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# STABILITY ANALYSIS OF TRANSPORTATION NETWORKS WITH MULTISCALE DRIVER DECISIONS*

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**Abstract.** Stability of Wardrop equilibria is analyzed for dynamical transportation networks in which the drivers' route choices are influenced by information at multiple temporal and spatial scales. The considered model involves a continuum of nonatomic indistinguishable drivers commuting between a common origin-destination pair in an acyclic transportation network. The drivers' route choices are affected by their relatively infrequent perturbed best responses to global information about the current network congestion levels, as well as their instantaneous local observation of the immediate surroundings as they transit through the network. A novel model is proposed for driver route choice behavior, exhibiting local consistency with their preference toward globally less congested paths as well as myopic decisions in favor of locally less congested paths. The simultaneous evolution of the traffic congestion on the network and of the aggregate path preference is modeled by a system of coupled ordinary differential equations. The main result shows that if the frequency of updates of path preferences is sufficiently small as compared to the frequency of the traffic flow dynamics, then the state of the transportation network ultimately approaches a neighborhood of the Wardrop equilibrium. The presented results may be read as further evidence in support of Wardrop's postulate of equilibrium, showing robustness of it with respect to nonpersistent perturbations. The proposed analysis combines techniques from singular perturbation theory, evolutionary game theory, and cooperative dynamical systems.

**Key words.** transportation networks, Wardrop equilibrium, traffic flows, evolutionary game dynamics, route choice behavior, multiscale decisions

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## 1. Introduction.

As transportation demand is fast approaching its infrastructure capacity, a rigorous understanding of the relationship between the macroscopic properties of transportation networks and realistic driver route choice behavior is attracting renewed research interest. Such an analysis is essential for, among other things, appropriate design of incentives influencing drivers' behavior in order to induce a desired socially optimal usage of the transportation infrastructure. A particularly relevant issue is the impact of drivers' en route responses to *unexpected events* on the overall transportation network dynamics. This issue is particularly significant in modern transportation network settings, where recent technological advancements in

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intelligent traveler information devices have enabled drivers to be much more flexible in selecting their routes to destination even while en route. While there has been a significant research effort to investigate the effect of such technologies on the route choice behavior of drivers (e.g., see [26, 22]), the analytical study of the dynamical properties of the whole network under such behavior has attracted very little attention.

This paper is focused on the stability analysis of transportation networks in a setup where the drivers have access to traffic information at multiple temporal and spatial scales and they have the flexibility to switch their route to the destination at every intermediate traffic intersection. Specifically, we consider a model in which the drivers choose their routes while having access to relatively infrequent global information about the network congestion state and to real-time local information as they transit through the network. Drivers route choice behavior is then influenced by relatively slowly evolving path preferences as well as myopic responses to the instantaneous observation of the local congestion levels at the intersections. This setup captures many real-life scenarios where unexpected events observed en route might cause drivers to take a temporary detour, but not necessarily to change their path preferences. Such path preferences may instead be updated, e.g., on a daily, weekly, or longer time basis in response to information about the global congestion state of the different origin-destination paths collected from drivers' personal experience, opinion exchanges with peers, and information media. However, since traffic dynamics is significantly influenced by drivers' responses to real-time local information, such responses can influence drivers' path preferences, thereby modifying their global route choice behavior in the long run. We propose and analyze a novel model for driver route choice behavior that combines relatively infrequent information about the global congestion status of the network with real-time local observations, as explained below.

In our model, the network is represented by a directed acyclic graph with one origin and one destination. A continuous constant flow of nonatomic indistinguishable drivers enters from the origin and flows through the network until reaching the destination node. Traffic parameters, such as average speed, traffic density, and flow, are modeled as homogeneous quantities on every link, related to each other by functional dependencies representative of the links' congestion properties. The dynamics of such traffic parameters is governed by the law of conservation of mass, as well as driver route choice behavior. In turn, the driver route choice behavior is assumed to be influenced by two factors: the aggregate path preference, measuring the relative appeal of the different routes to the drivers, and local observations of the current congestion levels. The path preference dynamics evolve at a slow time scale (as compared to the traffic dynamics), following a perturbed best response to global information, embodied by the current congestion levels on the whole network. When traversing an intermediate node in the network, drivers behave according to their path preference, if this is consistent with the current, locally observed, aggregate behavior of the other drivers. On the other hand, when there is a discrepancy between the aggregate path preference and the locally observed aggregate behavior, then drivers tend to compensate by myopically preferring routes which appear to be locally less congested.

The model described above gives rise to a double feedback dynamics, governed by a finite-dimensional system of coupled ordinary differential equations. Such a dynamical system has two natural time scales, characterizing the dynamics of the drivers' aggregate path preference and of the traffic parameters on the different links, respectively. We study the long-time behavior of this dynamical system: our main result shows that in the limit of a small update rate of the aggregate path preferences, a state of approximate *Wardrop equilibrium* [27] is approached. The latter is a configuration in which the delay associated to any source-destination path chosen by a nonzero fraction of drivers does not exceed the delay associated to any other path. Our results provide a stronger evidence in support of the significance of Wardrop's postulate of equilibrium for a transportation network. They may also be read as a sort of robustness of such equilibrium notion with respect to nonpersistent perturbations of the network.

The analytical arguments we propose mainly rely on three ideas: adopting a singular perturbation approach [16] by considering the aggregate path preference as "quasi-static" when studying the fast-scale dynamics of the traffic parameters and the traffic parameters as "almost equilibrated" when analyzing the slow-scale dynamics of the aggregate path preference; exploiting the inherent cooperative[1] dependence of the route choice function on the local traffic parameters in order to establish exponential stability of the fast-scale dynamics of the traffic parameters; and adapting results from evolutionary population games [15, 24] in order to establish stability properties of the slow-scale perturbed best response dynamics of the aggregate path preference.

Our work is naturally related to two streams of literature on transportation networks. On the one hand, traffic flows on networks have been widely analyzed with fluid-dynamical and kinetic models: see, e.g., [11] and references therein. As compared to these models (typically described by integro- or partial differential equations), ours significantly simplifies the evolution of the traffic parameters (treating them as homogeneous quantities on the links, representative of spatial averages), whereas it highlights the role of driver route choice behavior with its double feedback dynamics, which is typically neglected in that literature.

On the other hand, transportation networks have been studied from a decision-theoretic perspective within the framework of *congestion games* [3, 23]. In these models, drivers make sequential myopic route choice decisions in pursuit of minimizing their personal travel times and in response to complete information about the whole network. Congestion games are known to belong to the class of *potential games* [19], a consequence of which is that best responses of the drivers are *aligned* with the gradient of a common potential function and hence the system eventually converges to a critical point of this potential function, which under appropriate monotonicity conditions of the congestion properties of the links of the network corresponds to a Wardrop equilibrium. Such an approach has been used, for example, in [18]. Dynamical systems frameworks for stability analysis of transportation equilibria have also been developed in [25, 5, 20]. The stability of Wardrop equilibrium in the context of communication networks has been studied in [4]. It is important to note that the two salient features of a typical congestion game setup are that information is available to the drivers at a single temporal and spatial scale and that the dynamics of traffic parameters are completely neglected by assuming that they are instantaneously equilibrated.

In contrast, we study the stability of Wardrop equilibrium in a setting where the dynamics of the traffic parameters are not neglected and driver route choice decisions are affected by relatively infrequent global information as well as real-time local information as they transit through the network. As a consequence, classic results of evolutionary game theory and population dynamics [15, 24] are not directly applicable to our framework, and novel analytical tools have to be developed, particularly for the analysis of the fast-scale dynamics of the traffic parameters. For such dynamics, the most novel technical feature of our approach consists in proving local contraction properties which follow from the cooperative nature and other structural properties of the system.

[1]Here, the adjective "cooperative" is intended in the sense of Hirsch [12, 13].

The rest of the paper is organized as follows. In section 2, we formulate the model and state the main result. Section 3 is a technical section that contains the proofs for the main result, including intermediate results. In section 4, we report results from illustrative numerical experiments. Finally, we conclude in section 5 and mention potential future research directions.

Before proceeding, we establish some notation to be used throughout the paper. Let $\mathbb{R}$ be the set of reals and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ be the set of nonnegative reals, Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets. Then, $|\mathcal{A}|$ will denote the cardinality of $\mathcal{A}$, $\mathbb{R}^{\mathcal{A}}$ (respectively, $\mathbb{R}_+^{\mathcal{A}}$) the space of real-valued (nonnegative-real-valued) vectors whose components are labeled by elements of $\mathcal{A}$, and $\mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ the space of matrices whose real entries are labeled by pairs of elements in $\mathcal{A} \times \mathcal{B}$. The transpose of a matrix $M \in \mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ will be denoted by $M' \in \mathbb{R}^{\mathcal{B}\times\mathcal{A}}$, while $I$ will be an identity matrix and $\mathbf{1}$ the all-one vector, whose size will be clear from the context. We shall use the notation $\Phi := I - |\mathcal{A}|^{-1}\mathbf{1}\mathbf{1}' \in \mathbb{R}^{\mathcal{A}\times\mathcal{A}}$ to denote the projection matrix on the space orthogonal to $\mathbf{1}$. The simplex of probability vectors over a finite set $\mathcal{A}$ will be denoted by $\mathcal{S}(\mathcal{A}) := \{x \in \mathbb{R}_+^{\mathcal{A}} : \mathbf{1}'x = 1\}$. If $\mathcal{B} \subseteq \mathcal{A}$, $\mathbb{1}_{\mathcal{B}} : \mathcal{A} \to \{0,1\}$ will stand for the indicator function of $\mathcal{B}$ with $\mathbb{1}_{\mathcal{B}}(a) = 1$ if $a \in B$ and $\mathbb{1}_{\mathcal{B}}(a) = 0$ if $a \in \mathcal{A} \setminus \mathcal{B}$. For $p \in [1,\infty]$, $\|\cdot\|_p$ is the $p$-norm. By default, let $\|\cdot\| := \|\cdot\|_2$ denote the Euclidean norm. Let $\mathrm{int}(\mathcal{X})$ be the interior of a set $\mathcal{X} \subseteq \mathbb{R}^d$, and let $\partial\mathcal{X}$ denote its boundary. Let $\mathrm{sgn} : \mathbb{R} \to \{-1,0,1\}$ be the sign function, defined by $\mathrm{sgn}(x) = 1$ if $x > 0$, $\mathrm{sgn}(x) = -1$ if $x < 0$, and $\mathrm{sgn}(x) = 0$ if $x = 0$. By convention, we shall assume the identity $\mathrm{d}|x|/\mathrm{d}x = \mathrm{sgn}(x)$ to be valid for every $x \in \mathbb{R}$, including $x = 0$. Finally, we shall adopt the convention that the gradient $\nabla f$ of a function $f : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subseteq \mathbb{R}^{\mathcal{A}}$, is a column vector in $\mathbb{R}^{\mathcal{A}}$, while $\tilde{\nabla} f := \Phi\nabla f$ will stand for the projected gradient on $\mathcal{S}(\mathcal{A})$.

**2. Model formulation and main result.** In this section, we formulate the problem and state the main result. In our formulation, we represent the dynamics of the traffic and the route choice behavior on a transportation network as a system of coupled ordinary differential equations with two time scales representative of route choice behavior influenced by the two levels of information. The dynamics of the physical variables, i.e., density and flow on each link, evolve at the fast time scale and are driven by local information on the current physical state of the network, whereas the aggregate path preferences evolve at the slow time scale in response to global information on the current physical state of the network.

The key components of our model are network topology, congestion properties of the links, path preference dynamics, and nodewise route choice decision. We next describe each of these components in detail.

**2.1. Network characteristics.** Let the topology of the transportation network be described by a directed graph (in short, *di-graph*) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of (directed) links. For every node $v \in \mathcal{V}$, we shall denote by $\mathcal{E}_v^-$ and $\mathcal{E}_v^+$ the sets of its incoming and, respectively, outgoing links. A length-$l$ (directed) path from $u \in \mathcal{V}$ to $v \in \mathcal{V}$ is an $l$-tuple of consecutive links $\{(v_{j-1}, v_j) \in \mathcal{E} : 1 \leq j \leq l\}$ with $v_0 = u$ and $v_l = v$. A cycle is path of length $l \geq 1$ from a node $v$ to itself. Throughout this paper, we shall assume the following.

*Assumption* 1. The di-graph $\mathcal{G}$ contains no cycles, has a unique origin (i.e., some $v \in \mathcal{V}$ such that $\mathcal{E}_v^- = \emptyset$), and has a unique destination (i.e., some $v \in \mathcal{V}$ such that $\mathcal{E}_v^+ = \emptyset$). Moreover, there exists a path to the destination node from every other node in $\mathcal{V}$, as well as from the origin node to any other node in $\mathcal{V}$.

Assumption 1 implies that one can find a (not necessarily unique) topological ordering of the node set $\mathcal{V}$ (see, e.g., [8]). We shall assume to have fixed one such ordering, identifying $\mathcal{V}$ with the integer set $\{0, 1, \ldots, n\}$, where $n := |\mathcal{V}| - 1$, in such a way that

$$\mathcal{E}_v^- \subseteq \bigcup_{0 \le u < v} \mathcal{E}_u^+ \qquad \forall v = 0, \ldots, n\,.$$

We shall model the traffic parameters as time-varying quantities which are homogeneous over each link of the network. Specifically, for every link $e \in \mathcal{E}$ and time instant $t \ge 0$, we shall denote the current traffic density and flow by $\rho_e(t)$ and $f_e(t)$, respectively, while

$$\rho(t) := \{\rho_e(t) : \, e \in \mathcal{E}\}\,, \qquad f(t) := \{f_e(t) : e \in \mathcal{E}\}$$

will stand for the vectors of all traffic densities and flows, respectively. Current traffic flow and density on each link are related by a functional dependence

$$f_e = \mu_e(\rho_e)\,, \qquad e \in \mathcal{E}\,. \tag{2.1}$$

Such functional dependence models the drivers' speed and lane adjustment behavior in response to traffic density on a particular segment of a road. It will be assumed to satisfy the following.

*Assumption* 2. For every link $e \in \mathcal{E}$, the flow-density function $\mu_e : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, strictly increasing, and strictly concave and is such that

$$\mu_e(0) = 0\,, \qquad \lim_{\rho_e \downarrow 0} \frac{\mathrm{d}}{\mathrm{d}\rho_e} \mu_e(\rho_e) < \infty\,.$$

*Remark* 1. Flow-density functions commonly used in transportation theory typically are not globally increasing but rather have a $\cap$-shaped graph [11]: $\mu_e(\rho_e)$ increases from $\mu_e(0) = 0$ until achieving a maximum $C_e = \mu_e(\tilde{\rho}_e)$ and then decreases for $\rho_e \ge \tilde{\rho}_e$. Assumption 2 remains a good approximation of this setting, provided that $\rho_e$ stays in the interval $[0, \tilde{\rho}_e)$. It should be noted that the fact that the support of the flow function is unbounded, i.e., that the density can grow as large as possible, prevents the (fast time scale) dynamics (2.11) of the physical variables to take into account backward propagation of perturbations.

For every link $e \in \mathcal{E}$, let

$$C_e := \sup\{\mu_e(\rho_e) : \, \rho_e \ge 0\} = \lim_{\rho_e \to \infty} \mu_e(\rho_e)$$

be its *maximum flow capacity*. Moreover, let

$$\mathcal{F}_v := \prod_{e \in \mathcal{E}_v^+} [0, C_e)\,, \qquad \mathcal{F} := \prod_{e \in \mathcal{E}} [0, C_e)$$

be the sets of local and, respectively, global feasible flow vectors. Observe that our formulation allows for both the cases of bounded and unbounded maximum flow capacities. As the flow $f_e$ is the product of speed and density, it is natural to introduce the delay function

$$T : \mathbb{R}_+^{\mathcal{E}} \to [0, +\infty]^{\mathcal{E}}\,, \qquad T_e(f_e) := \begin{cases} +\infty & \text{if} \quad f_e \ge C_e, \\ \mu_e^{-1}(f_e)/f_e & \text{if} \quad f_e \in (0, C_e), \\ 1/\frac{\mathrm{d}\mu_e}{\mathrm{d}\rho_e}(0) & \text{if} \quad f_e = 0\,, \end{cases} \tag{2.2}$$

whose components measure the flow-dependent time taken to traverse the different links.[2]

*Example* 1. A flow-density function that satisfies Assumption 2 is given by

$$\mu_e(\rho_e) = C_e\left(1 - e^{-\theta_e\rho_e}\right) \quad \forall e \in \mathcal{E}, \tag{2.3}$$

where $C_e > 0$ and $\theta_e > 0$. The corresponding delay function is

$$T_e(f_e) = \frac{1}{\theta_e f_e}\log\frac{C_e}{C_e - f_e}\,.$$

We shall denote by $\mathcal{P}$ the set of distinct paths in $\mathcal{G}$ from the origin node 0 to the destination node $n$. Let

$$A \in \mathbb{R}^{\mathcal{E}\times\mathcal{P}}\,, \qquad A_{ep} = \begin{cases} 1 & \text{if} \quad e \in p, \\ 0 & \text{if} \quad e \notin p \end{cases}$$

be the link-path incidence matrix of $\mathcal{G}$. The relative appeal of the different paths to the drivers will be modeled by a time-varying probability vector over $\mathcal{P}$, which will be referred to as the current *aggregate path preference* and denoted by $\pi(t)$. If one assumes, as we shall do throughout this paper, a constant unit in-flow in the origin node, it is natural to consider the vector

$$f^\pi := A\pi$$

of the flows associated to the current aggregate path preference. Indeed, the $e$th entry, $f^\pi_e = \sum_p A_{ep}\pi_p$, represents the total traffic flow that a link $e \in \mathcal{E}$ would sustain in a static condition in which the fraction of drivers choosing any path $p \in \mathcal{P}$ is given by $\pi_p$. Now, let

$$\Pi := \{\pi \in \mathcal{S}(\mathcal{P}) \,:\, f^\pi_e < C_e \,\forall e \in \mathcal{E}\}$$

be the set of *feasible path preferences*. Here, the term "feasible" refers to the fact that the flow vector $f^\pi$ associated to any $\pi \in \Pi$ satisfies the capacity constraint $f^\pi_e < C_e$ for every $e \in \mathcal{E}$. Observe that whenever $C_e > 1$ for every $e \in \mathcal{E}$ (or when link capacities are infinite), the set of feasible path preferences $\Pi$ coincides with the whole simplex $\mathcal{S}(\mathcal{P})$. In contrast, when $C_e \le 1$ for some $e \in \mathcal{E}$, $\Pi \subset \mathcal{S}(\mathcal{P})$ is a strict inclusion. On the other hand, the following result shows that whether $\Pi$ is empty depends solely on the value of the min-cut capacity of the network [1, Ch. 4]. Let

$$C^* := \min_{\substack{\mathcal{U}\subseteq\mathcal{V}:\\ 0\in\mathcal{U}, n\notin\mathcal{U}}} C_{\mathcal{U}}\,, \qquad C_{\mathcal{U}} := \sum_{e\in\mathcal{E}^+_{\mathcal{U}}} C_e\,,$$

where $\mathcal{E}^+_{\mathcal{U}} := \{e = (u, v) \in \mathcal{E} \,:\, u \in \mathcal{U},\, v \in \mathcal{V}\setminus\mathcal{U}\}$.

PROPOSITION 2.1. *The set* $\Pi$ *is nonempty if and only if* $C^* > 1$.

*Proof.* Fix a cut-set $\mathcal{U} \subseteq \mathcal{V}$ such that $0 \in \mathcal{U}$ and $n \notin \mathcal{U}$. Then, every path $p \in \mathcal{P}$ contains exactly one link $(u, v) \in p$ such that $u \in \mathcal{U}$ and $v \in \mathcal{V}\setminus\mathcal{U}$. Hence, for every $\pi \in \Pi$, one has that

$$C_{\mathcal{U}} = \sum_{e\in\mathcal{E}^+_{\mathcal{U}}} C_e > \sum_p \sum_{e\in\mathcal{E}^+_{\mathcal{U}}} A_{ep}\pi_p = \sum_p \pi_p = 1\,.$$

Minimizing over all cut-sets $\mathcal{U}$ shows that $C^* > 1$ is necessary for $\Pi$ to be nonempty.

[2]Here, it has implicitly been assumed without any loss of generality that all the links are of unit length.

For the inverse implication, consider a network with the same topology $\mathcal{G}$ and link capacities $c_e = \max\{C_e - |\mathcal{E}|^{-1}(C^* - 1), 0\}$. The min-cut capacity of this network satisfies $c^* \geq C^* - (C^* - 1) = 1$. Note also that from our construction, $C_e > c_e \geq 0$. Therefore, the max-flow min-cut theorem (see, e.g., [1, Thm. 4.1]) implies that there exists some $\pi \in \Pi$, thus proving that $\Pi$ is nonempty. $\square$

In the case when $C^* \leq 1$ it is not hard to show that the system will grow unstable, i.e., $\rho_e(t)$ is unbounded as $t$ grows large, for some link $e \in \mathcal{E}$. Therefore, throughout this paper, we shall confine ourselves to transportation networks satisfying the following.

*Assumption* 3. The min-cut capacity satisfies $C^* > 1$.

**2.2. Route choice behavior and traffic dynamics.** We now describe driver route choice behavior and traffic dynamics on the network. We envision a continuum of indistinguishable drivers traveling through the network. Drivers enter the network from the origin node 0 at a constant unit rate, travel through it, and leave the network from the destination node $n$. While inside the network, drivers occupy some link $e \in \mathcal{E}$. The time required by the drivers to traverse link $e$ and the current flow on such link are governed by its congestion properties, as given by (2.2) and (2.1), respectively. When entering the network from the origin node $v = 0$, as well as when reaching the head node $v \in \{1, 2, \ldots, n-1\}$ of some link $e \notin \mathcal{E}_n^-$, the drivers instantaneously join some link $e \in \mathcal{E}_v^+$. In this paper, we shall model the choice of such a new link to depend on infrequently updated perturbed best responses of the drivers to global information about the congestion status of the whole network as well as on their instantaneous observation of the local congestion levels. We next describe these two aspects of the model in detail.

**Aggregate path preference dynamics.** The drivers' aggregate path preference $\pi(t)$, already introduced in section 2.1, models the relative appeal of the different paths to the drivers' population. It is updated as drivers access global information about the current congestion status of the whole network. This occurs at some rate $\eta > 0$, which could be thought of as being small with respect to the time scale of the network flow dynamics. Information about the current status of the network is embodied by the current traffic flow vector $f(t)$. From $f(t)$, drivers can evaluate the vector $A'T(f(t))$, whose $p$th entry, $\sum_e A_{ep} T_e(f_e(t))$, coincides with the total delay a driver expects to incur on path $p$ assuming that the congestion levels on that path won't change during her journey.[3] Drivers are modeled as reacting to such global information by updating their path preferences independently at rate $\eta$ according to some feasible path preference $F^h(f(t)) \in \Pi$, so that the aggregate path preference $\pi(t)$ evolves as

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(t) = \eta \left(F^h(f(t)) - \pi(t)\right) . \qquad (2.4)$$

Here $F^h : \mathcal{F} \to \Pi$ is a *perturbed* (or *smoothed*) *best response* function, as per Assumption 4 formulated below. First, let us introduce the notion of *admissible perturbation.*

[3]The delay that a driver would actually incur taking path $p$ at time $t$ would in fact possibly differ from $\sum_e A_{ep} T_e(f_e(t))$, since by the time $t' \geq t$ the driver reaches a certain link $e \in p$, the delay on that link, $T_e(f_e(t'))$, might well have changed from its value $T_e(f_e(t))$ at time $t$, as a result of the fast-scale dynamics of the physical variables $\rho_e(t)$ and $f_e(t)$.

Definition 2.2. *An admissible perturbation is a function $h : \Pi_h \to \mathbb{R}$, where $\Pi_h \subseteq \Pi$ is a closed convex set, $h(\,\cdot\,)$ is strictly convex, twice differentiable in* $\mathrm{int}(\Pi_h)$, *and is such that* $\lim_{\pi \to \partial \Pi_h} ||\tilde{\nabla} h(\pi)|| = \infty$.

*Example* 2. Let $\mathcal{D} := \{e \in \mathcal{E} : C_e \leq 1\}$ be the possibly empty set of links with capacity not exceeding 1. For a $\varepsilon \in (0, \min_e C_e)$ and $\beta > 0$, let

$$\Pi_h := \{\pi \in \Pi : f_d^\pi \leq C_d - \varepsilon \ \forall d \in \mathcal{D}\}$$

and define $h : \Pi_h \to \mathbb{R}$ by

$$h(\pi) := \beta^{-1} \sum_p \pi_p \log \pi_p + \beta^{-1} \sum_d (C_d - \varepsilon - f_d^\pi) \log(C_d - \varepsilon - f_d^\pi)\,,$$

where the summation indices $p$ and $d$ run over the sets $\mathcal{P}$ and $\mathcal{D}$, respectively, and the standard convention $0 \log 0 := 0$ is adopted. It can be readily verified that $\Pi_h \subseteq \Pi$ is a nonempty convex polytope, and $\lim_{\pi \to \partial \Pi_h} ||\tilde{\nabla} h(\pi)|| = \infty$. Hence, $h$ is an admissible perturbation. Observe that if $C_e > 1$ for all $e \in \mathcal{E}$, then $\mathcal{D}$ is empty, $\Pi_h = \Pi$, and $h(\pi) = \beta^{-1} \sum_p \pi_p \log \pi_p$ reduces to the standard negative entropy function.

Observe that compactness and convexity of $\Pi_h$, together with strict convexity of $h(\omega)$, imply existence and uniqueness of a minimizer of $\omega' A' T(f) + h(\omega)$ in $\Pi_h$. This supports the following.

*Assumption* 4. The function $F^h : \mathcal{F} \to \Pi$ is a perturbed best response, i.e.,

$$F^h(f) := \operatorname*{argmin}_{\omega \in \Pi_h} \left\{\omega' A' T(f) + h(\omega)\right\}, \qquad f \in \mathcal{F}\,, \tag{2.5}$$

where $h : \Pi_h \to \mathbb{R}$ is an *admissible perturbation* (as per Definition 2.2).

In fact, Assumption 4 and Definition 2.2 imply that $F^h(f) \in \mathrm{int}(\Pi_h)$ and that $F^h(f)$ is continuously differentiable on $\mathcal{F}$. The perturbed best response function $F^h(f)$ provides an idealized description of the behavior of drivers whose decisions are based on inexact information about the state of the network. In particular, it can be shown that the form of $F^h(f)$ given in (2.5) is equivalent to the minimization, over paths $p \in \mathcal{P}$, of the expected delay $\sum_e A_{ep} T_e(f_e)$ corrupted by some (admissible) stochastic perturbation (see, e.g., [14]). Moreover, it is well known [24] that as $\|h\|_\infty \downarrow 0$ and $\Pi_h \uparrow \overline{\Pi}$, the perturbed best response $F^h(f)$ converges to the set $\mathrm{argmin}\{\omega' A' T(f) : \omega \in \Pi\}$ of best responses.[4]

*Example* 3. Assume that $C_e > 1$ for all $e \in \mathcal{E}$, and fix a noise parameter $\beta > 0$. Define a perturbed best response by putting $\Pi_h = \Pi$ and $h(\pi) = \beta^{-1} \sum_p \pi_p \log \pi_p$ for all $\pi \in \Pi$, as in the special case of Example 2. Then, the corresponding perturbed best response is the logit function

$$F_p^h(f) = \frac{\exp(-\beta(A'T(f))_p)}{\sum_{q \in \mathcal{P}} \exp(-\beta(A'T(f))_q)}\,, \qquad p \in \mathcal{P}\,. \tag{2.6}$$

For any fixed $f \in \mathcal{F}$, one has that $\lim_{\beta \to \infty} F^h(f)$, with $F^h(f)$ as defined in (2.6), is a uniform distribution over the set $\mathrm{argmin}\{(A'T(f))_p : p \in \mathcal{P}\}$. We refer the reader to [10, 15] for more on the connection between $F^h$ characterized by Assumption 4 and smoothed best response functions.

*Remark* 2. The microfoundations of the aggregate path dynamics in (2.4) can be heuristically justified by looking at it as the mean-field limit of a stochastic finite

[4]Here, the convergences $\Pi_h \uparrow \overline{\Pi}$ and $\{F^h(f)\} \to \mathrm{argmin}\{\omega' A' T(f) : \omega \in \Pi\}$ are intended to hold in the Hausdorff metric (see, e.g., [2, Def. 4.4.11]).

population model along the following lines. Consider a model with a large but finite driver population, with each driver updating her path preference at the clicking of an independent Poisson clock of rate $\eta$ by choosing a new preferred path $p$ with probability $F_p^h(f(t))$. Then one could show that the conditional average variation of the aggregate path preference from time $t$ to time $t+\varepsilon$, for small $\varepsilon > 0$, is given by $\eta(F^h(f(t)) - \pi(t))\varepsilon + o(\varepsilon)$. As the stochastic elements of the drivers' updating mechanisms are idiosynchratic, one may expect such stochastic influences to be averaged away as the population size grows large by appealing to some law of large numbers in the spirit of Kurtz's theorem [17], [9, Ch. 11]. We will not attempt to formally justify the microfoundations of the model discussed in this paper but rather leave it as a topic for future work.

*Remark* 3. In the evolutionary game theory literature, e.g., see [15, 24], the domain of an admissible perturbation function $h$, as well as that of the minimization in the right-hand side of (2.5), is typically assumed to be the whole simplex $\mathcal{S}(\mathcal{P})$, instead of a closed polytope $\Pi_h \subseteq \Pi \subseteq \mathcal{S}(\mathcal{P})$. Notice that, as already observed in section 2.1, when $C_e > 1$ for every $e \in \mathcal{E}$, $\Pi = \mathcal{S}(\mathcal{P})$ is a closed polytope, so that one can choose $\Pi_h = \Pi$. Therefore, in this case, Definition 2.2 does not introduce any additional restriction with respect to such theory.

On the other hand, when $C_e \leq 1$ for some $e \in \mathcal{E}$, then the inclusions of $\Pi_h \subset \Pi \subset \mathcal{S}(\mathcal{P})$ are both strict, so that Definition 2.2 does introduce additional restrictions on the admissible perturbations. However, it is worth observing that, in a classic evolutionary game theoretic framework, the dynamics of the aggregate path preference would be autonomous rather than coupled to the one of the actual flow. In particular, perturbed best response dynamics in that framework would read as

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(t) = F^h(f^\pi(t)) - \pi(t)\,, \tag{2.7}$$

rather than as in (2.4). For such dynamics, the fact that $T_e(f_e^\pi) = \infty$ whenever $f_e^\pi \geq C_e$ can be shown to imply that $\pi(t)$ reaches a compact $\Pi_h \subseteq \Pi$ in some finite time and never leaves it. In contrast, in the two-time-scale models of coupled dynamics considered in this paper (see (2.13)), such a more restrictive assumption is needed in order to ensure the same property for the trajectories of $\pi(t)$ (see Lemma 3.4).

**Local route decisions.** We now describe the local route decisions, characterizing the fraction of drivers choosing each link $e \in \mathcal{E}_v^+$ when traversing a nondestination node $v$. Such a fraction will be assumed to be a continuously differentiable function $G_e^v(f_{\mathcal{E}_v^+}, \pi)$ of the local traffic flow $f_{\mathcal{E}_v^+} := \{f_e : e \in \mathcal{E}_v^+\}$, as well as of the current aggregate path preference $\pi$. We shall refer to

$$G^v : \mathcal{F}_v \times \Pi \to \mathcal{S}(\mathcal{E}_v^+) \tag{2.8}$$

as the *local decision function* at node $v \in \{0, 1, \ldots, n-1\}$ and assume that it satisfies the following.

*Assumption* 5. For all $0 \leq v < n$ and $\pi \in \Pi$,

$$\left(\sum_{j \in \mathcal{E}_v^+} f_j^\pi\right) G_e^v\left(f_{\mathcal{E}_v^+}^\pi, \pi\right) = f_e^\pi \qquad \forall e \in \mathcal{E}_v^+\,.$$

*Assumption* 6. For all $0 \leq v < n$, $\pi \in \Pi$, and $f_{\mathcal{E}_v^+} \in \mathcal{F}_v$,

$$\frac{\partial}{\partial f_e} G_j^v(f_{\mathcal{E}_v^+}, \pi) \geq 0 \qquad \forall j \neq e \in \mathcal{E}_v^+\,.$$

Assumption 5 is a consistency assumption. It postulates that when the locally observed flow coincides with the one associated to the aggregate path preference $\pi$, drivers choose to join link $e \in \mathcal{E}_v^+$ with a frequency equal to the ratio between the flow $f_e^\pi$ and the total outgoing flow $\sum_{j\in\mathcal{E}_v^+} f_j^\pi$.

Assumption 6 instead models the drivers' myopic behavior in response to variations of the local congestion levels. It postulates that if the congestion on one link increases while the congestion on the other links outgoing from the same node is kept constant, the frequency with which each of the other outgoing links is chosen does not decrease. It is worth observing that Assumption 6 is reminiscent of Hirsch's notion of a *cooperative dynamical system* [12, 13].

*Example* 4. An example of a local decision function $G^v$ satisfying Assumptions 5 and 6 is the i-logit function. The i-logit route choice function with sensitivity $\gamma \geq 0$ is given by

$$G_e^v(f_{\mathcal{E}_v^+}, \pi) = \frac{f_e^\pi \exp(-\gamma(f_e - f_e^\pi))}{\sum_{j\in\mathcal{E}_v^+} f_j^\pi \exp(-\gamma(f_j - f_j^\pi))} \tag{2.9}$$

for every $e \in \mathcal{E}_v^+$, $0 \leq v < n$. Observe that in the extreme case $\gamma = 0$, (2.9) reduces to

$$G_e^v(f_{\mathcal{E}_v^+}, \pi) = \frac{f_e^\pi}{\sum_{j\in\mathcal{E}_v^+} f_j^\pi} \tag{2.10}$$

which models a situation where the drivers do not take into account the local observation on the current flow and always act in a way that is consistent with their aggregate path preference.

For every nondestination node $v \in \{0, 1, \ldots, n-1\}$ and outgoing link $e \in \mathcal{E}_v^+$, conservation of mass implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_e(t) = H_e(f(t), \pi(t)), \tag{2.11}$$

where

$$H_e(f,\pi) := \begin{cases} G_e^v(f_{\mathcal{E}_v^+}, \pi) - f_e & \text{if} \quad v = 0, \\ (\sum_{j\in\mathcal{E}_v^-} f_j) G_e^v(f_{\mathcal{E}_v^+}, \pi) - f_e & \text{if} \quad 1 \leq v < n \end{cases} \tag{2.12}$$

for all $\pi \in \Pi$ and $f \in \mathcal{F}$.

**2.3. Objective of the paper and main result.** The objective of this paper is to study the evolution of the coupled dynamics

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi(t) = \eta\left(F^h(f(t)) - \pi(t)\right), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\rho(t) = H(f(t), \pi(t)), \end{cases} \tag{2.13}$$

where $F^h$ is the perturbed best response function defined in (2.5); $\eta > 0$ is the rate at which global information becomes available; $H(f,\pi) = \{H_e(f,\pi) : e \in \mathcal{E}\}$ with $H_e$ defined in (2.11); and $f$ and $\rho$ are related by the functional dependence (2.1). In particular, our analysis will focus on the double limiting case of small $\eta$ and small $h$. We shall prove that in such a limiting regime, the long-time behavior of the system is approximately at Wardrop equilibrium [27, 21]. The latter is a configuration in which the delay is the same on all the paths chosen by a nonzero fraction of drivers. More formally, one has the following.

Definition 2.3 (Wardrop equilibrium). *A feasible flow vector* $f^W \in \mathcal{F}$ *is a* Wardrop equilibrium *if* $f^W = A\pi$ *for some* $\pi \in \Pi$ *such that for all* $p \in \mathcal{P}$,

$$\pi_p > 0 \quad \Longrightarrow \quad (A'T(A\pi))_p \leq (A'T(A\pi))_q \qquad \forall q \in \mathcal{P}\,. \tag{2.14}$$

Existence and uniqueness of a Wardrop equilibrium are guaranteed by the following standard result.

Proposition 2.4 (existence and uniqueness of Wardrop equilibrium). Let Assumptions 1–3 be satisfied. Then, there exists a unique Wardrop equilibrium $f^W \in \mathcal{F}$.

*Proof.* It follows from Assumption 2 that for every $e \in \mathcal{E}$, the delay function $T_e(f_e)$ is continuous, strictly increasing, and such that $T_e(0) > 0$. The proposition then follows by applying Theorems 2.4 and 2.5 from [21]. □

The following is the main result of this paper. It will be proved in section 3 using a singular perturbation approach.

Theorem 2.5. *Let Assumptions* 1–6 *be satisfied. Then, for every initial condition* $\pi(0) \in \mathrm{int}(\mathcal{S}(\mathcal{P}))$, $\rho(0) \in (0, +\infty)^{\mathcal{E}}$, *there exists a unique solution of* (2.13). *Moreover, there exists a perturbed equilibrium flow* $f^{(h)} \in \mathcal{F}$ *such that for all* $\eta > 0$,

$$\limsup_{t\to\infty} ||f(t) - f^{(h)}|| \leq \delta(\eta)\,, \tag{2.15}$$

*where* $\delta(\eta)$ *is a nonnegative real-valued, nondecreasing function of* $\eta > 0$ *such that* $\lim_{\eta\downarrow 0}\delta(\eta) = 0$. *Moreover, for every sequence of admissible perturbations* $\{h_k\}$ *such that* $\lim_k ||h_k||_\infty = 0$ *and* $\lim_k \Pi_{h_k} = \overline{\Pi}$, *one has*

$$\lim_k f^{(h_k)} = f^W\,. \tag{2.16}$$

Theorem 2.5 states that in the large time limit, the flow vector $f(t)$ approaches a neighborhood of the Wardrop equilibrium, whose size vanishes as both the time-scale ratio $\eta$ and the perturbation norm $||h||_\infty$ vanish. While a qualitatively similar result is known to hold [24] in a classic evolutionary game theoretic framework (i.e., neglecting the traffic dynamics and assuming it is instantaneously equilibrated, as in the ODE system (2.7)), the significance of the above is to show that an approximate Wardrop equilibrium configuration is expected to emerge also in our more realistic model of two-time-scale dynamics. Therefore, our results provide stronger evidence in support of the significance of Wardrop's postulate of equilibrium for a transportation network. In fact, they may be read as a sort of robustness of such an equilibrium notion with respect to nonpersistent perturbations.

## 3. Proofs.

In this section, Theorem 2.5 is proved. First, observe that, thanks to the continuous differentiability of $F^h$, $G^v$, and $\mu$, standard analytical arguments imply the existence and uniqueness of a solution of the initial value problem associated to the system (2.13) with initial condition $\rho(0) \in (0, +\infty)^{\mathcal{E}}$, $\pi(0) \in \mathrm{int}(\mathcal{S}(\mathcal{P}))$.

In order to prove the rest of the statement, we shall adopt a singular perturbation approach (e.g., see [16]), viewing the traffic density $\rho$ (or, equivalently, the traffic flow $f$) as a fast transient and the aggregate path preference $\pi$ as a slow component. Hence, we shall first think of $\pi$ as quasi-static (i.e., "almost a constant") while analyzing the fast-scale dynamics (2.11), and then assume that $f$ is "almost equilibrated," i.e., close to $f^\pi$ and study the slow-scale dynamics (2.4) as a perturbation of (2.7). We shall proceed by proving a series of intermediate technical results, gathered in the following subsections.

Before proceeding, we introduce some notation to be used throughout the section. Let

$$\rho_e^\pi := \mu_e^{-1}(f_e^\pi)\,, \qquad \sigma_e := \operatorname{sgn}(\rho_e - \rho_e^\pi) = \operatorname{sgn}(f_e - f_e^\pi)$$

denote, respectively, the density corresponding to the flow associated to the path preference $\pi$ and the sign of the difference between it and the actual density $\rho_e$. Finally, fix some $\alpha \in (0,1)$ and define

$$V(f,\pi) := \sum_{v=0}^{n-1} \alpha^v \sum_{e\in\mathcal{E}_v^+} |f_e - f_e^\pi|\,, \qquad W(\rho,\pi) := \sum_{v=0}^{n-1} \alpha^v \sum_{e\in\mathcal{E}_v^+} |\rho_e - \rho_e^\pi|. \tag{3.1}$$

**3.1. Stability of the fast-scale dynamics.** We gather here a few properties of the fast-scale dynamics. Our results will essentially amount to showing that $V(f,\pi)$ and $W(\rho,\pi)$ are Lyapunov functions for the fast-scale dynamics (2.11) with stationary path preference $\pi$.

The following result is a consequence of Assumptions 5 and 6 on the drivers' local decision function.

LEMMA 3.1. *For all* $\pi \in \Pi$, $v \in \{0,\dots,n-1\}$ *and* $f_{\mathcal{E}_v^+} \in \mathcal{F}_v$,

$$\sum_{e\in\mathcal{E}_v^+} \sigma_e \left(\lambda_v^\pi G_e^v(f_{\mathcal{E}_v^+},\pi) - f_e^\pi\right) \le 0\,,$$

*where* $\lambda_v^\pi := \sum_{e\in\mathcal{E}_v^+} f_e^\pi$.

*Proof.* Throughout this proof, the explicit dependence of $G_e^v$ on $\pi$ will be dropped. Define $\mathcal{J} := \{e \in \mathcal{E}_v^+ : f_e > f_e^\pi\}$, $\mathcal{K} := \{e \in \mathcal{E}_v^+ : f_e < f_e^\pi\}$ and let $G_\mathcal{J} := \sum_{j\in\mathcal{J}} G_j^v$, $G_\mathcal{K} := \sum_{k\in\mathcal{K}} G_k^v$, and $G_{\mathcal{J}^c} := \sum_{e\in\mathcal{E}_v^+\setminus\mathcal{J}} G_e^v$. First, observe that since $\sum_{e\in\mathcal{E}_v^+} G_e^v = 1$, one has that $\nabla G_\mathcal{J} = -\nabla G_{\mathcal{J}^c}$. Now, we are going to show that

$$G_\mathcal{J}(f^\pi_{\mathcal{E}_v^+}) - G_\mathcal{J}(f_{\mathcal{E}_v^+}) \ge 0 \tag{3.2}$$

by writing the difference above as a path integral of $\nabla G_\mathcal{J}(\,\cdot\,)$ first along the segment $S_\mathcal{J}$ from $f_{\mathcal{E}_v^+}$ to the point $f^* \in \mathbb{R}_+^{\mathcal{E}_v^+}$ with $f_j^* := f_j^\pi$, for $j \in \mathcal{J}$ and $f_e^* := f_e$ for $e \in \mathcal{E}_v^+ \setminus \mathcal{J}$, and then along the segment $S_\mathcal{K}$ from $f^*$ to $f^\pi$. In this way, one gets

$$\begin{aligned} G_\mathcal{J}(f^\pi_{\mathcal{E}_v^+}) - G_\mathcal{J}(f_{\mathcal{E}_v^+}) &= \int_{S_\mathcal{J}} \nabla G_\mathcal{J}(\tilde f_{\mathcal{E}_v^+}) \cdot \mathrm{d}\tilde f_{\mathcal{E}_v^+} + \int_{S_\mathcal{K}} \nabla G_\mathcal{J}(\tilde f_{\mathcal{E}_v^+}) \cdot \mathrm{d}\tilde f_{\mathcal{E}_v^+} \\ &= -\int_{S_\mathcal{J}} \nabla G_{\mathcal{J}^c}(\tilde f_{\mathcal{E}_v^+}) \cdot \mathrm{d}\tilde f_{\mathcal{E}_v^+} + \int_{S_\mathcal{K}} \nabla G_\mathcal{J}(\tilde f_{\mathcal{E}_v^+}) \cdot \mathrm{d}\tilde f_{\mathcal{E}_v^+}\,. \end{aligned} \tag{3.3}$$

Assumption 6 implies that $\partial G_{\mathcal{J}^c}/\partial\rho_j \ge 0$ for all $j \in \mathcal{J}$ and $\partial G_\mathcal{J}/\partial\rho_k \ge 0$ for all $k \in \mathcal{K}$. In turn, this implies that $\nabla G_{\mathcal{J}^c} \cdot \mathrm{d}\tilde f_{\mathcal{E}_v^+} \le 0$ along $S_\mathcal{J}$ and $\nabla G_\mathcal{J} \cdot \mathrm{d}\tilde f_{\mathcal{E}_v^+} \ge 0$ along $S_\mathcal{K}$. This and (3.3) prove (3.2). In a very similar fashion, one proves that

$$G_\mathcal{K}(f_{\mathcal{E}_v^+}) - G_\mathcal{K}(f^\pi_{\mathcal{E}_v^+}) \ge 0\,. \tag{3.4}$$

Now, observe that Assumption 5 implies that $\lambda_v^\pi G_e^v(f^\pi_{\mathcal{E}_v^+}, \pi) = f_e^\pi$. From this, (3.2), and (3.4), it follows that

$$
\begin{aligned}
0 &\geq \lambda_v^\pi \left( G_{\mathcal{J}}(f_{\mathcal{E}_v^+}) - G_{\mathcal{J}}(f^\pi_{\mathcal{E}_v^+}) \right) - \lambda_v^\pi \left( G_{\mathcal{K}}(f_{\mathcal{E}_v^+}) - G_{\mathcal{K}}(f^\pi_{\mathcal{E}_v^+}) \right) \\
&= \sum_{e \in \mathcal{E}_v^+} \sigma_e \left( \lambda_v^\pi G_e^v(f_{\mathcal{E}_v^+}) - \lambda_v^\pi G_e^v(f^\pi_{\mathcal{E}_v^+}) \right) \\
&= \sum_{e \in \mathcal{E}_v^+} \sigma_e \left( \lambda_v^\pi G_e^v(f_{\mathcal{E}_v^+}) - f_e^\pi \right) ,
\end{aligned}
$$

which proves the claim. ▯

We now proceed to analyzing, for a fixed global decision $\pi \in \Pi$, the fast-scale dynamics (2.11). Let

$$V_v^+(f, \pi) := \sum_{e \in \mathcal{E}_v^+} |f_e^\pi - f_e| , \qquad v = 0, 1, \ldots, n-1 ,$$

be the $l_1$-distance between the current flows on the outgoing links of $v$ and the flow associated to the aggregate path preference $\pi$, and let

$$V_v^-(f, \pi) := \left| \lambda_v^\pi - \lambda_v^- \right| , \qquad v = 1, 2, \ldots, n ,$$

with $\lambda_v^\pi := \sum_{e \in \mathcal{E}_v^-} f_e^\pi$ and $\lambda_v^- := \sum_{e \in \mathcal{E}_v^-} f_e$, be the absolute difference between the current flow incoming in node $v$ and the one associated to the aggregate path preference $\pi$. Also, let $V_0^-(f, \pi) := 0$.

Lemma 3.2. *For all $v = 0, 1, \ldots, n-1$, $\pi \in \Pi$, and $f \in \mathcal{F}$,*

$$\sum_{e \in \mathcal{E}_v^+} \sigma_e H_e(f, \pi) \leq -V_v^+(f, \pi) + V_v^-(f, \pi) .$$

*Proof.* Writing $G_e^v$ for $G_e^v(f_{\mathcal{E}_v^+}, \pi)$ and using Lemma 3.1, one gets that

$$
\begin{aligned}
\sum_{e \in \mathcal{E}_v^+} \sigma_e H_e(f, \pi) &= \sum_{e \in \mathcal{E}_v^+} \sigma_e \left( \lambda_v^- G_e^v - f_e \right) \\
&= \sum_{e \in \mathcal{E}_v^+} \sigma_e (\lambda_v^- - \lambda_v^\pi) G_e^v + \sum_{e \in \mathcal{E}_v^+} \sigma_e \left( \lambda_v^\pi G_e^v - f_e^\pi \right) + \sum_{e \in \mathcal{E}_v^+} \sigma_e \left( f_e^\pi - f_e \right) \\
&\leq |\lambda_v^- - \lambda_v^\pi| - \sum_{e \in \mathcal{E}_v^+} |f_e^\pi - f_e| \\
&= -V_v^+(f, \pi) + V_v^-(f, \pi) ,
\end{aligned}
$$

which proves the claim. ▯

By combining Lemma 3.2 and Assumption 1, one gets the result below. Recall the definition of $W(\rho, \pi)$ from (3.1) and that we are using the convention $\mathrm{d}|x|/\mathrm{d}x = \mathrm{sgn}(x)$ for all $x \in \mathbb{R}$.

Lemma 3.3. *For every $f = \mu(\rho) \in \mathcal{F}$ and $\pi \in \Pi$,*

$$\nabla_\rho W(\rho, \pi)' H(f, \pi) \leq -(1 - \alpha) V(f, \pi) .$$

*Proof.* Observe that thanks to the acyclicity of the graph as per Assumption 1, if $e \in \mathcal{E}_v^- \cap \mathcal{E}_w^+$ for some nodes $v$ and $w$, then necessarily $v \geq w + 1$. Since $\alpha < 1$, it follows that

$$\alpha^v \mathbb{1}_{\mathcal{E}_v^-}(e) \mathbb{1}_{\mathcal{E}_w^+}(e) \leq \alpha^{w+1} \mathbb{1}_{\mathcal{E}_v^-}(e) \mathbb{1}_{\mathcal{E}_w^+}(e)$$

for every $1 \leq v \leq n$ and $0 \leq w \leq n-1$. Hence,

$$\begin{aligned}
\sum_{0\leq v<n} \alpha^v V_v^-(f,\pi) &\leq \sum_{0\leq v<n} \sum_{e\in\mathcal{E}_v^-} \alpha^v \left| f_e - f_e^\pi \right| \\
&= \sum_{1\leq v<n} \sum_{0\leq w<n} \sum_{e\in\mathcal{E}} \alpha^v \mathbb{1}_{\mathcal{E}_v^-}(e) \mathbb{1}_{\mathcal{E}_w^+}(e) \left| f_e - f_e^\pi \right| \\
&\leq \sum_{0\leq w<n} \alpha^{w+1} \sum_{e\in\mathcal{E}} \mathbb{1}_{\mathcal{E}_w^+}(e) \left| f_e - f_e^\pi \right| \sum_{1\leq v<n} \mathbb{1}_{\mathcal{E}_v^-}(e) \\
&\leq \alpha \sum_{0\leq w<n} \alpha^w \sum_{e\in\mathcal{E}_w^+} \left| f_e - f_e^\pi \right| \\
&= \alpha V(f,\pi)\,,
\end{aligned}$$

where the last inequality follows from the fact that $\sum_{v=1}^{n-1} \mathbb{1}_{\mathcal{E}_v^-}(e) \leq \sum_{v=1}^{n} \mathbb{1}_{\mathcal{E}_v^-}(e) = 1$, and we recall (3.1) for the definition of $V(f,\pi)$. Thus, Lemma 3.2 implies that

$$\begin{aligned}
\nabla_\rho W(\rho,\pi)' H(f,\pi) &= \sum_{0\leq v<n} \alpha^v \sum_{e\in\mathcal{E}_v^+} \sigma_e H_e(f,\pi) \\
&\leq \sum_{0\leq v<n} \alpha^v V_v^-(f,\pi) - \sum_{0\leq v<n} \alpha^v V_v^+(f,\pi) \\
&\leq \alpha V(f,\pi) - V(f,\pi)\,,
\end{aligned}$$

which proves the claim. □

**3.2. Boundedness of the traffic densities.** We shall now prove a couple of results guaranteeing that the traffic density on every link remains bounded in time. We start with the following result, guaranteeing that on every link $e \in \mathcal{E}$, the flow associated to the current path preference, $f_e^\pi(t)$, stays eventually bounded away from the maximum flow capacity $C_e$. Its proof relies on Assumption 4. Recall that our formulation allows for both cases of finite and infinite maximum flow capacity on a link.

LEMMA 3.4. *For every admissible perturbation $h$, there exists $t_0 \in \mathbb{R}_+$ and, for every link $e \in \mathcal{E}$, a positive finite constant $\overline{C}_e$, dependent on $h$ but not on $\eta$, such that for every initial condition $\pi(0) \in \operatorname{int}(\mathcal{S}(\mathcal{P}))$, $\rho(0) \in (0,+\infty)^{\mathcal{E}}$,*

$$f_e^\pi(t) \leq \overline{C}_e < C_e \qquad \forall t \geq t_0 \qquad \forall\, e \in \mathcal{E}.$$

*Proof.* The fact that $f_e^\pi(t) \leq 1$ for all $e \in \mathcal{E}$ follows from the fact that the arrival rate at the origin is unitary. Therefore, for all $e \in \mathcal{E}$ with $C_e > 1$ (and hence also for $C_e = \infty$), the claim follows trivially with $\overline{C}_e = 1$ and $t_0 = 0$. We now prove the lemma for all $e \in \mathcal{E}$ with $C_e < 1$. Recall that by Definition 2.2, the domain of the admissible perturbation $h$ is a closed set $\Pi_h \subset \operatorname{int}(\Pi)$. This in particular implies that

$$\kappa_e := C_e - \sup\{(A\omega)_e : \omega \in \Pi_h\} > 0\,.$$

It follows from (2.5) that

$$\begin{aligned}
C_e - \kappa_e &= \sup\{(A\omega)_e : \omega \in \Pi_h\} \\
&\geq \sup\left\{\left(A \operatorname{argmin}\{\omega' A' T(f) + h(\omega) : \omega \in \Pi_h\}\right)_e : f \in \mathcal{F}\right\} \qquad (3.5)\\
&= \sup\left\{\left(AF^h(f)\right)_e : f \in \mathcal{F}\right\}\,.
\end{aligned}$$

Hence, one has

$$\frac{\mathrm{d}}{\mathrm{d}t} f_e^\pi(t) = \eta\left(A(F^h(f(t)) - \pi(t))\right)_e \leq \eta\left(C_e - \kappa_e - f_e^\pi\right).$$

This implies that

$$f_e^\pi(t) - C_e + \kappa_e \leq (f_e^\pi(0) - C_e + \kappa_e)e^{-\eta t} \leq e^{-\eta t}, \qquad t \geq 0, \tag{3.6}$$

where the last inequality follows from the fact that $f_e^\pi(0) = \sum_p A_{ep}\pi_p(0) \leq 1$ and $C_e \geq \kappa_e$. The lemma for $e \in \mathcal{E}$ with $C_e < 1$ now follows from (3.6) by choosing, e.g., $\overline{C}_e := C_e - \kappa/2$ with $\kappa := \min\{\kappa_e : e \in \mathcal{E} \text{ s.t. } C_e < 1\}$ and $t_0 := -\eta^{-1}\log(\kappa/2)$. ∎

The following result shows that the actual flow $f_e(t)$ also stays bounded away from the maximum flow capacity $C_e$.

LEMMA 3.5. *For every admissible perturbation $h$, there exists $\eta^* > 0$ and a positive finite constant $\tilde{C}_e$ for every $e \in \mathcal{E}$, dependent on $h$ but not on $\eta$, such that for every $\eta < \eta^*$ and every initial condition $\pi(0) \in \mathrm{int}(\mathcal{S}(\mathcal{P}))$, $\rho(0) \in (0, +\infty)^{\mathcal{E}}$,*

$$f_e(t) \leq \tilde{C}_e < C_e$$

*for all $t \geq 0$ and $e \in \mathcal{E}$.*

*Proof.* For $t \geq 0$, let us define

$$\zeta(t) := W(\rho(t), \pi(t)), \qquad \chi(t) := V(f(t), \pi(t)).$$

Observe that thanks to Lemma 3.4, there exists $t_0 \geq 0$ and a positive constant $\overline{C}_e$ for every $e \in \mathcal{E}$, such that for every $t \geq t_0$,

$$\rho_e^\pi(t) \leq \rho_e^*, \qquad \rho_e^* := \mu_e^{-1}(\overline{C}_e) \qquad \forall e \in \mathcal{E}. \tag{3.7}$$

Since $\rho_e^\pi(t) \geq 0$, the above implies that if $|\rho_e(t) - \rho_e^\pi(t)| \geq 2\rho_e^*$ for some $t \geq t_0$, then necessarily $\rho_e(t) \geq 2\rho_e^*$ for $t \geq t_0$. Hence, $f_e(t) - f_e^\pi(t) \geq \chi_e^*$ for all $t \geq t_0$, where $\chi_e^* := \mu_e(2\rho_e^*) - \overline{C}_e$. Observe that since $\mu_e$ is strictly increasing by Assumption 2, one has $\chi_e^* = \mu_e(2\rho_e^*) - \overline{C}_e > \mu_e(\rho_e^*) - \overline{C}_e = 0$. Now, let

$$\zeta^* := 2|\mathcal{E}|\max\{\rho_e^* : e \in \mathcal{E}\}, \qquad \chi^* := \alpha^{n-1}\min\{\chi_e^* : e \in \mathcal{E}\}.$$

Notice that

$$W(\rho, \pi) \leq |\mathcal{E}|\max\{|\rho_e - \rho_e^\pi)| : e \in \mathcal{E}\}, \qquad V(f, \pi) \geq \alpha^{n-1}|f_e - f_e^\pi| \qquad \forall e \in \mathcal{E}.$$

Therefore, it follows that for any $t \geq t_0$, if $\zeta(t) \geq \zeta^*$, then for some $e' \in \mathcal{E}$ we have that $|\rho_{e'}(t) - \rho_{e'}^\pi| \geq 2\rho_{e'}^*$ for $t \geq t_0$. This in turn implies that $\chi(t) \geq \chi_{e'}^* \geq \chi^*$. Therefore, in summary,

$$\zeta(t) \geq \zeta^* \quad \Longrightarrow \quad \chi(t) \geq \chi^* > 0 \qquad \forall t \geq t_0. \tag{3.8}$$

On the other hand, observe that (3.7) implies that there exists some $\ell > 0$ such that

$$\sum_{0 \leq v < n} \alpha^v \sum_{e \in \mathcal{E}_v^+} \frac{1}{\mu_e'(\rho_e^\pi(t))} \leq \ell \qquad \forall t \geq t_0.$$

By combining the above with Lemma 3.3, one finds that for any $u, t \geq t_0$,

$$
\begin{aligned}
\zeta(t) - \zeta(u) &= \int_u^t \sum_{0 \leq v < n} \alpha^v \sum_{e \in \mathcal{E}_v^+} \sigma_e \left( \frac{\mathrm{d}}{\mathrm{d}s}\rho_e - \frac{\mathrm{d}}{\mathrm{d}s}\rho_e^\pi \right) \mathrm{d}s \\
&\leq \int_u^t \nabla_\rho W(\rho, \pi)' H(f, \pi) \mathrm{d}s \\
&\quad + \int_u^t \sum_{0 \leq v < n} \alpha^v \sum_{e \in \mathcal{E}_v^+} \frac{\eta}{\mu_e'(\rho_e^\pi)} \left| (AF^h(f^\pi))_e - (A\pi)_e \right| \mathrm{d}s \\
&\leq \int_u^t \left( -(1-\alpha)\chi(s) + 2\eta\ell \right) \mathrm{d}s \,.
\end{aligned} \tag{3.9}
$$

Now, let us define $\eta^* := (1-\alpha)\chi^*/(2\ell)$. By contradiction, let us assume that $\limsup_{t\to\infty} f_e(t) \geq C_e$ for some $e \in \mathcal{E}$. Since $f_e(t) = \mu_e(\rho_e(t)) < C_e$ for every $t \geq 0$, this implies that $\limsup_{t\to\infty} \rho_e(t) = \infty$. This together with (3.7) implies that $\limsup_{t\to\infty} \zeta(t) = \infty$. Then, in particular, the set $\mathcal{T} := \{t > 0 : \zeta(t) > \zeta(s) \; \forall s < t\}$ is an unbounded union of open intervals with $\lim_{t\in\mathcal{T}, t\to\infty} \zeta(t) = \infty$. This and (3.8) imply that there exists a nonnegative constant $t^* \geq t_0$ such that

$$
\chi(t) \geq \chi^* \qquad \forall t \in \mathcal{T} \cap [t^*, \infty) \,.
$$

For every $\eta < \eta^*$, (3.9) and the above give

$$
\zeta(t) - \zeta(u) \leq \int_u^t \left( -(1-\alpha)\chi(s) + 2\eta\ell \right) \mathrm{d}s \leq \int_u^t \left( -(1-\alpha)\chi^* + 2\eta\ell \right) \mathrm{d}s < 0
$$

for every $t > u \geq t^*$ such that $t$ and $u$ belong to the same connected component of $\mathcal{T}$. But this contradicts the definition of the set $\mathcal{T}$. Hence, if $\eta < \eta^*$, then $\limsup_{t\to\infty} f_e(t) < C_e$ for every $e \in \mathcal{E}$. Since on every compact time interval $\mathcal{I} \subseteq \mathbb{R}_+$, one has $\sup_{t\in\mathcal{I}} f_e(t) = f_e(\hat{t}) < C_e$ for some $\hat{t} \in \mathcal{I}$, the foregoing implies the claim. ∎

The result below is a consequence of Lemma 3.5 and will prove useful in what follows.

Proposition 3.6. *There exist $K > 0$ and $t_1 \geq 0$ such that for every initial condition $\pi(0) \in \operatorname{int}(\mathcal{S}(\mathcal{P}))$, $\rho(0) \in (0, +\infty)^{\mathcal{E}}$, $||\tilde{\nabla}_\pi h(\pi(t))|| \leq K$ for all $t \geq t_1$.*

*Proof.* First, observe that thanks to Lemma 3.5, there exists $T^* > 0$ such that $||T(f(t))|| \leq T^*$ for all $t \geq 0$. Thanks to this and Assumption 4, one has that $F^h(f(t)) \in \operatorname{int}(\Pi_h)$ and $\tilde{\nabla}_\pi h(F^h(f(t))) = -\Phi A' T(f(t))$, where recall that $\Phi = I - |\mathcal{P}|^{-1}\mathbf{1}\mathbf{1}'$ is the projection matrix corresponding to the projected gradient with respect to $\pi$ on $\mathcal{S}(\mathcal{P})$. Hence, $||\tilde{\nabla}_\pi h(F^h(f(t)))|| \leq ||\Phi|| ||A'|| T^*$, which implies that there exists a convex compact $\mathcal{K} \subset \operatorname{int}(\Pi_h)$ such that $F^h(f(t)) \in \mathcal{K}$ for all $t \geq 0$. Define

$$
\Delta(t) := \frac{\eta}{1 - e^{-\eta t}} \int_0^t e^{-\eta(t-s)} F^h(f(s)) \mathrm{d}s \,.
$$

As $\Delta(t)$ is an average of elements of the convex set $\mathcal{K}$, necessarily $\Delta(t) \in \mathcal{K}$ for all $t \geq 0$. Then, $\pi(t) = e^{-\eta t}\pi(0) + (1 - e^{-\eta t})\Delta(t)$ approaches $\mathcal{K}$, which implies that for large enough $t$, $\pi(t) \in \mathcal{K}_1 \subset \operatorname{int}(\Pi_h)$, where $\mathcal{K}_1$ is a closed subset of $\operatorname{int}(\Pi_h)$ that contains $\mathcal{K}$. Hence, after large enough $t$, say, $t_1$, $\tilde{\nabla}_\pi h(\pi(t))$ stays bounded. ∎

**3.3. Estimating the distance between the current density and the one associated to the current path preference.** We analyze here the behavior in time of $W(\rho(t),\pi(t))$. First, we have the following result, characterizing the variation of $W(\rho,\pi)$ as a function of $\pi$. Recall that $\tilde{\nabla}_\pi = \Phi\nabla_\pi$ denotes the projected gradient with respect to $\pi$ on $\mathcal{S}(\mathcal{P})$.

LEMMA 3.7. *There exist $l>0$ and $t_0 \geq 0$ such that for every initial condition $\pi(0)\in \operatorname{int}(\mathcal{S}(\mathcal{P}))$, $\rho(0)\in(0,+\infty)^{\mathcal{E}}$,*

$$\tilde{\nabla}_\pi W(\rho(t),\pi(t))'(F^h(f(t))-\pi(t)) \leq \frac{2l}{1-\alpha} \qquad \forall t\geq t_0\,.$$

*Proof.* First, observe that thanks to Lemma 3.4, one has that there exists $t_0\geq 0$ such that $l_e := \sup\{1/\mu_e'(\rho_e^\pi(t)) : t\geq t_0\} < +\infty$. Put $l:=\max\{l_e : e\in\mathcal{E}\}$. Then, for every path $p\in\mathcal{P}$ and every $t\geq t_0$, one has

$$\begin{aligned}\left|\frac{\partial W(\rho,\pi)}{\partial \pi_p}\right| &= \left|-\sum_{0\leq v<n}\alpha^v\sum_{e\in\mathcal{E}_v^+}\sigma_e\frac{\partial}{\partial_{\pi_p}}\rho_e^\pi\right|\\ &= \left|\sum_{0\leq v<n}\alpha^v\sum_{e\in\mathcal{E}_v^+}\sigma_e\frac{\partial}{\partial_{\pi_q}}\mu_e^{-1}\left(\sum_q A_{ep}\pi_q\right)\right|\\ &\leq \sum_{0\leq v<n}\alpha^v\sum_{e\in\mathcal{E}_v^+}A_{ep}\frac{1}{\mu_e'(\rho_e^\pi)}\\ &\leq \sum_{0\leq v<n}\alpha^v\sum_{e\in\mathcal{E}_v^+}A_{ep}l_e\\ &\leq \frac{l}{1-\alpha}\,,\end{aligned} \tag{3.10}$$

where the third inequality follows from the fact that thanks to Assumption 1 on the acyclicity of the network, each path $p\in\mathcal{P}$ passes through at most one link $e\in\mathcal{E}_v^+$. Therefore,

$$\begin{aligned}\frac{2l}{1-\alpha} &\geq \sum_p F_p^h(f)\left|\frac{\partial}{\partial\pi_p}W(\rho,\pi)\right| + \sum_p \pi_p\left|\frac{\partial}{\partial\pi_p}W(\rho,\pi)\right|\\ &\geq \sum_p F_p^h(f)\frac{\partial}{\partial\pi_p}W(\rho,\pi) - \sum_p \pi_p\frac{\partial}{\partial\pi_p}W(\rho,\pi)\\ &= \tilde{\nabla}_\pi W(\rho,\pi)'(F^h(f)-\pi)\,,\end{aligned}$$

where the first inequality follows upon recalling that both $F^h(f)$ and $\pi$ are probability vectors over the path set $\mathcal{P}$ and by using (3.10). ∎

We can now combine Lemmas 3.3 and 3.7 in order to get the following estimate of the behavior in time of $W(\rho(t),\pi(t))$.

LEMMA 3.8. *There exist $l>0$, $L>0$, $\eta^*>0$, and $t_0\geq 0$ such that for every initial condition $\pi(0)\in\operatorname{int}(\mathcal{S}(\mathcal{P}))$, $\rho(0)\in(0,+\infty)^{\mathcal{E}}$,*

$$W(\rho(t),\pi(t)) \leq \frac{2\eta l L}{(1-\alpha)^2} + \left(W(\rho(t_0),\pi(t_0)) - \frac{2\eta l L}{(1-\alpha)^2}\right)e^{-(1-\alpha)(t-t_0)/L}$$

*for every $t\geq t_0$ and $\eta<\eta^*$.*

*Proof.* Define

$$\zeta(t) := W(\rho(t), \pi(t))\,.$$

Notice that thanks to Lemmas 3.4 and 3.5, there exist $L > 0$, $\eta^* > 0$, and $t_0 \geq 0$ such that for any $\eta < \eta^*$,

$$|\rho_e(t) - \rho_e^\pi(t)| \leq L|f_e(t) - f_e^\pi(t)| \qquad \forall e \in \mathcal{E}\,,\ t \geq t_0\,.$$

This in particular implies that

$$V(f(t), \pi(t)) \geq \frac{1}{L} W(\rho(t), \pi(t)) = \frac{1}{L}\zeta(t) \qquad \forall \eta < \eta^*\,,\ t \geq t_0\,.$$

Observe that $W(\rho, \pi)$ is a Lipschitz function of $\rho$ and $\pi$, while both $\rho(t)$ and $\pi(t)$ are Lipschitz on every compact time interval. Therefore, $\zeta(t)$ is Lipschitz on every compact time interval and thus absolutely continuous. Hence, $\mathrm{d}\zeta(t)/\mathrm{d}t$ exists for almost every $t \geq 0$, and, thanks to Lemmas 3.3 and 3.7, it satisfies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t) &= \frac{\mathrm{d}}{\mathrm{d}t} W(\rho(t), \pi(t)) \\
&= \nabla_\rho W(\rho, \pi)' H(f, \pi) + \eta \tilde{\nabla}_\pi W(\rho, \pi)'(F^h(f) - \pi) \\
&\leq -(1-\alpha) V(f, \pi) + \frac{2\eta l}{1-\alpha} \\
&\leq -\frac{(1-\alpha)}{L}\zeta(t) + \frac{2\eta l}{1-\alpha}\,.
\end{aligned}$$

Then, the claim follows by integrating both sides. ∎

**3.4. Proof of Theorem 2.5.** We now proceed to proving Theorem 2.5. Let us introduce the function

$$\Theta : \Pi \to \mathbb{R}_+\,, \qquad \Theta(\pi) := \sum_{e \in \mathcal{E}} \int_0^{f_e^\pi} T_e(s)\, ds \tag{3.11}$$

and observe that

$$\tilde{\nabla}\Theta(\pi) = \Phi A' T(f^\pi) \qquad \forall \pi \in \operatorname{int}(\Pi)\,. \tag{3.12}$$

In game-theoretic terminology, (3.12) implies that $\Theta(\pi)$ is the potential function [19] for the continuous-population congestion game with action space $\mathcal{P}$ and payoff vector function $-A'T(f^\pi)$.[5]

Observe that since $T_e(f_e)$ is increasing, one has that each term $\int_0^{f_e^\pi} T_e(f_e)\, \mathrm{d}f_e$ is convex in $f_e^\pi$. Hence, the composition with the linear map $\pi \mapsto f_e^\pi = \sum_p A_{ep}\pi_p$ is

[5]In fact, (3.12) is equivalent to

$$\sum_e A_{eq} T_e(f_e^\pi) - \sum_e A_{ep} T_e(f_e^\pi) = \frac{\partial}{\partial \pi_q}\Theta(\pi) - \frac{\partial}{\partial \pi_p}\Theta(\pi)$$

for every $p, q \in \mathcal{P}$, i.e., the difference between the total delays associated to the flow $f^\pi$ on paths $q$ and $p$ equals the limit incremental ratio of $\Theta(\pi)$ with respect to an infinitesimal mass transfer in $\pi$ from path $p$ to path $q$. Intuitively, if a nonatomic driver, whose weight is infinitesimal in the continuum population model, switches path from $p$ to $q$, the potential $\Theta$ increases by an infinitesimal amount equal to the product of the increase in the driver's delay cost times the driver's weight.

convex in $\pi$, which in turn implies convexity of $\Theta$ over $\Pi$. Then, for any admissible perturbation $h : \Pi_h \to \mathbb{R}_+$, Definition 2.2 implies strict convexity of $\Theta(\pi) + h(\pi)$. Therefore, since $\Pi_h$ is compact and convex, there exists a unique minimizer

$$\pi^h := \operatorname{argmin}\{\Theta(\pi) + h(\pi) : \pi \in \Pi_h\} . \tag{3.13}$$

Let $f^{(h)} := f^{\pi^h}$. Then, we have the following.

LEMMA 3.9. *Let $\{h_k\}$ be any sequence of admissible perturbation functions such that* $\lim_k ||h_k||_\infty = 0$, $\lim_k \Pi_{h_k} = \overline{\Pi}$. *Then,*

$$\lim_{k\to\infty} f^{(h_k)} = f^W .$$

*Proof.* Write $\pi^k$ for $\pi^{h_k}$, $F^k$ for $F^{h_k}$, and $\Pi_k$ for $\Pi_{h_k}$. Since $\{A\pi^k\} \subseteq A\Pi$, and $A\overline{\Pi}$ is compact, there exists a converging subsequence $\{A\pi^{k_j} : j \in \mathbb{N}\}$. Let us denote by $f^* := \lim_j A\pi^{k_j} \in A\overline{\Pi}$ its limit and choose some $\pi^* \in \overline{\Pi}$ such that $f^* = A\pi^*$. Notice that since $\sup\{T_e(f_e^\pi) : \pi \in \Pi_h\} < +\infty$ for all $e \in \mathcal{E}$, Definition 2.2 implies that the minimizer in (3.13) has to be in the interior of $\Pi_h$. As a consequence, one finds that necessarily $\tilde{\nabla}_\pi h(\pi^{k_j}) = -\Phi A' T(A\pi^{k_j})$, which in turn implies that $F^{k_j}(A\pi^{k_j}) = \pi^{k_j}$. Then, using (2.5), one finds that

$$(A\pi^{k_j})' T(A\pi^{k_j}) + h_{k_j}(\pi^{k_j}) \le (A\pi^{k_j})' T(A\pi^{k_j}) + h_{k_j}(\omega) \tag{3.14}$$

for all $\omega \in \Pi_{k_j}$. Now, fix any $\pi \in \Pi$. Since $\Pi_k \xrightarrow{k} \overline{\Pi}$, one has that there exists a sequence $\{\tilde{\pi}^j\}$ such that $\tilde{\pi}^j \in \Pi_{k_j}$ for all $j$ and $\lim_j \tilde{\pi}^j = \pi$. Hence, taking $\omega = \tilde{\pi}^j$ in (3.14) and passing to the limit as $j$ grows large, one finds that

$$(\pi^*)' A' T(A\pi^*) \le \pi' A' T(A\pi^*) \qquad \forall \pi \in \Pi .$$

In turn, the above can be easily shown to be equivalent to the condition (2.14) characterizing Wardrop equilibria. From the uniqueness of the Wardrop equilibrium, it follows that necessarily $f^* = f^W$. Then the claim follows from the arbitrariness of the accumulation point $f^*$. ∎

We shall now estimate the time derivative of $\Theta_h(\pi)$ along trajectories of our dynamical system. For this, define

$$\Gamma(t) := \Theta(\pi(t)) + h(\pi(t)) , \qquad \psi(t) := \Phi A' T(f^\pi(t)) + \tilde{\nabla}_\pi h(\pi(t)) . \tag{3.15}$$

Then, using (3.12), one has

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t) &= \left(\tilde{\nabla}_\pi \Theta + \tilde{\nabla} h(\pi(t))\right)' \frac{\mathrm{d}}{\mathrm{d}t}\pi \\ &= \eta \psi(t)' \left(F^h(f(t)) - \pi(t)\right) \\ &= \eta \psi(t)' \left(F^h(f^\pi(t)) - \pi(t)\right) + \eta \psi(t)' \left(F^h(f(t)) - F^h(f^\pi(t))\right) . \end{aligned} \tag{3.16}$$

Lemma 3.8 implies that there exists $t_2 \ge 0$, $\eta^* > 0$, and $M_1 > 0$ such that for any $\eta < \eta^*$, $W(\rho(t), \pi(t)) \le \eta M_1$ for all $t \ge t_2$. From the definition of $W$, it also follows that $W(\rho, \pi) \ge \alpha^{n-1} \|\rho - \rho^\pi\|_1$ for all $\rho, \pi$. Moreover, following Assumption 2, with $\overline{L} := \max\{\mathrm{d}\mu_e/\mathrm{d}\rho_e(0) : e \in \mathcal{E}\}$, we also have that $\|f - f^\pi\|_1 \le \overline{L} \|\rho - \rho^\pi\|_1$ for all $f = \mu(\rho)$ and $\pi$. Combining all these relationships, one can see that there exists a $M > 0$ such that for any $\eta < \eta^*$,

$$||f(t) - f^\pi(t)|| \le \eta M \qquad \forall\, t \ge t_2. \tag{3.17}$$

Moreover, recall that $F^h$ is differentiable on $\mathcal{F}$ and that thanks to Lemmas 3.4 and 3.5, for $\eta < \eta^*$, both $f(t)$ and $f^\pi(t)$ are eventually confined in a compact $\mathcal{K} \subseteq \mathcal{F}$. This implies that

$$||F^h(f(t)) - F^h(f^\pi(t))|| \leq K_1\eta$$

for some positive constant $K_1$, $\eta < \eta^*$ and sufficiently large value of $t$. On the other hand, Lemma 3.4 and Proposition 3.6 imply that both $T(f^\pi(t))$ and $\tilde{\nabla}_\pi h(\pi(t))$ are eventually bounded, so that $||\psi(t)|| \leq K_2$ for some positive constant $K_2$ and large enough $t$. It follows that the second addend in the last line of (3.16) can be bounded as

$$\eta\psi(t)'\left(F^h(f(t)) - F^h(f^\pi(t))\right) \leq K\eta^2 \qquad \forall \eta < \eta^*, \quad \forall\, t \geq t_3 \tag{3.18}$$

for some sufficiently large but finite value of $t_3$, where $K = K_1K_2$. Now, observe that for every $\pi \in \Pi$,

$$\Phi A' T(f^\pi) = -\tilde{\nabla}_\pi h\left(F^h(f^\pi)\right),$$

so that the first addend in the last line of (3.16) may be rewritten as

$$\psi(t)'\left(F^h(f^\pi(t)) - \pi(t)\right) = -\Upsilon(\pi(t)), \tag{3.19}$$

where

$$\Upsilon(\pi) := \left(\tilde{\nabla}_\pi h(F^h(f^\pi)) - \tilde{\nabla}_\pi h(\pi)\right)'\left(F^h(f^\pi) - \pi\right).$$

It follows from (3.16), (3.18), and (3.19) that for $\eta < \eta^*$ and $t \geq t_3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t) \leq -\eta\Upsilon(\pi(t)) + M\eta^2. \tag{3.20}$$

From the strict convexity of $h(\pi)$ on the simplex $\Pi$, one finds that $\Upsilon(\pi) \geq 0$ for all $\pi$, with equality if and only if $\pi = \pi^h$. Now, put

$$\delta(x) := \begin{cases} \sup\{||f^\pi - f^{(h)}|| : \Upsilon(\pi) \leq Mx\} + Mx & \text{if} \quad 0 \leq x < \eta^*, \\ \tilde{C}\sqrt{|\mathcal{E}|} & \text{if} \quad x \geq \eta^*, \end{cases} \tag{3.21}$$

where $\tilde{C} := \max\{1, \tilde{C}_e : e \in \mathcal{E}\}$, with $\tilde{C}_e$ as defined in Lemma 3.5. It can be verified that $\delta(x)$ is nondecreasing. Moreover, it is right-continuous, so that in particular $\lim_{\eta\downarrow 0}\delta(\eta) = \delta(0) = 0$. Then, (3.17) and (3.20) imply that for $\eta < \eta^*$,

$$\limsup_{t\to\infty} ||f(t) - f^{(h)}|| \leq \delta(\eta). \tag{3.22}$$

Observe that since $f(t) \in [0, \tilde{C}]^{\mathcal{E}}$ by Lemma 3.5 and $f^{(h)} \in A\Pi \subseteq [0,1]^{\mathcal{E}}$, one has that $|f_e(t) - f_e^{(h)}| \leq \max\{\tilde{C}_e, 1\} \leq \tilde{C}$ for all $e \in \mathcal{E}$, so that $||f(t) - f^{(h)}||^2 \leq |\mathcal{E}|\tilde{C}^2$. Hence, (3.22) holds also for $\eta \geq \eta^*$, since in that range $\delta(x) = \sqrt{|\mathcal{E}|}\tilde{C}$. Together with Lemma 3.9, this completes the proof of Theorem 2.5.

*Remark* 4. Observe that since $\Upsilon(\pi^h) = 0$ is a minimum of $\Upsilon(\pi)$, one has that $\Upsilon(\pi) = \pi' H \pi + O(||\pi - \pi^h||^3)$ as $\pi \to \pi^h$, where $H$ is symmetric nonnegative definite. From this and (3.21) one finds that $\delta(\eta) \asymp \sqrt{\eta}$ as $\eta \downarrow 0$.

**4. Numerical simulations.** In this section, we present results from numerical experiments. We performed several experiments with different graph topologies and for values of $\eta$ ranging from 0.01 to 100. In all the cases, we found that the trajectories converge exactly to the perturbed Wardrop equilibrium, i.e., $\delta(\eta)$ in Theorem 2.5 was estimated to be uniformly zero. We suspect that this might be because of the exponential convergence also of the slow-scale dynamics. Additionally, we compared the convergence of the trajectories corresponding to an i-logit local decision function as in Example 4 with sensitivity $\gamma > 0$ and $\gamma = 0$. As already argued, the latter corresponds to the case when the drivers do not take into account the local observation on the currently observed flow and always act in a way that is consistent with their aggregate path preference. We found no significant difference between the convergence rates.

We demonstrate these findings through an illustrative example. For this example, the parameters were selected as follows:

- graph topology $\mathcal{G}$ as shown in Figure 4.1;
- linkwise flow functions as given by (2.3) with $C_1 = 2$ and $\theta_e = 1$ for all $e \in \mathcal{E}$;
- $F^h$ as in (2.6) with $\beta = 1$,
- $G$ as in (2.9) with $\gamma = 1$,
- initial conditions: $\pi_e(0) = 1/15$ for all $e \in \mathcal{E}$, $\rho_{e_1}(0) = \rho_{e_{12}}(0) = 5$, $\rho_{e_2}(0) = \rho_{e_6}(0) = \rho_{e_8}(0) = 7$, $\rho_{e_3}(0) = \rho_{e_7}(0) = 3$, $\rho_{e_4}(0) = 6$, $\rho_{e_5}(0) = 1$, $\rho_{e_9}(0) = 9$, $\rho_{e_{10}}(0) = 10$, $\rho_{e_{13}}(0) = 12$, $\rho_{e_{14}}(0) = 4$, $\rho_{e_{15}}(0) = 8$.
- $\eta = 0.1$.

For these values, $\rho^h$ was numerically calculated by first numerically computing $\pi^h$ as the equilibrium of $\dot{\pi} = F^h(f^\pi) - \pi$ with the function $F^h$ as defined in (2.6) and then setting $\rho^h = \mu^{-1}(A\pi^h)$. We considered the i-logit local route choice decision function as in Example 4. The evolution of the 1-norm distance of $\rho$ from $\rho^h$ is plotted on a log-linear scale in Figure 4.2 for two values of the sensitivity: (i) $\gamma > 0$ and (ii) $\gamma = 0$.

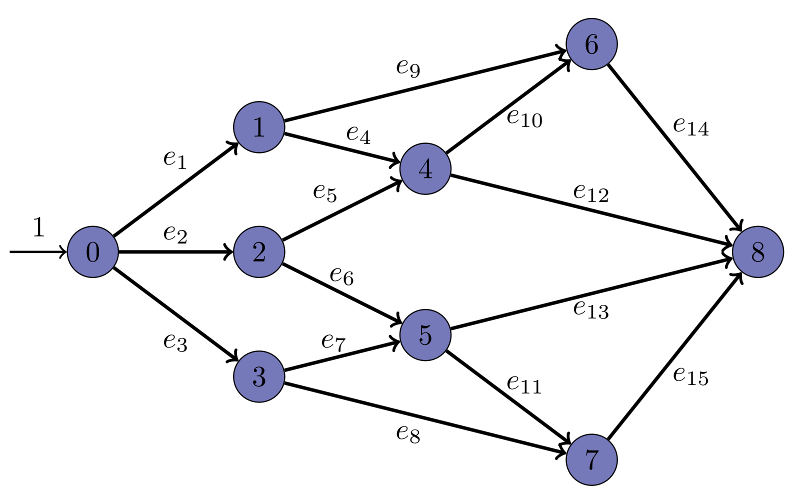


FIG. 4.1. *The graph topology used in simulations.*

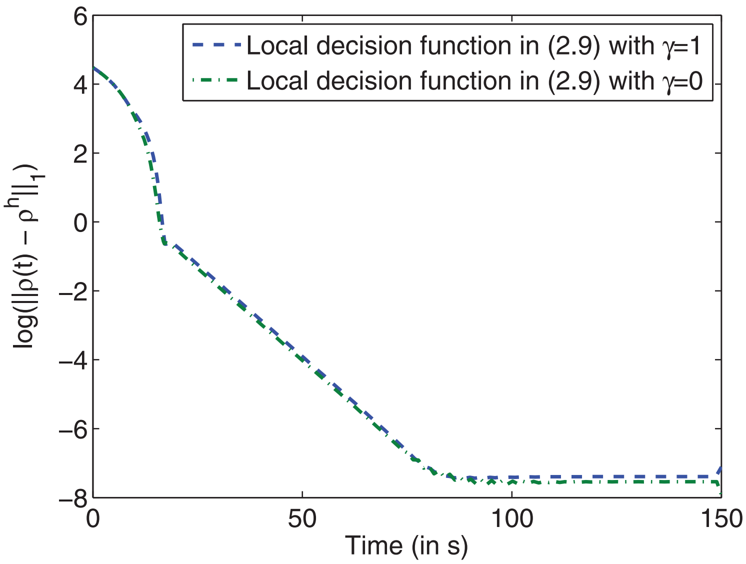


FIG. 4.2. *Log-linear plot for comparison of the evolution of* $\|\rho(t) - \rho^h\|_1$ *for the local decision function of Example* 4 *with sensitivity* $\gamma > 0$ *and* $\gamma = 0$.

Figure 4.2 also shows that there is no significant difference between the convergence of trajectory corresponding to the two values of the sensitivity.

**5. Conclusion.** In this paper, we have analyzed the stability of Wardrop equilibria in dynamical transportation networks characterized by dual temporal and spatial scales of driver route choice behavior. This is affected by the relatively infrequent drivers' perturbed best responses to global information about the current network congestion levels, as well as their instantaneous local observation of the immediate surroundings as they transit through the network.

We showed that if the frequency of updates of path preferences is sufficiently small, then the state of the transportation network ultimately approaches a neighborhood of the Wardrop equilibrium. The technical approach relied on establishing relevant properties for the resultant two-time-scale dynamics independently using tools from evolutionary game dynamics and cooperative dynamical systems and then using singular perturbation techniques to establish sufficient conditions for the stability of the Wardrop equilibrium for the coupled system. Our results contribute to providing stronger evidence supporting the significance of Wardrop's equilibrium postulate for a transportation network. They may be read as a sort of robustness of such equilibrium notion with respect to nonpersistent perturbations of the network.

There are several possible directions for future work. In a related work [6, 7] we have studied the effect of persistent, and possibly adversarial, perturbations on the traffic dynamics under a driver behavior model similar to the one considered in this paper. Moreover, we plan to provide microfoundations of our dynamical model by formally deriving it as the mean-field limit of a finite population stochastic process (see Remark 2). Additionally, it would be very interesting to extend our analysis by relaxing the assumptions of a single origin-destination pair, acyclic topology, increasing flow function, and unbounded density. In particular, removing the last assumption would be crucial in order to take into account backward propagation of perturbations in the fast-scale dynamics (cf. Remark 1).

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