{p+1}}] \\ &= z_0[z_1|\tilde{z_2}| \dots |\tilde{z_{p+1}}] \end{aligned}$$

Now since M preserves intersection numbers with F we have

$$\#\{ \{\tilde{z}_2, \dots, \tilde{z}_{p+1}\} \cap F\} \geq \#\{ \{z_2, \dots, z_{p+1}\} \cap F\}$$

and therefore $\#\{ \{z_1, \tilde{z}_2, \dots, \tilde{z}_{p+1}\} \cap F\} \geq \#\{ \{z_1, z_2, \dots, z_{p+1}\} \cap F\}$.

Corollary 6.25 Since any simplicial operator S is a sum of monotonic operators, corollary 6.24 holds on each summand of S.

Proposition 6.26 $h's_0$ preserves (or increases) intersection numbers with F (for any $F \subseteq \mathbf{R}^n$).

Proof: Let $z_0[z_1|\dots|z_k] \in W_k(\mathbb{Z}^n)$.

$$\begin{aligned} h's_0(z_0[z_1|\dots|z_k]) &= h'(z_0[0|z_1|\dots|z_k]) \\ &= h'(z_0 * z_0[z_1|\dots|z_k]) \\ &= z_0 * h(z_0[z_1|\dots|z_k]) \end{aligned}$$

h preserves the intersection numbers with F and joining with a vertex certainly does too.
We can now prove (iii).

Corollary 6.27 Recall that

$$\Phi_q(\omega_q) = -\Phi'_{q-1}(\omega_q) + h's_0(\omega_q)$$

and let $F \subseteq \mathbf{R}^{n-1}$. Assume Φ_{q-1} preserves (or increases) intersection numbers with F. Then by corollary 6.25, Φ'_{q-1} preserves (or increases) intersection numbers with F since Φ is frontal. Combining this with proposition 6.26, and noting that the assertion holds for $\Phi_0 = 0$ (and for $\Phi_1 = h'_0 s_0$) completes the proof.

6.9 A chain homotopy equivalence in dimension 1

In this section we describe another (smaller) resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}]$ and an explicit chain homotopy equivalence with $W_*(\mathbb{Z})$.

Definition 6.28 Any CW decomposition of S^1 induces a CW decomposition of its universal cover $\mathbf{R}$, and the cellular chain complex of the induced complex is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}]$ (see [7] for the details). The open cells "upstairs" are simply the connected components of the inverse images (under the covering map) of the cells downstairs.
Let $C_*^{CW}(S^1)$ denote the CW decomposition of S^1 consisting of a single 0-cell and a single 1-cell. In the induced complex upstairs which we denote by $\widetilde{C_*^{CW}}(S^1)$, the 0-cells are the integral points of $\mathbf{R}$ and the 1-cells are the line segments $\{(z, z+1) : z \in \mathbb{Z}\}$. We now describe an explicit chain homotopy equivalence

$$W_*(\mathbb{Z}) \underset{\eta}{\overset{g}{\rightleftarrows}} \widetilde{C_*^{CW}}(S^1). \tag{49}$$

In degrees $i > 1$ both maps are 0.

<u>**degree 0:**</u>
$\widetilde{C_0^{CW}}(S^1)$ is generated by $\{z : z \in \mathbb{Z}\}$.
$W_0(\mathbb{Z})$ is generated by $\{z[\ ] : z \in \mathbb{Z}\}$.
g_0 and η_0 are the obvious bijections.

<u>**degree 1:**</u>
$\widetilde{C_1^{CW}}(S^1)$ is generated by $\{(z, z+1) : z \in \mathbb{Z}\}$
$W_1(\mathbb{Z})$ is generated by $\{z_0[z_1] : z_0, z_1 \in \mathbb{Z}\}$.
Define

$$\eta_1(z, z+1) = z[1]$$

and

$$g_1(z_0[z_1]) = \begin{cases} \sum_{i=0}^{z_1-1}(z_0+i, z_0+i+1) & \text{if } z_1 > 0 \\ -\sum_{i=0}^{|z_1|-1}(z_0+z_1+i, z_0+z_1+i+1) & \text{if } z_1 < 0 \\ 0 & \text{if } z_1 = 0 \end{cases}$$

Remark 6.29 If $z_1 < 0$ then $g_1(z_0[z_1]) = -g_1(\,(z_0+z_1)[\,|z_1|\,])$.

Proposition 6.30 g and η are chain maps.

Proof: In degree 0 it is completely trivial whereas in degree 1 it is an easy verification.

Proposition 6.31 $g \circ \eta = I$.

Proof: Simply plug in to the definitions.

Proposition 6.32 There is a degree +1 homotopy $\Psi : W_*(\mathbb{Z}) \to W_{*+1}(\mathbb{Z})$ such that

$$\partial\Psi + \Psi\partial = \eta \circ g - I. \tag{50}$$

Proof: In degree 0, $\eta \circ g = I$. $W_*(\mathbb{Z})$ is both acyclic (since $\mathbb{R}$ is contractible) and free and hence the existence of Ψ is guaranteed by the Acyclic Models Theorem. We describe Ψ explicitly in degrees 0 and 1 nonetheless :
Set

$$\Psi_0 = 0, \text{ and}$$

$$\Psi_1(z_0[z_1]) = \begin{cases} \sum_{i=1}^{z_1-1} z_0[\,i\,|\,1\,] & \text{if } z_1 > 0 \\ -\sum_{i=1}^{|z_1|-1} (z_0 + z_1)[\,i\,|\,1\,] & \text{if } z_1 < 0 \\ 0 & \text{if } z_1 = 0 \end{cases}$$

Remark 6.33 As was the case for the map g_1, if $z_1 < 0$, then

$$\Psi_1(z_0[z_1]) = -\Psi_1(\,(z_0 + z_1)[-z_1]).$$

<u>**Degree 0:**</u> (50) holds trivially.
<u>**Degree 1:**</u> First, suppose $z_1 > 0$. Then

$$\eta \circ g(z_0[z_1]) = \sum_{i=0}^{z_1-1} (z_0 + i)[1].$$

Also, $\Psi_0 \partial = 0$ (since $\Psi_0 = 0$).
Now,

$$\begin{aligned} \partial(\Psi(z_0[z_1])) &= \partial(\sum_{i=1}^{z_1-1} z_0[\,i\,|\,1\,]) \\ &= \sum_{i=1}^{z_1-1}(\,(z_0 + i)[\,1\,] - z_0[i+1] + z_0[\,i\,]) \\ &= \left(\sum_{i=1}^{z_1-1}(z_0 + i)[\,1\,]\right) + z_0[\,1\,] - z_0[z_1] \\ &= \underbrace{\sum_{i=0}^{z_1-1}(z_0 + i)[\,1\,]}_{\eta \circ g(z_0[z_1])} - z_0[z_1] \end{aligned}$$

Now note that from remarks 6.29 and 6.33 it follows that (50) holds when $z_1 < 0$ too. When $z_1 = 0$, everything is 0.
Note: $\widetilde{C_*^{CW}}(S^1)$ is also a resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}]$. $\mathbb{Z}$ acts by translation namely :

$$z_0 \cdot z = z_0 + z \quad \text{and} \quad z_0 \cdot (z, z+1) = (z_0 + z, z_0 + z + 1).$$

Finally note that the maps g and η are maps of $\mathbb{Z}$-modules.

7. The homology of a torus revisited

We formally construct the chain complex of section 4 and use the results of the previous sections to prove that it does in fact compute the homology with twisted coefficients of a torus. We do so by constructing an explicit chain-homotopy equivalence which we will make heavy use of in section 8. Then we define a chain map from the bar complex of $\mathbb{Z}^n$ to the *P.L.*-geometric chain complex of the torus (both with local coefficients) and prove that it too is a quasi-isomorphism. This map will also play an important role in section 8.

7.1 The map from $\bar{W}_*(\mathbb{Z}^n;\mathcal{L})$ to $C_*^{CW}(\mathcal{T}^n;\mathcal{L})$

By composition of the chain-homotopy equivalence (43) with that of section 6.9, we obtain a chain-homotopy equivalence

$$W_*(\mathbb{Z}^n) \underset{\nabla\circ\eta}{\overset{g\circ f}{\rightleftarrows}} \widetilde{C_*^{CW}}(S^1)\otimes W_*(\mathbb{Z}^{n-1}) \tag{51}$$

and by repeated application of the same, we obtain a chain-homotopy equivalence

$$W_*(\mathbb{Z}^n) \underset{\Gamma}{\overset{G}{\rightleftarrows}} \underbrace{\widetilde{C_*^{CW}}(S^1)\otimes\ldots\otimes\widetilde{C_*^{CW}}(S^1)}_{n\ times} \tag{52}$$

In both cases the composition from right to left and back is equal to the identity.

$\mathbb{Z}^n$ acts on $\widetilde{C_*^{CW}}(S^1)\otimes\ldots\otimes\widetilde{C_*^{CW}}(S^1)$ componentwise namely for $z=(a_1,\ldots,a_n)\in\mathbb{Z}^n$ and $c_1\otimes\ldots\otimes c_n\in\widetilde{C_*^{CW}}(S^1)\otimes\ldots\otimes\widetilde{C_*^{CW}}(S^1)$,

$$z\cdot(c_1\otimes\ldots\otimes c_n)=a_1\cdot c_1\otimes\ldots\otimes a_n\cdot c_n. \tag{53}$$

The maps G and Γ are maps of $\mathbb{Z}^n$-modules (*i.e.* preserve the $\mathbb{Z}^n$ action) and hence for any $\mathbb{Z}^n$-module $\mathbf{V}$, we have

Theorem 7.1 There is a chain homotopy equivalence

$$\mathbf{V} \otimes_{\mathbf{Z}[\mathbf{Z}^n]} W_\bullet(\mathbf{Z}^n) \underset{I \otimes_{\mathbf{Z}[\mathbf{Z}^n]} \Gamma}{\overset{I \otimes_{\mathbf{Z}[\mathbf{Z}^n]} G}{\rightleftarrows}} \mathbf{V} \otimes_{\mathbf{Z}[\mathbf{Z}^n]} \big(\bigotimes_{i=1}^{n} \widetilde{C_\bullet^{CW}}(S^1)\big) \tag{54}$$

□

Now note that the complex on the left is $W_\bullet(\mathbf{Z}^n;\mathcal{L})$ and the complex on the right is precisely the complex $C_\bullet^{CW}(\mathcal{T}^n;\mathcal{L})$ constructed in section 4.

7.2 The map from $\bar{W}_*(\mathbf{Z}^n;\mathcal{L})$ to $C_*(\mathcal{T}^n;\mathcal{L})$

In this section we define a chain map

$$\psi : \bar{W}_\bullet(\mathbf{Z}^n;\mathcal{L}) \to C_\bullet(\mathcal{T}^n;\mathcal{L})$$

where $\bar{W}_\bullet$ denotes the <u>normalized</u> bar complex and $C_\bullet$ denotes the complex of *P.L.*-geometric chains. We will later prove that it induces an isomporphism on homology.

In formally defining this map there is an abundance of technical details which need to be taken care of. In the process the fairly intuitive geometric idea behind the definition is somewhat obscured. Thus before plunging into the formal haze, we give an informal intuitive description of the map.

7.2.1 The informal description

Let C_0 be the standard unit cube in the positive n-tant of $\mathbf{R}^n$, minus its faces which do not contain the origin. C_0 is a fundamental domain for the n-torus, and $\mathbf{R}^n$ is partitioned into the sets

$$\{C_z = z + C : z \in \mathbf{Z}^n\}.$$

Let $\sigma \subset \mathbf{R}^n$ be a k-simplex and let $\alpha \in \mathbf{V}$ be a coefficient. Denote by σ_z the intersection of σ with each set C_z. For each z for which σ_z is k-dimensional, translate (the closure of) σ_z over to C_0 and "adjust" the coefficient by replacing α with $(-z)\cdot\alpha$. Finally - project the sum of these chains down to the torus.

Now, for any $\alpha \in \mathbf{V}$ and nondegenerate $\omega_k = z_0[z_1|\dots|z_k]$, map $\alpha \otimes \omega_k \in W_\bullet(\mathbf{Z}^n;\mathcal{L})$ to the image under the procedure described above of the simplex $\sigma \in \mathbf{R}^n$ whose vertices are those of ω_k, with coefficient α.

7.2.2 The formal description

First :

Definition 7.2 Let $C_*^c(\mathbf{R}^n)$ denote the complex of compactly supported *P.L.*-geometric chains on $\mathbf{R}^n$. Define a map

$$\tilde{\psi} : W_*(\mathbf{Z}^n) \to C_*^c(\mathbf{R}^n)$$

as follows :

Let $\omega_k = z_0[z_1|\ldots|z_k] \in W_*(\mathbf{Z}^n)$.
If ω_k is not weakly degenerate (and hence not strongly degenerate), then define $\tilde{\psi}(\omega_k)$ to be the oriented simplex in $\mathbf{R}^n$ whose vertices are the vertices of ω_k (namely $(z_0, z_0 + z_1, \ldots, z_0 + z_1 + \ldots + z_k)$), with the orientation induced by this ordering of the vertices.
Otherwise set $\tilde{\psi}(\omega_k) = 0$.

Proposition 7.3 $\tilde{\psi}$ is a chain map.

Proof: By the definition of $W_*(\mathbf{Z}^n)$ it is clear that if ω_k is not weakly degenerate then
(i) none of the summands in $\partial\omega_k$ are weakly degenerate.
(ii) $\tilde{\psi}(\partial\omega_k) = \partial(\tilde{\psi}(\omega_k))$.

Thus we have only to show the following

Lemma 7.4 If ω_k is degenerate (*i.e.* $\dim(\text{span}_{\mathbf{R}}\{z_1, \ldots, z_k\}) < k$) then $\tilde{\psi}(\partial\omega_k) = 0$.

Proof: Let $\omega_k = z_0[z_1|\ldots|z_k] \in W_k(\mathbf{Z}^n)$ be degenrate and assume first that

$$\dim(\text{span}_{\mathbf{R}}\{z_1, \ldots, z_k\}) = k - 1.$$

Let Δ be the standard k-simplex in $\mathbf{R}^k$, and let $e_i = (0, \ldots, 1, 0, \ldots, 0)$ be the i^{th} vertex of Δ. Denote the i-skeleton of Δ by Δ^i for $i = 0 \ldots k$.
Consider the linear map

$$l : \Delta \to \mathbf{R}^n$$

defined by mapping each e_i to $z_0 + \ldots + z_i$.
Let $S_*(\Delta)$ be the simplicial chain complex of Δ. Then l induces a chain map

$$l_\# : S_*(\Delta) \to C_*^c(\mathbf{R}^n)$$

and it is clear that
(i) $l_\#(\Delta) = \tilde{\psi}(\omega_k)$ and
(ii) $l_\#(\Delta^{k-1}) = \tilde{\psi}(\partial\omega_k)$.

Remark 7.5 The following observation about *P.L.* geometric chains was made in [2] and somewhat more elaborately in [4]:
If $\mathbf{X}$ is a *P.L.* space and $B \subset A \subset \mathbf{X}$ are *P.L.* subspaces with $\dim(A) = k$ and $\dim(B) = k-1$, then denoting by $C_k(A, B)$ the set of k-dimensional P.L. geometric chains on $\mathbf{X}$ which are supported on A and whose boundaries are supported on B, one has :

$$C_k(A, B) = H_k(A, B).$$

Now let $A \subset \mathbf{R}^n$ be the support of $l_\#(\Delta)$ (and of $\tilde{\psi}(\omega_k)$) and for $0 \leq i < k$ let A^i be the support of $l_\#(\Delta^i)$.
There is a commutative diagram :

$$\begin{array}{ccc} \mathbf{Z} = H_k(\Delta, \Delta^{k-1}) & \xrightarrow{l_\#} & H_k(A, A^{k-1}) \\ \downarrow \partial & & \downarrow \partial \\ H_{k-1}(\Delta^{k-1}, \emptyset) & \xrightarrow{l_\#} & H_{k-1}(A^{k-1}, \emptyset) \end{array} \tag{55}$$

where the vertical maps are the connecting homomorphisms from the long exact sequences of the appropriate triples.
Our assertion now reduces to the claim that the image of the generator $1 \in H_k(\Delta, \Delta^{k-1})$ under $l_\# \circ \partial$ is 0. This follows from the commutativity of the diagram and from the observation that since $\dim(A) = k-1$, $A = A^{k-1}$, and thus $H_k(A, A^{k-1}) = 0$.

Finally if $\dim(\mathrm{span}_{\mathbf{R}}\{z_1, \ldots, z_k\}) \leq k-2$, then all faces of ω_k are degenerate too and hence $\tilde{\psi}(\partial\omega_k) = 0$ by definition. □

Now note the following two remarks concerning $C^c_\bullet(\mathbf{R}^n)$:

Remark 7.6
(i) The action (by translation) of $\mathbf{Z}^n$ on $C^c_\bullet(\mathbf{R}^n)$ makes the latter a free $\mathbf{Z}^n$-module.
(ii) Given any (locally finite) triangulation $\tilde{T}'$ of $\mathbf{R}^n$, there is a refinement $\tilde{T}$ of $\tilde{T}'$ and a triangulation T of the n torus T^n so that with these triangulations the covering map

$$proj : \mathbf{R}^n \rightarrow T^n$$

is simplicial.

Corollary 7.7 It follows from (*ii*) that *proj* induces a chain map

$$proj_{\#} : C^c_\bullet(\mathbb{R}^n) \to C_\bullet(\mathcal{T}^n).$$

Corollary 7.8 In view of (*ii*) above, and since $\tilde{\psi}$ preserves the $\mathbb{Z}^n$ action, there is a chain map

$$I \otimes_{\mathbb{Z}[\mathbb{Z}^n]} \tilde{\psi} : \mathbf{V} \otimes_{\mathbb{Z}[\mathbb{Z}^n]} W_\circ(\mathbb{Z}^n) \to \mathbf{V} \otimes_{\mathbb{Z}[\mathbb{Z}^n]} C^c_\bullet(\mathbb{R}^n). \qquad (56)$$

In (56) the complex on the left is $\bar{W}_\bullet(\mathbb{Z}^n; \mathcal{L})$ and

Proposition 7.9 The complex on the right can be canonically identified with $C_\bullet(\mathcal{T}^n, \mathcal{L})$.

Proof: For any chain in $\mathbf{V} \otimes_{\mathbb{Z}[\mathbb{Z}^n]} C^c_\bullet(\mathbb{R}^n)$, choose a representative

$$\sum \alpha_i \otimes c_i \in \mathbf{V} \otimes_{\mathbb{Z}} C^c_\bullet(\mathbb{R}^n)$$

and a fine enough triangulation $\tilde{T}$ so that :
(*i*) All of the c_i's are simplices of $\tilde{T}$ supported on the unit cube in $\mathbb{R}^n$ and
(*ii*) There is a triangulation T of $\mathcal{T}^n$ so that the map *proj* is simplicial.
Then by applying $I \otimes proj$ we obtain a chain in $C_\bullet(\mathcal{T}^n, \mathcal{L})$.
By a similar process (in reverse) it is easy to see that any chain in $C_\bullet(\mathcal{T}^n, \mathcal{L})$ can be obtained in this way. □

With these comments we can define :

Definition 7.10 The definition of ψ.

$$\psi = (I \otimes_{\mathbb{Z}[\mathbb{Z}^n]} proj) \circ (I \otimes_{\mathbb{Z}[\mathbb{Z}^n]} \tilde{\psi}) : \bar{W}_\bullet(\mathbb{Z}^n; \mathcal{L}) \longrightarrow C_\bullet(\mathcal{T}^n, \mathcal{L}).$$

7.2.3 ψ is a quasi-isomorphism

Theorem 7.11 The map $\psi : \bar{W}_\bullet(\mathbb{Z}^n; \mathcal{L}) \to C_\bullet(\mathcal{T}^n; \mathcal{L})$ is a quasi-isomorphism.

Proof: Consider the following diagram :

$$\begin{array}{ccc} C^{CW}_\bullet(\mathcal{T}^n; \mathcal{L}) & \xrightarrow{\Gamma} & \bar{W}_\bullet(\mathbb{Z}^n; \mathcal{L}) \\ & {\scriptstyle i} \searrow \quad \swarrow {\scriptstyle \psi} & \\ & C_\bullet(\mathcal{T}^n; \mathcal{L}) & \end{array}$$

where i can be considered a natural inclusion since each cell in $C_*^{CW}(\mathcal{T}^n)$ is an elementary chain (the interior of a cell is contractible and hence simply connected).

Proposition 7.12 The diagram commutes.

Proof: This follows from the second part of remark 6.18, since the image under $\psi \circ \Gamma$ of any chain in $C_*^{CW}(\mathcal{T}^n; \mathcal{L})$ is simply a subdivision of its image under i.

It is an elementary fact that for a smooth manifold equipped with a CW-decomposition, the inclusion of the cellular chain complex into the complex of $P.L.$-geometric chains induces an isomorphism on homology.

Finally, Γ is a chain-homotopy equivalence and hence a quasi-isomorphism and the proof is thus complete. □

8 The main construction

Let $\mathcal{K} \subset \mathbb{R}^n$ be a cone complex with dual polytope $\mathcal{P}$, and let $\mathbf{X}$ denote the associated toric variety. Recall the diagram from section 2.1 :

$$\begin{array}{ccc} \hat{\mathbf{X}} & \xrightarrow{\hat{p}} & \mathbf{X} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{P} & \end{array}$$

Let $\Sigma = p^{-1}(\partial\mathcal{P})$ denote the "singular set".
Let $\mathbf{V}$ be a rational vector space. A local system (with fiber $\mathbf{V}$) for intersection homology of $\mathbf{X}$ consists of a $\mathbf{V}$-bundle over $\mathbf{X} \setminus \Sigma = \text{int}(\mathcal{P}) \times \mathcal{T}^n$. Any such bundle is uniquely determined (up to equivalence of bundles) by its restriction to $\{x\} \times \mathcal{T}^n$, where x is some fixed point in $\text{int}(\mathcal{P})$, for example the barycenter of $\mathcal{P}$.
So, let $\mathcal{L}$ be a local system on $\text{int}(\mathcal{P}) \times \mathcal{T}^n$ and assume WOLOG that it is the trivial extension to $\text{int}(\mathcal{P}) \times \mathcal{T}^n$ of a local system (which we also denote by $\mathcal{L}$) on $\{b(\mathcal{P})\} \times \mathcal{T}^n$ where $b(\mathcal{P})$ denotes the barycenter of $\mathcal{P}$.

For any $x \in \mathbf{X} \setminus \Sigma$, denote by $\mathbf{V}_x$ the fiber of $\mathcal{L}$ over x.

8.1 $\tilde{D}_*(\mathbf{X};\mathcal{L})$

Let $\mathcal{P}' = sd(\mathcal{P})$ denote the barycentric subdivision of $\mathcal{P}$. Let $S_*(\mathcal{P}')$ denote the simplicial chain complex of $\mathcal{P}'$, and let $S_*(\partial\mathcal{P}')$ be its restriction to $\partial\mathcal{P}$. Denote by $S^0_*(\mathcal{P})$ the relative chain complex $S_*(\mathcal{P}', \partial\mathcal{P}')$. $S^0_*(\mathcal{P})$ is generated by the collection of simplices

$$\{\sigma \in \mathcal{P}' \mid \sigma \not\subset \partial\mathcal{P}'\} = \{\sigma \in \mathcal{P}' \mid \text{The barycenter of } \mathcal{P} \text{ is a vertex of } \sigma\}.$$

Each simplex $\sigma \in S^0_*(\mathcal{P})$ corresponds to an increasing chain of faces of $\mathcal{P}$ ending in the unique n-dimensional face, and hence, dually, it corresponds to a flag of cones

$$K_{-1} \subset K_0 \subset K_1 \subset \ldots \subset K_{m-1} = K_\sigma$$

where K_{-1} denotes the unique 0-dimensional cone and $m = \dim(\sigma)$.
We say that σ *is carried by* K_σ as well as by any cone K of which K_σ is a face.

Setting $F_i = \mathrm{span}_{\mathbf{R}}(K_i)$, σ determines a partial flag of subspaces in $\mathbf{R}^n$:

$$\{0\} \subset F_0 \subset F_1 \subset \ldots \subset F_{m-1}.$$

Denote this flag by $FL(\sigma)$.

Definition 8.1 Let $\bar{W}_\bullet(\mathbb{Z}^n;\mathcal{L})$ be the <u>normalized</u> bar complex. Define a chain complex

$$\tilde{D}_\bullet(\mathbf{X};\mathcal{L}) = S^0_\bullet(\mathcal{P}) \otimes \bar{W}_\bullet(\mathbb{Z}^n;\mathcal{L}).$$

8.2 Allowability and the subcomplex $ID_*(\mathbf{X};\mathcal{L})$

Definition 8.2 Let $\bar{p}$ be a perversity and let $\bar{\omega} = [z_1|\ldots|z_k] \in \bar{W}_\bullet(\mathbb{Z}^n)$ and $\sigma \in S^0_\bullet(\mathcal{P})$.
For each F_i in $FL(\sigma)$, set

$$\lambda_i = 2\dim(F_i) - \bar{p}(2\dim(F_i)) - i - 1.$$

ω is *allowable over* σ if for each F_i in $FL(\sigma)$,

$$\omega|_{F_i} \geq \lambda_i. \tag{57}$$

(Recall that $\omega|_{F_i}$ is defined to be $\#\{\ \{z_1,\ldots,z_k\} \cap F_i\}$).
In the special case of $\bar{p}$ = middle perversity, $\bar{p}(2\dim(F_i)) = \dim(F_i) - 1$, and thus (57) takes on the simpler form :

$$\omega|_{F_i} \geq \dim(F_i) - i.$$

Definition 8.3 Allowability in $\tilde{D}_\bullet(\mathbf{X};\mathcal{L})$. A chain

$$\sum \sigma_i \otimes (\alpha_i \otimes \omega_i) \in \tilde{D}_\bullet(\mathbf{X};\mathcal{L})$$

is allowable if and only if for each i for which $\alpha_i \neq 0$, ω_i is allowable over σ_i.

Definition 8.4 Define $ID_\bullet(\mathbf{X};\mathcal{L})$ to be the subcomplex of $\tilde{D}_\bullet(\mathbf{X};\mathcal{L})$ consisting of all allowable chains with allowable boundaries.

8.3 The map $\varphi : ID_*(X;\mathcal{L}) \to IC_*(X;\mathcal{L})$

We define a map

$$\tilde{\varphi} : \tilde{D}_\bullet(\mathbf{X};\mathcal{L}) \to \tilde{C}_\bullet(\mathbf{X};\mathcal{L}),$$

which preserves allowability and which commutes with the boundary homomorphism when restricted to allowable chains, and hence it induces a chain map

$$\varphi : ID_\bullet(\mathbf{X};\mathcal{L}) \to IC_\bullet(\mathbf{X};\mathcal{L}).$$

The main theorem of this thesis is that the map φ is a quasi-isomorphism (*i.e.* induces an isomorphism on homology).

Let $\psi : \bar{W}_\bullet(\mathcal{I}^n;\mathcal{L}) \to C_\circ(\mathcal{T}^n;\mathcal{L})$ be the map defined in section 7.2. The map $\tilde{\varphi}$ is defined to be the composition $\varphi_2\circ\varphi_1$ of the maps φ_1, φ_2 defined below.
Set

$$\varphi_1 = I \otimes \psi : \tilde{D}_\bullet(\mathbf{X};\mathcal{L}) \to S^0_\circ(\mathcal{P}) \otimes C_\bullet(\mathcal{T}^n;\mathcal{L}).$$

Note that φ_1 is a chain map.

Now define the map $\varphi_2 : S^0_\bullet(\mathcal{P}) \otimes C_\bullet(\mathcal{T}^n;\mathcal{L}) \to \tilde{D}_\bullet(\mathbf{X};\mathcal{L})$ as follows.
Let $\sigma \in S^0_\bullet(\mathcal{P})$ and let $\alpha \otimes \xi \in C_\bullet(\mathcal{T}^n;\mathcal{L})$ be an elementary chain, with $\alpha \in \mathbf{V}_{x_\xi}$, for some designated point $x_\xi \in \text{int}(\xi)$. Let x_σ be some (any) interior point of σ, and set $y = \hat{p}(\{x_\sigma\} \times \{x_\xi\})$.
Now $(\hat{p}(\sigma \times \xi), \hat{p}(\partial(\sigma \times \xi)))$ is an elementary chain in $\mathbf{X}$, and by our assumptions on $\mathcal{L}$, we can consider α to be in $\mathbf{V}_y$.
Define

$$\varphi_2(\sigma \otimes (\alpha \otimes \xi)) = \alpha \cdot \hat{p}(\sigma \times \xi) \in \tilde{C}_\bullet(\mathbf{X};\mathcal{L}),$$

and set

$$\tilde{\varphi} = \varphi_2\circ\varphi_1.$$

Proposition 8.5 $\tilde{\varphi}$ maps allowable chains to allowable chains.

Proof: Let $\sigma\otimes(\alpha\otimes\omega) \in \tilde{D}_{m+k}(\mathbf{X};\mathcal{L})$ be allowable with $\omega = [z_1|\ldots|z_k]$ and $\dim(\sigma) = m$. Let σ correspond to the flag of cones

$$K_{-1} \subset K_0 \subset K_1 \subset \ldots \subset K_{m-1}.$$

For each $0 \le i \le m-1$, denote: The face of $\mathcal{P}$ dual to K_i by $K_i^\bullet$; the stratum dual to K_i by $S_i^\bullet$; the subspace in $FL(\sigma)$ corresponding to K_i by F_i and the dimension of K_i (and of F_i) by d_i. Note that the $S_i^\bullet$'s are precisely the strata which $\tilde{\varphi}(\sigma\otimes(\alpha\otimes\omega))$ meets.

If ω is degenerate then $\tilde{\varphi}(\sigma\otimes(\alpha\otimes\omega)) = 0$ and there is nothing to prove. Otherwise, the k-simplex in $\mathbb{R}^n$ whose vertices are $\{\vec{0}, z_1, z_1+z_2, \ldots, z_1+z_2+\ldots+z_k\}$ spans a k-plane U which satisfies

$$\dim(U \cap F_i) \ge \lambda_i, \quad \forall F_i \in FL(\sigma) \tag{58}$$

because of the allowability condition on ω. Let

$$proj : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = \mathcal{T}^n$$

be the projection. For each $0 \leq i \leq m-1$, let τ_i be the (d_i-dimensional) subtorus

$$\tau_i = proj(F_i) \subset \mathcal{T}^n$$

and set

$$\tau_U = proj(U).$$

$\dim(\tau_U) = k$, and in view of (58) we have that

$$\dim(\tau_U \cap \tau_i) \geq \lambda_i, \qquad \text{for } 0 \leq i \leq m-1. \tag{59}$$

By definition, $\tilde{\varphi}(\sigma \otimes (\alpha \otimes \omega))$ is supported on an $(m+k)$-dimensional $P.L.$-subset of the $(m+k)$-dimensional $P.L.$-subspace $\hat{p}(\sigma \times \tau_U)$ and thus it suffices to prove that the latter is allowable.

For each i, the torus τ_i is collapsed to a point over every point in K_i^*. Thus, in view of (59) we have

$$\dim(\hat{p}(\{x\} \times \tau_U)) \leq k - \lambda_i$$

for every point $x \in K_i^*$, and therefore

$$\dim(\hat{p}(\sigma \times \tau_U) \cap S_i^*) \leq \dim(\sigma \cap K_i^*) + k - \lambda_i.$$

But $\sigma \cap K_i^*$ is simply the simplex of $sd(\mathcal{P})$ corresponding to the flag of faces of $\mathcal{P}$:

$$K_{m-1}^* \subset K_{m-2}^* \subset \ldots \subset K_i^*$$

which is a simplex of dimension $(m-1)-i$. Thus we have

$$\begin{aligned} \dim(\hat{p}(\sigma \times \tau_U) \cap S_i^*) &\leq ((m-1)-i) + (k-\lambda_i) \\ &= m+k-\lambda_i-i-1 \\ &= m+k-2d_i+\bar{p}(2d_i). \end{aligned} \tag{60}$$

On the other hand, the allowability condition is :

$\hat{p}(\sigma \times \tau_U)$ is allowable $\Leftrightarrow$

$$\dim(\hat{p}(\sigma \times \tau_U) \cap S_i^*) \leq \underbrace{\dim(\hat{p}(\sigma \times \tau_U))}_{m+k} - \text{codim}(S_i^*) + \bar{p}(\text{codim}(S_i^*)). \tag{61}$$

Now since S_i^* is dual to K_i, its dimension is $2(n - d_i)$, whence its codimension is $2d_i$. Thus the right hand side of inequality (61) is $m + k - 2d_i + \bar{p}(2d_i)$, and thus

$$\hat{p}(\sigma \times \tau_U) \text{ is allowable } \iff \dim(\hat{p}(\sigma \times \tau_U) \cap S_i^*) \leq m + k - 2d_i + \bar{p}(2d_i) \qquad (62)$$

and the right hand side holds by (60). □

Corollary 8.6 The restriction of $\tilde{\varphi}$ to allowable chains with allowable boundaries gives a chain map

$$\varphi : ID_\bullet(\mathbf{X}; \mathcal{L}) \to IC_\bullet(\mathbf{X}; \mathcal{L}).$$

□

The main theorem of this thesis is :

Theorem 8.7 φ is a quasi-isomorphism.

8.4 The proof that φ is a quasi-isomorphism

We filter the toric variety $\mathbf{X}$ by inverse images of a filtration of the underlying polytope $\mathcal{P}$. Correspondingly we filter the two chain complexes $ID_\bullet(\mathbf{X}; \mathcal{L})$ and $IC_\bullet(\mathbf{X}; \mathcal{L})$ in such a way so that the map φ is filtration preserving. Consequently it induces a map of spectral sequences from the spectral sequence corresponding to the filtered complex $ID_\bullet(\mathbf{X}; \mathcal{L})$ to that of $IC_\bullet(\mathbf{X}; \mathcal{L})$. We prove that the induced map is an isomorphism on the E^1 terms and therefore by the spectral sequence comparison theorem, it follows that φ induces an isomorphism on homology.

8.4.1 The filtrations

<u>**The filtration of $\mathcal{P}$.**</u>

The cone complex $\mathcal{K}$ incuces a filtration of $\mathcal{P}$

$$\mathcal{P}^0 \subset \mathcal{P}^1 \subset \ldots \subset \mathcal{P}^n \qquad (63)$$

as follows :
Let $K \in \mathcal{K}$. Set

$$\mathcal{P}_K = \bigcup_{\sigma \text{ carried by } K} |\sigma|$$

(where $|\sigma|$ denotes the topological closure of σ), and for all $0 \leq k \leq n$ set

$$\mathcal{P}^k = \bigcup_{K : \dim(K) = k} \mathcal{P}_K.$$

Also set

$$\mathcal{P}_{\partial K} = \bigcup_{\sigma \text{ carried by } \partial K} |\sigma|$$

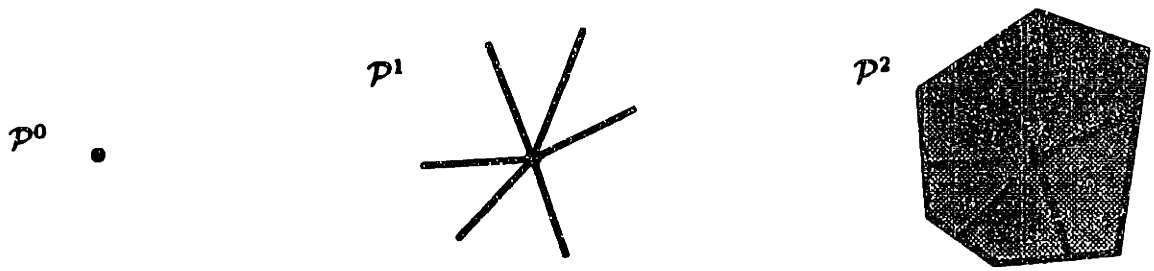

Figure 24: The filtration of the polytope $\mathcal{P}$ from figure 1

The filtration of X .

Set

$$\mathbf{X}^k = p^{-1}(\mathcal{P}^k).$$

We will also refer to the sets

$$\mathbf{X}_K = p^{-1}(\mathcal{P}_K) \quad \forall K \in \mathcal{K}$$

and

$$\mathbf{X}_{\partial K} = p^{-1}(\mathcal{P}_{\partial K}) \quad \forall K \in \mathcal{K}.$$

Note that

$$\mathbf{X}^k = \bigcup_{K:\dim(K)=k} \mathbf{X}_K.$$

Corresponding to the filtration of $\mathcal{P}$, the following subcomplexes of $S^0_\bullet(\mathcal{P})$ are defined in the obvious way :

$$S^0_\bullet(\mathcal{P}_K),\ \ S^0_\bullet(\mathcal{P}_{\partial K}),\ \ S^0_\bullet(\mathcal{P}^k). \tag{64}$$

To simplify notation, we abbreviate the the first two complexes : $S^0_\bullet(K) = S^0_\bullet(\mathcal{P}_K)$ and $S^0_\bullet(\partial K) = S^0_\bullet(\mathcal{P}_{\partial K})$. In addition various relative complexes are defined such as

$$S^0_\bullet(\mathcal{P}_K, \mathcal{P}_{\partial K}) = S^0_\bullet(\mathcal{P}_K)/S^0_\bullet(\mathcal{P}_{\partial K}) \tag{65}$$

etc.

The filtrations of the chain complexes .

The filtration of $\mathbf{X}$

$$\mathbf{X}^0 \subset \mathbf{X}^1 \subset \ldots \subset \mathbf{X}^n = \mathbf{X} \tag{66}$$

induces a filtration of $IC_\bullet(\mathbf{X};\mathcal{L})$:

$$IC_\bullet(\mathbf{X}^0;\mathcal{L}) \subset IC_\bullet(\mathbf{X}^1;\mathcal{L}) \subset \ldots \subset IC_\bullet(\mathbf{X}^n;\mathcal{L}) = IC_\bullet(\mathbf{X};\mathcal{L}). \quad (67)$$

For any of the subcomplexes of $S_\bullet^0(\mathcal{P})$ as in (64) and (65) we define the associated subcomplex of $ID_\bullet(\mathbf{X};\mathcal{L})$ as in the following example :

$$ID_\bullet(\mathbf{X}_K;\mathcal{L}) \stackrel{def}{=} ID_\bullet(\mathbf{X};\mathcal{L}) \cap (S_\bullet^0(K) \otimes \bar{W}(\mathbb{Z}^n;\mathcal{L})). \quad (68)$$

Note that if K_1 is a face of K_2 then $ID_\bullet(\mathbf{X}_{K_1};\mathcal{L}) \subset ID_\bullet(\mathbf{X}_{K_2};\mathcal{L})$.
We thus obtain the filtration of $ID_\bullet(\mathbf{X};\mathcal{L})$:

$$ID_\bullet(\mathbf{X}^0;\mathcal{L}) \subset ID_\bullet(\mathbf{X}^1;\mathcal{L}) \subset \ldots \subset ID_\bullet(\mathbf{X}^n;\mathcal{L}) = ID_\bullet(\mathbf{X};\mathcal{L}). \quad (69)$$

8.4.2 The proof

To prove theorem 8.7 we must show that φ induces an isomorphism

$$\varphi_\bullet : H_\bullet(ID_\bullet(\mathbf{X}^k,\mathbf{X}^{k-1};\mathcal{L})) \xrightarrow{\cong} H_\bullet(IC_\bullet(\mathbf{X}^k,\mathbf{X}^{k-1};\mathcal{L})) = IH_\bullet(\mathbf{X}^k,\mathbf{X}^{k-1};\mathcal{L}). \quad (70)$$

Now note that by the definition of a cone complex, for any two k-dimensional cones $K_1 \neq K_2$, the intersection $K_1 \cap K_2$ is a <u>proper</u> face of each. It follows that

$$\mathbf{X}_{K_1} \cap \mathbf{X}_{K_2} \subset \mathbf{X}^{k-1}.$$

Therefore,

$$ID_\bullet(\mathbf{X}^k,\mathbf{X}^{k-1};\mathcal{L}) = \bigoplus_{K:\dim(K)=k} ID_\bullet(\mathbf{X}_K,\mathbf{X}_K \cap \mathbf{X}^{k-1};\mathcal{L}).$$

But note that

$$\mathbf{X}_K \cap \mathbf{X}^{k-1} = \mathbf{X}_{\partial K}.$$

Thus, in order to show that (70) holds it suffices to prove

Proposition 8.8 For any cone $K \in \mathcal{K}$, φ induces an isomorphism

$$\varphi_\bullet : H_\bullet(ID_\bullet(\mathbf{X}_K,\mathbf{X}_{\partial K};\mathcal{L})) \xrightarrow{\cong} IH_\bullet(\mathbf{X}_K,\mathbf{X}_{\partial K};\mathcal{L}). \quad (71)$$

Thus the proof of the main theorem is reduced to proving proposition 8.8. The rest of this section is devoted to this proof.
Proof: The proof is by induction on $n = \dim(\mathcal{K}) = \dim_{\mathbb{C}}(\mathbf{X})$. The base case $n = 1$ follows from the following lemma which we will use in the inductive step as well.

Lemma 8.9 Let $\mathcal{K} \subset \mathbf{R}^n$ be a cone complex and let $K \in \mathcal{K}$ be a cone. then (71) holds if either of the following two conditions holds :
(i) $\dim(K) = 0$
(ii) $\dim(K) = n$ and (71) holds for all $K' \subset \partial K$.

proof of lemma 8.9:
(i) $\mathbf{X}^0 = \mathcal{T}^n$, and it does not meet any of the singular strata. Thus the assertion is precisely that of theorem 7.11.

(ii) Let K be n-dimensional.

Fact 8.10

$$\mathbf{X}_K = c(\mathbf{X}_{\partial K}) \tag{72}$$

where for any sratified pseudomanifold $\mathbf{Y}$, $c(\mathbf{Y})$ denotes the topological cone on $\mathbf{Y}$, stratified by the cones on the strata of $\mathbf{Y}$, along with the apex of the cone as an additional 0-dimensional stratum.

Thus we have as in the coning lemma (3.21) a quasi-isomorphism $q \circ coning$:

$$IC_{\tau_{\bullet \geq r}}(\mathbf{X}_{\partial K};\mathcal{L}) \xrightarrow{coning} IC_{\bullet+1}(\mathbf{X}_K;\mathcal{L}) \xrightarrow{q} IC_{\bullet+1}(\mathbf{X}_K,\mathbf{X}_{\partial K};\mathcal{L})$$

where q is the quotient map, τ is the truncation operator defined in 3.6.3 and

$$r = \dim(\mathbf{X}_K) - \bar{p}(\dim(\mathbf{X}_K)) - 1 = 2n - \bar{p}(2n) - 1.$$

Note that if $\bar{p}$ = middle-perversity, then $r = n$.
We define an analogous chain map on $ID_\bullet(\mathbf{X}_{\partial K};\mathcal{L})$ as follows:
Let $\sigma \in S^0_\bullet(K)$ be a simplex which is not in $S^0_\bullet(\partial K)$. Then $\sigma = c(\sigma')$ for some $\sigma' \in S^0_\bullet(\partial K)$, where the 'apex' of the cone is $K^* =$ the vertex of $\mathcal{P}$ dual to K. The composition

$$\begin{array}{ccccc} S^0_\bullet(\partial K) & \longrightarrow & S^0_{\bullet+1}(K) & \longrightarrow & S^0_{\bullet+1}(K,\partial K) \\ \sigma & \longmapsto & c(\sigma) & \longmapsto & [c(\sigma)] \end{array}$$

is clearly a chain-isomorphism.
Consequently, we define

$$\begin{array}{ccccc} \tilde{D}_\bullet(\mathbf{X}_{\partial K};\mathcal{L}) & \xrightarrow{coning} & \tilde{D}_{\bullet+1}(\mathbf{X}_K;\mathcal{L}) & \xrightarrow{q} & \tilde{D}_{\bullet+1}(\mathbf{X}_K,\mathbf{X}_{\partial K};\mathcal{L}) \\ \sigma\otimes(\alpha\otimes\omega) & \longmapsto & c(\sigma)\otimes(\alpha\otimes\omega) & \longmapsto & [c(\sigma)\otimes(\alpha\otimes\omega)] \end{array}$$

and again the composition is a chain-isomorphism.

Proposition 8.11

$$c(\sigma)\otimes(\alpha\otimes\omega) \in \tilde{D}_{i+1}(\mathbf{X}_K;\mathcal{L}) \text{ is allowable } \Leftrightarrow \begin{cases} \sigma\otimes(\alpha\otimes\omega) \in \tilde{D}_r(\mathbf{X}_{\partial K};\mathcal{L}) \text{ is allowable} \\ \text{and} \\ i \geq r \end{cases}$$

Proof: Let $\omega = [z_1|\ldots|z_k]$ and let σ be an m-simplex with $m + k = i$. Let

$$FL(\sigma) = F_0 \subset \ldots \subset F_{m-1}.$$

Then

$$FL(c(\sigma)) = F_0 \subset \ldots \subset F_{m-1} \subset F_m = \mathbb{R}^n.$$

In order for $c(\sigma)\otimes(\alpha\otimes\omega)$ to be allowable we must have that

$$\omega|_{F_j} \geq \lambda_j$$

for all $0 \leq j \leq m$. For $j \leq m-1$ this holds if and only if $\sigma\otimes(\alpha\otimes\omega)$ is allowable. For $j = m$ it holds if and only if

$$k \;=\; \#\{\,\{z_1,\ldots,z_k\}\cap\mathbb{R}^n\} \geq \lambda_m$$

i.e. if and only if $i \;=\; m+k \;\geq\; m+\lambda_m \;=\; m+(2n-\bar{p}(2n)-m-1) \;=\; r.$ □

Corollary 8.12 The composition

$$ID_{\tau_* \geq r}(\mathbf{X}_{\partial K};\mathcal{L}) \xrightarrow{coning} ID_{*+1}(\mathbf{X}_K;\mathcal{L}) \xrightarrow{q} ID_{*+1}(\mathbf{X}_K,\mathbf{X}_{\partial K};\mathcal{L}) \tag{73}$$

is defined and is a quasi-isomorphism. □

Finally, there is a commutative diagram

$$\begin{array}{ccc} ID_{\tau_* \geq r}(\mathbf{X}_{\partial K};\mathcal{L}) & \xrightarrow{q\circ coning} & ID_{*+1}(\mathbf{X}_K,\mathbf{X}_{\partial K};\mathcal{L}) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ IC_{\tau_* \geq r}(\mathbf{X}_{\partial K};\mathcal{L}) & \xrightarrow{q\circ coning} & IC_{*+1}(\mathbf{X}_K,\mathbf{X}_{\partial K};\mathcal{L}) \end{array} \tag{74}$$

in which the horizontal maps are quasi-isomorphisms by the previous discussion. As for the map on the left, it is a quasi-isomorphism for the following reason : If we restrict the spectral sequences to the subspace $\mathbf{X}_{\partial K}$, the hypothesis in (*ii*) is precisely stating that (71) holds for all $K' \subset \partial K$, and hence that φ induces an isomorphism on the E^1 terms of the restricted spectral sequences. Thus by the spectral sequence comparison theorem, the restricted map

$$\varphi : ID_\bullet(\mathbf{X}_{\partial K};\mathcal{L}) \to IC_\circ(\mathbf{X}_{\partial K};\mathcal{L})$$

is a quasi-isomorphism. Applying the truncation operator τ does not change this.

Corollary 8.13 The map on the right has no choice but to be a quasi-isomorphism as well.

□(lemma 8.9)

Corollary 8.14 (of lemma 8.9). Proposition 8.8 holds if $\dim(\mathcal{K}) = 1$.

Proof: There is only one cone comlex in $\mathbf{R}^1$. It has only 0 and 1-dimensional (*i.e.* n-dimensional) cones and hence both parts of lemma 8.9 apply. □

We proceed by induction on n, so suppose that proposition 8.8 holds for all cone complexes of dimension $\leq n-1$, and let $\mathcal{K} \subset \mathbf{R}^n$ be an n-dimensional cone complex, $\mathcal{P}$ the dual polytope, $\mathbf{X}$ the associated toric variety, and $\mathcal{L}$ an appropriate local system with fiber $\mathbf{V}$.
Let $K \in \mathcal{K}$.

Case 1: $\dim(K) = 0$. In this case, (71) holds by part (i) of lemma 8.9.

Case 2: $1 \leq \dim(K) \leq n-1$. This is the difficult case. The proof immediately follows the case $\dim(K) = n$.

Case 3: $\dim(K) = n$. As a consequence of case 1 and case 2, part (ii) of lemma 8.9 applies and (71) holds.

The proof of case 2.

Loosely speaking, the proof goes as follows : To use the inductive hypothesis, we "factor out" a circle from the space $\mathbf{X}_K$. As an algebraic analogue, we use the results of section 7 about the Eilenberg-Zilber theorem to algebraically "factor out a circle" from $ID_\bullet(\mathbf{X};\mathcal{L})$ allowing us to express the latter complex in terms of a related complex corresponding to a toric variety of one lower dimension. Finally we apply the combing lemma to complete the proof.

So, let $K \in \mathcal{K}$ be a k-dimensional cone with $1 \leq k \leq n-1$, and set $F = \mathrm{span}_{\mathbb{R}} K$. Then $F \cap \mathbb{Z}^n$ is a k-dimensional sublattice of $\mathbb{Z}^n$.
Choose a basis $\{z_1, \ldots, z_k\} \subset \mathbb{Z}^n$ of this sublattice and complete it to a basis $\{z_1, \ldots, z_n\}$ of $\mathbb{Z}^n$.
Set

$$\mathbb{Z}^{n-1} = \mathrm{span}_{\mathbb{Z}}\{z_1, \ldots, z_{n-1}\}$$

and

$$\mathbb{Z} = \mathrm{span}_{\mathbb{Z}}\{z_n\},$$

whence

$$\mathbb{Z}^n = \mathbb{Z} \oplus \mathbb{Z}^{n-1}.$$

Correspondingly we have the decompositions :

$$\mathbf{R}^n = \mathbf{R} \oplus \mathbf{R}^{n-1}$$

and

$$\mathcal{T}^n = S^1 \times \mathcal{T}^{n-1}.$$

Accordingly, set

$$\mathcal{L}' = \mathcal{L}|_{\{s\} \times \mathcal{T}^{n-1}} \quad \text{and} \quad \mathcal{L}'' = \mathcal{L}|_{S^1 \times \{t\}}$$

where s and t are respective base points of S^1 and of $\mathcal{T}^{n-1}$.
K is a k-dimensional rational polyhedral cone in $\mathbf{R}^{n-1}$. Completing it arbitrarily to a cone complex $\mathcal{K}' \subset \mathbf{R}^{n-1}$ we have the associated toric variety $\mathbf{X}'$ which is $(n-1)$-dimensional (over $\mathbf{C}$), and for which :

(*i*) $\mathcal{P}'$ can be constructed so that $\mathcal{P}'_K = \mathcal{P}_K$ and $\mathcal{P}'_{\partial K} = \mathcal{P}_{\partial K}$ (see remark 2.4).
(*ii*) $\mathbf{X}_K = S^1 \times \mathbf{X}'_K$, and
(*iii*) $\mathbf{X}_{\partial K} = S^1 \times \mathbf{X}'_{\partial K}$.

Furthermore,

(*iv*) $S^0_\bullet(K)$, $S^0_\bullet(\partial K)$, and $S^0_\bullet(K, \partial K)$ are the same for both spaces, and
(*v*) For any $\sigma \in S^0_\bullet(K)$, the condition of allowability over σ is essentially identical in $\tilde{D}_\bullet(\mathbf{X}; \mathcal{L})$ and in $\tilde{D}_\bullet(\mathbf{X}'; \mathcal{L}')$ since $FL(\sigma) \subseteq \mathbf{R}^{n-1}$.

By the inductive hypothesis we have that

$$\varphi' : ID_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}') \to IC_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}') \tag{75}$$

is a quasi-isomorphism. (We use φ' to denote the map φ from theorem 8.7 for the toric variety $\mathbf{X}'$).

As in (51) in section 7, we compose the chain-homotopy equivalence (43) of section 6.7 with that of section 6.9, to obtain a chain-homotopy equivalence

$$W_\bullet(\mathbb{Z}^n) \underset{\nabla\circ\eta}{\overset{g\circ f}{\rightleftarrows}} \widetilde{C^{CW}_\bullet}(S^1) \otimes W_\bullet(\mathbb{Z}^{n-1}). \tag{76}$$

By tensoring with $S^0_\bullet(K,\partial K)$ on both sides and rearranging the terms on the right (using the cummutativity of the tensor product) we have a chain homotopy equvalence

$$S^0_\bullet(K,\partial K) \otimes W_\bullet(\mathbb{Z}^n) \underset{\Theta_2}{\overset{\Theta_1}{\rightleftarrows}} \widetilde{C^{CW}_\bullet}(S^1) \otimes (S^0_\bullet(K,\partial K) \otimes W_\bullet(\mathbb{Z}^{n-1})). \tag{77}$$

Where $\Theta_1 = rearrange\circ(I \otimes (g\circ f))$ and
$\Theta_2 = (I \otimes (\nabla\circ\eta))\circ unrearrange.$

Remark 8.15
(*i*) Since all the maps involved are the identity on the $S^0_\bullet(K,\partial K)$ component, all the properties pertaining to allowability (section 6.8) continue to hold independently of this component.
(*ii*) Considering each $S^0_i(K,\partial K)$ as a trivial $\mathbb{Z}^n$-module on the left hand side and as a trivial $\mathbb{Z}^{n-1}$-module on the right hand side, both sides are complexes of $\mathbb{Z}^n$-modules and the maps preserve the $\mathbb{Z}^n$-action.
Recall that the $\mathbb{Z}^n$-action on the right is the following :
For $z = (z'', z')$ (with $z'' \in \mathbb{Z}$ and $z' \in \mathbb{Z}^{n-1}$),

$$z \cdot (c \otimes (\sigma \otimes \omega)) = (z'' \cdot c) \otimes (\sigma \otimes (z' \cdot \omega)).$$

Corollary 8.16 Tensoring both sides of (77) over $\mathbb{Z}[\mathbb{Z}^n]$ with $\mathbf{V}$ we have a chain-homotopy equivalence

$$\mathbf{V} \otimes_{\mathbb{Z}[\mathbb{Z}^n]} (S^0_\bullet(K,\partial K) \otimes W_\bullet(\mathbb{Z}^n)) \underset{\Theta_4}{\overset{\Theta_3}{\rightleftarrows}} \tag{78}$$

$$\mathbf{V} \otimes_{\mathbb{Z}[\mathbb{Z}^n]} (\widetilde{C^{CW}_\bullet}(S^1) \otimes (S^0_\bullet(K,\partial K) \otimes W_\bullet(\mathbb{Z}^{n-1}))).$$

Where $\Theta_3 = I_{\mathbf{V}} \otimes_{\mathbb{Z}[\mathbb{Z}^n]} \Theta_1$ and $\Theta_4 = I_{\mathbf{V}} \otimes_{\mathbb{Z}[\mathbb{Z}^n]} \Theta_2$.

Now note that the complex on the left is isomorphic to $\tilde{D}_\circ(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$ (by rearranging the terms and since $S^0_\bullet(K, \partial K)$ is a trivial $\mathbb{Z}^n$-module), whereas for the complex on the right we have, in view of the remarks in section 6.5 and the description of the $\mathbb{Z}^n$-action :

Proposition 8.17

$$\mathbf{V} \otimes_{\mathbb{Z}[\mathbb{Z}^n]} (\widetilde{C^{CW}_\bullet}(S^1) \otimes (S^0_\bullet(K,\partial K) \otimes W_\bullet(\mathbb{Z}^{n-1}))) =$$

$$\begin{aligned}
&= [\mathbf{V} \otimes (\widetilde{C^{CW}_\bullet}(S^1) \otimes (S^0_\bullet(K,\partial K) \otimes W_\bullet(\mathbb{Z}^{n-1})))]_{\mathbb{Z}^n} \\
&= [\, [\mathbf{V} \otimes (\widetilde{C^{CW}_\bullet}(S^1) \otimes (S^0_\bullet(K,\partial K) \otimes W_\bullet(\mathbb{Z}^{n-1})))]_{\mathbb{Z}^{n-1}}]_{\mathbb{Z}} \\
&\cong [\widetilde{C^{CW}_\bullet}(S^1) \otimes (\mathbf{V} \otimes_{\mathbb{Z}[\mathbb{Z}^{n-1}]} (S^0_\bullet(K,\partial K) \otimes W_\bullet(\mathbb{Z}^{n-1})))]_{\mathbb{Z}} \\
&\cong [\widetilde{C^{CW}_\bullet}(S^1) \otimes (S^0_\bullet(K,\partial K) \otimes (\mathbf{V} \otimes_{\mathbb{Z}[\mathbb{Z}^{n-1}]} W_\bullet(\mathbb{Z}^{n-1})))]_{\mathbb{Z}} \\
&= \widetilde{C^{CW}_\bullet}(S^1) \otimes_{\mathbb{Z}[\mathbb{Z}]} (S^0_\bullet(K,\partial K) \otimes \tilde{W}_\bullet(\mathbb{Z}^{n-1}; \mathcal{L}')) \\
&= \widetilde{C^{CW}_\bullet}(S^1) \otimes_{\mathbb{Z}[\mathbb{Z}]} \tilde{D}_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')
\end{aligned}$$

where $[\,]_G$ denotes the co-invariants functor as described in section 6.5.

Proof: The first two equalities follow from (36) in section 6.5.
The two isomorphisms which follow are simply a rearrangement of terms, after using the fact that $\widetilde{C^{CW}_\bullet}(S^1)$ and $S^0_\bullet(K, \partial K)$ are both trivial modules over $\mathbb{Z}^{n-1}$. The final two equalities hold by the definitions of $\tilde{W}_\bullet(\mathbb{Z}^{n-1}; \mathcal{L}')$ and of $\tilde{D}_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')$. □

For the sake of clarity we state once again the final homotopy equivalence :

$$\tilde{D}_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L}) \underset{\Theta_4}{\overset{\Theta_3}{\rightleftarrows}} \widetilde{C^{CW}_\bullet}(S^1) \otimes_{\mathbb{Z}[\mathbb{Z}]} \tilde{D}_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}'). \tag{79}$$

and note that the $\mathbb{Z}$-action on $\tilde{D}_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')$ is :

$$z \cdot (\sigma \otimes (\alpha \otimes \omega)) = \sigma \otimes ((z \cdot \alpha) \otimes \omega).$$

We now show that (79) continues to hold when the complexes $\tilde{D}_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$ and $\tilde{D}_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')$ are restricted to their respective subcomplexes $ID_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$ and $ID_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')$.

First note that the chain complex $\widetilde{C^{CW}_\bullet}(S^1) \otimes_{\mathbf{Z}[\mathbf{Z}]} \tilde{D}_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')$ can be described explicitly as follows :
Let c_0 and c_1 denote respective representatives of the $\mathbf{Z}$-orbits in $\widetilde{C^{CW}_0}(S^1)$ and $\widetilde{C^{CW}_1}(S^1)$, which we can identify with the cells of $C^{CW}_\bullet(S^1)$ (the complex "downstairs"). Then for any $k \geq 0$, the k-chains of $\widetilde{C^{CW}_\bullet}(S^1) \otimes_{\mathbf{Z}[\mathbf{Z}]} \tilde{D}_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')$ all have the form

$$c_0 \otimes \gamma_k + c_1 \otimes \gamma_{k-1} \qquad (80)$$

where $\gamma_k \in \tilde{D}_k(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')$ and $\gamma_{k-1} \in \tilde{D}_{k-1}(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')$.
The boundary homomorphism is given by :

$$\partial(c_0 \otimes \gamma_k + c_1 \otimes \gamma_{k-1}) = c_0 \otimes \partial\gamma_k + c_0 \otimes (1 \cdot \gamma_{k-1} - \gamma_{k-1}) - c_1 \otimes \partial\gamma_{k-1} \qquad (81)$$

(where 1 is not the identity but rather the generator of $\mathbf{Z}$).

Remark 8.18 The following two remarks follow from theorem 6.20 and from part (i) of remark 8.15.

(i) Let $\xi \in \tilde{D}_k(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$ be allowable and let $\Theta_3(\xi) = c_0 \otimes \gamma_k + c_1 \otimes \gamma_{k-1}$. then γ_k and γ_{k-1} are allowable.
(ii) Conversely, if γ_k and γ_{k-1} are allowable then $\Theta_4(c_0 \otimes \gamma_k)$ and $\Theta_4(c_1 \otimes \gamma_{k-1})$ are both allowable.

Proposition 8.19 Let $\xi \in ID_k(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$ and let $\Theta_3(\xi) = c_0 \otimes \gamma_k + c_1 \otimes \gamma_{k-1}$. Then $\gamma_k, \gamma_{k-1} \in ID_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$.

Proof: By remark 8.18 both γ_k and γ_{k-1} are allowable. Thus we have only to show that their respective boundaries are too.
By (81)

$$\partial(\Theta_3(\xi)) = c_0 \otimes (\partial\gamma_k + 1 \cdot \gamma_{k-1} - \gamma_{k-1}) - c_1 \otimes \partial\gamma_{k-1}.$$

Now, Θ_3 is a chain map. Thus $\partial(\Theta_3(\xi)) = \Theta_3(\partial(\xi))$ and since $\xi \in ID_k(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$, $\partial(\xi)$ is allowable. It then follows from part (i) of remark 8.18, that $\partial\gamma_{k-1}$ is allowable as well as $\partial\gamma_k + (1 \cdot \gamma_{k-1} - \gamma_{k-1})$.
Now to see that $\partial\gamma_k$ is allowable note that γ_{k-1} was allowable to begin with, and therefore $1 \cdot \gamma_{k-1}$ is allowable since the $\mathbf{Z}$-action only affects the coefficients and in particular it does not introduce any new non-zero coefficients. Thus the chain $\zeta =$

$(1_{\cdot}\, \gamma_{k-1} - \gamma_{k-1})$ is allowable, and since $\partial\gamma_k + \zeta$ is allowable, it follows that $\partial\gamma_k$ is allowable too. □

Conversely we have

Proposition 8.20 Let $\gamma \in ID_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')$. Then both $\Theta_4(c_0 \otimes \gamma)$ and $\Theta_4(c_1 \otimes \gamma)$ are in $ID_k(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$.

Proof: This follows immediately from part (ii) of remark 8.18 and from the fact that Θ_4 is a chain map. □

Corollary 8.21 Restricting Θ_3 and Θ_4 there is a chain-homotopy equivalence :

$$ID_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L}) \underset{\Theta_4}{\overset{\Theta_3}{\rightleftarrows}} \widetilde{C^{CW}_\bullet}(S^1) \otimes_{\mathbf{Z}[\mathbf{Z}]} ID_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}'). \tag{82}$$

with

$$\Theta_3 \circ \Theta_4 = I.$$

Proof: It remains only to be shown that the homotopy, which we denote by

$$\Lambda : \tilde{D}_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L}) \to \tilde{D}_{\bullet+1}(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$$

maps $ID_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$ to $ID_{\bullet+1}(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$.
In part (iii) of theorem 6.20 it was shown that the "original" homotopy $\Phi : W_\bullet(\mathbf{Z}^n) \to W_{\bullet+1}(\mathbf{Z}^n)$ preserves itersection numbers with any subspace $F \subseteq \mathbf{R}^{n-1}$. The homotopy Λ was derived from Φ by various operations which do not affect this property (*e.g.* tensoring with various identity maps) and since the allowability in $\tilde{D}_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$ is determined by intersection numbers with subspaces of $\mathbf{R}^{n-1}$, it follows that Λ preserves allowability. Λ is not a chain map so this does not immediately imply that $\partial(\Lambda(\xi))$ is allowable whenever $\partial(\xi)$ is allowable, however note that

$$\partial(\Lambda(\xi)) = \Theta_4 \circ \Theta_3(\xi) - \xi - \Lambda(\partial(\xi))$$

since Λ is a homotopy between $\Theta_4 \circ \Theta_3$ and I, and each of the summands on the right is allowable when $\xi \in ID_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$.

□

The final step.

The following proposition completes the proof :

Proposition 8.22 There is a commutative diagram

$$\begin{array}{ccc} \widetilde{C^{CW}_{\circ}}(S^1)\otimes_{\mathbb{Z}[\mathbb{Z}]} ID_{\circ}(\mathbf{X}'_K,\mathbf{X}'_{\partial K};\mathcal{L}') & \xrightarrow{\ \Theta_4\ } & ID_{\circ}(\mathbf{X}_K,\mathbf{X}_{\partial K};\mathcal{L}) \\ \Big\downarrow j & & \Big\downarrow \varphi \\ IC_{\bullet}(\mathbf{X}'_K,\mathbf{X}'_{\partial K};\mathcal{L}')\oplus\overline{susp}(IC_{\circ}(\mathbf{X}'_K,\mathbf{X}'_{\partial K};\mathcal{L}')) & \xhookrightarrow{\ \imath\ } & IC_{\bullet}(\mathbf{X}_K,\mathbf{X}_{\partial K};\mathcal{L}) \end{array}$$

and the maps $\imath$, j, Θ_4 are all quasi-isomorphisms.

Recall that $\mathbf{X}_K = S^1\times\mathbf{X}'_K$.

Proof of proposition 8.22:
Θ_4 is a chain-homotopy equivalence and hence a quasi-isomorphism.
$\imath$ is a quasi-isomorphism by the corollary of the combing lemma (corollary 3.19).

<u>The map j:</u> Recall that by the inductive hypothesis, the map

$$\varphi' : ID_{\bullet}(\mathbf{X}'_K,\mathbf{X}'_{\partial K};\mathcal{L}') :\longrightarrow IC_{\bullet}(\mathbf{X}'_K,\mathbf{X}'_{\partial K};\mathcal{L}')$$

is a quasi-isomorphism. Thus, in view of the description (80), we define

$$j(c_0\otimes\gamma_k + c_1\otimes\gamma_{k-1}) \stackrel{def}{=} (\varphi'(\gamma_k),\ \overline{susp}(\varphi'(\gamma_{k-1}))).$$

It is not immediate that j is a chain map. The subtle point is that it is not clear that j commutes with the respective $\mathbb{Z}$-actions on $ID_{\bullet}(\mathbf{X}'_K,\mathbf{X}'_{\partial K};\mathcal{L}')$ and on $IC_{\bullet}(\mathbf{X}'_K,\mathbf{X}'_{\partial K};\mathcal{L}')$ (where the $\mathbb{Z}$-action on the latter is the one described in section 3.6.2). However the commutativity of the diagram (which we prove next) will imply that j is a chain map since $\varphi\circ\Theta_4$ is a chain map as the composition of chain maps, and $\imath$ is simply an inclusion.
A careful examination of the reason given in proposition 7.12 for why the diagram of section 7.2.3 commutes shows that the same reasoning holds here namely that the (simplicial) image under $\varphi\circ\Theta_4$ of any chain in $\widetilde{C^{CW}_{\bullet}}(S^1)\otimes_{\mathbb{Z}[\mathbb{Z}]} ID_{\bullet}(\mathbf{X}'_K,\mathbf{X}'_{\partial K};\mathcal{L}')$ is simply a subdivision of its image under $\imath\circ j$.

Proposition 8.23 ***j is a quasi-isomorphism.***

Proof: For simplicity, denote

$$D_{\bullet} = ID_{\circ}(\mathbf{X}'_K,\mathbf{X}'_{\partial K};\mathcal{L}')\ .$$

and

$$C_\bullet = IC_\circ(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}').$$

Then for each k, the degree k chain group in $\widetilde{C^{CW}_\bullet}(S^1) \otimes_{\mathbb{Z}[\mathbb{Z}]} ID_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}')$ is canonically isomorphic to

$$D_k \oplus D_{k-1}$$

while the degree k chain group in $IC_\bullet(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}') \oplus \overline{susp}(IC_\circ(\mathbf{X}'_K, \mathbf{X}'_{\partial K}; \mathcal{L}'))$ is canonically isomorphic to

$$C_k \oplus C_{k-1}.$$

Correspondingly, the map j in degree k can be written as :

$$j_k = \varphi'_k \oplus \varphi'_{k-1}.$$

Thus, by filtering the two complexes respectively as

$$D_\bullet \subset D_\bullet \oplus D_{\bullet+1}$$

and

$$C_\bullet \subset C_\bullet \oplus C_{\bullet+1}$$

and using the fact that φ' is a quasi-isomorphism, a simple spectral sequence argument (as in corollary 3.19) implies that j is in fact a quasi-isomorphism. □prop. 8.23

□prop. 8.22

Corollary 8.24

$$\varphi : ID_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L}) \to IC_\bullet(\mathbf{X}_K, \mathbf{X}_{\partial K}; \mathcal{L})$$

is a quasi-isomorphism.

□

□prop. 8.8

□thm. 8.7

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