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LAGRANGIAN FIBRATIONS OF HYPERKÄHLER FOURFOLDS

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Abstract. The base surface B of a Lagrangian fibration $X \twoheadrightarrow B$ of a projective, irreducible symplectic fourfold X is shown to be isomorphic to $\mathbb{P}^2$.

For a morphism $X \twoheadrightarrow B$ from a projective, irreducible symplectic manifold X onto a (normal) variety B, Matsushita [Mat99, Mat01] proves that only three situations can occur: The morphism is generically finite, constant, or it describes a Lagrangian fibration.

Moreover, it has been generally conjectured that the normal base B of any connected Lagrangian fibration $X \twoheadrightarrow B$ of a compact, irreducible symplectic manifold X of dimension $2n$ is isomorphic to $\mathbb{P}^n$, cf. [Huy01, Sec. 21.4]. The conjecture has been verified for deformations of Hilbert schemes of K3 surfaces by Markman [Mar14] and for the case that X is projective and B is smooth by Hwang [Hwa08]. In dimension four and assuming smoothness of the base, the conjecture follows easily from the ampleness of ω_B^*, the observation that the Hodge index theorem on X implies $b_2(B) = \rho(B) = 1$, and the classification theory of surfaces, see [Mar95, Mat99].

By building upon work of Ou [Ou18], the present paper completes the verification of the conjecture in dimension four:

Theorem 0.1. *Assume $X \twoheadrightarrow B$ is a connected Lagrangian fibration of a projective, irreducible symplectic fourfold X over a normal surface B. Then $B \simeq \mathbb{P}^2$.*

To prove this result, we study the local situation and exclude the case of E_8-singularities:

Theorem 0.2. *Assume $X \twoheadrightarrow B$ is a projective Lagrangian fibration of a quasi-projective symplectic fourfold over a normal algebraic surface B. If B is locally analytically at a point $0 \in B$ of the form $\mathbb{C}^2/G$ for a finite subgroup $G \subset \mathrm{GL}(2, \mathbb{C})$, then G is not the binary icosahedral group.*

In general, the normal surface B is known to be $\mathbb{Q}$-factorial with at most log-terminal singularities [Mat99]. Thus, locally analytically, B is isomorphic to a quotient of the form $\mathbb{C}^2/G$ for some finite subgroup $G \subset \mathrm{GL}(2, \mathbb{C})$ (not containing quasi-reflections), cf. [Mats02, Ch. 4.6]. As the quotient by the binary icosahedral group is the only factorial quotient singularity [Mum61],

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Theorem 0.2 can be expressed equivalently by saying that all singular points of B are non-factorial. Note, once the morphism is known to be flat, the base is automatically smooth. In fact, by miracle flatness, smoothness of B is equivalent to flatness of the morphism.

The second theorem implies the first one. Indeed, according to [Ou18], for X projective either $B \simeq \mathbb{P}^2$ or B is a specific Fano surface with exactly one singular point, which, moreover, is an E_8-singularity. Note that in the local situation non-factorial singularities do occur, which makes the proof of the conjecture in dimension four an interesting mix of global and local arguments.

The proof of Theorem 0.2 makes use of results of Halle–Nicaise [HaNi18], building upon work of Berkovich [Ber90], and results of Nicaise–Xu [NiXu16]. The former classifies the essential skeleton of semiabelian degenerations of abelian surfaces, which can be seen as a variant of the results of Kulikov and Persson, cf. [FrMo83]. The latter allows one to identify the essential skeleton and the dual complex of the degeneration. Another crucial input for our arguments is Alexeev's work [Ale02].

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While working on this project, F. Bogomolov and N. Kurnosov published [BoKu18] which uses results by J.-M. Hwang and K. Oguiso [HwOg11] to exclude the case of E_8-singularities. We are grateful to F. Bogomolov for explanations concerning their arguments.

1. Actions of the binary icosahedral group

Let G be the binary icosahedral group, which by definition fits into a non-split, short exact sequence $1 \longrightarrow \{\pm 1\} \longrightarrow G \longrightarrow A_5 \longrightarrow 1$ with A_5 the group of alternating permutations of five letters. We exploit the properties of G to study its actions in two settings: First, we let G act as a group of homeomorphisms on a low-dimensional topological manifold. Later this arises as a G-action on the quotient of the essential skeleton of an abelian surface. Second, we study G-actions on complex varieties that are dominated by semiabelian surfaces.

1.1. We begin with the topological setting. Consider a finite extension $1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$.

Proposition 1.1. *Assume Γ acts by homeomorphisms on a topological manifold S, which is either $S^1 \times S^1$, S^1, or a point. Then the induced action of G on S/Γ_0 is trivial.*

Proof. There is nothing to prove in the last case. To deal with the case $S \simeq S^1 \times S^1$, we first observe that if the action of G is not trivial, then the image of $\rho_G \colon G \longrightarrow \mathrm{Homeo}(S/\Gamma_0)$ is either

A_5 or G and, in any case, admits a surjection onto A_5. To see this, recall that $\{\pm 1\}$ is the only non-trivial normal subgroup of G and that A_5 is simple.

We will also use the fact that for any finite subgroup $H \subset \mathrm{Homeo}(S)$ of orientation-preserving homeomorphisms of a compact real surface S, there exists a complex structure on S with respect to which H is a group of biholomorphic automorphisms [Edm85, pp. 340–341]. In our situation, the complex structure defines a complex curve $E \simeq \mathbb{C}/\Lambda$ of genus one. Now, the group of biholomorphic automorphisms of E is a semi-direct product of the abelian group E acting by translations and the group of automorphisms of E as an elliptic curve. The latter is a subgroup of $\mathrm{SL}(2, \mathbb{Z})$, which only contains finite subgroups of order at most six. Hence, H contains an abelian subgroup of index at most six.

Let us apply this to the image of the given action $\rho_\Gamma \colon \Gamma \longrightarrow \mathrm{Homeo}(S)$. Thus, if $\mathrm{Im}(\rho_\Gamma)$ contains only orientation-preserving homeomorphisms, it contains an abelian subgroup of index at most six. The same then holds for its image under the surjection $\mathrm{Im}(\rho_\Gamma) \twoheadrightarrow \mathrm{Im}(\rho_G) \twoheadrightarrow A_5$. However, the only non-trivial subgroups of A_5 of index at most six are isomorphic to A_4 or D_{10}, which are both not abelian. If $\mathrm{Im}(\rho_\Gamma) \subset \mathrm{Homeo}(S)$ does not only contain orientation-preserving homeomorphisms, then the above applies to a certain index two subgroup $\mathrm{Im}(\rho_\Gamma)' \subset \mathrm{Im}(\rho_\Gamma)$. However, as A_5 does not contain any subgroup of index two, the image of $\mathrm{Im}(\rho_\Gamma)'$ under $\mathrm{Im}(\rho_\Gamma) \twoheadrightarrow \mathrm{Im}(\rho_G) \twoheadrightarrow A_5$ is still A_5 and the arguments above yield again a contradiction.

In the second case, the quotient $S/\Gamma_0 \simeq S^1/\Gamma_0$ is either S^1 or the closed interval $[0,1]$. Recall that any finite subgroup of the group $\mathrm{Homeo}(S^1)$ of homeomorphisms of the circle is either cyclic or dyhedral, cf. [Ghy01, Sec. 4]. In the case that S/Γ_0 is again a circle, we apply this to the image of $\rho \colon G \longrightarrow \mathrm{Homeo}(S^1)$. However, as the binary icosahedral group does not admit any non-trivial homomorphism onto a cyclic group, the image has to be trivial. Hence, any action of G on S^1 is actually trivial. If G acts by homeomorphisms on the interval $[0,1]$, then it leaves invariant the boundary and hence, by gluing the boundary points, acts by homeomorphisms on S^1 as well. However, as before, on S^1 the action must be trivial and, hence, G acts trivially on the open interval, which suffices to conclude. □

1.2. Let us now turn to actions of G on low-dimensional varieties.

Proposition 1.2. *Let T be a variety of dimension at most two that is rationally dominated by a surface that is either abelian, rational, or of the form $\mathbb{P}^1 \times E$ with E an elliptic curve. Then, the binary icosahedral group G does not act freely on T.*

Proof. If T is a curve, its normalization is of genus at most one. However, $\mathbb{P}^1$ does not admit any non-trivial, free group action and, according to Proposition 1.1, G does not act (freely) on any elliptic curve. Hence, G does not act freely on T.

Let now T be a surface. Then, by universality of the minimal resolution, the action of G on T lifts to an action of G on its minimal resolution $\tilde{T} \longrightarrow T$. Moreover, the action on $\tilde{T}$ is still free. Now pick a G-equivariant minimal model $\tilde{T} \longrightarrow T_0$. Note that T_0 is indeed minimal unless, possibly, when it has negative Kodaira dimension. Suppose the induced G-action on T_0 has a non-trivial stabilizer G_x at some point $x \in T_0$. Then G_x leaves invariant the exceptional curve in $\tilde{T}$ over x that is blown-down last. However, this exceptional curve is a $\mathbb{P}^1$, which does not admit any non-trivial, free group action. Therefore, also the G-action on T_0 must be free.

If T is a rational surface with a non-trivial, free G-action, then the quotient of the free G-action on the smooth rational surface T_0 is again a smooth rational surface, contradicting the fact that such a surface is simply connected. Assume now that the equivariant minimal model T_0 of T is a blow-up of a ruled surface over a curve $E_0 := \mathrm{Alb}(T_0)$ of genus one. Then the action of G on T_0 covers an action of G on the base E_0 of its ruling. However, again using Proposition 1.1, G does not admit any non-trivial action on the elliptic curve E_0. Hence, G acts on the fibres of the ruling $T_0 \longrightarrow E_0$, but as before there are no non-trivial free group actions on $\mathbb{P}^1$.

Assume now that the minimal model T_0 of T is isomorphic, as a complex manifold, to an abelian surface. It suffices to show that the induced action of G on T_0 is trivial. The G-action on T_0 naturally induces an action on the Albanese variety $\mathrm{Alb}(T_0)$, which now is an abelian surface non-canonically isomorphic to T_0. As $\{\pm 1\}$ is the only non-trivial normal subgroup of G, the image of $\tau\colon G \longrightarrow \mathrm{Aut}(\mathrm{Alb}(T_0))$ is either G, A_5, or trivial. Furthermore, the action on the one-dimensional $H^{2,0}(T_0)$ is trivial, for G does not admit any non-trivial cyclic quotients. Hence, the action on $H^{1,0}(T_0)$ is special. Then, according to [Fuj88, Lem. 3.3], the elements of $\mathrm{Im}(\tau)$ have order $1, 2, 3, 4$, or 6, but both groups, A_5 and G, contain elements of order five. See also [Kat87, Cor. 3.17]. Hence, τ is trivial and, therefore, the action of G on T_0 factors through the abelian group of translations. However, G has only trivial abelian quotients.

Let us next consider the case that T_0 is a K3 or an Enriques surface. Now use that $\chi(\mathcal{O}_{T_0}) = 1$ or 2, to show that T_0 does not admit any free group action of any group of order > 2.

Lastly, assume T_0 is a bielliptic surface $(E_1 \times E_2)/G_0$ and consider the action of the natural extension $1 \longrightarrow G_0 \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$ on $E_1 \times E_2$. As above, if this action on $H^{1,0}(E_1 \times E_2)$ is special, then $\tilde{G}$ acts by translations (note that $\tilde{G}$ again contains an element of order five). Hence, the action of G on T_0 factors through an abelian group and, therefore, is trivial. If the action is not special, then replace $\tilde{G}$ by the kernel of the induced surjection $\tilde{G} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$, which still surjects onto G, and conclude as before. □

2. Semiabelian degenerations

The generic fibre X_t of the Lagrangian fibration $X \twoheadrightarrow B$ is a smooth variety isomorphic to an abelian surface. Indeed, its cotangent bundle is isomorphic to its normal bundle and, therefore, trivial. Thus, the fibration can be viewed as a degeneration of abelian surfaces to more singular fibres. Very little can be said about arbitrary degenerations of abelian surfaces, so we will have to construct a new family that has only mildly singular fibres and, in order to exploit the properties of the binary icosahedral group G, that comes with a G-action. However, only one-dimensional degenerations of abelian surfaces can be studied by means of essential skeleta and dual complexes. So, in a second step, we will pass from the two-dimensional family of complex abelian surfaces to a one-dimensional family of abelian surface over a function field.

2.1. Assume B is étale locally at $0 \in B$ of the form $V := U/G$ with $G \subset \mathrm{GL}(2, \mathbb{C})$ an arbitrary finite group acting on a smooth surface U with a unique fixed point $0 \in U$ and such that the action is free on $U \setminus \{0\}$. To prove the theorem, we may actually reduce to the case $B = V = U/G$, which we will henceforth assume.

The action of G on U naturally lifts to an action on the fibre product $X_U := U \times_V X$ and, by functoriality, to an action on its normalization $Y \longrightarrow X_U$. Note that, by construction, $Y|_{U\setminus\{0\}} \simeq X_{U\setminus\{0\}}$ and, due to the following lemma, Y is in fact smooth.

Lemma 2.1. *The composition $\varphi\colon Y \longrightarrow X_U \longrightarrow X$ is étale and can be identified with the quotient of the action of G on Y, which is free:*

$$\varphi\colon Y \twoheadrightarrow Y/G \simeq X.$$

Proof. Indeed, as the action of G is free on the complement of $0 \in U$, the morphism $\varphi\colon Y \longrightarrow X$ is étale on the complement of the subscheme $Y_0 \subset Y$, which as a Lagrangian subvariety has codimension two [Mat00]. However, as Y is normal and X is smooth, the ramification locus is pure of codimension one and hence empty, i.e. φ is étale. The induced morphism $Y/G \longrightarrow X$ is injective and birational and, using normality of both varieties, in fact an isomorphism. □

From the lemma one immediately deduces the following key fact.

Corollary 2.2. *The induced action $G \times Y_0 \longrightarrow Y_0$ on the central fibre $Y_0 \subset Y$ is free.* □

Eventually, the free action on the fibre Y_0 will lead to a contradiction if the group G is the binary icosahedral group. This will be done in two steps:

(i) The G-action on Y_0 leaves invariant a certain subvariety T.
(ii) The subvariety T does not admit a free G-action.

2.2. Information about Y_0 together with its G-action is difficult to obtain. We will instead work with a functorial extension of the smooth part of the family $Y \longrightarrow U$ to a family of stable semiabelic pairs [Ale02]. The extension has to be produced in a G-equivariant fashion.

In a first step, we turn the smooth fibres Y_t of $Y \longrightarrow U$ into stable semiabelic pairs and then later extend this smooth family to a family over the entire U. To realize this first step, we need to choose, in a uniform way, for each smooth fibre Y_t an effective ample divisor $D_t \subset Y_t$ and an origin $o(t) \in Y_t$ that makes Y_t an abelian surface. The first is easy to arrange, uniformly over U, by picking an ample divisor $D_X \subset X$. If necessary, shrink V to ensure that D_X does not contain any fibres, so that its restriction to all fibres is a divisor. Its pull-back under φ yields a G-invariant relative ample effective divisor $D := \varphi^{-1}(D_X) \subset Y$. To turn the smooth part of $Y \longrightarrow U$ into a family of abelian surfaces, which amounts to choosing uniformly a zero-section, we need to pass to an appropriate Galois cover $U' \longrightarrow U$ (shrink U if necessary), e.g. the Galois closure of a multisection of $Y \to U$. Then the smooth part of $Y' := U' \times_U Y \to U'$ together with the induced section $o\colon U' \to Y'$ constitutes a family of abelian surfaces and the pull-back of $D \subset Y$ is a relative ample effective divisor $D' \subset Y'$. Furthermore, the G-action on $Y \to U$ lifts to an action of G' on $Y' \to U'$, where G' is an extension $1 \to H \to G' \to G \to 1$ of G by the Galois group H of the cover $U' \to U$. We summarize the construction above by the following diagram

$$\begin{array}{ccccccc}
 & \overset{G'}{\circlearrowleft} & & \overset{G}{\circlearrowleft} & & & \\
D' \subset & Y' & \longrightarrow & Y & \longrightarrow X_U & \longrightarrow & X \simeq Y/G \\
 & o \uparrow\downarrow & & \downarrow & \downarrow & & \downarrow \\
 & U' & \xrightarrow[/H]{} & U & = U & \xrightarrow[/G]{} & V \simeq U/G. \\
 & \underset{G'}{\circlearrowleft} & & & & &
\end{array}$$

Note that when passing to U', the origin $0 \in U$ gets replaced by an orbit $\{0_1, \ldots, 0_n\}$ of the action of H or, equivalently, of the action of G'.

Let $\Delta \subset U'$ be the discriminant locus of $Y' \to U'$. Then the restriction of Y', D', and the section o to its complement $U' \setminus \Delta$ describes a family of abelian surfaces together with an effective polarization of a certain degree d. It corresponds to a morphism to the moduli stack $\mathcal{AP}_{2,d}$ of such pairs, which admits a coarse quasi-projective moduli space $AP_{2,d}$:

$$U' \setminus \Delta \to \mathcal{AP}_{2,d} \to AP_{2,d}. \tag{2.1}$$

According to [Ale02], the main component $\overline{\mathcal{AP}}_{2,d}$ of the moduli stack of stable semiabelic pairs admits a coarse moduli space $\overline{AP}_{2,d}$, which is a proper algebraic space and, in fact, a projective variety [KoPa17]. Thus, $\overline{AP}_{2,d}$ provides a compactification of $AP_{2,d}$. As the surface U' is normal, we may shrink U' to an open, invariant neighbourhood of the H-orbit $\{0_i\}$, such that (2.1) extends to a morphism

$$U' \setminus \{0_i\} \to \overline{AP}_{2,d}. \tag{2.2}$$

This determines a stable semiabelic pair for each point $t \in U' \setminus \{0_i\}$, but not quite a family yet. In order to complete our family over $U' \setminus \Delta$ to a family of stable semiabelic varieties over Δ and, in particular, over the points 0_i, we need to modify U'. We do this in two steps. Firstly, resolve the indeterminacies of the rational map (2.2) by passing to a (multiple) blow-up $U' \longleftarrow \mathrm{Bl}(U') \longrightarrow \overline{AP}_{2,d}$, which can be done in a G'-equivariant manner (with the trivial action on $\overline{AP}_{2,d}$). However, a morphism to $\overline{AP}_{2,d}$ does still not provide us with a family. But there exists a Galois cover $U'' \to \mathrm{Bl}(U')$, with some Galois group H', by an irreducible normal surface U' (to simplify, you may shrink U' further) together with a lift $\tilde{\phi}$ of $\phi\colon \mathrm{Bl}(U') \to \overline{AP}_{2,d}$ to the stack $\overline{\mathcal{AP}}_{2,d}$:

$$
\begin{array}{ccccccc}
\overset{G''}{\circlearrowleft} & & \overset{G'}{\circlearrowleft} & & \overset{G'}{\circlearrowleft} & & \\
U'' & \xrightarrow[/H']{} & \mathrm{Bl}(U') & \xrightarrow{\sigma} & U' & \hookleftarrow & U' \setminus \{0_i\} \\
\downarrow{\scriptstyle \tilde{\phi}} & & & \searrow^{\phi} & \vdots & & \downarrow \\
\overline{\mathcal{AP}}_{2,d} & & \longrightarrow & & \overline{AP}_{2,d} & \hookleftarrow & AP_{2,d}.
\end{array}
$$

Here, G'' is the natural extension $1 \longrightarrow H' \longrightarrow G'' \longrightarrow G' \longrightarrow 1$ of G' by the Galois group H'. Alternatively, G'' can be seen as a finite extension

$$1 \longrightarrow K \longrightarrow G'' \longrightarrow G \longrightarrow 1 \tag{2.3}$$

of the original G by a finite group K, which in turn is an extension of H by H'. Again, $\tilde{\phi}$ is G''-equivariant, this time with respect to a non-trivial action on the target.

Pulling-back the universal family over $\overline{\mathcal{AP}}_{2,d}$ via $\tilde{\phi}$ to U'' yields a family $(\mathcal{A}, \mathcal{B}) \longrightarrow U''$ of stable semiabelic varieties. Here, $\mathcal{A} \longrightarrow U''$ is a semiabelian scheme with an action on the projective family $\mathcal{B} \longrightarrow U''$. We are suppressing the effective relative ample divisor, which is of no relevance to us. The existence of the semiabelian extension $\mathcal{A}$ is due to Grothendieck and Mumford [Gro72, Exp. IX, Prop. 3.5] and Faltings–Chai [FaCh90, Ch. 6], cf. [CaMa13].

Note that over the pre-image under $U'' \longrightarrow U'$ of the open subset $U' \setminus \Delta$ the two families $\mathcal{A}$ and $\mathcal{B}$ coincide with the pull-back $Y'' := U'' \times_{U'} Y'$ of $Y' \longrightarrow U'$. Clearly, the action of G' on Y' lifts to an action of G'' on Y'', which in turn yields an action of G'' on the restriction of $(\mathcal{A}, \mathcal{B})$ to the pre-image of $U' \setminus \Delta'$. In fact, as $\overline{\mathcal{AP}}_{2,d}$ is separated, the action of G'' extend to all of $(\mathcal{A}, \mathcal{B}) \longrightarrow U''$ over the given action on U''.

2.3. Consider the equivariant birational morphism $\sigma \colon \mathrm{Bl}(U') \longrightarrow U'$ with respect to the action of the stabilizer $G'_1 := \mathrm{Stab}_{G'}(0_1)$ of the point $0_1 \in U'$. Then, running the equivariant MMP for surfaces, one always finds a closed point $t_1 \in \sigma^{-1}(0_1)$ fixed under the G'_1-action or an irreducible G'_1-invariant curve $C_1 \subset \sigma^{-1}(0_1)$. This can also be seen as a special case of a more general result due to Hogadi–Xu [HoXu09, Thm. 1.3], cf. [LiXu19, Prop. 3.6]. In the case of a fixed point t_1, we blow-up once more to reduce to the case of a G'_1-invariant exceptional curve C_1. Note, for later reference, that G'_1 is a finite extension $1 \longrightarrow K_1 \longrightarrow G'_1 \longrightarrow G \longrightarrow 1$, i.e. the projection induces a surjection $G'_1 \twoheadrightarrow G$. Note that in general the inclusion $K_1 \subset H$ is proper.

Next, we pick an irreducible curve $C \subset U''$ dominating C_1 with its scheme-theoretic generic point $\eta_C \in C$ and consider the subgroup $H'_C \subset H'$ leaving it invariant. Then there exists a subgroup $G''_C \subset G''$ given by an extension $1 \longrightarrow H'_C \longrightarrow G''_C \longrightarrow G'_1 \longrightarrow 1$, whose induced action on $(\mathcal{A}, \mathcal{B}) \longrightarrow U''$ leaves $C \subset U''$ invariant.

Consider now a morphism $\mathrm{Spec}(R) \longrightarrow U''$ from the spectrum of a DVR R with its closed point 0_R mapped to η_C and its generic point η_R mapped to the generic point $\eta_{U''}$ of U''. We assume that both extensions $k(\eta_C) \subset k(0_R)$ and $k(\eta_{U''}) \subset k(\eta_R)$ are finite and Galois. Geometrically, up to finite cover, we think of $\mathrm{Spec}(R)$ as a curve in U'' intersecting the curve C in its generic point.

The pull-back of $(\mathcal{A},\mathcal{B})$ to $\mathrm{Spec}(R)$ shall be denoted by $(\mathcal{A}_R,\mathcal{B}_R)$. Then, $\mathcal{A}_R \longrightarrow \mathrm{Spec}(R)$ describes a degeneration of the abelian surface $A := \mathcal{A}_{\eta_R}$ to the semiabelian variety $A_0 := \mathcal{A}_{0_R}$. After passing to a further finite extension of R, we may assume that the irreducible components of $\mathcal{B}_{0_R}$ are geometrically irreducible. For later use, we state the following observation.

Lemma 2.3. *Let $W_0 \subset \mathcal{B}_{0_R}$ be an irreducible component of the closed fibre of $\mathcal{B}_R \longrightarrow \mathrm{Spec}(R)$. Then the $k(0_R)$-variety W_0 is either an abelian surface, or birational to $\mathbb{P}^1 \times E$ with E an elliptic curve, or a rational surface.*

Proof. Indeed, the pair $(\mathcal{A}_{0_R},\mathcal{B}_{0_R})$ forms a stable semiabelic variety. In particular, the semiabelian surface $\mathcal{A}_{0_R}$ acts with only finitely many orbits on $\mathcal{B}_{0_R}$. The three cases correspond to the three possibilities for the dimension $0 \leq d \leq 2$ of the toric part of $\mathcal{A}_{0_R}$. □

2.4. Note that the normal surface U''/K comes with a birational morphism $U''/K \longrightarrow U$, where K is given by (2.3). Also, the two quotients $\mathcal{A}/K$ and $\mathcal{B}/K$ over U''/K are birational to $Y \longrightarrow U$. In fact, they coincide over the pre-image of the complement of the discriminant locus of the latter.

The action of the group G''_C on $(\mathcal{A},\mathcal{B})$ lifts to an action of a group Γ on $(\mathcal{A}_R,\mathcal{B}_R) \longrightarrow \mathrm{Spec}(R)$. Here, Γ is as an extension $1 \longrightarrow \mathrm{Gal} \longrightarrow \Gamma \longrightarrow G''_C \longrightarrow 1$ by the finite Galois group of the extension $k(\eta_{U''}) \subset k(\eta_R)$. The action of Γ respects the closed fibre $\mathcal{B}_{0_R}$. However, as it involves the action of Gal, it is not an action on the $k(0_R)$-variety $\mathcal{B}_{0_R}$. For later use, we remark that Γ can also be regarded as a finite extension of G, say

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

The situation is pictured by the following diagram

$$\begin{array}{ccccc}
\overset{\Gamma}{\circlearrowleft} & & \overset{G''_C}{\circlearrowleft} & & \overset{G'_1}{\circlearrowleft} \\
\mathrm{Spec}(k(0_R)) & \longrightarrow & C & \xrightarrow{/H'_C} & C_1 \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma \curvearrowright \mathrm{Spec}(R) & \longrightarrow & U'' & \xrightarrow{/H'} & \mathrm{Bl}(U') \\
\uparrow & & \uparrow & & \\
\mathrm{Spec}(k(\eta_R)) & \longrightarrow & \mathrm{Spec}(k(\eta_{U''})). & &
\end{array}$$

Let $K_C := K \cap G''_C$, which sits in an exact sequence $1 \longrightarrow K_C \longrightarrow G''_C \longrightarrow G \longrightarrow 1$, and let $\Gamma_C \subset \Gamma$ be its pre-image, for which we have an exact sequence $1 \longrightarrow \mathrm{Gal} \longrightarrow \Gamma_C \longrightarrow K_C \longrightarrow 1$. Then, taking quotients of the semiabelic family $\mathcal{B}$ yields inclusions

$$\mathcal{B}_{0_R}/\mathrm{Gal} \subset \mathcal{B}_C \text{ and } \mathcal{B}_{0_R}/\Gamma_C \subset \mathcal{B}_C/K_C \subset \mathcal{B}/K \dashrightarrow Y.$$

In particular, the closure of the image of an irreducible $k(0_R)$-subvariety $W_0 \subset \mathcal{B}_{0_R}$ yields a $\mathbb{C}$-subvariety $\overline{W}_0 \subset \mathcal{B}_C$ fibred over C.

Remark 2.4. Assume $W_0 \subset \mathcal{B}_{0_R}$ is invariant under a subgroup $\Gamma' \subset \Gamma$ which under the projection $\Gamma \twoheadrightarrow G$ surjects onto G. Then $\overline{W}_0 \subset \mathcal{B}_C$ is G''_C-invariant and

$$W \coloneqq \overline{W}_0/K_C \subset \mathcal{B}_C/K_C \subset \mathcal{B}/K \dashleftarrow\dashrightarrow Y$$

is G-invariant. Also note that the varieties $W = \overline{W}_0/K_C \subset \mathcal{B}_C/K_C$ are both fibred over the curve $C/K_C = C_1/K_1$, which is an exceptional curve of $\mathrm{Bl}(U')/H \longrightarrow U$.

Remark 2.5. Note that if the $k(0_R)$-variety W_0 is of dimension two, i.e. an irreducible component of the special fibre $\mathcal{B}_{0_R}$, then $W = \overline{W}_0/K_C$ is of complex dimension three and, hence, a divisor in $\mathcal{B}/K$. Therefore, W rationally dominates a subvariety T of the closed fibre Y_0.

More concretely, as $C/K_C \subset U''/K$ is blown-down under the birational map $U''/K \simeq \mathrm{Bl}(U')/H \longrightarrow U$ and Y is a family over U, any fibre W_t of $W \longrightarrow C/K_C$ for t in a dense open subset of C/K_C rationally dominates T. Now, specializing the $k(0_R)$-variety W_0 to the complex variety W_t, Lemma 2.3 allows one to conclude that T is dominated by a surface that is either abelian, rational, or isomorphic to $\mathbb{P}^1 \times E$ with E an elliptic curve.

3. Action on the skeleton and the fibre

In this final section, we apply the results of Section 1.1 to the G-action on the essential skeleton of the one-dimensional degeneration over $\mathrm{Spec}(R)$ constructed above. This will provide us with a G-invariant subvariety of the special fibre. Then, combining the results of Sections 1.2 and 2.2, leads to a contradiction.

3.1. The next result is a consequence of [HaNi18, Cor. 4.3.3], building upon [Ber90, Sect. 6.5], adapted to our situation. As in the previous section, A denotes the generic fibre $\mathcal{A}_{\eta_R}$ of the semiabelian family $\mathcal{A}_R \longrightarrow \mathrm{Spec}(R)$ over a DVR. By construction, the function field $k(\eta_R)$ of R is a finite Galois extension of the function field $k(\eta_{U''})$ of the complex surface U''. In the course of our discussion, we will tacitly replace R by a suitable finite extensions when necessary. Passing to the algebraic closure of $k(0_R)$ and the completion of R, the base change of A yields an abelian surface over a complete discretely valued field with algebraically closed residue field of characteristic zero. In this situation, the Kontsevich–Soibelman essential skeleton, a subspace of the Berkovich space associated with the abelian surface, was studied by Mustaţă and Nicaise [MuNi15], see also [Nic17] for a survey. Suppressing the passage to the completion in the notation, we will write $\mathrm{Sk}(A)$ for the essential skeleton of the base change of A. Note that $\mathrm{Sk}(A)$ can be computed in terms of any sncd model [MuNi15, Thm. 4.5.5] and is, therefore, a finite CW complex. In particular, it can in fact be computed over a finite extension of R.

Proposition 3.1. *The group* Γ *acts continuously on the essential skeleton* $\mathrm{Sk}(A)$ *of the fibre* $A = \mathcal{A}_{\eta_R}$. *The homeomorphism type of* $\mathrm{Sk}(A)$ *is given by one of the three possibilities:*

$$\text{(i) } \mathrm{Sk}(A) \simeq \{\mathrm{pt}\}, \ \text{(ii) } \mathrm{Sk}(A) \simeq S^1, \ \text{or (iii) } \mathrm{Sk}(A) \simeq S^1 \times S^1. \tag{3.1}$$

Proof. The description of the homeomorphism type of $\mathrm{Sk}(A)$ is [HaNi18, Cor. 4.3.3]. For the first assertion, we use again that $\mathrm{Sk}(A)$ can be computed in terms of an sncd model and of the dual complex of its closed fibre. Such a model can always be equivariantly constructed, starting with $\mathcal{B}_R$ and possibly enlarging the group Γ further, so that the group action on A can be extended to the closed fibre. Although the group action on the closed fibre does not respect the structure of the closed fibre as a variety over $k(0_R)$, it still acts continuously on its dual complex. □

Combined with Proposition 1.1, one obtains the following result.

Corollary 3.2. *If* G *is the binary icosahedral group, then the induced action on* $\mathrm{Sk}(A)/\Gamma_0$ *is trivial.* □

We will also need the identification of the essential skeleton with the dual complex of an appropriate sncd model. The result is a consequence of [NiXu16] and [KNX18]. Recall that the family $\mathcal{B}_R \twoheadrightarrow \mathrm{Spec}(R)$, with generic fibre $A = \mathcal{A}_{\eta_R} \simeq \mathcal{B}_{\eta_R}$, was constructed in a way such that all irreducible components of the closed fibre $\mathcal{B}_{0_R}$ are geometrically irreducible. However, $\mathcal{B}_{0_R} \subset \mathcal{B}_R$ is usually not an snc divisor and its dual complex is not well defined. To remedy the situation, we first observe the following.

Lemma 3.3. *The family* $\mathcal{B}_R \twoheadrightarrow \mathrm{Spec}(R)$ *satisfies the following conditions:* $\mathcal{B}_{0_R}$ *is reduced, the canonical bundle* $K_{\mathcal{B}_R}$ *is trivial, and the pair* $(\mathcal{B}_R, \mathcal{B}_{0_R})$ *is log-canonical.*

Proof. By construction $\mathcal{B}_{0_R}$ is a semiabelic variety and, therefore, by definition reduced. The other two assertions follow from the explicit construction of the degeneration, see [Ale02, Thm. 5.7.1] and [Mum61]. See also [AlNa99, Lem. 4.1&4.2] for the fact that $\mathcal{B}_{0_R}$ is Gorenstein and $K_{\mathcal{B}_{0_R}}$ trivial and [Ale96, Lem. 3.7&3.8] combined with [Kol, Thm. 4.9(2)] for the fact that $(\mathcal{B}_R, \mathcal{B}_{0_R})$ is log-canonical. □

Now we use the existence of a Γ-equivariant dlt modification $\pi \colon \tilde{\mathcal{B}} \twoheadrightarrow \mathcal{B}_R$ of the pair $(\mathcal{B}_R, \mathcal{B}_{0_R})$, cf. [Kol, Thm. 1.34]. In particular, $(\tilde{\mathcal{B}}, (\tilde{\mathcal{B}}_{0_R})_{\mathrm{red}} = \pi_*^{-1}\mathcal{B}_{0_R} + E)$ is dlt. Here, $\pi_*^{-1}\mathcal{B}_{0_R}$ is the strict transform of the (reduced) fibre $\mathcal{B}_{0_R}$ and E is the reduced exceptional divisor of π. Hence, the dual complex $\Delta((\tilde{\mathcal{B}}_{0_R})_{\mathrm{red}})$ of $(\tilde{\mathcal{B}}_{0_R})_{\mathrm{red}}$ is well defined [dFKX17]. As $(\mathcal{B}_R, \mathcal{B}_{0_R})$ is log-canonical, the dlt modification satisfies $\pi^*(K_{\mathcal{B}_R} + \mathcal{B}_{0_R}) = K_{\tilde{\mathcal{B}}} + \pi_*^{-1}\mathcal{B}_{0_R} + E$. As $K_{\mathcal{B}_R}$ and $\mathcal{B}_{0_R}$ are both trivial divisors on $\mathcal{B}_R$, also $K_{\tilde{\mathcal{B}}} + \pi_*^{-1}\mathcal{B}_{0_R} + E$ is trivial. Hence, both assumptions of [KNX18, Thm. 24] are fulfilled, which immediately yields the next result.

Corollary 3.4. *There exists a* Γ*-equivariant homeomorphism* $\mathrm{Sk}(A) \simeq \Delta((\tilde{\mathcal{B}}_{0_R})_{\mathrm{red}})$ *between the essential skeleton of* A *and the dual complex of the closed fibre* $(\tilde{\mathcal{B}}_{0_R})_{\mathrm{red}}$. □

Let now $W_0 \subset \mathcal{B}_{0_R}$ be an irreducible component and let $\Gamma' \subset \Gamma$ be the subgroup that leaves W_0 invariant. Combining Corollary 3.2, Corollay 3.4, and Remark 2.4, we have proved the following consequence.

Corollary 3.5. *The composition $\Gamma' \subset \Gamma \twoheadrightarrow G$ is surjective and, therefore, $W = \overline{W}_0/K_C$ is a G-invariant subvariety of $\mathcal{B}_C/K_C \subset \mathcal{B}/K$.* □

3.2. As the G-actions on $\mathcal{B}/K$ and Y are compatible under the birational map $\mathcal{B}/K \dashrightarrow Y$, the subvariety $T \subset Y_0$ corresponding to W, cf. Remark 2.5, is G-invariant. However, due to Proposition 1.2, the action of G on T cannot be free, which contradicts Corollary 2.2.

This concludes the proof of Theorem 0.2 and, at the same time, of Theorem 0.1.

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