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YANGIANS VS MINIMAL W-ALGEBRAS: A SURPRIZING COINCIDENCE

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ABSTRACT. We prove that the singularities of the R-matrix $R(k)$ of the minimal quantization of the adjoint representation of the Yangian $Y(\mathfrak{g})$ of a finite dimensional simple Lie algebra $\mathfrak{g}$ are the opposite of the roots of the monic polynomial $p(k)$ entering in the OPE expansions of quantum fields of conformal weight 3/2 of the universal minimal affine W-algebra at level k attached to $\mathfrak{g}$.

1. INTRODUCTION

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$, different from $sl(2)$. Let $Y(\mathfrak{g})$ be Drinfeld's Yangian associated to $\mathfrak{g}$ and $W^k(\mathfrak{g}, \theta)$ the universal minimal affine W-algebra at level k. The purpose of this paper is to explain a remarkable coincidence arising when considering, on the one hand, the minimal quantization to $Y(\mathfrak{g})$ of the adjoint representation of $\mathfrak{g}$, and, on the other hand, the OPE expansion for primary fields of $W^k(\mathfrak{g}, \theta)$ of conformal weight 3/2. To explain more precisely this coincidence we need some recollections. The algebra $Y(\mathfrak{g})$ is a Hopf algebra deformation of $U(\mathfrak{g}[t])$ which has been introduced in the famous paper [Dr] to construct solutions of the quantum Yang-Baxter equation. Its presentation involves generators X and $J(X), X \in \mathfrak{g}$. In [Dr, Theorem 8] Drinfeld explains how to quantize the adjoint representation of $\mathfrak{g}$ to $Y(\mathfrak{g})$: the "minimal" way of getting this quantization is to consider the space $V = \mathfrak{g} \oplus \mathbb{C}$ and let the generators X, $X \in \mathfrak{g}$ act in the natural way, and the generators $J(X)$, $X \in \mathfrak{g}$ act by (2.4). Formula (2.4) involves a certain constant δ, which depends just on the choice of the bilinear invariant form on $\mathfrak{g}$. Expanding on Drinfeld's work, Chari and Pressley [CP] studied the R-matrix $R(k)$ associated to V, and found that the blocks of this matrix corresponding the trivial and the adjoint isotypic components have as singularities 1, $h^\vee/2$, and the roots r_1, r_2 of a degree two monic polynomial in k. Here and further $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. It is implicit in their analysis that $\delta = -\frac{1}{2}r_1r_2$. See Remark 5.2.

Kac, Roan and Wakimoto [KRW] associated a vertex algebra $W^k(\mathfrak{g}, f)$, called a *universal affine W-algebra*, to each triple $(\mathfrak{g}, f, k)$, where f is a nilpotent element of $\mathfrak{g}$ viewed up to conjugation, and $k \in \mathbb{C}$, by applying the quantum Hamiltonian reduction functor to the affine vertex algebra $V^k(\mathfrak{g})$. In particular, it was shown that, for $k \neq -h^\vee$, $W^k(\mathfrak{g}, f)$ has a set of free generators, including a Virasoro vector ω. A more explicit presentation has been obtained in [KW] when f is an element from the minimal non-zero nilpotent orbit. Since $e_{-\theta}$, a root vector attached to the minimal root $-\theta$, is such an element, we will denote this vertex algebra by $W^k(\mathfrak{g}, \theta)$. A further improvement has been obtained in [AKMPP], where it has been proved that the OPE expansion of quantum fields of conformal weight 3/2 depends on a canonical monic quadratic polynomial $p(k)$. The surprizing fact is that its roots are $-r_1$ and $-r_2$.

Our approach to the explanation is essentially Lie-theoretic, even if information coming from the structure of vertex operator algebras generated by fields of low conformal weight is needed. Our first step is to provide a proof of Theorem 8 in [Dr] convenient for our goals. This is based on the analysis of certain $\mathfrak{g}$-equivariant maps $G_2 : \bigwedge^2 \mathfrak{g} \to S^3(\mathfrak{g}^*)$ and $G_3 : \bigwedge^3 \mathfrak{g} \to S^3(\mathfrak{g}^*)$ (cf. (2.5)), which arise naturally when considering Drinfeld's formula (2.4). The crucial Lemma 2.2 has been suggested to us by Corrado De Concini. Along the way we obtain a uniform formula for δ, see (4.18), (4.20), which easily specialize to Drinfeld's expressions for δ in each type of $\mathfrak{g}$. Remark that the handier formula (4.18) is given in terms of the grading (3.1) on $\mathfrak{g}$ associated to an element from the minimal nilpotent orbit.

On the W-algebra side, we consider the grading (3.1) and investigate the possible vertex algebras generated by fields L, J^v with $v \in \mathfrak{g}^\natural$ (cf. (3.2)), G^u with $u \in \mathfrak{g}_{-1/2}$, with the following λ-brackets: L is a Virasoro vector with central charge $\frac{k\,\dim\mathfrak{g}}{k+h^\vee} - 6k + h^\vee - 4$, J^u are primary of conformal weight 1, G^v are primary of conformal weight $\frac{3}{2}$, the J^u generate an affine vertex algebra, and no other constraints. The existence of such vertex algebras is guaranteed by [KW]. The final outcome is that imposing Jacobi identity, up to an overall multiplicative constant, one obtains precisely the relations given by [KW]: see Proposition 5.8. Coming back to the explanation of the coincidence, one substitutes auxiliary relations popping up in the proof of Proposition 5.8 (cf. (5.31), (5.32)) in formula (4.18) to get the desired result: see Theorem 5.9.

As a byproduct of our analysis, we get the following results.

(1) *Uniform fomulas for* $\dim\mathfrak{g}$*:* in Proposition 5.5 we get uniform formulas expressing the dimension of $\mathfrak{g}$ in terms of canonical data associated to the minimal grading (3.1).
(2) *Application to the Deligne exceptional series:* in particular, we can view the simple Lie algebras in the Deligne exceptional series in this framework (cf. Remark 5.6), providing a characterization in terms of the minimal grading which yields yet another uniform derivation of the dimension formulas.
(3) *OPE expansions of quantum fields of conformal weight* $3/2$*:* in Proposition 5.8, we refine [AKMPP, Lemma 3.1] by providing a precise expression for the 0-th product in the OPE expansions of quantum fields of conformal weight $3/2$ in $W^k(\mathfrak{g},\theta)$.

2. Yangians

2.1. **Setup and basic relations.** Let $\mathfrak{g}$ be a simple Lie algebra different from $sl(2)$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a set Δ_+ of positive roots for the $(\mathfrak{g},\mathfrak{h})$-root system Δ. Let Π be the corresponding set of simple roots. For $\alpha \in \Delta$ we let $\mathfrak{g}_\alpha$ denote the corresponding root space. Choose a nondegenerate invariant symmetric form $(\cdot,\cdot)$ on $\mathfrak{g}$. Denote by $\alpha_i, \omega_i, \theta$ the simple roots, the fundamental weights and the highest root, respectively. Set $\theta = \sum_i n_i\alpha_i$. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an orthonormal basis of $\mathfrak{g}$.

As noticed in the Introduction, we will focus on the case when $\mathfrak{g}$ is different from $sl(2)$. We recall the definition of the Yangian in this case.

Definition 2.1 ([Dr])**.** The Yangian $Y(\mathfrak{g})$ is the unital associative $\mathbb{C}$-algebra generated by the set of elements $\{X, J(X) : X \in \mathfrak{g}\}$ subject to the defining relations

$$XY - YX = [X,Y]_{\mathfrak{g}}, \quad J([X,Y]) = [J(X),Y], \tag{2.1}$$

$$J(cX + dY) = cJ(X) + dJ(Y), \tag{2.2}$$

(2.3)
$$[J(X),[J(Y),Z]]-[X,[J(Y),J(Z)]]=\sum_{\lambda,\mu,\nu\in\Lambda}([X,X_\lambda],[[Y,X_\mu],[Z,X_\nu]])\{X_\lambda,X_\mu,X_\nu\},$$

for all $X,Y,Z,W\in\mathfrak{g}$ and $c,d\in\mathbb{C}$, where $\{x_1,x_2,x_3\}=\frac{1}{24}\sum_{\pi\in\mathfrak{S}_3}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}$ for all $x_1,x_2,x_3\in Y(\mathfrak{g})$.

Remark 2.2. When $\mathfrak{g}=sl(2)$ relation (2.3) follows from (2.1) and (2.2), but a further complicated relation is needed: see [Dr], [GRW, Theorem 2.6], [GNW, 3.2] for details.

2.2. **Drinfeld's Theorem on the minimal quantization of the adjoint representation.** In the following we provide a uniform approach to Drinfeld's description of the minimal quantization of the adjoint representation of $Y(\mathfrak{g})$.

The following statement sums up the content of Theorems 7 and 8 from [Dr].

Theorem 2.1 (Drinfeld)**.** *Let $\mathfrak{g}$ be a simple Lie algebra, different from $sl(2)$. Let $\mathcal{V}=\mathfrak{g}\oplus\mathbb{C}$.*

(a). There exists a unique constant $\delta\in\mathbb{C}$ such that the natural action of $\mathfrak{g}$ on $\mathcal{V}$ extends to an action of $Y(\mathfrak{g})$ by setting

$$J(x)(y,\alpha)=(\delta\alpha x,(x,y)). \tag{2.4}$$

(b). If either $n_i=1$ or $n_i=(\theta,\theta)/(\alpha_i,\alpha_i)$, then the fundamental representation V_{ω_i} of $\mathfrak{g}$ extends to a $Y(\mathfrak{g})$-representation by letting $J(x)$ act as 0.

Remark 2.3. For $\mathfrak{g}=sl(2)$ relation 2.4 holds for any δ.

To prove Theorem 2.1 we need some preliminary work. Consider the maps $G_2:\bigwedge^2\mathfrak{g}\to S^3(\mathfrak{g}^*)$ and $G_3:\bigwedge^3\mathfrak{g}\to S^3(\mathfrak{g}^*)$ defined by setting

$$G_2(X\wedge Y)(a)=([[X,a],a],[Y,a]),\ \ G_3(X\wedge Y\wedge Z)(a)=([[X,a],[Y,a]],[Z,a]). \tag{2.5}$$

Let $\partial_p(X_1\wedge\ldots\wedge X_p)=\sum_{i<j}(-1)^{i+j+1}[X_i,X_j]\wedge X_1\wedge\ldots\wedge\widehat{X_i}\wedge\ldots\wedge\widehat{X_j}\wedge\ldots X_p$ be the usual boundary operator for the Lie algebra homology. The next lemma has been suggested to us by C. De Concini.

Lemma 2.2.
(1)
$$G_3=\tfrac{1}{3}G_2\circ\partial_3.$$
(2) The maps G_2,G_3 are $\mathfrak{g}$-equivariant.

Proof. To prove (1) we start with the Jacobi identity:

$$[[X,Y],a],a]=[[X,a],Y],a]+[[X,[Y,a]],a]=2[[X,a],[Y,a]]+[X,[[Y,a],a]]-[Y,[[X,a],a]],$$

so

$$\begin{aligned}G_2\circ\partial_3(X\wedge Y\wedge Z)&=6G_3(X\wedge Y\wedge Z)\\&+([X,[[Y,a],a]]-[Y,[[X,a],a]],[Z,a])+([Z,[[X,a],a]]-[X,[[Z,a],a]],[Y,a])\\&+([Y,[[Z,a],a]]-[Z,[[Y,a],a]],[X,a]).\end{aligned}$$

Using the invariance of the form we have

$$G_2\circ\partial_3(X\wedge Y\wedge Z)-6G_3(X\wedge Y\wedge Z)=(Z,R(X,Y,a)),$$

where

$$\begin{aligned}R(X,Y,a)=&-[[X,[[Y,a],a]],a]+[[Y,[[X,a],a]],a]+[[X,a],a],[Y,a]]\\&+[a,[a,[X,[Y,a]]]]-[a,[a,[Y,[X,a]]]]-[[[Y,a],a],[X,a]]).\end{aligned}$$

Since

$$[[X,[[Y,a],a]],a]=[[X,a],[[Y,a],a]]+[X,[[[Y,a],a],a]]$$

and

$$[[Y,[[X,a],a]],a]=[[Y,a],[[X,a],a]]+[Y,[[[X,a],a],a]],$$

we can rewrite $R(X,Y,a)$ as

$$\begin{aligned}&-[X,[[[Y,a],a],a]]+[Y,[[[X,a],a],a]]+[a,[a,[X,[Y,a]]]]-[a,[a,[Y,[X,a]]]]\\&=-[X,[[[Y,a],a],a]]+[Y,[[[X,a],a],a]]+[a,[a,[[X,Y],a]]]].\end{aligned}$$

Thus

$$\begin{aligned}&G_2\circ\partial_3(X\wedge Y\wedge Z)-6G_3(X\wedge Y\wedge Z)=(Z,R(X,Y,a))\\&=(Z,-[X,[[[Y,a],a],a]]+[Y,[[[X,a],a],a]]+[a,[a,[[X,Y],a]]]])\\&=-([[[Z,X],a],a],[Y,a])-([[[Y,Z],a],a],[X,a])-([[[X,Y],a],a],[Z,a])\\&=-G_2\circ\partial_3(X\wedge Y\wedge Z).\end{aligned}$$

To prove (2) observe that

$$\begin{aligned}G_2(\theta(x)(X\wedge Y)(a)&=([[x,X],a],a],[Y,a])+((([X,a],a],[[x,Y],a])\\&=([x,[[X,a],a],[Y,a])+((([X,a],a],[x,[Y,a]])\\&+([[X,[x,a]],a],[Y,a])+([[X,a],[x,a]],[Y,a])+([[X,a],a],[Y,[x,a]])\\&=([[X,[x,a]],a],[Y,a])+([[X,a],[x,a]],[Y,a])+([[X,a],a],[Y,[x,a]]).\end{aligned}$$

Since ∂_3 is $\mathfrak{g}$-equivariant, it follows from (1) that G_3 is $\mathfrak{g}$-equivariant. □

Identify $S^3(\mathfrak{g}^*)$ and $S^3(\mathfrak{g})$ using the form $(\cdot,\cdot)$. Set

$$\phi_i=ad\circ Symm\circ G_i:\bigwedge^i\mathfrak{g}\to End(\mathfrak{g}),\tag{2.6}$$

where $Symm:S(\mathfrak{g})\to U(\mathfrak{g})$ is the symmetrization map, ad is the extension to $U(\mathfrak{g})$ of the adjoint representation $ad:\mathfrak{g}\to End(\mathfrak{g})$. If $X\in\wedge^i\mathfrak{g}$ then, clearly, the map $\phi_i(X)(U,V)=(\phi_i(X)(U),V)$ is bilinear in U,V so ϕ_i defines a map $g_i:\wedge^i\mathfrak{g}\to\mathfrak{g}^*\otimes\mathfrak{g}^*$.

Lemma 2.3. *The maps g_i are alternating, thus they define maps $g_i:\wedge^i\mathfrak{g}\to\wedge^2\mathfrak{g}^*$.*

Proof. By Lemma 2.2, in order to prove the first statement, we need only to prove that, if $U,V\in\mathfrak{g}$, then

$$(g_2(X\wedge Y)(U),V)=-(g_2(X\wedge Y)(V),U).$$

Explicitly

$$\begin{aligned}&(g_2(X\wedge Y)(U),V)=\sum_\sigma\sum_{p_1,p_2,p_3}([[X,a_{p_1}],a_{p_2}],[Y,a_{p_3}])\,([a^{p_{\sigma(1)}},[a^{p_{\sigma(2)}},[a^{p_{\sigma(3)}},U]]],V)=\\&-\sum_\sigma\sum_{p_1,p_2,p_3}([[X,a_{p_1}],a_{p_2}],[Y,a_{p_3}])\,(U,[a^{p_{\sigma(3)}},[a^{p_{\sigma(2)}},[a^{p_{\sigma(1)}},V]]]).\end{aligned}$$

Set $\tau=\sigma\circ(13)$; then

$$\begin{aligned}&\sum_\sigma\sum_{p_1,p_2,p_3}([[X,a_{p_1}],a_{p_2}],[Y,a_{p_3}])\,([a^{p_{\sigma(1)}},[a^{p_{\sigma(2)}},[a^{p_{\sigma(3)}},U]]],V)=\\&-\sum_\tau\sum_{p_1,p_2,p_3}([[X,a_{p_1}],a_{p_2}],[Y,a_{p_3}])\,([a^{p_{\tau(1)}},[a^{p_{\tau(2)}},[a^{p_{\tau(3)}},V]]],U)\end{aligned}$$

as required. □

Extend $(\cdot,\cdot)$ to an invariant bilinear form on $\wedge^2\mathfrak{g}$ (by determinants) and identify $\wedge^2\mathfrak{g}^*$ with $\wedge^2\mathfrak{g}$ using this form. In particular we can view the maps g_i as maps from $\wedge^i\mathfrak{g}$ to $\wedge^2\mathfrak{g}$.

Lemma 2.4. *The map g_2 is symmetric:*

$$(g_2(X\wedge Y), U\wedge V) = (X\wedge Y, g_2(U\wedge V)).$$

Proof. By unwinding all the identifications we find

$$\begin{aligned}(g_2(X\wedge Y), U\wedge V) &= \sum_\sigma \sum_{p_1,p_2,p_3} ([[X,a_{p_1}],a_{p_2}],[Y,a_{p_3}])\,([a^{p_{\sigma(1)}},[a^{p_{\sigma(2)}},[a^{p_{\sigma(3)}},U]]],V)\\ &= \sum_\sigma \sum_{p_1,p_2,p_3} ([a_{p_{\sigma^{-1}(2)}},[a_{p_{\sigma^{-1}(1)}},[a_{p_{\sigma^{-1}(3)}},X]]]],Y)\,([U,a^{p_1}],a^{p_2}],[V,a^{p_3}]).\end{aligned}$$

Set $\tau=\sigma^{-1}\circ(12)$; then

$$\begin{aligned}(g_2(X\wedge Y), U\wedge V) &= \sum_\tau \sum_{p_1,p_2,p_3} ([a_{p_{\tau(1)}},[a_{p_{\tau(2)}},[a_{p_{\tau(3)}},X]]]],Y)\,([U,a^{p_1}],a^{p_2}],[V,a^{p_3}])\\ &= (X\wedge Y, g_2(U\wedge U)).\end{aligned}$$

□

Lemma 2.5. *There is a unique costant $k\in\mathbb{C}$ such that*

$$g_3 = k\,\partial_3. \tag{2.7}$$

Proof. Since $\mathfrak{g}\neq sl(2)$, recall that by [Ko] we have orthogonal decompositions

$$\bigwedge^2\mathfrak{g} = \mathbf{d}\mathfrak{g}\oplus U_2, \tag{2.8}$$

$$\bigwedge^3\mathfrak{g} = Ker\,\partial_3\oplus Im\,\mathbf{d} = Ker\,\partial_3\oplus\mathbf{d}(U_2), \tag{2.9}$$

where $\mathbf{d}$ is the Chevalley-Eilenberg differential for Lie algebra cohomology, U_2 is the subspace of $\bigwedge^2\mathfrak{g}$ generated by 2-tensors $x\wedge y$ with $[x,y]=0$.

Moreover, again by [Ko], $Hom_{\mathfrak{g}}(\mathfrak{g},U_2)=0$ and U_2 is irreducible for $\mathfrak{g}\neq sl(n)$, while, if $\mathfrak{g}=sl(n)$, U_2 decomposes as $U_2=V_1\oplus V_2$ with V_1, V_2 inequivalent irreducibles with $V_2=V_1^*$.

Since

$$\phi_2 = ad\circ Symm\circ G_2 : \bigwedge^2\mathfrak{g}\to End(\mathfrak{g})$$

is $\mathfrak{g}$-equivariant, by the invariance of the form, g_2 is also equivariant. It follows that $g_2(U_2)\subset U_2$. If $\mathfrak{g}\neq sl(n)$, then

$$(g_2)_{|U_2} = k'I \text{ for some } k'\in\mathbb{C}. \tag{2.10}$$

Note that $(Im\,\partial_3)^\perp = Ker\,\mathbf{d}$. Since $H^2(\mathfrak{g})=0$, $Ker\,\mathbf{d}=\mathbf{d}\mathfrak{g}$. It follows that $Im\,\partial_3=U_2$. Since $g_3=\frac{1}{3}g_2\circ\partial_3$, formula (2.7) is proven in this case by setting

$$k=\tfrac{k'}{3}. \tag{2.11}$$

If $\mathfrak{g}=sl(n)$, by the same argument, we have that $g_2(V_i)\subset V_i$, hence there is k such that $(g_2)_{|V_1}=kI_{V_1}$. Let $x\in V_2$ and $y=v_1+v_2\in U_2$ with $v_i\in V_i$. Then

$$(g_2(x),y) = (g_2(x),v_1+v_2) = (g_2(x),v_1) = (x,g_2(v_1)) = k(x,v_1) = k(x,y).$$

Since the form is nondegenerate when restricted to U_2, (2.7) holds in this case too. □

2.2.1. *Proof of Theorem 2.1.* By (2.1) we must have

$$[x, J(y)](u, 0) = -(0, (y, [x, u])) = (0, ([x, y], u)) = J([x, y])(u, 0)$$

and

$$[x, J(y)](0, 1) = (\delta[x, y], 0) = J([x, y])(0, 1),$$

which holds for all δ. It is clear that both sides of (2.3) act on $(0, 1)$ trivially.

Define $f : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to End(\mathfrak{g})$ by setting

$$([J(X), [J(Y), Z]] - [X, [J(Y), J(Z)]])(U, 0) = (f(X, Y, Z)(U), 0).$$

Then

$$\begin{aligned}(f(X,Y,Z)(U), W) &= \delta(([Y,Z],U)(X,W) - (X,U)([Y,Z],W) - (Z,U)([X,Y],W)) \\ &+ \delta((Y,U)([X,Z],W) + (Z,[X,U])(Y,W) - (Y,[X,U])(Z,W)) \\ &= \delta([X,Y]\wedge Z, U\wedge W) - ([X,Z]\wedge Y, U\wedge W) + ([Y,Z]\wedge X, U\wedge W)) \\ &= \delta(\partial_3(X\wedge Y\wedge Z), U\wedge W).\end{aligned}$$

We let the R.H.S. of (2.3) act on $(U, 0)$:

$$\begin{aligned}&\sum_{\lambda,\mu,\nu\in\Lambda} ([X, X_\lambda], [[Y, X_\mu], [Z, X_\nu]])\{X_\lambda, X_\mu, X_\nu\}(U, 0) \\ &= (\tfrac{1}{24}\sum_\sigma \sum_{p_1,p_2,p_3} ([X, X_{p_1}], [[Y, X_{p_2}], [Z, X_{p_3}]])[X_{p_{\sigma(1)}}, [X_{p_{\sigma(2)}}, [X_{p_{\sigma(3)}}, U], 0) \\ &= (\tfrac{1}{24}\phi_3(X\wedge Y\wedge Z)(U), 0),\end{aligned}$$

so we must have $\delta(\partial_3(X \wedge Y \wedge Z), U \wedge W) = (\frac{1}{24}g_3(X \wedge Y \wedge Z), U \wedge W)$. Thus, by Lemma 2.5, relation (2.3) holds if and only if

$$\delta = \tfrac{k}{24}. \tag{2.12}$$

This proves claim (a) of the theorem. To prove claim (b), set

$$g_i^j = \rho_j \circ Symm \circ G_i : \bigwedge^i \mathfrak{g} \to gl(V_{\omega_j}), \tag{2.13}$$

where $\rho_j : \mathfrak{g} \to gl(V_{\omega_j})$ is the j-th fundamental representation of $\mathfrak{g}$. Then, as shown in the next table, U_2 does not appear in $V_{\omega_j} \otimes V_{\omega_j}^*$, since its highest weight $2\theta - \bar{\alpha}$ is not less than or equal than $\omega_j - w_0(\omega_j)$ (here $\bar{\alpha}$ is a simple root not orthogonal to θ and w_0 is the longest element in the Weyl group). In the exceptional cases we display the coordinates w.r.t. the choice of the simple roots from Bourbaki.

Type of $\mathfrak{g}$, j	$2\theta - \bar{\alpha}$	$\omega_j - w_0(\omega_j)$
A_n, $1 \le j \le [n+1]/2$	$\epsilon_1 + \epsilon_2 - 2\epsilon_{n+1}, 2\epsilon_1 - \epsilon_n - \epsilon_{n+1}$	$\sum_{h=1}^{j}(\epsilon_h - \epsilon_{n+2-h})$
B_n, $j = 1$	$2\epsilon_1 + \epsilon_2 + \epsilon_3$	$2\epsilon_1$
B_n, $j = n$	$2\epsilon_1 + \epsilon_2 + \epsilon_3$	$\sum_{i=1}^{n}\epsilon_i$
C_n, $1 \le j \le n$	$2\epsilon_1 + \epsilon_2$	$\sum_{i=1}^{j}\epsilon_i$
D_n, $j = 1$	$2\epsilon_1 + \epsilon_2 + \epsilon_3$	$2\epsilon_1$
D_n, $j = n-1\,(n)$	$2\epsilon_1 + \epsilon_2 + \epsilon_3$	$\sum_{i=1}^{n-1}\epsilon_i$
E_6, $j = 1$ (6)	$(2, 3, 4, 6, 4, 2)$	$(2, 2, 3, 4, 3, 2)$
E_7, $j = 7$	$(3, 4, 6, 8, 6, 4, 2)$	$(2, 3, 4, 6, 5, 4, 3)$
F_4, $j = 4$	$(3, 6, 8, 4)$	$(1, 2, 3, 2)$
G_2, $j = 1$	$(6, 3)$	$(2, 1)$

3. Minimal $\frac{1}{2}\mathbb{Z}$-grading of a simple Lie algebra

Choose root vectors $x_{\pm\theta} \in \mathfrak{g}_{\pm\theta}$ so that $(x_\theta|x_{-\theta}) = \frac{1}{2}$. Set $x = [x_\theta, x_{-\theta}]$. The eigenspace decomposition of $ad\,x$ defines the *minimal* $\frac{1}{2}\mathbb{Z}$-grading:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \tag{3.1}$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C}x_{\pm\theta}$. Furthermore, one has

$$\mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x, \quad \mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}. \tag{3.2}$$

Note that $\mathfrak{g}^\natural$ is the centralizer of the triple $\{x_{-\theta}, x, x_\theta\}$. We can choose $\mathfrak{h}^\natural = \{h \in \mathfrak{h} \mid (h|x) = 0\}$ as a Cartan subalgebra of $\mathfrak{g}^\natural$, so that $\mathfrak{h} = \mathfrak{h}^\natural \oplus \mathbb{C}x$. Set, for $u, v \in \mathfrak{g}_{-1/2}$,

$$\langle u, v\rangle = (x_\theta|[u, v])$$

and note that $\langle\cdot,\cdot\rangle$ is a $\mathfrak{g}^\natural$–invariant symplectic form on $\mathfrak{g}_{-1/2}$.

We will use the following terminology.

Definition 3.1. We say that an ideal in $\mathfrak{g}^\natural$ is *irreducible* if it is simple or 1-dimensional. We call such an ideal a *component* of $\mathfrak{g}^\natural$.

Write

$$\mathfrak{g}^\natural = \bigoplus_{i=1}^{r} \mathfrak{g}^\natural_i$$

with $\mathfrak{g}^\natural_i$ irreducible. Recall that $r = 1, 2$ or 3. For a simple Lie algebra $\mathfrak{a}$ we let $h^\vee_{\mathfrak{a}}$ to be its dual Coxeter number and, if $\mathfrak{a}$ is abelian, we set $h^\vee_{\mathfrak{a}} = 0$. Set $h^\vee = h^\vee_{\mathfrak{g}}$ and $h^\vee_i = h^\vee_{\mathfrak{g}^\natural_i}$. Let ν_i be the ratio of the normalized invariant form of $\mathfrak{g}$ restricted to $\mathfrak{g}^\natural_i$ and the normalized form on $\mathfrak{g}^\natural_i$. Set finally $\bar{h}^\vee_i = h^\vee_i/\nu_i$. For reader's convenience, we display the relevant data in the following Table (although we proceed uniformly, so we do not need to use them).

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$	$\bar{h}^\vee_i$
$sl(3)$	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}^*$	3	0
$sl(n), n \geq 4$	$gl(n-2)$	$\mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*$	n	$0, n-2$
$so(n), n > 6, n \neq 8$	$sl(2) \oplus so(n-4)$	$\mathbb{C}^2 \otimes \mathbb{C}^{n-4}$	$n-2$	$2, n-6$
$so(8)$	$sl(2) \oplus sl(2) \oplus sl(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$	6	$2, 2, 2$
$sp(2n), n \geq 2$	$sp(2n-2)$	$\mathbb{C}^{2n-2}$	$n+1$	n
G_2	$sl(2)$	$S^3\mathbb{C}^2$	4	$4/3$
F_4	$sp(6)$	$\bigwedge_0^3 \mathbb{C}^6$	9	4
E_6	$sl(6)$	$\bigwedge^3 \mathbb{C}^6$	12	6
E_7	$so(12)$	$spin_{12}$	18	10
E_8	E_7	$\dim = 56$	30	18

Consider now the involution $\sigma_x = e^{2\pi\sqrt{-1}ad(x)}$. Since $\alpha_i(x) \geq 0$ for all simple roots α_i and $\theta(x) = 1$, it follows that the set $\{1 - \theta(x)\} \cup \{\alpha_i(x) \mid \alpha_i \text{ simple root}\}$ is the set of Kac parameters for the automorphism σ_x. In particular, since σ_x is an involution, either there is a unique simple root α_{i_0} such that $\alpha_{i_0}(x) \neq 0$ or there are exactly two simple roots $\alpha_{i_0}, \alpha_{i_1}$ such that $\alpha_{i_j}(x) \neq 0$. Let s be the number of simple roots not orthogonal to θ. If $s = 1$ then $\alpha_{i_0}(x) = \frac{1}{2}$ and $n_{i_0} = 2$. If $s = 2$, then $\alpha_{i_0}(x) = \alpha_{i_1}(x) = \frac{1}{2}$, $n_{i_0} = n_{i_1} = 1$.

Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the eigenspace decomposition of σ_x. We observe that

$$\mathfrak{k} = span(x_\theta, x, x_{-\theta}) \oplus \mathfrak{g}^\natural \simeq sl(2) \times \mathfrak{g}^\natural, \quad \mathfrak{p} = \mathfrak{g}_{1/2} \oplus \mathfrak{g}_{-1/2}.$$

One can choose the set of positive roots for $\mathfrak{k}$ so that the corresponding set of simple roots is $\{-\theta\}\cup\{\alpha\in\Pi\mid\alpha(x)=0\}=\{-\theta\}\cup\{\alpha\in\Pi\mid(\alpha|\theta)=0\}$.

Consider the case $s=1$. Then $\mathfrak{g}^\natural$ is semisimple and the number of simple ideals of $\mathfrak{g}^\natural$ equals the number of roots attached to α_{i_0}. Moreover $\mathfrak{p}$ is irreducible and its highest weight as a $\mathfrak{k}$-module is $-\alpha_{i_0}$. Since $\alpha_{i_0}(\theta^\vee)=1$, we see that $\mathfrak{p}=V_{sl(2)}(\omega_1)\otimes V_{\mathfrak{g}^\natural}(-(\alpha_{i_0})_{|\mathfrak{h}^\natural})$. (Here $V_{\mathfrak{a}}(\lambda)$ denotes the irreducible finite dimensional $\mathfrak{a}$-module of highest weight λ). If U is a $ad(x)$-stable space we let U_k denote the eigenspace corresponding to the eigenvalue k. Since $V_{sl(2)}(\omega_1)=V_{sl(2)}(\omega_1)_{1/2}\oplus V_{sl(2)}(\omega_1)_{-1/2}$ we see that, as $\mathfrak{g}^\natural$-module,

$$\mathfrak{g}_{1/2}\simeq\mathfrak{g}_{-1/2}\simeq V_{\mathfrak{g}^\natural}(-(\alpha_{i_0})_{|\mathfrak{h}^\natural}).$$

In particular $\mathfrak{g}_{\pm1/2}$ are irreducible as $\mathfrak{g}^\natural$-modules.

If $s=2$ we have $\mathfrak{g}^\natural_0=\mathbb{C}\varpi$ with

$$\varpi=\omega^\vee_{i_0}-\omega^\vee_{i_1}. \tag{3.3}$$

Moreover $\mathfrak{p}=V_{\mathfrak{k}}(-\alpha_{i_0})\oplus V_{\mathfrak{k}}(-\alpha_{i_1})$. Arguing as above we obtain that

$$\mathfrak{g}_{1/2}\simeq\mathfrak{g}_{-1/2}\simeq V_{\mathfrak{g}^\natural}(-(\alpha_{i_0})_{|\mathfrak{h}^\natural})\oplus V_{\mathfrak{g}^\natural}(-(\alpha_{i_1})_{|\mathfrak{h}^\natural}).$$

Since ϖ acts on $V_{\mathfrak{g}^\natural}(-(\alpha_{i_j})_{|\mathfrak{h}^\natural})$ as $-(-1)^jI$ we see that $\mathfrak{g}_{-1/2}$ is the sum of two inequivalent $\mathfrak{g}^\natural$-modules.

As shown in [CMPP, Proposition 4.8], $V_{\mathfrak{k}}(-\alpha_{i_0})^*\simeq V_{\mathfrak{k}}(-\alpha_{i_1})$, hence,

$$V_{\mathfrak{g}^\natural}(-(\alpha_{i_0})_{|\mathfrak{h}^\natural})^*\simeq V_{\mathfrak{g}^\natural}(-(\alpha_{i_1})_{|\mathfrak{h}^\natural}).$$

We now turn to the study of $\wedge^2\mathfrak{g}_{-1/2}$ as a $\mathfrak{g}^\natural$-module. Let $\mathbf{d},\mathbf{d}_{\mathfrak{k}}$ be coboundary operators for the Lie algebra cohomology of $\mathfrak{g}$, $\mathfrak{k}$ respectively and set $\mathbf{d}_1=\mathbf{d}-\mathbf{d}_{\mathfrak{k}}$. By [P, Prop. 4.3], $\wedge^2\mathfrak{p}=\mathbf{d}_1\mathfrak{k}\oplus V'$, where

$$V'=span(u\wedge v\mid u,v\in\mathfrak{p},\ [u,v]=0). \tag{3.4}$$

We observe that $\wedge^2\mathfrak{g}_{-1/2}=(\wedge^2\mathfrak{p})_{-1}$, so

$$\wedge^2\mathfrak{g}_{-1/2}=\mathbf{d}_1(\mathfrak{k})_{-1}\oplus V'_{-1}.$$

As $\mathfrak{k}=span(x_\theta,x,x_{-\theta})\oplus\mathfrak{g}^\natural$, we see that $\mathbf{d}_1(\mathfrak{k})_{-1}=\mathbb{C}\mathbf{d}_1(x_{-\theta})\simeq V_{\mathfrak{g}^\natural}(0)$. Set $\Pi_j=\{\alpha\in\Pi\mid(\alpha,\alpha_{i_j})\neq0\}\cup\{-\theta\}$, $\mathcal{W}'_j=\{-\alpha_{i_j}-s_{\alpha_{i_j}}(\alpha)\mid\alpha\in\Pi_j\}$, and

$$\mathcal{W}'=\begin{cases}\mathcal{W}'_0 & \text{if } s=1,\\ \mathcal{W}'_0\cup\mathcal{W}'_1 & \text{if } s=2 \text{ and } (\alpha_{i_0},\alpha_{i_1})\neq0,\\ \mathcal{W}'_0\cup\mathcal{W}'_1\cup\{-\alpha_{i_0}-\alpha_{i_1}\} & \text{if } s=2 \text{ and } (\alpha_{i_0},\alpha_{i_1})=0.\end{cases}$$

Recall from [CMP] that

$$V'=\oplus_{\lambda\in\mathcal{W}'}V_{\mathfrak{k}}(\lambda)$$

and that, if $\lambda=-\alpha_{i_j}-s_{\alpha_{i_j}}(\alpha)$, then the highest weight vector is $x_{-\alpha_{i_j}}\wedge x_{-s_{\alpha_{i_j}}(\alpha)}$. Set explicitly $\{\lambda_1,\dots,\lambda_p\}=\{\lambda\in\mathcal{W}'\mid\lambda(x)=-1\}$, so that, the (-1)-eigenspace of $ad(x)$ is

$$V'_{-1}=\bigoplus_{i=1}^{p}V_{\mathfrak{g}^\natural}(\lambda_{i|\mathfrak{h}^\natural}).$$

Consider the map $\Phi:\bigwedge^2\mathfrak{g}_{-1/2}\to S^2((\mathfrak{g}^\natural)^*)$ defined by

$$\Phi(u\wedge v)(a)=\langle[u,a],[v,a]\rangle. \tag{3.5}$$

It is easy to check that the map Φ is $\mathfrak{g}^\natural$-equivariant.

Proposition 3.1. *Let $P : \bigwedge^2 \mathfrak{g}_{-1/2} \to S^2((\mathfrak{g}^\natural)^*)$ be a $\mathfrak{g}^\natural$-equivariant map. Then there are constants $f_1, \dots, f_p$ such that*

$$P_{|V_{\mathfrak{g}^\natural}(\lambda_i)} = f_i \ \Phi_{|V_{\mathfrak{g}^\natural}(\lambda_i)}$$

for $1 \le i \le p$.

Proof. By [R, Proposition 2.1], $S^2(\mathfrak{g}^\natural)$ decomposes with multiplicity one. Since the same happens to V' (cf. (3.4)), it suffices to prove that if $P_{|V_{\mathfrak{g}^\natural}(\lambda_i)}$ is nonzero then $\Phi_{|V_{\mathfrak{g}^\natural}(\lambda_i)}$ is nonzero.

Assume that $\mathfrak{g}^\natural$ is semisimple. In this case we prove that $Ker\Phi = \{0\}$. It is enough to check that $\Phi(x_{-\alpha_{i_j}} \wedge x_{-s_{\alpha_{i_j}}(\alpha)}) \ne 0$ for all $\alpha \in \Pi_j$ such that $(\alpha_{i_j} + s_{\alpha_{i_j}}(\alpha))(x) = 1$. Since $\alpha_{i_j}(x) = 1/2$ and $\alpha(x) = 0$, we see that $s_{\alpha_{i_j}}(\alpha) = \alpha + \alpha_{i_j}$. It follows that

$$\Phi(x_{-\alpha_{i_j}} \wedge x_{-s_{\alpha_{i_j}}(\alpha)})(a) = \langle [x_{-\alpha_{i_j}}, a], [x_{-\alpha_{i_j}-\alpha}, a] \rangle. \tag{3.6}$$

Assume first that there is a unique simple root α_{i_0} not orthogonal to θ. If α_{i_0} is a short root then $x_{-\theta+2\alpha_{i_0}} \in \mathfrak{g}^\natural$. Since $-\theta + \alpha_{i_0} - \alpha, -\alpha_{i_0} + \alpha$ are not roots, we obtain, taking $a = x_{-\theta+2\alpha_{i_0}} + x_\alpha$,

$$\langle [x_{-\alpha_{i_0}}, a], [x_{-\alpha_{i_0}-\alpha}, a] \rangle = \langle [x_{-\alpha_{i_0}}, x_{-\theta+2\alpha_{i_0}}], [x_{-\alpha_{i_0}-\alpha}, x_\alpha] \rangle \ne 0.$$

We can therefore assume that α_{i_0} is a long root. Assume α short. The fact that α_{i_0} is long implies that $(\alpha|\alpha_{i_0}) = -1$, so

$$(\theta - \alpha_{i_0} - \alpha|\alpha) = (-\alpha_{i_0} - \alpha|\alpha) = 1 - (\alpha|\alpha) \ge 0.$$

Since $\theta - \alpha_{i_0} = \theta - \alpha_{i_0} - \alpha + \alpha$ is a root, it follows that $\theta - \alpha_{i_0} - 2\alpha$ is a positive root. Since

$$(\theta - \alpha_{i_0} - 2\alpha|\alpha_{i_0}) = 1 - 2 + 2 = 1,$$

$\theta - 2\alpha_{i_0} - 2\alpha$ is a positive root as well. We choose $a = x_{-\alpha} + x_{-\theta+2\alpha_{i_0}+2\alpha}$, so

$$\begin{aligned}
\langle [x_{-\alpha_{i_0}}, a], [x_{-\alpha_{i_0}-\alpha}, a] \rangle &= \langle [x_{-\alpha_{i_0}}, x_{-\alpha}], [x_{-\alpha_{i_0}-\alpha}, x_{-\theta+2\alpha_{i_0}+2\alpha}] \rangle \\
&+ \langle [x_{-\alpha_{i_0}}, x_{-\theta+2\alpha_{i_0}+2\alpha}], [x_{-\alpha_{i_0}-\alpha}, x_{-\alpha}] \rangle \\
&= \langle [x_{-\alpha_{i_0}}, x_{-\alpha}], [x_{-\alpha_{i_0}-\alpha}, x_{-\theta+2\alpha_{i_0}+2\alpha}] \rangle \\
&+ \langle x_{-\alpha_{i_0}}, [x_{-\theta+2\alpha_{i_0}+2\alpha}, [x_{-\alpha_{i_0}-\alpha}, x_{-\alpha}]] \rangle \\
&= 2\langle [x_{-\alpha_{i_0}}, x_{-\alpha}], [x_{-\alpha_{i_0}-\alpha}, x_{-\theta+2\alpha_{i_0}+2\alpha}] \rangle \\
&+ \langle x_{-\alpha_{i_0}}, [x_{-\alpha_{i_0}-\alpha}, [x_{-\theta+2\alpha_{i_0}+2\alpha}, x_{-\alpha}]] \rangle.
\end{aligned}$$

If $-\theta + 2\alpha_{i_0} + \alpha$ is a root, then

$$(-\theta + 2\alpha_{i_0} + \alpha|\alpha_{i_0}) = -1 + 4 - 1 = 2,$$

contradicting the fact that α_{i_0} is long. It follows that $[x_{-\theta+2\alpha_{i_0}+2\alpha}, x_{-\alpha}] = 0$, hence

$$\langle [x_{-\alpha_{i_0}}, a], [x_{-\alpha_{i_0}-\alpha}, a] \rangle = 2\langle [x_{-\alpha_{i_0}}, x_{-\alpha}], [x_{-\alpha_{i_0}-\alpha}, x_{-\theta+2\alpha_{i_0}+2\alpha}] \rangle \ne 0.$$

We can therefore assume that α is long. Let Σ be the component of the Dynkin diagram of $\mathfrak{g}^\natural$ containing α. Let θ_Σ be its highest root. If $\alpha = \theta_\Sigma$, then $\mathfrak{g}^\natural$ is not simple, for, otherwise, $\mathfrak{g} \simeq sl(3)$. Let Σ' be the Dynkin diagram of another component of $\mathfrak{g}^\natural$. Then $\alpha + \alpha_{i_0} + \theta_{\Sigma'}$ is a positive root as well as $\theta - \alpha - \alpha_{i_0} - \theta_{\Sigma'}$. Since α_{i_0} is a long root, $(\theta_{\Sigma'}|\alpha_{i_0}) = -1$, hence

$$(\theta - \alpha - \alpha_{i_0} - \theta_{\Sigma'}|\alpha_{i_0}) = 1 + 1 - 2 + 1 = 1,$$

so that $\theta - \alpha - 2\alpha_{i_0} - \theta_{\Sigma'}$ is a positive root. We choose $a = x_{-\theta_{\Sigma'}} + x_{-\theta+\alpha+2\alpha_{i_0}+\theta_{\Sigma'}}$.

Arguing as above, we conclude that

$$\langle [x_{-\alpha_{i_0}}, a], [x_{-\alpha_{i_0}-\alpha}, a]\rangle = 2\langle [x_{-\alpha_{i_0}}, x_{-\theta_{\Sigma'}}], [x_{-\alpha_{i_0}-\alpha}, x_{-\theta+\alpha+2\alpha_{i_0}+\theta_{\Sigma'}}]\rangle \neq 0.$$

We can therefore assume both α and α_{i_0} long roots and $\alpha \neq \theta_\Sigma$. Since $(\alpha_{i_0}|\alpha) < 0$, we have that $\alpha_{i_0} + \theta_\Sigma$ is a root. Since $(\theta|\alpha_{i_0} + \theta_\Sigma) = (\theta|\alpha_{i_0}) > 0$, we see that $\theta - \alpha_{i_0} - \theta_\Sigma$ is a positive root.

Since α_{i_0} is long, by Lemma 5.7 of [CMP], α has coefficient 1 in θ_Σ. We now prove that

$$(\theta_\Sigma, \alpha) = 0. \tag{3.7}$$

Indeed, if (3.7) holds, $(\theta - \alpha_{i_0} - \theta_\Sigma, \alpha) = 1$ so $\theta - \alpha_{i_0} - \alpha - \theta_\Sigma$ is a root. Since $(\theta - \alpha_{i_j} - \theta_\Sigma - \alpha, \alpha_{i_0}) = 1 - 2 + 1 + 1 = 1 > 0$, we obtain that $\theta - 2\alpha_{i_0} - \theta_\Sigma - \alpha$ is a positive root. Choosing $a = x_{-\theta_\Sigma} + x_{-\theta+\alpha+2\alpha_{i_0}+\theta_\Sigma}$ and arguing as above, we conclude that

$$\langle [x_{-\alpha_{i_0}}, a], [x_{-\alpha_{i_0}-\alpha}, a]\rangle = 2\langle [x_{-\alpha_{i_0}}, x_{-\theta_\Sigma}], [x_{-\alpha_{i_0}-\alpha}, x_{-\theta+\alpha+2\alpha_{i_0}+\theta_\Sigma}]\rangle \neq 0.$$

To prove (3.7) we need to use definitions and results from [CMPP]. If (3.7) does not hold, then $\alpha \notin A(\Sigma)$ (see [CMPP, Definition 4.1] for the definition of $A(\Sigma)$). Then $A(\Sigma) = \widehat{\Pi} \setminus \Sigma$ and by [CMPP, Proposition 4.2], $\theta - \theta_\Sigma$ is supported outside Σ so

$$\theta = \sum_{\eta \notin \Sigma, \eta \neq \alpha_{i_0}} n_\eta \eta + 2\alpha_{i_0} + \theta_\Sigma.$$

This contradicts the fact that $\theta - \alpha - \alpha_{i_0}$ is a root since $\alpha \neq \theta_\Sigma$ and the coefficient of α in θ_Σ is 1.

We are left with the case when $\mathfrak{g}^\natural$ ha a 1-dimensional center generated by ϖ (cf. (3.3)). We can assume that $(\alpha_{i_0}|\alpha_{i_1}) = 0$. If this is not the case, then $\mathfrak{g} = sl(3)$ and $V' = \{0\}$. Suppose σ is a simple roots not orthogonal to α_{i_0}. Then $(-2\alpha_{i_0} - \sigma)(\varpi) = -2$. In particular $V_{\mathfrak{g}^\natural}((-2\alpha_{i_0} - \sigma)_{|\mathfrak{h}^\natural})$ does not occur in $S^2(\mathfrak{g}^\natural)$. It follows that $P_{|V_{\mathfrak{g}^\natural}((-2\alpha_{i_0}-\sigma)_{|\mathfrak{h}^\natural})} = \Phi_{|V_{\mathfrak{g}^\natural}((-2\alpha_{i_0}-\sigma)_{|\mathfrak{h}^\natural})} = 0$. Note that the highest weight vector of It remains only to check that $\Phi_{|V_{\mathfrak{g}^\natural}((-\alpha_{i_0}-\alpha_{i_1})_{|\mathfrak{h}^\natural}))}$ is nonzero. $V_{\mathfrak{g}^\natural}((-\alpha_{i_0} - \alpha_{i_1}))$ is $x_{-\alpha_{i_0}} \wedge x_{-\alpha_{i_1}}$. We need to find a such that

$$\langle [x_{-\alpha_{i_0}}, a], [x_{-\alpha_{i_1}}, a]\rangle \neq 0.$$

Let Σ be a component of $\mathfrak{g}^\natural$ attached to α_{i_0}, so that $\alpha_{i_0} + \theta_\Sigma, \theta - \alpha_{i_0} - \theta_\Sigma$ are both roots. Since $(\theta_\Sigma|\alpha_{i_1}) \leq 0$, hence $(\theta - \alpha_{i_0} - \theta_\Sigma|\alpha_{i_1}) > 0$ and in turn $\theta - \alpha_{i_0} - \theta_\Sigma - \alpha_{i_1}$ is a positive root or zero. Choosing $a = x_{-\theta_\Sigma} + x_{-\theta+\alpha_{i_0}+\theta_\Sigma+\alpha_{i_1}}$ in the first case and $a = x_{-\theta_\Sigma} + h_{\theta_\Sigma}$ in the second we are done. □

4. Calculation of δ

We note that for computing δ it is enough to compute

$$(g_2(a \wedge b), c \wedge d), \qquad a, b, c, d \in \mathfrak{g}, \tag{4.1}$$

where $[a, b] = [c, d] = 0$ and $(a \wedge b, c \wedge d) \neq 0$.

Indeed, recall from (2.10) that there is a constant k' such that $g_2(a \wedge b) = k' a \wedge b$. By (2.11) and (2.12) we obtain

$$\delta = \tfrac{k'}{72} = \frac{(g_2(a \wedge b), c \wedge d)}{72(a \wedge b, c \wedge d)}. \tag{4.2}$$

We first use (4.2) to determine the dependence of δ from the choice of the form $(\cdot, \cdot)$. Let us write δ as $\delta^{(\cdot,\cdot)}$ to emphasize this dependence.

Lemma 4.1.

$$\delta^{s(\cdot,\cdot)} = \frac{1}{s^3}\delta^{(\cdot,\cdot)}.$$

Proof. Let us write g_2 as $g_2^{(\cdot,\cdot)}$ to emphasize the dependence on $(\cdot,\cdot)$. Then,

$$s^2(g_2^{s(\cdot,\cdot)}(a\wedge b), c\wedge d) = \sum_{t_1,t_2,t_3,\sigma} s([[a,x_{t_1}],x_{t_2}],[b,x_{t_3}])s([\frac{x^{t_{\sigma(1)}}}{s},[\frac{x^{t_{\sigma(2)}}}{s},[\frac{x^{t_{\sigma(3)}}}{s},c]]],d)$$

hence

$$(g_2^{s(\cdot,\cdot)}(a\wedge b), c\wedge d) = \frac{1}{s^3}(g_2^{(\cdot,\cdot)}(a\wedge b), c\wedge d).$$

It follows that

$$\delta^{s(\cdot,\cdot)} = \frac{s^2(g_2^{s(\cdot,\cdot)}(a\wedge b), c\wedge d)}{72s^2(a\wedge b, c\wedge d)} = \frac{1}{s^3}\frac{(g_2^{(\cdot,\cdot)}(a\wedge b), c\wedge d)}{72(a\wedge b, c\wedge d)} = \frac{1}{s^3}\delta^{(\cdot,\cdot)}.$$

□

Lemma 4.1 allows us to choose as invariant form the normalized one, that we denote by $(\cdot|\cdot)$. Recall that the normalized invariant form is defined by setting $(\theta|\theta)=2$. From now on by δ we mean $\delta^{(\cdot|\cdot)}$.

Choose a simple root α_{i_0} such that $(\theta|\alpha_{i_0})\neq 0$ (α_{i_0} is unique up to type A). Set $\gamma=\theta-\alpha_{i_0}$. Choose root vectors e_{i_0}, f_{i_0}, in $\mathfrak{g}_{\pm\alpha_{i_0}}$ such that $(e_{i_0}|f_{i_0}) = \frac{2}{(\alpha_{i_0},\alpha_{i_0})}$ and set $x_\gamma = [x_\theta, f_{i_0}]$, $x_{-\gamma} = [x_{-\theta}, e_{i_0}]$.

We compute δ by specializing (4.2) to the case where $(\cdot,\cdot)=(\cdot|\cdot)$ and $a=x_{-\theta}, b=x_{-\gamma}, c=x_\theta, d=x_\gamma$ so that

$$\delta = \frac{(g_2(x_{-\theta}\wedge x_{-\gamma})|x_\theta\wedge x_\gamma)}{72(x_{-\theta}\wedge x_{-\gamma}|x_\theta\wedge x_\gamma)} = \frac{(g_2(x_{-\theta}\wedge x_{-\gamma})|x_\theta\wedge x_\gamma)}{36(x_{-\gamma}|x_\gamma)}. \tag{4.3}$$

We choose a basis $\{u_i\}$ of $\mathfrak{g}_{-1/2}$ and let $\{u^i\}$ be its dual basis (i. e. $\langle u_i, u^j\rangle = \delta_{ij}$). We also choose an orthonormal basis $\{x_i\}$ of $\mathfrak{g}^\natural$. Then, as basis of $\mathfrak{g}$, we can choose

$$\{x_\theta\}\cup\{[x_\theta,u_i]\}\cup\{x_i\}\cup\{x\}\cup\{u_i\}\cup\{x_{-\theta}\}. \tag{4.4}$$

The corresponding dual basis (w.r.t. $(\cdot|\cdot)$) is

$$\{2x_{-\theta}\}\cup\{u^i\}\cup\{x_i\}\cup\{2x\}\cup\{-[x_\theta,u^i]\}\cup\{2x_\theta\}. \tag{4.5}$$

We choose an orthonormal basis $\{x_i^r\}$ for each component $\mathfrak{g}_r^\natural$ so that we can set $\{x_i\} = \cup_r\{x_i^r\}$.

Let $C_{\mathfrak{g}^\natural} = \sum_i (x_i)^2$ be the Casimir element of $\mathfrak{g}^\natural$ and $C_{\mathfrak{g}_0}$ the Casimir element of $\mathfrak{g}_0$. Since $C_{\mathfrak{g}^\natural} = C_{\mathfrak{g}_0} - 2x^2$ by Lemma 5.1 of [KW] we have that

$$\sum_i [x_i,[x_i,v]] = (h^\vee - \tfrac{3}{2})v. \tag{4.6}$$

Recall also that it follows from Lemma 5.1 of [KW] that

$$\dim\mathfrak{g}_{-1/2} = \dim\mathfrak{g}_{1/2} = 2h^\vee - 4. \tag{4.7}$$

We extend $\langle\cdot,\cdot\rangle$ on $\wedge^2\mathfrak{g}_{-1/2}$ by determinants:

$$\langle u\wedge v, w\wedge z\rangle = \langle u,w\rangle\langle v,z\rangle - \langle u,z\rangle\langle v,w\rangle.$$

We collect various formulas in the following lemma.

Lemma 4.2. *If* $u, v, w, z \in \mathfrak{g}_{-1/2}$ *and* $a \in \mathfrak{g}_r^\natural$, *then*

$$\sum_p [[x_p^s, v], [a, x_p^s]] = \delta_{rs}(\bar{h}_r^\vee)[v, a], \tag{4.8}$$

$$\sum_p [x_p^s, [v, [a, x_p^s]]] = -\delta_{rs}(\bar{h}_r^\vee)[v, a], \tag{4.9}$$

$$\sum_p [x_p, [a, [v, x_p]]] = (h^\vee - 3/2 - \bar{h}_r^\vee)[v, a], \tag{4.10}$$

$$[[x_\theta, u], [x_\theta, v]] = -\langle u, v\rangle x_\theta, \tag{4.11}$$

$$[u, v] = 2\langle u, v\rangle x_{-\theta}, \tag{4.12}$$

$$[[x_\theta, u], v] = \sum_i \langle u, [v, x_i]\rangle x_i + \langle u, v\rangle x, \tag{4.13}$$

$$[v, [x_\theta, u]] = \sum_i \langle [v, x_i], u\rangle x_i + \langle v, u\rangle x, \tag{4.14}$$

$$\sum_i [u, x_i] \wedge [v, x_i] = -\frac{\langle u, v\rangle}{2} \sum_r u_r \wedge u^r - \tfrac{1}{2} u \wedge v, \tag{4.15}$$

$$\begin{aligned} &\sum_{i,j} \langle [u, x_i], [v, x_j]\rangle \langle [w, x_i], [z, x_j]\rangle - \sum_{i,j} \langle [w, x_i], [v, x_j]\rangle \langle [u, x_i], [z, x_j]\rangle \\ &= (h^\vee - 1)\langle u, w\rangle \langle v, z\rangle + \tfrac{1}{4}\langle u \wedge w, v \wedge z\rangle, \end{aligned} \tag{4.16}$$

Proof. If $r \neq s$, then (4.8) and (4.9) are obvious. If $r = s$ then, on one hand,

$$\begin{aligned} \sum_p [x_p^r, [v, [a, x_p^r]]] &= \sum_{p,j} [x_p^r, [v, ([a, x_p^r]|x_j^r)x_j^r]] = -\sum_{p,j} [x_p^r, [v, (x_p^r|[a, x_j^r])x_j^r]] = \\ &= -\sum_j [[a, x_j^r], [v, x_j^r]] = -\sum_p [[x_p^r, v], [a, x_p^r]]. \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_p [x_p^r, [v, [a, x_p^r]]] &= \sum_p [[x_p^r, v], [a, x_p^r]] + \sum_p [v, [x_p^r, [a, x_p^r]]] \\ &= \sum_p [[x_p^r, v], [a, x_p^r]] - 2(\bar{h}_r^\vee)[v, a], \end{aligned}$$

so

$$\sum_p [[x_p^r, v], [a, x_p^r]] - 2(\bar{h}_r^\vee)[v, a] = -\sum_p [[x_p^r, v], [a, x_p^r]],$$

hence

$$\sum_p [[x_p^r, v], [a, x_p^r]] = (\bar{h}_r^\vee)[v, a], \quad \sum_p [x_p^r, [v, [a, x_p^r]]] = -(\bar{h}_r^\vee)[v, a],$$

and

$$\sum_p [x_p, [a, [v, x_p]]] = \sum_p [x_p, [[a, v], x_p]]] + \sum_p [x_p^r, [v, [a, x_p^r]]] = (h^\vee - 3/2 - \bar{h}_r^\vee)[v, a].$$

This proves (4.8), (4.9), and (4.10). Formulas (4.11), (4.12), (4.13), (4.14) are straightforward.

We now prove (4.15). Note that $\mathfrak{g}_{-1/2}\wedge\mathfrak{g}_{-1/2}=\mathbb{C}\sum_i u_i\wedge u^i\oplus V_2$ with $V_2=span(u\wedge v\mid [u,v]=0)$. If $u,v\in\mathfrak{g}_{-1/2}$, then the corresponding decomposition of $u\wedge v$ is

$$u\wedge v=\frac{\langle u,v\rangle}{\dim\mathfrak{g}_{1/2}}\sum_i u_i\wedge u^i+s,\quad s\in V_2.$$

If $w,z\in\mathfrak{g}_{-1/2}$ then

$$\sum_n[w,x_n]\wedge[z,x_n]=\tfrac{1}{2}(C_{\mathfrak{g}^\natural}(w\wedge z)-C_{\mathfrak{g}^\natural}w\wedge z-w\wedge C_{\mathfrak{g}^\natural}z).$$

Assume now $[w,z]=0$ and set $C_{sl(2)}=2(x^2+x_\theta x_{-\theta}+x_{-\theta}x_\theta)$ to be the Casimir element of $span(x_\theta,x,x_{-\theta})\simeq sl(2)$. By [P], $(C_{sl(2)}+C_{\mathfrak{g}^\natural})(w\wedge z)=2h^\vee(w\wedge z)$. Since

$$C_{sl(2)}(w\wedge z)=C_{sl(2)}w\wedge z+w\wedge C_{sl(2)}z+4xw\wedge xz,$$

$C_{sl(2)}w=\frac{3}{2}w$, and $4xw\wedge xz=w\wedge z$, we obtain $C_{sl(2)}(w\wedge z)=4w\wedge z$ so that $C_{\mathfrak{g}^\natural}(w\wedge z)=(2h^\vee-4)w\wedge z$. Using (4.6), the final outcome is that

$$\sum_n[w,x_n]\wedge[z,x_n]=\tfrac{1}{2}(2h^\vee-4-2(h^\vee-3/2))(w\wedge z)=-\tfrac{1}{2}(w\wedge z),$$

By (4.7), (4.6), and the above formula applied to s, we find

$$\sum_i[u,x_i]\wedge[v,x_i]=-\frac{\langle u,v\rangle}{2}\frac{2h^\vee-3}{2h^\vee-4}\sum_r u_r\wedge u^r-\tfrac{1}{2}(u\wedge v-\frac{\langle u,v\rangle}{2h^\vee-4}\sum_i u_i\wedge u^i)$$

hence (4.15).

Finally, we prove (4.16):

$$\sum_i\langle[u,x_i]\wedge[w,x_i],v\wedge z\rangle=-\frac{\langle u,w\rangle}{2}\sum_r(\langle u_r,v\rangle\langle u^r,z\rangle-\langle u^r,v\rangle\langle u_r,z\rangle)-\tfrac{1}{2}\langle u\wedge w,v\wedge z\rangle$$
$$=-\langle u,w\rangle\langle v,z\rangle-\tfrac{1}{2}\langle u\wedge w,v\wedge z\rangle.$$

It follows that

$$\begin{aligned}&\sum_{i,j}\langle[u,x_i],[v,x_j]\rangle\langle[w,x_i],[z,x_j]\rangle-\sum_{i,j}\langle[w,x_i],[v,x_j]\rangle\langle[u,x_i],[z,x_j]\rangle\\&=-\sum_j(\langle u,w\rangle\langle[v,x_j],[z,x_j]\rangle-\tfrac{1}{2}\sum_j\langle u\wedge w,[v,x_j]\wedge[z,x_j]\rangle\\&=(h^\vee-3/2)\langle u,w\rangle\langle v,z\rangle-\tfrac{1}{2}\sum_j\langle u\wedge w,[v,x_j]\wedge[z,x_j]\rangle\\&=(h^\vee-1)\langle u,w\rangle\langle v,z\rangle+\tfrac{1}{4}\langle u\wedge w,v\wedge z\rangle,\end{aligned}$$

which is precisely (4.16). □

We choose the basis $\{a_i\}$ for $\mathfrak{g}$ to be the basis displayed in (4.4). Then the dual basis $\{a^i\}$ is the basis given in (4.5). We start our computation of δ by computing

$$G_2(x_{-\theta}\wedge x_{-\gamma})=\sum_{i,j,k}([[x_{-\theta},a_i],a_j]|[x_{-\gamma},a_k])a^ia^ja^k. \tag{4.17}$$

We have to compute expressions of type $([[x_{-\theta},b_1],b_2]|[x_{-\gamma},b_3])$ with $b_i\in\mathfrak{g}_{d_i}$. Such an expression can be non-zero only if $d_3-1/2=1-d_1-d_2$ i. e. $d_1+d_2+d_3=3/2$. The possibilities are

d_1	d_2	d_3	d_1	d_2	d_3	d_1	d_2	d_3
-1/2	1	1	0	1/2	1	0	1	1/2
1/2	0	1	1/2	1/2	1/2	1/2	1	0
1	-1/2	1	1	0	1/2	1	1/2	0
1	1	-1/2						

For each of the cases above let $S(d_1, d_2, d_3)$ be the corresponding summand in the expression of $G_2(x_{-\theta} \wedge x_{-\gamma})$ given in (4.17). Then direct computations yield

$$
\begin{aligned}
S(-1/2,1,1) &= 4\sum_i ([[x_{-\theta},u_i],x_\theta]|[x_{-\gamma},x_\theta])(-[x_\theta,u^i])x_{-\theta}^2 = 0,\\
S(0,1/2,1) &= 2\sum_j \langle u_j, x_{-\gamma}\rangle u_j x x_{-\theta} = 2xx_{-\gamma}x_{-\theta},\\
S(0,1,1/2) &= 2\sum_j \langle u_j, x_{-\gamma}\rangle u^j x x_{\theta} = 2xx_{-\gamma}x_{-\theta},\\
S(1/2,0,1) &= \sum_i [x_i,x_{-\gamma}]x_i x_{-\theta} + x_{-\gamma}xx_{-\theta},\\
S(1/2,1/2,1/2) &= -\tfrac{1}{2}\sum_{j,s}[u_j,x_s]u^j[x_s,x_{-\gamma}] - \tfrac{1}{4}\sum_j u_j u^j x_{-\gamma},\\
S(1/2,1,0) &= \sum_i [x_i,x_{-\gamma}]x_i x_{-\theta} - xx_{-\gamma}x_{-\theta},\\
S(1,-1/2,1) &= -2[x_\theta,x_{-\gamma}]x_{-\theta}^2 = -e_{i_0}x_{-\theta}^2,\\
S(1,0,1/2) &= 0,\\
S(1,1/2,0) &= \sum_i [x_i,x_{-\gamma}]x_i x_{-\theta} - xx_{-\gamma}x_{-\theta},\\
S(1,1,-1/2) &= 4\sum_j \langle x_{-\gamma},u_j\rangle [x_\theta,u^j]x_{-\theta}^2 = -4[x_\theta,x_{-\gamma}]x_{-\theta}^2 = -2e_{i_0}x_{-\theta}^2.
\end{aligned}
$$

Summing up we find

$$
\begin{aligned}
G_2(x_{-\theta}\wedge x_{-\gamma}) = 3xx_{-\gamma}x_{-\theta} &+ 3\sum_i [x_i,x_{-\gamma}]x_i x_{-\theta} - 3e_{i_0}x_{-\theta}^2\\
&- \tfrac{1}{2}\sum_{j,s}[u_j,x_s]u^j[x_s,x_{-\gamma}] - \tfrac{1}{4}\sum_j u_j u^j x_{-\gamma}.
\end{aligned}
$$

Note that $\sum_j u_j u^j([x_\theta,u]) = \langle u,u\rangle = 0$, so

$$G_2(x_{-\theta}\wedge x_{-\gamma}) = 3xx_{-\gamma}x_{-\theta} + 3\sum_i [x_i,x_{-\gamma}]x_i x_{-\theta} - 3e_{i_0}x_{-\theta}^2 - \tfrac{1}{2}\sum_{j,r}[u_j,x_r]u^j[x_r,x_{-\gamma}].$$

We now compute $g_2(x_{-\theta}\wedge x_{-\gamma})|x_\theta\wedge x_\gamma) = ((Symm(G_2(x_{-\theta}\wedge x_{-\gamma}))(x_\theta)|x_\gamma)$. We start by computing $\sum_{j,r}((Symm([u_j,x_r]u^j[x_r,x_{-\gamma}])(x_\theta)|x_\gamma)$. For this we note that, if $u,v,w\in\mathfrak{g}_{-1/2}$,

$$([u,[v,[w,x_\theta]]]|x_\gamma) = ([v,[x_\theta,w]]|[u,x_\gamma]) = \sum_i \langle v,[w,x_i]\rangle\langle u,[f_{i_0},x_i]\rangle + \tfrac{1}{2}\langle v,w\rangle\langle u,f_{i_0}\rangle,$$

so, letting γ_s denote the eigenvalue of $C_{\mathfrak{g}_s^\natural}$ on $\mathfrak{g}_{-1/2}$,

$$\begin{aligned}
&\sum_{r,j}([[u_j,x_r],[u^j,[[x_r,x_{-\gamma}],x_\theta]]]|x_\gamma)\\
&=\sum_{i,j,r}\langle u^j,[[x_r,x_{-\gamma}],x_i]\rangle\langle[u_j,x_r],[f_{i_0},x_i]\rangle+\tfrac12\sum_{j,r}\langle u^j,[x_r,x_{-\gamma}]\rangle\langle[u_j,x_r],f_{i_0}\rangle\\
&=\sum_{i,r}\langle[x_r,[f_{i_0},x_i]],[[x_r,x_{-\gamma}],x_i]\rangle+\tfrac12\sum_{r}\langle[x_r,f_{i_0}],[x_r,x_{-\gamma}]\rangle\\
&=\sum_{i,r,s}\langle[x_i,[x_r^s,[f_{i_0},x_i]]],[x_r^s,x_{-\gamma}]\rangle-\tfrac12\sum_{r}\langle[x_r,[x_r,f_{i_0}],x_{-\gamma}]\rangle\\
&=\sum_{s}(h^\vee-3/2-\bar h_s^\vee)\sum_r\langle[f_{i_0},x_r^s],[x_r^s,x_{-\gamma}]\rangle-\tfrac12(h^\vee-3/2)\langle f_{i_0},x_{-\gamma}\rangle\\
&=\sum_{s}(h^\vee-3/2-\bar h_s^\vee)\gamma_s\langle f_{i_0},x_{-\gamma}\rangle-\tfrac12(h^\vee-3/2)\langle f_{i_0},x_{-\gamma}\rangle\\
&=\sum_{s}(h^\vee-2-\bar h_s^\vee)\gamma_s\langle f_{i_0},x_{-\gamma}\rangle=\sum_{s}(h^\vee-2-\bar h_s^\vee)\gamma_s(x_\gamma|x_{-\gamma}).
\end{aligned}$$

Similarly

$$\begin{aligned}
\sum_{r,j}([[u_j,x_r],[[x_r,x_{-\gamma}],[u^j,x_\theta]]]|x_\gamma)&=\sum_{s}(h^\vee-1-\bar h_s^\vee)\gamma_s(x_\gamma|x_{-\gamma}),\\
\sum_{r,j}([u^j,[[u_j,x_r],[[x_r,x_{-\gamma}],x_\theta]]]|x_\gamma)&=\sum_{s}(h^\vee-2-\bar h_s^\vee)\gamma_s(x_\gamma|x_{-\gamma}),\\
\sum_{r,j}([u^j,[[x_r,x_{-\gamma}],[[u_j,x_r],x_\theta]]]|x_\gamma)&=\sum_{s}(h^\vee-1-\bar h_1^\vee)\gamma_s(x_\gamma|x_{-\gamma}).
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{r,j}([[x_r,x_{-\gamma}],[u^j,[[u_j,x_r],x_\theta]]]|x_\gamma)&+\sum_{r,j}([[x_r,x_{-\gamma}],[[u_j,x_r],[u^j,x_\theta]]]|x_\gamma)\\
&=\sum_{i,j,r}\langle u^j,[[u_j,x_r],x_i]\rangle\langle[x_r,x_{-\gamma}],[f_{i_0},x_i]\rangle+\tfrac12\sum_{j,r}\langle u^j,[u_j,x_r]\rangle\langle[x_r,x_{-\gamma}],f_{i_0}\rangle\\
&+\sum_{i,j,r}\langle[u_j,x_r],[u^j,x_i]\rangle\langle[x_r,x_{-\gamma}],[f_{i_0},x_i]\rangle+\tfrac12\sum_{j,r}\langle[u_j,x_r],u^j\rangle\langle[x_r,x_{-\gamma}],f_{i_0}\rangle\\
&=2\sum_{i,j,r}\langle[u_j,x_r],[u^j,x_i]\rangle\langle[x_r,x_{-\gamma}],[f_{i_0},x_i]\rangle.
\end{aligned}$$

Recall from (4.16) that

$$\begin{aligned}
&\sum_{i,j}\langle[u,x_i],[v,x_j]\rangle\langle[w,x_i],[z,x_j]\rangle-\sum_{i,j}\langle[w,x_i],[v,x_j]\rangle\langle[u,x_i],[z,x_j]\rangle\\
&=(h^\vee-1)\langle u,w\rangle\langle v,z\rangle+\tfrac14\langle u\wedge w,v\wedge z\rangle.
\end{aligned}$$

so

$$\sum_{r,j}([[x_r,x_{-\gamma}],[u^j,[[u_j,x_r],x_\theta]]]|x_\gamma)+\sum_{r,j}([[x_r,x_{-\gamma}],[[u_j,x_r],[u^j,x_\theta]]]|x_\gamma)$$
$$=2\sum_{i,j,r}\langle[u_j,x_r],[u^j,x_i]\rangle\langle[x_r,x_{-\gamma}],[f_{i_0},x_i]\rangle$$
$$=-2(h^\vee-1)\sum_j\langle u_j,x_{-\gamma}\rangle\langle u^j,f_{i_0}\rangle-\tfrac12\sum_j\langle u_j\wedge x_{-\gamma},u^j\wedge f_{i_0}\rangle$$
$$+2\sum_{i,j,r}\langle[x_{-\gamma},x_r],[u^j,x_i]\rangle\langle[x_r,u_j],[f_{i_0},x_i]\rangle$$
$$=-2(h^\vee-1)\langle x_{-\gamma},f_{i_0}\rangle-\tfrac12\sum_j\langle u_j,u^j\rangle\langle x_{-\gamma},f_{i_0}]\rangle+\tfrac12\sum_j\langle u_j,f_{i_0}\rangle\langle x_{-\gamma},u^j\rangle$$
$$+2\sum_{i,r,s}\langle[x_{-\gamma},x_r^s],[x_i,[x_r^s,[f_{i_0},x_i]]]\rangle$$
$$=(2(h^\vee-1)+\tfrac12(\dim\mathfrak{g}_{1/2}-1))(x_\gamma|x_{-\gamma})+2\sum_s(h^\vee-3/2-\bar h_s^\vee)\sum_r\langle[x_{-\gamma},x_r^s],[f_{i_0},x_r^s]\rangle$$
$$=2(h^\vee-1)+\tfrac12(2h^\vee-4-1))(x_\gamma|x_{-\gamma})+2\sum_s(h^\vee-3/2-\bar h_s^\vee)\gamma_s(x_\gamma|x_{-\gamma})$$
$$=2\sum_s(h^\vee-\bar h_s^\vee)\gamma_s(x_\gamma|x_{-\gamma}).$$

The final outcome is that

$$\sum_{j,r}((Symm([u_j,x_r]u^j[x_r,x_{-\gamma}])(x_\theta)|x_\gamma)=6\sum_s(h^\vee-1-\bar h_s^\vee)\gamma_s(x_\gamma|x_{-\gamma}).$$

The other terms are easier:

$$(Symm(xx_{-\gamma}x_{-\theta})(x_\theta)|x_\gamma)=-3/4(x_\gamma|x_{-\gamma}),$$
$$(Symm(e_{i_0}x_{-\theta}^2)(x_\theta)|x_\gamma)=3(x_\gamma|x_{-\gamma}),$$
$$\sum_j(Symm([x_j,x_{-\gamma}]x_jx_{-\theta})(x_\theta)|x_\gamma)=-3/2(h^\vee-3/2)(x_\gamma|x_{-\gamma}).$$

Summing up

$$(g_2(x_\theta\wedge x_\gamma)|x_{-\theta}\wedge x_{-\gamma})=(-3\sum_s(h^\vee+1/2-\bar h_s^\vee)\gamma_s-45/4)(x_\gamma|x_{-\gamma}).$$

Using (4.3) we obtain the following result

Proposition 4.3.

$$\delta=-\frac{12\sum_s(h^\vee+1/2-\bar h_s^\vee)\gamma_s+45}{144}. \tag{4.18}$$

Remark 4.1. Relation (4.18) easily gives Drinfeld's formulas for δ: if κ denotes the Killing form of $\mathfrak{g}$, then

$$\delta^\kappa = \begin{cases} -\frac{1}{32n^2} & \text{if } \mathfrak{g} = sl(n), \\ -\frac{n-4}{16(n-2)^3} & \text{if } \mathfrak{g} = so(n), \\ -\frac{n+2}{64(n+1)^3} & \text{if } \mathfrak{g} = sp(2n), \\ -\frac{5}{144(\dim\mathfrak{g}+2)} & \text{if } \mathfrak{g} \text{ is of exceptional type.} \end{cases} \tag{4.19}$$

Remark 4.2. It is possible to give an alternative formula for δ without using the minimal gradation. We already observed that we are reduced to evaluate (4.1). We'll do it by choosing $a = c = u, b = d = v$ with $u, v \in \mathfrak{h}$. Let $\{x_\alpha\}$ be a set of root vectors with $[x_\alpha, x_\beta] = N_{\alpha,\beta}x_{\alpha+\beta}$ for $\alpha, \beta \in \Delta, \alpha \neq -\beta$ and $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. Then

$$(g_2(u \wedge v), u \wedge v) = 6 \sum_{\alpha\in\Delta^+,\beta\in\Delta} \alpha(u)(\alpha+\beta)(v)(\alpha(v)\beta(u) - \alpha(u)\beta(v))N^2_{\alpha,\beta}. \tag{4.20}$$

The computation of (4.19) starting from (4.20) is possible but quite less handy than using (4.18).

5. Minimal W-algebras

It is known by [KW] that for $k \neq -h^\vee$ there is a vertex algebra W strongly and freely generated by fields L, J^v with $v \in \mathfrak{g}^\natural$, G^u with $u \in \mathfrak{g}_{-1/2}$ with the following λ-brackets: L is a Virasoro element with central charge $\frac{k\dim\mathfrak{g}}{k+h^\vee} - 6k + h^\vee - 4$, J^u are primary of conformal weight $\Delta = 1$, G^v are primary of conformal weight $\Delta = \frac{3}{2}$ and

1. $[J^v_\lambda J^w] = J^{[v,w]} + \lambda\delta_{ij}(k + \frac{h^\vee - \bar{h}^\vee_i}{2})(v|w)$ for $v \in \mathfrak{g}^\natural_i$, $w \in \mathfrak{g}^\natural_j$;
2. $[J^v_\lambda G^u] = G^{[v,u]}$ for $u \in \mathfrak{g}_{-1/2}$, $v \in \mathfrak{g}^\natural$;
3. $[G^u_\lambda G^v] = A(u,v,k) + \lambda B(u,v,k) + \frac{\lambda^2}{2}C(u,v,k)$ for $u, v \in \mathfrak{g}_{-1/2}$ with $C(u,v,k) \in \mathbb{C}$, and conformal weights of $\Delta(B(u,v,k)) = 1$ and $\Delta(A(u,v,k)) = 2$.

To simplify notation, we will not record the dependence on k in the functions A, B, C. We choose the basis $\{x_i\}$ to be the union of orthonormal bases of $\mathfrak{g}^\natural_r$. Let $T = L_{-1}$ be the translation operator of W.

If $p = a \otimes b \in \mathfrak{g}^\natural \otimes \mathfrak{g}^\natural$ write $: p :=: J^a J^b :$. We extend $: \cdot :$ linearly to obtain a map $: \cdot :$ from $\mathfrak{g}^\natural \otimes \mathfrak{g}^\natural$ to W. Consider $S^2(\mathfrak{g}^\natural) = \{a \otimes b + b \otimes a \mid a, b \in \mathfrak{g}^\natural\} \subset \mathfrak{g}^\natural \otimes \mathfrak{g}^\natural$. Since $: J^u J^v :=: J^v J^u : + TJ^{[u,v]}$ and since the elements J^{x_i}, G^{u_i}, L strongly and freely generate W, we see that there exist maps

$$P : \mathfrak{g}_{-1/2} \times \mathfrak{g}_{-1/2} \to S^2(\mathfrak{g}^\natural), \quad K, H : \mathfrak{g}_{-1/2} \times \mathfrak{g}_{-1/2} \to \mathfrak{g}^\natural, \quad Q : \mathfrak{g}_{-1/2} \times \mathfrak{g}_{-1/2} \to \mathbb{C}$$

such that $A(u,v)$ can be uniquely written as

$$A(u,v) =: P(u,v) : + TJ^{K(u,v)} + Q(u,v)L.$$

and

$$B(u,v) = J^{H(u,v)}.$$

By skewsymmetry $[G^u_\lambda G^v] = -[G^v_{-\lambda-T} G^u]$ so

1. $C(u,v) = -C(v,u)$;
2. $H(u,v) = H(v,u)$;
3. $P(u,v) = -P(v,u)$, $K(u,v) = -K(v,u) + H(v,u)$, $Q(u,v) = -Q(v,u)$,

hence $C(\cdot,\cdot)$ and $Q(\cdot,\cdot)$ are symplectic forms on $\mathfrak{g}_{-1/2}$, and

$$P : \bigwedge^2 \mathfrak{g}_{-1/2} \to S^2(\mathfrak{g}^\natural), \quad H : S^2(\mathfrak{g}_{-1/2}) \to \mathfrak{g}^\natural.$$

By applying the axioms of vertex algebra, (§ 1.5 of [DK]) we find for $a \in \mathfrak{g}^\natural$, $v, w \in \mathfrak{g}_{-1/2}$:

$$C([a,v],w) = -C(v,[a,w]).$$

Since $\mathfrak{g}_{1/2}$, as a $\mathfrak{g}^\natural$–module, is either irreducible or a sum $U \oplus U^*$ with U irreducible and U inequivalent to U^*, we see that, up to a constant, there is a unique symplectic $\mathfrak{g}^\natural$-invariant nondegenerate bilinear form on $\mathfrak{g}_{-1/2}$. Since

$$\langle u, v\rangle := (x_\theta | [u,v])$$

is such a form, we have that

$$C(\cdot,\cdot) = \Gamma(k)\langle\cdot,\cdot\rangle \tag{5.1}$$

for some constant $\Gamma(k)$.

For $b \in \mathfrak{g}^\natural$, let $b_i^\natural$ denote the orthogonal projection of b onto $\mathfrak{g}_i^\natural$. Write

$$P(v,w) = \sum_{i,j,r,s} k_{i,j}^{r,s}(v,w)(x_i^r \otimes x_j^s)$$

with $k_{i,j}^{r,s} = k_{j,i}^{s,r}$.

5.1. Jacobi identities between two G and one J.

By Jacobi identity $[J_\lambda^x[G_\mu^v G^w]] - [G_\mu^v[J_\lambda^a G^w]] = [[J_\lambda^a G^v]_{\lambda+\mu} G^w]$. Explicitly

$$\begin{aligned}
&[J_\lambda^a : P(v,w) :] + [J_\lambda^a T J^{K(v,w)}] + \lambda Q(v,w) J^a + \mu [J_\lambda^a J^{H(v,w)}] \\
&- : P(v,[a,w]) : - T J^{K(v,[a,w])} - Q(v,[a,w]) L - \mu J^{H(v,[a,w])} - \tfrac{\mu^2}{2} c(v,[a,w]) \\
&=: P([a,v],w) : + T J^{K([a,v],w)} + Q([a,v],w) L + (\lambda+\mu) J^{H([a,v],w)} + \tfrac{(\lambda+\mu)^2}{2} c([a,v],w).
\end{aligned}$$

Using Wick formula ((1.37) [DK]) and sesquilinearity we compute explicitly $[J_\lambda^a : P(v,w) :]$ and $[J_\lambda^a T J^{K(v,w)}]$. Then, equating the coefficients in λ, μ, we find

Proposition 5.1.

$$H(v,w) = \sum_r \frac{2\Gamma(k)}{2k + h^\vee - \bar h_r^\vee} [[x_\theta, v], w]_r^\natural. \tag{5.2}$$

$$K(v,w) = \tfrac{1}{2} H(v,w). \tag{5.3}$$

$$Q([a,v],w) = -Q(v,[a,w]). \tag{5.4}$$

$$P([a,v],w) = -P(v,[a,w]) + ad(a)P(v,w). \tag{5.5}$$

$$\begin{aligned}
H([a,v],w) &= Q(v,w)a + [a, K(v,w)] \\
&+ \sum_{i,j,r,s} k_{i,j}^{r,s}(v,w)(2k + h^\vee - \bar h_r^\vee)(a|x_i^r) x_j^s + \sum_{i,j,r,s} k_{i,j}^{r,s}(v,w)[[a,x_i^r], x_j^s].
\end{aligned} \tag{5.6}$$

$$\sum_{j,r} k_{j,j}^{r,r}(v,w)(2k + h^\vee - \bar h_r^\vee) + Q(v,w)\left(\frac{k \dim \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4\right) = 3\Gamma(k)\langle v, w\rangle. \tag{5.7}$$

In particular, Q is an invariant symplectic form on $\mathfrak{g}_{-1/2}$, hence

$$Q(u,v) = D(k)\langle u, v\rangle. \tag{5.8}$$

5.2. **Jacobi identities between three G.**
We need auxiliary formulas. By Wick formula [DK, (1.37)]

$$[G^u_\lambda(:J^yJ^z:)] =: G^{[u,y]}J^z : + : J^yG^{[u,z]} : +\lambda G^{[[u,y],z]}$$
$$=: G^{[u,y]}J^z : + : G^{[u,z]}J^y : +TG^{[y,[u,z]]} + \lambda G^{[[u,y],z]},$$

moreover, by sesquilinearity,

$$[G^u_\lambda TJ^a] = T[G^u_\lambda J^a] + \lambda[G^u_\lambda J^a] = TG^{[u,a]} + \lambda G^{[u,a]}.$$

By Jacobi identity $[G^u_\lambda[G^v_\mu G^w]] - [G^v_\mu[G^u_\lambda G^w]] = [[G^u_\lambda G^v]_{\lambda+\mu}G^w]$. We compute each term.

$$[[G^u_\lambda G^v]_{\lambda+\mu}G^w] = [(:P(u,v): + \tfrac{1}{2}TJ^{H(u,v)} + Q(u,v)L + \lambda J^{H(u,v)} + \tfrac{\lambda^2}{2}c(u,v))_{\lambda+\mu}G^w] =$$
$$-[G^w_{-\lambda-\mu-T} : P(u,v) :] - \tfrac{1}{2}(\lambda+\mu)G^{[H(u,v),w]} + Q(u,v)(TG^w + \tfrac{3}{2}(\lambda+\mu)G^w) + \lambda G^{[H(u,v),w]}$$
$$= -[G^w_{-\lambda-\mu-T} : P(u,v) :] + \tfrac{1}{2}(\lambda-\mu)G^{[H(u,v),w]} + Q(u,v)(TG^w + \tfrac{3}{2}(\lambda+\mu)G^w)$$

$$[G^u_\lambda[G^v_\mu G^w]] = [G^u_\lambda(:P(v,w): + \tfrac{1}{2}TJ^{H(v,w)} + Q(v,w)L + \mu J^{H(v,w)} + \tfrac{\mu^2}{2}c(v,w))]$$
$$= [G^u_\lambda : P(v,w) :] + \tfrac{1}{2}TG^{[u,H(v,w)]} + \tfrac{1}{2}\lambda G^{[u,H(v,w)]}$$
$$+ \tfrac{1}{2}Q(v,w)TG^u + \tfrac{3}{2}Q(v,w)\lambda G^u + \mu G^{[u,H(v,w)]}$$

so

$$[G^v_\mu[G^u_\lambda G^w]] = [G^v_\mu : P(u,w) :]$$
$$+ \tfrac{1}{2}TG^{[v,H(u,w)]} + \tfrac{1}{2}\mu G^{[v,H(u,w)]} + \tfrac{1}{2}Q(u,w)TG^v + \tfrac{3}{2}Q(u,w)\mu G^v + \lambda G^{[v,H(u,w)]}.$$

Equating the coefficients of λ, μ and the constant term we find

Proposition 5.2.

$$\tfrac{1}{2}G^{[H(u,v),w]} + \tfrac{3}{2}Q(u,v)G^w + \sum_{i,j,r,s} k^{r,s}_{i,j}(u,v)G^{[[w,x^r_i],x^s_j]}$$
$$= \tfrac{1}{2}G^{[u,H(v,w)]} + \tfrac{3}{2}Q(v,w)G^u + \sum_{i,j,r,s} k^{r,s}_{i,j}(v,w)G^{[[u,x^r_i],x^s_j]} - G^{[v,H(u,w)]} \tag{5.9}$$

$$-\tfrac{1}{2}G^{[H(u,v),w]} + \tfrac{3}{2}Q(u,v)G^w + \sum_{i,j,r,s} k^{r,s}_{i,j}(u,v)G^{[[w,x^r_i],x^s_j]}$$
$$= G^{[u,H(v,w)]} - \tfrac{1}{2}G^{[v,H(u,w)]} - \tfrac{3}{2}Q(u,w)G^v - \sum_{i,j,r,s} k^{r,s}_{i,j}(u,w)G^{[[v,x^r_i],x^s_j]} \tag{5.10}$$

$$Q(u,v)TG^w + 2\sum_{i,j} k_{i,j}(u,v)TG^{[[w,x_i],x_j]} = \tag{5.11}$$
$$\tfrac{1}{2}TG^{[u,H(v,w)]} + \tfrac{1}{2}Q(v,w)TG^u - \sum_{i,j,r,s} k^{r,s}_{i,j}(v,w)TG^{[[u,x^r_i],x^s_j]}$$
$$-\tfrac{1}{2}TG^{[v,H(u,w)]} - \tfrac{1}{2}Q(u,w)TG^v + \sum_{i,j,r,s} k^{r,s}_{i,j}(u,w)TG^{[[v,x^r_i],x^s_j]}$$

$$-\sum_{i,j,r,s} k_{i,j}^{r,s}(u,v) :G^{[w,x_i]}J^{x_j}: \tag{5.12}$$
$$=\sum_{i,j,r,s} k_{i,j}^{r,s}(v,w) :G^{[u,x_i]}J^{x_j}: -\sum_{i,j,r,s} k_{i,j}^{r,s}(u,w) :G^{[v,x_i]}J^{x_j}:.$$

Recall from Section 3 that $\mathfrak{g}_{-1/2}\wedge\mathfrak{g}_{-1/2}=\mathbb{C}\oplus V'$, where V' is the $\mathfrak{g}^\natural$–module generated by bivectors $u\wedge v$ with $[u,v]=0$, and that V' decomposes with multiplicity one and no component is trivial. By Proposition 3.1, $P_{|V_{\mathfrak{g}^\natural}(\lambda_h)}=f_h(k)\Phi_{|V_{\mathfrak{g}^\natural}(\lambda_h)}$ for some costant $f_h(k)\in\mathbb{C}$. Thus, if $u\wedge v\in V_{\mathfrak{g}^\natural}(\lambda_h)$,

$$k_{i,j}^{r,s}(u,v)=f_h(k)(\langle[[u,x_i^r],[v,x_j^s]]\rangle+\langle[[u,x_j^s],[v,x_i^r]]\rangle).$$

If $[v,w]=0$, (5.6) becomes

$$H([a,v],w)=[a,\sum_r\frac{\Gamma(k)}{2k+h^\vee-\bar h_r^\vee}[[x_\theta,v],w]_r^\natural]$$
$$+\sum_{i,j,r,s}k_{i,j}^{r,s}(v,w)(2k+h^\vee-\bar h_r^\vee)(a|x_i^r)x_j^s+\sum_{i,j,r,s}k_{i,j}^{r,s}(v,w)[[a,x_i^r],x_j^s].$$

Now compute

$$(H([x_i^r,v],w)|x_j^s)$$
$$=([x_i^r,\tfrac12 H(v,w)]|x_j^s)+\sum_{n,m,r',s'}k_{n,m}^{r',s'}(v,w)(2k+h^\vee-\bar h_{r'}^\vee)(x_i^r|x_n^{r'})(x_m^{s'}|x_j^s)$$
$$+\sum_{n,m,r',s'}k_{n,m}^{r',s'}(v,w)([[x_i^r,x_n^{r'}],x_m^{s'}]|x_j^s)$$
$$=([x_i^r,\tfrac12 H(v,w)]|x_j^s)+k_{i,j}^{r,s}(v,w)(2k+h^\vee-\bar h_r^\vee)+\delta_{r,s}\sum_{n,m}k_{n,m}^{r,s}(v,w)([[x_i^r,x_n^r],x_m^s]|x_j^s).$$

Since $[a,H(v,w)]=H([a,v],w)+H(v,[a,w])$ we can rewrite the above relations as

$$(H([x_i^r,v],w)|x_j^s)-(H(v,[x_i^r,w])|x_j^s)$$
$$=2f_h(k)(\langle[[v,x_i^r],[w,x_j^s]]\rangle+\langle[[v,x_j^s],[w,x_i^r]]\rangle)(2k+h^\vee-\bar h_r^\vee)$$
$$+\delta_{r,s}\sum_{n,m}2f_h(k)(\langle[v,x_n^r],[w,x_m^s]\rangle+\langle[v,x_m^s],[w,x_n^s]\rangle)([[x_i^r,x_n^r],x_m^s]|x_j^s).$$

More precisely

$$-\frac{\Gamma(k)}{2k+h^\vee-\bar h_s^\vee}(\langle[v,x_i^r],[w,x_j^s]\rangle+\langle[v,x_j^s],[w,x_i^r]\rangle)$$
$$=f_h(k)(\langle[[v,x_i^r],[w,x_j^s]]\rangle+\langle[[v,x_j^s],[w,x_i^r]]\rangle)(2k+h^\vee-\bar h_r^\vee) \tag{5.13}$$
$$+\delta_{r,s}\sum_{n,m}f_h(k)(\langle[v,x_n^r],[w,x_m^s]\rangle+\langle[v,x_m^s],[w,x_n^s]\rangle)([[x_i^r,x_n^r],x_m^s]|x_j^s).$$

If there are at least two simple components in $\mathfrak{g}^\natural$, we have

$$f_h(k)=-\frac{\Gamma(k)}{(2k+h^\vee-\bar h_s^\vee)(2k+h^\vee-\bar h_r^\vee)}, \tag{5.14}$$

which is in particular independent from h. So we are reduced to determine $f_h(k)$ when $\mathfrak{g}^\natural$ is simple or one-dimensional. Recall from the explicit description of the decomposition of $\mathfrak{g}_{-1/2} \wedge \mathfrak{g}_{-1/2}$ given in Section 3, that in this case V' is simple. We can therefore drop the superscript from $x_i^r, k_{i,j}^{r,s}$ and we denote f_h simply by f. Choosing in (5.13) v, w, i, j such that $\langle [v, x_i], [w, x_j]\rangle + \langle [v, x_j], [w, x_i]\rangle \neq 0$ we can write

$$f(k) = \frac{\Gamma(k)}{(\xi k + \eta)(2k + h^\vee - \bar{h}_1^\vee)}$$

with $\xi \neq 0$. In particular

$$\begin{aligned}&-(\xi k + \eta)(\langle [v, x_i], [w, x_j]\rangle + \langle [v, x_j], [w, x_i]\rangle)\\&= (\langle [v, x_i], [w, x_j]\rangle + \langle [v, x_j], [w, x_i]\rangle)(2k + h^\vee - \bar{h}_1^\vee)\\&+ \sum_{n,m}(\langle [v, x_n], [w, x_m]\rangle + \langle [v, x_m], [w, x_n]\rangle)([[x_i, x_n], x_m]|x_j),\end{aligned}$$

so $\xi = -2$ and

$$\begin{aligned}&-(\eta + (h^\vee - \bar{h}_1^\vee))(\langle [v, x_i], [w, x_j]\rangle + \langle [v, x_j], [w, x_i]\rangle)\\&= \sum_{n,m}(\langle [v, x_n], [w, x_m]\rangle + \langle [v, x_m], [w, x_n]\rangle)([[x_i, x_n], x_m]|x_j).\end{aligned}$$

To compute η we first observe that

$$\begin{aligned}&\sum_{n,m}(\langle [v, x_n], [w, x_m]\rangle + \langle [v, x_m], [w, x_n]\rangle)([[x_i, x_n], x_m]|x_j)\\&= -\sum_{n,m}(\langle [v, x_n], [w, x_m]\rangle + \langle [v, x_m], [w, x_n]\rangle)([[x_i, x_n], x_j]|x_m)\\&= -\sum_{n}(\langle [v, x_n], [w, [[x_i, x_n], x_j]]\rangle + \langle [v, [[x_i, x_n], x_j]], [w, x_n]\rangle),\end{aligned}$$

which implies, for any $a, b \in \mathfrak{g}^\natural$

$$\begin{aligned}&(\eta + (h^\vee - \bar{h}_1^\vee))(\langle [v, a], [w, b]\rangle + \langle [v, b], [w, a]\rangle)\\&= \sum_{n}(\langle [v, x_n], [w, [[a, x_n], b]]\rangle + \langle [v, [[a, x_n], y]], [w, x_n]\rangle).\end{aligned}$$

Next we need some formulas: let $C_{\mathfrak{g}^\natural} = \sum_i x_i^2$ be the Casimir element of $\mathfrak{g}^\natural$ and $C_{\mathfrak{g}_0}$ the Casimir element of $\mathfrak{g}_0$. Since $C_{\mathfrak{g}^\natural} = \mathbb{C}_{\mathfrak{g}_0} - 2x^2$, by Lemma 5.1 of [KW] we have that $\sum_i [x_i, [x_i, v]] = (h^\vee - \frac{3}{2})v$. Now a lengthy computation yields

$$\begin{aligned}&\sum_{n}(x_\theta|[[v, x_n], [w, [[a, x_n], b]]] + [[v, [[a, x_n], b]], [w, x_n]])\\&= (h^\vee - 3/2 - 2\bar{h}_1^\vee)(\langle [v, b], [w, a]\rangle + \langle [v, a], [w, b]\rangle)\\&+ \sum_{n}(\langle [b, [v, x_n]], [a, [w, x_n]]\rangle + \langle [a, [v, x_n]], [b, [w, x_n]]\rangle).\end{aligned}$$

Consider the map $\Psi : \wedge^2\mathfrak{g}_{-1/2} \to S^2(\mathfrak{g}^\natural)^*$ defined by polarizing (3.5):

$$\Psi(v \wedge w)(a, b) = \langle [v, a], [w, b]\rangle + \langle [v, b], [w, a]\rangle,$$

and note that

$$\sum_n(\langle[b,[v,x_n]],[a,[w,x_n]]\rangle+\langle[a,[v,x_n]],[b,[w,x_n]]\rangle)=\Psi(\sum_n[v,x_n]\wedge[w,x_n])(a,b).$$

By Lemma 4.2, (4.15), we have

$$\sum_n(\langle[b,[v,x_n]],[a,[w,x_n]]\rangle+\langle[a,[v,x_n]],[b,[w,x_n]]\rangle)=-\tfrac{1}{2}(\langle[b,v],[a,w]\rangle+\langle[a,v],[b,w]\rangle).$$

The outcome is that

$$\begin{aligned}&\sum_n(\langle[v,x_n],[w,[[a,x_n],b]]\rangle+\langle[v,[[a,x_n],b]],[w,x_n]\rangle)\\&=(h^\vee-2-2\bar h_1^\vee)(\langle[v,b],[w,a]\rangle+\langle[v,a],[w,b]\rangle).\end{aligned}$$

Thus $\eta=-2-\bar h_1^\vee$. In particular

$$f(k)=-\frac{\Gamma(k)}{4(k+\frac{h^\vee-\bar h_1^\vee}{2})(k+\frac{\bar h_1^\vee}{2}+1)}. \tag{5.15}$$

This ends the computation of the proportionality factor $f_h(k)$.

It remains to compute $Q(v,w)$ and $P(v,w)$ with $[v,w]\neq 0$. To this end introduce

$$TR_r(v,w):=\sum_j k_{j,j}^{r,r}(v,w).$$

Relation (5.6) gives

$$\begin{aligned}(H([x_i^r,v],w)|x_i^r)&=Q(v,w)+k_{i,i}^{r,r}(v,w)(2k+h^\vee-\bar h_r^\vee)\\&\quad+\sum_{n,m}k_{n,m}^{r,r}(v,w)(([[x_i^r,x_n^r],x_m^r]|x_i^r).\end{aligned}$$

So

$$\begin{aligned}\sum_i(H([x_i^r,v],w)|x_i^r)&=\dim\mathfrak g_r^\natural Q(v,w)+TR_r(v,w)(2k+h^\vee-\bar h_r^\vee)\\&\quad+\sum_i\sum_{n,m}k_{n,m}^{r,r}(v,w)([[x_i^r,x_n^r],x_m^r]|x_i^r).\end{aligned}$$

Using the relation $\sum_i[x_i^r,[x_i^r,a]]=2(\bar h_r^\vee)a$ for $a\in\mathfrak g_r^\natural$ we obtain

$$\sum_i(H([x_i^r,v],w)|x_i^r)=\dim\mathfrak g_r^\natural Q(v,w)+TR_r(v,w)(2k+h^\vee+\bar h_r^\vee).$$

Since

$$\sum_i(H([x_i^r,v],w)|x_i^r)=\tfrac{2\Gamma(k)\gamma_r}{2k+h^\vee-h_r^\vee}\langle v,w\rangle,$$

we have, recalling that $Q(v,w)=D(k)\langle v,w\rangle$,

$$TR_r(v,w)=\frac{1}{2k+h^\vee+\bar h_r^\vee}\left(-\dim\mathfrak g_r^\natural D(k)+\tfrac{2\Gamma(k)\gamma_r}{2k+h^\vee-h_r^\vee}\right)\langle v,w\rangle. \tag{5.16}$$

Relation (5.7) becomes

$$\sum_r \frac{2k+h^\vee-\bar h_r^\vee}{2k+h^\vee+\bar h_r^\vee}\left(-\dim\mathfrak{g}_r^\natural D(k)+\tfrac{2\Gamma(k)\gamma_r}{2k+h^\vee-\bar h_r^\vee}\right)+D(k)(\frac{k\,\dim\mathfrak{g}}{k+h^\vee}-6k+h^\vee-4)=3\Gamma(k).$$

Solving for $D(k)$ we find that

$$D(k)=\Gamma(k)E(k),$$

where $E(k)$ is a complicated but explicit rational function in $k, h^\vee, \bar h_r^\vee, \gamma_r, \dim\mathfrak{g}, \dim\mathfrak{g}_r^\natural$. In turn, substituting in (5.16), TR_r can be expressed as

$$TR_r(v,w)=\Gamma(k)E_r(k)\langle v,w\rangle. \tag{5.17}$$

Let $\{u_i\}$, $\{u^i\}$ be dual bases of $\mathfrak{g}_{-1/2}$: $\langle u_i,u^j\rangle=\delta_{i,j}$. We have to compute $P(\sum_i u_i\wedge u^i)$. If $u,v\in\mathfrak{g}_{-1/2}$, then

$$u\wedge v=\frac{\langle u,v\rangle}{\dim\mathfrak{g}_{1/2}}\sum_i u_i\wedge u^i+s,\quad s\in V_2.$$

By covariance of P, we have $P(\sum_i u_i\wedge u^i)=\sum_{i,r}\alpha_r x_i^r\otimes x_i^r$, thus

$$\sum_{i,r,s,m,n}k_{m,n}^{r,s}(u_i,u^i)x_m^r\otimes x_n^s=\sum_{i,r}\alpha_r x_i^r\otimes x_i^r,$$

hence $\sum_i k_{m,n}^{r,s}(u_i,u^i)=\delta_{r,s}\delta_{m,n}\alpha_r$. In particular $\dim\mathfrak{g}_r^\natural\alpha_r=\sum_{i,j}k_{j,j}^{r,r}(u_i,u^i)$ hence

$$\dim\mathfrak{g}_r^\natural\alpha_r=\sum_i TR_r(u_i,u^i)=\Gamma(k)E_r(k)\sum_i\langle u_i,u^i\rangle=\Gamma(k)E_r(k)\dim\mathfrak{g}_{-1/2}.$$

Since Φ is equivariant we have likewise

$$\sum_{i,r,s,m,n}(\langle[u_i,x_m^r],[u^i,x_n^s]\rangle+\langle[u_i,x_m^s],[u^i,x_m^r]\rangle)x_m^r\otimes x_n^s=\sum_{i,r}\beta_r x_i^r\otimes x_i^r,$$

hence

$$\sum_i(\langle[u_i,x_m^r],[u^i,x_n^s]\rangle+\langle[u_i,x_n^s],[u^i,x_m^r]\rangle)=\delta_{r,s}\delta_{m,n}\beta_r$$

and

$$\begin{aligned}\dim\mathfrak{g}_r^\natural\beta_r&=\sum_{i,m}(\langle[u_i,x_m^r],[u^i,x_m^r]\rangle+\langle[u_i,x_m^r],[u^i,x_m^r]\rangle)=-2\sum_{i,m}\langle[x_m^r,[x_m^r,u_i]],u^i\rangle\\&=-2\gamma_r\sum_i\langle u_i,u^i\rangle=-2\gamma_r\dim\mathfrak{g}_{-1/2}.\end{aligned}$$

Since $P(s)=f(k)\Phi(s)$,

$$P(s)=\sum_{n,m,r,s}f(k)(\langle[u,x_m^r],[v,x_n^s]\rangle+\langle[u,x_n^s],[v,x_m^r]\rangle)(x_m^r\otimes x_n^s)$$

$$-\frac{\langle u,v\rangle}{\dim\mathfrak{g}_{1/2}}f(k)\sum_{i,m,n,r,s}(\langle[u_i,x_m^r],[u^i,x_n^s]\rangle+\langle[u_i,x_n^s],[u^i,x_m^r]\rangle)(x_m^r\otimes x_n^s)=$$

$$\sum_{m,n,r,s}f(k)(\langle[u,x_m^r],[v,x_n^s]\rangle+\langle[u,x_n^s],[v,x_m^r]\rangle)(x_m^r\otimes x_n^s)-\frac{\langle u,v\rangle}{\dim\mathfrak{g}_{1/2}}f(k)\sum_{i,r}\beta_r(x_i^r\otimes x_i^r)=$$

$$\sum_{m,n,r,s}f(k)(\langle[u,x_m^r],[v,x_n^s]\rangle+\langle[u,x_m^s],[v,x_m^r]\rangle)(x_m^r\otimes x_n^s)+\langle u,v\rangle f(k)\sum_{m,r}\frac{2\gamma_r}{\dim\mathfrak{g}_r^\natural}(x_m^r\otimes x_m^r).$$

The outcome is that

$$P(u \wedge v) = \langle u, v\rangle \sum_{r,m} \frac{\Gamma(k)E_r(k) + 2\gamma_r f(k)}{\dim \mathfrak{g}_r^\natural}(x^r_m \otimes x^r_m) + \sum_{m,n,r,s} f(k)(\langle [u, x^r_m], [v, x^s_n]\rangle + \langle [u, x^s_n], [v, x^r_m]\rangle)(x^r_m \otimes x^s_n). \tag{5.18}$$

Observe from (5.14) that f does not depend on the choice of r, s, hence $\{\bar{h}^\vee_r\}$ has at most two elements, and if there are more than two components, $\{\bar{h}^\vee_r\}$ is a singleton.

We now deal with the case in which $\mathfrak{g}^\natural$ has three components. Suppose that $\mathfrak{g}^\natural$ has a nontrivial center. Then $\bar{h}^\vee_1 = 0$, also $\bar{h}^\vee_2, \bar{h}^\vee_3$ vanish and $\mathfrak{g}^\natural$ is 3-dimensional abelian. This is not possible, hence $\mathfrak{g}^\natural$ is semisimple, α_{i_0} is long and is a node of degree 3 in the Dynkin diagram of $\mathfrak{g}$. Therefore one of the components, say $\mathfrak{g}^\natural_1$, has to be $sl(2)$. In particular $\bar{h}^\vee_1 = h^\vee_1 = 2$. By the above remark, we have $\bar{h}^\vee_2 = \bar{h}^\vee_3 = 2$ and indeed $\nu_2 = \nu_3 = 1$. Hence all components are isomorphic to $sl(2)$ and this forces $\mathfrak{g}$ to be of type D_4.

Set

$$p(k) = \begin{cases} (k + \frac{h^\vee - \bar{h}^\vee_1}{2})(k + \frac{h^\vee - \bar{h}^\vee_2}{2}) & \text{if } \mathfrak{g}^\natural \text{ has two components,} \\ (k + \frac{h^\vee - \bar{h}^\vee_1}{2})(k + \frac{\bar{h}^\vee_1}{2} + 1) & \text{otherwise.} \end{cases}$$

Observe that, combining (5.14) and (5.15), we have $f(k) = -\frac{\Gamma(k)}{4p(k)}$ in all cases. We summarize our findings in the following proposition.

Proposition 5.3. *There are explicitly computable rational functions $a_r(k)$, $b(k)$, $c(k)$, $d_r(k)$ such that, up to a constant C,*

$$[G^v{}_\lambda G^w] = C\left(\langle v, w\rangle \sum_{i,r} a_r(k) :J^{x^r_i} J^{x^r_i}: + b(k) \sum_{i,j,r} (\langle [v, x^r_i], [w, x^r_j]\rangle + \langle [v, x^r_j], [w, x^r_i]\rangle) :J^{x^r_i} J^{x^r_j}: \right) + C\left(c(k)\langle v, w\rangle L + \sum_r d_r(k)\left(\tfrac{1}{2} T J^{[[x_\theta, v], w]^\natural_r} + \lambda J^{[[x_\theta, v], w]^\natural_r}\right) + \tfrac{\lambda^2}{2}\langle v, w\rangle\right).$$

More precisely

$$b(k) = -\frac{1}{4p(k)}, \quad d_r(k) = \frac{1}{k + \frac{h^\vee - \bar{h}^\vee_r}{2}},$$

while $a_r(k)$ and $c(k)$ are certain rational functions of degree respectively -2 and -1.

If we set $\varphi(u, v)(w, z) = \sum_{i,j,r,s} k^{r,s}_{i,j}(u, v)\langle [[w, x^r_i], x^s_j], z\rangle$ then φ is alternating in w, z so it defines a map $\varphi : \bigwedge^2 \mathfrak{g}_{-1/2} \to \bigwedge^2 \mathfrak{g}_{1/2}$ and it is $\mathfrak{g}^\natural$-equivariant, since $\varphi = \pi \circ Symm \circ \Phi$ where π is the action of $\mathfrak{g}^\natural$ on $\mathfrak{g}_{-1/2}$.

Relations (5.9), (5.10), (5.11) then become

$$\frac{1}{2}\langle [H(u,v),w],z\rangle+\frac{3}{2}Q(u,v)\langle w,z\rangle+\varphi(u,v)(w,z) \tag{5.19}$$
$$=\frac{1}{2}\langle [u,H(v,w)],z\rangle+\frac{3}{2}Q(v,w)\langle u,z\rangle+\varphi(v,w)(u,z)-\langle [v,H(u,w)],z\rangle,$$

$$-\frac{1}{2}\langle [H(u,v),w],z\rangle+\frac{3}{2}Q(u,v)\langle w,z\rangle+\varphi(u,v)(w,z) \tag{5.20}$$
$$=\langle [u,H(v,w)],z\rangle-\frac{1}{2}\langle [v,H(u,w)],z\rangle-\frac{3}{2}Q(u,w)\langle v,z\rangle-\varphi(u,w)(v,z),$$

$$Q(u,v)\langle w,z\rangle+2\varphi(u,v)(w,z)= \tag{5.21}$$
$$\frac{1}{2}\langle [u,H(v,w)],z\rangle+\frac{1}{2}Q(v,w)\langle u,z\rangle-\varphi(v,w)(u,z)$$
$$-\frac{1}{2}\langle [v,H(u,w)],z\rangle-\frac{1}{2}Q(u,w)\langle v,z\rangle+\varphi(u,w)(v,z).$$

Lemma 5.4. *Assume $C\neq 0$ for almost all k and set*

$$R(k)=\begin{cases}-\frac{\sum_r d_r(k)\|(h_{\alpha_{i_0}})_r^\natural\|^2+3c(k)+2\sum_r a_r(k)\gamma_r}{2b(k)} & \text{if } \mathfrak{g}^\natural \text{ has two components,}\\ -\frac{3/2d_1(k)+3c(k)+(2h^\vee-3)a_1(k)}{2b(k)} & \text{otherwise.}\end{cases}$$

Then $R(k)$ does not depend on k. More precisely

$$R(k)=\begin{cases}\frac{3-4h^\vee+2\sum_r \bar h_r^\vee\|(h_{\alpha_{i_0}})_r^\natural\|^2}{2} & \text{if } \mathfrak{g}^\natural \text{ has two components,}\\ \frac{3-4h^\vee+3\bar h_1^\vee}{2} & \text{otherwise.}\end{cases}$$

Proof. Choose $v=u$ in (5.19). Then we obtain

$$\frac{1}{2}(\langle [H(u,u),w]+[u,H(u,w)],z\rangle)-\frac{3}{2}Q(u,w)\langle u,z\rangle=\varphi(u,w)(u,z). \tag{5.22}$$

Using the explicit formulas

$$H(v,w)=C\sum_r d_r(k)[[x_\theta,v],w]_r^\natural,\quad Q(v,w)=Cc(k)\langle v,w\rangle$$

and (4.16) we find

$$\frac{C}{2}\sum_r d_r(k)\sum_i(\langle u,[u,x_i^r]\rangle\langle [x_i^r,w],z\rangle+\langle u,[w,x_i^r]\rangle\langle [u,x_i^r],z\rangle)-\frac{3C}{2}c(k)\langle u,w\rangle\langle u,z\rangle=$$
$$\varphi(u,w)(u,z).$$

We first evaluate the right hand side of (5.22): recall that

$$k_{i,j}^{r,s}(u,v)=Cb(k)(\langle [u,x_i^r],[v,x_j^s]\rangle+\langle [u,x_j^s],[v,x_i^r]\rangle)+C\delta_{i,j}\delta_{r,s}a_r(k)\langle u,v\rangle,$$

so

$$\varphi(u,w)(u,z)=\sum_{i,j,r,s}Cb(k)(\langle [u,x_i^r],[w,x_j^s]\rangle+\langle [u,x_j^s],[w,x_i^r]\rangle)\langle [[u,x_i^r],x_j^s],z\rangle \tag{5.23}$$
$$+C\sum_{i,r}a_r(k)\langle [[u,x_i^r],x_i^r],z\rangle\langle u,w\rangle$$
$$=\sum_{i,j,r,s}Cb(k)(\langle [u,x_i^r],[w,x_j^s]\rangle+\langle [u,x_j^s],[w,x_i^r]\rangle)\langle [[u,x_i^r],x_j^s],z\rangle$$
$$+C\sum_r a_r(k)\gamma_r\langle u,z\rangle\langle u,w\rangle.$$

Take $u=f_{i_0}, w=z=[x_{-\theta},e_{i_0}]$. Passing to dual bases $\{x_i^r\},\{x^i_r\}$ of $\mathfrak{g}^\natural_r$ and using (5.23), relation (5.22) becomes

(5.24)

$$\frac{C}{2}\sum_{r,i} d_r(k)\left(\langle f_{i_0},[f_{i_0},x_i^r]\rangle\langle[x^i_r,[x_{-\theta},e_{i_0}]],[x_{-\theta},e_{i_0}]\rangle+\langle f_{i_0},[[x_{-\theta},e_{i_0}],x_i^r]\rangle\langle[f_{i_0},x^i_r],[x_{-\theta},e_{i_0}]\rangle\right)$$
$$-\frac{3C}{2}c(k)\langle f_{i_0},[x_{-\theta},e_{i_0}]\rangle^2=$$
$$\sum_{i,j,r,s} Cb(k)\left(\langle[f_{i_0},x_i^r],[[x_{-\theta},e_{i_0}],x_j^s]\rangle+\langle[f_{i_0},x_j^s],[[x_{-\theta},e_{i_0}],x_i^r]\rangle\right)\langle[[f_{i_0},x_i^r],x_j^s],[x_{-\theta},e_{i_0}]\rangle$$
$$+C\sum_r a_r(k)\gamma_r\langle f_{i_0},[x_{-\theta},e_{i_0}]\rangle^2.$$

We now evaluate the left hand side of (5.24). Assume first that $\mathfrak{g}^\natural$ has two components, then $-\theta+2\alpha_{i_0}$ is not a root. By weight considerations, $\langle f_{i_0},[f_{i_0},x_i^r]\rangle$ can be non zero only if x_i^r has weight $-\theta+2\alpha_{i_0}$. Arguing in the same way, we also conclude that in the second sum x_i^r should belong to $\mathfrak{h}$, so that the left hand side of (5.24) simplifies to

$$\frac{C}{2}(\sum_r d_r(k)\sum_i\langle f_{i_0},[[x_{-\theta},e_{i_0}],h_i^r]\rangle\alpha_{i_0}(h_i^r)-3c(k)\langle f_{i_0},[x_{-\theta},e_{i_0}]\rangle)\langle f_{i_0},[x_{-\theta},e_{i_0}]\rangle,$$

where $\{h_i^r\}$ is an orthonormal basis of $\mathfrak{h}^\natural_r$. The above formula can be further reduced to

$$-\frac{C}{2}(\sum_r d_r(k)\|(h_{\alpha_{i_0}})^\natural_r\|^2+3c(k)))\langle f_{i_0},[x_{-\theta},e_{i_0}]\rangle^2. \tag{5.25}$$

Assume now that $\mathfrak{g}^\natural$ has only one component (i.e., it is simple or 1-dimensional). Then, if $u,v\in\mathfrak{g}_{-1/2}$,

$$[[x_\theta,u],v]=[[x_\theta,u],v]^\natural+\langle u,v\rangle x.$$

In this case (5.22) becomes

$$\frac{C}{2}(d_1(k)(\langle[[[x_\theta,u],u]^\natural,w]+[u,[[x_\theta,u],w]^\natural],z\rangle)-\frac{3C}{2}Q(u,w)\langle u,z\rangle=\varphi(u,w)(u,z).$$

Since

$$[[[x_\theta,u],u]^\natural,w]+[u,[[x_\theta,u],w]^\natural]=[[[x_\theta,u],u],w]+[u,[[x_\theta,u],w]]-\langle u,w\rangle[u,x]$$

and $[u,[[x_\theta,u],w]]=-[[[x_\theta,u],u],w]+[[x_\theta,u],[u,w]]$, we see that

$$\begin{aligned}[[[x_\theta,u],u]^\natural,w]+[u,[[x_\theta,u],w]^\natural]&=[[x_\theta,u],[u,w]]-\tfrac{1}{2}\langle u,w\rangle u\\&=2\langle u,w\rangle[[x_\theta,u],x_{-\theta}]-\tfrac{1}{2}\langle u,w\rangle u=-\tfrac{3}{2}\langle u,w\rangle u.\end{aligned}$$

The upshot is that

$$-\frac{C}{2}(\frac{3}{2}d_1(k)+3c(k))\langle u,w\rangle\langle u,z\rangle=\varphi(u,w)(u,z).$$

Substituting $u=f_{i_0}$ and $w=z=[x_{-\theta},e_{i_0}]$ the left hand side of (5.24) becomes

$$-\frac{C}{2}(\frac{3}{2}d_1(k)+3c(k))\langle f_{i_0},[x_{-\theta},e_{i_0}]\rangle^2. \tag{5.26}$$

Finally, if $\mathfrak{g}^\natural$ has three components, since $d_1(k)=d_2(k)=d_3(k)$, formula (5.23) becomes

$$-\frac{C}{2}(d_1(k)\|(h_{\alpha_{i_0}})^\natural\|^2+3c(k)))\langle f_{i_0},[x_{-\theta},e_{i_0}]\rangle^2$$

which is indeed (5.26). To evaluate the right hand side of (5.23) in the current case, notice that γ_r does not depend on r (see also (5.27) below). From this and relation (5.18) we deduce that $a_r(k)$ does not depend on r. It follows, using (5.28) below, that

$$C\sum_r a_r(k)\gamma_r\langle f_{i_0},[x_{-\theta},e_{i_0}]\rangle^2=\tfrac{C}{2}a_1(k)(2h^\vee-3)\langle f_{i_0},[x_{-\theta},e_{i_0}]\rangle^2,$$

as in the case when $\mathfrak{g}^\natural$ has only one component.

The final outcome is that

$$\sum_{i,j,r,s}(\langle[f_{i_0},x_i^r],[[x_{-\theta},e_{i_0}],x_j^s]\rangle+\langle[f_{i_0},x_j^s],[[x_{-\theta},e_{i_0}],x_i]\rangle)\langle[[f_{i_0},x_i^r],x_j^s],[x_{-\theta},e_{i_0}]\rangle$$
$$=R(k)\langle f_{i_0},[x_{-\theta},e_{i_0}]\rangle^2,$$

where

$$R(k)=\begin{cases}-\frac{\sum_r d_r(k)\|(h_{\alpha_{i_0}})^\natural_r\|^2+3c(k)+2\sum_r a_r(k)\gamma_r}{2b(k)} & \text{if } \mathfrak{g}^\natural \text{ has two components,}\\ -\frac{3/2d_1(k)+3c(k)+(2h^\vee-3)a_1(k)}{2b(k)} & \text{otherwise.}\end{cases}$$

It follows that $R(k)$ does not depend on k, hence it equals the value of its limit for $k\to\infty$. This limit is

$$\lim_{k\to\infty}R(k)=\begin{cases}\frac{3-4h^\vee+2\sum_r\bar h_r^\vee\|(h_{\alpha_{i_0}})^\natural_r\|^2}{2} & \text{if } \mathfrak{g}^\natural \text{ has two components,}\\ \frac{3-4h^\vee+3\bar h_1^\vee}{2} & \text{otherwise.}\end{cases}$$

□

There are several relations among the values $\gamma_i,\dim\mathfrak{g}_i^\natural,\dim\mathfrak{g},\bar h_i^\vee$ and $h^\vee$. Indeed, if $\mathfrak{g}_i^\natural$ is abelian, then $\mathfrak{g}_i^\natural=\mathbb{C}\varpi$ with ϖ as in (3.3). As noted in Section 3, ϖ acts on $\mathfrak{g}_{-1/2}$ as $\pm I$, so the eigenvalue of $C_{\mathfrak{g}_i^\natural}=\frac{\varpi^2}{(\varpi|\varpi)}$ is $\frac{1}{(\varpi|\varpi)}$. On the other hand $(\varpi|\varpi)=\frac{tr(ad(\varpi)^2)}{2h^\vee}=\frac{2\dim\mathfrak{g}_{-1/2}}{2h^\vee}$, so we conclude that $\gamma_i=\frac{\dim\mathfrak{g}_i^\natural h^\vee}{2(h^\vee-2)}$. Since $h_i^\vee=0$, this formula can be written as

$$\gamma_i=\frac{\dim\mathfrak{g}_i^\natural(h^\vee-\bar h_i^\vee)}{2(h^\vee-2)}. \tag{5.27}$$

By [P, (2.2)], the index $ind_{\mathfrak{g}_i^\natural}(\mathfrak{g}_{1/2}\oplus\mathfrak{g}_{-1/2})$ is $(h^\vee-\bar h_i^\vee)/\bar h_i^\vee$. The same index can be computed as $ind_{\mathfrak{g}_i^\natural}(\mathfrak{g}_{1/2})+ind_{\mathfrak{g}_i^\natural}(\mathfrak{g}_{-1/2})=2\,ind_{\mathfrak{g}_i^\natural}(\mathfrak{g}_{-1/2})$ and this last quantity is computed by [P, (1.3)] to be $\frac{\gamma_i\dim\mathfrak{g}_{-1/2}}{2\bar h_i^\vee\dim\mathfrak{g}_i^\natural}$. Since $\dim\mathfrak{g}_{-1/2}=2(h^\vee-2)$, it follows that (5.27) holds in these cases too.

By (4.6),

$$\sum_r\gamma_r=h^\vee-3/2 \tag{5.28}$$

so, if $\mathfrak{g}^\natural$ has two components one can solve for $\dim\mathfrak{g}_2^\natural$ and obtain that

$$\dim\mathfrak{g}_2^\natural=\frac{\dim\mathfrak{g}_1^\natural(\bar h_1^\vee-h^\vee)+2(h^\vee)^2-7h^\vee+6}{h^\vee-\bar h_2^\vee}.$$

Moreover, by (4.7),

$$\dim\mathfrak{g}=4h^\vee-5+\dim\mathfrak{g}^\natural. \tag{5.29}$$

Our analysis provides more refined relations.

Proposition 5.5. *If $\mathfrak{g}^\natural$ has two components, then*

$$\dim\mathfrak{g} = \frac{(h^\vee+1)\left(2(h^\vee)^2 + h^\vee(\bar h_1^\vee - 2) - \bar h_1^\vee(\bar h_1^\vee+2)\right)}{(\bar h_1^\vee+2)(h^\vee - \bar h_1^\vee)}, \quad \bar h_1^\vee + \bar h_2^\vee = h^\vee - 2. \tag{5.30}$$

Otherwise

$$\dim\mathfrak{g} = \frac{2\left(5(h^\vee)^2 - h^\vee - 6\right)}{h^\vee+6}, \quad \bar h_1^\vee = \frac{2(h^\vee-3)}{3} \tag{5.31}$$

or

$$\dim\mathfrak{g} = 2(h^\vee)^2 - 3h^\vee + 1, \quad \bar h_1^\vee = h^\vee - 1. \tag{5.32}$$

Moreover, (5.32) *occurs if and only if $\mathfrak{g}^\natural$ is simple and α_{i_0} is short.*

Proof. Write explicitly the rational function $R(k) - \lim_{k\to\infty} R(k)$ as P/Q with P, Q polynomials in $k, h^\vee, \bar h_i^\vee$ and $\dim\mathfrak{g}$. Since P is identically zero, by equating its coefficients to zero and solving the system of equations with respect to $\dim\mathfrak{g}$ and $h_i^\vee/\nu_i$ we get the above formulas.

To finish the proof we show that $\bar h_1^\vee = h^\vee - 1$ if and only if $\mathfrak{g}^\natural$ is simple and α_{i_0} is short. Let Σ be the set of simple roots of $\mathfrak{g}^\natural$ and θ_Σ its highest root. Write $\theta = \sum_{\alpha\in\Pi} m_\alpha\alpha$, $\theta_\Sigma = \sum_{\alpha\in\Sigma} n_\alpha\alpha$ and note that $n_\alpha \le m_\alpha$ for all $\alpha\in\Sigma$. If $\bar h_1^\vee$, since $\nu_1 = 2/(\theta_\Sigma|\theta_\Sigma)$, we have

$$\bar h_1^\vee = \tfrac12((\theta_\Sigma|\theta_\Sigma) + \sum_{\alpha\in\Sigma}(\alpha|\alpha)n_\alpha = \tfrac12\sum_{\alpha\in\Pi}(\alpha|\alpha)m_\alpha = (\alpha_{i_0}|\alpha_{i_0}) + \tfrac12\sum_{\alpha\in\Sigma}(\alpha|\alpha)m_\alpha,$$

hence

$$(\theta_\Sigma|\theta_\Sigma) - 2(\alpha_{i_0}|\alpha_{i_0}) = \sum_{\alpha\in\Sigma}(\alpha|\alpha)(m_\alpha - n_\alpha) \ge 0.$$

It follows that α_{i_0} is short. Then $\theta - 2\alpha_{i_0}$ is a root of $\mathfrak{g}^\natural$ and this forces $\mathfrak{g}^\natural$ to be simple, otherwise the support of $\theta - 2\alpha_{i_0}$ would be disconnected.

Assume now that $\mathfrak{g}^\natural$ is simple and α_{i_0} is short. Then $\theta - 2\alpha_{i_0}$ is a root of $\mathfrak{g}^\natural$. This forces $\theta_\Sigma = \theta - 2\alpha_{i_0}$ and $(\theta|\alpha_{i_0}^\vee) = 2$, so that $(\alpha_{i_0}|\alpha_{i_0}) = 1$. We have

$$\begin{aligned} h^\vee - 1 &= \sum_{\alpha\in\Delta}\tfrac{(\alpha|\alpha)}{2}m_\alpha = \sum_{\alpha\in\Sigma}\tfrac{(\alpha|\alpha)}{2}n_\alpha + (\alpha_{i_0}|\alpha_{i_0}) = \tfrac{(\theta_\Sigma|\theta_\Sigma)}{2}\sum_{\alpha\in\Sigma}\tfrac{(\alpha|\alpha)}{(\theta_\Sigma|\theta_\Sigma)}n_\alpha + (\alpha_{i_0}|\alpha_{i_0}) \\ &= \bar h_1^\vee - \tfrac{(\theta_\Sigma|\theta_\Sigma)}{2} + (\alpha_{i_0}|\alpha_{i_0}) = \bar h_1^\vee, \end{aligned}$$

since $(\theta_\Sigma|\theta_\Sigma) = (\theta - 2\alpha_{i_0}|\theta - 2\alpha_{i_0}) = 2 - 4(\theta|\alpha_{i_0}) + 4 = 2$. □

Remark 5.6. *A brief inspection of the Dynkin diagrams shows that $\mathfrak{g}^\natural$ does not have exactly two irreducible ideals precisely when $\mathfrak{g}$ belongs to the Deligne's series $A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$ or it is of type C_n. The first case occurs exactly when there is a long simple root not orthogonal to the highest root.*

Remark 5.7. *It is worthwhile to recall that* $\dim\mathfrak{g} = r(h+1)$, *where r is the rank of $\mathfrak{g}$ and h is the Coxeter number of $\mathfrak{g}$.*

Proposition 5.8. *Let $\{x_i\}$ be an orthonormal basis of $\mathfrak{g}^\natural$. Then*

$$[G^v{}_\lambda G^w] =$$
$$C\left(\tfrac{1}{4p(k)}\langle u,v\rangle\sum_i :J^{x_i}J^{x_i}: -\tfrac{1}{4p(k)}\sum_{i,j}(\langle[v,x_i],[w,x_j]\rangle+\langle[v,x_j],[w,x_i]\rangle):J^{x_i}J^{x_j}:\right)+$$
$$C\left(-\tfrac{k+h^\vee}{2p(k)}\langle u,v\rangle L+\sum_r\frac{1}{k+\frac{h^\vee-\bar h_r^\vee}{2}}(\tfrac12 TJ^{[[x_\theta,v],w]^\natural_r}+\lambda J^{[[x_\theta,v],w]^\natural_r})+\tfrac{\lambda^2}{2}\langle u,v\rangle\right).$$

Proof. Substitute the values (5.31), (5.32), (5.30) in the explicit expressions for $a(k),b(k),c(k),d(k)$. □

Remark 5.1. Choosing $C=4p(k)$ one obtains a refinement of the formula (1.1) of [AKMPP] which is in turn an improvement of the original formula of Kac and Wakimoto [KW].

Indeed, recall that $(x_\theta|[u_r,u^s])=\delta_{r,s}$. As in [KW], we let $\langle\cdot,\cdot\rangle_{ne}$ be the invariant form on $\mathfrak{g}_{1/2}$ defined by setting $\langle v,w\rangle_{ne}=(x_{-\theta}|[v,w])$. Note that

$$\langle[x_\theta,u_r],[x_\theta,u^s]\rangle_{ne}=-\tfrac12\delta_{r,s}.$$

In fact,

$$\langle[x_\theta,u_r],[x_\theta,u^s]\rangle_{ne}=(x_{-\theta}|[[x_\theta,u_r],[x_\theta,u^s]])=\tfrac12(u_r|[x_\theta,u^s])=-\tfrac12(x_\theta|[u_r,u^s])$$

It follows that $\{[x_\theta,u^r]\}$ gives a basis of $\mathfrak{g}_{1/2}$ and that $\{2[x_\theta,u_s]\}$ is its dual basis.

If $u\in\mathfrak{g}_{-1/2}$ then

$$[u,[x_\theta,u_s]]^\natural=\sum_i([u,[x_\theta,u_s]]|x_i)x_i=-\sum_i(x_\theta|[[x_i,u_s],u])x_i,$$
$$[[x_\theta,u^r],v]^\natural=\sum_i([[x_\theta,u^r],v]|x_i)x_i=\sum_i(x_\theta|[[x_i,u^r],v])x_i,$$

so

$$\begin{aligned}2\sum_r :J^{[u,[x_\theta,u_r]]^\natural}J^{[[x_\theta,u^r],v]^\natural}: &= -2\sum_{i,j,r}(x_\theta|[[x_i,u_r],u])(x_\theta|[[x_j,u^r],v]):J^{x_i}J^{x_j}:\\ &= -2\sum_{i,j,r}(x_\theta|[u_r,[u,x_i]])(x_\theta|[u^r,[v,x_j]]):J^{x_i}J^{x_j}:\\ &= -2\sum_{i,j}(x_\theta|[[u,x_i],[v,x_j]]):J^{x_i}J^{x_j}:.\end{aligned}$$

Theorem 5.9. *If $\mathfrak{g}^\natural$ has one or three components, then*

$$\delta=-\frac12\left(\frac{h^\vee-\bar h_1^\vee}{2}\right)\left(\frac{\bar h_1^\vee}{2}+1\right),$$

while if $\mathfrak{g}^\natural$ has two components, then

$$\delta=-\frac12\left(\frac{h^\vee-\bar h_1^\vee}{2}\right)\left(\frac{h^\vee-\bar h_2^\vee}{2}\right)$$

and $\bar h_1^\vee+\bar h_2^\vee=h^\vee-2$. In particular, in both cases,

$$p(k)=k^2+\left(\frac{h^\vee}{2}+1\right)k-2\delta. \tag{5.33}$$

Proof. Just substitute (5.27), (5.31), (5.32), and (5.30) in formula (4.18). □

Remark 5.2. Let V be a finite dimensional $Y(\mathfrak{g})$-module. Using the (Hopf algebra) automorphism τ_u, $u \in \mathbb{C}$ defined by $\tau_u(x) = x$, $\tau_u(J(x)) = J(x) + ux$, $x \in \mathfrak{g}$, the representation V can be pulled back to give a one-parameter family of representations $V(u)$.

Recall now that the R-matrix associated to $Y(\mathfrak{g})$-modules V, W is of the form $R_{V,W}(u) = I_{V,W}(u)\sigma$, where σ is the switch automorphism and $I_{V,W}(u) : V \otimes W(u) \to W(u) \otimes V$ is the unique intertwining operator which preserves the tensor product of the highest weight vectors in V, W. Since $I_{V,W}(u)$ is a $\mathfrak{g}$-module map, it must preserve the isotypic components in $V \otimes W$. Denote by $(V \otimes W(u))_{\mathfrak{g}}$, $(W(u) \otimes V)_{\mathfrak{g}}$ the isotypic components corresponding to the adjoint representation and by $(V \otimes W(u))_0$, $(W(u) \otimes V)_0$ the isotypic components corresponding to the trivial representation. Set also

$$P_{\mathfrak{g}} = I_{V,W}(u)_{|(V\otimes W(u))_{\mathfrak{g}}}, \quad P_0 = I_{V,W}(u)_{|(V\otimes W(u))_0}.$$

Assume $\mathfrak{g}$ is not $sl(n)$. In the special case when $V = W = \mathcal{V} = \mathfrak{g} \oplus \mathbb{C}$, the adjoint representation occurs in $\mathcal{V} \otimes \mathcal{V}$ with multiplicity three and the trivial representation with multiplicity two.

As in Section 5.4 of [CP], choose the following bases for the $\mathfrak{g}$-highest weight spaces of $\mathcal{V} \otimes \mathcal{V}$ of weight θ and 0:

$$\{x_\theta \otimes 1 + 1 \otimes x_\theta, \delta[x_\theta \otimes 1, C_{\mathfrak{g}}], x_\theta \otimes 1 - 1 \otimes x_\theta\}, \qquad \{1 \otimes 1, 1 \otimes 1 - \delta C_{\mathfrak{g}}\}.$$

One the main results of [CP] is the explicit computation of the matrices of $P_{\mathfrak{g}}$ and P_0 in the bases given above. The final outcome, as far as we are concerned, is that the entries of these matrices are rational functions of u whose denominator is either $g(u) = u^2 - (\frac{h^\vee}{2} + 1)u - 2\delta = p(-u)$ (using (5.33)), or $u - 1$, or $(h^\vee/2 - u)g(u)$.

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