HAMILTON'S LAW OF VARYING ACTION: ITS INTERPRETATION AND APPLICATION IN DYNAMICS PROBLEMS OF LUMPED-PARAMETER SYSTEMS

by

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ABSTRACT

Relationships between Hamilton's Law of Varying Action (HLVA), Hamilton's Principle and Hamilton's Principle of Stationary Action are discussed. A conceptual interpretation of HLVA is presented with an illustrative example. The application of HLVA to obtain an approximate solution to dynamics problems of lumped-parameter systems via trial solutions with undetermined parameters, namely the HLVA method, is evaluated and extended to nonholonomic systems. The approximation accuracy of the HLVA method is investigated. Comparison of the HLVA method with Galerkin's method is made. This comparison shows that the former provides the identical approximation accuracy as the latter when identical trial solutions are used, but requires more computational labor than the latter. The role of the HLVA method in engineering analysis is also discussed.

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To my parents

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1 INTRODUCTION

1.1 Background of Hamilton's Law of Varying Action

Hamilton's *Law of Varying Action* was presented by Hamilton in his two classical *Essays* [1-2] on a general energy method in dynamics in 1834 and 1835.

In his first *Essay*, Hamilton set forth two basic forms of his *Law of Varying Action* for lumped-parameter holonomic conservative systems. The first form expresses the variation of a function, which he called the *characteristic function*, in terms of the variations of initial and final configurations, and of the variation of the *Hamiltonian* of the system. The *characteristic function*, defined as the time integral of twice the kinetic coenergy along a motion, is now called the *Maupertuis Action* [6]. Transformed from the first form, the second form of Hamilton's Law of Varying Action expresses the variation of a function, which he called the *principal function*, in terms of the variations of initial and final configurations and of the variation of the final instant. The *principle function*, defined as the time integral of the *lagrangian* along a motion, is now called the *lagrangian action* [6].

The second form of Hamilton's *Law of Varying Action* has been widely referred to as *Hamilton's Law of Varying Action* due to the applications made by Hamilton in his second *Essay*. In the second Essay, Hamilton systematically developed the use of the *principle function* and established the *canonical equations* of motion. In addition, Hamilton pointed out that the variation of the *principle function* enjoys the double property of giving the integrals of the *canonical equations* when the configurations of the system at end instant are varying, and of giving the *Lagrange's Equations* of motion when the configurations of the system at terminal instants are fixed. When the configurations of the system at the terminal instants are fixed, Hamilton's Law of Varying Action is what is now called *Hamilton's Principle of Stationary Action* [3].

Since Hamilton, conceptual interpretation of Hamilton's Principle of Stationary Action has been given in [3, p32-34]. Conceptual interpretation of Hamilton's Law of Varying Action, however, has not been seen in literature. (One of the goals of this thesis is to give a conceptual interpretation of Hamilton's Law of Varying Action)

The primary application of an extended form of Hamilton's Principle of Stationary Action, called *Hamilton's Principle*†, has been that of deriving the differential equations of motion. A recent application is that of using Hamilton's Law of Varying Action as a variational formulation for an approximate solution to dynamics problem [10-12].

Bailey [10-11] demonstrated the application of Hamilton's Law of Varying Action to obtain direct solutions to initial value problems for particular holonomic systems. By direct solutions, he meant that the solution is achieved without the use of differential equation theory or reference to the equations of motion of the system.

Oz and Adiguzel [12] applied the procedure demonstrated by Bailey to a generic holonomic dynamics problem and called the procedure the "formulation of algebraic equations of motion" ††.

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† *Hamilton's Principle of Stationary Action* is for holonomic systems under conservative forces, whereas *Hamilton's Principle* applies to holonomic systems under nonconservative forces as well as conservative forces. In addition, *Hamilton's Principle* can be extended for nonholonomic systems as well. These two principles are discussed in chapter 2 of this thesis.

†† The confusing phrase 'algebraic equations of motion' will not be used hereafter in this thesis.

1.2 Objective of the Study

Although the application of Hamilton's Law of Varying Action (HLVA) to obtain a direct solution to dynamics problems [10] has triggered recent debate and investigation on HLVA [9, 11, 12], important questions remain unanswered. For example, the relationship among *Hamilton's Law of Varying Action*, *Hamilton's Principle* [5], and *Hamilton's Principle of Stationary Action* [3], and their scopes of application have not been discussed satisfactorily in terms of the types of systems (e.g. holonomic or nonholonomic, conservative or nonconservative) to which these principles apply. In addition, some important issues regarding the method of directly solving dynamics problems via HLVA have not been clearly addressed. For example, whether a direct solution via HLVA can be obtained for a nonholonomic system, and whether this application of HLVA is an efficient way to obtain approximate solutions to dynamics problems as compared with other technique remain unanswered.

In view of the above issues, the objectives of this study are two fold.

The first objective is to discuss the relations of HLVA with Hamilton's Principle and Hamilton's Principle of Stationary Action through illustrations, instead of through advanced mathematical treatments. The approach involves interpreting conceptually HLVA through discussions of the role of the time boundary terms, and demonstrating the difference between HLVA for holonomic systems and HLVA for nonholonomic systems with an illustrative example.

The second objective of this thesis is to evaluate a recent application of HLVA; that is, the application of HLVA to obtain direct (approximate) solutions to dynamics problems. The approach involves discussing how HLVA is used in this application and how to extend this application of HLVA to nonholonomic systems, comparing this method with weighted residual methods for dynamics problems, and investigating the role of this application of HLVA in dynamics.

This study is confined to lumped-parameter systems. In addition, the system is assumed to be such that the constraints can be analytically expressed by Pfaffian equations† [5, p13-16].

1.3 Organization of the Thesis

Chapter 1 gives an introduction of the thesis. In chapter 2, HLVA is introduced and relations of HLVA with Hamilton's Principle and Hamilton's Principle of Stationary Action are discussed. Then chapter 3 presents a conceptual interpretation of HLVA for holonomic systems and demonstrates the difference between HLVA for holonomic systems and HLVA for nonholonomic systems. Chapter 4 starts with a review of the method of applying HLVA to obtain direct (approximate) solution of dynamics problem. Following the review, some numerical issues pertinent to this method are discussed, and then the possibility of applying this method to nonholonomic systems is demonstrated. The

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† An expression is said to be a *Pfaffian equation* if it takes the following form:

$$\sum_{i=1}^{n} A_i dx_i + A_0 dt = 0$$

where A_i and A_0 are known functions of $x_1, x_2, \cdots, x_n$ and t, and have continuous first derivatives in a domain of $x_1, x_2, \cdots, x_n$ and t.

difference between HLVA and Hamilton's Principle in obtaining direct solutions to dynamics problems is addressed toward the end of chapter 4. In chapter 5, this method is positioned in the context of engineering analysis and compared with Galerkin's method for dynamics problems. Finally, conclusions of this study are given in chapter 6.

Notation in this thesis follows that of [3], with the major terminology sources being both [3] and [5]. However, the former is our source of first priority.

2 THEORY OF HAMILTON'S LAW OF VARYING ACTION

2.1 Introduction

This chapter presents the underlying theory of HLVA and two special cases, namely, Hamilton's Principle and Hamilton's Principle of Stationary Action. The chapter begins with a derivation of HLVA from the so-called fundamental equation of motion for a dynamic system. Then the two special cases of HLVA are discussed. Also the three variational principles of Hamilton are compared briefly.

2.2 Hamilton's Law of Varying Action

Consider a general dynamical system of N particles m_i (i = 1, ..., N), as indicated in Fig. 2.2.1.

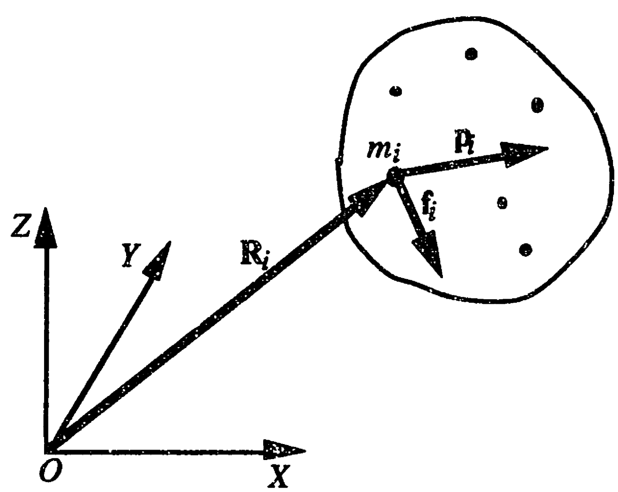

Fig. 2.2.1 System of N particles where $OXYZ$ represents an inertial frame.

The dynamic behavior of this system is governed by *the fundamental equation* [5, p28]

$$\boxed{\sum_{i=1}^{N} \left(\mathbf{f}_i - \frac{d\mathbf{p}_i}{dt} \right) \circ \delta\mathbf{R}_i = 0}, \tag{2.2.1}$$

where $\mathbf{f}_i$ is the resultant given force, which is the totality of all forces, excluding the forces of constraints†, acting on the particle m_i, δ is the contemporaneous variational operator, and $\mathbf{p}_i$, $\mathbf{R}_i$ and $\delta\mathbf{R}_i$ are the momentum of the ith particle, the position vector, and the admissible variation of the position vector, respectively.

This equation, discovered by Lagrange in the 1760's, is a combination and generalization of the *principle of virtual work* and *d'Alembert's principle*. We shall call it *the fundamental equation* following [5, p28], although many different names have been given to it††. The fundamental equation states that *the total work increment of the given forces plus the inertia forces of the system vanishes for arbitrary admissible variations of the configuration of the system.* It should be noted that the admissible variation $\delta\mathbf{R}_i(t)$ is the displacement of the ith particle measured between a point on a specified trajectory of the particle at instant t and the contemporaneous point on a neighboring trajectory, as illustrated in Fig. 2.2.2.

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† The forces of constraint are eliminated because they do no work during an arbitrary admissible variation [5, p22-27].

†† For example, it is called *d'Alembert's principle* [Goldstein p18], *generalized principle of d'Alembert* [Meirovitch p65], and *Lagrange's form of d'Alembert's principle* [Rosenberg p126].

Now, let $\xi_1, \ldots, \xi_n$ be a complete and independent set of *generalized coordinates* and let $\delta\xi_1, \ldots, \delta\xi_n$ be their associated *variational variables*†. In addition, let $V(\xi_1, \ldots, \xi_n; t)$ be the *potential energy* of the conservative forces, and let Ξ_j be the *generalized force* associated with the generalized coordinate ξ_j due to the nonconservative forces. Then, the total work increment of all given forces acting on the system can be divided into a contribution from conservative forces and a contribution from nonconservative forces as follows [3, p130]:

$$\sum_{i=1}^{N} \mathbf{f}_i \cdot \delta\mathbf{R}_i = -\,\delta V + \sum_{j=1}^{n} \Xi_j\, \delta\xi_j. \tag{2.2.2}$$

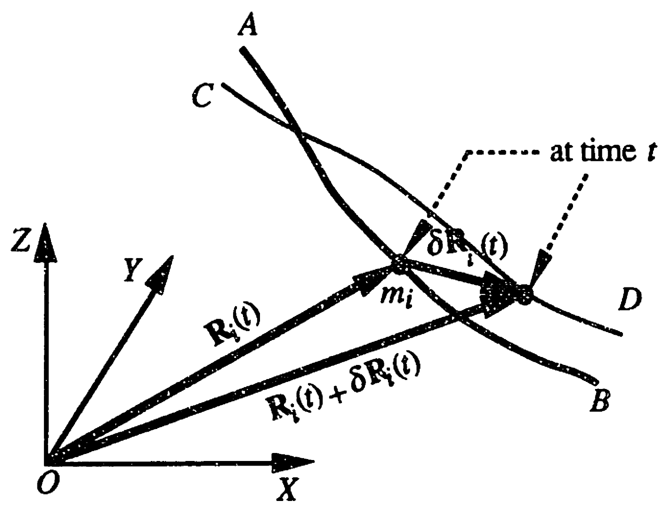

Fig. 2.2.2 The variation $\delta\mathbf{R}_i(t)$ between two trajectories. *OXYZ* represents an inertial frame. Curve *AB* is a specified trajectory of particle m_i. Curve *CD* is a neighboring trajectory.

† Note that the variational variables $\delta\xi_1, \cdots, \delta\xi_n$ are independent if the system is holonomic, but are not independent if the system is nonholonomic.

The total work increment of the inertia forces acting on the masses of the system can be expressed as follows [3, p130]:

$$\sum_{i=1}^{N} -\frac{d\mathbf{p}_i}{dt} \circ \delta\mathbf{R}_i = \delta T^* - \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{\xi}_j} \delta\xi_j\right), \qquad (2.2.3)$$

where T^* is the *kinetic coenergy* of the system, $T^* = \sum_{i=1}^{N} \frac{1}{2} m_i \frac{d\mathbf{R}_i}{dt} \circ \frac{d\mathbf{R}_i}{dt}$, and $\dot{\xi}_j$ is the *generalized velocity* associated with the generalized coordinate ξ_j. The relation (2.2.3) is called by Hamel [8, p233-p234] as the *Central Principle*.

Through the Central Principle (2.2.3) and the expression (2.2.2), *the fundamental equation* (2.2.1) is transformed into the following form:

$$\delta T^* - \delta V + \sum_{j=1}^{n} \Xi_j \,\delta\xi_j = \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{\xi}_j} \delta\xi_j\right). \qquad (2.2.4)$$

The time derivative term on the right hand side of (2.2.4) is not convenient to evaluate. In an attempt to eliminate the derivative quantity, integrating the above equation with respect to time over an *arbitrary* interval from $t = t_1$ to $t = t_2 (t_1 < t_2)$ gives

$$\boxed{\int_{t_1}^{t_2} \left(\delta L + \sum_{j=1}^{n} \Xi_j \,\delta\xi_j\right) dt = \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j} \delta\xi_j\right]_{t_1}^{t_2}}, \qquad (2.2.5)$$

where $L = T^* - V$ is the *lagrangian* of the system. We shall refer to the above equation as the *general form of Hamilton's Law of Varying Action*. (For reference, Table 2.2.1 shows the terminologies associated with this equation and its slightly different forms in the literature.) In view that equation (2.2.5) is a direct transformation from the fundamental equation (2.2.1), equation (2.2.5) can be stated as follows: *The total work increment of the system over any finite time interval vanishes for arbitrary admissible variations of the configurations of the system.*

In order to help interpret the general form of HLVA, let us introduce some terminologies. The phrase *geometrically admissible motion* (or *admissible motion* for short) [3, p24] will be used to denote a motion which is always geometrically compatible but which does not necessarily satisfy the dynamic-force requirements at all times. The terminology *natural motion* [3, p30] will be used to mean a geometrically admissible motion *which also satisfies the dynamic-force requirements*. Another class of motion, the class of *varied motions*, is not defined based on the geometric admissibility requirements or the dynamic-force requirements of the system as the admissible motions and natural motions are; instead, it is obtained by pure mathematical construction. Assume a natural motion of a system is represented by $\xi_1, \cdots, \xi_n$, then at each instant during the natural motion there is an arbitrary admissible variation $\delta\xi_1, \cdots, \delta\xi_n$ associated with this instant. The sequence of configurations $\xi_1 + \delta\xi_1, \cdots, \xi_n + \delta\xi_n$ as time changes is called the *varied motion* [3, p417] [5, p34]†.

† *Varied motion* is called *varied path* in [Pars p34]

Table 2.2.1 **Terminologies associated with equation (2.2.5)**

Equation	*Terminology in literature*	*Terminology in this thesis*
$\int_{t_1}^{t_2}\left(\delta L + \sum_{j=1}^{n} \Xi_j \delta\xi_j\right)dt = \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j}\delta\xi_j\right]_{t_1}^{t_2}$	Hamilton's Principle in its most general form [13]	general form of Hamilton's Law of Varying Action
" "	General virtual variational equation [9]	"
$\delta\int_{t_1}^{t_2} L dt = \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j}\delta\xi_j\right]_{t_1}^{t_2}$	Law of Varying Action [1]	Hamilton's Law of Varying Action
" "	(Hamilton's) Principle of Varying Action [7]	"
" "	Hamilton's Principle of Varying Action [14]	"

* Hamilton's original Law of Varying Action includes the variation of the end time t_2 (the terminal time t_1 is fixed) in addition to the variations of the terminal configurations of the system. Here we assume that the two terminal instants are fixed, hence the variations of the terminal times vanish.

Note that if $\xi_1, \cdots, \xi_n$ are not subject to any constraint, the admissible variation $\delta\xi_1, \cdots, \delta\xi_n$ is arbitrary; otherwise, the admissible variation is defined by the constraint equations† of the system. It should also be noted that a varied motion in general is not an admissible motion if the system is nonholonomic, but is always an admissible motion if the system is holonomic. This point will be illustrated in Section 3.3.

With the introduction of the above terminologies, the general form of HLVA (2.2.5) can be stated as follows: *A varied motion of the system over an arbitrary finite time interval $t_1 \leq t \leq t_2$ is a natural motion if, and only if, the variational indicator*

$$\text{V.I.} = \int_{t_1}^{t_2} \left(\delta L + \sum_{j=1}^{n} \Xi_j \, \delta\xi_j \right) dt - \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j} \delta\xi_j \right]_{t_1}^{t_2}, \qquad (2.2.6)$$

vanishes for arbitrary geometrically admissible variations of the motion within this time interval.

Remarks.

1. The fundamental equation (2.2.1) and the general form of HLVA (2.2.5) are equivalent since the latter is a mathematical transformation of the former. Both are variational

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† For example, if a generic constraint equation is in the form $a_1 d\xi_1 + a_2 d\xi_2 + a_3 d\xi_3 + a_0 = 0$, where a_1, a_2, a_3, and a_0 are functions of ξ_1, ξ_2, ξ_3, and time t, then the relation that defines an admissible variation ($\delta\xi_1$, $\delta\xi_2$, $\delta\xi_3$) is obtained by changing $d\xi_1$, $d\xi_2$, and $d\xi_3$ in the constraint equation to $\delta\xi_1$, $\delta\xi_2$, and $\delta\xi_3$, and setting a_0 to zero. Therefore, the equation that defines an admissible variation for this case is $a_1\delta\xi_1 + a_2\delta\xi_2 + a_3\delta\xi_3 = 0$.

formulations of the general dynamics problem for holonomic as well as nonholonomic systems. Both state the dynamics-force requirements of a system from an energy viewpoint.

The fundamental equation (2.2.1) and the general form of HLVA (2.2.5) can be interpreted differently. The former states the dynamic-force requirements on an instant-to-instant basis, whereas the latter states the dynamic-force requirements over a time interval. In other words, the fundamental equation compares the configurations from one instant to another, whereas the HLVA compares complete time histories over a time interval.

2. It is important to notice that the general form of HLVA (2.2.5) holds for arbitrary terminal instants t_1 and t_2 ($t_1 < t_2$) after the system's initial time, and it holds for both holonomic and nonholonomic systems.

3. Equation (2.2.5) and equation (2.2.6) are different forms of the same statement of the general form of HLVA.

When all given forces acting on a system are conservative, that is, they can be derived from a potential function, the general HLVA (2.2.5) becomes

$$\int_{t_1}^{t_2} \delta L \, dt = \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j} \delta \xi_j \right]_{t_1}^{t_2}, \tag{2.2.7}$$

If, in addition, the system is holonomic, then the varied motion is an admissible motion, and the operations of integration and variation in (2.2.7) are commutative†. Thus,

$$\int_{t_1}^{t_2} \delta L \, dt = \delta \int_{t_1}^{t_2} L dt \, . \tag{2.2.8}$$

Hence, combining (2.2.7) and (2.2.8) gives HLVA for a holonomic system under conservative forces as follows:

$$\boxed{\delta A = \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j} \delta \xi_j \right]_{t_1}^{t_2}} , \tag{2.2.9}$$

where A denotes the *lagrangian action* of an admissible motion in the interval $t_1 \leq t \leq t_2$,

that is, $A = \int_{t_1}^{t_2} L \, dt$. The above equation is Hamilton's *Law of Varying Action* presented in [1, p307]††. It expresses the variation of the action A of the natural motion of the system over a time interval $t_1 \leq t \leq t_2$ as a linear function of the variations of the configurations of the system at t_1 and t_2. This interpretation of HLVA (2.2.9) is illustrated in Section 3.2.

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† This is a consequence of the calculus of variations. See Section 4.2 and Appendix B for more discussion on the commutability of the operations of integration and variation.

†† Actually, Hamilton's original Law of Varying Action includes the variation of the end time t_2 in addition to the the variations of the end configurations of the system. Here we assume that the end instants t_1 and t_2 are fixed. Therefore, equation (2.2.9) is a reduced form of original Hamilton's Law of Varying Action.

2.3 Two Special Cases of Hamilton's Law of Varying Action

2.3.1 Hamilton's Principle

In the general form of HLVA (2.2.5), the time history of the natural motion is compared with those of the neighboring varied motions over the interval $t_1 \le t \le t_2$. If we restrict the comparison to the family of varied motions that share the same end configurations as the natural motion at $t = t_1$ and $t = t_2$, (that is, $\delta\xi_j(t_1) = 0$, $\delta\xi_j(t_2) = 0$, $j = 1, ..., n$), then the right hand side of (2.2.5) (which we shall refer to as the *time boundary terms*) vanishes. Hence, the general form of HLVA (2.2.5) becomes

$$\boxed{\int_{t_1}^{t_2} \left[\delta L + \sum_{j=1}^{n} \Xi_j \, \delta\xi_j \right] dt = 0}, \qquad (2.3.1)$$

which is called *Hamilton's Principle* [3, p35].

In general, for both holonomic and nonholonomic systems, Hamilton's principle provides a criterion to determine the natural motion among the *varied motions* having the same end configurations as the natural motion. Therefore, Hamilton's principle for both holonomic and nonholonomic systems can be stated as follows: *A varied motion*† *of a*

† Since varied motions in general are not admissible motions for nonholonomic systems (see Section 3.3.1) and are always admissible motions for holonomic systems, the term *varied motion* instead of *admissible motion* is used in the statement of Hamilton's Principle in order to cover nonholonomic as well as holonomic systems.

dynamic system between specified configurations at t_1 and t_2 is a natural motion if, and only if, the variational indicator

$$\text{V.I.} = \int_{t_1}^{t_2} \left[\delta L + \sum_{j=1}^{n} \Xi_j \, \delta\xi_j \right] dt \tag{2.3.2}$$

vanishes for arbitrary admissible variations $\delta\xi_1, \ldots, \delta\xi_n$.

Like the general form of HLVA, Hamilton's Principle applies to general holonomic and nonholonomic systems. Unlike the general form of HLVA, Hamilton's Principle compares the natural motion of the system with the family of varied motions that share the same terminal configurations as the natural motion at the two terminal instants $t = t_1$ and $t = t_2$.

Again, the two terminal instants t_1 and t_2 ($t_1 < t_2$) are arbitrary instants after the system's initial time.

2.3.2 Hamilton's Principle of Stationary Action

If all given forces of the system can be accounted for in the potential energy function V (and hence in the lagrangian L), Hamilton's Principle becomes

$$\int_{t_1}^{t_2} \delta L \, dt = 0. \tag{2.3.3}$$

If, in addition, the system is holonomic, then the variation and integration operators commute. Hence the above statement of Hamilton's Principle becomes

$$\boxed{\delta A = 0}, \tag{2.3.4}$$

where $A = \int_{t_1}^{t_2} L\, dt$ is the lagrangian action. Equation (2.3.4) is *Hamilton's Principle of Stationary Action* [3, p31-34]. This principle may be stated as follows: *For a dynamic system in which the work of all forces is accounted for in the lagrangian, an admissible motion between specified configurations at t_1 and t_2 is a natural motion if, and only if, the action A is stationary for arbitrary admissible variations.*

Hamilton's Law of Varying Action (2.2.9) and Hamilton's Principle of Stationary Action (2.3.4) both apply to holonomic systems under conservative forces. The difference between them lies in the different groups of admissible motions with which the natural motion is compared. The former compares the action A of a natural motion with *all neighboring admissible motions*, whereas the later compares the action A of a natural motion with *the family of admissible motions that share the same configurations as the natural motion at the two end instants.* The comparison of *Hamilton's Law of Varying Action* and *Hamilton's Principle of Stationary Action* is also illustrated in Section 3.2.

2.4 Summary

Of the variational principles of Hamilton discussed in this chapter, Hamilton's Principle (2.3.1) has been widely used as an energy method to formulate differential equations of motion for systems in various engineering fields [3]. However, the general form of HLVA can be used also to derive the differential equations of motion (See Appendix A).

The variational principles of Hamilton are summarized in Table 2.4.1.

Table 2.4.1 **Summary of variational principles of Hamilton**

Principles	*Equations**	*Systems*	*Requirements that govern motion*** (*Geometric requirements*)
HLVA (general form) (2.2.5)	$\int_{t_1}^{t_2}\left(\delta\mathcal{L} + \sum_{j=1}^{n} \Xi_j \delta\xi_j\right)dt = \left[\sum_{j=1}^{n} \frac{\partial\mathcal{T}^*}{\partial\dot{\xi}_j}\delta\xi_j\right]_{t_1}^{t_2}$	holonomic	satisfied by the choice of $\xi_1, \cdots, \xi_n$ alone.
		nonholonomic	In addition to the choice of $\xi_1, \cdots, \xi_n$, the variational variables $\delta\xi_1, \cdots, \delta\xi_n$ must satisfy the equations of nonholonomic constraints.
Hamilton's Principle (2.3.1)	$\int_{t_1}^{t_2}\left(\delta\mathcal{L} + \sum_{j=1}^{n} \Xi_j \delta\xi_j\right)dt = 0$; $\delta\xi_j\vert_{t_1} = 0,\ \delta\xi_j\vert_{t_2} = 0,\ j = 1, \ldots, n$	holonomic	satisfied by the choice of $\xi_1, \cdots, \xi_n$ alone.
		nonholonomic	In addition to the choice of $\xi_1, \cdots, \xi_n$, the variational variables $\delta\xi_1, \cdots, \delta\xi_n$ must satisfy the equations of nonholonomic constraints.
HLVA (2.2.9)	$\delta\int_{t_1}^{t_2} L\,dt = \left[\sum_{j=1}^{n} \frac{\partial\mathcal{T}^*}{\partial\dot{\xi}_j}\delta\xi_j\right]_{t_1}^{t_2}$	holonomic with conservative forces	satisfied by the choice of $\xi_1, \cdots, \xi_n$ alone.
Hamilton's Principle of Stationary Action (2.3.4)	$\delta\int_{t_1}^{t_2} L\,dt = 0$; $\delta\xi_j\vert_{t_1} = 0,\ \delta\xi_j\vert_{t_2} = 0,\ j = 1, \ldots, n$	holonomic with conservative forces	satisfied by the choice of $\xi_1, \cdots, \xi_n$ alone.

* In these equations $\xi_1, \cdots, \xi_n$ constitutes a complete and independent set of generalized coordinates of a system.

** Only *geometric requirements* on the motions are discussed for each principle in the table. The *dynamic-force requirements* are embodied in the corresponding variational criterion. The *constitutive relations* are embodied in the kinetic coenergy, the potential energy and the variational work expressions of each principle.

3 INTERPRETATION OF HAMILTON'S LAW OF VARYING ACTION

3.1 Introduction

In this chapter a conceptual interpretation of HLVA for a holonomic system under conservative forces is presented using a simple harmonic oscillator. The interpretation is accomplished by the illustrations of how the (lagrangian) action of the natural motion varies when the natural motion is compared with neighboring admissible motions. Also, HLVA for nonholonomic systems is discussed. An example is used to demonstrate that varied motions in general are not admissible motions for nonholonomic systems. Comments on the difference between HLVA for nonholonomic systems and HLVA for holonomic systems are also given.

3.2 Interpretation of HLVA For Holonomic Systems

As stated in Section 2.2, HLVA (2.2.9) expresses the variation of action A of the natural motion of a system over a time interval $t_1 \leq t \leq t_2$ as a linear function of the configurations of the system at the terminal instants $t = t_1$ and $t = t_2$. In this section, an example is employed to interpret HLVA and to illustrate this statement of HLVA. The example, which is the same as that used to illustrate Hamilton's Principle of Stationary Action in [3, p33-34], is the simple harmonic oscillator shown in Fig. 3.2.1.

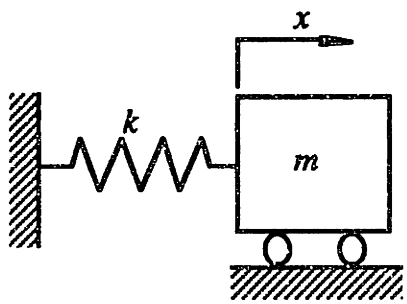

Fig 3.2.1. Simple undamped harmonic oscillator. Initial conditions: x = 0 and $dx/dt = v_0$ at $t = 0$.

For this (holonomic and conservative) system, geometrically admissible trajectories are represented by the single-valued function $x(t)$ provided the spring elongation is equal to x and the mass velocity is equal to dx/dt. Consider trajectories which emanate from $x = 0$ at $t = t_1 = 0$ and which end at various configurations at $t = t_2$, as indicated in Fig. 3.2.2. These trajectories can be categorized into families of trajectories with each family consisting of those trajectories emanating from $x = 0$ at $t = 0$ and converging to a common configuration at $t = t_2$. In particular, if $x(t)$ is the natural motion of the system, a family of admissible trajectories consists of those trajectories emanating from $x = 0$ at $t = 0$ and converging to $x = x(t_2) + \delta x|_{t_2}$ at $t = t_2$, where $\delta x|_{t_2}$ is an admissible variation from the configuration $x(t_2)$ at $t = t_2$. Therefore, it may be said that with each value of the end variation $\delta x|_{t_2}$ there is associated a family of admissible trajectories.

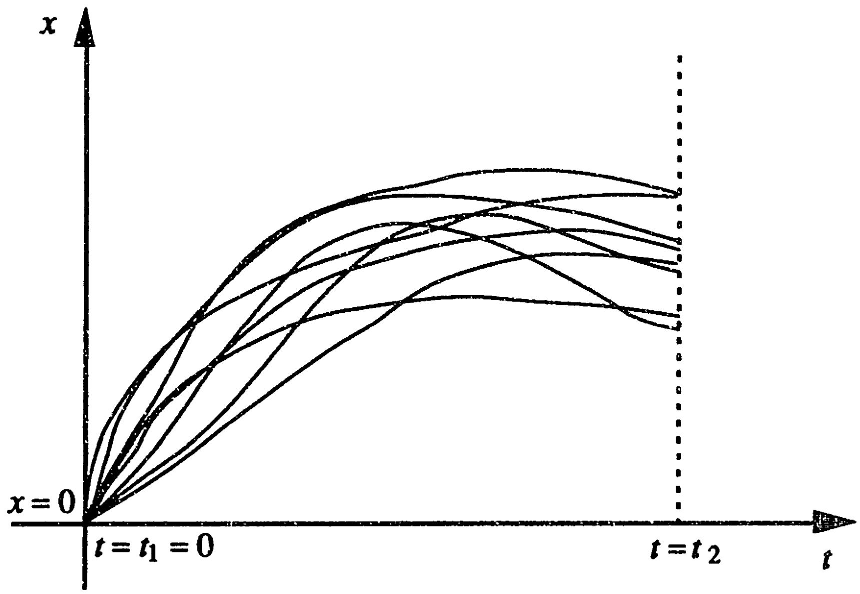

Fig. 3.2.2. Admissible trajectories.

The action A associated with each admissible trajectory of this system is

$A = \int_{t_1}^{t_2} \left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right]dt$, where $t_1 = 0$, t_2 can be any time after t_1, and $\dot{x} \equiv dx/dt$.

According to HLVA (2.2.9), an admissible trajectory x(t) is a natural trajectory of the system if, and only if, it satisfies the equation

$$\delta A = \left[\frac{\partial T^*}{\partial \dot{x}}\,\delta x\right]_{t_1}^{t_2} = \left[m\dot{x}\ \delta x\right]_{t_1}^{t_2} \tag{3.2.1}$$

for an arbitrary admissible variation $\delta x(t)$ over the time interval $t_1 \le t \le t_2$, where T^* is the kinetic coenergy of the system, $T^* = \frac{1}{2}m\dot{x}^2$. Since t_1 has been taken to be the initial time of the system and the configuration of the system at $t_1 = 0$ is known, the variation $\delta x|_{t_1}$ vanishes. Therefore, equation (3.2.1) becomes

$$\delta A = [m\dot{x}]_{t_2} \cdot \delta x|_{t_2}. \tag{3.2.2}$$

HLVA (3.2.2) says that for a family of admissible trajectories associated with a fixed value of $\delta x|_{t_1}$ (for example, $\delta x|_{t_2} = \varepsilon$, where ε is a first-order differential quantity), the *variation of the action A is a constant* when the natural trajectory is compared with this restricted family of admissible trajectories. The constant is nothing else but the product of the momentum of the system at $t = t_2$ and the admissible variation of the end configuration $\delta x|_{t_2}$. Fig. 3.2.3 shows several admissible trajectories from the family of admissible trajectories associated with a value of $\delta x|_{t_2} = \varepsilon$.

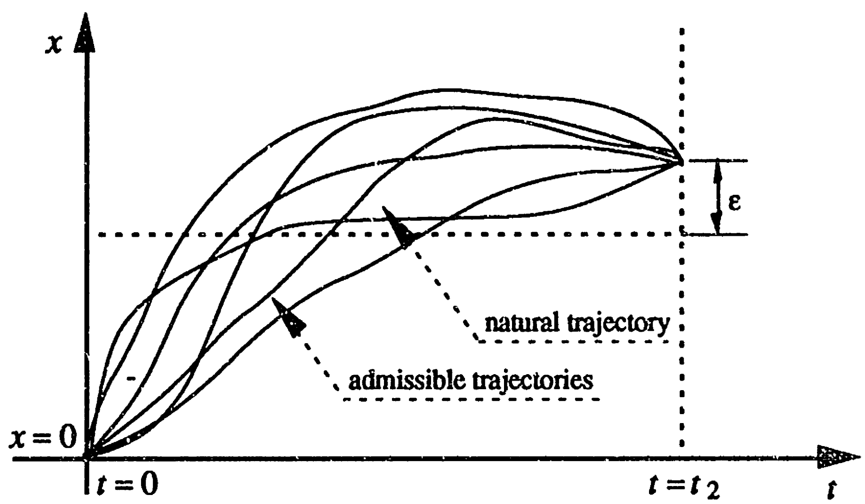

Fig. 3.2.3. Constant variation of action A, $\delta A = [m\dot{x}]_{t_2}\varepsilon$, as the natural trajectory is compared with a family of admissible trajectories in the neighborhood of the natural trajectory.

A special family of admissible trajectories is the family associated with the special value $\delta x|_{t_2} = 0$. This family consists of the admissible trajectories that share the same configurations with the natural trajectory at $t = t_1$ and $t = t_2$. For this special family, the variation of the action A is zero, or in other words, the action A is stationary for the natural motion. Therefore, when HLVA compares the natural trajectory with this special family of admissible trajectories, HLVA becomes Hamilton's Principle of Stationary Action [3, p33-34].

When the natural trajectory is compared with different families of admissible trajectories, HLVA (3.2.2) says that the *variation of action A* varies in proportion to the admissible variation of the end configuration $\delta x|_{t_2}$. The proportional coefficient is nothing else but the momentum of the system at $t = t_2$. Fig. 3.2.3 shows the linear relationship between the variation of the action A, δA, and the variation of the end configuration, $\delta x|_{t_2}$.

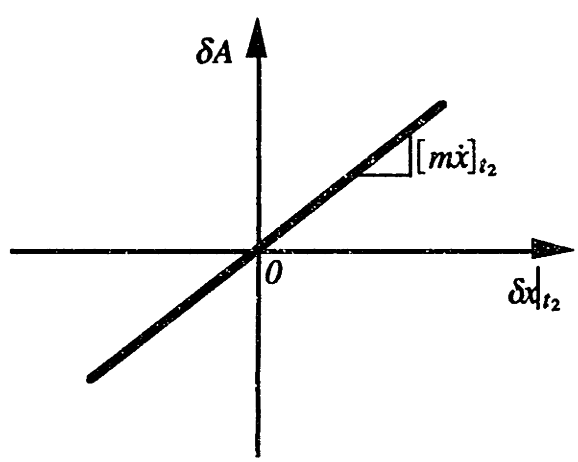

Fig. 3.2.4. Linear relationship between δA and $\delta x|_{t_2}$.

3.3 Interpretation of HLVA for Nonholonomic Systems

3.3.1 Varied Motion vs Admissible Motion

In Section 2.2, terminologies such as admissible motion, natural motion and varied motion have been introduced. In this section a nonholonomic example, the constrained motion of a boat [3, p119-120], is used to demonstrate that varied motions in general are not admissible motions for nonholonomic systems. The procedure of the demonstration is as follows: first, find a natural motion of the boat; second, construct a varied motion that is indeed admissible; and third, construct another varied motion that is not admissible.

1. *A natural motion of the boat*

In Fig. 3.3.1, the water surface is the plane of the sketch. The boat is modeled so that motions which are transverse to the keel are completely prohibited. In other words, the translation of the boat must always be parallel to the instantaneous heading of the keel. The analytical expression of this constraint is

$$dy - dx\tan\theta = 0. \tag{3.3.1}$$

Geometrically admissible motions of the system are those sets of functions $(x(t), y(t), \theta(t))$ that satisfy the constraint equation (3.3.1) provided the angle of the keel with respect to the x axis is taken equal to θ and the displacement components in the x-direction and the y-direction are taken equal to x and y, respectively. Fig. 3.3.2 shows several geometrically admissible motions.

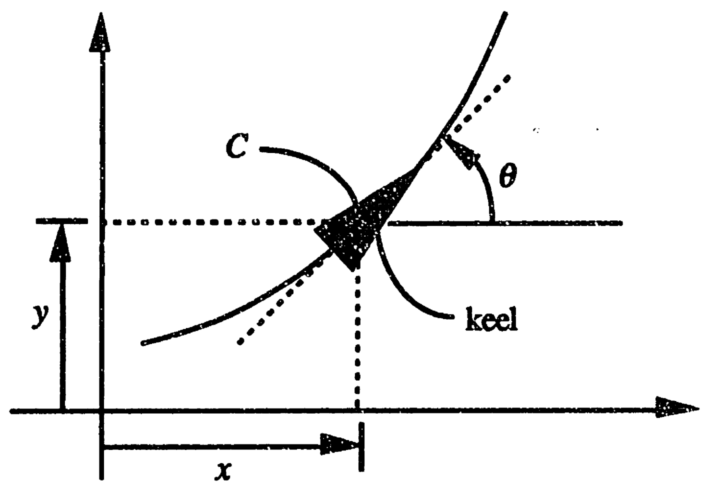

Fig 3.3.1. Constrained motion of boat [3, p119].

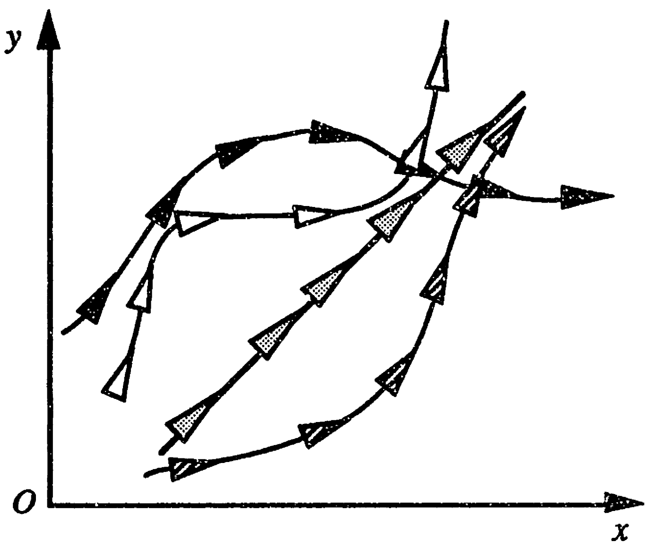

Fig. 3.3.2. Admissible motions. The pointed triangles represent the heading of the keel. Curves represent projections of the trajectories in configuration space onto the *oxy* plane.

Assume the boat is not subject to any force or torque, then the natural motion of the boat under the particular set of initial conditions: $x(0) = y(0) = 0$, $\theta(0) = 45°$, $\dot{x}(0) = \dot{y}(0) = 1$, and $\dot{\theta}(0) = 0$, is (see Appendix C)

$$\left.\begin{aligned} x^* &= t \\ y^* &= t \\ \theta^* &= 45° \end{aligned}\right\}, \qquad t \geq 0 \tag{3.3.2}$$

as shown in Fig. 3.3.3. Equation (3.3.2) is a natural motion because it is obtained by solving the equations of motion of the system together with the equation of constraint (3.3.1), which indicates that the motion represented by (3.3.2) satisfies not only the geometric constraint but also the dynamic-force requirements. Note that to avoid confusion, the particular natural motion of the boat has been represented by the generalized coordinates x, y, and θ with the superscript '*' in (3.3.2). For the same purpose, these coordinates with a tilde '~' on top will be used to represent a particular varied motion in this example.

2. *A varied motion that is admissible*

Since a varied motion is constructed by adding to the natural motion an admissible variation, then the first step of constructing a varied motion would be to construct an admissible motion. Let $(\delta x^*, \delta y^*, \delta\theta^*)$ denote an admissible variation from the natural motion (3.3.2). Based on the equation of constraint (3.3.1), the admissible variation $(\delta x^*, \delta y^*, \delta\theta^*)$ should satisfy (see Section 2.2)

$$\delta y^* - \delta x^* \tan\theta^* = 0. \tag{3.3.3}$$

To construct the varied motion that is admissible, choose a particular *admissible variation* to be as follows:

$$\left.\begin{aligned} \delta x^* &= \varepsilon \sin\frac{\pi}{5}t \\ \delta y^* &= \varepsilon \sin\frac{\pi}{5}t \\ \delta\theta^* &= 0 \end{aligned}\right\}, \qquad t \geq 0 \qquad (3.3.4)$$

where ε is a positive first-order differential quantity with unit of displacement. With this admissible variation, the associated varied motion is

$$\left.\begin{aligned} \tilde{x} &\equiv x^* + \delta x^* = t + \varepsilon \sin(\pi/5)t \\ \tilde{y} &\equiv y^* + \delta y^* = t + \varepsilon \sin(\pi/5)t \\ \tilde{\theta} &\equiv \theta^* + \delta\theta^* = 45° \end{aligned}\right\}, \qquad t \geq 0 \qquad (3.3.5)$$

as shown in Fig. 3.3.4.

Comparing the varied trajectory in Fig. 3.3.4 and the natural trajectory in Fig. 3.3.2 reveals that although this varied trajectory is geometrically the same as the natural trajectory, the boat traverses these trajectories with different velocities. If proper forces are applied to the boat, the boat is indeed capable of actually moving along this varied trajectory without violating the geometric constraint. Therefore, the varied motion represented by (3.3.5) is an admissible motion.

It should be noted that the varied motion (3.3.5) can also be mathematically verified to be admissible. This verification is accomplished by testing whether $\tilde{x}$, $\tilde{y}$, and $\tilde{\theta}$ in

(3.3.5) satisfy the geometric constraint (3.2.1). Since the equation $d\tilde{y} - d\tilde{x} \tan \tilde{\theta} = 0$ indeed holds for $t \geq 0$, then the varied motion (3.3.5) is an admissible motion.

3. *A varied motion that is not admissible*

To construct the varied motion that is not admissible, choose a particular *admissible variation* to be as follows:

$$\left.\begin{aligned} \delta x^* &= 0 \\ \delta y^* &= 0 \\ \delta\theta^* &= \varepsilon \sin\frac{\pi}{5}t \end{aligned}\right\}, \qquad t \geq 0 \qquad (3.3.6)$$

where ε is a positive first-order differential quantity with degrees as its unit. With this chosen admissible variation, the associated varied motion is

$$\left.\begin{aligned} \tilde{x} &\equiv x^* + \delta x^* = t \\ \tilde{y} &\equiv y^* + \delta y^* = t \\ \delta\theta^* &\equiv \theta^* + \delta\theta^* = 45° + \varepsilon \sin\frac{\pi}{5}t \end{aligned}\right\}, \qquad t \geq 0 \qquad (3.3.7)$$

as shown in Fig. 3.3.5.

Inspecting the varied trajectory in Fig. 3.3.5 reveals that this varied trajectory dictates that the boat move along a straight line at 45 degrees to the x-axis while twisting its keel direction back and forth in the neighborhood of 45 degrees. No matter what forces or torques acting upon the boat, it can never move in such a fashion because the boat is constrained to translate only in the instantaneous direction of the keel. Therefore, the varied motion represented by (3.3.7) is not admissible.

It should be noted that the varied motion (3.3.7) can also be mathematically verified to be not admissible. Since the varied motion $\tilde{x}$, $\tilde{y}$, and $\tilde{\theta}$ in (3.3.7) does not satisfy the geometric constraint (3.2.1), that is, $d\tilde{y} - d\tilde{x} \tan \tilde{\theta} = dt\,[1 - \tan(45° + \varepsilon \sin\frac{\pi}{5}t)] \neq 0$, the varied motion (3.3.7) is not admissible.

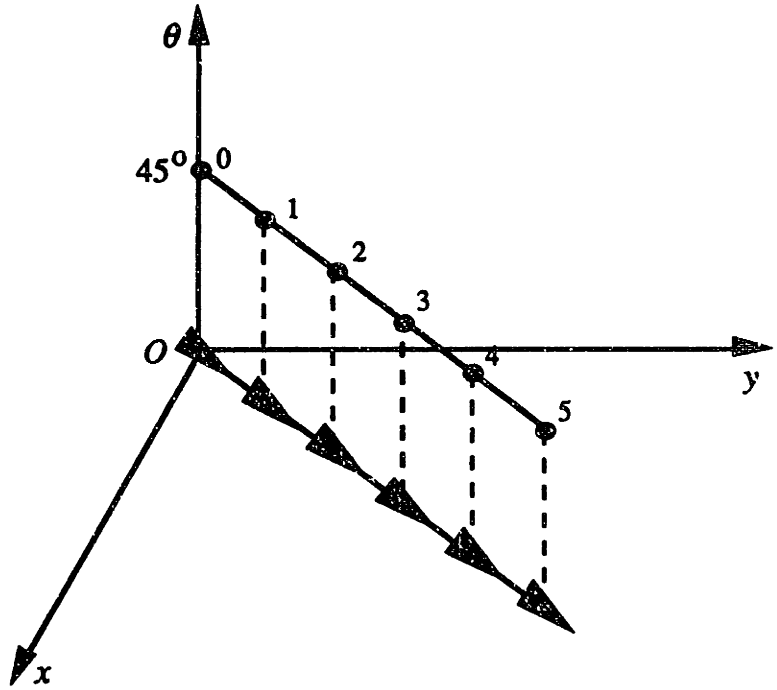

Fig. 3.3.3. Natural motion represented by (3.3.2). Each dot and attached number represent location of boat in configuration space at successive times. Pointed triangles and lines connecting them in *oxy* plane represent the natural motion of the boat on water surface.

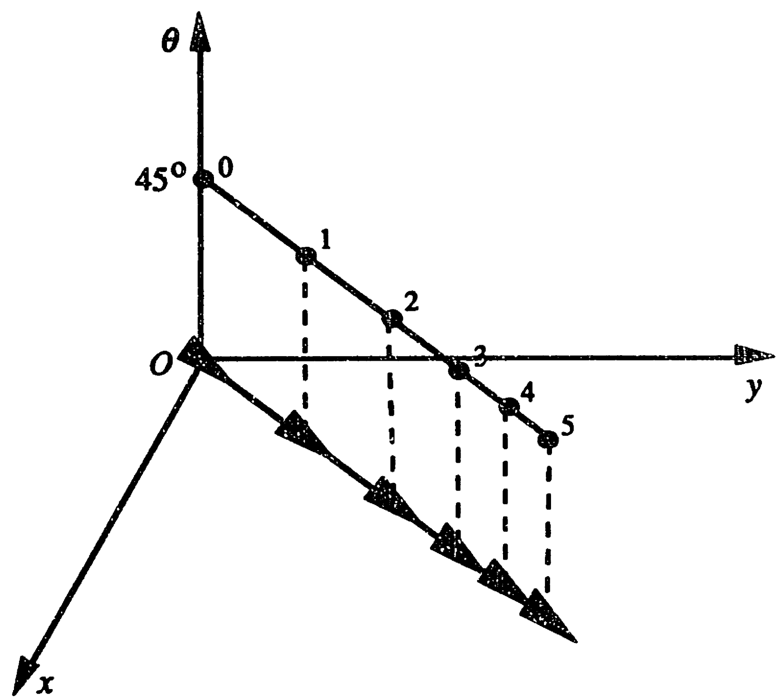

Fig. 3.3.4. Varied motion represented by (3.3.5). It is admissible because the translation of the boat is always in the instantaneous direction of the keel. Numbers attached to the dots (representing boat's locations in configuration space) are time units.

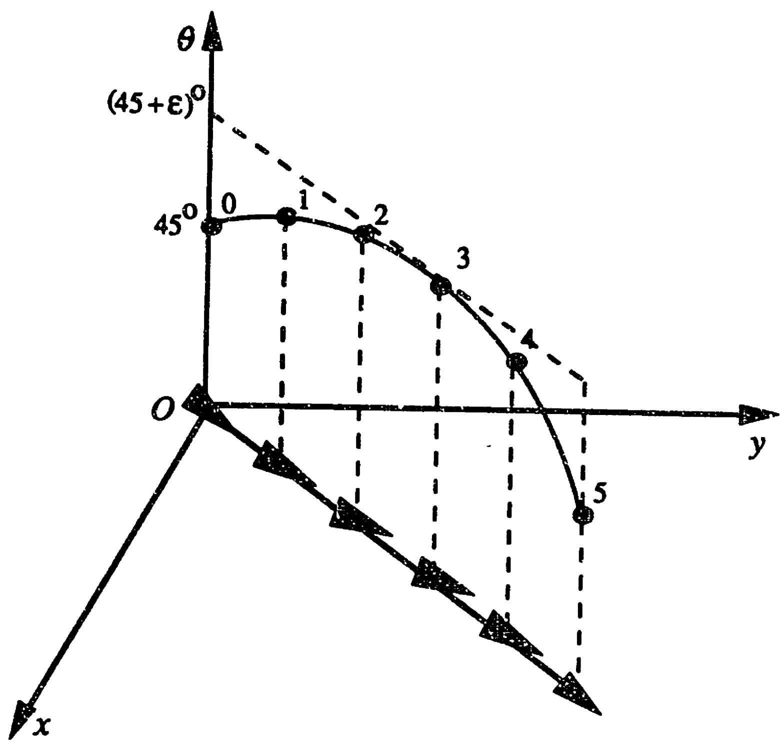

Fig. 3.3.5. Varied motion represented by (3.3.7). It is not admissible because the translation of the boat is not always in the instantaneous direction of the keel. Numbers attached to the dots (representing the boat's locations in configuration space) are time units.

3.3.2 Comments

There are two aspects of difference between HLVA for holonomic systems and that for nonholonomic systems. First, for nonholonomic systems, the variational variables are dependent on each other. The dependency is defined by the nonholonomic constraints of the system. For the "boat" example in section 3.3.1, the constraint equation (3.3.1) defines the relation that the three variational variables δx, δy, $\delta\theta$ should satisfy so that $(\delta x, \delta y, \delta\theta)$ is an admissible variation. Second, for nonholonomic systems the integration and variation operators are no longer commutative due to the fact that varied motions in general are not admissible motions (see Appendix B).

4 APPLICATION TO OBTAIN APPROXIMATE SOLUTION OF DYNAMICS PROBLEM

4.1 Introduction

Previous studies, [10-11], have shown that HLVA can be applied to obtain an approximate solution to dynamics problems of holonomic systems with satisfactory accuracy. In order to evaluate this application of HLVA (which we shall refer to as *the HLVA method.*), this chapter starts with a general description of this method for a generic one degree-of-freedom system in section 4.2. Then this method is illustrated with a particular example in section 4.3. Following the description of the HLVA method, some numerical aspects of the method are discussed in section 4.4. In section 4.5, the possibility of applying this method to nonholonomic systems is explored. In section 4.7, Hamilton's Principle is discussed in terms of its potential to be applied to obtain a direct approximate solution to dynamics problems is explained.

4.2 The HLVA Method

The HLVA method is an approximate procedure for "solving" a dynamics problem via HLVA. Like many other approximate procedures for "solving" problems and dynamics problems, a basic step in this method is the choice of a trial solution which, because of undetermined parameters, actually represents a family of possible approximations. Once the family is fixed, the HLVA method provides a criterion for picking out the "best" approximation within the family.

To better explain the HLVA method, consider a general one degree-of-freedom system (without loss of generality). The HLVA method starts with setting up the variational indicator associated with HLVA as follows:

$$\text{V.I.} = \int_{t_1}^{t_2} \left(\delta L + \Xi\, \delta\xi\right) dt - \left[\frac{\partial T^*}{\partial \dot{\xi}}\, \delta\xi\right]_{t_1}^{t_2}. \tag{4.2.1}$$

Then the method's second step is the choice of a trial solution having the following linear form:

$$\xi(t) = \varphi_0(t) + \sum_{i=1}^{r} c_i\, \varphi_i(t), \tag{4.2.2}$$

where the basis functions $\varphi_i(t)$'s are linearly independent known continuous functions with continuous first derivatives in the time interval from t_1 to t_2, the c_i's are undetermined parameters, and r is the number of undetermined parameters. Trial solutions must satisfy the initial conditions of the system. This is accomplished by choosing the functions φ_0 and $\varphi_i(t)$'s in (4.2.2) so as to satisfy the following initial conditions of the system:

$$\varphi_0(t_1) = x_0, \qquad \left.\frac{d\varphi_0(t)}{dt}\right|_{t_1} = v_0,$$

$$\varphi_i(t_1) = 0, \qquad \left.\frac{d\varphi_i(t)}{dt}\right|_{t_1} = 0\,, \qquad i = 1, \ldots, r.$$

where $x_0 \equiv \xi(t_1)$ and $v_0 \equiv \dot{\xi}(t_1)$.

The next step of the HLVA method is to insert the trial solution (4.2.2) directly into the V.I. (4.2.1). After appropriate manipulations (see Appendix D for demonstration), the V.I. can be arranged in the following format:

$$\text{V.I.} = \sum_{i=1}^{r} \left\{ \sum_{j=1}^{r} A_{ij}c_j + b_i + p_i(c_1, \cdots, c_r) \right\} \delta c_i \tag{4.2.3}$$

where the A_{ij}'s and the b_i's are constants, and the p_i's are generally quadratures containing the undetermined parameters. Note that the arrangement of the terms inside the curly braces in (4.2.3) can be made differently. The form (4.2.3) takes advantage of the possibility of separating linear terms from nonlinear ones in adjustable parameters in the V.I., and makes it easy to apply standard techniques for solving the set of algebraic equations that the HLVA method gives in the step that follows.

The method's next step is to extract a set of algebraic equations in adjustable parameters from the V.I. (4.2.3) based on HLVA. Because the undetermined parameters c_i's are arbitrary, according to HLVA, the V.I. (4.2.3) vanishes if, and only if, the coefficient of every δc_i in (4.2.3) vanishes, i.e.

$$\sum_{j=1}^{r} A_{ij}c_j + b_i + p_i(c_1, \cdots, c_r) = 0, \qquad i = 1, \ldots, r, \tag{4.2.4}$$

This result consists of a set of r algebraic equations in the r undetermined parameters c_i's.

The final step of this method is, naturally, to solve for the undetermined parameters from the above equations via standard techniques (see Appendix E). Once the adjustable parameters are known, the analytical expression of $\xi(t)$ in (4.2.2) constitutes an approximate solution to the dynamics problem of the system over the time interval $t_1 \leq t \leq t_2$.

For an outline of the HLVA method for a general dynamic system, see Appendix F.

4.3 An Example: Simple Harmonic Oscillator

The HLVA method outlined in the previous section is illustrated with the simple harmonic oscillator shown in Fig. 4.3.1.

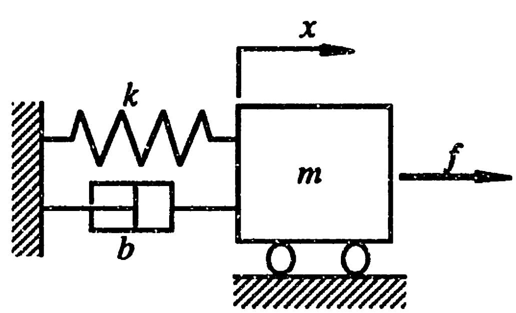

Fig. 4.3.1 Simple harmonic oscillator subjected to prescribed force f. Initial conditions of the system are: $x = x_0$, $\frac{dx}{dt} = v_0$ at $t = 0$.

The first step is to set up the variational indicator associated with HLVA for the above system. With $L = T^* - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ as the lagrangian, (where $\dot{x} \equiv dx/dt$), and

$\Xi\delta x = [-b\dot{x} + f]\delta x$ as the work increment due to nonconservative given forces, the V.I. (4.2.1), after the variation is carried out, can be written as

$$\frac{\text{V.I.}}{m} = \int_{t_1}^{t_2} \left[\dot{x}\delta\dot{x} - \left(\omega_n^2 x + 2\zeta\omega_n\dot{x} - \frac{f}{m}\right)\delta x\right]dt - \left[\dot{x}\delta x\right]_{t_1}^{t_2}, \tag{4.3.1}$$

where the natural frequency ω_n and damping ratio ζ are defined as $\omega_n = \sqrt{k/m}$, and $\zeta = b/2m\omega_n$, t_1 and t_2 are the initial and final instants of the interval for which the dynamics solution is desired.

Next a trial function for $x(t)$ must be selected according to (4.2.2). Although many choices exist for the set of basis functions φ_i's, the simplest basis functions that lead to a simple power series expansion of $x(t)$ in time is selected as follows:

$$x(t) = x_0 + v_0(t - t_1) + \sum_{i=1}^{r} c_i(t - t_1)^{i+1}, \qquad t_1 \le t \le t_2. \tag{4.3.2}$$

where the first two terms are to ensure satisfaction of initial conditions.

Now, inserting the above trial solution directly into the V.I. (4.3.1) and following the detailed steps in Appendix D give

$$\frac{\text{V.I.}}{m} = \sum_{i=1}^{r}\left(\sum_{j=1}^{r} A_{ij}c_j + b_i + p_i\right)\delta c_i \tag{4.3.3}$$

where

$$A_{ij} = -\left[\frac{j(j+1)}{T^2(i+j+1)} + \frac{2\zeta\omega_n(j+1)}{T(i+j+2)} + \frac{\omega_n^2}{(i+j+3)}\right]T^{i+j+3},$$

$$b_i = -\left[\frac{2\zeta\omega_n v_0}{(i+2)} + \frac{\omega_n^2 x_0}{(i+2)} + \frac{\omega_n^2 v_0 T}{(i+3)}\right]T^{i+2},$$

$$p_i = \int_0^T \frac{f(\tau + t_1)}{m}\tau^{i+1}d\tau,$$

$$T \equiv t_2 - t_1, \qquad x_0 \equiv x(t_1), \qquad v_0 \equiv \dot{x}(t_1).$$

Since the undetermined parameters c_i's are arbitrary, the V.I. (4.3.3) vanishes if, and only if, the following set of nonhomogeneous algebraic equations holds:

$$\sum_{j=1}^{r} A_{ij}c_j + b_i + p_i = 0, \qquad i = 1, ..., r, \tag{4.3.4}$$

When the system parameters (m, k, b), the forcing function $f(t)$, the instants t_1 and t_2, the system conditions at $t = t_1$, and the number of adjustable parameters r in the trial solution are specified, the time history $x(t)$ can be obtained through the solution of the set of algebraic equations (4.3.4).

For a demonstration of the approximation accuracy, free vibration and harmonically forced vibrations are obtained (see Appendix D) for the interval $0 \leq t \leq 3$ sec† with the

† At present, the time interval is kept small so that good accuracy can be obtained with only a few adjustable parameters. For an arbitrarily large time interval, the HLVA method can also give approximations of high accuracy if a numerical treatment is applied to the method, which is discussed in Section 4.4.

following quantities: $m = 20$ kg, $k = 80$ N/m, $b = 16$ N-s/m, $x_0 = 1$ m, $v_0 = 1$ m/s. The results are summarized in the following table.

Table 4.3.1 Dynamics solutions for simple harmonic oscillator

($\omega_n = 2$ rad/s, $\zeta = 0.2$, $x_0 = 1$ m, $v_0 = 1$ m/s, $0 \le t \le 3$ sec)

Case	*Forcing* $f(t) = F_0 \sin\omega t$ N	r*	*Maximum error e%* ** *(Order of magnitude)*
1	$f(t) = 0$	12	10^{-4}
2	$f(t) = 40\sin 4t$	12	10^{-2}
3	$f(t) = 40\sin 0.5t$	10	10^{-2}

* r is the number of undetermined parameters included in the trial solution.

** $e = (|x_{approximate} - x_{true}| / |x_{true}|)100\%$

For *Case* 1 in the above table, the accuracy of $10^{-4}\%$ of the true solution is satisfactory for most engineering problems. The accuracy for *Case* 2 and *Case* 3 can be improved by increasing the number of adjustable parameters, or by dividing this 3-second time interval into 3 or 4 time segments and apply the HLVA method to each segment with the final conditions of one segment being the initial conditions of the next time segment. The advantages and disadvantages as well as the accuracy of the solution due to such a treatment of the time interval are discussed in Section 4.4 and Appendix G.

4.4 Numerical Aspects of The HLVA Method

This section discusses numerical issues pertinent to the HLVA method; namely: (1) Initial and final instants of the time interval $t_1 \leq t \leq t_2$; (2) trial functions and approximation accuracy; (3) nondimensional time in trial solution.

(1) *Initial and final instants of the time interval* $t_1 \leq t \leq t_2$

As we have seen in the previous two sections, the time interval $t_1 \leq t \leq t_2$ for which an approximate dynamics solution is desired should be chosen based on the following two considerations: (i) the initial time t_1 of the time interval $t_1 \leq t \leq t_2$ should be such that the conditions of the system (i.e., the generalized coordinates and generalized velocities) at t_1 are somehow known (e.g., initial conditions of the system); (ii) the length of the interval, that is, $T \equiv t_2 - t_1$, should be kept relatively small, although theoretically it can be arbitrary. This is because for increasing T, the higher the number of adjustable parameters is required in the trial solution to maintain a specified approximation accuracy.

In practice, a large time interval is divided into small time segments. For any given time segment, the initial conditions are the final conditions from the previous segment, and the finial conditions will be used as initial conditions for the next segment. When the HLVA method is applied singly from the first segment to the last segment, the approximate solution for the entire time interval is obtained. Note that this propagative process† is possible only when the displacement and velocity of every particle of the system is continuous (see Appendix G for demonstration).

† This method of treating large time intervals is not confined to the HLVA method.

Experiences have shown that the duration of a time segment, T, depends on the system's characteristic times. Usually T should not be larger than the smallest characteristic time of the system if a polynomial trial solution with fewer than 10~12 undetermined parameters is used. For example, for a lumped-parameter system with a largest natural frequency ω_n and subjected to a harmonic forcing of frequency ω, then according the author's experience, T should not be larger than $\frac{1}{2}\left[\frac{2\pi}{\max(\omega, \omega_n)}\right]$. If the solution is not satisfactory, a decrease in T shall improve the approximation accuracy (at least theoretically).

Although the propagative application of the HLVA method to segments of a large time interval allows a solution to an arbitrarily large time interval when the computing facility is limited, the error of solution in one time segment is also propagated to the solution of the next time segment. Experiences with both linear and nonlinear systems (see Appendices G and H) have shown that the error propagation is slow when proper number of undetermined parameters are included in trial solution and a proper segment duration is chosen. For example, Table 4.4.1 shows the solutions to the simple harmonic oscillator discussed in Section 4.3. for the time interval $0 \leq t \leq 30$ sec. For all the three cases shown in Table 4.4.1 the maximum errors occur during the transient times of the system's corresponding responses (refer to figures in Appendix G). After the system enters steady state the errors appear to be steady also.

Table 4.3.1 Dynamics solutions for simple harmonic oscillator

(ω_n= 2 rad/s, ζ = 0.2, x_0 = 1 m, v_0 = 1 m/s, $0 \le t \le 30$ sec)

Case	*Forcing* $f(t) = F_0 \sin\omega t$ N	T (sec) / r *	*Maximum error* e% ** *(Order of magnitude)*
1	$f(t) = 0$	1 / 10	10^{-6}
2	$f(t) = 40\sin 4t$	0.2 / 8	10^{-2}
3	$f(t) = 40\sin 0.5t$	0.2 / 8	10^{-4}

* T is the duration of each time segment, r is the number of undetermined parameters included in the trial solution for each time segment.

** $e = (|x_{\text{approximate}} - x_{\text{true}}| / |x_{\text{true}}|)100\%$

(2) *Trial functions and approximation accuracy*

It is understood that the trial functions should satisfy the initial conditions of the system . However, no restriction on the state of the system at $t = t_2$ can be made since it would depend on both the initial conditions at $t = t_1$ and the forcing acting on the system during the time interval. Thus the trial functions should allow for some "freedom" at $t = t_2$.

As seen previously, the choice of basis functions is truly arbitrary. However, the approximation accuracy depends heavily on which basis functions are used in constructing the trial solutions. Good basis functions can give a good approximation with a small number of adjustable parameters. Usually the polynomial trial solution is chosen based on the following reasons: (i) The convergence of the polynomial trial solution to the true solution is assured by Weierstrass's theorem [Rektorys]†. (ii) The polynomial trial solution has been shown to give satisfactory approximate dynamics solutions to a wide range of systems (see [10-11], Appendix H and Appendix I).

For other basis functions, it is conjectured that, if the number of independent adjustable parameters in a trial solution is increased without limit, then the corresponding approximate solution obtained by the HLVA method would converge (at least under certain conditions††) to the true solution [4, p153]. Once the trial solution is known to be convergent, then more adjustable parameters in the trial solution means higher accuracy of approximation by the HLVA method.

(3) *Nondimensional time in trial solutions*

For the polynomial trial solutions, a nondimensional time is usually introduced into the trial solution in order to reduce the risk of having an ill-conditioned matrix when solving the set of algebraic equations. The procedure of the HLVA method with trial solutions in nondimensional time is demonstrated in Appendix H with a nonlinear harmonic oscillator.

---

† **Weierstrass's Theorem:** It is possible to approximate uniformly in $[a, b]$ with an arbitrary accuracy every function continuous in $[a, b]$ by means of a sequence of polynomials, that is, to every $\varepsilon > 0$ there exists a polynomial $P_n(x)$ such that $|f(x) - P_n(x)| < \varepsilon$ for all $a \leq x \leq b$.

†† For example, the existence and uniqueness of the solution to the dynamics problem.

4.5 The HLVA Method for a Nonholonomic System

The constrained motion of a boat, shown in Fig. 4.5.1, is used as an example to demonstrate the HLVA method for nonholonomic systems. The basic steps of setting up the set of algebraic equations associated with the HLVA method is summarized below. For details of each step, see Appendix I.

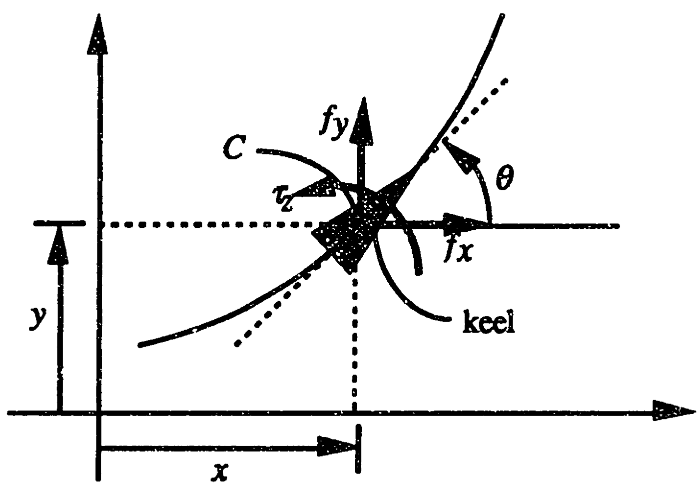

Fig. 4.5.1. Constrained motion of boat under prescribed forcings† : fx, fy, and τ_z. The equation of constraint is $dy - dx \tan\theta = 0$.

† Note that no matter what the prescribed forcings fx, fy, and τ_z are, the motion of the boat conforms to the equation of constraint

$$dy - dx \tan\theta = 0. \tag{1}$$

The mechanism by which this is achieved is the forces of constraint. During the motion of the boat, the forces of constraint is called into play and they so adjust themselves that the motion under the action of the given external forces and the constraint forces together satisfies the equation of constraint (1). Note that the variational work increment of these forces of constraint under an admissible variation of the system vanishes.

1. *Set up variational indicator associated with HLVA*

If the lagrangian and work expression of the system are inserted into the variational indicator of HLVA (2.2.6) and integrations by parts for terms containing $\delta\dot{x}$, $\delta\dot{y}$, or $\delta\dot{\theta}$ are conducted, the variational indicator (2.2.6) for this system is as follows:

$$\text{V.I.} = -\int_{t_1}^{t_2}\left[\left(m\frac{d^2x}{dt^2} - f_x\right)\delta x + \left(m\frac{d^2y}{dt^2} - f_y\right)\delta y + \left(I_0\frac{d^2\theta}{dt^2} - \tau_z\right)\delta\theta\right] dt\,, \qquad (4.5.1)$$

where m is the mass of the boat, I_c is the moment of inertia of the boat about the axis perpendicular to water surface and passing through the boat's centroid.

2. *Eliminate dependent variational variable from V.I.*

In the V.I. (4.5.1), the variation of the generalized coordinates, δx, δy and $\delta\theta$ are constrained by the relation $\delta y - \delta x\tan\theta = 0$, and any two of them can be chosen as independent variational variables. Arbitrarily, we choose δx and $\delta\theta$ as the two independent variational variables. The elimination of δy is accomplished by embedding the equation of constraint into the variational indicator, then the V.I. (4.5.1) is transformed to the following form (see Appendix I):

$$\text{V.I.} = -\int_{t_1}^{t_2}\left[\left(m\frac{d^2x}{dt^2}\left(1+\tan^2\theta\right) + m\frac{dx}{dt}\frac{d\theta}{dt}\sec^2\theta\tan\theta - f_x - f_y\tan\theta\right)\delta x + \left(I_0\frac{d^2\theta}{dt^2} - \tau_z\right)\delta\theta\right]dt$$

(5.4.2)

Note that usually the generalized coordinate y can not be eliminated also from the V.I., although for this particular example y has been eliminated from the V.I. via a different form of the constraint equation (see Appendix I).

3. *Choose trial solutions*

The trial solutions should be chosen for those generalized coordinates associated with the independent variational variables. For this particular problem these generalized coordinates are x and θ. The simple polynomial trial solutions for x and θ are selected as follows:

$$x(t) = x_0 + v_0(t - t_1) + \sum_{i=1}^{r} c_i(t - t_1)^{i+1} \tag{4.5.3a}$$

$$\theta(t) = \theta_0 + \omega_0(t - t_1) + \sum_{i=1}^{r} k_i(t - t_1)^{i+1} \tag{4.5.3b}$$

where $x_0 \equiv x(t_1)$, $v_0 \equiv \left.\frac{dx}{dt}\right|_{t_1}$, $\theta_0 \equiv \theta(t_1)$, and $\omega_0 \equiv \left.\frac{d\theta}{dt}\right|_{t_1}$.

4. *Obtain set of algebraic equations in adjustable parameters*

Inserting the trial solutions into the V.I. (4.5.2) and collecting terms (see Appendix I) lead to the following set of algebraic equations:

$$\sum_{j=1}^{r} A_{ij}\alpha_j + p_{xi} = 0 \qquad i = 1, \ldots, r \tag{4.5.4a}$$

$$\sum_{j=1}^{r} B_{ij}\beta_j + p_{\theta i} = 0 \qquad i = 1, \ldots, r \tag{4.5.4b}$$

where the coefficients A_{ij}, p_{xi}, B_{ij}, and $p_{\theta i}$ are specified in Appendix I. This is a set of nonlinear algebraic equations in the undetermined parameters α_j's and β_j's.

The approximate solutions of $x(t)$ and $\theta(t)$ are obtained when the adjustable parameters are obtained from the above set of algebraic equations. The solution of the generalized coordinate $y(t)$ can then be determined from the equation of constraint $\frac{dy}{dt} = \frac{dx}{dt}\tan\theta$ †.

Remarks. The amount of labor involved in deriving the set of algebraic equations for this two degree-of-freedom nonholonomic system is much larger than that for a holonomic system of the same number of degrees of freedom. In addition, the process of numerically solving the set of highly nonlinear algebraic equations (4.5.4) (see Appendix I) is extremely inefficient when using the method of *iteration by total steps* (see Appendix E). For a most general nonholonomic system, the HLVA method does not render a set of algebraic equations, instead it gives a set of equations that consist of both differential and algebraic equations in undetermined parameters (see Appendix F).

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† The constraint equation $dy - dx\tan\theta = 0$ is written for a displacement $(dx, dy, d\theta)$ in an infinitesimal time interval dt. If we divide both sides this constraint equation by dt and let dt approach zero, this equation of constraint, after rearranging terms, is transformed to $\frac{dy}{dt} = \frac{dx}{dt}\tan\theta$.

4.6 HLVA vs Hamilton's Principle†

In section 2.4 we pointed out that HLVA can be used to derive differential equations of motion and also can be used to obtain a direct approximate solution to dynamics problems. A question we now may ask is the following: Can Hamilton's Principle also be used to obtain a direct approximate solution to dynamics problems? The answer is no, and we explain it hereafter.

Since Hamilton's Principle is HLVA under the assumption that the varied motions are coterminal with the natural motion at the terminal times $t = t_1$ and $t = t_2$, if any trial solution is to satisfy Hamilton's Principle it has to be one that does not allow any variation at the terminal times. In fact, Hamilton's Principle ca be written as follows:

$$\left.\begin{aligned} &\text{V.I.} = \int_{t_1}^{t_2} \left(\delta\mathcal{L} + \Xi\, \delta\xi\right) dt \\ &\delta\xi\big|_{t_1} = 0, \;\; \delta\xi\big|_{t_2} = 0 \end{aligned}\right\}. \tag{4.6.1}$$

Equation (4.6.1) says that if t_1 is the system's initial time and t_2 is a fixed time after t_1, then the solution to the dynamics problem via trial solutions with undetermined parameters can be obtained for the time interval $t_1 < t \leq t_2$ only when the configurations of the system at t_1 and t_2 are known in advance. In view of the the fact that for most dynamics problems the

† For convenience and without loss of generality, a one degree-of-freedom system is used in the discussions throughout this section.

configurations of the system after the initial time can not be predicted in advance, Hamilton's Principle cannot be used to obtain approximate solutions to dynamics problems via trial solutions with undetermined parameters as HLVA can, except in those rare cases in which the configurations of the system at both terminal times $t = t_1$ and $t = t_2$ are known.

In contrast, HLVA allows the varied motions to be arbitrary in any interval $t_1 \leq t \leq t_2$ after the system's initial time, then it permits a trial solution to vary not only at the times within the interval but also at the terminal times $t = t_1$ and $t = t_2$ if the configurations of the system at these instants are not known in advance. Particularly, the following trial solution used in section 4.2

$$\left.\begin{aligned} &\xi(t) = \varphi_0(t) + \sum_{i=1}^{r} c_i\, \varphi_i(t) \\ &\varphi_0(t_1) = \xi(0), \qquad \left.\frac{d\varphi_0(t)}{dt}\right|_{t_1} = \left.\frac{d\xi}{dt}\right|_{t=t_1} \\ &\varphi_i(t_1) = 0, \qquad \left.\frac{d\varphi_i(t)}{dt}\right|_{t_1} = 0 \end{aligned}\right\} \tag{4.6.2}$$

can vary anywhere in the interval $t_1 \leq t \leq t_2$ except at the initial time $t = t_1$ since the configuration of the system at this instant is fixed by the system's initial conditions.

5 DISCUSSIONS ON THE APPLICATION OF HAMILTON'S LAW OF VARYING ACTION

5.1 Introduction

As seen in Chapter 4, the HLVA method reduces the dynamics problem to solving a set of simultaneous algebraic equations via trial solutions with undetermined parameters. In this chapter, the position of the HLVA method in the context of engineering analysis† is discussed. Comparison between the HLVA method with Galerkin's method for dynamics problems is made, and the scope of application of the former is also briefly discussed.

5.2 The HLVA Method in Engineering Analysis

If we recall the HLVA method discussed in Chapter 4 and recall some procedures of solving engineering problems covered in most engineering analysis textbooks (e.g., [4]), we observe that: in terms of the mathematical model for an engineering problem, the HLVA method is based on an energy formulation of a dynamics problem; in terms of the procedure that reduces a mathematical problem to a numerical procedure, the HLVA method belongs to the category of procedures that solve the mathematical problem via trial solutions with undetermined parameters.

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† The term *engineering analysis* as used here is defined in [4, preface], means the performance of the following steps within the framework of an engineering problem:

1. Construction of a mathematical model for a physical situation.
2. Reduction of the mathematical model to a numerical procedure.

In order to indicate the relation of the HLVA method with other similar techniques, Table 5.2.1 summarizes the methods of trial solutions with undetermined parameters for both lumped-parameter systems and continuous systems. These methods are categorized into two groups in terms of the mathematical formulations of the problems: methods base on a differential formulation and methods based an on energy formulation. From Table 5.2.1, it is seen that the HLVA method fills the void of an energy-based method for dynamics problems for both lumped-parameter and continuous systems. In this sense, it may be said that the HLVA method bears the same relationship to the weighted-residual methods (see Appendix J) for dynamics problems as the Ritz method bears to the weighted-residual methods for equilibrium problems of continuous systems.

Table 5.2.1. **Methods of trial solutions with undetermined parameters in engineering analysis**

	Lumped-parameter systems		*Continuous systems*	
Problem Category	*Equilibrium problems*	*Dynamics problems*	*Equilibrium problems*	*Dynamics problems*
Methods based on differential formulation	N/A*	Collocation Subdomain Galerkin's Least squares	Collocation Subdomain Galerkin's Least squares	Collocation Subdomain Galerkin's Least squares
Methods based on energy formulation	N/A*	**HLVA method**	Ritz method	**HLVA method**

* 'N/A' is used because the equilibrium problem of a lumped-parameter system is formulated as a set of algebraic equations.

5.3 Comparison With Galerkin's Method

In terms of accuracy, we compare the HLVA method with Galerkin's method because the later is widely used in engineering problems as compared with other weighted-residual methods. We find that both Galerkin's and the HLVA methods produce the same set of algebraic equations in adjustable parameters if identical trial solutions are used (see Appendix K). Therefore, the approximation accuracy of the HLVA method is always as good as that of Galerkin's method given the same set of trial solutions.

In terms of the computational labor required, since for both methods the same set of algebraic equations has to be solved, the only difference is in the way the coefficients of the algebraic equations are derived. In fact we found that the HLVA method demands more labor in constructing the set of algebraic equations than Galerkin's method does (see Appendix K).

5.4 The Application Scope

In general, the HLVA method applies for a dynamic system whose generalized coordinates and generalized velocities are continuous. These continuity requirements are necessary to ensure (1) the convergence of trial solutions, and (2) the proper initial conditions of the time segments of a large time interval.

6 CONCLUSIONS

Relationships between the general form of HLVA (2.2.5), Hamilton's Principle (2.3.1) and Hamilton's Principle of Stationary Action (2.3.4) have been discussed. The general form of HLVA (for both holonomic and nonholonomic systems) is the general variational principle associated with Hamilton when the two terminal times $t = t_1$ and $t = t_2$ are fixed. Hamilton's Principle (for both holonomic and nonholonomic systems) and Hamilton's Principle of Stationary Action (for holonomic conservative systems only) are two special cases of the general form of HLVA when the natural motion of the system is compared with neighboring varied motions that share the same configurations as the natural motion at $t = t_1$ and $t = t_2$. While both HLVA and Hamilton's Principle can be used to derive differential equations of motion of a system, only HLVA can be used to obtain an approximate solution to dynamics problems directly via trial solutions with undetermined parameters.

A conceptual interpretation of HLVA for conservative systems has been presented with an illustrative example. The illustration shows that the variation of the action of a system in a time interval $t_1 \le t \le t_2$ is a linear function of the admissible variations of the system at the terminal instants $t = t_1$ and $t = t_2$. This interpretation is consistent with that of Hamilton's Principle of Stationary Action [3, p31-34]. It has also been illustrated that varied motions in general are not admissible motions for nonholonomic systems, when demonstrating the differences between HLVA for nonholonomic systems and HLVA for holonomic systems.

The application of HLVA to obtain an approximate solution to a dynamics problem via trial solutions with undetermined parameters, namely the HLVA method, has been evaluated. It has been shown that the HLVA method can be extended to nonholonomic systems. For a general nonholonomic system a set of both algebraic and differential equations, instead of algebraic only as for holonomic systems, must be solved. A particular nonholonomic system of three independent generalized coordinates and two degrees of freedom has been used to demonstrate the application of the HLVA method for nonholonomic systems.

The approximation accuracy of the HLVA method has been investigated. For a lumped-parameter system, to obtain a solution of accuracy of at least $10^{-2}\%$ when using a polynomial trial solution with about 10 undetermined parameters, the required duration of the time segment for the HLVA method has been estimated to be less than or equal to one half of the smallest characteristic time of the system. If the desired accuracy is not achieved, the number of undetermined parameters included in the trial solution must be increased or the duration of the time segment must be decreased.

The approximation accuracy of the HLVA method has also been compared with that of Galerkin's method. It has been found that the former provides the identical approximation accuracy as the latter when identical trial solutions are used, but requires more computational labor than the latter. Therefore, the HLVA method does not display any significant advantage over Galerkin's method for lumped-parameter systems.

The HLVA method has been compared with other methods of trial solutions with undetermined parameters for different categories of engineering problems. The comparison shows that the HLVA bears the same relationship to the weighted-residual methods for dynamics problems as the Ritz method bears to the weighted-residual methods for equilibrium problems of continuous systems. This is due to the following two reasons: (1) both the HLVA and the Ritz methods are based on energy formulations of physical systems and use trial solutions with undetermined parameters; and (2) each of them gives an identical approximation with that of the corresponding Galerkin's method under the same trial solution.

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APPENDIX A

Derivation of Lagrange's Equations From Hamilton's Law of Varying Action

This Appendix shows how the time boundary terms will be canceled out exactly in deriving Lagrange's Equations of motion from Hamilton's Law of Varying Action.

Let $\xi_1, \ldots, \xi_n$ be a complete and independent set of generalized coordinates for a dynamic system, and let $\delta\xi_1, \ldots, \delta\xi_n$ be their associated variational variables. In addition, let T^* be the kinetic coenergy of the system, V be the potential energy of the conservative forces, and Ξ_j be the generalized force associated with the generalized coordinate ξ_j due to nonconservative forces.

The general form of Hamilton's Law of Varying Action for both holonomic and nonholonomic systems states that *A varied motion of the system during an arbitrary finite time interval* $t_1 \leq t \leq t_2$ *is a natural motion if, and only if, the variational indicator*

$$\text{V.I.} = \int_{t_1}^{t_2} \left(\delta L + \sum_{j=1}^{n} \Xi_j \, \delta\xi_j \right) dt - \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j} \delta\xi_j \right]_{t_1}^{t_2}, \tag{A.1}$$

(where $L = T^* - V$ is the lagrangian of the system.) *vanishes for arbitrary admissible variations of the motion within this time interval.*

Because the lagrangian of a system is a function of the generalized coordinates, the generalized velocities and time, that is, $L = L(\xi_1, \ldots, \xi_n; \dot{\xi}_1, \ldots, \dot{\xi}_n; t)$, the variation of the lagrangian can be written as

$$\delta L = \sum_{j=1}^{n} \frac{\partial L}{\partial \xi_j} \delta \xi_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{\xi}_j} \delta \dot{\xi}_j . \qquad \text{(A.2)}$$

Due to the commutativity relation

$$\frac{d}{dt}\left(\delta \xi_j\right) = \delta\left(\frac{d}{dt}\, \xi_j\right), \qquad \text{(A.3)}$$

and the notation $\dot{\xi}_j = \frac{d}{dt}\, \xi_j$, the second term on the right hand side of (A.2) is equivalent to the expression $\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{\xi}_j} \frac{d}{dt}\left(\delta \xi_j\right)$, which, by the product differentiation rule of calculus, can be written as

$$\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{\xi}_j} \frac{d}{dt}\left(\delta \xi_j\right) = \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j} \delta \xi_j\right) - \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) \delta \xi_j . \qquad \text{(A.4)}$$

Then inserting (A.4) into (A.2) gives

$$\delta L = \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j} \delta \xi_j\right) - \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) \delta \xi_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \xi_j} \delta \xi_j . \qquad \text{(A.5)}$$

Inserting the lagrangian expression (A.5) into the variational indicator (A.1), the V.I. (A.1) becomes

$$\text{V.I.} = \left[\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{\xi}_j}\delta\xi_j\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{j=1}^{n} \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} - \Xi_j\right\} \delta\xi_j dt - \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j}\delta\xi_j\right]_{t_1}^{t_2}. \tag{A.6}$$

Since the potential energy V is a function of the generalized coordinates $\xi_1, \ldots, \xi_n$, and time t only, then the relation $\partial L / \partial \dot{\xi}_j = \partial T^* / \partial \dot{\xi}_j$ holds, which means that the first term and the last term in (A.6) are identical. Therefore, the variational indicator associated with Hamilton's Law of Varying Action (A.1), which is valid for both holonomic and nonholonomic systems, becomes

$$\text{V.I.} = -\int_{t_1}^{t_2} \sum_{j=1}^{n} \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} - \Xi_j\right\} \delta\xi_j \, dt \, . \tag{A.7}$$

If the system is holonomic, then the admissible variations $\delta\xi_1, \ldots, \delta\xi_n$ are independent. Therefore, the variational indicator (A.7) vanishes if, and only if the following equations hold

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} = \Xi_j, \qquad j = 1, \ldots, n \tag{A.8}$$

which are *Lagrange's Equations* for holonomic systems.

For nonholonomic systems, the $\delta\xi_j$'s are not independent. Lagrange multipliers can be used to obtain Lagrange's Equations from (A.7), which can be found in [5, p77].

APPENDIX B

Comments on $\int_{t_1}^{t_2} \delta L\, dt$ and $\delta \int_{t_1}^{t_2} L\, dt$

Without loss of generality, this appendix discusses the difference between the expression $\delta \int_{t_1}^{t_2} L\, dt$ and the expression $\int_{t_1}^{t_2} \delta L\, dt$ via Hamilton's Principle for systems under conservative forces.

To begin with, we shall state Hamilton's Principle for a dynamic system (holonomic or nonholonomic) under conservative forces. Let $\xi_1, \ldots, \xi_n$ be a complete and independent set of generalized coordinates for a dynamic system, and let $\delta\xi_1, \ldots, \delta\xi_n$ be their associated variational variables. In addition, let T^* be the kinetic coenergy of the system, V be the potential energy of the conservative forces. Hamilton's Principle asserts that *A varied motion of the system during an arbitrary finite time interval $t_1 \le t \le t_2$ is a natural motion if, and only if, the following relations holds*

$$\left.\begin{aligned} &\int_{t_1}^{t_2} \delta L dt = 0 \\ &\delta\xi_j\big|_{t_1} = 0,\ \delta\xi_j\big|_{t_2} = 0,\ j = 1, \ldots, n \end{aligned}\right\}, \tag{B.1}$$

(where $L = T^{*} - V$ is the lagrangian of the system.) *for arbitrary admissible variations of the motion within this time interval.* In (B.1), the operator δ refers to a displacement from the point $(\xi_1, \ldots, \xi_n)$ on the natural trajectory in configuration space to a contemporaneous point $(\xi_1 + \delta\xi_1, \ldots, \xi_n + \delta\xi_n)$ on the varied trajectory. The essential point to keep in mind is that $(\delta\xi_1, \ldots, \delta\xi_n)$ are always *admissible variations*, whereas the varied trajectory, $(\xi_1 + \delta\xi_1, \ldots, \xi_n + \delta\xi_n)$ may not be admissible.

If the system is holonomic, the varied motions are always admissible. Then Hamilton's Principle for holonomic systems compares the natural motion with neighboring admissible motions. According to the calculus of variation, the natural motion that makes $\int_{t_1}^{t_2} \delta L \, dt$ vanish when compared with neighboring admissible motions also makes the integral $\int_{t_1}^{t_2} L \, dt$ stationary, i.e., $\delta \int_{t_1}^{t_2} L dt = 0$. Therefore, if the system is holonomic the variation and integration operators in (B.1) are commutative, that is, Hamilton's Principle for holonomic systems can be written as

$$\left.\begin{aligned} &\delta \int_{t_1}^{t_2} L dt = 0 \\ &\delta\xi_j\big|_{t_1} = 0, \; \delta\xi_j\big|_{t_2} = 0, \; j = 1, \ldots, n \end{aligned}\right\} \tag{B.2}$$

If the system is nonholonomic, however, the forms (B.1) and (B.2) of Hamilton's Principle are no longer equivalent, and we must adhere to the original form (B.1). In fact, for nonholonomic systems, it is with the ***varied motions*** that Hamilton's Principle, $\int_{t_1}^{t_2} \delta L dt = 0$, compares the natural motion, and these varied motions are in general ***not admissible***. Since the expression $\delta \int_{t_1}^{t_2} L\, dt$ always implies comparison of the integral $\int_{t_1}^{t_2} L\, dt$ for the natural motion with neighboring ***admissible motions***, the motion that satisfies $\int_{t_1}^{t_2} \delta L dt = 0$ does not necessarily make $\int_{t_1}^{t_2} L\, dt$ stationary in the class of ***geometrically admissible motions.***

APPENDIX C

Constrained Motion of A Boat

In this appendix, the equations of motion for the nonholonomic system shown in Fig C.1 [3, p119] is derived via Lagrange's Equations. The solution of the equations of motion when the system is under a particular forcing condition is obtained.

Fig. C.1 shows a boat on a body of water whose surface is in the plane of sketch. The boat's motion is modeled such that the translation of the boat must always be parallel to the instantaneous heading of the keel. This requirement implies the following constraint relation:

$$\delta y - \delta x \tan\theta = 0 \qquad \text{(C.1)}$$

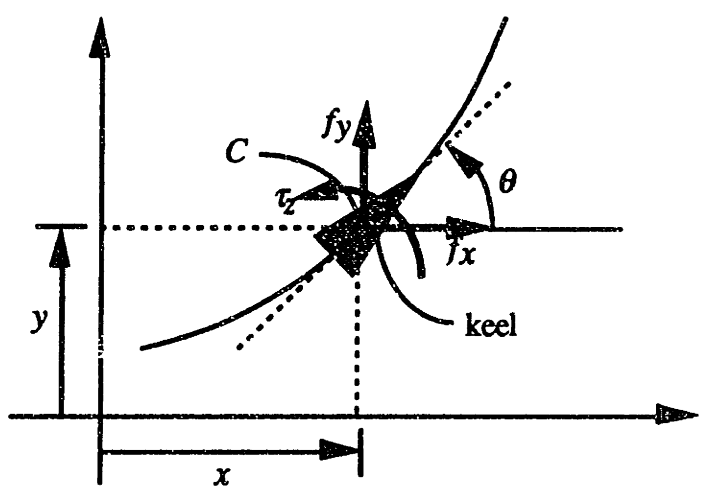

Fig. C.1 Constrained motion of a boat. Action of keel is considered as a nonholonomic constraint.

Equations of motion

Let us first set up the lagrangian of the system and the work expression as follows:

$$L = T^* - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I_0\dot{\theta}^2, \qquad \sum_{j=1}^{3} \Xi_j\, \delta\xi_j = f_x\, \delta x + f_y\, \delta y + \tau_z\, \delta\theta\,,$$

where, m is the mass of the boat, I_c is the moment of inertia of the boat about the axis perpendicular to the water surface and passing through the boat's centroid; f_x and f_y are the components of the total external forces acting on the boat in the x-direction and y - direction, respectively; τ_z is the torque on the boat about the axis perpendicular to the water surface and passing through the boat's centroid; $\dot{x} \equiv dx/dt$, $\dot{y} \equiv dy/dt$, and $\dot{\theta} \equiv d\theta/dt$, all inertial.

Lagrange's Equations for this system is [5, p75]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} = \Xi_j + \lambda B_j, \qquad j = 1, 2, 3 \qquad \text{(C.2)}$$

where B_j (j = 1, 2, 3) are the coefficients of δx, δy, $\delta\theta$ in the constraining relation (C.1), respectively, and λ is the *Lagrange multiplier* associated with the constraint equation (C.1). Therefore, inserting the lagrangian and the generalized forces into (C.2) gives the Lagrange's Equations of motion for this system as follows:

$$\left.\begin{aligned} \frac{d}{dt}(m\dot{x}) &= f_x - \lambda \tan\theta \\ \frac{d}{dt}(m\dot{y}) &= f_y + \lambda \\ \frac{d}{dt}\left(I_0\dot{\theta}\right) &= \tau_z \end{aligned}\right\}. \qquad \text{(C.3a)}$$

With the equations (C.3a) we must associate the equation of constraint (C.1), i.e.,

$$\dot{y} - \dot{x}\tan\theta = 0 \qquad \text{(C.3b)}^{\dagger}$$

in order to determine the four unknowns, x, y, θ and λ, for a given set of initial conditions and forcing of the system. Therefore, equations (C.3a) and (C.3b) constitute the equations of motion of the system.

Transformations of the equations of motion

The set of equations of motion (C.3) can be transformed by eliminating the Lagrange multiplier λ. Then the equations of motion for the system becomes

$$\left.\begin{aligned} m\ddot{x} &= f_x - \tan\theta\left[m\ddot{y} - f_y\right] \\ I_0\ddot{\theta} &= \tau_z \\ \dot{y} &= \dot{x}\tan\theta \end{aligned}\right\}, \qquad \text{(C.4)}$$

where $\ddot{x} \equiv d^2x/dt^2$, $\ddot{y} \equiv d^2y/dt^2$, and $\ddot{\theta} \equiv d^2\theta/dt^2$, all inertial.

Equations (C.4) can be further transformed by eliminating the $\ddot{y}$ term in the first equation of (C.4). This is accomplished by first differentiating the constraint equation (C.3.b), and then inserting the results of differentiation into the first equation of (C.4).

† Since the infinitesimal motion of the boat in an infinitesimal time dt is represented by dy, dx, and $d\theta$ which satisfy the relation $dy - dx\tan\theta = 0$, then dividing both sides of this equation by dt and letting dt approach zero will give the equivalent form of the constraint equation as in (C.3b).

Hence the set of equations in (C.4) becomes

$$\left.\begin{aligned} \ddot{x} &= \frac{f_x - m\dot{x}\,\dot{\theta}\tan\theta\sec^2\theta + \tan\theta f_y}{m\left(1+\tan^2\theta\right)} \\ \ddot{\theta} &= \frac{\tau_z}{I_0} \\ \dot{y} &= \dot{x}\tan\theta \end{aligned}\right\}, \tag{C.5}$$

which is another form of the differential equations of motion of the system.

<u>Solution under a particular set of conditions</u>

If the system is unforced, then the equations of motion (C.5) become

$$\left.\begin{aligned} \ddot{x} &= \frac{- m\dot{x}\,\dot{\theta}\tan\theta\sec^2\theta}{m\left(1+\tan^2\theta\right)} \\ \ddot{\theta} &= 0 \\ \dot{y} &= \dot{x}\tan\theta \end{aligned}\right\}. \tag{C.6}$$

In addition, if the initial conditions of the system are $x(0) = y(0) = 0$, $\theta(0) = 45°$, $\dot{x}(0) = \dot{y}(0) = 1$, and $\dot{\theta}(0) = 0$, the solution of the above equations can be found to be

$$\left.\begin{aligned} x &= t \\ y &= t \\ \theta &= 45° \end{aligned}\right\} \qquad t \geq 0 \tag{C.7}$$

In fact, from the second equation in (C.6) and the initial conditions $\theta(0) = 45°$ and $\dot{\theta}(0) = 0$, the solution of θ is found to be $\theta = 45°$. Inserting this solution of θ into the first equation of (C.6) gives $\ddot{x} = 0$. From the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$, the solution of x is found to be $x = t$. Now that x and θ are all known, from the third equation of (C.6) with the initial condition $y(0) = 0$, the solution of y can be found to be $y = t$. Note that the initial condition $\dot{y}(0) = 1$ is not used in solving the equations of motion because this initial condition is not independent. It is determined by the initial conditions $\dot{x}(0) = 1$ and $\theta(0) = 45°$.

APPENDIX D

Detailed Procedure of the HLVA Method for Simple Harmonic Oscillator

The HLVA method is a procedure for obtaining an approximate solution to dynamics problems via trial solution with undetermined parameters. This appendix gives the detailed steps of the HLVA method with a simple harmonic oscillator shown in Fig. D.1.

To begin with, let us state Hamilton's Law of Varying Action (HLVA) for a one degree-of-freedom holonomic system as follows: *an admissible motion of the system during an arbitrary finite time interval,* $t_1 \le t \le t_2$, *is a natural motion if, and only if, the variational indicator*

$$\text{V.I.} = \int_{t_1}^{t_2} \left(\delta L + \Xi\, \delta\xi \right) dt - \left[\frac{\partial T^*}{\partial \dot{\xi}}\, \delta\xi \right]_{t_1}^{t_2} \tag{D.1}$$

vanishes for arbitrary admissible variations of the motion within this time interval. In (D.1), ξ is the generalized coordinate of the system, $\delta\xi$ is the associated variational variable, $T^* = T^*(\xi, \dot{\xi}, t)$ is the kinetic coenergy of the system, $V = V(\xi, t)$ is the potential energy of the conservative forces, and Ξ is the generalized force associated with the generalized coordinate ξ due to nonconservative forces acting in the system.

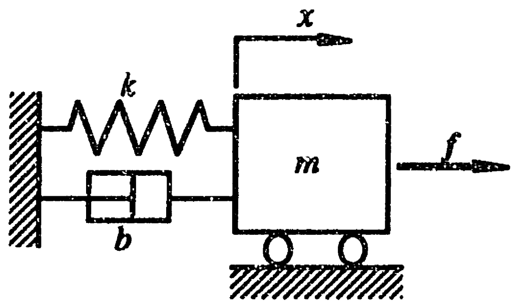

Fig. D.1 Simple harmonic oscillator subjected to prescribed external force f.

Assume the solution for the above simple harmonic oscillator is desired for the interval $t_1 \leq t \leq t_2$. The HLVA method for this problem consists of the following steps:

1. Construct the variational indicator (D.1) for the particular system

The lagrangian of the system is $L = T^* - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$, (where $\dot{x} \equiv dx/dt$, inertial), and the variational work due to the nonconservative forces is $\Xi\delta x = (-b\dot{x} + f)\delta x$. Inserting these expressions into the V.I (D.1) and carrying out the variation and differentiation give the V.I. as follows:

$$\frac{\text{V.I.}}{m} = \int_{t_1}^{t_2} \left[\dot{x}\delta\dot{x} - \left(\omega_n^2 x + 2\zeta\omega_n\dot{x} - \frac{f}{m}\right)\delta x\right]dt - \left[\dot{x}\delta x\right]_{t_1}^{t_2}, \quad \text{(D.2)}$$

where $\omega_n^2 = \sqrt{k/m}$, and $2\zeta\omega_n = b/m$.

Inserting the commutativity relation $\delta\dot{x} = \frac{d}{dt}(\delta x)$ into the V.I. (D.2) and integrating the term $\dot{x}\delta\dot{x} = \dot{x}\frac{d}{dt}(\delta x)$ by parts transform the V.I. (D.2) into the following form:

$$\frac{\text{V.I.}}{m} = -\int_{t_1}^{t_2} \left(\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x - \frac{f}{m}\right)\delta x dt\,, \tag{D.3}$$

where $\ddot{x} \equiv \frac{dx^2}{d^2t}$, inertial.

2. <u>Choose trial solution</u>

Choose the trial solution to be a family of trial functions of the following linear form

$$x(t) = \varphi_0(t) + \sum_{j=1}^{r} c_j\,\varphi_j(t), \qquad t_1 \le t \le t_2 \tag{D.4}$$

where the basis functions $\varphi_j(t)$ are chosen to be linearly independent continuous functions and have continuous first derivatives in the time interval from t_1 to t_2, the c_j's are undetermined parameters, and r is the number of undetermined parameters. In addition, the basis functions $\varphi_j(t)$ should be chosen such that $\varphi_0(t)$ ensures the satisfaction of the initial conditions of the system, i.e. $\varphi_0(t_1) = x(t_1)$, $\left.\frac{d\varphi_0(t)}{dt}\right|_{t_1} = \left.\frac{dx(t)}{dt}\right|_{t_1}$, $\varphi_j(t_1) = 0$, and $\left.\frac{d\varphi_j(t)}{dt}\right|_{t_1} = 0$.

3. Insert the trial solution (D.4) directly into the V.I. (D.3)

Replacing $x(t)$ and its derivatives in (D.3) by the trial function (D.4), and $\delta x(t)$ in (D.3) by the variation of the trial function, i.e.

$$\delta x(t) = \sum_{j=1}^{r} \varphi_j(t)\delta c_j, \tag{D.5}$$

then the V.I. (D.3) becomes

$$\frac{V.I.}{m} = -\int_{t_1}^{t_2} \left[\left(\ddot{\varphi}_0 + \sum_{j=1}^{r} c_j\ddot{\varphi}_j\right) + 2\zeta\omega_n\left(\dot{\varphi}_0 + \sum_{j=1}^{r} c_j\dot{\varphi}_j\right) + \omega_n^2\left(\varphi_0 + \sum_{j=1}^{r} c_j\varphi_j\right) - \frac{f}{m}\right]\sum_{i=1}^{r} \varphi_i\delta c_i dt$$

$$= -\int_{t_1}^{t_2} \left[\left(\ddot{\varphi}_0 + 2\zeta\omega_n\dot{\varphi}_0 + \omega_n^2\varphi_0\right) + \sum_{j=1}^{r} \left(\ddot{\varphi}_j + 2\zeta\omega_n\dot{\varphi}_j + \omega_n^2\varphi_j\right)c_j - \frac{f}{m}\right]\sum_{i=1}^{r} \varphi_i\delta c_i\, dt$$

(D.6)

Note that in (D.6) the dummy index of the trial solution of $x(t)$ is different from that of $\delta x(t)$ to ensure that the multiplication $x(t)\delta x(t)$ is conducted properly after the insertion of the trial solutions into the V.I.

4. Organize the V.I. into the form: $\sum_{i=1}^{r} f_i(c_1, \cdots, c_r)\delta c_i$

The organization step includes interchanges of the orders of the integration operator and the summation operators. These interchanges are possible because the integration interval is finite and the number of terms of summation is also finite. Then interchanging the order of $\sum_{i=1}^{r}$ and $\int_{t_1}^{t_1}$ and bringing the δc_i's out of the integration in (D.6) (this is possible because the δc_i's do not depend on time) transform the V.I. (D.6) into the following form:

$$\frac{\text{V.I.}}{m} = -\left\{\int_{t_1}^{t_2}\left(\ddot{\varphi}_0 + 2\zeta\omega_n\dot{\varphi}_0 + \omega_n^2\varphi_0\right)\varphi_i dt + \right.$$

$$\left.\sum_{j=1}^{r}\left[\int_{t_1}^{t_2}\left(\ddot{\varphi}_j + 2\zeta\omega_n\dot{\varphi}_j + \omega_n^2\varphi_j\right)\varphi_i dt\right]c_j - \int_{t_1}^{t_2}\frac{f}{m}\varphi_i dt\right\}\delta c_i \ . \tag{D.7}$$

Now, denote

$$b_i \equiv \int_{t_1}^{t_2}\left(\ddot{\varphi}_0 + 2\zeta\omega_n\dot{\varphi}_0 + \omega_n^2\varphi_0\right)\varphi_i dt \tag{D.8a}$$

$$p_i \equiv -\int_{t_1}^{t_2}\frac{f}{m}\varphi_i dt \tag{D.8b}$$

$$A_{ij} \equiv \int_{t_1}^{t_2}\left(\ddot{\varphi}_j + 2\zeta\omega_n\dot{\varphi}_j + \omega_n^2\varphi_j\right)\varphi_i dt \ , \tag{D.8c}$$

then the variational indicator (D.7) becomes

$$\frac{\text{V.I.}}{m} = -\sum_{i=1}^{r}\left(\sum_{j=1}^{r} A_{ij}c_j + b_i + p_i\right)\delta c_i. \tag{D.9}$$

5. Identify the set of algebraic equations in adjustable parameters

Since there are no restrictions on the undetermined parameters c_i ($i = 1, \ldots, r$), the variational indicator (D.9) vanishes if, and only if, the coefficient of each δc_i is identical to zero, i.e.

$$\sum_{j=1}^{r} A_{ij}c_j + b_i + p_i = 0, \qquad i = 1, \ldots, r, \qquad \text{(D.10)}$$

which constitutes a set of r algebraic equations with r unknowns, i.e. the c_j's. These adjustable parameters can be found by standard numerical techniques.

6. Solve for the adjustable parameters from the set of algebraic equations (D.10)

Since the number of unknowns is the same as the number of equations in (D.9), the adjustable parameters c_i's can be determined through the solution of this set of algebraic equations. Once the c_i's are determined, the approximate solution of the dynamics problem for the simple harmonic oscillator for $t_1 \leq t \leq t_2$ is given in the form of analytical expressions (D.4).

Expressions of A_{ij}, b_i and p_i for a particular trial function

The analytical solution of the dynamics problem for a general set of basis functions of the trial family is derived above. Now, a particular set of basis functions is chosen as follows:

$$\varphi_0(t) = x_0 + v_0(t - t_1), \qquad \varphi_i(t) = (t - t_1)^{i+1}. \qquad \text{(D.11)}$$

Then the trial solution is a polynomial in time as follows:

$$x(t) = x_0 + v_0(t - t_1) + \sum_{i=1}^{r} c_i (t - t_1)^{i+1}, \qquad t_1 \le t \le t_2. \tag{D.12}$$

where $x_0 \equiv x(t_1)$, and $v_0 \equiv \left.\frac{dx}{dt}\right|_{t_1}$. Inserting the basis functions (D.11) into (D.8) and carrying out the integrations will give explicit expressions for A_{ij}, b_i and p_i. This process is demonstrated as follows.

The nonhomogeneous term b_i

Substitution of the basis functions (D.11) into (D.8a) gives

$$b_i = \int_{t_1}^{t_2} \left[2\zeta\omega_n v_0 + \omega_n^2 x_0 + \omega_n^2 v_0 (t - t_1)\right](t - t_1)^{i+1} dt .$$

Denote $T \equiv t_2 - t_1$ and change the dummy integration variable t by introducing $\tau = t - t_1$ in the above expression. Then,

$$b_i = \int_0^T \left[2\zeta\omega_n v_0 + \omega_n^2 x_0 + \omega_n^2 v_0 \tau\right]\tau^{i+1} d\tau$$

$$= \left[\frac{2\zeta\omega_n v_0}{(i+2)} + \frac{\omega_n^2 x_0}{(i+2)} + \frac{\omega_n^2 v_0 T}{(i+3)}\right] T^{i+2} . \tag{D.13a}$$

The nonhomogeneous term p_i

In much the same way as in obtaining b_i above, the p_i is found to be

$$p_i = -\int_{t_1}^{t_2} \frac{f(t)}{m}(t - t_1)^{i+1}dt$$

$$= -\int_0^T \frac{f(\tau + t_1)}{m}\tau^{i+1}d\tau. \qquad \text{(D.13b)}$$

Once the forcing function $f(t)$ is specified, the p_i, presently in quadrature, may be integrated directly.

The coefficients A_{ij}

Similarly, after introducing $T \equiv t_2 - t_1$ and changing the dummy integration variable to $\tau = t - t_1$, the coefficients A_{ij} become

$$A_{ij} = \int_0^T \left[j\,(j+1)\tau^{i+j} + 2\zeta\omega_n(j+1)\tau^{i+j+1} + \omega_n^2\tau^{i+j+2}\right]d\tau$$

$$= \left[\frac{j\,(j+1)}{T^2(i+j+1)} + \frac{2\zeta\omega_n(j+1)}{T\,(i+j+2)} + \frac{\omega_n^2}{(i+j+3)}\right]T^{i+j+3}. \qquad \text{(D.13c)}$$

Sample Solutions

Solutions for the system ($m = 20$ kg, $k = 80$ N/m, $b = 16$ N-s/m, $x_0 = 1$ m, $v_0 = 1$ m/s) under the three forcing conditions in the relatively short interval $0 \leq t \leq 3$ sec are conducted:

1. Free vibration: $f(t) = 0$ N. (Fig. D.2)
2. Harmonically forced vibration with the forcing frequency larger than the natural frequency: $f(t) = 40\sin 4t$ N. (Fig. D.3)
3. Harmonically forced vibration with the forcing frequency smaller than the natural frequency: $f(t) = 40\sin 0.5t$ N. (Fig. D.4)

For the free vibration, the maximum error of $10^{-4}\%$ indicates that approximation with twelve adjustable parameters (that is, twelve c_i's) in the trial solution (D.12) for a time interval of 3 seconds is satisfactory (at least for most engineering applications). For the two forced vibrations, the approximation can be improved when more adjustable parameters are included in the trial solution, or by dividing the 3-second time interval into 3 or 4 time segments and applying the HLVA method to each segment with the final conditions of one segment being the initial conditions of the next time segment.

The approximation error is evaluated by the relative error defined by $e = (|x_{\text{approximate}} - x_{\text{true}}| / |x_{\text{true}}|)100\%$. The true solution of the system under a generic harmonic forcing $f(t) = F_0 \sin \omega t$ is given by the following closed form expression:

$$x_{\text{true}} = e^{-\zeta\omega_n t}[C_1 \cos \omega_d t + C_2 \sin \omega_d t] + X(\omega)\sin(\omega t - \varphi)$$

where

$$C_1 = x_0 + X(\omega)\sin\varphi$$

$$C_2 = \frac{v_0 + \zeta\omega_n x_0 + X(\omega)\left[\zeta\omega_n \sin\varphi - \omega\cos\varphi\right]}{\omega_d}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

$$X(\omega) = \frac{F_0/k}{\left[\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}}}$$

$$\varphi = \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

This solution is found by standard techniques of solving harmonically forced vibrations.

Note that the peaks in the relative errors in these figures are caused by a small $|x_{true}|$ value. In fact, when $|x_{true}|$ is smaller than 5% of its maximum value over this time interval, the denominator of the error expression is taken to be 5% of $|x_{true}|_{max}$.

Table D.1 summarizes the results of the three cases.

Table D.1 Dynamics solutions for simple harmonic oscillator
(ω_n= 2 rad/s, ζ = 0.2, x_0 = 1 m, v_0 = 1 m/s, $0 \le t \le 3$ sec)

Case	*Forcing* $f(t) = F_0\sin\omega t$ N	r*	*Maximum error % (Order of magnitude)*
1	$f(t) = 0$	12	10^{-4}
2	$f(t) = 40\sin 4t$	12	10^{-2}
3	$f(t) = 40\sin 0.5t$	10	10^{-2}

* r is the number of undetermined parameters c_i's included in the trial solution (D.12).

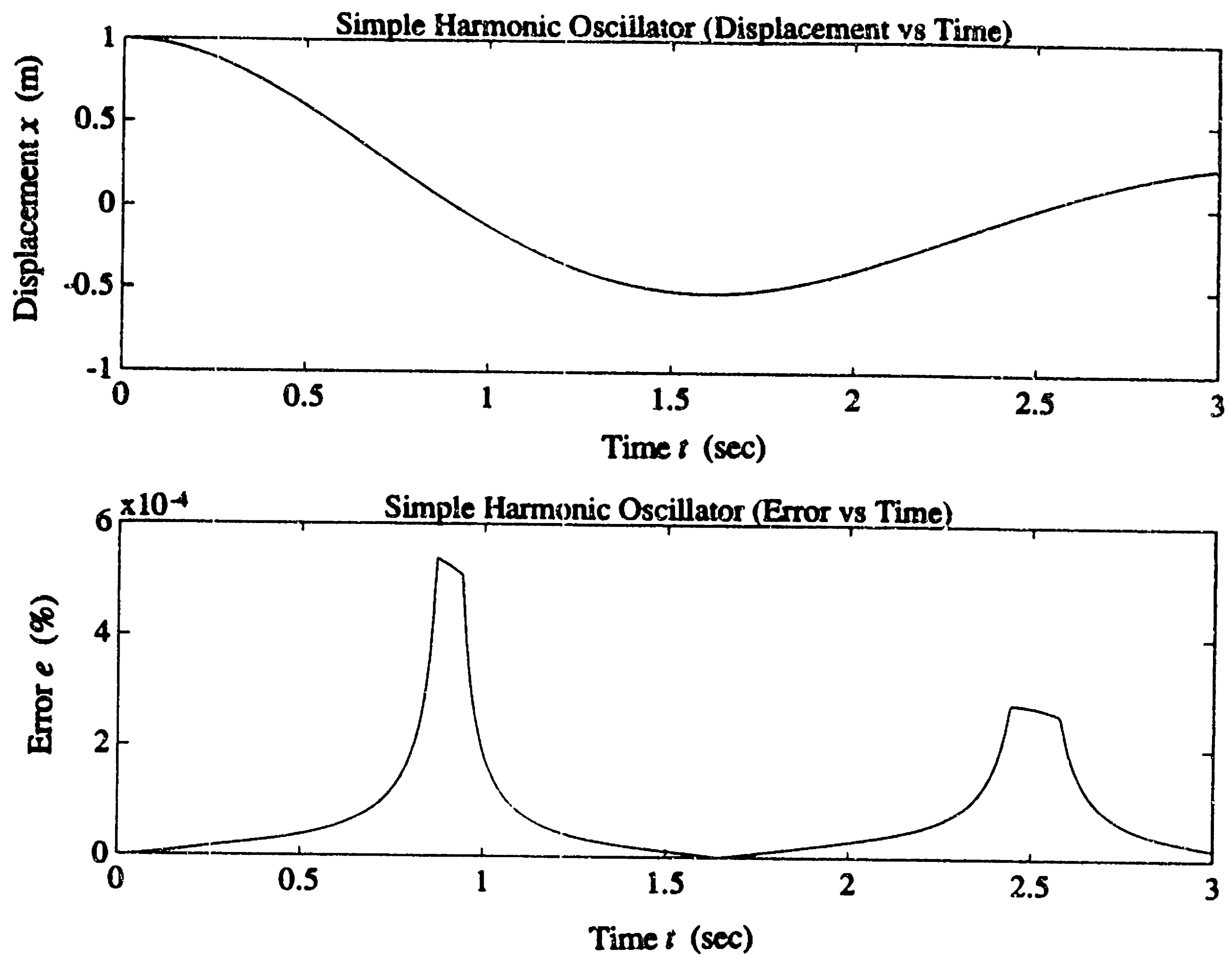

Fig. D.2. Free vibration calculated using $r = 12$ showing (*a*) displacement x versus time t and (*b*) error, $e = (|x_{approximate} - x_{true}| / |x_{true}|)100\%$, versus time t. Note that since there is no appropriate characteristic displacement available for free vibration, the displacement x is not nondimensionalized.

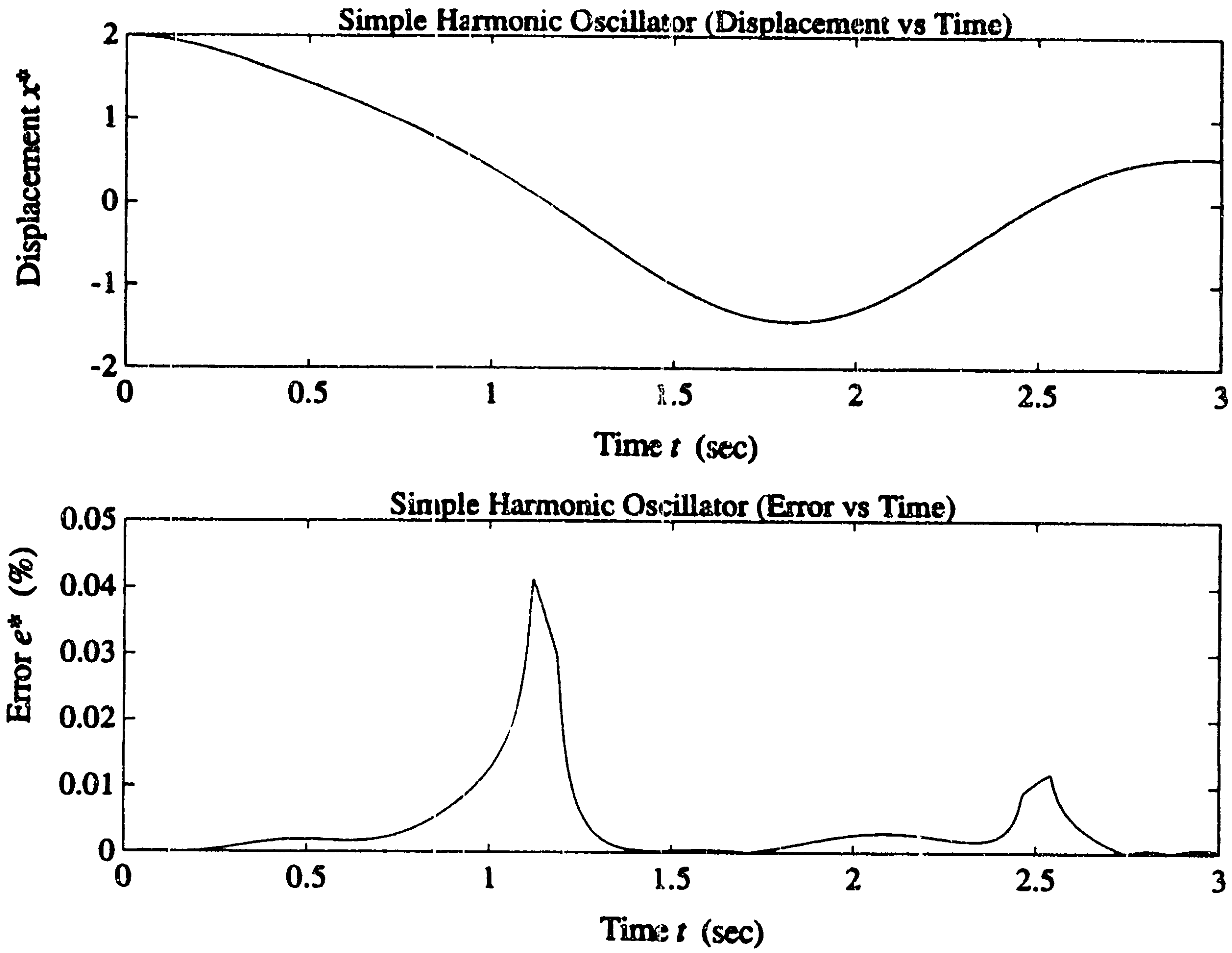

Fig. D.3. Forced vibration ($f(t) = 40\sin 4t$ N) calculated using $r = 12$ showing

(a) nondimensional displacement, $x^* = \frac{x}{F_0/k}$, versus time t and

(b) nondimensional error, $e^* = \left(\left|x^*_{approximate} - x^*_{true}\right| / \left|x^*_{true}\right|\right)100\%$, versus time t.

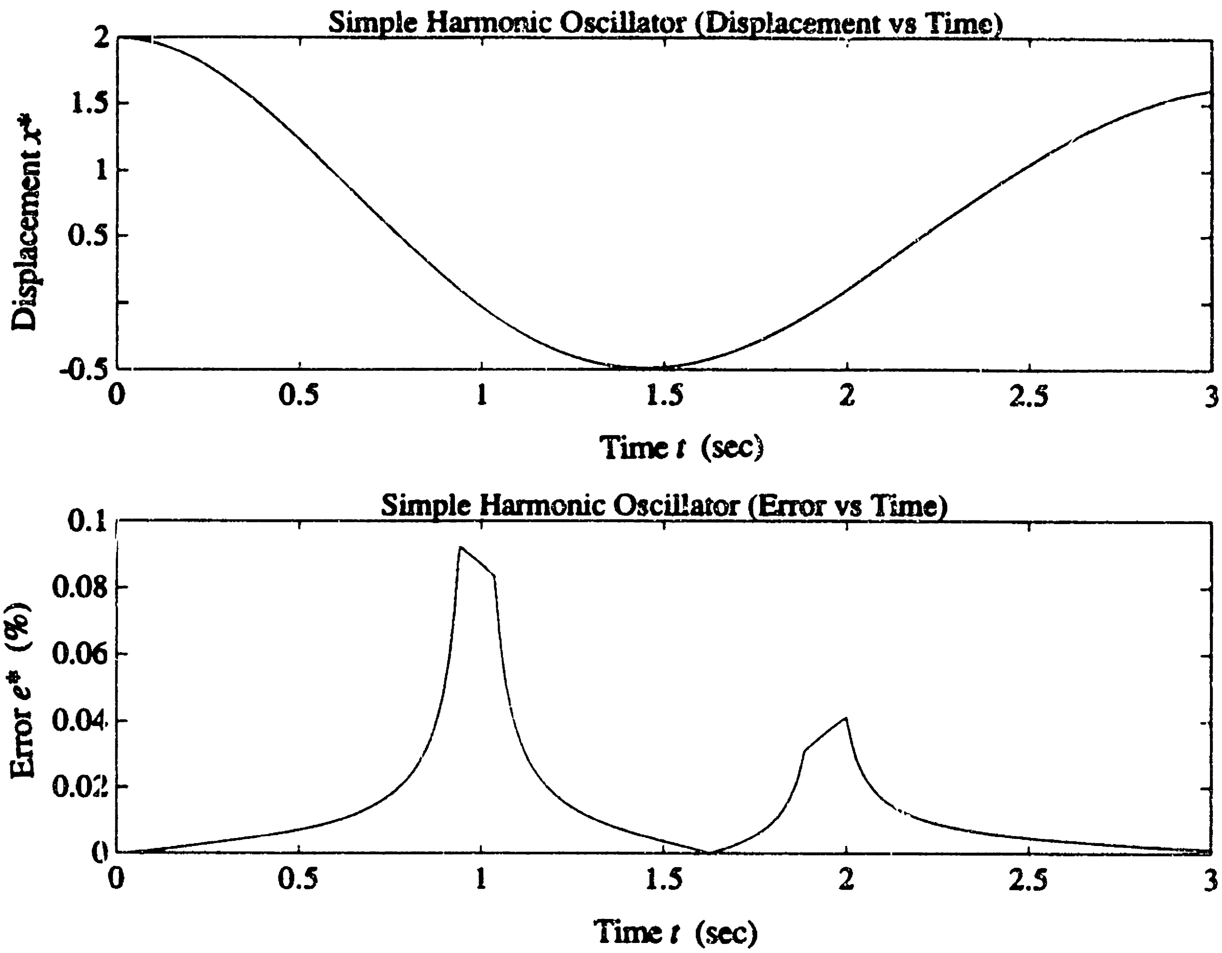

Fig. D.4. Forced vibration ($f(t) = 40\sin 0.5t$ N) calculated using $r = 10$ showing

(*a*) nondimensional displacement, $x^* = \dfrac{x}{F_0/k}$, versus time t and

(*b*) nondimensional error, $e^* = \left(\left|x^*_{\text{approximate}} - x^*_{\text{true}}\right| / \left|x^*_{\text{true}}\right|\right)100\%$, versus time t.

APPENDIX E

Solving A Set of Algebraic Equations via Iteration

This appendix outlines the procedure of *iteration by total steps* [4, p41] for solving a set of algebraic equations of the following form

$$\sum_{j=1}^{r} A_{ij}c_j + b_i + p_i(c_1,\cdots,c_r) = 0, \qquad i = 1, \ldots, r, \tag{E.1}$$

where the c_j's are the unknown parameters to be found, the A_{ij}'s and b_i's are known constants, and the p_i's are known functions in terms of the unknown parameters.

To better describe this successive approximation procedure, the above set of equations is written in the following the matrix form:

$$[A]\{c\} + \{b\} + \{p(\{c\})\} = 0, \tag{E.2}$$

where $[A]$ is an $r \times r$ matrix whose elements are the A_{ij}'s, $\{c\}$ is a column vector whose elements are the undetermined parameter the c_i's, and $\{b\}$ and $\{p\}$ are column vectors whose elements are the b_i's and p_i's in (E.1), respectively. Then the solution of the vector of undetermined parameters $\{c\}$ can be obtained through the following iterative formula:

$$\{c\}^{(k)} = -[A]^{-1}\left[\{b\} + \{p(\{c\}^{(k-1)})\}\right], \qquad k = 1, \ldots. \tag{E.3}$$

where k is the iteration cycle number and $\{c\}^{(k)}$ represents the approximate solution of $\{c\}$ after k cycles. The iteration process can start with an arbitrary initial guess of the undetermined parameter vector $\{c\}^{(0)}$. For simplicity $\{c\}^{(0)} = \{0\}$ is usually taken as the initial guess. The iteration process stops when two adjacent approximations of $\{c\}$ are "close" enough. The "closeness" of $\{c\}^{(k)}$ and $\{c\}^{(k-1)}$ can be measured by the *norm* of their difference, which is denoted by $\|e\|$ and defined as follows:

$$\|e\| \equiv \sqrt{\sum_{i=1}^{r} \left[c_i^{(k)} - c_i^{(k-1)}\right]^2} \,. \tag{E.4}$$

If the expression in (E.4) is less than a given tolerance, then the iteration process ends with the approximate solution of $\{c\}$ being $\{c\}^{(k)}$.

It should be noted that matrix $[A]$ and vector $\{b\}$ need only to be evaluated once since they do not depend on the undetermined parameters. It is the vector $\{p\}$ that needs to be evaluated during each cycle of the iteration.

For a thorough treatment of solving a set of algebraic equations, refer to [4, p39-47].

APPENDIX F

Outline of The HLVA Method for General Dynamic Systems

The HLVA method is a procedure for obtaining an approximate solution to dynamics problems via trial solutions containing undetermined parameters. This appendix outlines this method for both holonomic and nonholonomic systems.

To begin with, let us state Hamilton's Law of Varying Action (HLVA) for a general dynamic system. Let $\xi_1, \ldots, \xi_n$ be a complete and independent set of generalized coordinates for a dynamic system, and let $\delta\xi_1, \ldots, \delta\xi_n$ be their associated variational variables. In addition, define T^* as the kinetic coenergy of the system, V as the potential energy of the conservative forces, and Ξ_j as the generalized force associated with the generalized coordinate ξ_j due to nonconservative forces acting on the system.

The general form of Hamilton's Law of Varying Action states that *A varied motion of the system during an arbitrary finite time interval $t_1 \le t \le t_2$ is a natural motion if, and only if, the general variational indicator*

$$\text{V.I.} = \int_{t_1}^{t_2} \left(\delta L + \sum_{j=1}^{n} \Xi_j \, \delta\xi_j \right) dt - \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j} \delta\xi_j \right]_{t_1}^{t_2} \tag{F.1}$$

vanishes for arbitrary admissible variations of the motion within this time interval, where $L = T^* - V$ is the lagrangian of the system.

Assume the solution for a system is desired for the interval $t_1 \leq t \leq t_2$, and the state of the system (that is, the generalized coordinates and generalized velocities) at $t = t_1$ is known.

The HLVA Method for Holonomic Systems

The HLVA method for a holonomic system consists of the steps listed and discussed in this section.

1. Construct variational indicator

The lagrangian and the generalized forces of the system should be identified first. Then inserting them into the variational indicator (F.1) and carrying out the variations give the variational indicator in the following form:

$$\text{V.I.} = \int_{t_1}^{t_2} \left\{ \sum_{j=1}^{n} \left[\frac{\partial L}{\partial \dot{\xi}_j} \delta \dot{\xi}_j + \left(\frac{\partial L}{\partial \xi_j} + \Xi_j \right) \delta \xi_j \right] \right\} dt - \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial \dot{\xi}_j} \delta \xi_j \right]_{t_1}^{t_2} . \tag{F.2}$$

Inserting the commutativity relation $\delta \dot{\xi}_j = \frac{d}{dt}(\delta \xi_j)$ into the V.I. (F.2) and integrating the term $\dot{\xi}_j \delta \dot{\xi}_j = \dot{\xi}_j \frac{d}{dt}(\delta \xi_j)$ by parts transform the V.I. (F.2) into the following form:

$$\text{V.I.} = - \int_{t_1}^{t_2} \left\{ \sum_{j=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j} \right) - \frac{\partial L}{\partial \xi_j} - \Xi_j \right] \delta \xi_j \right\} dt . \tag{F.3}†$$

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† The transformation of the V.I. (F.2) into (F.3) is optional.

2. Choose trial solutions

Consider the following trial family for each generalized coordinate of the system

$$\xi_j(t) = \varphi_{j0}(t) + \sum_{i=1}^{r} c_{ji}\, \varphi_{ji}(t), \qquad j = 1, \ldots, n, \qquad t_1 \le t \le t_2, \qquad \text{(F.4)}$$

where, for each trial family, the basis functions $\varphi_{j1}, \varphi_{j2}, \ldots, \varphi_{jr}$ are linearly independent known continuous functions and have continuous first derivatives in the time interval $t_1 \le t \le t_2$, the $c_{j1}, c_{j2}, \ldots, c_{jr}$ are adjustable parameters associated with the generalized coordinate ξ_j, and r is the number of adjustable parameters associated with the generalized coordinate ξ_j.

In addition, the basis functions $\varphi_{j1}, \varphi_{j2}, \ldots, \varphi_{jr}$ are chosen such that φ_{j0} ensures the satisfaction of the initial conditions associated with the generalized coordinate ξ_j, i.e.,

$$\varphi_{j0}(t_1) = \xi_j(t_1) \qquad \varphi_{ji}(t_1) = 0 \qquad i = 1, \ldots, n, \quad j = 1, \ldots, r,$$

$$\dot{\varphi}_{j0}(t_1) = \dot{\xi}_j(t_1) \qquad \dot{\varphi}_{ji}(t_1) = 0 \qquad i = 1, \ldots, n, \quad j = 1, \ldots, r,$$

3. Insert the trial solutions (F.4) directly into the V.I. (F.3)

This step is to replace every ξ_j by its associated trial solution in (F.4), every $\delta\xi_j$ by the variation of the associated trial solution, that is,

$$\delta\xi_j(t) = \sum_{i=1}^{r} \varphi_{ji}(t)\delta c_{ji}, \qquad j = 1, \ldots, n. \tag{F.5}$$

in the variational indicator (F.3). After the insertion, the variational indicator can be written as follows:

$$\text{V.I.} = -\int_{t_1}^{t_2} \left\{ \sum_{j=1}^{n} R_j(c_{11}, \cdots, c_{nr}, t) \sum_{i=1}^{r} \varphi_{ji}\, \delta c_{ji} \right\} dt \tag{F.6}$$

where

$$R_j(c_{11}, \cdots, c_{nr}, t) \equiv \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} - \Xi_j \right]_{\xi_j(t) = \varphi_{j0}(t) + \sum_{i=1}^{r} c_{ji}\, \varphi_{ji}(t)}$$

are known functions of the undetermined parameters and time t.

4. Organize the V.I.

The organization step includes interchanges of the orders of the integration operator $\int_{t_1}^{t_2}$ and the summation operators $\sum_{i=1}^{r}$ and $\sum_{j=1}^{n}$. These interchanges are possible because the integration interval is finite and the numbers of terms of the two summations are also finite. After appropriate manipulations, the V.I. of (F.6) can always be written as a linear function of the variations of the undetermined parameters δc_{ji}, that is,

$$\text{V.I.} = -\sum_{i=1}^{r}\sum_{j=1}^{n}\left[\int_{t_1}^{t_2} R_j(c_{11},\cdots,c_{nr},t)\varphi_{ji}dt\right]\delta c_{ji}\ . \tag{F.7}$$

5. Identify set of algebraic equations in undetermined parameters

Since there are no restrictions on the undetermined parameters the c_{ji}'s, the variational indicator (F.7) vanishes if, and only if, the coefficient of each δc_{ji} is identical to zero, i.e.

$$\int_{t_1}^{t_2} R_j(c_{11},\cdots,c_{nr},t)\varphi_{ji}dt = 0, \quad j = 1, \ldots, r, \quad i = 1, \ldots, n. \tag{F.8}$$

6. Solve for undetermined parameters

Since the number of unknowns is the same as the number of equations in (F.8), the adjustable parameters the c_{ji}'s can be determined through the solution of this set of algebraic equations. Once the c_{ji}'s are determined, the approximate solution of the dynamics problem for a holonomic system from $t = t_1$ to $t = t_2$ is given in the form of analytical expressions (F.4).

The HLVA Method for Nonholonomic Systems

For convenience (and without loss of generality), the HLVA method for nonholonomic systems is described for a system of three generalized coordinates ξ_1, ξ_2, ξ_3, and two degrees of freedom $\delta\xi_1$, and $\delta\xi_2$. Assume the equation of nonholonomic constraint of the system has the following general form [5, P15]:

$$a_1\dot{\xi}_1 + a_2\dot{\xi}_2 + a_3\dot{\xi}_3 + a_0 = 0 \tag{F.9}$$

where a_1, a_2, a_3, a_0 are known continuous functions and have continuous first derivatives. From the definition of admissible variations [5, p14-18 and 3, p5], an admissible variation ($\delta\xi_1$, $\delta\xi_2$, $\delta\xi_3$) for such a general form satisfies

$$a_1\delta\xi_1 + a_2\delta\xi_2 + a_3\delta\xi_3 = 0. \tag{F.10}$$

The HLVA method for a nonholonomic system consists of the steps which follows.

Step 1 Construct variational indicator associated with HLVA

The variational indicator for this system is

$$\text{V.I.} = \int_{t_1}^{t_2} \sum_{j=1}^{3} \left(\frac{\partial L}{\partial \dot{\xi}_j} \delta\dot{\xi}_j + \frac{\partial L}{\partial \xi_j} \delta\xi_j + \Xi_j \, \delta\xi_j \right) dt - \left[\sum_{j=1}^{3} \frac{\partial T^*}{\partial \dot{\xi}_j} \, \delta\xi_j \right]_{t_1}^{t_2}. \tag{F.11}$$

After inserting the relation $\delta\left(\frac{d\xi_j}{dt}\right) = \frac{d}{dt}(\delta\xi_j)$ into the V.I. (F.11) and integrating the first term in the integrand of (F.11) by parts, the V.I. can be transformed into the following form:

$$\text{V.I.} = -\int_{t_1}^{t_2} \sum_{j=1}^{3} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} - \Xi_j\right]\delta\xi_j dt\,. \qquad \text{(F.12)}$$

Step 2 Eliminate dependent variable $\delta\xi_3$ from V.I.

The elimination of the dependent variable $\delta\xi_3$ from the V.I. (F.12) is accomplished by embedding the constraint equation into the V.I. The embedding consists of two steps. First, express the dependent variational variable $\delta\xi_3$ in terms of the two independent variational variables, $\delta\xi_1$ and $\delta\xi_2$, by rearranging the equation of constraint (F.10) into the following form:

$$\delta\xi_3 = -\frac{a_1}{a_3}\delta\xi_1 - \frac{a_2}{a_3}\delta\xi_2. \qquad \text{(F.13)}$$

Second, insert (F.13) for $\delta\xi_3$ into the V.I. (F.12). Collecting terms according to $\delta\xi_1(t)$ and $\delta\xi_2(t)$ gives the V.I. in the following form:

$$\text{V.I.} = -\int_{t_1}^{t_2} R_1(\xi_1, \xi_2, \xi_3, t)\delta\xi_1 + R_2(\xi_1, \xi_2, \xi_3, t)\delta\xi_2 dt\,, \qquad \text{(F.14)}$$

where

$$R_1(\xi_1, \xi_2, \xi_3, t) \equiv \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_1}\right) - \frac{\partial L}{\partial \xi_1} - \Xi_1\right] + \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_3}\right) - \frac{\partial L}{\partial \xi_3} - \Xi_3\right]\left(-\frac{a_1}{a_3}\right)$$

$$R_2(\xi_1, \xi_2, \xi_3, t) \equiv \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_2}\right) - \frac{\partial L}{\partial \xi_2} - \Xi_2\right] + \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_3}\right) - \frac{\partial L}{\partial \xi_3} - \Xi_3\right]\left(-\frac{a_2}{a_3}\right)$$

Step 3 Select trial solutions for generalized coordinates associated with independent variational variables

Because of the constraint, the trial solutions for $\xi_1(t)$ and $\xi_2(t)$ only must be chosen. The trial solutions selected here are of the form

$$\xi_1(t) = \varphi_0(t) + \sum_{i=1}^{r} c_i\, \varphi_i(t), \tag{F.15a}$$

$$\xi_2(t) = \psi_0(t) + \sum_{i=1}^{r} k_i\, \psi_i(t), \tag{F.15b}$$

where the basis functions $\varphi_i(t)$ and $\psi_i(t)$ are two sets of linearly independent continuous functions and have continuous first derivatives in the time interval from t_1 to t_2, the c_i's and k_i's are undetermined parameters, and r is the number of undetermined parameters. In addition, the basis functions $\varphi_i(t)$ and $\psi_i(t)$ should be chosen such that $\varphi_0(t)$ and $\psi_0(t)$ ensure the satisfaction of the initial conditions of the system, that is,

$$\left.\begin{array}{ll} \varphi_0(t_1) = \xi_1(t_1), & \left.\dfrac{d\varphi_0(t)}{dt}\right|_{t_1} = \left.\dfrac{d\xi_1(t)}{dt}\right|_{t_1} \\ \varphi_j(t_1) = 0, & \left.\dfrac{d\varphi_j(t)}{dt}\right|_{t_1} = 0 \end{array}\right\}$$

$$\left.\begin{array}{ll}\psi_0(t_1) = \xi_2(t_1), & \left.\dfrac{d\psi_0(t)}{dt}\right|_{t_1} = \left.\dfrac{d\xi_2(t)}{dt}\right|_{t_1} \\[2ex] \psi_j(t_1) = 0, & \left.\dfrac{d\psi_j(t)}{dt}\right|_{t_1} = 0\end{array}\right\}$$

Then, through the constraint equation (F.9), ξ_3 can be expressed (at least functionally) as a function of the adjustable parameters as

$$\xi_3 = \xi_3(c_1, ..., c_r; k_1, ..., k_r; t). \qquad \text{(F.15c)}$$

<u>Step 4 Insert trial solutions into V.I. and arrange V.I. into special form</u>

Inserting the expressions (F.15) into the V.I. (F.14) gives

$$\text{V.I.} = -\int_{t_1}^{t_2} \left[R_1(c_1, ..., c_r; k_1, ..., k_r; t) \sum_{i=1}^{r} \varphi_i \delta c_i + R_2(c_1, ..., c_r; k_1, ..., k_r; t) \sum_{i=1}^{r} \psi_i \delta k_i \right] dt \qquad \text{(F.16)}$$

where $R_1(c_1, \cdots, c_r; k_1, \cdots, k_r; t)$ and $R_2(c_1, \cdots, c_r; k_1, \cdots, k_r; t)$ are obtained by inserting the trial solutions (F.15) into $R_1(\xi_1, \xi_2, \xi_3, t)$ and $R_2(\xi_1, \xi_2, \xi_3, t)$. Since the number of adjustable parameters is finite, the integration operator $\int_{t_1}^{t_2}$ and the summation operator $\sum_{i=1}^{r}$ are commutative. In addition, since the variation of adjustable parameters δc_j's and δk_j's are independent of time t, the δc_j's and the δk_j's can be taken out of the integration. Then after exchanging the order of the two operators and bringing the δc_j's and the δk_j's out of the integration in (F.16), the variational indicator becomes

$$\text{V.I.} = -\sum_{i=1}^{r}\left\{\int_{t_1}^{t_2} R_1(c_1,\cdots,c_r;k_1,\cdots,k_r;t)\varphi_i dt\right\}\delta c_i - \sum_{i=1}^{r}\left\{\int_{t_1}^{t_2} R_2(c_1,\cdots,c_r;k_1,\cdots,k_r;t)\psi_i dt\right\}\delta k_i \qquad \text{(F.17)}$$

Step 5 Identify set of algebraic equations in adjustable parameters

Since the adjustable parameters are arbitrary, the variational indicator (F.17) vanishes if, and only if, each coefficient associated with δc_i and δk_i vanishes identically, that is,

$$\int_{t_1}^{t_2} R_1(c_1,\cdots,c_r;k_1,\cdots,k_r;t)\varphi_i dt = 0 \qquad i = 1, ..., r \qquad \text{(F.18a)}$$

$$\int_{t_1}^{t_2} R_2(c_1,\cdots,c_r;k_1,\cdots,k_r;t)\psi_i dt = 0 \qquad i = 1, ..., r \qquad \text{(F.18b)}$$

This is a set of $2r$ algebraic equations in $2r$ undetermined parameters.

Step 6 Solve the set of algebraic equations

The above set of algebraic equations can be solved via standard techniques. Once the adjustable parameters c_i's and k_i's are known, the expressions in (F.15) constitute the solution to the dynamics problem of the system.

Note: In case the equation of constraint is such that an expression like (F.15c) can not be obtained, then ξ_3(t) would be an additional unknown in the V.I. (F.16). In this case, the solution of the adjustable parameters would be obtained through the following set of equations:

$$\int_{t_1}^{t_2} R_1(c_1, \cdots, c_r; k_1, \cdots, k_r; \xi_3, \dot{\xi}_3, \ddot{\xi}_3, t)\varphi_i dt = 0 \qquad i = 1, ..., r \qquad \text{(F.19a)}$$

$$\int_{t_1}^{t_2} R_2(c_1, \cdots, c_r; k_1, \cdots, k_r; \xi_3, \dot{\xi}_3, \ddot{\xi}_3\, t)\psi_i dt = 0 \qquad i = 1, ..., r \qquad \text{(F.19b)}$$

$$f(c_1, \cdots, c_r; k_1, \cdots, k_r; \xi_3, \dot{\xi}_3, t) = 0 \qquad \text{(F.19c)}$$

where (F.19c) would be obtained by inserting the trial solution of $\xi_1(t)$ and $\xi_2(t)$ directly into the equation of constraint (F.9). Equation (F.19c) is a first order differential equation with undetermined parameters. Equations (F.19a) and (F.19b) are $2r$ algebraic equations in undetermined parameters. The solution of this set of equations generally requires techniques for numerical solutions of differential equations, which can be found in textbooks concerning numerical methods for solving differential equations.

APPENDIX G

Numerical Issues of The HLVA Method

The HLVA method is a procedure for obtaining an approximate solution to dynamics problems via trial solutions with undetermined parameters. In this appendix, we discuss three numerical issues pertinent to this method. They are: (1) *nondimensional time in polynomial trial solution* and (2) *treatment of arbitrarily large time intervals.*

Nondimensional Time in Polynomial Trial Solution

When polynomial trial solutions are used in the HLVA method, the set of algebraic equations in adjustable parameters is often ill-conditioned when large numbers of undetermined parameters are involved. To avoid this situation, a nondimensional time is introduced into the polynomial trial solution. The handling of the nondimensional time is demonstrated with the HLVA method for the simple harmonic oscillator shown in Fig. G.1.

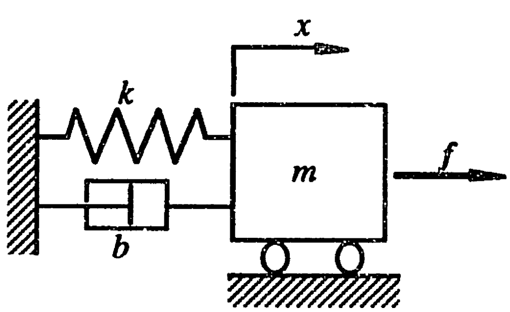

Fig. G.1 Simple harmonic oscillator subjected to prescribed external force *f*.

Assume an approximate solution is desired in the interval $t_1 \le t \le t_2$, and at $t = t_1$, the state of the system is known, that is, $x(t_1) = x_0$ and $\left.\frac{dx}{dt}\right|_{t_1} = v_0$. Then a polynomial trial solution for $x(t)$ may be selected

$$x(t) = x_0 + v_0(t - t_1) + \sum_{i=1}^{r} c_i(t - t_1)^{i+1} \qquad t_1 \le t \le t_2. \tag{G.1}$$

Introduce the nondimensional time α

$$\sigma = \frac{t - t_1}{T}, \qquad 0 \le \sigma \le 1, \tag{G.2}$$

(where $T \equiv t_2 - t_1$) in place of the physical time t, $t_1 \le t \le t_2$, into the trial solution (G.1). Then the trial solution (G.1) becomes

$$x(\sigma) = x_0 + (v_0 T)\sigma + \sum_{i=1}^{r} \alpha_i \sigma^{i+1}, \qquad 0 \le \sigma \le 1. \tag{G.3}$$

If $\alpha_i \equiv c_i T^{i+1}$, in stead of c_i $(i = 1, \ldots, r)$ are taken to be undetermined parameters in the nondimensional-time trial solution (G.3), the corresponding basis functions in (G.3) can be identified to be

$$\varphi_0(\sigma) = x_0 + (v_0 T)\sigma, \quad \varphi_i(\sigma) = \sigma^{i+1}, \qquad 0 \le \sigma \le 1. \tag{G.4}$$

Once the trial solution in nondimensional time is set up, the remaining steps of the HLVA method are the same as those when nondimensional time is not introduced. Therefore, following the steps of the HLVA method, the set of algebraic equations in the α_i's can be obtained

$$\sum_{j=1}^{r} A_{ij}\alpha_j + b_i + p_i = 0 \qquad i = 1, ..., r, \qquad \text{(G.5)}$$

where

$$b_i = \frac{2\zeta\omega_n v_0}{(i+2)} + \frac{\omega_n^2 x_0}{(i+2)} + \frac{\omega_n^2 v_0 T}{(i+3)}$$

$$p_i = -\int_0^1 \frac{f(\tau T + t_1)}{m}\tau^{i+1}d\tau$$

$$A_{ij} = \frac{j(j+1)}{T^2(i+j+1)} + \frac{2\zeta\omega_n(j+1)}{T(i+j+2)} + \frac{\omega_n^2}{(i+j+3)}$$

$$\alpha_i \equiv c_i T^{i+1}$$

$$T \equiv t_2 - t_1$$

$$i = 1, ..., r, \quad j = 1, ..., r.$$

and $\omega_n = \sqrt{k/m}$, $\zeta = b/2m\omega_n$.

Thus the parameters α_i's can be solved from the set of algebraic equations (G.5). Once the α_i's are obtained, the adjustable parameters c_i's are given by $c_i = \alpha_i / T^{i+1}$. With the now known adjustable parameters c_i's, the approximate solution of the displacement of the mass m is given by (G.1).

Treatment of Arbitrarily Large Time Intervals

Usually, if the dynamics solution is desired for an arbitrarily large time interval, a large number of adjustable parameters must be included in the trial solution in order for the approximate solution to maintain a specified accuracy. However, in practical applications the number of adjustable parameters is usually quite limited due to limited computational facilities. A practical treatment to an arbitrarily large time interval is to divide the large time interval into smaller time segments and apply the HLVA method† to each time segment. Without loss of generality, we use the simple harmonic oscillator shown in Fig. G.1 to demonstrate this treatment for a large time interval.

Assume that the approximate solution of the dynamics problem for this system is desired in the interval $t_i \leq t \leq t_f$ $(0 \leq t_i < t_f)$. In addition, assume that the time history of the generalized coordinate of the system, i.e., $x(t)$, is continuous and has continuous first derivatives throughout the interval. Then this time interval can be divided into S time segments: $t_k \leq t \leq t_{k+1}$, $k = 1, ..., S$, where $t_1 = t_i$ and $t_{S+1} = t_f$. Note that these segments should be sufficiently small such that good accuracy of approximation can be obtained for each segment with only a convenient number of adjustable parameters.

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† Note that this treatment for an arbitrarily large time interval is not confined to the HLVA method.

For given initial conditions of the system, the HLVA method can be readily applied to the first time segment $t_1 \le t \le t_2$ and the approximate solution of $x(t)$ for this time segment can thus be obtained. Taking the then known $x(t_2)$ and $\dot{x}(t_2)$ as the initial conditions for the next time segment $t_2 \le t \le t_3$, the HLVA method is able to provide a solution to $x(t)$ for this time segment. By following this iterative procedure, the dynamics solution for the arbitrarily large time interval $t_i \le t \le t_f$ can be obtained with a specified accuracy.

Solutions for the system $(m = 20$ kg, $k = 80$ N/m, $b = 16$ N-s/m, $x_0 = 1$ m, $v_0 = 1$ m/s$)$ under the following three forcing conditions in the time interval $0 \le t \le 30$ sec (about the duration of 10 cycles of free vibration), are obtained:

(1) Free vibration. $f(t) = 0$ N. (Fig. G.2)

(2) Harmonically forced vibration with the forcing frequency larger than the natural frequency: $f(t) = 40\sin 4t$ N. (Fig. G.3)

(3) Harmonically forced vibration with the forcing frequency smaller than the natural frequency: $f(t) = 40\sin 0.5t$ N. (Fig. G.4)

Fig. G.2 shows that maximum approximation error for the free vibration is about $10^{-6}\%$ of the true solution and it occurs at the transient stage of the response. The error decreases rapidly as the system approaches the steady state.

Fig. G.3 shows that when the harmonic force has higher frequency than the system's natural frequency, the approximation error is of the order of $10^{-2}\%$ at the transient stage of the vibration, and is of the order of $10^{-3}\%$ when the system is in steady-state forced vibration. The approximation accuracy can be improved by increasing the number of adjustable parameters in the trial function or decreasing the duration of the time segments.

Fig. G.4 shows that when the harmonic force has lower frequency than the system's natural frequency, the approximation error is of the order of $10^{-4}\%$ throughout the vibration in this 30-second time interval.

The approximation error is evaluated by the relative error defined by $e = (|x_{\text{approximate}} - x_{\text{true}}| / |x_{\text{true}}|)100\%$. The true solution of the system under a generic harmonic forcing $f(t) = F_0 \sin \omega t$ is given by the following closed form expression:

$$x_{\text{true}} = e^{-\zeta \omega_n t}[C_1 \cos \omega_d t + C_2 \sin \omega_d t] + X(\omega)\sin(\omega t - \varphi)$$

where

$$C_1 = x_0 + X(\omega)\sin\varphi,$$

$$C_2 = \frac{v_0 + \zeta\omega_n x_0 + X(\omega)\left[\zeta\omega_n \sin\varphi - \omega\cos\varphi\right]}{\omega_d},$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

$$X(\omega) = \frac{F_0/k}{\left[\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}}},$$

$$\varphi = \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}.$$

This solution is found by standard techniques of solving harmonically forced vibrations.

Note that in all the above three cases, the errors during the transient time are larger than those during the steady-state time by about one order of magnitude. Note also that the peaks in the errors in these figures are caused by a small $|x_{true}|$ value. In fact, when $|x_{true}|$ is smaller than 5% of its maximum value over this time interval, the denominator of the error expression is taken to be 5% of $|x_{true}|_{max}$.

The following table shows the summary of the results.

Table G.1 Dynamics solutions for simple harmonic oscillator

(ω_n= 2 rad/s, ζ = 0.2, x_0 = 1 m, v_0 = 1 m/s, $0 \le t \le 30$ sec)

Case	*Forcing* $f(t) = F_0 \sin \omega t$ N	T (sec) /r *	*Maximum error %* *(Order of magnitude)*
1	$f(t) = 0$	1 / 10	10^{-6}
2	$f(t) = 40\sin 4t$	0.2 / 8	10^{-2}
3	$f(t) = 40\sin 0.5t$	0.2 / 8	10^{-4}

* T is the duration of each time segment, r is the number of undetermined parameters included in the trial solution for each time segment.

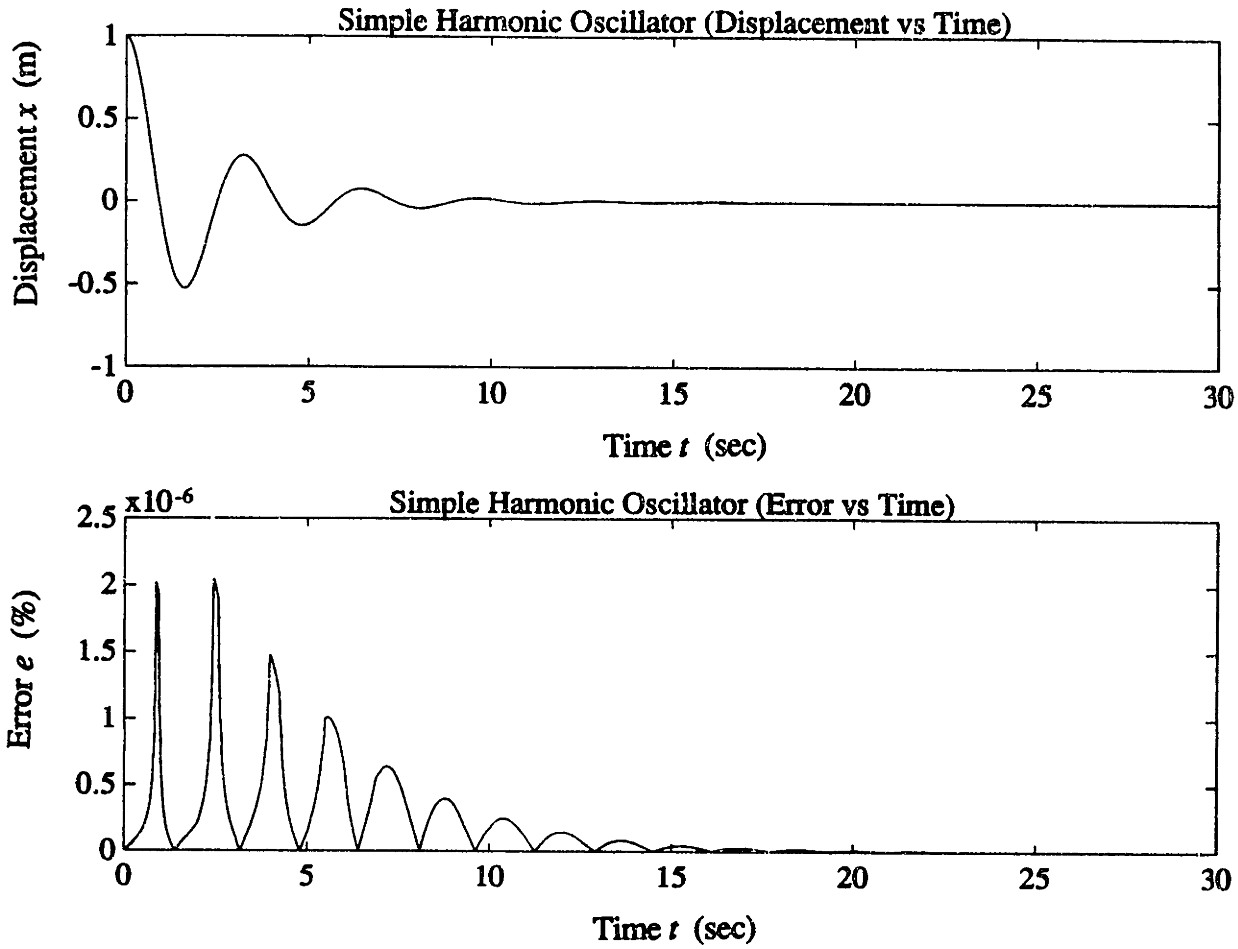


Fig. G.2 Free vibration calculated using $r = 10$ showing (*a*) displacement x versus time t (*b*) error, $e = (|x_{\text{approximate}} - x_{\text{true}}| / |x_{\text{true}}|)100\%$, versus time t. Note that since there is no appropriate characteristic displacement available for free vibration, the displacement x is not nondimensionalized.

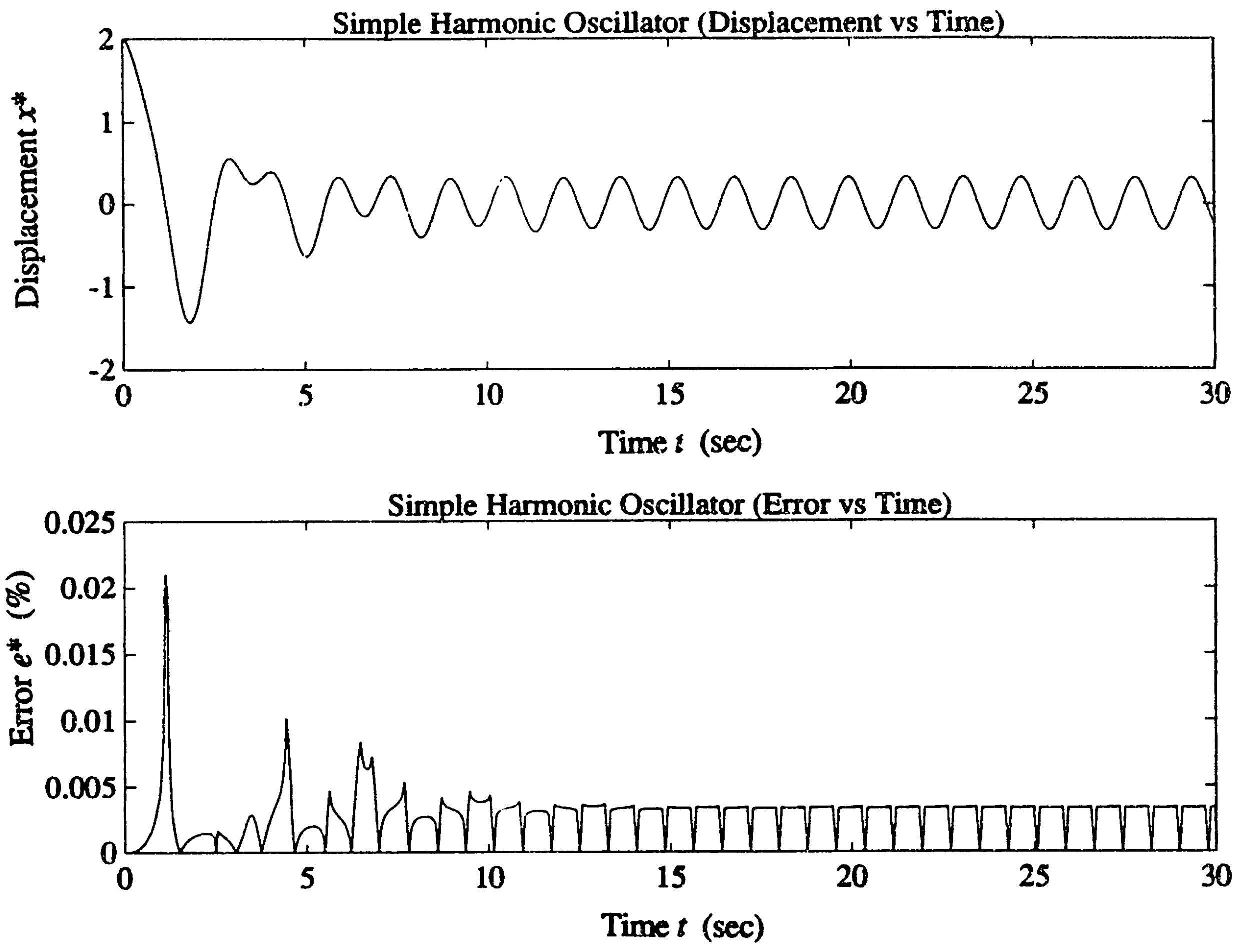
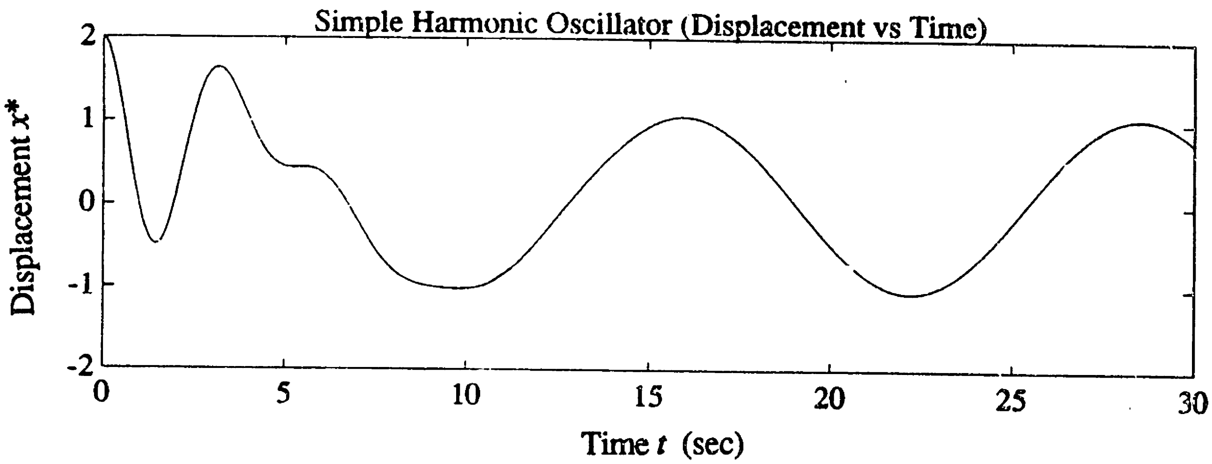


Fig. G.3 Forced vibration ($f(t) = 40\sin 4t$ N) calculated using $r = 8$ showing

(a) nondimensional displacement, $x^* = \frac{x}{F_0/k}$, versus time t and

(b) nondimensional error, $e^* = \left(\left|x^*_{\text{approximate}} - x^*_{\text{true}}\right| / \left|x^*_{\text{true}}\right|\right)100\%$, versus time t.



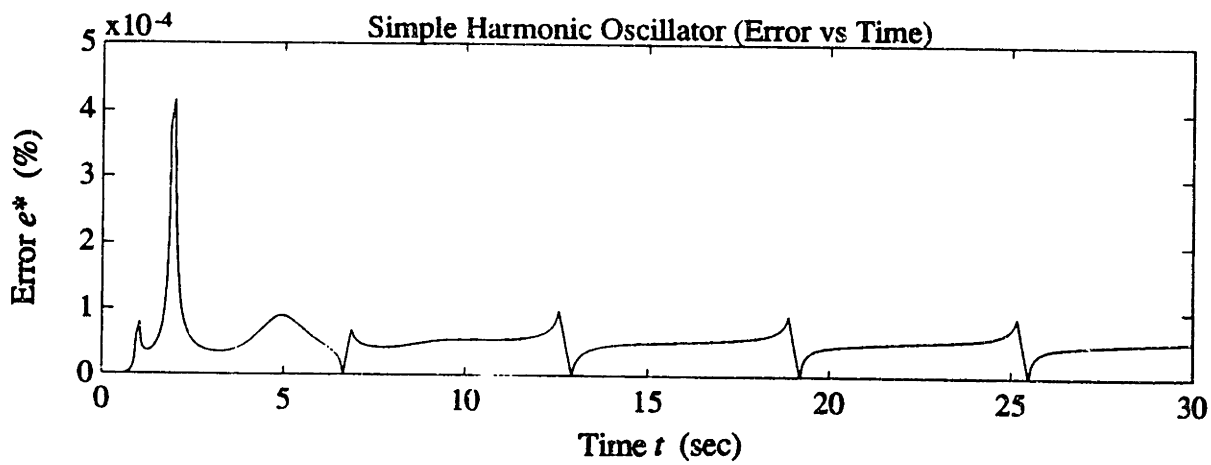


Fig. G.4 Forced vibration ($f(t) = 40\sin 0.5t$ N) calculated using $r = 10$ showing

(a) nondimensional displacement, $x^* = \frac{x}{F_0/k}$, versus time t

(b) nondimensional error, $e^* = \left(\left|x^*_{approximate} - x^*_{true}\right| / \left|x^*_{true}\right|\right)100\%$, versus time t.

APPENDIX H

The HLVA Method For A Nonlinear System

The HLVA method is a procedure for obtaining an approximate solution to dynamics problems via trial solutions with undetermined parameters. This appendix demonstrates the HLVA method for nonlinear systems via the nonlinear oscillator shown in Fig. H.1.

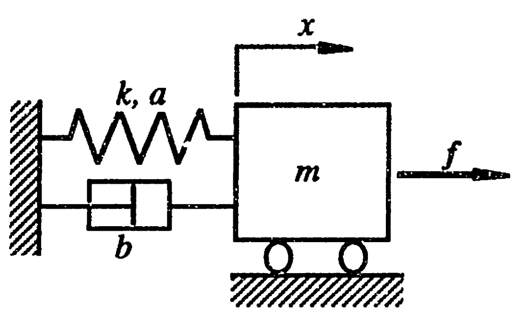


Fig. H.1 Nonlinear oscillator subjected to prescribed external force f. The spring has nonlinear constitutive relation $f_s = kx(1 + ax^2)$, where the parameters a and k are positive.

Assume the dynamics solution is desired for the interval $t_1 \leq t \leq t_2$, and the displacement and velocity of the mass are known to be x_0 and v_0 at $t = t_1$. The HLVA method consists of the steps which follows.

Step 1 Set up variational indicator associated with HLVA

For the system shown in Fig. H.1 the Lagrangian is $L = T^* - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2(1 + \frac{1}{2}ax^2)$ and the work expression is $\Xi\delta x = [-b\dot{x} + f]\,\delta x$, where $\dot{x} \equiv dx/dt$, inertial. Then the variational indicator associated with HLVA for this system is

$$\frac{\text{V.I.}}{m} = \int_{t_1}^{t_2} \left[\dot{x}\,\delta\dot{x} - \left(\omega_n^2 x\,(1 + ax^2) + 2\zeta\omega_n\dot{x} - \frac{f}{m}\right)\delta x\right]dt - \left[\dot{x}\,\delta x\right]_{t_1}^{t_2}, \tag{H.1}$$

where the natural frequency ω_n and the damping ratio ζ are defined as $\omega_n = \sqrt{k/m}$, $2\zeta\omega_n = b/m$.

Inserting the relation $\delta\left(\frac{dx}{dt}\right) = \frac{d(\delta x)}{dt}$ into the V.I. (H.1) and integrating by parts of the first term of the integrand in (H.1) transform (H.1) into the following form:

$$\frac{\text{V.I.}}{m} = -\int_{t_1}^{t_2} \left[\left(\frac{d^2x}{dt^2} + \omega_n^2 x\,(1 + ax^2) + 2\zeta\omega_n\frac{dx}{dt} - \frac{f}{m}\right)\delta x\right]dt\,. \tag{H.2}$$

Step 2 Choose trial solution

Instead of going one further step beyond (H.2) to obtain the differential equation of motion of the system, we insert a trial solution into the V.I. (H.2). The assumed polynomial trial solution is as follows:

$$x(t) = x_0 + v_0(t - t_1) + \sum_{i=1}^{r} c_i(t - t_1)^{i+1}\,, \qquad t_1 \le t \le t_2. \tag{H.3}$$

Step 3 Introduce nondimensional time into trial solution

Although it is not a basic step of the HLVA method, introducing nondimensional time into the trial solution helps to reduce the risk that the set of algebraic equations in undetermined parameters is ill-conditions. Introduction of the following nondimensional time

$$\sigma = \frac{t - t_1}{T}, \qquad 0 \le \sigma \le 1 \qquad \text{(H.4)}$$

(where $T \equiv t_2 - t_1$) into the trial solution (H.3) yields the trial solution in nondimensional time σ as follows:

$$x(\sigma) = x_0 + (v_0 T)\sigma + \sum_{i=1}^{r} \alpha_i \sigma^{i+1}, \qquad 0 \le \sigma \le 1 \qquad \text{(H.5)}$$

where $\alpha_i \equiv c_i T^{i+1}$ $(i = 1, ..., r)$.

Step 4 Insert the trial solution (H.5) into the V.I. (H.2)

First, the independent integration variable t in (H.2) must be transformed into the nondimensional time σ using the following relations:

$$dt = T d\sigma, \qquad \frac{dx}{dt} = \frac{1}{T}\frac{dx}{d\sigma}, \qquad \text{and } \frac{d^2x}{dt^2} = \frac{1}{T^2}\frac{d^2x}{d\sigma^2}.$$

Then the V.I. (H.2) becomes

$$\frac{\text{V.I.}}{m} = -\int_0^1 \left[\left(\frac{1}{T^2}\frac{d^2x}{d\sigma^2} + \omega_n^2 x\,(1 + ax^2) + 2\zeta\omega_n \frac{1}{T}\frac{dx}{d\sigma} - \frac{f(T\sigma + t_1)}{m}\right)\delta x\right] d\sigma. \qquad \text{(H.6)}$$

Denote the coefficient of δx in the integrand of (H.6) by $R(x, \sigma)$, i.e.,

$$R(x, \sigma) \equiv \left(\frac{1}{T^2}\frac{d^2x}{d\sigma^2} + \omega_n^2 x\,(1 + ax^2) + 2\zeta\omega_n \frac{1}{T}\frac{dx}{d\sigma} - \frac{f(T\sigma + t_1)}{m}\right). \tag{H.7}$$

then substituting the trial solution (H.5) directly into the V.I. (H.6) gives

$$\frac{\text{V.I.}}{m} = -\int_0^1 \left[R(\alpha_1, \cdots, \alpha_r; \sigma)\sum_{i=1}^{r} \sigma^{i+1}\delta\alpha_i\right]d\sigma \tag{H.8}$$

where $R(\alpha_1, \cdots, \alpha_r; \sigma)$ is the $R(x, \sigma)$ evaluated at $x = x_0 + (v_0T)\sigma + \sum_{i=1}^{r} \alpha_i\sigma^{i+1}$, i.e.,

$$R(\alpha_1, \cdots, \alpha_r; \sigma) = 2\zeta\omega_n v_0 + \omega_n^2 x_0 + \omega_n^2 v_0 T\sigma +$$

$$\left\{\sum_{j=1}^{r}\left[\frac{j(j+1)}{T^2}\sigma^{j-1} + 2\zeta\omega_n\frac{(j+1)}{T}\sigma^j + \omega_n^2\sigma^{j+1}\right]\alpha_j\right\} +$$

$$a\omega_n^2\left[x_0 + v_0T\sigma + \sum_{j=1}^{r} \alpha_j\sigma^{j+1}\right]^3 - \frac{f(T\sigma + t_1)}{m}.$$

Step 5 Manipulate the V.I. (H.8) into a special form

Since the number of adjustable parameters is finite (i.e., r is finite), the integration operator $\int_0^1$ and the summation operator $\sum_{i=1}^{r}$ are commutative. In addition, since the variations of adjustable parameters $\delta\alpha_i$'s are independent of time t, the $\delta\alpha_i$'s can be taken out of the integration. Then after interchanging the order of the two operators and bringing the $\delta\alpha_i$'s out of the integration in (H.8), the V.I. (H.8) becomes

$$\frac{\text{V.I.}}{m} = -\sum_{i=1}^{r}\left\{\int_0^1 \left[R(\alpha_1, \cdots, \alpha_r; \sigma)\sigma^{i+1}\right]d\sigma\right\}\delta\alpha_i. \tag{H.9}$$

Step 6 Identify set of algebraic equations in adjustable parameters

Since the adjustable parameters are arbitrary, the variational indicator (H.9) vanishes if, and only if, each coefficient associated with $\delta\alpha_i$ vanishes identically, that is,

$$\int_0^1 \left[R(c_1, \cdots, c_r; \sigma)\sigma^{i+1}\right]d\sigma = 0 \qquad i = 1, \ldots, r. \tag{H.10}$$

This is a set of r algebraic equations in r undetermined parameters.

After carrying out the integration with respect to time, the set of algebraic equations (H.10) becomes

$$\sum_{j=1}^{r} A_{ij}\alpha_j + b_i + p_i = 0 \qquad i = 1, \ldots, r \tag{H.11}$$

where

$$A_{ij} = \frac{j(j+1)}{T^2(i+j+1)} + \frac{2\zeta\omega_n(j+1)}{T(i+j+2)} + \frac{\omega_n^2}{(i+j+3)}$$

$$b_i = \frac{2\zeta\omega_n v_0}{(i+2)} + \frac{\omega_n^2 x_0}{(i+2)} + \frac{\omega_n^2 v_0 T}{(i+3)}$$

$$p_i = \int_0^1 a\omega_n^2\left[x_0 + v_0 T\sigma + \sum_{j=1}^{r} \alpha_j \sigma^{j+1}\right]^3 \sigma^{i+1} d\sigma - \int_0^1 \frac{f(T\sigma + t_1)}{m}\sigma^{i+1} d\sigma$$

$i = 1, \ldots, r; \qquad j = 1, \ldots, r.$

Step 7 Solve for adjustable parameters

The set of algebraic equations (H.11) can be solved for the α_j's via standard techniques. Once the α_j's are obtained, the adjustable parameters c_j's can be readily determined from the relation $c_j = \alpha_j / T^{j+1}$ ($j = 1, \ldots, r$). Therefore, the expression (H.3) with the now known parameters c_j's constitutes the approximate solution to the dynamics problem for the desired time interval $t_1 \leq t \leq t_2$.

Sample Solutions

Solutions for the system (m = 20 kg, k = 80 N/m, a = 1.5, b = 16 N-s/m, x_0 = 1 m, v_0 = 1 m/s) under three forcing conditions in the time interval $0 \leq t \leq 20$ sec (which is for a duration of about 6.5 cycles of the natural vibration of the corresponding linear system) are conducted.

(1) Free vibration. $f(t) = 0$ N. (Fig. H.2)

(2) Harmonically forced vibration with the forcing frequency larger than the corresponding linear system natural frequency: $f(t) = 40\sin 4t$ N. (Fig. H.3)

(3) Harmonically forced vibration with the forcing frequency smaller than the corresponding linear system natural frequency: $f(t) = 40\sin 0.5t$ N. (Fig. H.4)

Since this nonlinear system has no available closed-form solution, the degree of error of the approximate solutions is indicated by the residuals of the differential equation (obtained by substituting the solution into the differential equations that represent the summation of forces at any instant). For this system, the differential equation of motion is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx\,(1 + ax^2) - f = 0.$$

Then the residual R is

$$R = \left[m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx\,(1 + ax^2) - f\right]_{x_{\text{approximate}}} \tag{H.12}$$

For fair evaluation, a normalized residual is defined as follows: (i) for free vibration, $f = 0$, the normalized residual R^* is defined as $R^* = \frac{R}{mg}100\%$; (ii) for the two forced vibrations, $f(t) = F_0 \sin \omega t$, the normalized residual R^* is defined as $R^* = \frac{R}{F_0}100\%$.

Fig. H.2 shows that maximum residual for the free vibration of this nonlinear system is of about $10^{-3}\%$ of the weight of the mass and it occurs during the transient portion of the response. The residual error decreases rapidly and becomes about $10^{-8}\%$ to $10^{-10}\%$ of the weight of the mass when the system response approaches the steady state.

Fig. H.3 shows that when the harmonic force has higher frequency than the corresponding linear system's natural frequency, the residual error ranges from $10^{-6}\%$ to $10^{-3}\%$ of the magnitude of the forcing function.

Fig. H.4 shows that when the harmonic force has lower frequency than the corresponding linear system's natural frequency, the residual error is of the order of 10^{-3}% throughout the vibration in this 20-second time interval.

Note that although the approximate solutions of the displacement shown in these figures are continuous, the residuals are not. The discontinuity of the residuals, signified by the spikes in these figures, is due to the noncontinuous acceleration of the approximate solution during the solution time interval. The discontinuity occurs at the instants that are the final or initial instants of the time segments consisting of the time interval $0 < t < 20$. Since for each segment there is a different trial solution, and any two trial solutions for adjacent time segments are connected by enforcing only continuous displacement and velocity at the adjoining instants, then the acceleration at these adjoining instants generally can not be continuous.

Table H.1 summarizes the results of the three cases.

Table H.1 Dynamics solutions for nonlinear harmonic oscillator
(ω_n= 2 rad/s, $\zeta = 0.2$, $a = 1.5\ m^{-2}$, $x_0 = 1$ m, $v_0 = 1$ m/s, $0 \le t \le 20$ sec)

Case	*Forcing* $f(t) = F_0 \sin\omega t$ N	T (sec) / r *	*Maximum error %*** *(Order of magnitude)*
1	$f(t) = 0$	0.5 / 10	10^{-3}
2	$f(t) = 40\sin 4t$	0.4 / 10	10^{-3}
3	$f(t) = 40\sin 0.5t$	0.5 / 10	10^{-3}

* T is the duration of each time segments, r is the number of undetermined parameters included in the trial solution for each time segment.

** Percent of residual error with respect to the weight of the mass for *Case* 1 (free vibration), and with respect to the magnitude of the forcing function for *Cases* 2 and 3 (forced vibrations).

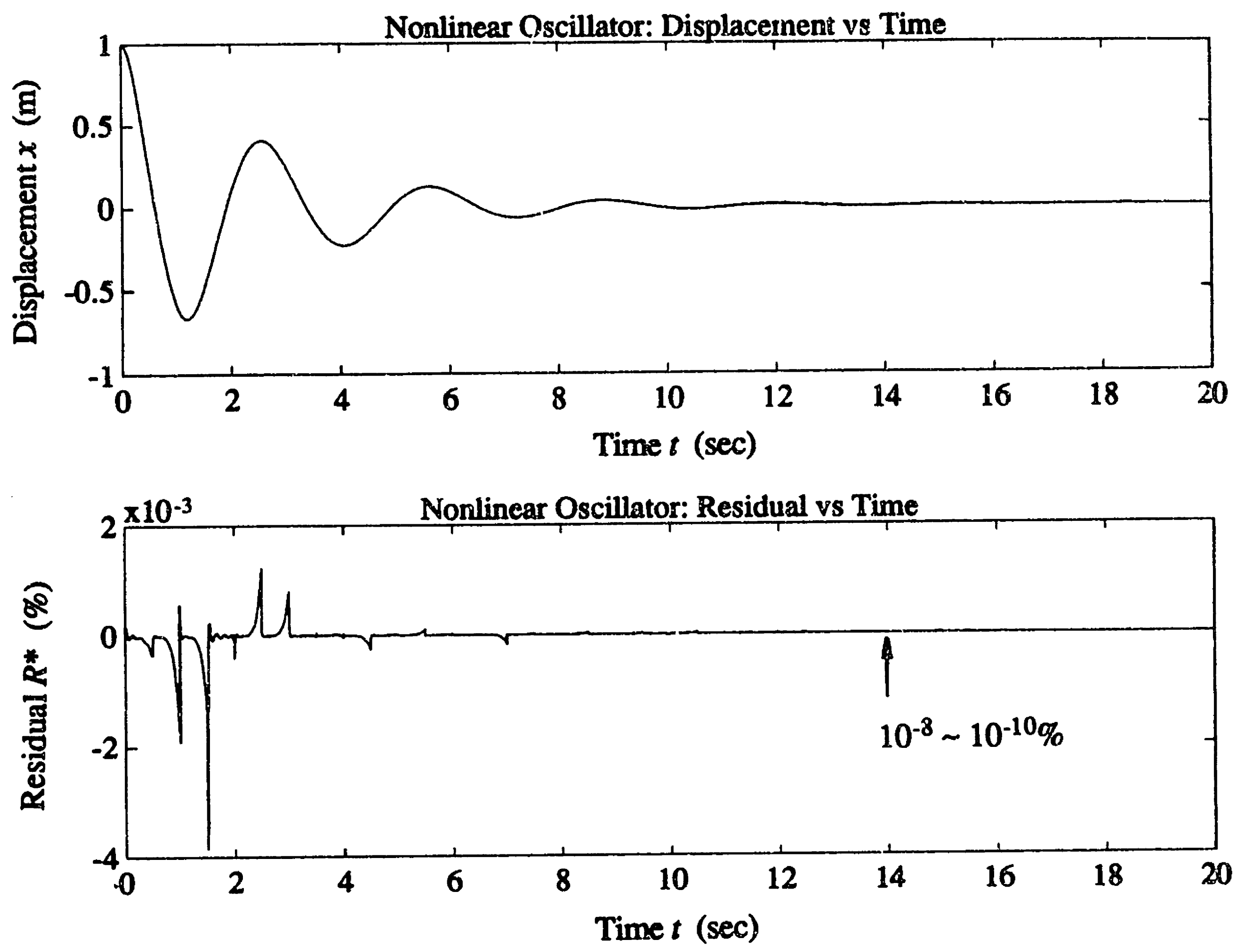


Fig. H.2 Free vibration calculated using $r = 10$ showing (*a*) displacement x versus time t and (*b*) residual, $R^* = \frac{R}{mg}100\%$, versus time t. Note that since there is no appropriate characteristic displacement available for free vibration, the displacement x is not nondimensionalized.

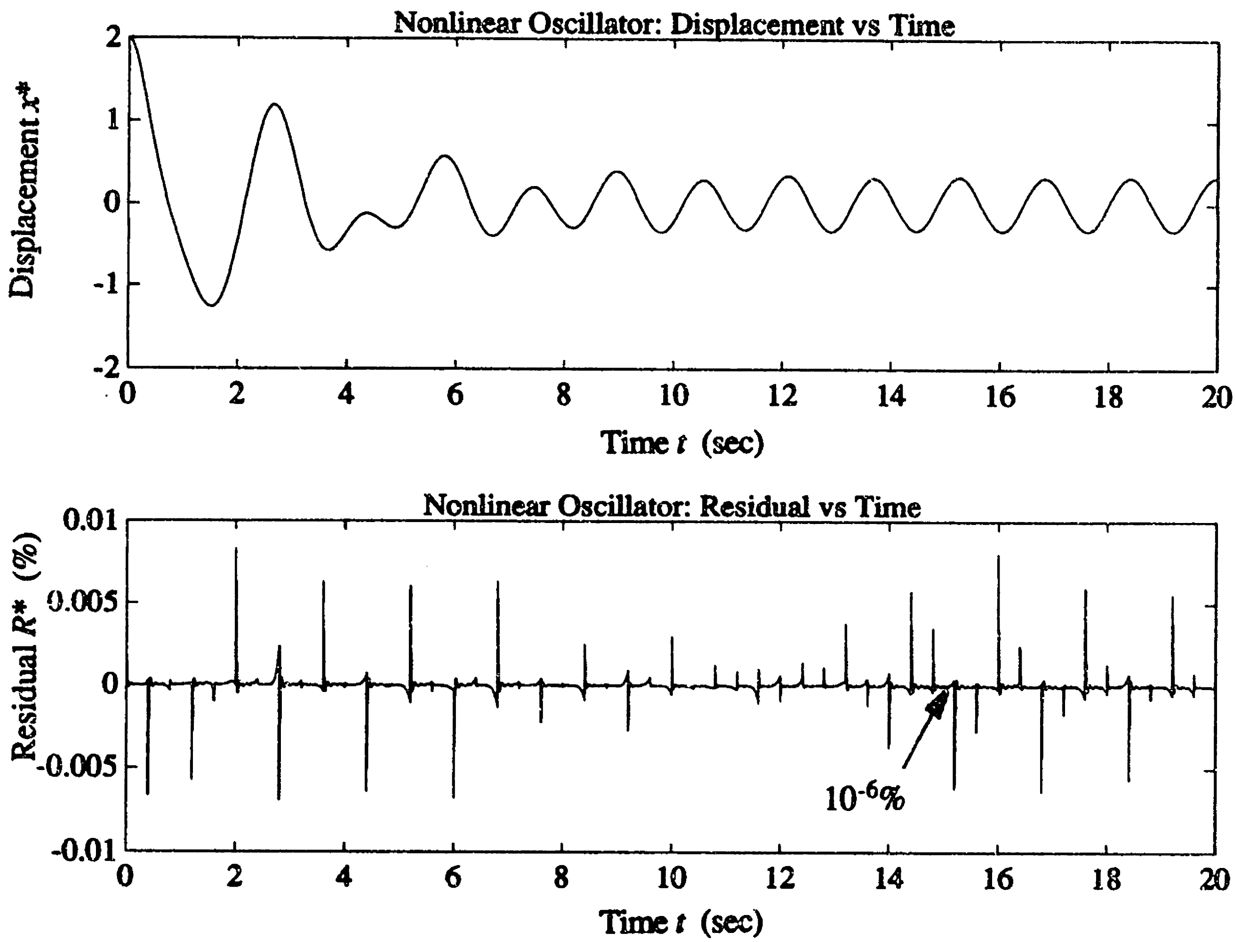


Fig. H.3 Forced vibration ($f(t) = 40\sin 4t$ N) calculated using r = 10 showing (a) nondimensional displacement, $x^* = \frac{x}{F_0/k}$, versus time t and (b) nondimensional residual, $R^* = \frac{R}{F_0}100\%$, versus time t.

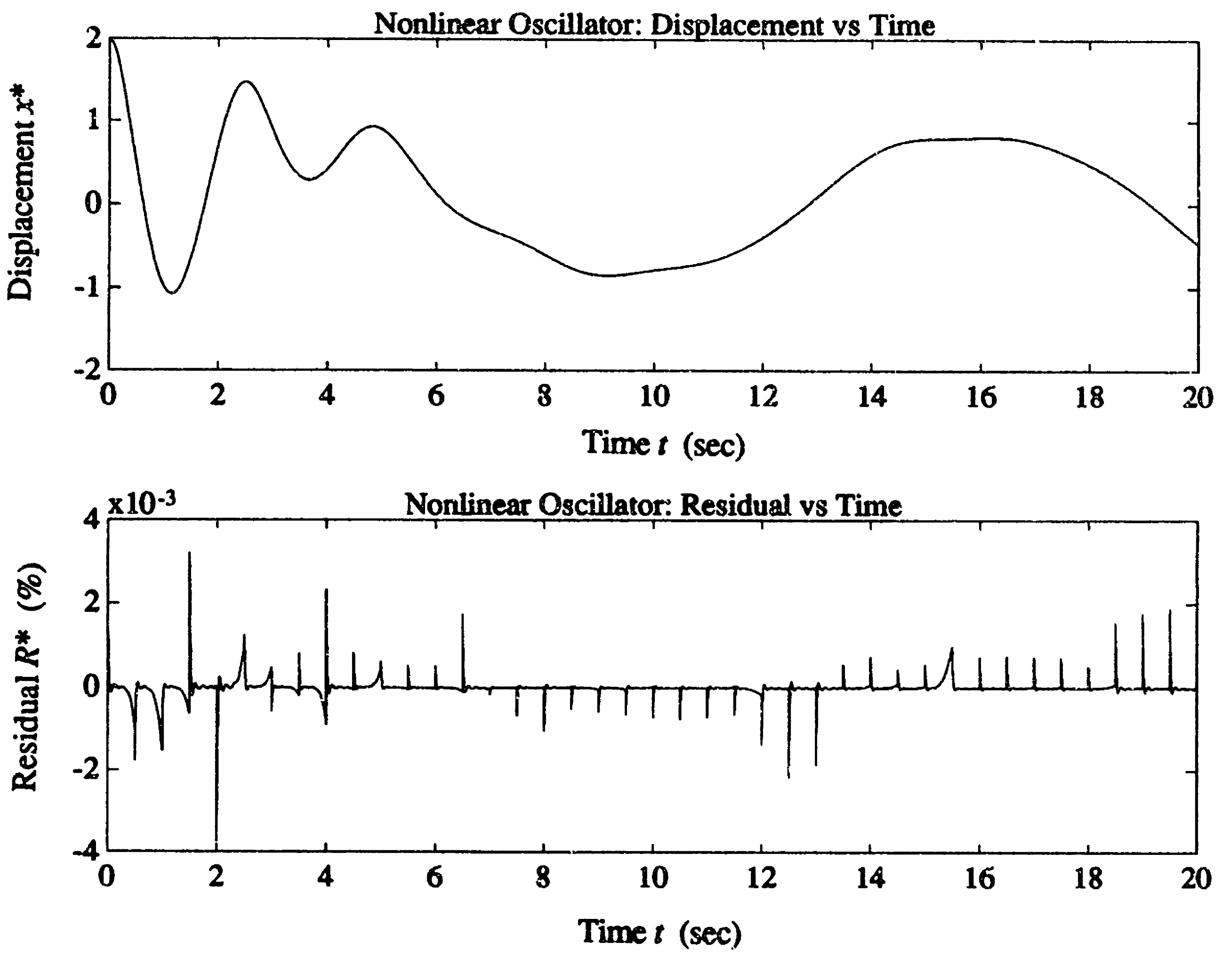


Fig. H.4 Forced vibration ($f(t) = 40\sin 0.5t$ N) calculated using $r = 10$ showing (*a*) nondimensional displacement, $x^* = \frac{x}{F_0/k}$, versus time t and (*b*) nondimensional residual, $R^* = \frac{R}{F_0}100\%$, versus time t.

APPENDIX I

The HLVA Method For A Nonholonomic System

The HLVA method is a procedure for obtaining an approximate solution to dynamics problems via trial solutions containing undetermined parameters. This appendix demonstrates the HLVA method for a nonholonomic system via the particular nonholonomic system show in Fig. I.1 [3, p121].

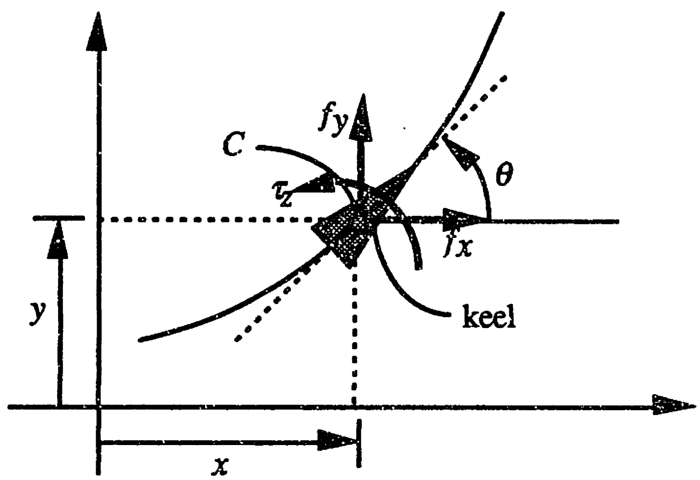


Fig. I.1. Constrained motion of a boat.

Fig. I.1 shows the motion of a boat on water surface. The boat is acted on by prescribed external forces f_x, f_y, and by a prescribed external torque τ_z along the axis perpendicular to the water surface and passing through the centroid of the boat. The motion of the boat is such that any translation must be in the instantaneous direction of the keel. The analytical expression of this constraint is

$$dy - dx \tan\theta = 0 \tag{I.1}$$

Assume the dynamics solution is desired for the interval -. The HLVA method for an approximate solution to the dynamics problem of this system consists of the steps listed below.

Step 1 Set up variational indicator associated with HLVA

The Lagrangian of the system is $L = T^* = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] + \frac{1}{2}I_c\left(\frac{d\theta}{dt}\right)^2$, where m is the mass of the boat, I_C is the moment of inertia of the boat about the axis perpendicular to the water surface and passing through the boats centroid C . The variational work expression is $\sum_{j=1}^{3} \Xi_j\delta\xi_j = f_x\delta x + f_y\delta y + \tau_z\delta\theta$. Then the variational indicator associated with HLVA is

$$\text{V.I.} = \int_{t_1}^{t_2} \sum_{j=1}^{3} \left(\frac{\partial T^*}{\partial \dot{\xi}_j}\delta\dot{\xi}_j + \frac{\partial T^*}{\partial \xi_j}\delta\xi_j + \Xi_j\,\delta\xi_j\right)dt - \left[\sum_{j=1}^{3} \frac{\partial T^*}{\partial \dot{\xi}_j}\,\delta\xi_j\right]_{t_1}^{t_2}. \tag{I.2}$$

Inserting the relation $\delta\left(\frac{d\xi_j}{dt}\right) = \frac{d}{dt}(\delta\xi_j)$ into the V.I. (I.2) and integrating by parts with respect to time for the first term in the integral in (I.2) transform (I.2) into

$$\text{V.I.} = -\int_{t_1}^{t_2} \sum_{j=1}^{3} \left[\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{\xi}_j}\right) - \frac{\partial T^*}{\partial \xi_j} - \Xi_j\right]\delta\xi_j\,dt\,. \tag{I.3}$$

Substituting the kinetic coenergy and work expressions into (I.3) gives

$$\text{V.I.} = -\int_{t_1}^{t_2}\left[\left(m\frac{d^2x}{dt^2} - f_x\right)\delta x + \left(m\frac{d^2y}{dt^2} - f_y\right)\delta y + \left(I_0\frac{d^2\theta}{dt^2} - \tau_z\right)\delta\theta\right]dt\,. \tag{I.4}$$

Step 2 Eliminate dependent variational variable from V.I.

The three variations δx, δy, and $\delta\theta$ in (I.4) are not independent; they are subject to the following constraint:

$$\delta y = \delta x \tan\theta \qquad \text{(I.5)}^{\dagger}$$

Without loss of generality, choose δx and $\delta\theta$ to be the two independent variational variables. Hence, δy can be eliminated from (I.4) by inserting the relation (I.5) into the V.I. (I.4). Then the variational indicator (I.4) becomes

$$\text{V.I.} = -\int_{t_1}^{t_2}\left\{\left[\left(m\frac{d^2x}{dt^2} - f_x\right) + \left(m\frac{d^2y}{dt^2} - f_y\right)\tan\theta\right]\delta x + \left(I_0\frac{d^2\theta}{dt^2} - \tau_z\right)\delta\theta\right\}dt\,. \tag{I.6}$$

---

† If a generic constraint equation is in the form $a_1 d\xi_1 + a_2 d\xi_2 + a_3 d\xi_3 + a_0 = 0$, where a_1, a_2, a_3, and a_0 are functions of ξ_1, ξ_2, ξ_3, and time t, then the relation that defines an admissible variation ($\delta\xi_1$, $\delta\xi_2$, $\delta\xi_3$) is obtained by changing $d\xi_1, d\xi_2$, and $d\xi_3$ in the constraint equation to $\delta\xi_1, \delta\xi_2$, and $\delta\xi_3$, and setting a_0 to zero. Therefore, the equation that defines an admissible variation for this case is $a_1\delta\xi_1 + a_2\delta\xi_2 + a_3\delta\xi_3 = 0$, where $d\xi_1 = \delta x$, $d\xi_2 = \delta y$, $d\xi_3 = \delta\theta$, $a_1 = \tan\theta$, $a_2 = -1$, and $a_3 = 0$.

Although usually not possible, the generalized coordinate y can be eliminated from the V.I. (I.6) for this particular problem. This elimination is accomplished by first writing the constraint equation (I.1) into the following form:

$$\frac{dy}{dt} = \frac{dx}{dt}\tan\theta\,, \tag{I.7}††$$

then differentiating (I.7) to give

$$\frac{d^2y}{dt^2} = \frac{d^2x}{dt^2}\tan\theta + \frac{dx}{dt}\frac{d\theta}{dt}\sec^2\theta\,. \tag{I.8}$$

When (I.8) is inserted into the V.I. (I.6) and terms associated with δx and $\delta\theta$ are collected together, respectively, the V.I. (I.6) is converted to

$$\text{V.I.} = -\int_{t_1}^{t_2}\left[\left(m\frac{d^2x}{dt^2}\left(1+\tan^2\theta\right)+m\frac{dx}{dt}\frac{d\theta}{dt}\sec^2\theta\tan\theta - f_x - f_y\tan\theta\right)\delta x + \left(I_0\frac{d^2\theta}{dt^2} - \tau_z\right)\delta\theta\right]dt \tag{I.9}$$

<u>Step 3 Choose trial solutions for x and θ</u>

Select the polynomial trial solutions as

$$x(t) = x_0 + v_0(t - t_1) + \sum_{i=1}^{r} c_i(t - t_1)^{i+1} \tag{I.10a}$$

$$\theta(t) = \theta_0 + \omega_0(t - t_1) + \sum_{i=1}^{r} k_i(t - t_1)^{i+1} \tag{I.10b}$$

†† The constraint equation (I.1) is written for a displacement $(dx, dy, d\theta)$ in an infinitesimal time interval dt. Now if we divide both sides of (I.1) by dt and let dt approach zero, the equation of constraint (I.1) is transformed into the form (I.7).

where x_0 and v_0 are the x-components of the displacement and velocity of the boat at $t = t_1$, respectively, and θ_0 and ω_0 are the heading angle and rate of change of the heading angle at $t = t_1$, respectively.

To reduce the risk that the set of algebraic equations produced by the HLVA method be ill-conditioned, introduce nondimensional time $\sigma = (t - t_1)/T$, $0 \leq \sigma \leq 1$, (where $T \equiv t_2 - t_1$) into the above two trial solutions. Then, the trial solutions in nondimensional time are as follows:

$$x(\sigma) = x_0 + (v_0 T)\sigma + \sum_{i=1}^{r} \alpha_i \sigma^{i+1} \qquad 0 \leq \sigma \leq 1 \qquad \text{(I.11a)}$$

$$\theta(\sigma) = \theta_0 + (\omega_0 T)\sigma + \sum_{i=1}^{r} \beta_i \sigma^{i+1} \qquad 0 \leq \sigma \leq \qquad \text{(I.11b)}$$

where $\alpha_i \equiv c_i T^{i+1}$ and $\beta_i \equiv k_i T^{i+1}$ ($i = 1, \ldots, r$).

Step 4 Obtain set of algebraic equations in adjustable parameters

First transform the independent integration variable t in (I.9) into the nondimensional time α using the following relations:

$$dt = T d\sigma, \qquad \frac{dx}{dt} = \frac{1}{T}\frac{dx}{d\sigma}, \qquad \frac{d^2x}{dt^2} = \frac{1}{T^2}\frac{d^2x}{d\sigma^2}.$$

Then the variational indicator in nondimensional time is

$$\text{V.I.} = -\int_0^1 \left[R_x(x, \theta, \sigma)\delta x + R_\theta(x, \theta, \sigma)\delta\theta\right] d\sigma, \qquad \text{(I.12)}$$

where

$$R_x(x,\ \theta,\ \sigma) \equiv \frac{m}{T^2}\frac{d^2x}{d\sigma^2}\left(1+\tan^2\theta\right) + \frac{m}{T^2}\frac{dx}{d\sigma}\frac{d\theta}{d\sigma}\sec^2\theta\,\tan\theta - f_x(T\sigma+t_1) - f_y(T\sigma+t_1)\tan\theta$$

$$R_\theta(x,\ \theta,\ \sigma) \equiv \frac{I_0}{T^2}\frac{d^2\theta}{dt^2} - \tau_z(T\sigma+t_1).$$

<u>Step 5 Insert trial solutions into V.I. and rearrange V.I. into special form</u>

Now inserting the trial solutions in nondimensional time into the V.I. (I.12) gives

$$\text{V.I.} = -\int_0^1 \Bigg[R_x(\alpha_1,\cdots,\alpha_r;\ \beta_1,\cdots,\beta_r;\ \sigma)\sum_{i=1}^{r}\sigma^{i+1}\delta\alpha_i +$$

$$R_\theta(\alpha_1,\cdots,\alpha_r;\ \beta_1,\cdots,\beta_r;\ \sigma)\sum_{i=1}^{r}\sigma^{i+1}\delta\beta_i\Bigg]d\sigma \qquad \text{(I.13)}$$

where $R_x(\alpha_1,\cdots,\alpha_r;\ \beta_1,\cdots,\beta_r;\ \sigma)$ and $R_\theta(\alpha_1,\cdots,\alpha_r;\ \beta_1,\cdots,\beta_r;\ \sigma)$ are obtained by inserting the trial solutions (I.10) directly into $R_x(x,\ \theta,\ \sigma)$ and $R_\theta(x,\ \theta,\ \sigma)$.

Since the number of adjustable parameters is finite (that is, r is finite), the integration operator $\int_0^1$ and the summation operator $\sum_{i=1}^{r}$ are commutative. In addition, since the variation of adjustable parameters $\delta\alpha_i$'s and $\delta\beta_i$'s are independent of time t, the $\delta\alpha_i$'s and the $\delta\beta_i$'s can be taken out of the integration. Then, after interchanging the order of the two operators and bringing the $\delta\alpha_i$'s and the $\delta\beta_i$'s out of the integral in (I.13), the V.I can be written in the following form:

$$\text{V.I.} = -\sum_{i=1}^{r}\left\{\int_0^1 R_x(\alpha_1,\cdots,\alpha_r;\beta_1,\cdots,\beta_r;\sigma)\sigma^{i+1}d\sigma\right\}\delta\alpha_i - \sum_{i=1}^{r}\left\{\int_0^1 R_\theta(\alpha_1,\cdots,\alpha_r;\beta_1,\cdots,\beta_r;\sigma)\sigma^{i+1}d\sigma\right\}\delta\beta_i \quad \text{(I.14)}$$

Step 6 Identify set of algebraic equations in adjustable parameters

Since the adjustable parameters are arbitrary, the variational indicator (I.14) vanishes if, and only if, each coefficient associated with $\delta\alpha_i$ and $\delta\beta_i$ vanishes identically; that is,

$$\int_0^1 R_x(\alpha_1,\cdots,\alpha_r;\beta_1,\cdots,\beta_r;\sigma)\sigma^{i+1}d\sigma = 0 \qquad i = 1, ..., r \quad \text{(I.15a)}$$

$$\int_0^1 R_\theta(\alpha_1,\cdots,\alpha_r;\beta_1,\cdots,\beta_r;\sigma)\sigma^{i+1}d\sigma = 0 \qquad i = 1, ..., r. \quad \text{(I.15b)}$$

This is a set of $2r$ algebraic equations in $2r$ undetermined parameters.

Step 7 Solve set of algebraic equations

After the integrations in (I.15) are conducted, the set of equations (I.15) becomes

$$\sum_{j=1}^{r} A_{ij}\alpha_j + p_{xi} = 0 \qquad i = 1, ..., r \quad \text{(I.16a)}$$

$$\sum_{j=1}^{r} B_{ij}\beta_j + p_{\theta i} = 0 \qquad i = 1, ..., r \quad \text{(I.16b)}$$

where

$$A_{ij} = \frac{j(j+1)}{T^2(i+j+1)}$$

$$p_{xi} = \frac{1}{T^2}\int_0^1 \left(\frac{d^2x}{d\sigma^2}\tan^2\theta + \frac{m}{T^2}\frac{dx}{d\sigma}\frac{d\theta}{d\sigma}\sec^2\theta\tan\theta\right)\sigma^{i+1}d\sigma +$$

$$+\int_0^1 \left[-\frac{f_x(T\sigma+t_1)}{m} - \frac{f_y(T\sigma+t_1)}{m}\tan\theta\right]\sigma^{i+1}d\sigma$$

$$B_{ij} = \frac{j(j+1)}{T^2(i+j+1)}$$

$$p_{\theta i} = \int_0^1 \left[-\frac{\tau_z(T\sigma+t_1)}{I_0}\right]\sigma^{i+1}d\sigma$$

$i = 1, ..., r; \quad j = 1, ..., r.$

Once the α_j's and β_j's are obtained by solving equations (I.16), the adjustable parameters c_j's and k_j's can be readily obtained from the relation $c_j = \alpha_j/T^{j+1}$ and $k_j = \beta_j/T^{j+1}$ ($j = 1, ..., r$). Therefore, the expressions (I.10) with the now known parameters c_j's and k_j's form the approximate solutions to $x(t)$ and $\theta(t)$ for the desired time interval $t_1 \le t \le t_2$. And $y(t)$ can be obtained by integrating the constraint equation (I.7) once $x(t)$ and $\theta(t)$ are obtained.

A Sample Solution

Fig. I.2 shows a solution of the system ($m = 100$ kg, $I_0 = 1600$ kgm^2) under the following simple but nontrivial conditions:

Initial conditions:	$x(0) = 0,$	$y(0) = 0,$	$\theta(0) = 0$
	$\frac{dx}{dt}(0) = 0,$	$\frac{dy}{dt}(0) = 0,$	$\frac{d\theta}{dt}(0) = 0;$
Forcing:	$f_x = 0.2mg,$	$f_y = 0.5mg,$	$\tau_z = 0.1mg\sqrt{I_0/m}.$
Time interval:	$0 \le t \le 1.6$ sec.		

Unfortunately, solving the set of highly nonlinear algebraic equations (I.16) under the above forcing conditions is an extremely slow process, which explains why only a very short time interval is shown in Fig. I.2. (In fact, it took about 40 minutes to produce the results shown in Fig. I.2. The main cause of the sluggish speed of solution is the slow rate of convergence of the solution to the set of nonlinear algebraic equations (I.16) using the technique of iteration by total steps.) For the interval $0 \le t \le 1.6$ sec, Fig. I.2 shows the residual† of the differential equation in the x direction (an equation of force balance in the x

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† For this system the differential equations in x and θ can be found from the V.I. (I.9) to be

$m\ddot{x} - \frac{f_x - m\dot{x}\,\dot{\theta}\tan\theta\sec^2\theta + f_y\tan\theta}{(1+\tan^2\theta)} = 0$ and $I_c\ddot{\theta} - \tau_z = 0$, respectively. Then the residuals of these two equations are

$$R_x = \left[m\ddot{x} - \frac{f_x - m\dot{x}\,\dot{\theta}\tan\theta\sec^2\theta + f_y\tan\theta}{(1+\tan^2\theta)}\right]_{(x_{approximate},\ \theta_{approximate})}$$

and

$$R_z = \left[I_c\ddot{\theta} - \tau_z\right]_{\theta_{approximate}},$$

and the corresponding normalized residuals are $R_x^* = \frac{R_x}{|f_x|_{max}}$ and $R_\theta^* = \frac{R_\theta}{|\tau_z|_{max}}$.

direction) to be about $5\times10^{-2}\%$ of the magnitude of the applied forced in the x direction, and the residual of the differential equation in the z direction (an equation of torque balance in the z direction) to be about $10^{-8}\%$ of the magnitude of the applied torque in the z direction.

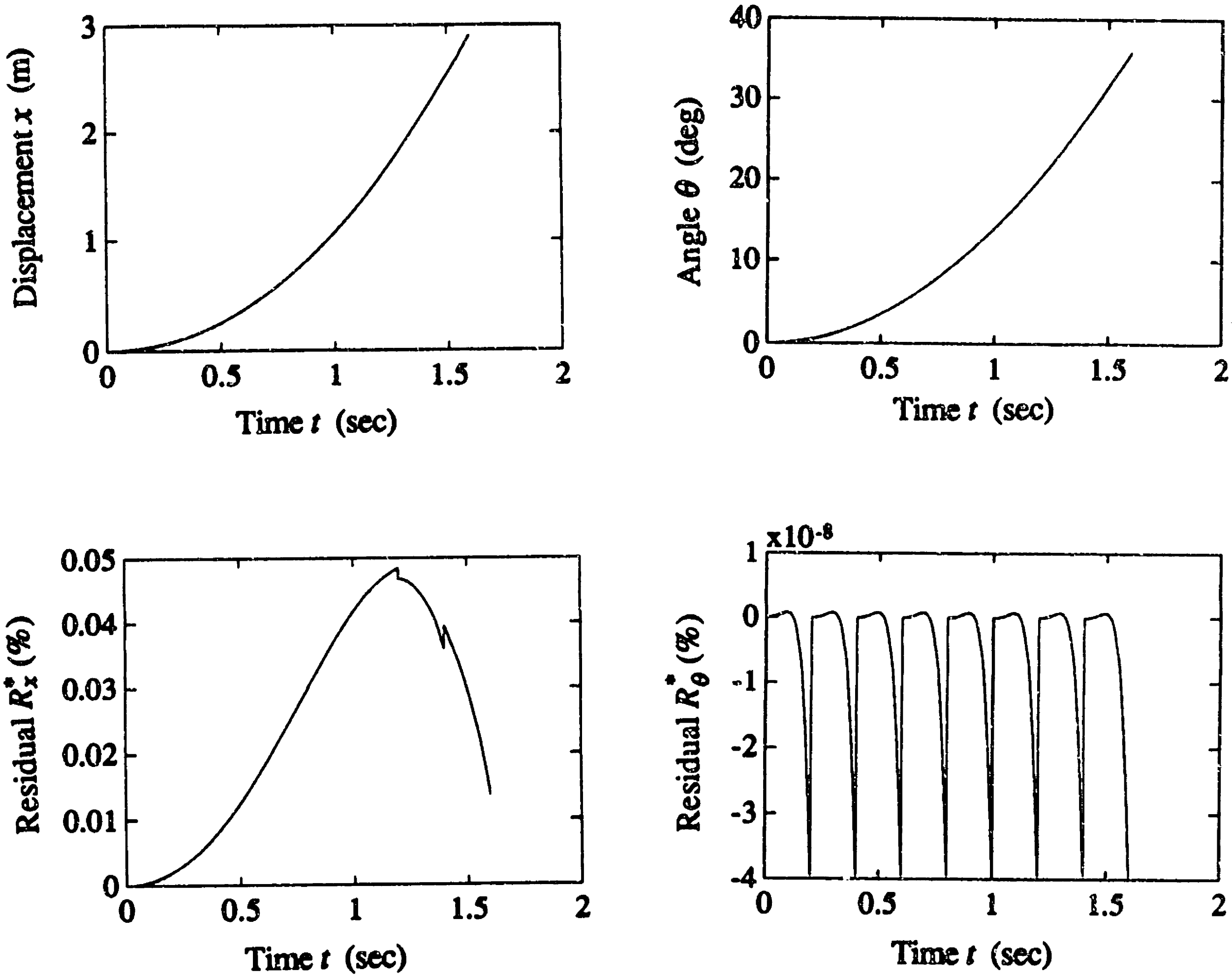

Fig. I.2. Motion of boat under constant forcings calculated with duration of time segments $T = 0.2$ sec and number of adjustable parameters in trial solutions $r = 6$:

(*a*) the x-component of displacement versus time t,

(*b*) angle θ versus time t,

(*c*) nondimensional residual R_x^*, $R_x^* = \frac{R_x}{|f_x|_{max}}$, versus time t, and

(*d*) nondimensional residual R_θ^*, $R_\theta^* = \frac{R_\theta}{|\tau_z|_{max}}$, versus time t.

APPENDIX J

Weighted-residual Methods for Dynamics Problems of Lumped-parameter Systems

The weighted-residual methods [4, p147-153] are procedures that produce approximate solutions to dynamics problems in the form of analytical expressions. These methods are based on the differential formulation of dynamics problems. They generally consist of the steps listed below.

1. Formulate dynamics problem by set of differential equations

The general form of the formulation for a system with n degrees of freedom is

$$\left.\begin{aligned} \frac{dx_j}{dt} &= f_j(x_1, x_2, \cdots, x_n, t) \qquad & t > 0 \\ x_j &= x_j(0) \qquad & t = 0 \end{aligned}\right\} \qquad j = 1, \ldots, n. \tag{J.1}$$

2. Choose trial solutions

A set of n trial functions, each containing adjustable parameters, must be constructed. The trial functions should be sufficiently simple so that the operations required can be easily performed. The parameters should be inserted into the trial solutions in such a way as to give a wide variation of possible solutions all of which satisfy the initial conditions.

It is common to take trial solutions of the form

$$x_j = \varphi_{j0}(t) + c_{j1}\varphi_{j1}(t) + \cdots + c_{jr}\varphi_{jr}(t) \qquad j = 1, ..., n. \tag{J.2}$$

where r is the number of adjustable parameters, and the $\varphi_{jk}(t)$ $(k = 1, ..., r)$ are known functions (for example, polynomial, trigonometric or exponential functions) which satisfy the following initial conditions:

$$\left.\begin{array}{ll} \varphi_{j0}(0) = x_j(0) & \\ \varphi_{jk}(0) = 0 & k = 1, \cdots, r \end{array}\right\} \qquad j = 1, ..., n. \tag{J.3}$$

<u>3. Apply weighted residual criteria to obtain set of algebraic equations in adjustable parameters</u>

When the trial solutions (J.2) are substituted into the governing equations (J.1), n equation *residuals*, $R_j(t)$, $R_j(t) = \frac{dx_j}{dt} - f_j(x_1, x_2, \cdots, x_n, t)$, $(j = 1, ..., n)$, are formed. For the true solution $x_j(t)$ $(j = 1, ..., n)$ the equation residuals vanishes identically. For the restricted family of trial solutions (J.2), the weighted-residual criteria are procedures to pick out the best approximate solutions from the trial family by adjusting the undetermined parameters associated with each unknown function $x_j(t)$.

If r adjustable parameters were used in constructing x_j, then r conditions of the weighted-residual type must be satisfied by R_j in the desired interval $0 < t < T$. The conditions (criteria) for fixing the c_{jk} $(k = 1, ..., r)$ are listed below.

1. *Collocation.* We select as many locations within $0 < t < T$ as there are undetermined parameters and then adjust the parameters until the residual R_j vanishes at these locations. The presumption here is that the residual R_j does not get very far from zero in between the locations where it vanishes. Let t_k ($k = 1, ..., r$) denote r locations in the interval $0 < t < T$, then the set of equations given by the collocation criterion is

$$R_j(t_k) = 0, \qquad k = 1, ..., r. \tag{J.4}$$

2. *Subdomain Method.* We divide the desired interval $0 < t < T$ into as many subdomains as there are adjustable parameters and then adjust the parameters until the average value of the residual in each subdomain is zero. Let $t_k \le t \le t_{k+1}$ ($k = 1, ..., r$) denote r subdomains in the interval $0 < t < T$, the the set of equations given by the subdomain method is

$$\int_{t_k}^{t_{k+1}} R_j(t)dt = 0, \qquad k = 1, ..., r. \tag{J.5}$$

3. *Galerkin's Method.* We require that weighted averages of the residual over the desired interval $0 < t < T$ should vanish. The weighting functions are taken to be the same functions of t as were used in constructing the trial family; that is, the φ_{jk} ($k = 1, ..., r$). The set of r equations given by Galerkin's method is

$$\int_0^T R_j(t)\varphi_{jk}dt = 0, \qquad k = 1, ..., r. \tag{J.6}$$

4. *Method of Least Squares.* The parameters are here adjusted in such a way as to minimize the integral of the square of the residual over the desired interval $0 < t < T$. The set of adjustable parameters that minimize the integral $\int_0^T [R_j(t)]^2 dt$ can be determined from the following set of r equations:

$$\frac{1}{2}\frac{\partial}{\partial c_{jk}}\int_0^T [R_j(t)]^2 dt = 0, \qquad k = 1, \ldots, r. \tag{J.7}$$

4. Solve for adjustable parameters

When each of the n residuals is treated in the way describe above, enough equations are obtained to solve simultaneously for all the parameters. Once the adjustable parameters are obtained, the approximate solution of the dynamics problem is given by (J.2).

APPENDIX K

Relationship Between the HLVA Method and Galerkin's Method

This appendix proves that the approximate solution to a dynamics problem via the HLVA method is identical with that via Galerkin's method when identical trial solutions are used. Without loss of generality, the proof is given for a one degree-of-freedom system.

Let ξ be the generalized coordinate of a one degree-of-freedom system and let $\delta\xi$ be the associated variational variable. In addition, let $T^* = T^*(\xi, \dot{\xi}, t)$ be the kinetic coenergy of the system, $V(\xi, t)$ be the potential energy of conservative forces, and Ξ be the generalized force due to nonconservative forces acting on the system. Assume that the dynamics solution is desired for the time interval $t_1 \leq t \leq t_2$.

First, consider the HLVA method for this system. The variational indicator associated with HVLA for this system is

$$\text{V.I.} = \int_{t_1}^{t_2} \left(\delta L + \Xi\, \delta\xi\right) dt - \left[\frac{\partial T^*}{\partial \dot{\xi}}\, \delta\xi\right]_{t_1}^{t_2} \tag{K.1}$$

where $L = T^* - V$ is the lagrangian of the system. Carrying out the variation of the Lagrangian in the V.I. (K.1) gives

$$\text{V.I.} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{\xi}}\delta\dot{\xi} + \frac{\partial L}{\partial \xi}\delta\xi + \Xi\, \delta\xi\right) dt - \left[\frac{\partial T^*}{\partial \dot{\xi}}\, \delta\xi\right]_{t_1}^{t_2}. \tag{K.2}$$

By using the commutativity relation $\delta\dot{\xi} = \frac{d}{dt}(\delta\xi)$ and integrating the first term in the integrand of (K.2) by parts, the V.I. (K.2) can be transformed into the following form:

$$\text{V.I.} = -\int_{t_1}^{t_2} \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) - \frac{\partial L}{\partial \xi} - \Xi \right\} \delta\xi \, dt \,. \tag{K.3}$$

Now, assume the trial solution to be as follows:

$$\xi(t) = \varphi_0(t) + \sum_{j=1}^{r} c_j \, \varphi_j(t) \tag{K.4}$$

where the basis functions $\varphi_j(t)$'s are chosen to be linearly independent continuous functions and have continuous first derivatives in the time interval from t_1 to t_2, the c_j's are undetermined parameters, and r is the number of undetermined parameters. In addition, the basis functions $\varphi_j(t)$ should be chosen such that $\varphi_0(t)$ ensures the satisfaction of the initial conditions of the system, i.e. $\varphi_0(t_1) = \xi(t_1)$, $\left.\frac{d\varphi_0(t)}{dt}\right|_{t_1} = \left.\frac{d\xi(t)}{dt}\right|_{t_1}$, $\varphi_j(t_1) = 0$, and $\left.\frac{d\varphi_j(t)}{dt}\right|_{t_1} = 0$.

Inserting the above trial solution and its variation

$$\delta\xi(t) = \sum_{j=1}^{r} \varphi_j(t)\delta c_j \tag{K.5}$$

into the V.I. (K.3) gives the variational indicator as follows:

$$\text{V.I.} = \int_{t_1}^{t_2} R(c_1, \cdots, c_r; t) \sum_{j=1}^{r} \varphi_j(t) \delta c_j \, dt \,, \tag{K.6}$$

where

$$R(c_1, \cdots, c_r; t) \equiv \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) - \frac{\partial L}{\partial \xi} - \Xi \right]_{\xi(t) = \varphi_0(t) + \sum_{j=1}^{r} c_j \varphi_j(t)}$$

Since the trial solution consists of a finite number of adjustable parameters (i.e., r is finite), the integration and summation operators in (K.6) are commutative. In addition, since the variation of adjustable parameters δc_j's does not depend on time, the δc_j's can be taken out of the integration. Hence after exchanging the order of the integration and summation operators and bringing the δc_j's out of the integration in (K.6), we obtain

$$\text{V.I.} = \sum_{j=1}^{r} \left\{ \int_{t_1}^{t_2} R(c_1, \cdots, c_r; t) \varphi_j(t) dt \right\} \delta c_j \,. \tag{K.7}$$

Because the adjustable parameters are arbitrary, the variational indicator (K.7) vanishes if, and only if, the following r equations hold

$$\int_{t_1}^{t_2} R(c_1, \cdots, c_r; t) \varphi_j(t) dt = 0 \qquad j = 1, \ldots, r. \tag{K.8}$$

This set of equations states that the weighted averages of the function $R(c_1, \cdots, c_r; t)$ over the desired time interval $t_1 < t < t_2$ should vanish. The weighting functions are nothing else but the basis functions used to construct the trial family (K.4). Therefore, equation (K.8) is nothing but the Galerkin criterion [4, p149] that provides the conditions that the adjustable parameters should satisfy for the dynamics problem formulated by the differential equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) - \frac{\partial L}{\partial \xi} - \Xi = 0. \tag{K.9}$$

And the function $R(c_1, \cdots, c_r; t)$ is nothing but the *residual* formed by directly inserting the trial solution (K.4) into the differential equation (K.9).

Therefore, it is proved that the HLVA method produces the same set of algebraic equations in adjustable parameters as Galerkin's method does for identical trial solutions. Hence, we may conclude that the approximate solution to dynamics problems of lumped-parameter systems via the HLVA method is identical with those via Galerkin's method when identical trial solutions are used.