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FOURIER DIMENSION AND SPECTRAL GAPS FOR HYPERBOLIC SURFACES

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Abstract. We obtain an essential spectral gap for a convex co-compact hyperbolic surface $M = \Gamma\backslash\mathbb{H}^2$ which depends only on the dimension δ of the limit set. More precisely, we show that when $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that the Selberg zeta function has only finitely many zeroes s with $\operatorname{Re} s > \delta - \varepsilon_0$.

The proof uses the fractal uncertainty principle approach developed in Dyatlov–Zahl [DZ16]. The key new component is a Fourier decay bound for the Patterson–Sullivan measure, which may be of independent interest. This bound uses the fact that transformations in the group Γ are nonlinear, together with estimates on exponential sums due to Bourgain [Bou10] which follow from the discretized sum-product theorem in $\mathbb{R}$.

Let $M = \Gamma\backslash\mathbb{H}^2$ be a (noncompact) convex co-compact hyperbolic surface. The Selberg zeta function $Z_M(s)$ is a product over the set $\mathcal{L}_M$ of all primitive closed geodesics

$$Z_M(s) = \prod_{\ell\in\mathcal{L}_M}\prod_{k=0}^{\infty}\left(1 - e^{-(s+k)\ell}\right), \quad \operatorname{Re} s \gg 1,$$

and extends meromorphically to $s \in \mathbb{C}$. Patterson [Pa76] and Sullivan [Su79] proved that Z_M has a simple zero at the exponent of convergence of Poincaré series, denoted δ, and no other zeroes in $\{\operatorname{Re} s \geq \delta\}$. Naud [Na05], using the method originating in the work of Dolgopyat [Do98], showed that for $\delta > 0$, Z_M has only finitely many zeroes in $\{\operatorname{Re} s \geq \delta - \varepsilon\}$ for some $\varepsilon > 0$ depending on the surface. (See also Petkov–Stoyanov [PS10], Stoyanov [St11], and Oh–Winter [OW16].)

Our result removes the dependence of the improvement ε on the surface:

Theorem 1. *Let M be a convex co-compact hyperbolic surface with $\delta > 0$. Then there exists $\varepsilon_0 > 0$ depending only on δ such that $Z_M(s)$ has only finitely many zeroes in $\{\operatorname{Re} s > \delta - \varepsilon_0\}$.*

Remarks. 1. The proof of Theorem 1 uses the results of Dyatlov–Zahl [DZ16] and thus gives a resonance free strip with a polynomial resolvent bound, see [DZ16, (1.3)]. In the terminology used in [DZ16], Theorem 1 gives an *essential spectral gap* of size $\frac{1}{2} - \delta + \varepsilon_0$, improving over the Patterson–Sullivan gap $\frac{1}{2} - \delta$.

2. The Selberg zeta function Z_M has only finitely many zeroes in $\{\operatorname{Re} s > \frac{1}{2}\}$; that is, M has an essential spectral gap of size 0. Therefore, Theorem 1 only gives new

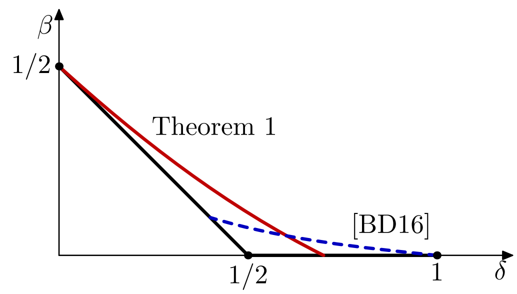

Figure 1. The dependence on δ of the essential spectral gap β (that is, a number such that Z_M has only finitely many zeroes in $\{\operatorname{Re} s > \frac{1}{2} - \beta\}$), showing curves representing the bounds of Theorem 1 and of [BD16]. These curves are for illustration purposes only, the actual size of the improvement is expected to be much smaller. The value of β from [BD16] depends on the surface M but the value given by Theorem 1 only depends on δ. The solid black line is the standard (Patterson–Sullivan and Lax–Phillips) gap $\beta = \max(0, \frac{1}{2} - \delta)$.

information when $\delta \leq \frac{1}{2} + \tilde{\varepsilon}$ for a small global constant $\tilde{\varepsilon} > 0$. In [BD16] the authors proved that there exists $\varepsilon > 0$ (depending on the surface M) such that Z_M only has finitely many zeroes in $\{\operatorname{Re} s > \frac{1}{2} - \varepsilon\}$. The latter result is only interesting when $\delta \geq \frac{1}{2}$. Therefore [BD16] and the present paper overlap only when $\delta \approx \frac{1}{2}$, and in the latter case the present paper gives a stronger result (since ε_0 depends only on δ). In view of the methods used in [BD16] a higher-dimensional extension of that result seems difficult at the present. See Figure 1.

3. The constant ε_1 can be chosen increasing in δ, and thus can be made continuous in δ – see the paragraph preceding §1.1.

4. In the more general setting of scattering on manifolds with hyperbolic trapped sets, the Patterson–Sullivan gap is replaced by the *pressure gap*, established by Ikawa [Ik88], Gaspard–Rice [GR89], and Nonnenmacher–Zworski [NZ09]. See the reviews of Nonnenmacher [No11] and Zworski [Zw17] for the history of the spectral gap question and [DZ16, DJ17] for an overview of more recent developments.

5. Dyatlov–Jin [DJ17] gave a bound on ε_0 depending only on δ and the regularity constant (that is, the constant C_Γ in Lemma 2.12), proving a fractal uncertainty principle for more general Ahlfors–David regular sets. Our proof removes the dependence of ε_0 on C_Γ by using the nonlinear nature of the transformations in the group Γ. In fact, the earlier work of Dyatlov–Jin [DJ16, Proposition 3.17] gives examples of Cantor sets with $\delta \in (0, 1/2]$ which are invariant under a group of linear transformations and do not satisfy the fractal uncertainty principle we derive for hyperbolic limit sets here (Propositions 4.1 and 4.3).

The key new component of the proof of Theorem 1, established in §3, is the following generalized Fourier decay bound for the Patterson–Sullivan measure:

Theorem 2. *Let M, δ be as in Theorem 1 and denote by μ the Patterson–Sullivan measure on the limit set $\Lambda_\Gamma \subset \mathbb{R}$. Assume that*

$$\varphi \in C^2(\mathbb{R}; \mathbb{R}), \quad g \in C^1(\mathbb{R}; \mathbb{C})$$

are functions satisfying the following bounds for some constant $C_{\varphi,g}$:

$$\|\varphi\|_{C^2} + \|g\|_{C^1} \le C_{\varphi,g}, \quad \inf_{\Lambda_\Gamma} |\varphi'| \ge C_{\varphi,g}^{-1}. \tag{1.1}$$

Then there exists $\varepsilon_1 > 0$ depending only on δ and there exists $C > 0$ depending on $M, C_{\varphi,g}$ such that

$$\left| \int_{\Lambda_\Gamma} \exp\big(i\xi\varphi(x)\big) g(x)\, d\mu(x) \right| \le C|\xi|^{-\varepsilon_1} \quad \text{for all } \xi, \quad |\xi| > 1. \tag{1.2}$$

Remarks. 1. By taking $\varphi(x) = x$, $g \equiv 1$ on Λ_Γ, we obtain the Fourier decay bound $\hat{\mu}(\xi) = \mathcal{O}(|\xi|^{-\varepsilon_1})$. This implies that the Fourier dimension $\dim_F \Lambda_\Gamma$ is positive, specifically $\dim_F \Lambda_\Gamma \ge 2\varepsilon_1$. The nonlinearity of transformations in Γ is crucial for obtaining Fourier decay, since there exist limit sets of linear transformations (for instance, the mid-third Cantor set) whose Fourier dimension is equal to zero – see [Ma95, §12.17]. Previously Jordan–Sahlsten [JS13] used a similar nonlinearity property to obtain Fourier decay for Gibbs measures for the Gauss map which have dimension greater than 1/2. (The method of the present paper can be adapted to prove [JS13, Theorem 1.3] without the dimensional assumption.)

2. The key tool in the proof of Theorem 2 is an estimate on decay of exponential sums established by the first author [Bou10], see Proposition 3.1 and the following remark. In particular our proof relies on the discretized sum-product theorem for $\mathbb{R}$.

3. The constant ε_1 can be chosen an increasing function of δ. Indeed, it is determined by the constants $\varepsilon_3, \varepsilon_4, k$ from Proposition 3.1, see (3.28) and the proof of Proposition 3.2. However, Proposition 3.1 holds for same $\varepsilon_3, \varepsilon_4, k$ and all larger values of δ since the condition (3.1) is stronger for larger values of δ_1 and we apply this proposition with $\delta_1 = \delta/24$.

Given Theorem 2, we establish a fractal uncertainty principle for the limit set Λ_Γ, see Propositions 4.1 and 4.3. Then Theorem 1 follows by combining the fractal uncertainty principle with the results of [DZ16], see §4. The value of ε_0 in Theorem 1 can be any number strictly less than $\varepsilon_1/4$, where ε_1 is obtained in Theorem 2, and thus can be chosen increasing as a function of δ.

1.1. **Extensions to higher dimensional situations.** While we do not pursue the case of higher-dimensional convex co-compact hyperbolic quotients in this paper, we briefly discuss a possible generalization of Theorem 1 to the case of three-dimensional quotients $M = \Gamma\backslash\mathbb{H}^3$ with $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ a Kleinian group.

The limit set Λ_Γ is contained in $\dot{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and it is invariant under the action of Γ on $\dot{\mathbb{C}}$ by complex Möbius transformations. The Patterson–Sullivan measure is equivariant under Γ similarly to (2.29).

Linearizing Möbius transformations leads to complex multiplication and the need of a complex analogue of our main tool, Proposition 3.1. In this analogue the measure μ_0 is supported on the annulus $\{z \in \mathbb{C} \colon 1/2 \leq |z| \leq 2\}$, the box dimension estimate (3.1) is replaced by

$$\sup_{x,\theta\in\mathbb{R}} \mu_0\big\{z \colon \operatorname{Im}(e^{i\theta}z) \in [x-\sigma, x+\sigma]\big\} < \sigma^{\delta_1} \tag{1.3}$$

and the conclusion (3.2) is replaced by

$$\left| \int \exp\big(2\pi i\eta \operatorname{Im}(e^{i\theta}z_1\cdots z_k)\big)\, d\mu_0(z_1)\cdots d\mu_0(z_k) \right| \leq N^{-\varepsilon_4}, \quad \theta\in\mathbb{R}.$$

This complex analogue of Proposition 3.1 can be shown by following the proof of [Bou10, Lemma 8.43] and replacing the real version of the sum-product theorem [Bou10, Theorem 1] by its complex version established in [BG12, Proposition 2].

However, the box dimension bound (1.3) is more subtle than in the case of surfaces. Indeed, in the case of a hyperbolic cylinder (i.e. when Γ is a co-compact subgroup of $\mathrm{SL}(2,\mathbb{R})$, with $\delta = 1$) the limit set Λ_Γ is equal to $\mathbb{R} \subset \mathbb{C}$ and the Patterson–Sullivan measure equals the Poisson measure $\pi^{-1}(1+x^2)^{-1}\,dx$. In this case, both (1.3) and the Fourier decay bound (1.2) fail.

In fact, for hyperbolic cylinders the specific fractal uncertainty principle [DZ16, Definition 1.1] used to establish the spectral gap still holds (and does recover the correct size of the spectral gap, equal to $\frac{1}{2}$), however the general fractal uncertainty principle (Proposition 4.1) fails if we take the phase function $\Phi(z,w) = \operatorname{Im}(zw)$ which restricts to 0 on $\Lambda_\Gamma \times \Lambda_\Gamma = \mathbb{R}^2 \subset \mathbb{C}^2$ but has nondegenerate matrix of mixed derivatives $\partial_{(z,\bar{z})}\partial_{(w,\bar{w})}\Phi$.

2. Structure of the limit set

In this section, we study limit sets of convex co-compact quotients, as well as the associated group action and Patterson–Sullivan measure, establishing their properties which form the basis for the proof of the Fourier decay bound in §3.

Let $M = \Gamma\backslash\mathbb{H}^2$ be a convex co-compact hyperbolic surface. Here $\mathbb{H}^2$ is the upper half-plane model of the hyperbolic plane and Γ is a convex co-compact (in particular, discrete) subgroup of $\mathrm{SL}(2,\mathbb{R})$ acting isometrically on $\mathbb{H}^2$ by Möbius transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \quad z \in \mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \quad \Longrightarrow \quad \gamma(z) = \frac{az+b}{cz+d}.$$

The action of $\mathrm{SL}(2,\mathbb{R})$ extends continuously to the compactified hyperbolic plane

$$\overline{\mathbb{H}^2} := \mathbb{H}^2 \cup \dot{\mathbb{R}}, \quad \dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}.$$

See for instance the book of Borthwick [Bor16, Chapter 2] for more details.

We assume that M is nonelementary and noncompact and introduce the following notation:

- $\delta \in (0,1)$, the exponent of convergence of Poincaré series, see [Bor16, §2.5.2];
- $\Lambda_\Gamma \subset \dot{\mathbb{R}}$, the limit set of the group Γ, see [Bor16, §2.2.1];
- μ, the Patterson–Sullivan measure (centered at $i \in \mathbb{H}^2$) which is a probability measure on $\dot{\mathbb{R}}$ supported on Λ_Γ, see [Bor16, §14.1].

2.1. **Schottky groups.** A *Schottky group* is a convex co-compact subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ constructed in the following way (see [Bor16, §15.1] and Figure 2):

- Fix nonintersecting closed half-disks $D_1, \dots, D_{2r} \subset \overline{\mathbb{H}^2}$ centered on the real line. Here $r \in \mathbb{N}$ and for the nonelementary cases studied here, we have $r \geq 2$.
- Put $\mathcal{A} := \{1, \dots, 2r\}$ and for each $a \in \mathcal{A}$, denote
$$\overline{a} := \begin{cases} a + r, & 1 \leq a \leq r; \\ a - r, & r + 1 \leq a \leq 2r. \end{cases}$$
- Fix transformations $\gamma_1, \dots, \gamma_{2r} \in \mathrm{SL}(2,\mathbb{R})$ such that for all $a \in \mathcal{A}$,
$$\gamma_a(\overline{\mathbb{H}^2} \setminus D^\circ_{\overline{a}}) = D_a, \quad \gamma_{\overline{a}} = \gamma_a^{-1}. \tag{2.1}$$
- Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be the free group generated by $\gamma_1, \dots, \gamma_r$.

Each convex co-compact group $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ can be represented in the above way for some choice of $D_1, \dots, D_{2r}$, $\gamma_1, \dots, \gamma_{2r}$, see [Bor16, Theorem 15.3]. We henceforth fix a Schottky structure for Γ.

Notation: In the rest of the paper, C_Γ denotes constants which only depend on the Schottky data $D_1, \dots, D_{2r}, \gamma_1, \dots, \gamma_{2r}$, whose exact value may differ in different places. The elements of Γ are indexed by words on the generators $\gamma_1, \dots, \gamma_{2r}$. We introduce some useful combinatorial notation:

- For $n \in \mathbb{N}_0$, define $\mathcal{W}_n$, the set of words of length n, by
$$\mathcal{W}_n := \{a_1 \dots a_n \mid a_1, \dots, a_n \in \mathcal{A}, \quad a_{j+1} \neq \overline{a_j} \quad \text{for } j = 1, \dots, n-1\}.$$
Denote by $\mathcal{W} := \bigcup_n \mathcal{W}_n$ the set of all words, and for $\mathbf{a} \in \mathcal{W}_n$, put $|\mathbf{a}| := n$. Denote the empty word by $\emptyset$ and put $\mathcal{W}^\circ := \mathcal{W} \setminus \{\emptyset\}$. For $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}$, put $\overline{\mathbf{a}} := \overline{a_n} \dots \overline{a_1} \in \mathcal{W}$. If $\mathbf{a} \in \mathcal{W}^\circ$, put $\mathbf{a}' := a_1 \dots a_{n-1} \in \mathcal{W}$. Note that $\mathcal{W}$ forms a tree with root $\emptyset$ and each $\mathbf{a} \in \mathcal{W}^\circ$ having parent $\mathbf{a}'$.
- For $\mathbf{a} = a_1 \dots a_n, \mathbf{b} = b_1 \dots b_m \in \mathcal{W}$, we write $\mathbf{a} \to \mathbf{b}$ if either at least one of $\mathbf{a}, \mathbf{b}$ is empty or $a_n \neq \overline{b_1}$. Under this condition the concatenation $\mathbf{ab}$ is a word.

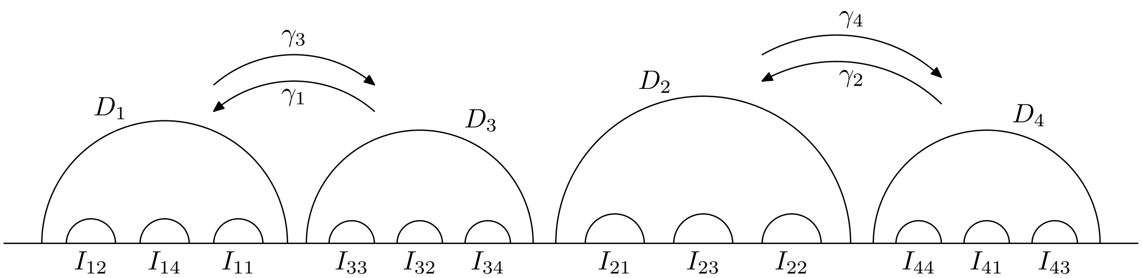

FIGURE 2. A Schottky structure with $r = 2$.

- For $\mathbf{a}, \mathbf{b} \in \mathcal{W}$, we write $\mathbf{a} \prec \mathbf{b}$ if $\mathbf{a}$ is a prefix of $\mathbf{b}$, that is $\mathbf{b} = \mathbf{ac}$ for some $\mathbf{c} \in \mathcal{W}$.
- For $\mathbf{a} = a_1 \dots a_n, \mathbf{b} = b_1 \dots b_m \in \mathcal{W}^\circ$, we write $\mathbf{a} \rightsquigarrow \mathbf{b}$ if $a_n = b_1$. Note that when $\mathbf{a} \rightsquigarrow \mathbf{b}$, the concatenation $\mathbf{a}'\mathbf{b}$ is a word of length $n + m - 1$.
- A finite set $Z \subset \mathcal{W}^\circ$ is called a *partition* if there exists N such that for each $\mathbf{a} \in \mathcal{W}$ with $|\mathbf{a}| \geq N$, there exists unique $\mathbf{b} \in Z$ such that $\mathbf{b} \prec \mathbf{a}$.

For each $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}$, define the group element $\gamma_{\mathbf{a}} \in \Gamma$ by

$$\gamma_{\mathbf{a}} := \gamma_{a_1} \cdots \gamma_{a_n}.$$

Note that each element of Γ is equal to $\gamma_{\mathbf{a}}$ for a unique choice of $\mathbf{a}$ and $\gamma_{\overline{\mathbf{a}}} = \gamma_{\mathbf{a}}^{-1}$, $\gamma_{\mathbf{ab}} = \gamma_{\mathbf{a}}\gamma_{\mathbf{b}}$ when $\mathbf{a} \to \mathbf{b}$.

To study the action of Γ on $\dot{\mathbb{R}}$, consider the intervals

$$I_a := D_a \cap \dot{\mathbb{R}} \ \subset \ \mathbb{R}.$$

For each $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}^\circ$, define the interval $I_{\mathbf{a}}$ as follows (see Figure 2):

$$I_{\mathbf{a}} := \gamma_{\mathbf{a}'}(I_{a_n}).$$

By (2.1), we have $I_{\mathbf{b}} \subset I_{\mathbf{a}}$ when $\mathbf{a} \prec \mathbf{b}$ and $I_{\mathbf{a}} \cap I_{\mathbf{b}} = \emptyset$ when $|\mathbf{a}| = |\mathbf{b}|$, $\mathbf{a} \neq \mathbf{b}$. The limit set is given by

$$\Lambda_\Gamma := \bigcap_n \bigsqcup_{\mathbf{a} \in \mathcal{W}_n} I_{\mathbf{a}}. \tag{2.2}$$

A finite set $Z \subset \mathcal{W}^\circ$ is a partition if and only if

$$\Lambda_\Gamma = \bigsqcup_{\mathbf{a} \in Z} (I_{\mathbf{a}} \cap \Lambda_\Gamma). \tag{2.3}$$

Denote by $|I|$ the size of an interval $I \subset \mathbb{R}$. The following contraction property is proved in §2.3:

$$\mathbf{a} \in \mathcal{W}^\circ,\ b \in \mathcal{A},\ \mathbf{a} \to b \quad \Longrightarrow \quad |I_{\mathbf{a}b}| \leq (1 - C_\Gamma^{-1})|I_{\mathbf{a}}|. \tag{2.4}$$

Note that (2.4) implies the bound

$$\mathbf{a}, \mathbf{b} \in \mathcal{W}^\circ,\ \mathbf{a} \prec \mathbf{b} \quad \Longrightarrow \quad |I_{\mathbf{b}}| \leq (1 - C_\Gamma^{-1})^{|\mathbf{b}| - |\mathbf{a}|}|I_{\mathbf{a}}| \tag{2.5}$$

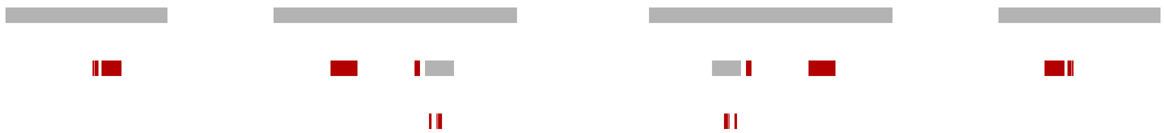

FIGURE 3. A numerically computed example of a partition $Z(\tau)$. The elements of the partition are in dark red and the preceding intervals on the tree are in light gray.

which gives exponential decay of the sizes of the intervals $I_{\mathbf{a}}$:

$$\mathbf{a} \in \mathcal{W}^\circ \quad \Longrightarrow \quad |I_{\mathbf{a}}| \leq C_\Gamma (1 - C_\Gamma^{-1})^{|\mathbf{a}|}. \tag{2.6}$$

We finally describe the collection of words discretizing to a certain resolution. For $\tau > 0$, let $Z(\tau) \subset \mathcal{W}^\circ$ be defined as follows:

$$Z(\tau) = \{\mathbf{a} \in \mathcal{W}^\circ \colon |I_{\mathbf{a}}| \leq \tau < |I_{\mathbf{a}'}|\}, \tag{2.7}$$

where we put $|I_\emptyset| := \infty$. It follows from (2.6) that $Z(\tau)$ is a partition. See Figure 3.

2.2. **Distortion estimates for Möbius transformations.** Let $\mathbf{a} = a_1 \dots a_n$ be a long word. Recall that $I_{\mathbf{a}} = \gamma_{\mathbf{a}'}(I_{a_n})$. In §2.3 below we study how the derivative $\gamma_{\mathbf{a}'}$ varies on the interval I_{a_n}, in particular how much it deviates from its average value $|I_{\mathbf{a}}|/|I_{a_n}|$. The results of §2.3 rely on several statements about general Möbius transformation which are proved in this section.

Let $\gamma \in \mathrm{SL}(2,\mathbb{R})$ and assume that $\gamma(I) = J$ for some intervals $I, J \subset \mathbb{R}$. Define the *distortion factor* of γ on I by

$$\alpha(\gamma, I) := \log \frac{\gamma^{-1}(\infty) - x_1}{\gamma^{-1}(\infty) - x_0} \in \mathbb{R} \quad \text{where } I = [x_0, x_1]. \tag{2.8}$$

If $\gamma^{-1}(\infty) = \infty$, then we put $\alpha(\gamma, I) := 0$. The transformation γ can be described in terms of I, J, and $\alpha(\gamma, I)$ as follows:

$$\gamma = \gamma_J\, \gamma_{\alpha(\gamma,I)}\, \gamma_I^{-1}, \quad \gamma_\alpha = \begin{pmatrix} e^{\alpha/2} & 0 \\ e^{\alpha/2} - e^{-\alpha/2} & e^{-\alpha/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{2.9}$$

Here $\gamma_I, \gamma_J \in \mathrm{SL}(2,\mathbb{R})$ are the unique affine transformations such that $\gamma_I([0,1]) = I$, $\gamma_J([0,1]) = J$. To see (2.9), it suffices to note that

$$\gamma_J \gamma_{\alpha(\gamma,I)} \gamma_I^{-1}(I) = J, \quad \gamma_J \gamma_{\alpha(\gamma,I)} \gamma_I^{-1}(\gamma^{-1}(\infty)) = \infty.$$

See Figure 4. The formula (2.9) implies the following identity:

$$\gamma'(x) = \gamma'_{\alpha(\gamma,I)}(\gamma_I^{-1}(x)) \cdot \frac{|J|}{|I|}. \tag{2.10}$$

Our first lemma states that as long as the distortion factor is controlled, the derivatives γ' at different points of I do not differ too much from each other and from the average:

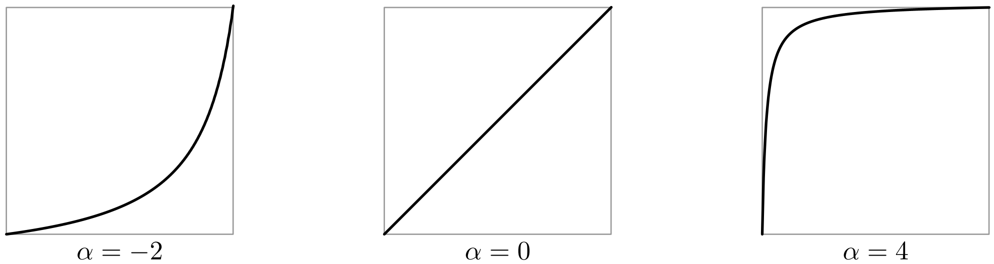

FIGURE 4. Graphs of the transformation γ_α for several different values of α, with the square being $[0,1]^2$.

Lemma 2.1. *Assume that* $\gamma(I) = J$ *as above. Then we have for all* $x, y \in I$

$$e^{-|\alpha(\gamma,I)|}\cdot\frac{|J|}{|I|} \leq \gamma'(x) \leq e^{|\alpha(\gamma,I)|}\cdot\frac{|J|}{|I|}, \tag{2.11}$$

$$\frac{\gamma'(x)}{\gamma'(y)} \leq \exp\Big(2e^{|\alpha(\gamma,I)|}\cdot\frac{|x-y|}{|I|}\Big). \tag{2.12}$$

Proof. We estimate for each $\alpha \in \mathbb{R}$

$$\gamma_\alpha'(x) = \frac{e^\alpha}{((e^\alpha - 1)x + 1)^2} \in [e^{-|\alpha|}, e^{|\alpha|}] \quad \text{for } x \in [0,1]$$

which together with (2.10) implies (2.11). Next, we have

$$\big|(\log\gamma_\alpha'(x))'\big| = \Big|\frac{2(1-e^\alpha)}{(e^\alpha - 1)x + 1}\Big| \leq 2e^{|\alpha|} \quad \text{for } x \in [0,1]$$

which gives

$$\frac{\gamma_\alpha'(x)}{\gamma_\alpha'(y)} \leq \exp\big(2e^{|\alpha|}\cdot|x-y|\big) \quad \text{for } x, y \in [0,1].$$

Combined with (2.10), this implies (2.12). □

As a corollary of (2.11) and the change of variable formula, we immediately obtain

Lemma 2.2. *Assume that* $\gamma(I) = J$ *as above and let* $I' \subset I$ *be a Borel subset. Then, denoting by* $|\bullet|$ *the Lebesgue measure on the line, we have*

$$e^{-|\alpha(\gamma,I)|}\cdot\frac{|I'|\cdot|J|}{|I|} \leq |\gamma(I')| \leq e^{|\alpha(\gamma,I)|}\cdot\frac{|I'|\cdot|J|}{|I|}. \tag{2.13}$$

The next lemma shows that transformations with different distortion factors have significantly different derivatives. It is an essential component of the proof of Theorem 2 which takes advantage of the nonlinearity of Möbius transformations.

Lemma 2.3. *Assume that* $\gamma_1, \gamma_2 \in \mathrm{SL}(2, \mathbb{R})$ *and* $I, J_1, J_2 \subset \mathbb{R}$ *are intervals such that* $\gamma_j(I) = J_j$. *Let* $L \subset \mathbb{R}$ *be an interval. Then the set of points* x *satisfying*

$$x \in I, \quad \log \frac{\gamma_1'(x)}{\gamma_2'(x)} \in L \tag{2.14}$$

is contained in an interval of size

$$\frac{e^{|\alpha(\gamma_1, I)| + |\alpha(\gamma_2, I)|} \cdot |I| \cdot |L|}{|\alpha(\gamma_1, I) - \alpha(\gamma_2, I)|}.$$

Proof. Denote $\alpha_j = \alpha(\gamma_j, I)$. For each $x \in I$ we have by (2.10)

$$\log \frac{\gamma_1'(x)}{\gamma_2'(x)} = \log \frac{\gamma_{\alpha_1}'(y)}{\gamma_{\alpha_2}'(y)} + \log \frac{|J_1|}{|J_2|}, \quad y := \gamma_I^{-1}(x).$$

Therefore, (2.14) corresponds to the set of all y such that

$$y \in [0, 1], \quad \log \frac{\gamma_{\alpha_1}'(y)}{\gamma_{\alpha_2}'(y)} \in \widetilde{L} \tag{2.15}$$

where $\widetilde{L}$ is some interval with $|\widetilde{L}| = |L|$. We compute

$$\partial_y \log \frac{\gamma_{\alpha_1}'(y)}{\gamma_{\alpha_2}'(y)} = \frac{2(1 - e^{\alpha_1})}{(e^{\alpha_1} - 1)y + 1} - \frac{2(1 - e^{\alpha_2})}{(e^{\alpha_2} - 1)y + 1} = \frac{2(e^{\alpha_2} - e^{\alpha_1})}{((e^{\alpha_1} - 1)y + 1)((e^{\alpha_2} - 1)y + 1)}.$$

We then have for all $y \in [0, 1]$

$$\left| \partial_y \log \frac{\gamma_{\alpha_1}'(y)}{\gamma_{\alpha_2}'(y)} \right| \geq 2e^{-|\alpha_1| - |\alpha_2|} \cdot |\alpha_1 - \alpha_2|.$$

It follows that the set of y satisfying (2.15) is an interval of size no more than

$$\frac{e^{|\alpha_1| + |\alpha_2|} \cdot |L|}{|\alpha_1 - \alpha_2|}$$

which finishes the proof. □

2.3. **Distortion estimates for Schottky groups.** We now return to the setting of Schottky groups introduced in §2.1. We start by estimating the distortion factors of transformations in Γ:

Lemma 2.4. *We have*

$$|\alpha(\gamma_{\mathbf{a}}, I_b)| \leq C_\Gamma \quad \textit{for all} \quad \mathbf{a} \in \mathcal{W},\ b \in \mathcal{A},\ \mathbf{a} \to b. \tag{2.16}$$

Proof. We may assume that $\mathbf{a} \in \mathcal{W}^\circ$. Let $\mathbf{a} = a_1 \dots a_n$. By (2.1), $\gamma_{\mathbf{a}}^{-1}(\infty) \in I_{\overline{a_n}}$. Moreover, $\overline{a_n} \neq b$ since $\mathbf{a} \to b$. It remains to recall the definition (2.8) and put

$$C_\Gamma := 2 \max \big\{ \big| \log |x - y| \big| : x \in I_a,\ y \in I_b,\ a, b \in \mathcal{A},\ a \neq b \big\}. \quad \square$$

Lemma 2.4 together with (2.11), (2.12), and (2.13) immediately gives

Lemma 2.5. *For all* $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}^\circ$ *and* $x, y \in I_{a_n}$*, we have*

$$C_\Gamma^{-1} |I_\mathbf{a}| \le \gamma'_{\mathbf{a}'}(x) \le C_\Gamma |I_\mathbf{a}|, \tag{2.17}$$

$$\frac{\gamma'_{\mathbf{a}'}(x)}{\gamma'_{\mathbf{a}'}(y)} \le \exp\big(C_\Gamma |x - y|\big). \tag{2.18}$$

Moreover, if $I' \subset I_{a_n}$ *is a Borel set, then*

$$C_\Gamma^{-1} |I_\mathbf{a}| \cdot |I'| \le |\gamma_{\mathbf{a}'}(I')| \le C_\Gamma |I_\mathbf{a}| \cdot |I'|. \tag{2.19}$$

Armed with Lemma 2.5, we give

Proof of (2.4). We write $\mathbf{a} = a_1 \dots a_n$. With $|\bullet|$ denoting the Lebesgue measure on the line, we compute

$$|I_{\mathbf{a}b}| = |\gamma_{\mathbf{a}'}(\gamma_{a_n}(I_b))| = |\gamma_{\mathbf{a}'}(I_{a_n})| - |\gamma_{\mathbf{a}'}(I_{a_n} \setminus \gamma_{a_n}(I_b))|.$$

Recall that $\gamma_{\mathbf{a}'}(I_{a_n}) = I_\mathbf{a}$. Using (2.19), we obtain the lower bound

$$|\gamma_{\mathbf{a}'}(I_{a_n} \setminus \gamma_{a_n}(I_b))| \ge C_\Gamma^{-1} |I_\mathbf{a}| \cdot |I_{a_n} \setminus \gamma_{a_n}(I_b)| \ge C_\Gamma^{-1} |I_\mathbf{a}|$$

finishing the proof. □

We next show several estimates on the sizes and positions of the intervals $I_\mathbf{a}$:

Lemma 2.6 (Parent-child ratio)**.** *We have*

$$C_\Gamma^{-1} |I_\mathbf{a}| \le |I_{\mathbf{a}b}| \le |I_\mathbf{a}| \quad \text{for all } \mathbf{a} \in \mathcal{W}^\circ,\ b \in \mathcal{A},\ \mathbf{a} \to b. \tag{2.20}$$

Proof. Denote $\mathbf{a} = a_1 \dots a_n$ and note that $I_{\mathbf{a}b} = \gamma_{\mathbf{a}'}(I')$ where $I' := \gamma_{a_n}(I_b) \subset I_{a_n}$. Then (2.20) follows from (2.19). □

Lemma 2.7 (Concatenation)**.** *We have*

$$C_\Gamma^{-1} |I_\mathbf{a}| \cdot |I_\mathbf{b}| \le |I_{\mathbf{a}'\mathbf{b}}| \le C_\Gamma |I_\mathbf{a}| \cdot |I_\mathbf{b}| \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathcal{W}^\circ,\ \mathbf{a} \rightsquigarrow \mathbf{b}. \tag{2.21}$$

Proof. This follows from (2.19) similarly to Lemma 2.6, using that $I_{\mathbf{a}'\mathbf{b}} = \gamma_{\mathbf{a}'}(I_\mathbf{b})$. □

Lemma 2.8 (Reversal)**.** *We have*

$$C_\Gamma^{-1} |I_\mathbf{a}| \le |I_{\overline{\mathbf{a}}}| \le C_\Gamma |I_\mathbf{a}| \quad \text{for all } \mathbf{a} \in \mathcal{W}^\circ. \tag{2.22}$$

Proof. Without loss of generality, we may assume that $|\mathbf{a}| \ge 3$. We write $\mathbf{a} = a_1 \dots a_n$ and denote $\mathbf{b} := a_2 \dots a_{n-1}$, so that $\mathbf{a} = a_1 \mathbf{b} a_n$. Since $I_\mathbf{a} = \gamma_{a_1}(I_{\mathbf{b}a_n})$ and $I_{\overline{\mathbf{a}}} = \gamma_{\overline{a_n}}(I_{\overline{\mathbf{b}}\overline{a_1}})$, it suffices to show that

$$C_\Gamma^{-1} |I_{\mathbf{b}a_n}| \le |I_{\overline{\mathbf{b}}\overline{a_1}}| \le C_\Gamma |I_{\mathbf{b}a_n}|. \tag{2.23}$$

Denote

$$I_{a_n} = [x_1, x_2], \quad I_{\overline{\mathbf{b}}\overline{a_1}} = [x_3, x_4], \quad I_{\mathbf{b}a_n} = [y_1, y_2], \quad I_{\overline{a_1}} = [y_3, y_4]$$

and remark that $\gamma_{\mathbf{b}}(x_j) = y_j$ and thus we have equality of cross ratios

$$\frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_1)(x_3 - x_2)} = \frac{(y_2 - y_1)(y_4 - y_3)}{(y_4 - y_1)(y_3 - y_2)}. \tag{2.24}$$

Now, $x_3, x_4 \in I_{\overline{a_{n-1}}}$ and $\overline{a_{n-1}} \neq a_n$. Therefore,

$$|x_2 - x_1|, |x_4 - x_1|, |x_3 - x_2| \in [C_\Gamma^{-1}, C_\Gamma].$$

Since $y_1, y_2 \in I_{a_2}$ we similarly bound $|y_4 - y_3|, |y_4 - y_1|, |y_3 - y_2|$. Then (2.23) follows from (2.24) and the fact that $|I_{\mathbf{b}a_n}| = y_2 - y_1$, $|I_{\overline{\mathbf{b}}\overline{a_1}}| = x_4 - x_3$. □

Lemma 2.9 (Separation)**.** *Assume that* $\mathbf{a}, \mathbf{b} \in \mathcal{W}^\circ$ *and* $\mathbf{a} \not\prec \mathbf{b}$, $\mathbf{b} \not\prec \mathbf{a}$. *Then*

$$|x - y| \geq C_\Gamma^{-1} \max\big(|I_{\mathbf{a}}|, |I_{\mathbf{b}}|\big) \quad \textit{for all } x \in I_{\mathbf{a}},\ y \in I_{\mathbf{b}}. \tag{2.25}$$

Proof. Since $\mathbf{a} \not\prec \mathbf{b}$, $\mathbf{b} \not\prec \mathbf{a}$, there exist

$$\mathbf{c} \in \mathcal{W},\ d, e \in \mathcal{A} \quad \text{such that} \quad \mathbf{c} \to d,\ \mathbf{c} \to e,\ \mathbf{c}d \prec \mathbf{a},\ \mathbf{c}e \prec \mathbf{b},\ d \neq e.$$

Without loss of generality we may assume that $\mathbf{c} \in \mathcal{W}^\circ$ and write $\mathbf{c} = c_1 \dots c_n$. Then

$$I_{\mathbf{a}} \subset I_{\mathbf{c}d} = \gamma_{\mathbf{c}'}(\gamma_{c_n}(I_d)), \quad I_{\mathbf{b}} \subset I_{\mathbf{c}e} = \gamma_{\mathbf{c}'}(\gamma_{c_n}(I_e)).$$

Since the distance between $\gamma_{c_n}(I_d)$ and $\gamma_{c_n}(I_e)$ is bounded below by C_Γ^{-1} and both these intervals are contained in I_{c_n}, we get by (2.17)

$$|x - y| \geq C_\Gamma^{-1}|I_{\mathbf{c}}| \geq C_\Gamma^{-1} \max\big(|I_{\mathbf{a}}|, |I_{\mathbf{b}}|\big) \quad \text{for all } x \in I_{\mathbf{a}},\ y \in I_{\mathbf{b}}$$

finishing the proof. □

We finally obtain estimates on the elements of the partition $Z(\tau)$ defined in (2.7):

Lemma 2.10. *For all* $\tau \in (0, 1]$ *and* $\mathbf{a} = a_1 \dots a_n \in Z(\tau)$, *we have*

$$C_\Gamma^{-1}\tau \leq |I_{\mathbf{a}}| \leq \tau, \tag{2.26}$$

$$C_\Gamma^{-1}\tau \leq |I_{\overline{\mathbf{a}}}| \leq C_\Gamma\,\tau, \tag{2.27}$$

$$C_\Gamma^{-1}\tau \leq \gamma'_{\mathbf{a}'} \leq C_\Gamma\,\tau \quad \textit{on } I_{a_n}. \tag{2.28}$$

Proof. Let $\mathbf{a} \in Z(\tau)$. Without loss of generality we may assume that $|\mathbf{a}| \geq 2$. We have $|I_{\mathbf{a}}| \leq \tau < |I_{\mathbf{a}'}|$ and by Lemma 2.6, $|I_{\mathbf{a}}| \geq C_\Gamma^{-1}|I_{\mathbf{a}'}|$. This gives (2.26). Now (2.27) follows from (2.22), and (2.28) follows from (2.17). □

2.4. **Patterson–Sullivan measure.** The Patterson–Sullivan measure μ is equivariant under the group Γ: for any bounded Borel function f on $\mathbb{R}$,

$$\int_{\Lambda_\Gamma} f(x)\,d\mu(x) = \int_{\Lambda_\Gamma} f(\gamma(x))|\gamma'(x)|_{\mathbb{B}}^{\delta}\,d\mu(x) \quad \text{for all } \gamma\in\Gamma \tag{2.29}$$

where $|\gamma'|_{\mathbb{B}}$ is the derivative of γ as a map of the ball model of the hyperbolic space:

$$|\gamma'(x)|_{\mathbb{B}} = \frac{1+x^2}{1+\gamma(x)^2}\,\gamma'(x),\quad x\in\dot{\mathbb{R}}.$$

See for instance [Bor16, Lemma 14.2]. Next, (2.29) implies

$$\int_{I_{\mathbf{a}b}} f(x)\,d\mu(x) = \int_{I_b} f(\gamma_{\mathbf{a}}(x))w_{\mathbf{a}}(x)\,d\mu(x) \quad \text{for all } \mathbf{a}\in\mathcal{W},\ b\in\mathcal{A},\ \mathbf{a}\to b \tag{2.30}$$

where the weight $w_{\mathbf{a}}$ is defined by

$$w_{\mathbf{a}}(x) := |\gamma_{\mathbf{a}}'(x)|_{\mathbb{B}}^{\delta}. \tag{2.31}$$

The Patterson–Sullivan measure of an interval $I_{\mathbf{a}}$ is estimated by the following

Lemma 2.11. *We have*

$$C_\Gamma^{-1}|I_{\mathbf{a}}|^\delta \le \mu(I_{\mathbf{a}}) \le C_\Gamma |I_{\mathbf{a}}|^\delta \quad \textit{for all } \mathbf{a}\in\mathcal{W}^\circ. \tag{2.32}$$

Proof. The formula (2.30) implies that for all $a,b\in\mathcal{A}$, $a\neq\bar{b}$, we have

$$\mu(I_a)\ge\mu(I_{ab}) = \int_{I_b} w_a(x)\,d\mu(x) \ge C_\Gamma^{-1}\mu(I_b).$$

Since μ is a probability measure, this implies that

$$C_\Gamma^{-1}\le\mu(I_a)\le 1 \quad \text{for all } a\in\mathcal{A}.$$

Denote $\mathbf{a}=a_1\dots a_n$. From (2.30) we have

$$\mu(I_{\mathbf{a}}) = \int_{I_{a_n}} w_{\mathbf{a}'}(x)\,d\mu(x).$$

By (2.17) we have

$$C_\Gamma^{-1}|I_{\mathbf{a}}|^\delta \le w_{\mathbf{a}'} \le C_\Gamma|I_{\mathbf{a}}|^\delta \quad \text{on } I_{a_n}$$

and (2.32) follows. □

Using Lemma 2.11, we give a self-contained proof of Ahlfors–David regularity of μ (see [Bor16, Lemma 14.13] for another proof):

Lemma 2.12. *Let $I\subset\mathbb{R}$ be an interval. Then*

$$\mu(I)\le C_\Gamma|I|^\delta. \tag{2.33}$$

If additionally $|I|\le 1$ and I is centered at a point in Λ_Γ, then

$$\mu(I)\ge C_\Gamma^{-1}|I|^\delta. \tag{2.34}$$

Proof. We first show the upper bound (2.33). Since μ is supported on Λ_Γ, replacing I with the intersections $I \cap I_a$ we may assume that $I \subset I_a$ for some $a \in \mathcal{A}$. Shrinking I without changing $\mu(I)$, we may also assume that its endpoints x_0, x_1 lie in Λ_Γ. If $I = \{x_0\}$ consists of one point, then by (2.2) we can find arbitrarily long words $\mathbf{a}$ such that $x_0 \in I_{\mathbf{a}}$; by (2.6) and (2.32), we have $\mu(I) = 0$.

Assume now that $x_0 < x_1$. By (2.6) there exists the longest word $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}^\circ$ such that $I \subset I_{\mathbf{a}}$. Then $x_0 \in I_{\mathbf{a}b}$, $x_1 \in I_{\mathbf{a}c}$ for two different $b, c \in \mathcal{A}$ such that $\mathbf{a} \to b$, $\mathbf{a} \to c$. By Lemma 2.9, the distance between $I_{\mathbf{a}b}$ and $I_{\mathbf{a}c}$ is bounded below by $C_\Gamma^{-1}|I_{\mathbf{a}}|$, therefore $|I| \geq C_\Gamma^{-1}|I_{\mathbf{a}}|$. Now (2.33) follows from (2.32):

$$\mu(I) \leq \mu(I_{\mathbf{a}}) \leq C_\Gamma |I_{\mathbf{a}}|^\delta \leq C_\Gamma |I|^\delta.$$

We next show the lower bound (2.34) where I is an interval of size $0 < |I| \leq 1$ centered at some $x \in \Lambda_\Gamma$. Using (2.6), take the shortest word $\mathbf{a} \in \mathcal{W}^\circ$ such that $x \in I_{\mathbf{a}} \subset I$. If $|\mathbf{a}| = 1$, then by (2.32) $\mu(I) \geq \mu(I_{\mathbf{a}}) \geq C_\Gamma^{-1}$. Assume now that $|\mathbf{a}| \geq 2$.

Since $x \in I_{\mathbf{a}'}$ and $I_{\mathbf{a}'} \not\subset I$, we have $|I_{\mathbf{a}'}| \geq \frac{1}{2}|I|$ and thus by (2.20) $|I_{\mathbf{a}}| \geq C_\Gamma^{-1}|I|$. Now (2.34) follows from (2.32):

$$\mu(I) \geq \mu(I_{\mathbf{a}}) \geq C_\Gamma^{-1}|I_{\mathbf{a}}|^\delta \geq C_\Gamma^{-1}|I|^\delta. \quad \square$$

As another corollary of Lemma 2.11, we estimate the number of elements in the partition $Z(\tau)$ defined in (2.7):

Lemma 2.13. *For $\tau \in (0, 1]$ we have*

$$C_\Gamma^{-1}\tau^{-\delta} \leq \#(Z(\tau)) \leq C_\Gamma\, \tau^{-\delta}. \tag{2.35}$$

Proof. Since $Z(\tau)$ is a partition, we have by (2.3)

$$1 = \mu(\Lambda_\Gamma) = \sum_{\mathbf{a} \in Z(\tau)} \mu(I_{\mathbf{a}}).$$

By (2.26) and (2.32), we have for all $\mathbf{a} \in Z(\tau)$

$$C_\Gamma^{-1}\tau^\delta \leq \mu(I_{\mathbf{a}}) \leq C_\Gamma\, \tau^\delta \tag{2.36}$$

which implies (2.35). $\square$

The following is an analogue of the upper bound of Lemma 2.11 where instead of the measure $\mu(I_{\mathbf{b}})$ we estimate the number of intervals of length at least τ in the subtree with root $I_{\mathbf{b}}$:

Lemma 2.14. *Assume that $\tau \in (0, 1]$, $\mathbf{b} \in \mathcal{W}^\circ$. Then*

$$\#\{\mathbf{a} \in \mathcal{W}^\circ \colon \mathbf{b} \prec \mathbf{a},\ |I_{\mathbf{a}}| \geq \tau\} \leq C_\Gamma\, \tau^{-\delta}|I_{\mathbf{b}}|^\delta. \tag{2.37}$$

Proof. We may assume that $|I_{\mathbf{b}}| \geq \tau$ since otherwise the left-hand side of (2.37) equals 0. By (2.6), the following sets are finite:

$$A := \{\mathbf{a} \in \mathcal{W}^\circ \colon \mathbf{b} \prec \mathbf{a},\ |I_{\mathbf{a}}| \geq \tau\}, \quad B := \{\mathbf{a} \in \mathcal{W}^\circ \colon \mathbf{b} \prec \mathbf{a},\ |I_{\mathbf{a}}| < \tau \leq |I_{\mathbf{a}'}|\}.$$

Then $\{I_{\mathbf{a}}\}_{\mathbf{a}\in B}$ is a disjoint collection of subintervals of $I_{\mathbf{b}}$. Therefore by (2.32)

$$\sum_{\mathbf{a}\in B} \mu(I_{\mathbf{a}}) \leq \mu(I_{\mathbf{b}}) \leq C_\Gamma |I_{\mathbf{b}}|^\delta.$$

On the other hand, by (2.20) and (2.32)

$$\mu(I_{\mathbf{a}}) \geq C_\Gamma^{-1}|I_{\mathbf{a}}|^\delta \geq C_\Gamma^{-1}\tau^\delta \quad \text{for all } \mathbf{a} \in B.$$

Therefore, the number of elements in B is bounded as follows:

$$\#(B) \leq C_\Gamma\, \tau^{-\delta}|I_{\mathbf{b}}|^\delta. \tag{2.38}$$

Next, $A \sqcup B$ forms a tree with root $\mathbf{b}$, where the parent of $\mathbf{a}$ is given by $\mathbf{a}'$. Moreover, B is the set of leaves of this tree and each element of A has exactly $2r-1$ children, where $2r \geq 4$ is the number of intervals in the Schottky structure. The number of edges of the tree is equal to both $\#(A) + \#(B) - 1$ and $(2r-1)\cdot\#(A)$, which implies

$$\#(A) = \frac{\#(B)-1}{2r-2} \leq \#(B).$$

Combining this with (2.38), we obtain (2.37). □

Arguing similarly to the proof of (2.33), we obtain from Lemma 2.14 the following

Lemma 2.15. *For all intervals J and all $C_0 \geq 2$ we have*

$$\#\big\{\mathbf{a} \in \mathcal{W}^\circ \colon \tau \leq |I_{\mathbf{a}}| \leq C_0\tau,\ I_{\mathbf{a}} \cap J \neq \emptyset\big\} \leq C_\Gamma\, \tau^{-\delta}|J|^\delta + C_\Gamma \log C_0. \tag{2.39}$$

Proof. Without loss of generality we may assume that J is contained in I_a for some $a \in \mathcal{A}$. Consider the finite set

$$X := \{\mathbf{a} \in \mathcal{W}^\circ \colon |I_{\mathbf{a}}| \geq \tau,\ I_{\mathbf{a}} \cap J \neq \emptyset\}.$$

Then X forms a tree with root a in the sense that $\mathbf{a} \in X \setminus \{a\}$ implies $\mathbf{a}' \in X$.

Take the longest word $\mathbf{b} \in X$ with the following property: for each $\mathbf{a} \in X$, we have $\mathbf{a} \prec \mathbf{b}$ or $\mathbf{b} \prec \mathbf{a}$. Then $\mathbf{b}$ cannot have exactly one child in X, leaving the following two options:

(1) $\mathbf{b}$ has no children in X. Then all $\mathbf{a} \in X$ satisfy $\mathbf{a} \prec \mathbf{b}$. By (2.5), we estimate the number of elements $\mathbf{a} \in X$ such that $|I_{\mathbf{a}}| \leq C_0\tau$ by $C_\Gamma \log C_0$.

(2) There exist $c, d \in \mathcal{A}$, $c \neq d$, $\mathbf{b} \to c$, $\mathbf{b} \to d$, such that $\mathbf{b}c, \mathbf{b}d \in X$. By Lemma 2.9 the distance between $I_{\mathbf{b}c}$ and $I_{\mathbf{b}d}$ is bounded below by $C_\Gamma^{-1}|I_{\mathbf{b}}|$, and both these intervals intersect J, therefore

$$|I_{\mathbf{b}}| \leq C_\Gamma |J|.$$

By (2.37), the number of elements $\mathbf{a} \in X$ such that $\mathbf{b} \prec \mathbf{a}$ is bounded above by $C_\Gamma \tau^{-\delta}|J|^\delta$. All other elements $\mathbf{a} \in X$ have to satisfy $\mathbf{a} \prec \mathbf{b}$, and arguing similarly to the previous case we see that the number of these with $|I_\mathbf{a}| \leq C_0\tau$ is bounded above by $C_\Gamma \log C_0$. □

We finally use Lemma 2.3 to obtain the following statement, which gives the positive box dimension estimate required in §3.3. This is the only statement which uses both Lemma 2.8 (via (2.27)) and the full power of Lemma 2.15. Recall the notation $\mathbf{a} \rightsquigarrow \mathbf{b}$ from §2.1. We introduce the following additional piece of notation:

$$\mathbf{a} \rightsquigarrow {}^{\mathbf{b}}_{\mathbf{c}} \rightsquigarrow \mathbf{d} \quad \text{if and only if} \quad \mathbf{a} \rightsquigarrow \mathbf{b} \rightsquigarrow \mathbf{d} \text{ and } \mathbf{a} \rightsquigarrow \mathbf{c} \rightsquigarrow \mathbf{d}. \tag{2.40}$$

Lemma 2.16. *Fix $\mathbf{a} \in Z(\tau)$ and for each $\mathbf{d} \in \mathcal{W}^\circ$ let $x_\mathbf{d}$ be the center of $I_\mathbf{d}$. Then we have for $0 < \tau \leq \sigma \leq 1$*

$$\begin{aligned}\#\big\{(\mathbf{b},\mathbf{c},\mathbf{d}) \in Z(\tau)^3 \colon \mathbf{a} \rightsquigarrow {}^{\mathbf{b}}_{\mathbf{c}} \rightsquigarrow \mathbf{d},\ |\gamma'_{\mathbf{a}'\mathbf{b}'}(x_\mathbf{d}) - \gamma'_{\mathbf{a}'\mathbf{c}'}(x_\mathbf{d})| \leq \tau^2\sigma\big\}& \\ \leq C_\Gamma\, \tau^{-3\delta}\sigma^{\delta/2}.&\end{aligned} \tag{2.41}$$

Proof. Without loss of generality, we may assume that τ is small enough so that $|\mathbf{c}| \geq 2$ for all $\mathbf{c} \in Z(\tau)$. For each $\mathbf{b} \in Z(\tau)$ such that $\mathbf{a} \rightsquigarrow \mathbf{b}$, we have

$$\#\big\{\mathbf{c} \in Z(\tau) \colon \mathbf{a} \rightsquigarrow \mathbf{c},\ |\gamma^{-1}_{\mathbf{a}'\mathbf{b}'}(\infty) - \gamma^{-1}_{\mathbf{a}'\mathbf{c}'}(\infty)| \leq \sqrt{\sigma}\big\} \leq C_\Gamma\, \tau^{-\delta}\sigma^{\delta/2}. \tag{2.42}$$

Indeed, denoting $\mathbf{e} := \overline{\mathbf{c}'}$, we have $\gamma^{-1}_{\mathbf{a}'\mathbf{c}'}(\infty) = \gamma_{\mathbf{e}\overline{\mathbf{a}'}}(\infty) \in I_\mathbf{e}$. Also, $C_\Gamma^{-1}\tau \leq |I_\mathbf{e}| \leq C_\Gamma\tau$ by (2.27) and (2.21). Therefore, the left-hand side of (2.42) is bounded by

$$2r \cdot \#\big\{\mathbf{e} \in \mathcal{W}^\circ \colon C_\Gamma^{-1}\tau \leq |I_\mathbf{e}| \leq C_\Gamma\tau,\ I_\mathbf{e} \cap J \neq \emptyset\big\}, \quad J := \gamma^{-1}_{\mathbf{a}'\mathbf{b}'}(\infty) + [-\sqrt{\sigma}, \sqrt{\sigma}].$$

Now (2.42) follows from (2.39).

By (2.42) and (2.35), the triples $(\mathbf{b},\mathbf{c},\mathbf{d})$ with $|\gamma^{-1}_{\mathbf{a}'\mathbf{b}'}(\infty) - \gamma^{-1}_{\mathbf{a}'\mathbf{c}'}(\infty)| \leq \sqrt{\sigma}$ contribute at most $C_\Gamma\tau^{-3\delta}\sigma^{\delta/2}$ to the left-hand side of (2.41). Therefore, it remains to show that for each $\mathbf{b},\mathbf{c} \in Z(\tau)$ such that $\mathbf{a} \rightsquigarrow \mathbf{b}$, $\mathbf{a} \rightsquigarrow \mathbf{c}$ and

$$|\gamma^{-1}_{\mathbf{a}'\mathbf{b}'}(\infty) - \gamma^{-1}_{\mathbf{a}'\mathbf{c}'}(\infty)| \geq \sqrt{\sigma}, \tag{2.43}$$

we have

$$\#\big\{\mathbf{d} \in Z(\tau) \colon \mathbf{b} \rightsquigarrow \mathbf{d},\ \mathbf{c} \rightsquigarrow \mathbf{d},\ |\gamma'_{\mathbf{a}'\mathbf{b}'}(x_\mathbf{d}) - \gamma'_{\mathbf{a}'\mathbf{c}'}(x_\mathbf{d})| \leq \tau^2\sigma\big\} \leq C_\Gamma\, \tau^{-\delta}\sigma^{\delta/2}. \tag{2.44}$$

Denote by b_n the last letter of $\mathbf{b}$; we may assume it is also the last letter of $\mathbf{c}$, since otherwise the left-hand side of (2.44) is zero.

By (2.26) and (2.21) we have $C_\Gamma^{-1}\tau^2 \leq |I_{\mathbf{a}'\mathbf{b}}| \leq C_\Gamma\tau^2$ and $C_\Gamma^{-1}\tau^2 \leq |I_{\mathbf{a}'\mathbf{c}}| \leq C_\Gamma\tau^2$. By (2.17) this gives $C_\Gamma^{-1}\tau^2 \leq \gamma'_{\mathbf{a}'\mathbf{b}'} \leq C_\Gamma\tau^2$ and $C_\Gamma^{-1}\tau^2 \leq \gamma'_{\mathbf{a}'\mathbf{c}'} \leq C_\Gamma\tau^2$ on I_{b_n}. Thus it suffices to show that for any given constant C_0 depending only on the Schottky data,

$$\#\Big\{\mathbf{d} \in Z(\tau) \colon \mathbf{b} \rightsquigarrow \mathbf{d},\ \mathbf{c} \rightsquigarrow \mathbf{d},\ \Big|\log \frac{\gamma'_{\mathbf{a}'\mathbf{b}'}(x_\mathbf{d})}{\gamma'_{\mathbf{a}'\mathbf{c}'}(x_\mathbf{d})}\Big| \leq C_0\sigma\Big\} \leq C_\Gamma\, \tau^{-\delta}\sigma^{\delta/2}. \tag{2.45}$$

By (2.4), (2.8), and (2.43), we have

$$|\alpha(\gamma_{\mathbf{a}'\mathbf{b}'}, I_{b_n})|, |\alpha(\gamma_{\mathbf{a}'\mathbf{c}'}, I_{b_n})| \le C_\Gamma, \quad |\alpha(\gamma_{\mathbf{a}'\mathbf{b}'}, I_{b_n}) - \alpha(\gamma_{\mathbf{a}'\mathbf{c}'}, I_{b_n})| \ge C_\Gamma^{-1}\sqrt{\sigma}.$$

By Lemma 2.3, there exists an interval $\widetilde{J}$ of size $C_\Gamma\sqrt{\sigma}$ depending on $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that for each $\mathbf{d}$ on the left-hand side of (2.45), the point $x_{\mathbf{d}}$ lies in $\widetilde{J}$ and thus $I_{\mathbf{d}} \cap \widetilde{J} \neq \emptyset$. Then by (2.39) and (2.26) we obtain (2.45), finishing the proof. □

2.5. **Transfer operators.** For a partition $Z \subset \mathcal{W}^\circ$, define the operator

$$\mathcal{L}_Z : \operatorname{Bor}(\mathcal{I}) \to \operatorname{Bor}(\mathcal{I}), \quad \mathcal{I} := \bigsqcup_{b\in\mathcal{A}} I_b,$$

where $\operatorname{Bor}(\mathcal{I})$ denotes the space of all bounded Borel functions on $\mathcal{I}$, as follows:

$$\mathcal{L}_Z f(x) = \sum_{\mathbf{a}\in Z,\, \mathbf{a}\rightsquigarrow b} f(\gamma_{\mathbf{a}'}(x)) w_{\mathbf{a}'}(x), \quad x \in I_b.$$

Here the weight $w_{\mathbf{a}'}(x)$ is defined in (2.31). The Patterson–Sullivan measure is invariant under the adjoint of $\mathcal{L}_Z$:

Lemma 2.17. *Assume that $Z \subset \mathcal{W}^\circ$ is a partition. Then we have for all $f \in \operatorname{Bor}(\mathcal{I})$,*

$$\int_{\Lambda_\Gamma} f\, d\mu = \int_{\Lambda_\Gamma} \mathcal{L}_Z f\, d\mu. \tag{2.46}$$

Proof. Since Z is a partition, we have by (2.3)

$$\int_{\Lambda_\Gamma} f\, d\mu = \sum_{b\in\mathcal{A}} \sum_{\mathbf{a}\in Z,\ \mathbf{a}\rightsquigarrow b} \int_{I_{\mathbf{a}}} f\, d\mu$$

which together with (2.30) gives (2.46). □

We will use the following corollary of Lemma 2.17:

$$\int_{\Lambda_\Gamma} f\, d\mu = \int_{\Lambda_\Gamma} \mathcal{L}_Z^k f\, d\mu, \quad f \in \operatorname{Bor}(\mathcal{I}),\ k \in \mathbb{N}. \tag{2.47}$$

Note that $\mathcal{L}_Z^k f$ is given by the formula

$$\mathcal{L}_Z^k f(x) = \sum_{\substack{\mathbf{a}_1,\dots,\mathbf{a}_k\in Z\\ \mathbf{a}_1\rightsquigarrow\dots\rightsquigarrow\mathbf{a}_k\rightsquigarrow b}} f(\gamma_{\mathbf{a}_1'\dots\mathbf{a}_k'}(x)) w_{\mathbf{a}_1'\dots\mathbf{a}_k'}(x), \quad x \in I_b. \tag{2.48}$$

3. Fourier decay bound

3.1. **Key combinatorial tool.** The key tool in the proof of Theorem 2 is the following result [Bou10, Lemma 8.43] (more precisely, its version in Proposition 3.3 below):

Proposition 3.1. *For all $\delta_1 > 0$, there exist $\varepsilon_3, \varepsilon_4 > 0$ and $k \in \mathbb{N}$ such that the following holds. Let μ_0 be a probability measure on $[\frac{1}{2}, 1]$ and let N be a large integer. Assume that for all $\sigma \in [N^{-1}, N^{-\varepsilon_3}]$*

$$\sup_x \mu_0\big([x-\sigma, x+\sigma]\big) < \sigma^{\delta_1}. \tag{3.1}$$

Then for all $\eta \in \mathbb{R}$, $|\eta| \sim N$,

$$\Big|\int \exp(2\pi i \eta x_1 \cdots x_k)\, d\mu_0(x_1) \dots d\mu_0(x_k)\Big| \leq N^{-\varepsilon_4}. \tag{3.2}$$

Remark. The main component of the proof of [Bou10, Lemma 8.43] is the discretized sum-product theorem [Bou10, Theorem 1]. Roughly speaking it states that for a finite set $A \subset [\frac{1}{2}, 1]$ of $\frac{1}{N}$-separated points which has box dimension $\geq \delta_1 > 0$, either the sum set $A+A$ or the product set $A \cdot A$ has size at least $N^{\varepsilon} \cdot \#(A)$, where $\varepsilon > 0$ depends only on δ_1. The box dimension condition is analogous to (3.1). We refer the reader to the papers by the first author [Bou03, Bou10] for history and applications of the sum-product theorem. For the passage from the sum-product theorem to the estimate (3.2) in the cleaner case of prime fields see Bourgain–Glibichuk–Konyagin [BGK06, Theorem 5]. See also the expository article of Green [Gr09].

The following is an adaptation of Proposition 3.1 to the case of several different measures with slightly relaxed assumptions:

Proposition 3.2. *Fix $\delta_0 > 0$. Then there exist $k \in \mathbb{N}$, $\varepsilon_2 > 0$ depending only on δ_0 such that the following holds. Let $C_0 > 0$ and $\mu_1, \dots, \mu_k$ be Borel measures on $[C_0^{-1}, C_0] \subset \mathbb{R}$ such that $\mu_j(\mathbb{R}) \leq C_0$. Let $\eta \in \mathbb{R}$, $|\eta| \geq 1$, and assume that for all $\sigma \in \big[C_0|\eta|^{-1}, C_0^{-1}|\eta|^{-\varepsilon_2}\big]$ and $j = 1, \dots, k$*

$$\mu_j \times \mu_j\big(\{(x, y) \in \mathbb{R}^2 \colon |x-y| \leq \sigma\}\big) \leq C_0 \cdot \sigma^{\delta_0}. \tag{3.3}$$

Then there exists a constant C_1 depending only on C_0, δ_0 such that

$$\Big|\int \exp(2\pi i \eta x_1 \cdots x_k)\, d\mu_1(x_1) \dots d\mu_k(x_k)\Big| \leq C_1 |\eta|^{-\varepsilon_2}. \tag{3.4}$$

Proof. We may assume that $|\eta|$ is large depending on C_0, δ_0. By breaking μ_j into pieces supported on $[2^\ell, 2^{\ell+1}]$ where $|\ell| \lesssim \log_2 C_0$ and rescaling η, we reduce to the case when each μ_j is supported on $[\frac{1}{2}, 1]$.

Put $\delta_1 := \delta_0/6$, choose $\varepsilon_3, \varepsilon_4, k$ as in Proposition 3.1, and put

$$\varepsilon_2 := \frac{\min(\varepsilon_4, \varepsilon_3\delta_0)}{10}.$$

We henceforth replace (3.3) with the following assumption:

$$\sup_x \mu_j\big([x-\sigma, x+\sigma]\big) \leq 2\sqrt{C_0} \cdot \sigma^{\delta_0/2}, \quad \sigma \in \big[C_0|\eta|^{-1}, (2C_0)^{-1}|\eta|^{-\varepsilon_2}\big] \tag{3.5}$$

which follows from (3.3) since $[x-\sigma,x+\sigma]^2\subset\{(x,y)\in\mathbb{R}^2\colon |x-y|\leq 2\sigma\}$.

We next claim that it suffices to consider the case $\mu_1=\cdots=\mu_k$. Indeed, denote

$$F(\mu_1,\dots,\mu_k):=\int \exp(2\pi i\eta x_1\cdots x_k)\,d\mu_1(x_1)\dots d\mu_k(x_k).$$

For $\lambda:=(\lambda_1,\dots,\lambda_k)\in[0,1]^k$, put

$$G(\lambda):=F(\mu_\lambda,\dots,\mu_\lambda),\quad \mu_\lambda:=\lambda_1\mu_1+\dots+\lambda_k\mu_k.$$

If $\mu_1,\dots,\mu_k$ satisfy (3.5), then the measure μ_λ satisfies (3.5) as well (with C_0 replaced by k^2C_0). Then the version of Proposition 3.2 for the case $\mu_1=\cdots=\mu_k$ implies that for some C_1' depending only on δ_0,C_0

$$\sup_{\lambda\in[0,1]^k}|G(\lambda)|\leq C_1'|\eta|^{-\varepsilon_2}.$$

Since G is a polynomial of degree k, we have for some C_1 depending only on δ_0,C_0

$$|F(\mu_1,\dots,\mu_k)|=\frac{1}{k!}|\partial_{\lambda_1}\dots\partial_{\lambda_k}G(0,\dots,0)|\leq C_1|\eta|^{-\varepsilon_2}$$

giving (3.4) in the general case.

We henceforth assume that $\mu_1=\cdots=\mu_k$. We consider two cases:

(1) $\mu_1(\mathbb{R})\geq|\eta|^{-\varepsilon_3\delta_0/10}$: define the probability measure μ_0 on $[\frac12,1]$ by

$$\mu_0:=\frac{\mu_1}{\mu_1(\mathbb{R})}.$$

Choose an integer N such that $N\leq|\eta|\leq 2N$. By (3.5) we have

$$\sup_x \mu_0\big([x-\sigma,x+\sigma]\big)<\sigma^{\delta_1},\quad \sigma\in[C_0N^{-1},N^{-\varepsilon_3}].$$

Same is true for $\sigma\in[N^{-1},C_0N^{-1}]$ by applying (3.5) to $\sigma:=C_0N^{-1}$. Then (3.4) follows from Proposition 3.1.

(2) $\mu_1(\mathbb{R})\leq|\eta|^{-\varepsilon_3\delta_0/10}$: the bound (3.4) follows from the triangle inequality. □

In the discrete probability case Proposition 3.2 gives the following statement which is used in the key step of the proof of Theorem 2 at the end of §3.3:

Proposition 3.3. *Fix $\delta_0>0$. Then there exist $k\in\mathbb{N}$, $\varepsilon_2>0$ depending only on δ_0 such that the following holds. Let $C_0,N_{\mathcal{Z}}\geq 0$ and $\mathcal{Z}_1,\dots,\mathcal{Z}_k$ be finite sets such that $\#(\mathcal{Z}_j)\leq C_0N_{\mathcal{Z}}$. Take some maps*

$$\zeta_j:\mathcal{Z}_j\to[C_0^{-1},C_0],\quad j=1,\dots,k.$$

Let $\eta\in\mathbb{R}$, $|\eta|>1$, and consider the sum

$$S_k(\eta)=N_{\mathcal{Z}}^{-k}\sum_{\mathbf{b}_1\in\mathcal{Z}_1,\dots,\mathbf{b}_k\in\mathcal{Z}_k}\exp\big(2\pi i\eta\zeta_1(\mathbf{b}_1)\cdots\zeta_k(\mathbf{b}_k)\big).$$

Assume that ζ_j *satisfy for all* $\sigma \in \big[|\eta|^{-1}, |\eta|^{-\varepsilon_2}\big]$ *and* $j = 1, \dots, k$

$$\#\{(\mathbf{b}, \mathbf{c}) \in \mathcal{Z}_j^2 \colon |\zeta_j(\mathbf{b}) - \zeta_j(\mathbf{c})| \le \sigma\} \le C_0 N_{\mathcal{Z}}^2 \cdot \sigma^{\delta_0}. \tag{3.6}$$

Then we have for some constant C_1 *depending only on* C_0, δ_0

$$|S_k(\eta)| \le C_1 |\eta|^{-\varepsilon_2}. \tag{3.7}$$

Proof. It suffices to apply Proposition 3.2 to the measures μ_j defined by

$$\mu_j(A) := N_{\mathcal{Z}}^{-1} \cdot \#\{\mathbf{b} \in \mathcal{Z}_j \colon \zeta_j(\mathbf{b}) \in A\}, \quad j = 1, \dots, k. \quad \square$$

3.2. **A combinatorial description of the oscillatory integral.** We now begin the proof of Theorem 2. We fix a Schottky representation for M as in §2.1. In this section C denotes constants which depend only on $C_{\varphi,g}$ and the Schottky data.

Put $\delta_0 := \delta/4$ and choose $k \in \mathbb{N}$, $\varepsilon_2 > 0$ from Proposition 3.3, depending only on δ. Let ξ be the frequency parameter in (1.2). Without loss of generality we may assume that $|\xi| \ge C$. Define the small number $\tau > 0$ by

$$|\xi| = \tau^{-2k-3/2}. \tag{3.8}$$

Let $Z(\tau) \subset \mathcal{W}^\circ$ be the partition defined in (2.7) and $\mathcal{L}_{Z(\tau)}$ be the associated transfer operator, see §2.5. Recall from (2.35) that

$$\#(Z(\tau)) \le C\tau^{-\delta}. \tag{3.9}$$

Moreover, by (2.28) and (2.31) we have for each $\mathbf{a} = a_1 \dots a_n \in Z(\tau)$,

$$w_{\mathbf{a}'} \le C\tau^{\delta} \quad \text{on } I_{a_n}. \tag{3.10}$$

We introduce some notation used throughout this section:

- we denote
$$\mathbf{A} = (\mathbf{a}_0, \dots, \mathbf{a}_k) \in Z(\tau)^{k+1}, \quad \mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_k) \in Z(\tau)^k;$$
- we write $\mathbf{A} \leftrightarrow \mathbf{B}$ if and only if $\mathbf{a}_{j-1} \rightsquigarrow \mathbf{b}_j \rightsquigarrow \mathbf{a}_j$ for all $j = 1, \dots, k$;
- if $\mathbf{A} \leftrightarrow \mathbf{B}$, then we define the words $\mathbf{A} * \mathbf{B} := \mathbf{a}_0'\mathbf{b}_1'\mathbf{a}_1'\mathbf{b}_2' \dots \mathbf{a}_{k-1}'\mathbf{b}_k'\mathbf{a}_k'$ and $\mathbf{A}\#\mathbf{B} := \mathbf{a}_0'\mathbf{b}_1'\mathbf{a}_1'\mathbf{b}_2' \dots \mathbf{a}_{k-1}'\mathbf{b}_k'$;
- denote by $b(\mathbf{A}) \in \mathcal{A}$ the last letter of $\mathbf{a}_k$;
- for each $\mathbf{a} \in \mathcal{W}^\circ$, denote by $x_{\mathbf{a}}$ the center of $I_{\mathbf{a}}$;
- for $j \in \{1, \dots, k\}$ and $\mathbf{b} \in Z(\tau)$ such that $\mathbf{a}_{j-1} \rightsquigarrow \mathbf{b} \rightsquigarrow \mathbf{a}_j$, define
$$\zeta_{j,\mathbf{A}}(\mathbf{b}) := \tau^{-2}\gamma'_{\mathbf{a}_{j-1}'\mathbf{b}'}(x_{\mathbf{a}_j}) \tag{3.11}$$
and note that $\zeta_{j,\mathbf{A}}(\mathbf{b}) \in [C^{-1}, C]$ by the chain rule and (2.28).

Using the functions φ, g from the statement of Theorem 2, define

$$f(x) := \exp(i\xi\varphi(x))g(x), \quad x \in \Lambda_\Gamma. \tag{3.12}$$

By (2.47) and (2.48) the integral in (1.2) can be written as follows:

$$\int_{\Lambda_\Gamma} f\,d\mu = \int_{\Lambda_\Gamma} \mathcal{L}_{Z(\tau)}^{2k+1} f\,d\mu = \sum_{\mathbf{A},\mathbf{B}\colon \mathbf{A}\leftrightarrow\mathbf{B}} \int_{I_{b(\mathbf{A})}} f(\gamma_{\mathbf{A}*\mathbf{B}}(x)) w_{\mathbf{A}*\mathbf{B}}(x)\,d\mu(x). \tag{3.13}$$

We use Hölder's inequality and approximations for the weight $w_{\mathbf{A}*\mathbf{B}}$ and the amplitude g to get the following bound. Note that (2.28) and (3.8) imply that the function $e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))}$ below oscillates at frequencies $\sim \tau^{-1/2}$.

Lemma 3.4. *We have*

$$\Big|\int_{\Lambda_\Gamma} f\,d\mu\Big|^2 \leq C\tau^{(2k-1)\delta} \sum_{\mathbf{A},\mathbf{B}\colon \mathbf{A}\leftrightarrow\mathbf{B}} \Big|\int_{I_{b(\mathbf{A})}} e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))} w_{\mathbf{a}'_k}(x)\,d\mu(x)\Big|^2 + C\tau^2. \tag{3.14}$$

Proof. Take arbitrary $x \in I_{b(\mathbf{A})}$, then

$$w_{\mathbf{A}*\mathbf{B}}(x) = w_{\mathbf{A}\#\mathbf{B}}(\gamma_{\mathbf{a}'_k}(x)) w_{\mathbf{a}'_k}(x).$$

Now, $\gamma_{\mathbf{a}'_k}(x)$ lies in $I_{\mathbf{a}_k}$, which by (2.7) is an interval of size no more than τ. By (2.18)

$$\exp(-C\tau) \leq \frac{w_{\mathbf{A}\#\mathbf{B}}(\gamma_{\mathbf{a}'_k}(x))}{w_{\mathbf{A}\#\mathbf{B}}(x_{\mathbf{a}_k})} \leq \exp(C\tau). \tag{3.15}$$

Moreover, by (3.10) and the chain rule

$$w_{\mathbf{A}^*\mathbf{B}}(x) \leq C\tau^{(2k+1)\delta}, \quad w_{\mathbf{A}\#\mathbf{B}}(x_{\mathbf{a}_k}) \leq C\tau^{2k\delta}. \tag{3.16}$$

Recall that $\|g\|_{C^1} \leq C$ by (1.1). Since $\gamma_{\mathbf{A}*\mathbf{B}}(x) \in I_{\mathbf{a}_0}$, by (2.7) we have

$$|f(\gamma_{\mathbf{A}*\mathbf{B}}(x)) - e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))} g(x_{\mathbf{a}_0})| \leq C\tau. \tag{3.17}$$

Put

$$g_{\mathbf{A},\mathbf{B}} := w_{\mathbf{A}\#\mathbf{B}}(x_{\mathbf{a}_k}) g(x_{\mathbf{a}_0}).$$

Combining (3.15)–(3.17), we obtain

$$|f(\gamma_{\mathbf{A}*\mathbf{B}}(x)) w_{\mathbf{A}*\mathbf{B}}(x) - g_{\mathbf{A},\mathbf{B}} e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))} w_{\mathbf{a}'_k}(x)| \leq C\tau^{(2k+1)\delta+1}.$$

Therefore by (3.13) and (3.9)

$$\Big|\int_{\Lambda_\Gamma} f\,d\mu - \sum_{\mathbf{A},\mathbf{B}\colon \mathbf{A}\leftrightarrow\mathbf{B}} g_{\mathbf{A},\mathbf{B}} \int_{I_{b(\mathbf{A})}} e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))} w_{\mathbf{a}'_k}(x)\,d\mu(x)\Big| \leq C\tau. \tag{3.18}$$

Using Hölder's inequality, (3.9), and (3.16), we get

$$\begin{aligned}&\Big|\sum_{\mathbf{A},\mathbf{B}:\ \mathbf{A}\leftrightarrow\mathbf{B}} g_{\mathbf{A},\mathbf{B}}\int_{I_{b(\mathbf{A})}} e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))}w_{\mathbf{a}'_k}(x)\,d\mu(x)\Big|^2\\ &\leq C\tau^{(2k-1)\delta}\sum_{\mathbf{A},\mathbf{B}:\ \mathbf{A}\leftrightarrow\mathbf{B}}\Big|\int_{I_{b(\mathbf{A})}} e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))}w_{\mathbf{a}'_k}(x)\,d\mu(x)\Big|^2.\end{aligned}\tag{3.19}$$

Combining (3.18) and (3.19) finishes the proof. □

To handle the first term on the right-hand side of (3.14), we estimate using (3.10)

$$\begin{aligned}&\sum_{\mathbf{A},\mathbf{B}:\ \mathbf{A}\leftrightarrow\mathbf{B}}\Big|\int_{I_{b(\mathbf{A})}} e^{i\xi\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))}w_{\mathbf{a}'_k}(x)\,d\mu(x)\Big|^2\\ &=\sum_{\mathbf{A}}\int_{I^2_{b(\mathbf{A})}} w_{\mathbf{a}'_k}(x)w_{\mathbf{a}'_k}(y)\sum_{\mathbf{B}:\ \mathbf{A}\leftrightarrow\mathbf{B}} e^{i\xi(\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))-\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(y)))}\,d\mu(x)d\mu(y)\\ &\leq C\tau^{2\delta}\sum_{\mathbf{A}}\int_{I^2_{b(\mathbf{A})}}\Big|\sum_{\mathbf{B}:\ \mathbf{A}\leftrightarrow\mathbf{B}} e^{i\xi(\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))-\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(y)))}\Big|\,d\mu(x)d\mu(y).\end{aligned}\tag{3.20}$$

The next statement bounds the integral $\int f\,d\mu$ by an expression which can be analyzed using Proposition 3.3, by linearizing the phase φ. Recall the definition (3.11) of $\zeta_{j,\mathbf{A}}(\mathbf{b})$.

Lemma 3.5. *Denote*

$$J_\tau := \{\eta\in\mathbb{R}:\ \tau^{-1/4}\leq|\eta|\leq C\tau^{-1/2}\}\tag{3.21}$$

where C is sufficiently large. Then

$$\Big|\int_{\Lambda_\Gamma} f\,d\mu\Big|^2\leq C\tau^{(2k+1)\delta}\sum_{\mathbf{A}}\sup_{\eta\in J_\tau}\Big|\sum_{\mathbf{B}:\ \mathbf{A}\leftrightarrow\mathbf{B}} e^{2\pi i\eta\zeta_{1,\mathbf{A}}(\mathbf{b}_1)\cdots\zeta_{k,\mathbf{A}}(\mathbf{b}_k)}\Big|+C\tau^{\delta/4}.$$

Proof. Fix $\mathbf{A}$. Take $x,y\in I_{b(\mathbf{A})}$ and put

$$\tilde x:=\gamma_{\mathbf{a}'_k}(x),\ \ \tilde y:=\gamma_{\mathbf{a}'_k}(y)\ \in\ I_{\mathbf{a}_k}.$$

Assume that $\mathbf{A}\leftrightarrow\mathbf{B}$. Since $\gamma_{\mathbf{A}*\mathbf{B}}(x)=\gamma_{\mathbf{A}\#\mathbf{B}}(\tilde x)$, $\gamma_{\mathbf{A}*\mathbf{B}}(y)=\gamma_{\mathbf{A}\#\mathbf{B}}(\tilde y)$, we have

$$\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(y))-\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x))=\int_{\tilde x}^{\tilde y}(\varphi\circ\gamma_{\mathbf{A}\#\mathbf{B}})'(t)\,dt.$$

By the chain rule, for each $t\in I_{\mathbf{a}_k}$ there exist $s_j\in I_{\mathbf{a}_j}$, $j=0,\dots,k$, such that

$$(\varphi\circ\gamma_{\mathbf{A}\#\mathbf{B}})'(t)=\varphi'(s_0)\gamma'_{\mathbf{a}'_0\mathbf{b}'_1}(s_1)\cdots\gamma'_{\mathbf{a}'_{k-1}\mathbf{b}'_k}(s_k).$$

By (2.7), we have $|s_j-x_{\mathbf{a}_j}|\leq\tau$. Then by (1.1) and (2.18), we have for all $t\in I_{\mathbf{a}_k}$

$$\exp(-C\tau)\leq\frac{(\varphi\circ\gamma_{\mathbf{A}\#\mathbf{B}})'(t)}{\tau^{2k}\varphi'(x_{\mathbf{a}_0})\zeta_{1,\mathbf{A}}(\mathbf{b}_1)\cdots\zeta_{k,\mathbf{A}}(\mathbf{b}_k)}\leq\exp(C\tau).$$

Since $|\varphi'(x_{\mathbf{a}_0})|, \zeta_{j,\mathbf{A}}(\mathbf{b}_j) \in [C^{-1}, C]$ and $|\tilde{x} - \tilde{y}| \leq \tau$, it follows that

$$\big|\varphi(\gamma_{\mathbf{A}*\mathbf{B}}(x)) - \varphi(\gamma_{\mathbf{A}*\mathbf{B}}(y)) - \tau^{2k}\varphi'(x_{\mathbf{a}_0})\zeta_{1,\mathbf{A}}(\mathbf{b}_1)\cdots\zeta_{k,\mathbf{A}}(\mathbf{b}_k)(\tilde{x} - \tilde{y})\big| \leq C\tau^{2k+2}. \quad (3.22)$$

Denote

$$\eta := \frac{\operatorname{sgn}\xi}{2\pi}\tau^{-3/2}\varphi'(x_{\mathbf{a}_0}) \cdot (\tilde{x} - \tilde{y})$$

and note that by (1.1) and (2.28)

$$C^{-1}\tau^{-1/2}|x - y| \leq |\eta| \leq C\tau^{-1/2}|x - y|.$$

We have by Lemma 3.4, (3.20), (3.22), and (3.9), recalling that $|\xi| = \tau^{-2k-3/2}$ by (3.8)

$$\Big|\int_{\Lambda_\Gamma} f\, d\mu\Big|^2 \leq C\tau^{(2k+1)\delta}\sum_{\mathbf{A}}\int_{I^2_{b(\mathbf{A})}}\Big|\sum_{\mathbf{B}\colon \mathbf{A}\leftrightarrow\mathbf{B}} e^{2\pi i\eta\zeta_{1,\mathbf{A}}(\mathbf{b}_1)\cdots\zeta_{k,\mathbf{A}}(\mathbf{b}_k)}\Big|\, d\mu(x)d\mu(y) + C\sqrt{\tau}.$$

Now, we remark that by (2.33), for each fixed constant C_0

$$\mu \times \mu\big\{(x, y) \in \Lambda_\Gamma^2\colon |x - y| \leq C_0\tau^{1/4}\big\} \leq C\tau^{\delta/4}.$$

Therefore, the double integral above can be taken over x, y such that $|x - y| \geq C_0\tau^{1/4}$, which for large enough C_0 implies that $\eta \in J_\tau$. This finishes the proof. □

3.3. **End of the proof of Theorem 2.** To apply Proposition 3.3 to the sum in Lemma 3.5, we need a positive box dimension estimate. To state it we recall the notation $\mathbf{a} \rightsquigarrow {\mathbf{b} \atop \mathbf{c}} \rightsquigarrow \mathbf{d}$ from (2.40) and the constant ε_2 fixed at the beginning of §3.2.

Lemma 3.6. *Define the set of* ***regular sequences*** $\mathcal{R} \subset Z(\tau)^{k+1}$ *as follows:* $\mathbf{A} \in \mathcal{R}$ *if and only if for all* $j = 1, \dots, k$ *and* $\sigma \in [\tau, \tau^{\varepsilon_2/4}]$ *we have*

$$\tau^{2\delta} \cdot \#\big\{(\mathbf{b}, \mathbf{c}) \in Z(\tau)^2\colon \mathbf{a}_{j-1} \rightsquigarrow {\mathbf{b} \atop \mathbf{c}} \rightsquigarrow \mathbf{a}_j,\ |\zeta_{j,\mathbf{A}}(\mathbf{b}) - \zeta_{j,\mathbf{A}}(\mathbf{c})| \leq \sigma\big\} \leq \sigma^{\delta/4}. \quad (3.23)$$

Then most sequences are regular, more precisely

$$\tau^{(k+1)\delta} \cdot \#(Z(\tau)^{k+1} \setminus \mathcal{R}) \leq C\tau^{\varepsilon_2\delta/20}. \quad (3.24)$$

Proof. For $\ell \in \mathbb{Z}$ with $\tau \leq 2^{-\ell} \leq 2\tau^{\varepsilon_2/4}$, define $\widetilde{\mathcal{R}}_\ell$ as the set of pairs $(\mathbf{a}, \mathbf{d}) \in Z(\tau)^2$ such that

$$\tau^{2\delta} \cdot \#\big\{(\mathbf{b}, \mathbf{c}) \in Z(\tau)^2\colon \mathbf{a} \rightsquigarrow {\mathbf{b} \atop \mathbf{c}} \rightsquigarrow \mathbf{d},\ |\gamma'_{\mathbf{a}'\mathbf{b}'}(x_{\mathbf{d}}) - \gamma'_{\mathbf{a}'\mathbf{c}'}(x_{\mathbf{d}})| \leq \tau^2 2^{-\ell}\big\} \leq 2^{-(\ell+1)\delta/4}.$$

For each $\sigma \in [\tau, \tau^{\varepsilon_2/4}]$, there exists ℓ such that $2^{-\ell-1} \leq \sigma \leq 2^{-\ell}$. By (3.11),

$$\bigcap_j\bigcap_\ell\big\{\mathbf{A} \in Z(\tau)^{k+1} \mid (\mathbf{a}_{j-1}, \mathbf{a}_j) \in \widetilde{\mathcal{R}}_\ell\big\} \subset \mathcal{R}.$$

It suffices to show that for each j, ℓ we have

$$\tau^{2\delta} \cdot \#(Z(\tau)^2 \setminus \widetilde{\mathcal{R}}_\ell) \leq C\tau^{\varepsilon_2\delta/16}. \quad (3.25)$$

By Chebyshev's inequality the left-hand side of (3.25) is bounded above by

$$2^{(\ell+1)\delta/4}\tau^{4\delta} \cdot \#\{(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \in Z(\tau)^4\colon \mathbf{a} \rightsquigarrow {\mathbf{b} \atop \mathbf{c}} \rightsquigarrow \mathbf{d},\ |\gamma'_{\mathbf{a}'\mathbf{b}'}(x_{\mathbf{d}}) - \gamma'_{\mathbf{a}'\mathbf{c}'}(x_{\mathbf{d}})| \leq \tau^2 2^{-\ell}\}$$

By Lemma 2.16 this is bounded above by

$$C2^{-\delta\ell/4} \le C\tau^{\varepsilon_2\delta/16}.$$

This gives (3.25), finishing the proof. □

We are now ready to finish the proof of Theorem 2. Using Lemma 3.5 and estimating the sum over $\mathbf{A} \in Z(\tau)^{k+1} \setminus \mathcal{R}$ by Lemma 3.6, we obtain

$$\left|\int_{\Lambda_\Gamma} f\,d\mu\right|^2 \le C\tau^{k\delta} \max_{\mathbf{A}\in\mathcal{R}} \sup_{\eta\in J_\tau} \left|\sum_{\mathbf{B}:\,\mathbf{A}\leftrightarrow\mathbf{B}} e^{2\pi i\eta\zeta_{1,\mathbf{A}}(\mathbf{b}_1)\cdots\zeta_{k,\mathbf{A}}(\mathbf{b}_k)}\right| + C\tau^{\varepsilon_2\delta/20}. \tag{3.26}$$

We estimate the first term on the right-hand side using Proposition 3.3. Fix $\mathbf{A} \in \mathcal{R}$ and define

$$\mathcal{Z}_j := \{\mathbf{b} \in Z(\tau) \colon \mathbf{a}_{j-1} \rightsquigarrow \mathbf{b} \rightsquigarrow \mathbf{a}_j\}, \quad j = 1, \dots, k.$$

By (2.35),

$$\#(\mathcal{Z}_j) \le CN_{\mathcal{Z}}, \quad N_{\mathcal{Z}} := \tau^{-\delta}.$$

Fix $\eta \in J_\tau$. Recall that $\delta_0 = \delta/4$. By (3.21) and (3.23) we have for all $j = 1, \dots, k$ and $\sigma \in \big[|\eta|^{-1}, |\eta|^{-\varepsilon_2}\big]$

$$\#\big\{(\mathbf{b},\mathbf{c}) \in \mathcal{Z}_j^2 \colon |\zeta_{j,\mathbf{A}}(\mathbf{b}) - \zeta_{j,\mathbf{A}}(\mathbf{c})| \le \sigma\big\} \le N_{\mathcal{Z}}^2 \cdot \sigma^{\delta_0}.$$

Therefore, condition (3.6) is satisfied. We also recall from (3.11) that $\zeta_{j,\mathbf{A}}(\mathbf{b}) \in [C^{-1}, C]$.

Applying Proposition 3.3, we obtain for all $\mathbf{A} \in \mathcal{R}$ and $\eta \in J_\tau$

$$\tau^{k\delta}\left|\sum_{\mathbf{B}:\,\mathbf{A}\leftrightarrow\mathbf{B}} e^{2\pi i\eta\zeta_{1,\mathbf{A}}(\mathbf{b}_1)\cdots\zeta_{k,\mathbf{A}}(\mathbf{b}_k)}\right| \le C|\eta|^{-\varepsilon_2} \le C\tau^{\varepsilon_2/4}. \tag{3.27}$$

From (3.26) and (3.27) we have

$$\left|\int_{\Lambda_\Gamma} f\,d\mu\right| \le C\tau^{\varepsilon_2\delta/40}.$$

Recalling (3.8) and the definition (3.12) of f, this gives Theorem 2 with

$$\varepsilon_1 := \frac{\varepsilon_2\delta}{40(2k+3/2)}. \tag{3.28}$$

4. Fractal uncertainty principle

In this section, we deduce Theorem 1 from Theorem 2 by establishing a fractal uncertainty principle (henceforth denoted FUP) and using the results of [DZ16]. Throughout this section we assume that $M, \delta, \Lambda_\Gamma, \mu$ are as in Theorem 2.

4.1. **FUP for the Patterson–Sullivan measure.** We first use Theorem 2 to obtain a fractal uncertainty principle with respect to the Patterson–Sullivan measure μ:

Proposition 4.1. *Assume that:*

- $U \subset \mathbb{R}^2$ *is an open set and* $V \subset U$ *is compact;*
- $\Phi \in C^3(U;\mathbb{R})$ *and* $G \in C^1(U;\mathbb{C})$, $\operatorname{supp} G \subset V$, *satisfy for some constant* $C_{\Phi,G}$

$$\|\Phi\|_{C^3} + \|G\|_{C^1} \le C_{\Phi,G}, \quad \inf|\partial^2_{xy}\Phi| \ge C^{-1}_{\Phi,G}. \tag{4.1}$$

Define for $0 < h < 1$ *the operator* $B(h) : L^2(\Lambda_\Gamma;\mu) \to L^2(\Lambda_\Gamma;\mu)$ *by*

$$B(h)u(x) = \int_{\Lambda_\Gamma} \exp\Big(\frac{i\Phi(x,y)}{h}\Big) G(x,y)u(y)\,d\mu(y). \tag{4.2}$$

Let $\varepsilon_1 = \varepsilon_1(\delta) > 0$ *be the constant from Theorem 2. Then*

$$\|B(h)\|_{L^2(\Lambda_\Gamma;\mu)\to L^2(\Lambda_\Gamma;\mu)} \le Ch^{\varepsilon_1/4}, \quad 0 < h < 1 \tag{4.3}$$

where the constant C *depends only on* $M, U, V, C_{\Phi,G}$.

Proof. We denote by C constants which depend only on $M, U, V, C_{\Phi,G}$. As in §2.1, we view Λ_Γ as a subset of $\mathbb{R}$. Using a partition of unity for G, we reduce to the case

$$U = I_1^\circ \times I_2^\circ, \quad V = J_1 \times J_2, \quad J_1 \subset I_1^\circ, \quad J_2 \subset I_2^\circ$$

for some intervals I_1, I_2, J_1, J_2. To prove (4.3) suffices to show that

$$\|B(h)B(h)^*\|_{L^2(\Lambda_\Gamma;\mu)\to L^2(\Lambda_\Gamma;\mu)} \le Ch^{\varepsilon_1/2}. \tag{4.4}$$

Note that $B(h)B(h)^*$ is an integral operator:

$$B(h)B(h)^* f(x) = \int_{\Lambda_\Gamma} \mathcal{K}(x,x')f(x')\,d\mu(x'),$$

where

$$\mathcal{K}(x,x') = \int_{\Lambda_\Gamma} \exp\Big(\frac{i}{h}\big(\Phi(x,y) - \Phi(x',y)\big)\Big) G(x,y)\overline{G(x',y)}\,d\mu(y).$$

By Schur's inequality, to show (4.4) it suffices to prove the bound

$$\sup_{x\in\Lambda_\Gamma} \int_{\Lambda_\Gamma} |\mathcal{K}(x,x')|\,d\mu(x') \le Ch^{\varepsilon_1/2}. \tag{4.5}$$

For $x, x' \in \Lambda_\Gamma \cap J_1$, define the functions $\varphi_{xx'}$, $g_{xx'}$ on I_2° as follows:

$$\Phi(x,y) - \Phi(x',y) = (x - x')\cdot\varphi_{xx'}(y), \quad g_{xx'}(y) = G(x,y)\overline{G(x',y)}.$$

Then

$$\mathcal{K}(x,x') = \int_{\Lambda_\Gamma} \exp\big(i\xi\varphi_{xx'}(y)\big) g_{xx'}(y)\,d\mu(y), \quad \xi := \frac{x - x'}{h}. \tag{4.6}$$

It follows from (4.1) that

$$\|\varphi_{xx'}\|_{C^2(I_2^\circ)} + \|g_{xx'}\|_{C^1(I_2^\circ)} \le C, \quad \inf_{I_2^\circ} |\partial_y\varphi_{xx'}| \ge C^{-1}$$

and we extend $g_{xx'}, \varphi_{xx'}$ to compactly supported functions on $\mathbb{R}$ so that

$$\|\varphi_{xx'}\|_{C^2(\mathbb{R})} + \|g_{xx'}\|_{C^1(\mathbb{R})} \leq C, \quad \inf_{\Lambda_\Gamma} |\partial_y \varphi_{xx'}| \geq C^{-1};$$

this is possible since $\Lambda_\Gamma \subset \mathbb{R}$ is compact.

Applying Theorem 2 and using (4.6) we get the bound

$$|\mathcal{K}(x,x')| \leq C\Big|\frac{x-x'}{h}\Big|^{-\varepsilon_1}, \quad x,x' \in \Lambda_\Gamma \cap J_1, \quad |x-x'| \geq h. \tag{4.7}$$

It remains to split the integral in (4.5) into two parts. The integral over $\{|x-x'| \leq h^{1/2}\}$ is bounded by $Ch^{\delta/2}$ by (2.33). The integral over $\{|x-x'| \geq h^{1/2}\}$ is bounded by $Ch^{\varepsilon_1/2}$ by (4.7). □

4.2. **FUP for the Lebesgue measure.** We now deduce from Proposition 4.1 a fractal uncertainty principle with respect to Lebesgue measure on a neighborhood

$$\Lambda_\Gamma(h) := \Lambda_\Gamma + [-h,h] \subset \mathbb{R}$$

of Λ_Γ. We use the following

Lemma 4.2. *For $0 < h < 1$, define the function $F_h(x)$ as the convolution of the Patterson–Sullivan measure μ with the rescaled uniform measure on $[-2h,2h]$:*

$$F_h(x) := \frac{1}{4h^\delta}\mu\big([x-2h, x+2h]\big). \tag{4.8}$$

Then for some constant $C_\Gamma > 0$ depending only on Γ,

$$F_h \geq C_\Gamma^{-1} \quad \text{on } \Lambda_\Gamma(h). \tag{4.9}$$

Proof. Let $x \in \Lambda_\Gamma(h)$. Then there exists $x_0 \in \Lambda_\Gamma$ such that $|x-x_0| \leq h$. We have $[x_0-h, x_0+h] \subset [x-2h, x+2h]$ and $\mu([x_0-h,x_0+h]) \geq C_\Gamma^{-1}h^\delta$ by (2.34). Therefore $F_h(x) \geq C_\Gamma^{-1}$. □

Our fractal uncertainty principle for the Lebesgue measure is the following

Proposition 4.3. *Let $\varepsilon_1 = \varepsilon_1(\delta) > 0$ be the constant from Theorem 2. Assume that U, V, Φ, G are as in Proposition 4.1. Define the operator $\mathcal{B}(h) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$\mathcal{B}(h)u(x) = (2\pi h)^{-1/2}\int_{\mathbb{R}} \exp\Big(\frac{i\Phi(x,y)}{h}\Big)G(x,y)u(y)\,dy. \tag{4.10}$$

Fix $\rho \in (0,1)$. Then

$$\|\, \mathbb{1}_{\Lambda_\Gamma(h^\rho)}\,\mathcal{B}(h)\,\mathbb{1}_{\Lambda_\Gamma(h^\rho)}\,\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leq Ch^{\beta-(1-\delta)(1-\rho)}, \quad \beta := \frac{1}{2} - \delta + \frac{\varepsilon_1}{4}. \tag{4.11}$$

Proof. Let F_{h^ρ} be the function defined in (4.8), with h replaced by h^ρ. By (4.9), it is enough to show the following estimate for each bounded Borel function u on $\mathbb{R}$:

$$\|\sqrt{F_{h^\rho}}\mathcal{B}(h)F_{h^\rho}u\|_{L^2(\mathbb{R})} \leq Ch^{\beta-(1-\delta)(1-\rho)}\|\sqrt{F_{h^\rho}}\,u\|_{L^2(\mathbb{R})}. \tag{4.12}$$

Define the shift operator ω_t on functions on $\mathbb{R}$ by

$$\omega_t v(x) = v(x-t), \quad t, x \in \mathbb{R}.$$

Then for each bounded Borel function v on $\mathbb{R}$,

$$\|\sqrt{F_{h^\rho}}\,v\|^2_{L^2(\mathbb{R})} = \frac{1}{4h^{\rho\delta}}\int_{-2h^\rho}^{2h^\rho}\|\omega_t v\|^2_{L^2(\Lambda_\Gamma;\mu)}\,dt.$$

Moreover

$$\omega_t\mathcal{B}(h)F_{h^\rho}u = \frac{1}{4\sqrt{2\pi}h^{1/2+\rho\delta}}\int_{-2h^\rho}^{2h^\rho}B_{ts}(h)\omega_s u\,ds$$

where

$$B_{ts}(h)v(x) = \int_{\Lambda_\Gamma}\exp\Big(\frac{i\Phi(x-t,y-s)}{h}\Big)G(x-t,y-s)v(y)\,d\mu(y).$$

By Proposition 4.1, we have for all $t, s \in [-2h^\rho, 2h^\rho]$,

$$\|B_{ts}(h)\|_{L^2(\Lambda_\Gamma;\mu)\to L^2(\Lambda_\Gamma;\mu)} \leq Ch^{\varepsilon_1/4}.$$

Then

$$\begin{aligned}\|\sqrt{F_{h^\rho}}\mathcal{B}(h)F_{h^\rho}u\|^2_{L^2(\mathbb{R})} &= \frac{1}{4h^{\rho\delta}}\int_{-2h^\rho}^{2h^\rho}\|\omega_t\mathcal{B}(h)F_{h^\rho}u\|^2_{L^2(\Lambda_\Gamma;\mu)}\,dt\\ &\leq h^{2\rho-3\rho\delta-1}\sup_{|t|\leq 2h^\rho}\int_{-2h^\rho}^{2h^\rho}\|B_{ts}(h)\omega_s u\|^2_{L^2(\Lambda_\Gamma;\mu)}\,ds\\ &\leq Ch^{2\rho-3\rho\delta-1+\varepsilon_1/2}\int_{-2h^\rho}^{2h^\rho}\|\omega_s u\|^2_{L^2(\Lambda_\Gamma;\mu)}\,ds\\ &= 4Ch^{2\rho-2\rho\delta-1+\varepsilon_1/2}\|\sqrt{F_{h^\rho}}\,u\|^2_{L^2(\mathbb{R})}\end{aligned}$$

which gives (4.12). □

4.3. **Proof of Theorem 1.** We use [DZ16, Theorem 3]. It suffices to show that Λ_Γ satisfies the fractal uncertainty principle with exponent $\beta = \frac{1}{2} - \delta + \frac{\varepsilon_1}{4}$ in the sense of [DZ16, Definition 1.1].

The paper [DZ16] uses the Poincaré disk model of the hyperbolic plane and the limit set there is a subset of the circle $\mathbb{S}^1 \subset \mathbb{C}$. To relate to our model, we use the standard transformation from the upper half-plane model to the disk model,

$$z \mapsto w = \frac{z-i}{z+i}. \tag{4.13}$$

Note that, with $|\bullet|$ denoting the Euclidean norm on $\mathbb{C}$, we have for $x, y \in \mathbb{R}$

$$|w(x) - w(y)|^2 = \frac{4(x-y)^2}{(1+x^2)(1+y^2)}.$$

Let $\chi \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$ satisfy $\operatorname{supp}\chi \cap \{w = w'\} = \emptyset$, and $\mathcal{B}_\chi(h)$ be the operator defined in [DZ16, (1.6)]. For the purpose of satisfying [DZ16, Definition 1.1] we may assume that χ is supported near Λ_Γ^2, in particular the pullback of χ to $\mathbb{R}^2$ by the square of the map (4.13) is supported in a compact subset of $\{(x, y) \in \mathbb{R}^2 \mid x \neq y\}$. Then the operator $\mathcal{B}_\chi(h)$ has the form (4.10) with

$$U \Subset \{(x, y) \in \mathbb{R}^2 \mid x \neq y\}, \quad \Phi(x, y) = 2\log|x - y| - \log(1 + x^2) - \log(1 + y^2),$$

and we have on U,

$$\partial^2_{xy}\Phi(x, y) = \frac{2}{(x-y)^2} \neq 0.$$

It remains to apply Proposition 4.3 to see that the fractal uncertainty principle [DZ16, Definition 1.1] holds, finishing the proof.

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