AN ACOUSTIC MEANS OF DETECTING
AN ELECTRON-PROTON CHARGE DIFFERENCE

by

Henry Frederick Dylla, III

Submitted in Partial Fulfillment of the
Requirements for the Degrees of
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"...full of sound and fury
signifying nothing."

Macbeth Act V, Scene v
ABSTRACT

An Acoustic Means of Detecting An Electron-Proton Charge Difference

Henry Frederick Dylla, III

Submitted to the Department of Physics in partial fulfillment of the requirements for the simultaneous degrees of Bachelor of Science and Master of Science.

A new experimental technique is described for testing the electrical neutrality of molecules. The measurement was made on SF₆ gas contained in a high Q acoustic cavity. An alternating electric field was applied across the volume of the cavity at a frequency corresponding to an acoustic normal mode of the gas. Phase sensitive detection of the sound energy at the fundamental frequency yielded a measurement of the charge per SF₆ molecule, Q; or dividing by the mass number of molecule (146) an upper limit for an electron-proton charge difference, 6q. (This last result assumes that the electron-proton charge difference equals the neutron charge). Detection of sound energy at twice the electric field frequency due to the polarizability of the gas yielded a calibration for the experiment. The results: Q ≤ 2 x 10⁻¹⁹e and 6q ≤ 1 x 10⁻²¹e, where e is the magnitude of the electronic charge, are consistent with the fundamental belief of zero charge imbalance. This technique is compared with previous neutrality measurements.

Thesis Supervisor: John G. King
Title: Professor of Physics
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SECTION I
INTRODUCTION

With the discovery and investigation of sub-atomic particles that has involved physicists since Thomson's isolation of the electron in 1897, came the conclusion that a fundamental unit of charge existed, and was universal despite wide variations in the physical characteristics of the charge carriers. Charge equality represents a symmetry in nature which corresponds to no known conservation law, but is based directly on a series of sensitive experimental measurements dating from 1925. Feinberg and Goldhaber in their 1959 review article on "absolute" conservation laws have shown that charge conservation applied to the known reactions among the elementary particles does not provide sufficient information to establish all the apparent charge equalities.\footnote{1} They conclude "that the apparent equality of the elementary particles involves some new physical principle." Dirac may have uncovered the existence of this new principle when he predicted the existence of a magnetic charge during the infancy of quantum electrodynamics.\footnote{2} He has shown that the existence of a magnetic monopole would quantize electric charge such that only integral multiples of a fundamental unit could occur. To date neither the magnetic monopole, nor the newly hypothesized fundamental unit, the quark, have been found. However, the fact remains with or without a supporting principle, that charge equality is an assumed property of nature, and as such should be supported by explicit experimental measurements.

There have been several reasons of lesser importance for searching
for an electron-proton charge difference, mainly as the result of predictions of various ad hoc theories. In 1924 Einstein suggested that a charge difference of 3 parts in $10^{19}$ of the electronic charge, $(e)$, would be sufficient to account for the magnetic fields of the earth and sun.\(^3\) In 1959, Lyttleton and Bondi calculated that a charge difference of 2 parts in $10^{18}$ e would balance the gravitational attraction of two hydrogen atoms, providing a mechanism for the observed rate of expansion of the universe on the basis of coulombic repulsion of the celestial bodies.\(^4\) However, both of these suggestions have been precluded by the null results of this experiment and several of the other experiments quoted in this paper, which have sensitivities several orders of magnitude greater than the above required charge difference. Any finite charge difference regardless of its magnitude would have consequences for the theory of elementary particles. For example, baryon conservation would follow simply from charge conservation, rather than being the basis of an independent conservation law.\(^5\) Again the sensitivity of the neutrality measurements offers little support for this suggestion.

The charge equality measurements in the literature have involved three different experimental techniques: 1) the gas-efflux method, 2) the isolated body method, and, 3) the molecular beam method. Methods 1 and 2 involve the measurement of the total charge on a macroscopic number of molecules. When care has been taken to eliminate the effect of ions and other charged particles, the findings of these experiments are consistent with the assumption that matter is neutral, i.e., the
measured charge per molecule is zero within the experimental error. This upper limit for the molecular charge is then used to establish upper limits on the electron-proton charge difference and the neutron charge.

Specifically, the gas-efflux method involves the measurement of the current that would flow to an insulated metal container as the hypothetically charged gas flows from the container. The method was pioneered by Piccard and Kessler in 1925, improved measurements were made by Hillas and Cranshaw in 1959 and King in 1960, and the best measurements were made by King in 1965, yielding an upper limit of $5 \times 10^{-23}$ e for the electron-proton charge difference. The isolated body method (2) involves the measurement of the motion of a macroscopic body in a uniform electric field. Millikan's famous oil drop experiments during the early part of the century were the first such experiments. Hopper and Laby improved the sensitivity of the measurements in 1941, but the method did not yield results comparable to the gas-efflux method until the recent measurements of Stover, Moran, and Trischka. This method is also plagued by stray charged particles, since one extra electron or univalent ion per mole of matter would be equivalent to an effective charge per molecule of a part in $10^{23}$ e. The isolated body method has an additional restriction concerning interpretation of the measurement as an upper limit on the charge per molecule. If a hypothetical molecular charge existed, a body of such matter of sufficient size would attract or lose an electron thus compensating for
any charge unbalance.

The third type of measurement that exists in the literature involves a microscopic measurement on individual molecules. The molecular beam method (3) involves the measurement of the deflection of the trajectory of a beam of molecules in a region of homogeneous electric field. The first such measurements were performed in 1957 by Hughes\textsuperscript{13} on the CsI molecule, and by Shapiro and Estulin\textsuperscript{14} on thermal neutrons. Improved measurements were obtained by Zorn, Chamberlain, and Hughes\textsuperscript{15} in 1963 with the use of Cs, K, H\textsubscript{2}, and D\textsubscript{2} molecular beams. The best measurements with this method were made by Fraser, Carlson, and Hughes\textsuperscript{16} in 1968 with Cs and K, and by Shull, Billman, and Wedgwood in 1966 with neutrons.\textsuperscript{17} The essential limitation of this method is the accuracy to which deflections of the beam profile can be measured.

A summary of the results of all three types of experiments is listed in Table 1.1. In all the experiments except the neutron measurements, a charge per molecule has been measured. This molecular charge is not immediately translated to an effective electron-proton charge difference, until the effect of the neutrons, which are also present in the molecular nuclei, are considered. The neutron charge has been treated in two ways to obtain an electron-proton charge difference from the measured molecular charge. When concurrent charge measurements have been made on two molecular species with different neutron-proton ratios, then a pair
of equations can be solved of the form:

\[ Q = Z \epsilon e + Nq_N \]  \hspace{1cm} (1.1)

where \( Q \) is the measured molecular charge, \( \epsilon e \) the hypothetical electron-proton charge difference, \((q_e + q_p)\), \( q_N \) the neutron charge, \( Z \), the number of protons, and \( N \), the number of neutrons. The solution to such a set of equations would yield independent results for \( \epsilon e \) and \( q_N \).

The second treatment of the neutron charge yields a smaller limit for \( \epsilon e \) and the basis for the values listed for this quantity in Table 1.1.

The assumption is that \( \epsilon e = q_N \), which follows from the assumption of charge conservation in the beta decay of the neutron \((n \rightarrow p + e^- + \bar{\nu})\), with the additional assumption of zero charge carried by the anti-neutrino. With these assumptions the effective electron-proton charge difference is thus the measured charge per molecule divided by the total number of nucleons per molecule, \( n = Z + N \).
| Method                          | Molecule | Upper Limit for $|q_1|$ | Upper Limit for $\epsilon$ |
|--------------------------------|----------|-----------------|--------------------------|
| Millikan $^{10}$               |          | -               | $1 \times 10^{-16}$     |
| Hopper and Laby $^{11}$         |          | -               | $5 \times 10^{-17}$     |
| Stover, Moran and Trischka $^{12}$ |          | -               | $1 \times 10^{-19}$     |
| Piccard and Kessler $^{3}$      | $CO_2$   | $2 \times 10^{-19}$ | $4 \times 10^{-21}$    |
| Hillas and Cranshaw $^{7}$      | $A$      | $8 \times 10^{-20}$ | $2 \times 10^{-21}$    |
|                                 | $N_2$    | $12 \times 10^{-20}$ | $5 \times 10^{-21}$    |
| King $^{8,9}$                   | $H_2$    | $2 \times 10^{-20}$ | $1 \times 10^{-20}$    |
|                                 | He       | $4 \times 10^{-20}$ | $1 \times 10^{-20}$    |
|                                 | $SF_6$   | $7 \times 10^{-21}$ | $5 \times 10^{-23}$    |
| Hugnes $^{13}$                  | CsI      | $4 \times 10^{-20}$ | $2 \times 10^{-15}$    |
| Zorn, Chamberlain and Hughes $^{15}$ | Cs      | $1 \times 10^{-17}$ | $5 \times 10^{-19}$    |
|                                 | K        | $4 \times 10^{-17}$ | $1 \times 10^{-18}$    |
|                                 | $H_2$    | $2 \times 10^{-15}$ | $1 \times 10^{-15}$    |
|                                 | $D_2$    | $3 \times 10^{-15}$ | $1 \times 10^{-15}$    |
| Fraser, Carlson and Hughes $^{16}$ | Cs      | $2 \times 10^{-18}$ | $1 \times 10^{-20}$    |
|                                 | K        | $1 \times 10^{-18}$ | $3 \times 10^{-20}$    |
| Shapiro and Estulin $^{14}$     | n        | $6 \times 10^{-12}$ | $6 \times 10^{-12}$    |
| Snell, Billman and Wedgwood $^{17}$ | n       | $2 \times 10^{-18}$ | $2 \times 10^{-18}$    |

Table 1.1 Comparison of Neutrality Measurements
In experiments of such sensitivity as claimed in the above table it is important to have results from experimental methods which are completely independent. The experiment described in this paper presents a new experimental method which is independent of the three existing methods previously described. The sensitivity of this experiment, as reported in the final section of this paper, has equaled the best published number for the upper limit of an electron-proton charge difference, \(1 \times 10^{-21}e\), but more important, this limit does not represent the ultimate sensitivity of the experiment. An improved sensitivity by a factor between \(10^2\) and \(10^3\) could be obtained with a reasonable amount of scaling of the hardware applied to this method.

This new method concerns the effect of an oscillating electric field on a gas contained within a high-Q acoustic cavity. The electric field is applied across the volume of the cavity at a frequency corresponding to an acoustic normal mode of the gas. It will be shown in Section 2 that if the gas molecules carry a charge, sound energy will be produced at the frequency of the electric field. Regardless of the presence of charge, the gas molecules have a finite polarizability, which will result in sound energy at twice the electric field frequency. Phase sensitive detection of sound energy at the fundamental frequency yields a measurement of the charge per molecule, and detection of the polarization signal yields a calibration for the experiment. Section 2 of this paper outlines the theory of the experiment in detail and concludes with an estimation of the experimental sensitivity. Section 3 describes the theoretical and experimental effort behind the design of a
nigh a acoustic cavity appropriate for the experiment. The cavity
design chosen was a sphere which exhibited Q's of over 1000 for a gas
pressure of 1 atm. The gas SF$_6$ was chosen for the final neutrality
measurements because of its acoustic properties and large mass number.
Section 4 describes the experimental apparatus in detail along with a
discussion of the major experimental problems which included: 1) elimi-
nation of electrostatic coupling and distortion of the electric field,
2) removal of ions from the gas, and 3) minimization of inherent
acoustic and electric noise sources. Section 5 describes the direct
calibration of the apparatus by calculation and measurement of the
sound produced by the polarization of the SF$_6$ molecule. The validity of
the polarization calculation is confirmed by comparison of the predicted
polarization signal for the N$_2$ molecule with the measured N$_2$ signal,
knowing the microphone response for an N$_2$ atmosphere. Section 6 describes
the final neutrality calculations and measurements with the results and
conclusions of the experiment. No signal at the fundamental frequency
was detected above the electrical Johnson noise of the detection cir-
cuitry. A charge per molecule of SF$_6$ of 2 x 10$^{-19}$ e would have been
detected with a signal to noise ratio of one in such a measurement.
This corresponds to an upper limit on the electron-proton charge differ-
ence of 1 x 10$^{-21}$ e.

This acoustic technique has several advantages over the other experi-
mental techniques previously discussed. The problem of ions and other
stray charged particles, which can seriously affect the measurements by
the gas-efflux and isolated body methods, is an easy problem to eliminate in the acoustic technique. Charged particles are easily removed from the gas in the acoustic cavity by application of a d.c. field prior to the neutrality measurements. In addition, if a number of ions were present in the cavity at the time of the neutrality measurements, their motion would couple only locally to the motion of the gas molecules. Thus, the effect of ions in the cavity is probably not important and need be considered only in the event of a non-null result. The acoustic technique is also free of the possible charge compensation effects which necessarily complicate the interpretation of the isolated body measurement.

It is doubtful that substantial improvements in the sensitivity of the experiments listed in Table 1.1 can be obtained without substantial improvements in "state of the art" electrical and physical measurement techniques, or without unreasonable expenditures of time, money, and effort. Thus, it can probably be said that the neutrality measurements exploiting these techniques have reached their "series limit" in terms of sensitivity. The author of this paper ventures the proposition that the results of this experiment have not reached the sensitivity limit of the acoustic technique. It will be shown that sensitivity of the experiment increases with the pressure of the gas, strength of electric field, and integrating time of the detection circuit. All three of these parameters were conservatively chosen for this initial experiment, and each could be increased by an order of magnitude with a reasonable amount of effort.
SECTION I

FOOTNOTES


5. G. Feinberg and M. Goldhaber, op. cit.


SECTION II
OUTLINE OF EXPERIMENT

The experiment is based on the effect of electrical forces on a collection of gas molecules, each of which is assumed to carry a small charge. Consider the contributions to the force per unit volume of a dielectric subjected to an electric field:

\[ F_v = gE - \frac{1}{\varepsilon \pi} E^2 \nabla \varepsilon + \frac{1}{\varepsilon \pi} \nabla \left( \varepsilon \frac{d\varepsilon}{dS_m} \right) \tag{2.1} \]

The first term is the ordinary electrostatic volume force when there exists a charge density, \( g \), in the medium. The second term, containing the gradient of the dielectric constant \( \varepsilon \), arises when there are inhomogenities in the medium. The third term arises when there exists inhomogenities in the applied electric field.

For the experiment being considered, a gas is held by some suitable container at a constant volume, pressure, and temperature. Also, experimental conditions are such that the ideal gas approximations are valid. Therefore, the medium is homogeneous and only the first and last terms of the volume force need be considered. The last term, known as the electrostrictive term, can be simplified further because the electrical equation of state of an ideal gas is a simple relation. The electrical equation of state of many substances is given by the Clausius-Mossotti equation:

\[ \frac{\varepsilon - 1}{\varepsilon + 2} = \frac{N_0 s_m}{3M} \alpha \tag{2.2} \]

The equation relates the bulk property given by the dielectric constant,
\( \chi \), to the microscopic quantity, \( \alpha \), the molecular polarizability of the substance. In the limit of an ideal gas, \( \chi \) differs from unity by only a few parts in \( 10^4 \), therefore equation (2.2) becomes:

\[
\chi - 1 = \frac{N_0 \sigma m}{M} \alpha
\]  
(2.3)

thus:

\[
\frac{\partial \chi}{\partial \sigma m} = \frac{N_0 \alpha}{M}
\]  
(2.4)

Substitution of the last expression into the electrostrictive term of equation (2.1), and using the fact that the mass density of ideal gas is uniform gives the following for the volume force on the gas:

\[
F_v = gE + \frac{1}{\beta n} \alpha \frac{N_0 \sigma m}{M} \nabla E^2
\]  
(2.5)

The factor \( \frac{N_0 \sigma m}{M} \) is just the number density, \( n \), of gas molecules.

If the charge density is assumed to be the result of a charge \( \epsilon \epsilon \) on each molecule (where \( \epsilon << 1 \), and \( \epsilon = 4.8 \times 10^{-10} \) e.s.u.) then \( g = n \epsilon \epsilon \). Therefore the force relation to be considered in the experiment is:

\[
F_v = n \epsilon \epsilon E + \frac{n}{\beta n} \alpha \nabla E^2
\]  
(2.6)

Now, assume that the applied electric field is harmonically time varying at the frequency \( \omega \), then the volume force will have an oscillating component at \( \omega \) due to the first term and a component at \( 2\omega \) due to the second term. Such time varying volume forces will give rise to harmonic pressure variations or sound waves in the gas. This can be shown as follows.
The gas can be characterized by an equilibrium value of the density, \( \rho_0 \), and its adiabatic compressibility, \( K_s \). We will look at relations between first order changes in the density, pressure, and velocity of the gas molecules, (denoted by \( g \), \( p \), and \( u \), respectively). By conservation of momentum:

\[
\rho_0 \frac{du}{dt} = -\nabla p + F_v
\]  

(2.7)

and the equation of continuity states:

\[
\rho_0 \nabla \cdot u = -\frac{dg}{dt}
\]  

(2.8)

Taking the divergence of the first equation, the time derivative of the second, and then adding the results, yields the following:

\[
\nabla^2 p = \frac{d^2 g}{dt^2} + \text{div} F_v
\]  

(2.9)

Substitution of \( g = \rho_0 K_s p \) into equation (2.9) results in a wave equation for first order pressure variations in the gas, where

\[
c = \frac{1}{\sqrt{\rho_0 K_s}}
\]

is the propagation velocity for the wave:

\[
\nabla^2 p - \frac{1}{c^2} \frac{d^2 p}{dt^2} = \text{div} F_v
\]  

(2.10)

Thus, a diverging volume force is seen to be the source term for sound waves in the gas. For the oscillating electric field previously considered, a sound wave will be produced at the fundamental frequency if the gas carries a charge, and at the second harmonic due to the polarizability of the gas molecules.
The basis of an experiment to define an upper limit for the charge of a molecule is now apparent. An apparatus is designed to detect the presence of sound waves of frequency $\omega$ in a gas that is subjected to an oscillating electric field of the same frequency.

1) If the medium is free of ions, 2) if care has been taken to prevent coupling between the electrical and acoustical circuits, and 3) if harmonic distortion in the electric field has been appropriately minimized to prevent coupling of the real signals at $2\omega$, then the detected sound wave at $\omega$ should be proportional to the charge on the molecule. Of course, a null result is expected. Therefore, the sensitivity of the experiment is limited by the ability to minimize the inherent noise sources (Johnson noise, background acoustics, etc.), and the effective elimination of three experimental problems mentioned above. A more complete discussion of these problems will be given with the details of the experimental apparatus.

To increase the sensitivity of the experiment the phenomenon of resonance is exploited. Until this point no mention has been made of the effect of the boundaries on the confined volume of gas. The boundaries establish a set of normal modes of oscillation in terms of which all sound wave motion can be represented. If sound energy is introduced to the cavity at a normal mode frequency, the oscillation would continue indefinitely if there were no losses in the cavity. However, there are loss mechanisms in the bulk fluid and at the boundary layers which cause the oscillation to decay. (These processes will be discussed in more detail in a succeeding section of this paper). A measure of the
losses in the cavity is the quality factor, or $Q$, of the system, defined as follows:

$$Q = \frac{\text{energy stored}}{\text{energy loss per radian}}$$  \hspace{1cm} (2.11)

The higher the $Q$, the smaller are the losses in the system. Also, an oscillation occurring at a cavity resonance is a factor of $Q$ larger in amplitude and narrower in band width than the oscillation occurring at a frequency far from a resonance.

Therefore, the experiment gains sensitivity by adjusting the frequency of the applied electric field to coincide with a cavity resonance and by discriminating against all but this single frequency in the detection circuitry. The important noise sources are wide band, so their minimization would be affected by the smallest practical band width in the detection circuitry. This is most easily accomplished using a phase sensitive detector which is referenced to the electric field frequency.

What kind of sensitivity can be expected ultimately? An order of magnitude calculation can be made using the Boltzmann factor. Assume the ambient pressure of the gas in the cavity to be $P_0$. When an alternating voltage $V$ is applied across the cavity, the pressure will adjust according to:

$$P = P_0 e^{-\frac{eMeV}{kT}}$$  \hspace{1cm} (2.12)

where, again, $eMe$ is the hypothetical charge on the gas molecules.
Since $\epsilon \ll 1$:

$$P \approx P_0 \left(1 - \frac{\epsilon M eV}{kT}\right)$$  \hspace{1cm} (2.13)

and

$$\Delta P = P_0 - P \approx \frac{P_0 \epsilon M eV}{kT}$$  \hspace{1cm} (2.14)

where $\Delta P$ is the magnitude of the pressure disturbance of the resulting sound wave. When resonance is exploited the pressure difference detected at the microphone diaphragm, $\Delta P_0$, will be $Q$ times $\Delta P$. Thus, the charge per molecule in units of the electronic charge is given by:

$$\epsilon M \leq \frac{\Delta P_0 kT}{P_0 Q eV}$$  \hspace{1cm} (2.15)

Dividing through by $M$ gives an expression for a fundamental electron-proton charge difference. To maximize the sensitivity of the experiment, $Q$, $P_0$ and $M$ should be maximized and $\Delta P_0$ (i.e. the minimum detectable sound), should be minimized. Assume the following to be attainable parameters in the experiment:

$$\Delta P_0 = 10^{-7} \text{ dynes/cm}^2$$

$$P_0 = 10^7 \text{ dynes/cm}^2 (=10 \text{ atm})$$

$$Q = 10^3$$

$$eV = 1.5 \times 10^{-8} \text{ ergs (V = 10kv)}$$

These values give an upper limit for the molecular charge of:

$$\epsilon M \leq 3 \times 10^{-23}.$$
SECTION II

FOOTNOTES

1. The experiment will be done at room temperature so $T$ is considered a constant.
SECTION III
DESIGN - HIGH Q CAVITY

The design problem requiring the most effort in the experiment was the design and understanding of a high Q acoustic cavity. In addition, the cavity design necessarily involved the maximization of the experimental parameters $P_0$ and $\mu$, since the type of gas and ambient gas pressure both affect the value of $Q$. Maximizing the $Q$ of an acoustic cavity necessitates an understanding of the important acoustic loss mechanisms in the cavity. The losses can be divided into three categories according to where they originate:

1) bulk fluid losses
2) boundary losses
3) microphone losses

The wave equation that was derived in the previous section did not include any of the effects of the thermal conductivity and viscosity of the gas molecules. These two transport phenomena are responsible for acoustic losses in the bulk fluid at most frequencies. In deriving the wave equation, the compressions were assumed to be purely adiabatic, i.e., there being no heat flow between regions of compression and rarefaction. This condition is only approximately satisfied in the bulk fluid and it is certainly not satisfied near the cavity boundary where $T = \text{constant}$. There will exist small temperature differentials between a compression and rarefaction with a resulting flow of heat between these two regions. (The heat flux, $J$, is given by the following where $K$ is the thermal conductivity of the gas: $J = -K\nabla T$.) As a result of
this thermal diffusion, the pressure change in the sound wave is not quite
in phase with the density change, and the energy is taken from the
sound wave during each cycle. To illustrate graphically, the energy
removed is equal to the area within the curve traced on the PV plane
by one cycle of the sound wave:

\[ P \]

\[ V \]

**Figure 3.1 Sound Wave Isotherms**

The lossless case where pressure is in phase with condensation is
illustrated by the dotted line. This diagram can be applied to the other
loss mechanisms in the bulk fluid. In general, any process causing a
phase difference between pressure and condensation will take energy from
the sound wave. In the case of thermal conductivity energy is trans-
formed from the orderly motion of the sound wave to the disorderly
motion of heat by thermal diffusion. A similar transfer occurs due to
the viscosity of the gas molecules. However, in this case the loss is
due to momentum diffusion. Viscosity displays itself as a gradient in
velocity perpendicular to the direction of motion.
Figure (3.2) shows two adjoining planes of gas molecules, with such a gradient in the $x$ direction.

\[ D_{xy} = -\mu \frac{\partial u_y}{\partial x} \]  \hspace{1cm} (3.1)

where $\mu$ is the coefficient of viscosity of the gas. Summation of the product of these shear stresses with the particle velocity yields the energy loss. To be more exact, the bulk viscosity, $\eta$, which is the resistance of the gas to compression, should be added to the stress terms, but $\eta$ is small enough to be neglected for mon-atomic gases so it will not be discussed further.

A wave equation must now be considered which includes the losses due to thermal conductivity and viscosity. Since the thermal conductivity and viscosity of a gas are relatively small, the loss of energy from the sound wave is sufficiently small to be described by the first order expressions. Thus, to this approximation the shape and nature of the
waveforms will be the same as the lossless case except for exponential decreases in the space or time coordinates. In the case of standing waves, which are of interest in this experiment, the damping in time of the waveform will be the important factor.

The mathematical model that includes thermal diffusion and momentum diffusion in the wave motion generates two types of "waves":

1) there is a propagational mode which obeys a damped wave equation; and
2) there are thermal and shear modes which both obey a diffusion equation. As is shown in Morse and Ingard\(^3\), the thermal and shear modes are important only near a boundary. If the propagational mode is adjusted to fit the acoustic impedance \( Z = \frac{P}{u} \) at the surface, it cannot at the same time satisfy the conditions of temperature and tangential velocity imposed by the boundary. The thermal and shear modes rapidly rise from zero at the boundary to cancel the temperature fluctuations and tangential velocity of the propagational mode within a distance given by:

\[
d_v = \sqrt{\frac{2\mu}{g_0\omega}} \quad \text{(3.2) for the shear mode, and}
\]

\[
d_h = \sqrt{\frac{2K}{g_0\omega C_p}} \quad \text{(3.3) for the thermal mode.}
\]

Outside these thin boundary layers (typically a fraction of a millimeter for air at standard conditions), the effect of these two modes is negligible. Thus, the thermal and shear modes can be considered as boundary loss mechanisms for the propagational mode. The energy loss per unit area because of thermal effects at the boundary is:
\[ \frac{L_{bh}}{\gamma - 1} = \frac{P_p}{2c_0 \gamma c^2} \omega d_k |P_p|^2 \]  

(3.4)

where \( P_p \) is the value of the propagational mode pressure at the boundary; and the loss per unit area because of viscous effects at the boundary is:

\[ L_{bv} = \frac{\rho_0}{2} \omega d_v |u_{\text{TAN}}|^2 \]  

(3.5)

If the bounding surface is rigid, (i.e. \( \rho_0 c / \gamma = 0 \)), then the above expressions integrated over the entire surface area give the total boundary loss. A perfectly rigid wall does not exist, but the condition can be approximated in experimental situations by the use of heavy wall materials or by thin walls surrounded by a medium whose acoustic impedance is largely mismatched from that of the cavity medium.

To return to the discussion of losses in the bulk fluid, only the familiar propagational mode need be considered. If the mode is to represent a traveling plane wave in space then the solution to the wave equation will contain a real \( \omega \) and complex \( k \) corresponding to attenuation in the spatial coordinate:

\[ p = P e^{-\alpha x - i(\omega t - kx)} \]  

(3.6)

where \( \alpha \) is called the attenuation coefficient of the medium. If the propagational mode is bound in space such that standing wave must be considered, then \( k \) is real, and \( \omega \) is complex corresponding to attenua-
tion in time:

\[ P = P_0 e^{-\gamma t - i(\omega t - kx)} \]

(3.7)

where \( \gamma \) is called the temporal decay constant. As long as frequencies are considered below \( 10^9 \) Hz, the propagation velocity is unchanged from the undamped case where purely adiabatic compressions were assumed. Thus the two constants are simply related by the magnitude of the velocity:

\[ c\alpha = \gamma \]

(3.8)

where \( c = \sqrt{\frac{1}{\rho_0}} k \) as given in equation (2.10).

Inspection of equations (3.6) and (3.7) show that when \( \alpha x = 1 \) or \( \gamma t = 1 \), the wave form has decayed by a factor of \( 1/e \). Near a resonance of an oscillatory system the \( Q \) can be defined as \( \pi \) times the number of cycles that occur while the system decays a factor of \( 1/e \).\(^4\) Thus in terms of the attenuation coefficient and decay constant the \( Q \) at the resonant frequency \( \omega_0 \) is equal to:

\[ Q = \pi \left( \frac{\omega_0}{2\pi} \frac{1}{c\alpha} \right) = \frac{\omega_0}{2c\alpha} \]

(3.9)

\[ = \frac{\omega_0}{2\gamma} \]

(3.10)

These expressions will be useful in calculating \( Q \) values when the attenuation coefficient is known, rather than starting with the more fundamental definition of \( Q \) given in Section 2.

The attenuation coefficient was first calculated for ideal gases
using kinetic theory by Stokes\textsuperscript{5} and Kirchhoff\textsuperscript{6}. In terms of the previously defined constants which describe a gas the classical attenuation coefficient is:

\[ \alpha = \left( \eta + \frac{4}{3} \mu \right) \frac{K}{2 c_0 c} \left[ 1 + \frac{K}{C_P} \frac{\gamma - 1}{\eta + \frac{4}{3} \mu} \right] \quad (3.11) \]

Table (3.1) lists values of \( \alpha \) calculated from equation (3.11), and the observed values for several common gases. (The contribution due to \( \eta \), the bulk viscosity, is ignored in this tabulation since it is negligible for monatomic gases and negligible at all but certain frequencies for the diatomic gases.) The second and third columns of the table list the separate contributions of thermal conductivity, \( \alpha \kappa \) and viscosity \( \alpha \mu \), to the total \( \alpha \). The agreement between calculated and observed values is reasonable only for the monatomic gases. Much of the discrepancy in the data of the diatomic gases can be explained by relaxation phenomena.
<table>
<thead>
<tr>
<th>GAS</th>
<th>$\alpha_k/\xi^2$</th>
<th>$\alpha_\mu/\xi^2$</th>
<th>$\alpha/\xi^2$</th>
<th>$\alpha_\text{obs}/\xi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$0.77 \cdot 10^{-11}$</td>
<td>$1.08 \cdot 10^{-11}$</td>
<td>$1.85 \cdot 10^{-11}$</td>
<td>$1.87 \cdot 10^{-11}$</td>
</tr>
<tr>
<td>He</td>
<td>0.22</td>
<td>0.31</td>
<td>0.53</td>
<td>0.54</td>
</tr>
<tr>
<td>O$_2$</td>
<td>0.47</td>
<td>1.14</td>
<td>1.61</td>
<td>1.92</td>
</tr>
<tr>
<td>N$_2$</td>
<td>0.39</td>
<td>0.96</td>
<td>1.35</td>
<td>1.64</td>
</tr>
<tr>
<td>AIR (DRY)</td>
<td>0.38</td>
<td>0.99</td>
<td>1.37</td>
<td>2.00</td>
</tr>
</tbody>
</table>


Up to this point purely classical kinetic theory has been used to discuss the bulk losses in fluids. However, for an adequate description of all the loss mechanisms certain quantum mechanical details must be included. A purely classical description is adequate for monatomic molecules where the only degree of freedom is translational. In such cases, the translational motion (and thus the pressure) adjusts immediately to the compression or rarefaction of a sound wave, thus there are no loss-incurring phase lags. More accurately, the translational states adjust in a time comparable to the collision times, $\tau_c$, between molecules. However, more complicated molecules will have various internal degrees of freedom corresponding to the vibrational and rotational quantum states. When a polyatomic gas is compressed, energy is fed much more slowly into these internal motions than the transla-
tional motion (i.e. in a time $\tau > \tau_c$). This phase lag results in less work being evolved on expansion than was absorbed during compression in a given volume of the gas. Therefore, the process transfers energy from the sound wave to the internal degrees of freedom, which in turn heats the gas. The effects of this loss mechanism will distinguish itself from the classical effects by its distinctive frequency dependence. The absorption will be a maximum when the period of the sound wave coincides with the relaxation time of a vibrational or rotational state of the molecule. When the frequency of the sound wave is much higher than the molecular relaxation frequency, then energy transfer is unable to occur and the gas behaves like a monatomic gas. If the frequency of the sound wave is much lower than the relaxation frequency, there may be enough time for the establishment of thermal equilibrium between the internal and external degrees of freedom, and thus no additional absorption. Therefore, relaxation phenomena as an important loss mechanism need be considered only when the frequencies are in the vicinity of a molecular rotational or vibrational state. The existence of such loss mechanisms in polyatomic gases was first proposed by Herzfeld and Rice.\(^7\)

Relaxation phenomena can be included in the attenuation coefficient by generalizing the specific heat of the gas to include the internal degrees of freedom. Recall that the speed of sound was computed to be

$$c = \sqrt{\frac{\kappa}{K_s} \rho_0},$$

(see Section 2). This can be written as

$$c = \sqrt{\frac{\kappa}{K_T} \rho_0} = \sqrt{\frac{P}{\rho_0}}$$

for an ideal gas where
\( \gamma \) is the ratio of specific heats, \( \frac{c_p}{c_v} = c_v + \frac{R}{c_v} \). Since the attenuation of a plane wave is given by the imaginary part of the wave number \( k = \omega/c \), knowledge of the dynamic behavior of \( \gamma \) (or \( c_v \)) is sufficient for the dynamic behavior of the attenuation factor.

The specific heat at constant volume, \( c_v \), can be considered as the sum of a part due to the external translational degrees of freedom, \( c_e \), and a part due to the internal rotational or vibrational degrees of freedom, \( c_i \).

In general, the specific heat for the \( n^{th} \) degree of freedom for an ideal gas is \( R/2 \) per mole times a function \( F_n(T) \), which varies from zero at \( T = 0 \) to an asymptotic value of one when \( T \) is much greater than \( T_n \), the characteristic temperature of the \( n^{th} \) degree of freedom:

![Graph](image_url)

**Figure 3.3** Specific Heat Function \( F_n(T) \)
The factor $F_n(T)$ and $T_n$ are calculated from the statistical mechanics of the gas. The three $F_n$'s for the three translational degrees of freedom have reached their asymptotic values long before the standard temperature regime. Thus, the external contribution to the specific heat, $C_e$, is just the constant $(3/2)k$. The $F_n$'s for the rotational states for all gases but hydrogen have also reached their asymptotic values by $0^\circ$C. However, for vibrational degrees of freedom, $F_n$ typically ranges between $0.1 - 0.1$ at this temperature. The heat capacity for the internal degrees of freedom, $C_i$, will be a sum over the rotational states ($2$ terms if diatomic and $3$ if polyatomic) plus a sum over the vibrational states ($3n-6$ terms for an $n$-atomic molecule).

To allow for the general frequency dependence of the specific heat due to the internal degrees of freedom, a dynamic heat capacity must be constructed which incorporates the relaxation frequency of each degree of freedom. A suitable dynamic heat capacity can be calculated with a simple harmonic temperature variation, such as that provided by a sound wave in the gas. When only one internal state is considered, the dynamic heat capacity has the form:

$$C_v(\omega) = C_e + \frac{C_i}{1 + i\omega\tau}$$  \hspace{1cm} (3.12)

where $\tau$ is the relaxation time of the state. Thus at low frequencies, such that $\omega\tau \ll 1$, $C_v(\omega)$ reduces to the equilibrium specific heat, $C_e + C_i$; and at high frequencies, such that $\omega\tau \gg 1$, the internal states are unaffected giving a specific heat of just $C_e$. 
The relaxation frequencies for rotational states are much higher than any frequency of concern in this experiment. The rotational states are more easily affected by collisions than the vibrational states, and as a result come into equilibrium with the translational states in a time not much larger than $\gamma_c$, ( $\gamma_c$ is approximately $10^{-9}$ sec for air at standard conditions). However, vibrational states have relaxation times of the order of milliseconds, which puts the corresponding relaxation frequencies in the midst of the audio spectrum. (In practical situations, small mole fractions of impurity gases can substantially reduce the relaxation frequencies of the host gas.) It will be shown for the audio frequencies that the effect of rotational relaxations contributes losses of the same order of magnitude as the thermal and viscous bulk losses, and the effects of vibrational relaxation need be considered only near the vicinity of a vibrational relaxation frequency.

For a simplified discussion only one internal state of the molecule will be considered. (The general discussion would include a sum over internal states). From equation (3.12), a dynamic value of $\gamma$ is given by:

$$\gamma' = \frac{C_\nu(\omega) + R}{C_\nu(\omega)} = 1 + \frac{R}{C_\nu(\omega)} = 1 + \frac{R}{C_e + \frac{C_i}{1 + i\omega \tau}}$$

(3.13)
This leads to a complex speed of sound, \( c' \):

\[
c' = \frac{c}{\sqrt{\varepsilon_0}} \left( 1 + \frac{R}{C_e + \frac{C_i}{1 + i\omega \gamma}} \right)^{1/2}
\]  

(3.14)

The attenuation coefficient is the imaginary part of \( \omega/c' \):

\[
\alpha = \text{Im} \frac{\omega}{c'} = \frac{\omega}{2c} \frac{RC_i}{C_e(C_e+R)} \left( \frac{\omega \gamma}{(1 + \omega^2 \gamma^2)} \right)
\]  

(3.15)

and the real speed of sound is the real part of equation (3.14):

\[
\nu = \frac{c}{\sqrt{\varepsilon_0}} \left( 1 + \frac{R[C_i + C_e(1 + \omega^2 \gamma^2)]}{C_v^3 + C_e^2 \omega^2 \gamma} \right)^{1/2}
\]  

(3.16)

Inspection of the last equation shows that molecular relaxations introduce dispersion into the medium. At low frequencies equation (3.16) reduces to \( c \), but at high frequencies equation (3.16) becomes:

\[
\nu(\omega \to \infty) = c \left[ \frac{C_v(C_e+R)}{C_e(C_v+R)} \right]^{1/2}
\]  

(3.17)

which is always greater than \( c \) since \( C_v > C_e \).

Equations (3.15) and (3.16) should describe the behavior of the sound attenuation and velocity for a diatomic molecule, since there is a single vibrational relaxation associated with this type of molecule. Figure (3.4) shows experimental data for the Fluorine molecule, \( T_n = 1284^\circ \text{K} \). The attenuation per unit wavelength, \( \alpha \lambda \), and velocity as functions of frequency are shown along with the theoretical curves which are calculated from equations (3.15) and (3.16) using a value 17kHz for the relaxation frequency, \( f_R = \frac{\omega R}{2\pi} \), and an Einstein model for the vibrational specific heat:
\[ C_i = \frac{R \left( \frac{T_n}{T} \right)^2 e^{\frac{T_n}{T}}}{(e^{\frac{T_n}{T}} - 1)^2} \] (3.18)

Figure 3.4 Sound Absorption and Dispersion in Fluorine at 102°C

It is important to notice that the excess attenuation at the relaxation frequency is approximately a factor \(10^3\) larger than the classical attenuation calculated from equation (3.11).\(^{10}\)

The experimental data for the oxygen molecule illustrates the effect of impurities on this single relaxation theory. Because of a high characteristic temperature \((T_n = 2239\,\text{K})\), the peak in the \(\alpha\)
vs f/p curve, and the frequency of this peak, $f_R$ are both small at room temperature. At one atmosphere, $\tau$ for the pure gas is 15.7 msec which corresponds to a relaxation frequency of only 10Hz. However, small amounts of water vapor in $O_2$ will shift $f_R$ to considerably higher frequencies. The explanation of this effect was first given by Tuesday and Boudart, and it involves the near coincidence in energy of the bending vibrational level of the water vapor molecule ($T_n = 2239^\circ K$) and the vibrational level of oxygen ($T_n = 2295^\circ K$). Since it is easier to exchange vibrational energy between molecules than to exchange vibrational and translational energy, the $H_2O$ vibrational levels provide a more rapid pathway for excitation and de-excitation of the $O_2$ molecules, thus shortening $\tau$ to $10^{-3}$ - $10^{-5}$ sec. and shifting $f_R$ to the range 1-100 KHz. Figure (3.5) shows the measured attenuation in dry air (b), air with 37% relative humidity (A) and the calculated classical attenuation (C):
Figure 3.5 Absorption Vs. Frequency in Air
The difference between curves (A) and (B) is due to the presence of water vapor and is presumably the result of the $\text{H}_2\text{O}$-$\text{O}_2$ resonance relaxations. (The nitrogen can be considered an inactive component of the mixture since its vibrational frequency is not in resonance with an $\text{H}_2\text{O}$ level and is far below the frequency range plotted in Figure (3.5)). The 50% discrepancy between curves (B) and (C) is apparently not accounted for by any other relaxation phenomena in $\text{O}_2$ or $\text{N}_2$, and may be due to the presence of impurities or other well hidden experimental errors. Even though the calculations on the attenuation of sound in air have been attempted since the days of Lord Rayleigh, many aspects of this common problem are yet to be understood.

One more aspect of relaxation phenomena need to be discussed before concluding the discussion of bulk acoustic losses. From the above remarks concerning vibrational relaxations, it is clear that vibrational effects are important only in the vicinity of the relaxation frequency and can be ignored at frequencies much larger than $f_R$. Usually $f_R$ is below the audio region, and as a result vibrational effects will be negligible in the range above .1 kHz.

The case of rotational relaxations involves the opposite frequency limit in the expression for the attenuation coefficient. The audio region is far below the rotational relaxation frequencies. Thus, taking the limit of $\omega \gamma << 1$ in equation (3.15) yields an attenuation per unit wavelength, $\alpha \lambda$, proportional to the relaxation time, $\gamma$:

$$\alpha \lambda = \left( \frac{2\pi}{3 Cp} \right) C_I \omega \gamma$$  \hspace{1cm} (3.19)
The small value of $\gamma$ means that the rotational states will come to equilibrium rapidly; therefore, the rotational heat capacity is proportional to a sum of $F_n$'s for the equilibrium distribution of $n$ rotational states. At $0^\circ C$ this summation has reached its asymptotic value of 2 for diatomic gases and 3 for polyatomic gases. The attenuation per wavelength for rotational effects is thus a number of the order of $\omega \gamma$. If the expressions for $\mu$ and $K/C_p$ in terms of the molecular mean free path, $l$, 

$$\mu = \frac{\rho g_0 c}{\sqrt{\gamma}}$$  \hspace{1cm} (3.20) 

$$\frac{K}{C_p} = \frac{1.6 \rho g_0 c C_N}{\sqrt{\gamma}}$$  \hspace{1cm} (3.21) 

are substituted into the classical attenuation coefficient, equation (3.11), then the following is obtained for $\alpha \lambda$:

$$\alpha \lambda = \frac{4}{3} \pi \frac{\omega}{\sqrt{\gamma}} \frac{l}{c} + \pi \frac{(\gamma-1)}{\gamma^{3/2}} \frac{\omega l}{c}$$  \hspace{1cm} (3.22) 

The factor $l/c$ is approximately the mean time between collisions $\gamma_c$, (since $c$ is approximately the molecular mean speed). Comparison of equation (3.19) to equation (3.22) shows that rotational relaxations contribute losses of the same magnitude as the bulk thermal and viscous losses since $\gamma$ is not much larger than $\gamma_c$.

The general results for the bulk fluid losses as given by equations (3.19), (3.21) and the boundary losses as given by equations (3.4), (3.5) will be applied to the calculation of the Q's for specific cavity geometries considered for the experiment. The total value of Q can be
calculated from the separate contributions, $Q_i$, of each loss mechanism by the following:

$$\frac{1}{Q} = \sum_i \frac{1}{Q_i} \tag{3.23}$$

More explicitly, this summation will include terms due to thermal boundary effects, $Q_{bh}$, viscous boundary effects, $Q_{DV}$, bulk viscous and thermal effects, $Q_{CL}$, relaxation effects $Q_K$, and microphone losses $Q_m$. As succeeding calculations will show, the boundary effects, $Q_{bh}$ and $Q_{DV}$, are the most significant contributions to the sum and the other effects can usually be ignored.

The two cavity geometries considered for the experiment were a cylinder and a sphere. As mentioned in Section 2, sound motion within a bounded region of space is completely described in terms of a complete set of eigenfunctions with the appropriate boundary conditions. Each eigenfunction represents a normal mode of vibration of sound within the cavity, and any arbitrary sound motion can be represented by a linear combination of the eigenfunctions (normal modes). The normal mode expansion for a cylinder is:

$$p(r, \phi, z) = \sum_{m,n,k} A_{m,n,k} J_m(k_{mn}r) \cos m\phi \sin k_z z e^{-i\omega t} \tag{3.24}$$

where $J_m$ is an $m^{th}$ order ordinary Bessel function. The Bessel functions represent radial normal modes, and the sine functions represent longitudinal normal modes. The constants, $k_{mn}$, $m$, and $k_z$ are determined from the boundary conditions imposed by the cylinder walls, where
the particle velocity must vanish. For a cylinder of length 1, and radius a, the boundary conditions are:

\[
\frac{\partial}{\partial r} P(a, \phi, z) = 0 = \frac{\partial}{\partial r} J_m(k m_n a) \quad (3.25)
\]

\[
\frac{\partial}{\partial z} P(r, \phi, 0) = \frac{\partial}{\partial z} P(r, \phi, 1) = 0 = \frac{\partial}{\partial z} \sin k_n z \quad (3.26)
\]

The normal mode expansion for a sphere is:

\[
P(r, \theta, \phi) = \sum \frac{A_m n s}{n} P_n^m(\cos \theta) \cos m \phi j_n(k n_s a) e^{i \omega t} \quad (3.27)
\]

where \( P_n^m \) is a Legendre polynomial of the second kind, and \( j_n \) an \( n^{th} \) order spherical Bessel function. Purely radial normal modes are represented by the spherical Bessel functions, and purely diametral normal modes are represented by the Legendre polynomials. As before, the constants \( k_{n_s} \) and \( m \) are determined from the condition that the particle velocity must vanish at the boundary of the sphere.

Two general observations on the Q of a cavity can be made from the knowledge of the normal mode expansion, and our previous results on loss mechanisms. Recall that the important acoustic losses increase with the frequency of the sound wave (directly with \( \omega \) for bulk losses, and as \( \sqrt{\omega} \) for boundary losses). Therefore, a cavity generally will have a higher Q for the lower frequency modes.

Also, it was shown that an important loss is the result of viscous shearing of longitudinal wave motion near a boundary. Such losses are absent in the purely radial modes of the spherical cavity. In this case,
wave motion is perpendicular to the cavity boundary and no viscous shearing can occur. Thus, the highest $Q$ is expected with the use of a spherical cavity whose first radial mode is excited. This fact will be illustrated by a specific calculation of the $Q$ for the first radial mode of a spherical cavity, and the first longitudinal mode of a cylindrical cavity. Experimental data for comparison with these calculations will be presented after the effects of microphone losses on actual $Q$ measurements are discussed.

The $Q$ of the longitudinal mode in the cylinder will be calculated first, since such a mode is a plane wave. The starting point is the definition of $Q$ given in equation (2.11).

$$Q = \frac{\text{energy stored}}{\text{energy loss per radian}} \quad (3.28)$$

The energy stored in the mode is the energy density integrated over the volume of the cylinder.

$$E_{\text{stored}} = \int_0^{2\pi} \int_0^a \int_0^L \left( \frac{p^2}{2\rho s^2} + \frac{\rho u^2}{2} \right) dz \, r \, dr \, d\phi \quad (3.29)$$

Substituting the spatial dependence of $p$ and $u$ for the first longitudinal mode:

$$p = P_0 \cos \frac{\pi}{L} z \quad (3.30)$$

$$u = u_0 \sin \frac{\pi}{L} z \quad (3.31)$$

and integrating yields:

$$E_{\text{stored}} = \frac{\pi a^2 L}{4} \left( \frac{\rho u_0^2}{2} + \frac{P_0^2}{\rho s^2} \right) \quad (3.32)$$
For a plane wave, the ratio of the magnitudes of the pressure and velocity is simply the acoustic impedance of the medium:

$$\frac{P_0}{u_0} = z = \frac{\rho_o c}{g_o} \quad (3.33)$$

Combining the last two expressions gives the energy stored in terms of the pressure magnitude:

$$E_{\text{stored}} = \frac{\pi \alpha^2 l}{2} \frac{P_0^2}{g_o c^2} \quad (3.34)$$

The total $Q$ is calculated from the separate contributions of boundary viscous losses, boundary thermal losses, and bulk losses. The power lost to the walls because of viscosity is given by the integral over the lateral cylinder area of equation (3.5)

$$P_{bv} = \int_0^l \int_0^{2\pi} u_o^2 \left( \frac{1}{2} g_o \omega d\Sigma \right) a d\phi d\Sigma \quad (3.35)$$

$$= \frac{\pi \alpha l}{2} g_o \omega d\Sigma = \frac{\pi \alpha l}{2} \sqrt{2 \mu g_o \omega} u_o^2$$

where the value of the boundary layer thickness, $d\Sigma$, is substituted from equation (3.2). The energy loss per radian is just the power loss divided by the circular frequency of the wave:

$$E_{\text{loss/rad}} = \frac{P_{bv}}{\omega} = \left( \frac{\pi \alpha l}{2} \right) \sqrt{\frac{2 \mu g_o}{\omega}} u_o^2 \quad (3.36)$$

Therefore, the contribution to the $Q$ because of viscous boundary losses:

$$\frac{1}{Q_{bv}} = \frac{\left( \frac{\pi \alpha l}{2} \right) \sqrt{\frac{2 \mu g_o}{\omega}} u_o^2}{\frac{\pi \alpha^2 l}{2} \frac{P_0^2}{g_o c^2}} = \frac{1}{\alpha} \sqrt{\frac{2 \mu}{\omega g_o}} = \frac{1}{\alpha} \frac{1}{\sqrt{\xi}} \quad (3.37)$$
where the last expression is valid for air at 1 atm. and 300°K. The power lost to the walls because of thermal conduction is the integral over the entire cylinder area of equation (3.4):

\[
P_{bh} = 2 \int_0^a \left( \frac{\gamma - 1}{2 \gamma \sigma c^2} \right) \omega dh \int_0^{2\pi} \cos^2 \frac{\pi z}{L} d\phi dz
\]

\[
+ \left( \frac{\gamma - 1}{2 \gamma \sigma c^2} \right) \omega dh \int_0^a P_0^2 \int_0^{2\pi} \cos^2 \frac{\pi z}{L} d\phi dz \int_0^{2\pi} \frac{\pi z}{L} d\phi dz
\]

\[
= P_0^2 \left( \frac{\gamma - 1}{2 \gamma \sigma c^2} \right) \omega dh \pi a (2a + L) \quad (3.38)
\]

Substituting for the thermal boundary layer thickness, \(d_h\), from equation (3.3) and dividing by the frequency gives the thermal energy loss per radian.

\[
E_{\text{loss}}/\omega = P_0^2 \left( \frac{\gamma - 1}{2 \gamma \sigma c^2} \right) \sqrt{\frac{2K}{\rho \omega c_p}} \pi a (2a + L) \quad (3.39)
\]

Thus, the contribution to the \(Q\) because of thermal boundary losses is:

\[
\frac{1}{Q_{bh}} = \left( \frac{2a + L}{aL} \right) (\gamma - 1) \sqrt{\frac{2K}{\rho \omega c_p}} (3.40)
\]

\[
= \left( \frac{2a + L}{aL} \right) (\gamma - 1) (1.25) \frac{1}{\sqrt{\nu}}
\]

where, again, the last expression is valid for air at 300°K and 1 atm.

The contribution to the \(Q\) because of bulk losses can be calculated from equation (3.10) and the measured value of the absorption coefficient for dry air given in Table (3.1):

\[
\frac{1}{Q_b} = \frac{1}{Q_{cl}} + \frac{1}{Q_R} = \frac{2ca}{\omega} = \frac{c}{\pi} \cdot 2.0 \cdot 10^{-13} \quad (c.g.s.)
\]

(3.41)

Thus, it can be seen that bulk effects contribute a negligible amount to the \(Q\) of an air filled cylinder in comparison to the boundary effects.
at audio frequencies. (Bulk effects are negligible for other gases except near the frequency of a molecular resonance.)

The experimental $Q$ measurements were made with a cylindrical cavity 38 cm. in length and 10 cm. in radius. For reasons that will become apparent in Section 4, a symmetric mode is best suited for the neutrality measurements, therefore the $Q$ measurements were made with the first symmetric mode or second longitudinal mode, $p = p_0 \sin \frac{2\pi z}{L}$. 

The frequency of this mode in a 38 cm. cylinder with air at 300 K and 1 atm is 900 Hz. Therefore, the $q$ of the cylinder at this frequency is:

$$Q = \frac{1}{Q_{bv}} + \frac{1}{Q_{bh}} \approx \frac{1}{7 \times 10^{-4} + 5 \times 10^{-4}} \approx 830$$

(3.42)

The above calculation is to be compared with the calculation of the $Q$ of the first radial mode of a spherical cavity. The energy stored in a sphere of radius $a$ is:

$$E_{stored} = \int_0^a \int_0^\pi \int_0^{2\pi} \left( \frac{p_0 u^2}{2} + \frac{p^2}{2 \rho c^2} \right) d\phi \sin \theta d\theta dr$$

(3.43)

The pressure in the first radial mode is:

$$p = p_0 j_0 (k_0 r) = p_0 \frac{\sin k_0 r}{k_0 r}$$

(3.44)

This is no longer a plane wave, but a spherical wave, so the appropriate relation between the pressure and velocity is:

$$u = - \frac{1}{i \omega p_0} \frac{dp}{dr}$$

(3.45)
Substituting equation (3.44) into the above equation, gives the following for the velocity in terms of pressure:

$$u = \frac{P_0 \sin k_{01} r}{\omega \rho_0 k_{01} r} \left(1 - \frac{i}{r}\right)$$  \hspace{1cm} (3.46)

Substituting this expression into equation (3.43) gives the following integrals for the energy stored:

$$E_{\text{STORED}} = 2\pi \frac{P_0^2}{\rho_0 c^2} \int_0^a \sin^2 k_{01} r \, dr \left(1 + \frac{1}{k_{01}^2}\right)$$

$$+ 2\pi \frac{P_0^2}{\rho_0 c^2} \int_0^a \frac{\sin^2 k_{01} r}{(k_{01})^2} \, dr$$

$$= 2\pi \frac{P_0^2}{\rho_0 c^2} \left\{ (1 + \frac{1}{k_{01}^2}) I_1 + I_2 \right\}$$  \hspace{1cm} (3.47)

The wave number $k_{c1}$ is determined from the boundary condition that $k_{01}^r$ equals the first zero of the function $j_1(k_{01} r) \sim \frac{d}{dr} \left(\sin k_{01} r/k_{01} r\right)$. For the 20 cm radius sphere used in the experiment, the equation$^{16}$

$20(k_{01}) = 4.49$ determines the wave number of the first radial mode as 0.228 and the frequency of the mode in air (at 300°K, 1 atm) as $ck_{01}/2\pi = 1270$ Hz. Evaluation of the integrals, $I_1$ and $I_2$, for $k_{01} = 0.288$ and $a = 20$ cm yields the following expression for the energy stored:

$$E_{\text{STORED}} \approx \frac{4\pi P_0^2}{2 \rho_0 c^2} (2 \cdot 10^2)$$  \hspace{1cm} (3.48)

There is now only one energy loss term to be evaluated, that being thermal conduction losses at the spherical boundary. The viscous losses vanish because of the purely radial particle velocities in the given
mode. The thermal energy loss per radian is:

\[
E_{\text{loss} / \text{Hz}} = \frac{1}{\omega} \int_{0}^{2\pi} \int_{0}^{\pi} \left( \frac{\gamma - 1}{2} \right) \omega \, d\omega \, \left( \frac{x^2 \sin^2 ka}{(ka)^2} \right) \sin \theta \, d\theta \, d\phi
\]

\[
= 4\pi \left( \frac{\gamma - 1}{2} \right) dh \, \rho_0^2 \frac{x^2 \sin^2 ka}{(ka)^2}
\]

(3.49)

and the boundary loss contribution to the Q of the first radial mode of the sphere with air at 300°K and 1 atm is:

\[
\frac{1}{Q_{bh}} = \frac{2(\gamma - 1) dh \sin^2 ka}{k \alpha} \approx 7 \cdot 10^{-6}
\]

(3.50)

The bulk loss contribution is not negligible at 1270 Hz in comparison to the above. It is calculated from equation (3.41):

\[
\frac{1}{Q_b} = \frac{c}{\pi} \cdot 2 \cdot 10^{-13} \gamma \approx 2.5 \cdot 10^{-6}
\]

(3.51)

Therefore the calculated value of the Q of the sphere is:

\[
Q = \frac{1}{Q_b} + \frac{1}{Q_{bh}} \approx \frac{1}{3.2 \cdot 10^{-6}} \approx 3 \cdot 10^5
\]

(3.52)

It is unlikely that a Q this large can be obtained in a practical situation for reasons that will be discussed with the data. However, the point remains to be emphasized, that a high-Q acoustic cavity can be made from a sphere that 1) is driven with a purely radial mode to eliminate viscous losses and 2) is large enough to minimize the thermal boundary losses owing to the 1/r dependence of spherical waves.

A high-Q acoustic cavity is useless as an acoustic amplifier if none of the stored energy is movable during each cycle of vibration. The microphones which are present in the cavity to detect the resonant sound energy contribute to the cavity losses in two ways: 1) the
microphones possess a radiation resistance since their diaphragms cannot be perfectly rigid surfaces, and 2) each microphone has an internal electrical and mechanical resistance in which sound energy is dissipated.

The microphones used in the experiment were chosen such that none of the above loss mechanisms were the limiting factors for the \( Q \) of the cavity. Two 2.5 cm diameter, 600 \( \Omega \) dynamic microphones were mounted on opposite ends of the cylinder, and on opposite sides of the sphere flush with the wall.

Some general remarks can be made concerning the pressure dependence of the microphones losses, but specific calculations of the contributions of these losses to the cavity \( Q \) will not be made. Instead, the effects of microphone losses are shown to be small by experimental measurements of the cavity \( Q \) vs. number of microphones, and \( Q \) vs. internal resistance of the microphones.

The energy dissipated in a microphone diaphragm of radius \( a \) because of its radiation resistance, \( R \), is first:

\[
E_{RR} = \frac{2n}{\omega} u_0^2 R(ka)
\]

(3.53)

where \( u_0 \) is velocity of the diaphragm.

An approximation for the radiation resistance is the resistance function, \( R_1 \), of a frictionless piston of radius \( a \), free to vibrate in a plane wall, with the acoustic medium on one side of the wall.
For small values of \( ka \), the first term of the expansion of \( R_1 \) can be used:

\[
R_1(ka) \approx \frac{90c}{ka} \frac{(ka)^2}{2} \quad \text{as} \quad ka \to 0.
\]  

(3.54)

The approximation \( ka \to 0 \) is valid with the chosen microphones, since the effective radius of the diaphragm, which is smaller than the radius of the microphone (\( a \approx 0.25 \text{cm} \)), is considerably smaller than the wavelengths of concern in the experiment, (\( \lambda_{\text{cyl}} = 38 \text{cm}, \lambda_{\text{sphere}} = 27 \text{cm} \)). If the mechanical impedance of the microphone diaphragm \( Z_m \) is known, then the diaphragm velocity can be written in terms of the sound pressure:

\[
u_o = \frac{P \cdot \pi a^2}{Z_m}.
\]  

(3.55)
Thus, the contribution to $Q$ because of radiation resistance would have the following form:

$$Q \sim \frac{\text{ENERGY STORED}}{E_{RR}} \sim \frac{\frac{p^2}{\rho c^2}}{\frac{P}{Z_m} \cdot \rho c} \sim \frac{Z_m^2}{\rho_0^2} \quad (3.56)$$

If the diaphragm were frictionless the mechanical impedance would reduce to the radiation impedance, which is proportional to $(\rho c \pi a^2)$. This shows that this contribution to the $Q$ is independent of the ambient pressure (i.e. independent of $\rho_0$), but such an approximation to the microphone radiation resistance cannot be used to calculate the magnitude of the contribution. The piston resistance function completely ignores the effect of the cavity on the diaphragm.

The semi-rigid nature of the microphone diaphragm prevents the radiation resistance from becoming a major energy loss, but semi-rigidity also requires dissipation in the mechanical resistance of the diaphragm. Energy is also dissipated in the resistance of the microphone coils. Both of these losses are independent of the ambient pressure, which means that the contributions to the $Q$ from these losses should depend on $1/\rho_0$ or vary inversely with the ambient pressure.

The quantitative effects of the radiation resistance and internal resistance of the microphones on the $Q$ of the cavity were shown to be small by the following measurements:

1) The $Q$ of the cylinder at the 900 Hz resonance with a 40 psi $N_2$ atmosphere, was measured with an increasing number of microphones attached to the cylinder end walls. One microphone was used as a sound source by
driving it with an audio oscillator, and a microphone on the opposite end of the cylinder detected the decay of the free resonant vibration following cut off of the source. Measurement of the decay time from an oscilloscope photograph of the output of the second microphone gave a measurement of the cavity Q. Additional microphones were then successively attached to the end walls, and their effect on the cavity Q was measured. Such a measurement shows not only the effect of the microphone radiation resistance on the Q, but also the effect of spatial asymmetries introduced in the cavity by the presence of the microphone housing. Plot (3.1) shows the results of these measurements for the successive addition of 10 microphones to the cylindrical cavity. The addition of one microphone is seen to affect the value of Q by no more than 10%.

2) The Q of the cylindrical cavity (under the same conditions as (1)) was measured with and without an additional 300 Ω of electrical resistance in the microphone circuit. The effect of this added resistance was measured by a measurement of the change in cavity band width.

The width of the resonance at half power points is just the resonant frequency, \( \omega_0 \), divided by the Q. For the 900 Hz resonance of the cylinder, this width is of the order of 1 Hz, (for 40 psi \( N_2 \)). If the cavity is excited by a signal at \( \omega_0 \) which is modulated by a signal of lower frequency, \( \omega_m \), the modulating frequency would have to be less than the width of the resonance to be detected in the cavity. Figure (3.7) shows the resonance spectrum superimposed on the spectrum of the exciting
carrier frequency and sidebands:

![Figure 3.7 Power Spectrum of Cavity, Carrier Signal, and Sidebands](image)

The cavity was excited with a carrier at $\omega_0$, modulated with a signal varying between 1 and 6 Hz. The decrease in per cent modulation was measured as the modulating frequency was increased beyond the effective band width of the cavity resonance. A second measurement was made with a 300 $\Omega$ resistor (which equals the d.c. resistance of one of the microphones) connected across the receiving microphone. The results of these two measurements are shown in Plot (3.2). No large change in the resonant band width is noted, even though the electrical dissipation of the microphone has been doubled in the second measurement. This most likely means that the electromagnetic coupling coefficient of the microphone is small.

Therefore, for the purposes of this experiment, microphone losses have been shown to have a small effect on the Q. Q measurements can now be compared with the previous predictions assuming boundary effects to
be the prime loss mechanism. $Q$ measurements were made as a function of pressure of two gases. The $Q$ of the cylinder and the sphere were measured with dry $N_2$, because the gas is free of relaxation effects at the frequencies of interest, and it has acoustic properties similar to air to facilitate comparison of the data with the theoretical calculations. In addition, the $Q$ of the sphere was measured with $SF_6$, since this was the final configuration chosen for the neutrality experiment. This gas is also free of relaxation effects in the vicinity of the first radial resonance\(^\text{18}\), (494 Hz) but more importantly the gas has a large mass number ($M = 146$) thus increasing the hypothetical charge per molecule.

The exact form of the pressure dependence of the $Q$ is difficult to predict because each of the various factors contributing to the total $Q$ has a different pressure dependence:

$$\frac{1}{Q} \sim A_{\text{rad}} + B P_{\text{mic. losses}} + C \frac{1}{\sqrt{P}} P_{\text{bound. losses}} + D \frac{1}{P} P_{\text{bulk losses}}$$

(3.57)

where the parameters $A$, $b$, $C$, $D$ are independent of $P$. The previous calculations and arguments have shown that the boundary loss terms (3.40) and (3.37) should dominate this sum for the cylinder, but the dominant terms in the case of the sphere are difficult to predict.

Plot (3.3) shows the results for the $Q$ vs $P$ measurements for $N_2$ in the cylindrical cavity, and Plot (3.4) shows the $Q$ vs $P$ measurements for the spherical cavity with both $N_2$ and $SF_6$ atmospheres. The cylinder was constructed from $1/2$" thick stainless steel enabling it to withstand
pressures to 120 psi. However, the sphere was constructed from 1/16" thick copper hemispheres soft-soldered together. The sphere could withstand a maximum pressure of only 16 psi. However, even at this low pressure the $Q$ of the sphere is as high as that of the cylinder at its maximum pressures, thus qualitatively confirming the theoretical predictions. The measured value of $Q$ for the cylinder at 1 atm agrees reasonably with the prediction of equation (3.42) and is consistent with the dominance of the boundary losses. Since such a limited range of pressure was investigated for the sphere, the linear increases with pressure are not unreasonable results. The measured values of the $Q$ for the sphere are considerably lower than the prediction of equation (3.52) for several reasons: 1) Viscous boundary losses are not entirely eliminated because of the presence of non-spherical asymmetries in the cavity. (The small electrode described in Section 4 was present in the cavity to obtain Q measurements valid for the neutrality measurement.) 2) Since boundary effects are much smaller than in the cylinder, microphone losses are more important and probably affect the $Q$ in a manner more dramatic than shown for the cylinder in Plots (3.1) and (3.2).
PLOT 3.3 Q vs. P MEASUREMENTS FOR CYLINDRICAL CAVITY
SECTION III

FOOTNOTES


2. For details pertaining to this model see P. M. Morse and K. U. Ingard op. cit., Sec. 6.4.


4. For a derivation of this formulation of $Q$, which is valid for $Q \gg 1$, see A. P. French, *Vibrations and Waves* (W. W. Norton and C., Inc. New York, 1971) p. 68.


10. For an approximate value of the classical absorption per unit wavelength, the value of $\alpha$ for a similar diatomic gas (N$_2$ or O$_2$) from Table 3.1 can be used. For an exact calculation using equation (3.11), the heat capacity of F$_2$ is given by L. G. Cole, H. Farber and G. W. Elverum, *J. Chem. Phys.* 20, 586 (1952), and the viscosity is given by E. Wicke and E. U. Franck, *Angew. Chem.* 66, 701 (1954).


14. The first radial mode is not the lowest frequency mode of a sphere. The first diametral mode, \( P_i(\phi) \sim P_j(\cos \phi) \) is more than an octave lower. See Lord Rayleigh, *The Theory of Sound*, pp. 265-267, for a more complete discussion of normal modes of a spherical cavity.

15. The radial modes appear in the normal mode spectrum at frequencies higher than the first two longitudinal modes for the given cylinder dimensions.


18. It should be noted that a gas also shows anomalous sound absorption in the vicinity of the critical point. Anomalous absorption in a regime extending approximately 1°C over the critical temperature, \( T_c = 45.2°C \), has been measured for SF\(_6\) by W. G. Schneider, *J. Chem. Phys.* 18, 1300 (1950).
SECTION IV
DESCRIPTION OF EXPERIMENTAL APPARATUS

The cavity used for the experiment was the copper sphere in which some of the measurements of the last section were made. In addition to the two microphones and gas intake on the periphery of the sphere, a high voltage electrode consisting of a 1" brass sphere was supported at the center of the sphere. Details of the sphere construction are shown in Figure (4.1).

The sphere was mounted on an elastic support in a large iron chamber which could be evacuated with a mechanical vacuum pump. Ports on the upper lid of the chamber provided exits for the microphone leads and entrances for the high voltage lead and gas intake to the sphere. All contacts to the sphere were made flexible to prevent coupling of external vibrations. The evacuation of the air around the sphere was an additional form of acoustic isolation. With a pressure of (100 \mu) Hg in the chamber, the microphones in the sphere were insensitive to room noises. A gas handling system attached to the gas intake port enabled the sphere to be evacuated or pressurized with any gas from standard high pressure cylinders. (See Figure (4.2) for details).

The Detection Circuit

The detection circuit begins with the microphones and ends with an analog signal on a strip chart recorder proportional to the pressure differential at the microphone diaphragm. The microphones are dynamic, (Lafayette Pa-74), microphones with an impedance of 600\Omega each at 1 KHz (300\Omega of which is d.c. resistance). They are wired in series
with the primary of a matching transformer such that the common mode sound signals in the circuit add. A circuit employing the addition of differential mode signals would probably discriminate against background acoustics and vibrations, but these noise sources are several orders of magnitude smaller than the pickup due to electrostatic coupling of the high voltage circuit. The common mode arrangement described here exploits the symmetry of the cavity to enable cancellation of the pickup with the addition of the sound signals. (Details of this property of the circuit can be found under the description of the high voltage driving circuit). The matching transformer matches the impedance of the microphones to the input of a Hewlett-Packard 466A pre-amplifier. The output of the pre-amplifier is matched to the input of a PAR JB-5 lock-in amplifier with an additional transformer. The reference frequency for the lock-in amplifier is provided by the same audio oscillator which provides the initial signal for the high voltage driving circuit. The output of the lock-in amplifier, which is the final form of the experimental data, is monitored and recorded on a Moseley 680 strip chart recorder. (See Figure 4.3).

The total electrical gain of the system was measured by substituting a 600Ω resistor for the microphones and introducing a small signal from a General Radio audio frequency microvolter. The measured value of the gain was 6.7 x 10^7 at 1270 Hz, which is the frequency of the fundamental radial resonance of the spherical cavity containing an atmosphere of N₂.
FIGURE 4.1  SPHERICAL CAVITY
Figure 4.2 Gas Handling System
Figure 4.3 Detection Circuit
There are two major noise sources in the detection circuit which must be reckoned with. There will be an electrical Johnson noise in the circuit due to the internal resistance of the microphones. Johnson noise arises from the particulate nature of the electronic charge and the resulting statistics which govern their movement. In a circuit of resistance $R$, with bandwidth $\Delta f$, and temperature $T$, the mean square noise voltage is given by:

$$\langle \delta V^2 \rangle = 4kTR\Delta f$$  \hspace{1cm} (4.1)

Thus, for a $1$ Hz bandwidth the noise voltage for the microphones is $\delta V = 1 \cdot 10^{-9}$ volts. This value is the largest source of electrical noise in the circuit, being several orders of magnitude larger than the noise sources in the pre-amplifier and lock-in amplifier.

However, an additional noise source is present in the system due to the random molecular impacts of the cavity gas on the microphone diaphragm. Using the acoustic radiation resistance, $R_f$, of a diaphragm of area $S$, an analogy can be made to the electrical Johnson noise:

$$\langle S \cdot P \rangle^2 = 4kTR_f\Delta f$$  \hspace{1cm} (4.2)

For an order of magnitude calculation, some approximations will be made. The radiation resistance of a rigid, massless piston of area, $a$, free to vibrate in an infinite, rigid wall, and exposed to the atmosphere on one side is as follows:

$$R_f = \pi a^2 \rho c \left[ 1 - \frac{J_1(2ka)}{ka} \right]$$  \hspace{1cm} (4.3)
where $J_1$ is a first order Bessel function. (This function is plotted in Figure (3.6)). For $a \ll \lambda$:

$$1 - \frac{J_1(2ka)}{ka} \approx \frac{(ka)^2}{2}$$

(4.4)

Therefore, $\bar{P}$, the rms noise pressure in the interval between $f_1$ and $f_2$ is:

$$\bar{P} = \left[ \int_{f_1}^{f_2} P^2 df \right]^{1/2} = \left[ \frac{8\pi g_o kT}{3c} \left( f_2^3 - f_1^3 \right) \right]^{1/2}$$

(4.5)

Assuming again a 1Hz bandwidth around 1270 Hz, the calculated value of $\bar{P}$ is $1.8 \times 10^{-7}$ dy/cm². The microphone response, $\delta V/\delta P_0$, must be measured at 1270 Hz to compare this noise source to the previous noise source. (The response will also be useful as an indirect means of calibrating the apparatus. See Section 5.)

To maintain simplicity in calculations microphones are calibrated either within a cavity whose acoustic impedances are known or under free-field conditions provided by an anechoic chamber. Two careful measurements were made on the response of the dynamic microphones in the anechoic chamber at the Research Laboratory of Electronics of M.I.T. (Since the acoustic impedance of the medium is an important factor in the response of a microphone, these measurements will be valid only for experiments involving air or nitrogen.) The first measurement was indirect, employing a calibration using the principle of reciprocity. ³ Since this particular work was done early in the evolution of the experiment, the measurements were done at the fundamental frequency of the
cylinder, (900 Hz). The measured response of two properly phased microphones at this frequency was $6.0 \times 10^{-4} \text{v/(rms)/dy/cm}^2$ using the calibrated microphone, (see Plot 4.1), and $5.3 \times 10^{-4} \text{v/(rms)/dy/cm}^2$ using the reciprocity measurement. Subsequent calibrations at cylinder resonances, and later, sphere resonances were made in a less sophisticated environment. A makeshift anechoic chamber was fashioned from a cardboard box (34cm x 42cm x 50cm) lined with several layers of fiberglass. Owing to the small size of the microphones (1" diameter), no significant differences were noted in reciprocity measurements in the chamber and the full-fledged anechoic chamber. (For the purposes of this measurement in the experiment, 10% accuracy is more than sufficient). The measured response of the microphones at the frequency of the $\pi_2$ sphere resonance (1270 Hz) was: $3.5 \times 10^{-4} \text{v/dy/cm}^2$. Multiplying this response by the calculated pressure of the acoustic Johnson noise gives an electrical noise of $5.7 \times 10^{-11} \text{volts}$ for a .1 Hz bandwidth, which is two orders of magnitude smaller than the electrical Johnson noise in the circuit.

There is one other noise source which must be considered before interference signals are discussed. The background acoustics could easily produce voltages in the detection circuitry larger than the inherent noise voltage if suitable precautions were not taken. The shock mount and evacuated chamber which house the sphere provide the necessary acoustic isolation. Plot (4.2) shows the output of the strip chart recorder with a 600Ω resistor in place of the microphones. It thus shows the approximate value of the Johnson noise in the circuitry for a .1 Hz bandwidth as $\delta V \approx 15 \times 10^{-9} \text{volts}$ referred to the input. Plot
PLOT 4.2 TOTAL ELECTRONIC NOISE WITH 600Ω RESISTOR IN PLACE OF MICROPHONES: $\delta V \approx 1.5 \cdot 10^{-9} V$
REFERRED TO INPUT ($f = 494 \text{ Hz}$, $\Delta f = 0.1 \text{ Hz}$)

PLOT 4.3 TOTAL NOISE WITH 12 psi SF₆ IN CAVITY
($f = 494 \text{ Hz}$, $\Delta f = 0.1 \text{ Hz}$)
(4.3) shows the output of the microphones with 12 psi SF\textsubscript{6} in the cavity. The output is essentially unchanged. Therefore, the background acoustic noise is below the level of the Johnson noise.

If the sensitivity of the experiment is limited by the Johnson noise, then the smallest detectable signal with a signal to noise ratio of one at a 0.1 Hz bandwidth would be 1 x 10\textsuperscript{-9} V. This would correspond to the detection of a 1270 Hz sound wave with a pressure differential of 1.4 x 10\textsuperscript{-5} dy/cm\textsuperscript{2} at the microphone diaphragm.

This sensitivity has importance in the experiment if there are no signals at the fundamental frequency upon application of the electric field to the cavity, that would mask the presence of the signal from the assumed charge difference. There are several such interfering signals that arise from the high voltage driving circuit that must be eliminated or minimized. The construction of this circuit will be discussed first.

**High Voltage Driving Circuit**

The production of high voltages at power or radio frequencies is not a difficult task using common electronic components. However, the production of high voltages at audio frequencies is difficult without specifically designed equipment which is often large in terms of physical size and power consumption. Of the practical designs considered, the cleanest and highest voltage signal was obtained from a 30 kv 60 Hz transformer which delivered 6 kv (rms) when driven at 500 Hz.

The audio signal must be a clean sinusoid for the following reason: any asymmetric distortion of the waveform such that a net d.c. or a
significant amount of second harmonic distortion is present will couple the polarization signal at $2\omega$ to the fundamental frequency, $\omega$. This is one type of interference signal which would mask a signal caused by a charge difference. Suppose there is a net d.c. offset in the driving voltage, $E_0\sin\omega t$, equal to $\delta E_0$, where $\delta$ is some fraction. The polarization signal, $P_{pol}$, was previously shown to be proportional to the square of the driving voltage (see equation (2.6)). Therefore:

$$P_{pol} \sim E_0^2 (\delta^2 + 2\delta \sin \omega t + \sin^2 \omega t)$$

$$= E_0^2 (\delta^2 + \frac{1}{2} + 2\delta \sin \omega t - \frac{1}{2} \cos 2\omega t) \quad (4.6)$$

Keeping only the a.c. terms in the above expression, since these are the only terms which give rise to sound waves:

$$P_{pol} \sim 2\delta E_0^2 \sin \omega t - \frac{1}{2} E_0^2 \cos 2\omega t \quad (4.7)$$

Thus, a d.c. offset as small as .25% of the a.c. voltage would couple 1% of the polarization signal to the fundamental frequency. Similar calculations were performed assuming distortion of the waveform by higher harmonics. As a result, the only harmonic which will couple the signal into the fundamental is the second. The coupling turns out to be approximately as strong as that caused by a d.c. offset, but in this case the coupled signal is $90^\circ$ out of phase from that produced by a charge difference. Assuming a fraction $\delta$ of the second harmonic in the driving voltage:

$$P_{pol} \sim E_0^2 (\delta \sin 2\omega t + \sin \omega t)^2 \quad (4.8)$$
Keeping only the a.c. terms:

\[ P_{\text{POL}} \sim E_0^2 \left( -\frac{b^2}{2} \cos 2\omega t + 2b \sin 2\omega t \sin \omega t - \frac{\cos 2\omega t}{2} \right) \]  \hspace{1cm} (4.9)

The first term of a Fourier sine or cosine expansion of the middle term in the above expression gives the component of the signal coupled into the fundamental:

\[ 2b \sin 2\omega t \sin \omega t = \sum_{m=1}^{\infty} A_m \sin m\omega t + \sum_{n=1}^{\infty} B_n \cos n\omega t \]  \hspace{1cm} (4.10)

\[ A_1 \] turns out to be zero, whereas \[ b_1 = \frac{1}{2} \]. Therefore, .5\% second harmonic distortion would couple 1\% of the polarization signal to the fundamental. However, this possible source of interference can be eliminated in the detection circuit. The lock-in amplifier is essentially a phase sensitive detector and can discriminate against the coupled signal which is 90° out of phase with the wanted signal.

The choice of a transformer for the high voltage circuit over a vacuum tube design considered was aided by the fact that no d.c. can leak through a transformer. This minimizes distortion of the first type that was considered. A vacuum tube amplifier is usually coupled to the electrode by means of a capacitor. Bothersome d.c. offsets could easily be built up between the capacitor and ground. However, even with the use of a transformer, a net d.c. could be present in the secondary waveform if some part of the secondary circuit causes an asymmetric distortion of the waveform. Arcing in the secondary circuit can cause such asymmetries. This resulted in the only significant distortion problem encountered in the experiment. Figure (4.4) shows a plot of the breakdown potential
versus pressure between plane parallel electrodes separated a distance, \( d \), in air: \(^4\)

![Graph showing breakdown voltage between plane parallel electrodes in air.](image)

**Figure 4.4** Breakdown Voltage between Plane Parallel Electrodes in Air

The minimum of the curve occurs at a pressure very near the obtainable pressure in the vacuum chamber when the vacuum pump has been pumping an hour or more. Therefore, it was not a surprising fact that experimental difficulties were encountered in preventing arcing in the high voltage connection through the vacuum chamber to the electrode on the cavity. In particular, minor arcing at 1000 volts, which produced none of the dramatic effects of complete breakdown, was nonetheless capable of sufficiently distorting the waveform for visibility on an oscilloscope. (This usually requires at least 5% distortion to be discernible with the eye). The arcing caused a series of small spikes riding on top of the waveform in the first quadrant, thus adding asymmetrically to the signal and giving rise to a net d.c. offset. The problem was eliminated by raising the vacuum chamber pressure to 10 mm, which was far enough from the minimum of the breakdown curve to prevent breakdown up to 5.0 kV, and still low
enough, surprisingly, to provide acoustic isolation. (One must consider that the acoustic impedance at this pressure is almost two orders of magnitude smaller than the impedance of the medium in the cavity).

The high voltage driving circuit in its final form is shown in Figure (4.5). The initial signal is provided by a Hewlett-Packard 202c audio oscillator, which also supplies the same signal for the reference and balance circuits discussed in the next section. The signal is amplified by a Macintosh HC75 audio amplifier and then fed into the primary of the high voltage transformer (M.E. Transformer Co. #8142). The secondary is connected to a Kovar on the vacuum chamber with 2m of 5kv insulated wire, and the Kovar feed-through on the chamber is connected to the electrode Kovar on the cavity by means of 40kv insulated wire. Also, large Teflon bushings with generous amounts of high voltage grease (Dow Corning 4 Compound) prevent any metallic part carrying high voltage from being exposed to the atmosphere within the vacuum chamber. The return path is from the cavity walls to the chamber walls, to an earth ground connection, through a switched, 260 volt battery to the low side of the secondary. The purpose of the battery is two-fold: 1) a d.c. voltage is needed to sweep the cavity of ions and charged particles that might interfere with the measurements; and 2) a d.c. voltage superimposed on the a.c. voltage produces a signal of the same frequency and phase as that produced by a hypothetical charge difference. Whereas, in a previous discussion of this fact it was considered a nuisance when occurring circumstantially, the effect will serve as a useful means of tuning the detection circuit when a d.c. voltage is intentionally applied.
balance and reference circuits

The last part of the experimental apparatus to be described concerns the circuits which eliminate the largest noise sources in the experiment. A balance circuit subtracts electrostatic and magnetic pickup of the high voltage signal by the microphone circuit, and an isolated reference signal, in conjunction with careful placement of other ground connections, prevent coupling of the driving signal to the detection circuit by means of ground loops.

With the high voltage electrodes in relative proximity to the microphones, which are placed in a circuit responding to voltages $10^{12}$ times smaller, electrostatic pickup is the most serious problem. The electrostatic coupling can be represented by a stray capacitance between the electrode and each end of the inductances in the microphones, as shown in Figure (4.6):

![Electrostatic Coupling Diagram](image)

Figure 4.6 Electrostatic Coupling Between High Voltage and Detection Circuits
Grounding of points 2 and 3 (to the cavity) for proper phasing to add the common mode sound signals, eliminates the stray capacitance to these points. This leaves capacitance to the opposite ends of the two microphones. Consider an instant when the voltage appearing on the electrode is positive. This will induce negative voltages at the points 1 and 4, that will cancel each other in the primary of the matching transformer if the circuit were completely symmetrical. Of course the symmetry is not perfect, distances are not exactly equal, and the impedances of the microphones differ slightly. Therefore, the cancellation is not complete and a slight amount of electrostatic pickup remains.

Stray magnetic fields from the audio amplifier and high voltage transformer are also capable of inducing detectable amounts of pickup in microphone-transformer loop. (Pickup can be identified as being magnetic rather than electrostatic if it is dependent on the area a particular circuit encloses rather than its distance from the source.) The magnetic pickup was minimized by placement of the cavity along a line of minimum stray magnetic field, and keeping the length of the microphone leads as short as possible.

The residual value of the combined pickup was typically $10^{-5}$ volts at the input to the lock-in amplifier. An equivalent amount of signal of the same frequency but opposite phase was added to the input of the amplifier to cancel the residual pickup. The circuit which provides this balance signal is shown in Figure (4.7). The operational amplifier is used to provide a signal which is adjustable in amplitude and either
in phase or out of phase with the input signal. The input of the balance circuit is isolated from the audio oscillator, therefore, it provides a convenient reference signal for the lock-in amplifier which is not coupled to the driving circuit.
FIGURE 4.7 BALANCE AND REFERENCE CIRCUIT
SECTION IV

FOOTNOTES

1. No improvement of the $Q$ of the cavity was noted with the evacuation of the air surrounding the cavity. This shows that sound radiation from the cavity, a loss mechanism not considered in the previous section, is not significant. This loss would not be negligible if a liquid, instead of a gas, were used as the acoustic medium within the cavity.


SECTION V
CALIBRATION

The experiment is calibrated by measurement of the polarization signal occurring at twice the fundamental frequency. As shown previously, this signal is the result of the second term in the volume force equation:

\[ F_{V(\text{POL})} = \frac{n}{\delta \pi} \alpha \nabla E^2 \]  \hspace{1cm} (5.1)

The field within the cavity when a voltage \( V \) is on the electrode is given by

\[ E = \frac{C V}{r^2} \]  \hspace{1cm} \text{where}  \hspace{1cm} (5.2)

\( C \) is the capacitance of the electrode and \( r \) the radial distance from the electrode center. Since the radius of the sphere \( a \) is considerably larger than the radius of the electrode \( r_0 \), the sphere as an outer boundary can be ignored and the capacitance is just that of the electrode.\(^\text{1}\) In c.g.s. units the capacitance of a sphere is equal to its radius. Therefore the volume force is:

\[ F_{V(\text{POL})} = \frac{n}{\delta \pi} \alpha r_0^2 V^2 \nabla \frac{1}{r^4} \]  \hspace{1cm} (5.3)

The wave equation for pressure must be solved with the above as the source term. The result will be combined with the measured polarization signal to obtain a calibration \( \frac{\delta V_{\text{sig}}}{\delta P_0} \) for the experiment. This calibration will be checked by a separate measurement of the microphone response and detection circuit gain.
The equation to be solved is:

\[
\nabla^2 p - \frac{1}{c^2} \frac{d^2 p}{dt^2} = \text{div} \vec{F}_v
\]

Substituting equation (5.3) for the volume force and taking the divergence yields

\[
\nabla^2 p - \frac{1}{c^2} \frac{d^2 p}{dt^2} = \frac{3n}{2\pi} \alpha \frac{r_0^2 V_0^2}{r^2} \left( \frac{1}{r^6} \right)
\]

The calibration is desired for the first radial mode of the spherical cavity, designated by the frequency \( \omega_{001} \). Therefore, the voltage will be driven at a frequency equal to \( \omega_{001}/2 \) to produce a polarization varying at \( \omega_{001} \). Assuming such a time dependence in the voltage, i.e. \( V = V_0 e^{-i \omega_0 t} \), the time dependence in the wave equation can be factored out. (Since the cavity has a high Q and the medium and transducer are assumed to be linear to the necessary approximation, only one mode in the normal mode expansion need be considered. In other words, if the cavity is driven at \( \omega \), only an oscillation at \( \omega \) will be excited.) The equation can be further reduced to one variable because the first radial mode of a spherical cavity possesses no angular dependence:

\[
(\nabla^2 + k^2) p(r) = \frac{3n}{2\pi} \alpha \frac{r_0^2 V_0^2}{r^2} \left( \frac{1}{r^6} \right) = B \left( \frac{1}{r^6} \right)
\]

The equation will be solved by a method which is somewhat more general than needed, but this solution was instructive for the author and hopefully will be so for the reader. The solution involved the Green's function for the particular geometry that was chosen for the cavity. A
Green's function \( G(R, R') \) is the solution at the point \( R \) to a differential equation, with a particular set of boundary conditions, when a unit source is present at the point \( R' \). For an arbitrary source distribution, say \( f(R') \), a particular solution to the equation is given by:

\[
P(R) = \int G(R, R') f(R') dR'
\] (5.7)

Therefore, if the Green's function is known, it offers a straightforward method of solution to an equation for quite arbitrary source functions.2

In the problem at hand, the Green's function is needed for the wave equation in a spherical interior with Neumann boundary conditions. Green's functions are usually expressed in terms of expansions of eigenfunctions satisfying the appropriate boundary conditions. Only one mode of the wave equation is being excited, hence, only one term of the Green's function is needed:

\[
G_{001} = \frac{2}{a^3} \frac{1}{j_0^2(k_0 a)} \frac{1}{k_{01}^2 - k^2} \left[ j_0(k_{01} r) j_0(k_{01} r') \right]
\] (5.8)

(The complete Green's function is given in the Appendix). The Green's function necessarily diverges at resonance (i.e. when \( k_{01} = k \)), because no damping has yet been included in the solution. An equation equivalent to equation (5.7) is now used to find the pressure for the polarization source term:

\[
P = \frac{2}{a^3} \frac{j_0(k_{01} r)}{j_0^2(k_{01} a)} \frac{1}{k_{01}^2 - k^2} \times \int_0^{2\pi} d\phi' \int_0^{\pi} \sin \theta' d\theta' \int_{r_0}^{a} r'^2 dr' \left[ j_0(k_{01} r') \right] \frac{B}{r'}
\] (5.9)
The primed (or source) coordinates span the volume between the spherical electrode and the outer spherical cavity. The angular integrals integrate immediately to \(4\pi\), while the radial integration yields the value \(8.48 \times 10^{-2} \ (k_{o1})^3\) with the limits of \(a = 19.7\) cm and \(r_0 = 1.27\) cm. The integrated expression is thus:

\[
P = \frac{8\pi}{a^3} \frac{j_0(k_{o1}r)}{j_0(k_{o1}a)} \frac{B(k_{o1})^3(8.48) \cdot 10^{-2}}{k_{o1}^2 - k^2}
\]  

(3.10)

For the pressure appearing at the microphones which are mounted on the periphery of the sphere \(r = a\) is substituted into the above:

\[
P(a) = \frac{8\pi}{a^3} \frac{B(8.48 \cdot 10^{-2})}{j_0(k_{o1}a)} \frac{(k_{o1})^3}{(k_{o1}^2 - k^2)}
\]  

(3.11)

The effects of damping must now be included to calculate the sound pressure at the resonant frequency. Our ignorance about the exact nature of the acoustic losses can be lumped into one parameter, the \(Q\) of the cavity. The sound pressure at zero frequency, \(P_0\) (a) is found by substituting \(k = 0\) into equation (3.11). The sound pressure at the resonant frequency is then given by the product of \(P_0\) (a) and \(Q\), where the value of \(Q\) is experimentally determined. (See Fig. 3.1).
Therefore, the sound wave due to the polarizability of the gas should have a pressure at the microphone given by:

$$P_0(\text{pol}) = Q \frac{P_0(a)}{a^3 \Delta o(k_o/a)} = \frac{Q \Delta \pi \times (8.48 \times 10^{-2}) \times (k_o/a) \times B}{\alpha \times n \times r_0 \times v^2}$$  \hspace{1cm} (5.12)

where $B = \frac{3}{2\pi} \alpha \times n \times r_0^2 \times v^2$

Typical values of the experimental parameters in the above expression are:

- $n = 2 \times 10^{19}$
- $v = 3600v \text{ (rms)} = 12 \text{ e.s.u.}$
- $r_o = 1.27 \text{ cm}$
- $a = 19.7 \text{ cm}$
\[ k = 0.223 \]

\[ \alpha = 6.54 \times 10^{-24} \text{cm}^3 \quad (\text{SF}_6) \]

\[ \eta = 1100 \]

Substitution of these values yields a pressure of \( 4 \times 10^{-3} \text{dy/cm}^2 \) for the polarization signal.

The use of the above calculation as a calibration for the experiment assumes that there are no additional signals produced at twice the driving frequency other than the polarization signal. This condition is not strictly satisfied because the electric field exerts mechanical forces on the electrode, and since these forces are proportional to the square of the induced charge density, the resulting sound waves also occur at twice the driving frequency. Fortunately, the strength of these sound waves can be shown to be several orders of magnitude smaller than the polarization signal, and thus be of no further concern.

First of all only the forces on the electrode need to be considered, since the electric field is of negligible strength at the cavity boundary. The force per unit area on the electrode is given by:

\[ \frac{F}{A} = 2\pi \sigma^2 = \frac{V^2}{8\pi \varepsilon^2} \]

(5.13)

where \( \sigma \) is the induced charged density. At 3600 volts (or 12 e.s.u.) \( F/A = 3.6 \text{ dynes/cm}^2 \). Using a value of \( 10^{11} \) for an approximate bulk modulus of the electrode, (brass sphere, 1" diameter), the change in radius under the influence of such a force can be calculated:
\[
\frac{P_2 - P_1}{\nu_2 - \nu_2} = 10^{11}
\]
(5.14)

\[\Delta V = (P_2 - P_1) \nu_1 \cdot 10^{-11} \approx 3 \cdot 10^{-10} \text{cm}^3\]
(5.15)

\[\Delta r = \frac{\Delta V}{4\pi \nu_0^2} \approx 1.5 \cdot 10^{-11} \text{cm}\]
(5.16)

If \(\nu \approx 500 \text{Hz}\), the velocity of the inner sphere, \(u_o\), is approximately \(\Delta r/\Delta t\), where \(\Delta t = 1/\nu = 5 \times 10^{-3} \text{s}\). Therefore:

\[u_o \approx \frac{\Delta r}{\Delta t} \approx 3 \cdot 10^{-9} \text{cm/s}\]
(5.17)

Since the size of the electrode is small compared to the wavelength and cavity dimensions, it can be considered as a simple source of sound. With this simplification, the pressure of the resulting sound wave can be calculated directly from the radial velocity of the electrode. For a simple source of radius, \(r_0\), pulsating with a velocity \(u_o e^{-i\omega t}\), the pressure disturbance radiated into the surrounding medium (free field) is given by:

\[p \approx \frac{-ikq_o c}{4\pi r^2} \cdot 4\pi \nu_0^2 u_o e^{ik(r-ct)}\]
(5.18)

Substituting the following values:

\[r \approx 20 \text{cm (cavity radius)}\]

\[q_o c \approx 10^3 \text{ (SF}_6\text{)}\]
\[ k = 0.228 \] (for first radial mode)

the magnitude of the pressure is: \[ |P| \sim 4.5 \cdot 10^{-9} \text{ dy/cm}^2. \]

The detected pressure, which includes the effects of the cavity, would be \( Q \) times this value or \( 4.5 \times 10^{-9} \) dy/cm\(^2\), but this pressure is still three orders of magnitude smaller than the polarization signal at the same voltage.

Plots of the polarization signal (\( V_{\text{sig}} \)) versus the driving voltage squared (\( V^2 \)) are shown in Plot (5.1). The only modification of the experimental apparatus described in the last section that was necessary for these measurements concerned the audio oscillator. The oscillator drove the high voltage current at half the fundamental frequency of the cavity, but it was necessary to tune the detection circuit to the fundamental frequency. A diode was connected between the oscillator and lock-in amplifier reference signal. Sufficient reference signal at the fundamental is provided by the second harmonic component of the rectified oscillator signal. The detected polarization signal was maximized first by adjustment of the audio oscillator, followed by adjustment of the lock-in amplifier frequency and phase. Plot (5.1) shows two separate runs with 12 psi SF\(_6\) in the cavity, and two runs with 12 psi \( N_2 \). The measured values of the \( Q \)'s of the cavity at the time of each run are also shown on the graph.

The 5% discrepancy in the slopes of the SF\(_6\) plots and the 20% discrepancies in the \( N_2 \) slopes can be accounted for by drift of the apparatus from the cavity resonance during the time of measurement (typically 10 minutes). These drifts are more apparent in a table of
the data showing consecutive measurements of the polarization signal. (See Table (5.1)). The most likely cause of these drifts is temperature gradients within the cavity. Since the resonant frequency, \( \nu_0 \), depends on the sound velocity, \( c = \sqrt{\frac{RT}{M}} \), a temperature change, \( \Delta T \), will shift the resonant frequency by the following amount:

\[
\Delta \nu \approx \frac{3\nu_0}{2T} \Delta T \approx \nu_0 \frac{\Delta T}{2T}
\]

(5.19)

Thus, a .1°C change will shift the \( \nu_2 \) resonance (1272 Hz) by more than .2 Hz, which is a non-negligible fraction of the width of this resonance (\( \sim 1.4 \) Hz).
<table>
<thead>
<tr>
<th>RUN 1 12 psi SF₆</th>
<th>RUN 2 12 psi SF₆</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>V²</td>
</tr>
<tr>
<td>1.0 kv (rms)</td>
<td>1.00 (kv)²</td>
</tr>
<tr>
<td>1.5</td>
<td>2.25</td>
</tr>
<tr>
<td>2.0</td>
<td>4.00</td>
</tr>
<tr>
<td>2.5</td>
<td>6.25</td>
</tr>
<tr>
<td>3.0</td>
<td>9.00</td>
</tr>
<tr>
<td>3.5</td>
<td>12.25</td>
</tr>
</tbody>
</table>

**SIGNAL PEAKED AT 494.4 Hz WITH SUBSEQUENT 1% TOTAL DRIFT**

<table>
<thead>
<tr>
<th>RUN 3 12 psi N₂</th>
<th>RUN 4 12 psi N₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>V²</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
</tr>
<tr>
<td>1.5</td>
<td>2.25</td>
</tr>
<tr>
<td>2.0</td>
<td>4.00</td>
</tr>
<tr>
<td>2.5</td>
<td>6.25</td>
</tr>
<tr>
<td>2.8</td>
<td>7.80</td>
</tr>
<tr>
<td>2.0</td>
<td>4.00</td>
</tr>
</tbody>
</table>

**SIGNAL PEAKED AT 1272 Hz WITH SUBSEQUENT 10% TOTAL DRIFT**

**SIGNAL PEAKED AT 1272 Hz WITH SUBSEQUENT 1.5% TOTAL DRIFT**

Table 5.1 Polarization Signal Data
RUN 9: $N_2$
\[ \frac{V_{sig}}{V^2} = 2.96 \]

$N_2$ RUNS
\[ \begin{cases} Q = 800 \pm 40 \\ f = 1272 \text{ Hz} \end{cases} \]

RUN 4: $N_2$
\[ \frac{V_{sig}}{V^2} = 2.32 \]

RUN 2: $SF_6$
\[ \frac{V_{sig}}{V^2} = .62 \]

RUN 1: $SF_6$
\[ \frac{V_{sig}}{V^2} = .59 \]

$SF_6$ RUNS
\[ \begin{cases} Q = 1050 \pm 40 \\ f = 494 \text{ Hz} \end{cases} \]
The average slope, $\frac{v_{\text{sig}}}{\alpha^2}$, of the SF$_6$ data will be used for apparatus calibration. Multiplication of the slope by the factor, $\frac{\alpha^2}{P_0(\text{pol})}$, evaluated from equation (5.12) for $p = 12$ psi, $Q = 1000$, and $\alpha = 6.54 \times 10^{-24}$ cm$^{-3}$ yields the desired calibration:

$$\frac{\partial v_{\text{sig}}}{\partial P_0} = \frac{\alpha^2}{P_0(\text{pol})} \cdot \frac{v_{\text{sig}}}{\alpha^2}$$

$$\simeq 1.0 \cdot 10^5 \frac{\text{(e.s.u.)}^2}{\text{dy} \cdot \text{cm}^2} \cdot 5.4 \cdot 10^{-2} \frac{\alpha}{\text{(e.s.u.)}^2}$$

$$\simeq 1.9 \cdot 10^3 \frac{\alpha}{\text{dy} \cdot \text{cm}^2}$$

Since great sensitivity is claimed in the experiment, the last result is crucial. This calibration depended on the calculation of the sound pressure produced in a polarizable gas with an applied alternating electric field. As a check on this calculation, the microphone calibration and measured detection circuit gain will be used to measure the polarization signal and compare this value with that predicted by equation (5.12).

This is the reason for the data on $\alpha_2$ shown on Plot (5.1). The microphones were calibrated in essentially an atmosphere of $\alpha_2(\text{air} = 0.78 \alpha_2)$ These measurements could not be applied to the performance of the microphones in an atmosphere of SF$_6$ where the radiation impedance is substantially changed. Therefore, this indirect calibration is performed with $\alpha_2$: the number $P_0(\text{pol})/\alpha^2$ is obtained from equation (5.12) substituting the experimental values$^6$ for the $\alpha_2$ run:

$p = 12$ psi, $\alpha = 1.71 \times 10^{-24}$ cm$^3$, and $Q = 800$. Multiplying $P_0(\text{pol})/\alpha^2$
by the measured microphone response, \( \kappa = 3.5 \times 10^{-4} \text{V/} \text{dy/cm}^2 \) at 1272 Hz (the frequency of the first radial mode with \( \bar{n}_2 \)), and the measured gain of the detection circuit at this frequency, \( G \), yields the calculated signal strength:

\[
\frac{V_{\text{SIG}}}{V^2} \text{ (CALCULATED)} = \frac{P_0 \text{ (POL)} R G}{(\text{e.s.u.})^2} = (5.9 \cdot 10^{-6})(3.5 \cdot 10^{-4})(6.7 \cdot 10^7) \frac{V}{(\text{e.s.u.})^2}
\]

\[
= 0.14 \frac{V}{(\text{e.s.u.})^2}
\]

The above number is to be compared to the observed polarization signal for \( \bar{n}_2 \), (using the average slope of the two \( \bar{n}_2 \) runs in Plot (5.1)):

\[
\frac{V_{\text{SIG}}}{V^2} \text{ (OBSERVED)} = 0.24 \frac{V}{(\text{e.s.u.})^2}
\]

The observed signal agrees with the predicted signal within a factor of two, which is sufficient agreement for the purposes of this experiment. Therefore, the use of the polarization calculation as a direct calibration is justified and assumed accurate, at least, within the percentage difference of the above two signals.
SECTION V

FOOTNOTES

1. Small corrections, (at least for the purposes of this experiment), due to the electrode lead are ignored.


3. Integration performed analytically by hand and checked with a numerical integration by computer.


5. The values of the polarizability of $\text{SF}_6$ and $\text{N}_2$ are taken from Landolt-Börnstein, Zahlenwerte und Funktionen, (Springer-Verlag, Berlin, 1955), Vol. II, part 6, p. 273. These values were measured at 1 atm and 25°C, but the polarizability is independent of pressure in the limit of an ideal gas.

6. See footnote 5.
SECTION VI

The Experiment

Most of the theoretical and experimental details of the experiment have been discussed in previous sections of this paper. It now remains to apply these results to the specific calculation of the neutrality of the SF₆ molecule. First of all, the wave equation must be solved with a source term given by the product of the charge density, \( q = neMe \), and the field within the cavity \( E = \frac{E}{V/r^2} \). This result, when combined with the calibration obtained in the preceding section, and a measurement of any sound energy at the fundamental frequency, yields the test for charge neutrality. Following the procedure that was developed in general, the starting point will be the wave equation for pressure:

\[
\left[ \nabla^2 + k^2 \right] \rho = \text{div} \, F_v
\]  \hspace{1cm} (6.1)

However, in this case the volume force, \( F_v = qE \), has no divergence owing to the \( 1/r^2 \) dependence of the electric field. The source of sound energy is entirely at the surfaces enclosing the volume because it is only at the boundary surface that the electric field possesses a divergence. The production of sound in this manner is completely analogous to the production of sound waves in an enclosed volume by physical oscillation of the container walls. Here, the volume force is zero, so the divergence of \( F_v \) is zero, but there is a non-zero divergence of \( F_v \) at the walls which forces the gas into motion.
This problem and the charge problem can both be solved by assuming \( \delta \)-function sources at the bounding surfaces. In the case of the charge problem the relevant wave equation is:

\[
[\nabla^2 + k^2] P(r) = gE \left[ \frac{\delta(r-r_0)}{4\pi r_0^2} + \frac{\delta(r-a)}{4\pi r_2^2} \right] 
\]

(6.2)

Since the form of the solution is known, this equation can be solved with the substitution:

\[
P = P_0 \cdot j_0(k_{01}r) = P_0 \frac{\sin k_{01}r}{k_{01}r}
\]

(6.3)

where \( P_0 \) is a constant to be determined, equal to the magnitude of the pressure wave. Substituting for \( P \), \( g \) and \( E \) in equation (6.2), and differentiating yields:

\[
P_0 \frac{\sin k_{01}r}{k_{01}r} = \frac{1}{k_{01}^2 - k^2} \frac{neM_e}{r^2} \left[ \frac{\delta(r-r_0)}{r_0^2} + \frac{\delta(r-a)}{r_2^2} \right]
\]

(6.4)

Multiplying both sides of the equation by \( \frac{\sin k_{01}r}{k_{01}r} \) and integrating from \( 0 \) to \( \infty \), gives an expression for \( P_0 \):
\[
\rho_0 \int_0^\infty \sin^2 k_{01} r \frac{k_{01} r}{(k_{01} r)^2} \, \frac{1}{(k_{01} r)} \, \frac{1}{r^2} \, d\gamma = \rho_0 \frac{\pi}{2} = \frac{k_{01}}{k_{01}^2 - k^2} \, n e M e V
\]

\[
x \left[ \int_0^\infty \sin k_{01} r \frac{r^2 dr}{k_{01} r^3} \delta(r - r_0) + \int_0^\infty \sin k_{01} r \frac{r^2 dr}{k_{01} r^3} \delta(r - a) \right] \quad (6.5)
\]

\[
\rho_0 = \frac{2k_{01}}{k_{01}^2 - k^2} \left( n e M e V \right) \left[ \frac{\sin k_{01} r_0}{k_{01} r_0^3} + \frac{\sin k_{01} a}{k_{01} a^3} \right]
\]

As before, the magnitude of the detected pressure is found by multiplying the pressure at \( k = 0 \) by the \( \phi \):

\[
\Delta P_D = Q \rho_0 \quad (k = 0)
\]

\[
= \frac{2}{n} \left( n e M e V \right) \left[ \frac{\sin k_{01} r_0}{k_{01} r_0^3} + \frac{\sin k_{01} a}{k_{01} a^3} \right] \quad (6.6)
\]

Substitution of \( P \text{(ambient)} / kT \) for the number density, \( n \), and rearrangement of terms yields an expression for the charge per molecule, \( \mu e \), that differs from the Boltzmann calculation in Section 2 (see equation (2.15)) only by a geometric factor:

\[
\mu e \leq \frac{\Delta P_D kT}{Q PeV} \left[ \frac{\pi}{2r_0} \frac{k_{01}}{k_{01} r_0^3} + \frac{\sin k_{01} a}{k_{01} a^3} \right] \quad (6.7)
\]

Evaluation of the geometric factor with the constants \( r_0 = 1.27 \) cm, \( a = 19.7 \) cm, and \( k = 0.228 \) cm\(^{-1} \) yields:

\[
\mu e \leq \frac{\Delta P_D kT}{Q PeV} \quad (0.45) \quad (6.8)
\]

Substitution of the calibration:

\[
\Delta P_D = \frac{3}{V_{510}} \frac{\Delta P_0}{V_{510}} \quad (6.9)
\]
results in the expression of the charge per molecule directly in terms of the chart recorder signal:

$$M_0 \leq \frac{\delta \Delta P_0}{\delta V_{siy}} \frac{kT (45) V_{siy}}{\Omega \beta eV}$$  \hspace{1cm} (6.10)

The procedure for the actual measurement of the charge signal will now be discussed. The effect of a d.c. offset added to the a.c. driving voltage was discussed in Section 4. Such a combined voltage couples a fraction of the polarization signal at $2\omega$ to the fundamental, in phase with the a.c. driving voltage. This coupled signal, having the same frequency and phase as a signal that would be produced by a charge, is useful for tuning the detection circuit. Recall, equation (4.7) gives the a.c. terms in the combined driving voltage if $\delta E_0$ is the d.c. voltage added to the maximum a.c. voltage, $E_0$:

$$P_{(pol)} \sim 2\delta E_0^2 \sin \omega t - \frac{1}{2} E_0^2 \cos 2\omega t$$  \hspace{1cm} (6.11)

Therefore, the coupled signal (termed $V_{d.c}(Pol)$) equals $4\delta$ times the actual polarization signal, $V_{siy}(Pol)$. The procedure for the charge measurement is as follows:

1) The detection circuit is tuned to the cavity resonance by applying the a.c. driving voltage combined with a d.c. offset. The signal is maximized by successive adjustment of the audio oscillator, lock-in amplifier frequency and phase controls. The d.c. also serves to remove ions and charged particles from the cavity.

2) The d.c. is turned off. The residual signal due to electrostatic and magnetic pickup is reduced to below the inherent noise level.
by adjustment of the balance circuit.

3) The d.c. is turned on again briefly to confirm that the adjustment procedure of step 2 did not detune the detection circuit from the cavity resonance.

4) The charge measurement is made following the decay of the coupled polarization signal of step 3.

5) The cavity, Q, pressure and temperature are measured.

Certain qualifying remarks need to be made concerning the validity of step 2 in the above procedure. It is crucial to establish that the residual signal which is cancelled is indeed pickup, and not a true acoustic signal. If the signal were a true acoustic signal it would show a marked resonance at the acoustic resonant frequency, $\omega_0$, whereas electrostatic and magnetic pickup would be slowly varying functions of frequency, showing no special behavior at $\omega_0$. Plot (6.3) shows the uncanceled magnitude of the residual signal in Run 1 as a function of frequency in the vicinity of $\omega_0$. The flat response proves that this residual signal is not acoustic. An acoustic signal would have peaked at $\omega_0$ at a value $\eta (1100)$ times larger than the signal several cycles removed from $\omega_0$.

Results

Recorder plots for two runs are shown in Plots (6.1) and (6.2). Experimental data for these two runs is as follows:

$P = 12 \text{ psi}$

$SF_0 = 8.2 \times 10^5 \text{dy/cm}^2$

$T = 300^\circ \text{K}$
\[ Q = 1040 \text{ (Run 1), } 1120 \text{ (Run 2)} \]

\[ V_{ac} = 3500 \text{ vrms} = 11.7 \text{ e.s.u.} \]

\[ V_{dc} = 280 \text{ v} \]

\[ RC = 10 \text{ sec} \]

Since \( \delta = \frac{V_{dc}}{V_{ac}} = 0.08 \), \( V_{dc}(\text{pol}) = 4(0.08)V_{SIG}(\text{pol}) \) from equation (6.11). From Plot (5.1) of the polarization signal in 12 psi \( SF_6 \), \( V_{SIG}(\text{pol}) \) at 3500v equals 7.5v. Thus \( V_{ac}(\text{pol}) = 4(0.08)7.5 = 2.4 \text{v} \) should be the observed values of the d.c. coupled signal if the experiment is on resonance. The value observed in Run 1 is 2.4v and in Run 2 is 2.6v, both values being within the uncertainties introduced by temperature drifts.

The plots show the decay of the coupled polarization signal to below the inherent noise level, thus revealing no charge signal greater than the noise. This noise is nothing but the Johnson noise in the detection circuit, as can be seen by comparing these data to the plots of background noise in Plots (4.2) and (4.3).

If a signal were present due to a molecular charge, it could easily be seen with a signal to noise ratio of approximately one if a signal were detected of magnitude .1v (i.e. offsetting the recorder signal by one small division). Substituting a value of \( V_{SIG} = .1 \text{v} \) into equation (6.10) yields an upper limit for the charge on the \( \text{SF}_6 \) molecule:

\[ Me \leq 1.9 \cdot 10^{-19} \quad (6.12) \]
Dividing the above by the mass number of $SF_6$ ($M = 146$) gives an upper limit for a fundamental electron-proton charge difference:

$$\epsilon \approx 1.3 \cdot 10^{-21}$$  \hspace{1cm} (6.13)

**Conclusions**

The experiment just described represents a sensitive new method of establishing upper limits on the charge carried by molecules and an electron-proton charge difference. The results are consistent with the assumption that a molecule carries no charge, thus confirming by an independent technique the results of previous neutrality measurements. The experimental technique avoids the problems of stray ions and charged particles, and charge compensation effects which complicate neutrality measurements by other macroscopic techniques.

The sensitivity of this initial experiment has equaled the sensitivity of the best published measurements of an upper limit for an electron-proton charge difference. However, the achieved sensitivity does not represent the ultimate sensitivity of the technique which could be obtained by reasonable improvements in the hardware. From equation (6.10) it is seen that the experiment gains sensitivity linearly with the cavity pressure, $P$, since the $Q$ also increases with pressure. Increases in $V$ and $P$ by an order of magnitude each, seem possible with reasonable improvements in the high voltage circuit and strength of the cavity. The important consideration which must be kept in mind when scaling the
the experimental parameters for increased sensitivity is the need for a stable and harmonic-free high voltage source, and stable balance and detection circuits. Higher voltages increase the likelihood of arcing which can cause unacceptable asymmetries in the waveform; and the higher cavity Q's obtainable with higher cavity pressures increase the susceptibility to frequency drift errors due to the narrower bandwidths. If the necessary stability is obtained in the electronic circuitry, then additional sensitivity can be gained from increasing the integration time of the detection circuit from the 10 second value used in the initial experiment. The final limitation in sensitivity is the Johnson noise in the detection circuit, and since this noise voltage is statistical, its magnitude decreases in proportion to the square root of the integration time. Considering all these factors a gain in sensitivity by a factor between $10^2$ and $10^3$ seems reasonably possible.
PLOT 6.1  NEUTRALITY MEASUREMENT: RUN 1
(f = 494 Hz, Δf = .1 Hz)
PLOT 6.2 NEUTRALITY MEASUREMENT: RUN 2
(f = 494 Hz, ∆f = .1 Hz)
APPENDIX

Green's Function for the Wave Equation

The eigenfunctions for the interior of a spherical cavity with Newmann boundary conditions are given by:

\[ \Psi_{mns} = \cos(m\phi) P_n^m(\cos\theta) j_n(k_{ns}r) + \sin(m\phi) P_n^m(\cos\theta) j_n(k_{ns}r) \]

\[ = \Psi_{e}^{mns} + \Psi_{o}^{mns} \]  

(A.1)

where m, n, and s are positive integers. If the eigenfunctions are to represent pressure waves in the cavity than the constant \(k_{ns}\) is determined such that the radial derivatives of the eigenfunctions vanish at the sphere radius, \(r = a\):

\[ \frac{d j_n(n\alpha)}{d\alpha} = 0 \]

(A.2)

Identifying \(\alpha_{ns}\) as the \(s^{th}\) root of equation (A.2) for the \(n^{th}\) spherical Bessel function, then: \(k_{ns} = n\alpha_{ns}/\alpha\)

The Green's function for the wave equation in terms of these eigenfunctions is:

\[ G(R,R') = \sum_{m,n,s} \frac{4\pi}{\Lambda_{mns}^2} \frac{1}{(k_{ns}^2 - k^2)} \cdot \]

\[ \left[ \Psi_{e}^{mns}(R') \Psi_{e}^{mns}(R) + \Psi_{o}^{mns}(R') \Psi_{o}^{mns}(R) \right] \]  

(A.3)

where the normalization constant is:

\[ \Lambda_{mns}^2 = \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^a \left[ \Psi_{e}^{mns} \right]^2 r^2 dr \]

\[ = \frac{2\pi a^3 \epsilon_m (n+m)!}{(2n+1)(n-m)!} \left[ \frac{(\pi \alpha_{ns})^2 - n(n-1)}{(\pi \alpha_{ns})^2} \right] j_n^2(\pi \alpha_{ns}) \]  

(A.4)
where:  \( \ell_m = 1 \) when \( m = 0 \)
\( \ell_m = 2 \) when \( m > 0 \)

For the purposes of the calculations in this paper only one term of the Green's function is needed. This term corresponds to the first radial mode of the sphere, described by the eigenfunction with \( m = 0, n = 0, \) and \( s = 1 \):

\[
\Psi_{001} = j_0(k_0 r)
\]  \( \text{(A.5)} \)

The normalization constant for this term is:

\[
\Lambda_{001}^2 = 2\pi a^3 j_0^2(k_0 a)
\]  \( \text{(A.6)} \)

The Green's function for the first radial mode is then:

\[
G_{001}(R,R') = \frac{4\pi}{2\pi a^3 j_0^2(k_0 a)} \frac{1}{k^2_0 - k^2} \left[ j_0(k_0 r) j_0(k_0 r') \right]
\]  \( \text{(A.7)} \)