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Symmetry-Protected Topological Orders in Interacting Bosonic Systems

Symmetry protected topological orders in interacting bosonic systems

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Symmetry protected topological (SPT) states are bulk gapped states with gapless edge excitations protected by certain symmetries. The SPT phases in free fermion systems, like topological insulators, can be classified by the K-theory. However, it is not known what SPT phases exist in general interacting systems. In this paper, we present a systematic way to construct SPT phases in interacting bosonic systems, which allows us to identify many new SPT phases, including three bosonic versions of topological insulators in three dimension and one in two dimension protected by particle number conservation and time reversal symmetry. Just as group theory allows us to construct 230 crystal structures in 3D, we find that group cohomology theory allows us to construct different interacting bosonic SPT phases in any dimensions and for any symmetry groups. In particular, we are going to show how topological terms in the path integral description of the system can be constructed from nontrivial group cohomology classes, giving rise to exactly soluble Hamiltonians, explicit ground state wave functions and symmetry protected gapless edge excitations.

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We used to believe that different phases of matter are different because they have different symmetries.[1–3] Recently, we see a deep connection between quantum phases and quantum entanglement[4–6] which allows us to go beyond this framework. First it was realized that even in systems without any symmetry there can be distinct quantum phases – topological phases[7,8] due to different patterns of long-range entanglement in the states.[6] For systems with symmetries, difference in long-range entanglement and in symmetry still lead to distinct phases. Moreover, even short-range entangled states with the same symmetry can belong to different phases. These symmetric short-range entangled states are said to contain a new kind of order – symmetry protected topological (SPT) order.[9] The SPT phases have symmetry protected gapless edge modes despite the bulk gap, which clearly indicates the topological nature of this order. On the other hand, the gapless edge modes disappear when the symmetry of the system is broken, indicating that this is a different type of topological order than that found in fractional quantum Hall systems[10,11] whose edge modes cannot be removed with any local perturbation.[12] Also, SPT orders have no factional statistics or fractional charges, while intrinsic topological orders from long range entanglement can have them. The discovery of SPT order hence greatly expands our original understanding of possible phases in many-body systems.

One central issue is to understand what SPT phases exist and much progress has been made in this regard. The first system with SPT order was discovered decades ago in spin-1 Haldane chains. The Haldane chain with antiferromagnetic interactions was shown to have a gapped bulk[13,14] and degenerate modes at the ends of the chain[15–17] which are protected by spin rotation or time reversal symmetry of the system.[9,18] This model has been generalized, leading to a complete classification of SPT orders in one dimension.[19–22] Topological insulator[23–28] with time reversal symmetry and particle number conservation protected gapless edge modes provides the first example of SPT order in higher dimensions. The non-interacting nature of fermions in this system allows a complete classification of such kind of SPT orders.[29,30]

However, the understanding of SPT orders in free fermion systems is not enough, because particles in many-body systems do interact and interaction could dramatically change the phase diagram obtained for free fermion systems. Examples have been found in one[31] and two[32–36] dimensions where different free fermion SPT phases become the same with interactions. Moreover, new interacting SPT phases can exist which cannot be realized by free fermion systems. Therefore, to have a complete understanding of SPT order, we need to answer the following question: what SPT phases exist in general interacting systems, with topological features stable under any symmetric interaction as long as no phase transition occurs?

In this paper, we answer this question by presenting a systematic construction of SPT phases in interacting bosonic systems in any dimension and for any symmetry. The stability of the order in the constructed model can be proven under any type of symmetric interaction (at least in one and two dimension). In one dimension, our construction reproduces the classification result already known. While no bosonic SPT phases were known previously in two and higher dimensions, our construction leads to the discovery of many new SPT phases in such systems, which are summarized in table I. As listed in the first row, we find one kind of bosonic topological insulator in 2D and three kinds in 3D with boson number conservation symmetry $U(1)$ and time reversal symmetry Z_2^T. If boson numbers are allowed to fluctuate but time reversal symmetry Z_2^T is preserved, then we find one

Symmetry	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_2^T \times D_2 = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$

TABLE I: (Color online) SPT phases in d-spatial dimensions protected by some simple symmetries (represented by the symmetry groups). Here $\mathbb{Z}_1$ means that our construction only gives rise to the trivial phase. $\mathbb{Z}_n$ means that the constructed non-trivial SPT phases plus the trivial phase are labeled by elements in $\mathbb{Z}_n$. Z_2^T represents time reversal symmetry, $U(1)$ represents boson number conservation symmetry, Z_n represents cyclic symmetry, *etc* . The red row is for bosonic topological insulators and the blue row is for bosonic topological superconductors.

kind of bosonic topological superconductor in every odd spatial dimension, as listed in the second row. More generally, for systems in d spatial dimension and with symmetry of group G, we are going to present a way to write down a quantized topological term in the path integral in $d+1$ dimensional space-time based on the nontrivial group cohomology of G. From the path integral, we can find the ground state wave function, identify the gapless edge states and understand how symmetry protects the gaplessness of the edge and hence the SPT order even against strong interactions. For simplicity, we are going to first present the construction in detail for the one dimensional Haldane chain, demonstrate the emergence of its SPT order from the nontrivial group cohomology of time reversal symmetry and then generalize to higher dimensions and to all other symmetries.

The fixed point ground state wave-function of the Haldane chain[9] takes a simple dimer form as shown in Fig.1, where each site contains two spin $1/2$'s which are connected into singlet pairs $|\uparrow_i^r \downarrow_{i+1}^l\rangle - |\downarrow_i^r \uparrow_{i+1}^l\rangle$ between neighboring sites.[46] Time reversal symmetry acts as $M(\mathcal{T}) = i\sigma_y K$ on each spin $1/2$, where K is complex conjugation and σ_y is the y component of the spin operator. The wave-function is invariant under the symmetry action. Note that for each spin $1/2$ $M(\mathcal{T})^2 = -I$ while on each site with two spins $(M(\mathcal{T}) \otimes M(\mathcal{T}))^2 = I$. So the states on each site form a representation of Z_2^T, the symmetry group generated by time reversal symmetry.

FIG. 1: (Color online)Dimer form of the ground state wave-function in Haldane chain. Each site (big oval) contains two spin $1/2$'s (small dot), which are connected into singlet pairs (connected dots) between neighboring sites.

The wave-function on a closed chain is the gapped ground states of anti-ferromagnetic Heisenberg interactions between each pair of spin $1/2$'s, $H = \sum_i \boldsymbol{\sigma}_i^r \cdot \boldsymbol{\sigma}_{i+1}^l$ where $\boldsymbol{\sigma}_i^l$ and $\boldsymbol{\sigma}_i^r$ are spin operators for the left and right spin $1/2$ on each site respectively. Obviously, the Hamiltonian is time reversal invariant. Therefore we have a situation where the ground state does not break any symmetry of the system, yet the system is far from a trivial phase which becomes evident when we put the system on an open chain. When the chain is open, the dangling spin $1/2$ at each end forms a nontrivial projective representation of Z_2^T with $M(\mathcal{T})^2 = -I$, which does not allow a one-dimensional representation (for definition of projective representations, see appendix A). Therefore, the degeneracy of the edge state is robust under any perturbation as long as time reversal symmetry is preserved.

Indeed, the nontrivial projective representation on the edge is the key to the existence of SPT order in the Haldane chain and applies in general to all one dimensional bosonic SPT phases.[19,22] A generic way to write projective representations of group G is to define symmetry operations $M(g)(g \in G)$ on group element labeled states $|g_0\rangle (g_0 \in G)$ as

$$M(g)|g_0\rangle = \nu_2^{s(g)}(g_0, g^{-1}g^*, g^*)|gg_0\rangle, g^* = E \quad (1)$$

where the 2-cocycle $\nu_2(g_0, g_1, g_2)$ is a function from three group elements to a $U(1)$ phase factor satisfying

$$\nu_2^{s(g)}(g_0, g_1, g_2) = \nu_2(gg_0, gg_1, gg_2), \ g \in G \quad (2)$$

$$\text{and} \quad \frac{\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)} = 1 \quad (3)$$

$s(g) = 1$ if g is unitary and $s(g) = -1$ if g is antiunitary. (We prove this in appendix A.) The projective representation on the edge of the Haldane chain is of exactly this form if we relabel the spin states with group elements. The time reversal symmetry group contains two elements $Z_2^T = \{E, \mathcal{T}\}$ with $\mathcal{T} \circ \mathcal{T} = E$. For the left spin $1/2$ on each site, label $|\uparrow\rangle/|\downarrow\rangle$ as $|E\rangle/|\mathcal{T}\rangle$ and for the right one, label $|\uparrow\rangle/|\downarrow\rangle$ as $|\mathcal{T}\rangle/-|E\rangle$. The total wave-function becomes

$$|\Phi_{dimer}\rangle = \prod_i |\mathcal{T}_i^r \mathcal{T}_{i+1}^l\rangle + |E_i^r E_{i+1}^l\rangle \quad (4)$$

Time reversal symmetry on the edge spin then acts as $M(\mathcal{T})|E\rangle = -|\mathcal{T}\rangle$ and $M(\mathcal{T})|\mathcal{T}\rangle = |E\rangle$, which indeed takes the form in Eq.(1) with the 2-cocycle ν_2 of time reversal symmetry given by

$$\nu_2(E, \mathcal{T}, E) = \nu_2(\mathcal{T}, E, \mathcal{T}) = -1 \quad (5)$$
$$\text{all other } \nu_2(g_0, g_1, g_2) = 1, \ g_i \in Z_2^T$$

This essential feature of 1D SPT phases can be reproduced from a simple construction of the low energy effective path integral as we show in the following for the Haldane phase. The path integral formulation not only provides understanding of the nontrivial physics from a
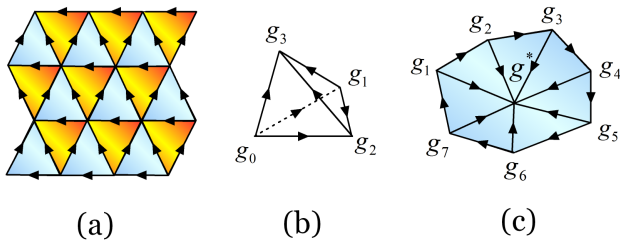
FIG. 2: (Color online) (a) A branched triangularization of space-time. (For details of braching see appendix D) (b) A tetrahedron – the simplest discrete closed surface. $\prod \nu^{s_{ijk}}(g_i, g_j, g_k) = 1$ on a tetrahedron is guaranteed by Eq.(3). Note that $s_{123} = s_{013} = 1$ and $s_{023} = s_{012} = -1$. (c) Discretized space-time manifold M_{ext} on an open disk with boundary manifold M. $g_i \in M$, g^* is in the interior of M_{ext}.

bulk perspective but also forms the basis for the generalization of SPT phases into higher dimensions.

To write the path integral for disordered phases, we first realize that the field of the system fluctuates strongly at all length scales and the low energy effective theory has no continuous limit. Therefore, we choose to write the path integral on discrete space-time. Imagine that we discretize the $(1+1)D$ space time of the Haldane chain with a branched triangularization as shown in Fig.2(a). On each vertex of the space-time complex, we put a $g_i \in Z_2^T$. Time reversal acts as complex conjugation K together with a mapping from g_i to $\mathcal{T}g_i$. The path integral is then written as

$$Z = |G|^{-N_v} \sum_{\{g_i\}} e^{-S(\{g_i\})}$$

$$e^{-S(\{g_i\})} = \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k) \qquad (6)$$

$|G|$ is the number of elements in G($|G| = 2$ for the time reversal symmetry), N_v is the number of vertices in the complex. Also, $s_{ijk} = \pm 1$ depending on the orientation of the triangle. Compared to the continuous formulation of path integral $Z = \int Dg \ e^{-\int d\boldsymbol{x} d\tau \ \mathcal{L}[g(\boldsymbol{x},\tau)]}$ g_i corresponds to the field $g(\boldsymbol{x}, t)$ and $\sum_{\{g_i\}}$ corresponds to the path integral $\int Dg$. $e^{-S(\{g_i\})}$ is the action-amplitude on the discretized space-time that corresponds to $e^{-\int d^d\boldsymbol{x} d\tau \ \mathcal{L}[g(\boldsymbol{x},\tau)]}$ of the continuous formulation and $\nu_2^{s_{ijk}}(g_i, g_j, g_k)$ corresponds to the action-amplitude $e^{-\int_{(i,j,k)} d\boldsymbol{x} d\tau \ \mathcal{L}[g(\boldsymbol{x},\tau)]}$ on a single triangle (i, j, k). The above term can be formally regarded as the discrete symmetry group and discrete space time generalization of topological θ-term for an $O(3)$ non-linear sigma model.(For details see appendix C.)

From the properties of $\nu_2(g_0, g_1, g_2)$, it can be checked that the path integral as defined is symmetric under time reversal symmetry with an action amplitude that is in a fixed point form and is always equal to 1 on a closed surface. First because $\nu_2(\mathcal{T}g_0, \mathcal{T}g_1, \mathcal{T}g_2) = \nu_2^{-1}(g_0, g_1, g_2)$ (see Eq.(2)), the path integral is invariant under time reversal. Secondly, Eq.(3) gives rise to a renormaliza-

tion flow under which the form of the action amplitude remains invariant (for details see appendix D). Finally, Eq.(3) also guarantees that the action amplitude on a closed space time surface is always 1. This can be easily verified on the simplest discrete closed surface – a tetrahedron, as shown in Fig.2(b). More complicated closed surfaces are obtained by putting tetrahedrons together, hence the action amplitude will always be 1. Therefore, we have constructed a quantized topological term for the path integral description of $(1+1)D$ systems with time reversal symmetry on discrete space time.

But how do we know that this quantized topological term describes the SPT order in Haldane chain? This can be made explicit by finding the ground state wave-function of the system from the quantized topological term. The ground state wave-function can be obtained by imaginary time evolution from time $-\infty$ until time 0. In our formulation, this is equivalent to imaginary time path integral on a space-time geometry with a boundary (at time 0). Denote the boundary as M and the whole manifold (a disk) as M_{ext}. As we are considering a fixed point path integral, it does not matter how big the interior of M_{ext} is and we can reduce it, for example, to just one point as shown in Fig.2 (c).

To obtain the ground state wave-function, we fix the degrees of freedom $\{g_i\}_M$ on M and choose an arbitrary configuration, for example $g^* = E$, for degrees of freedom in the interior of M_{ext}. We find

$$\Psi(\{g_i\}_M) = \prod_i \nu_2(g_i, g_{i+1}, g^*) \qquad (7)$$

where $\prod_i$ is product over all triangles on M_{ext} and for simplicity of notation we have chosen all triangles to be oriented clockwise. The wave-function on M does not depend on the choice of g^*. Note that time reversal acts as complex conjugate K together with a change of basis $|E\rangle \to |\mathcal{T}\rangle$, $|\mathcal{T}\rangle \to |E\rangle$ on each g_i and the wave-function is invariant under this action.
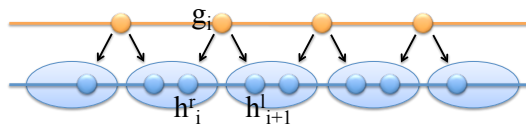

FIG. 3: (Color online) Duality transformation between wave-functions in Eq.(4) and Eq.(7).

This form of wave-function does not look exactly the same as the dimer form shown in Fig.1, but it will be after a duality transformation as shown in Fig.3. First expand each g_i into two degrees of freedom h_i^r and h_{i+1}^l such that $h_i^r = h_{i+1}^l = g_i$ and the amplitude of each configuration in the wave-function remains unchanged, $\Phi(\{h_i^r = h_{i+1}^l = g_i\}) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$. Then combine h_i^l and h_i^r into one site and apply a change of basis

on each site

$$|h_i^l, h_i^r\rangle' = \nu_2(h_i^l, h_i^r, g^*)|h_i^l, h_i^r\rangle = \nu_2(g_{i-1}, g_i, g^*)|h_i^l, h_i^r\rangle$$

The amplitude of all configurations in the new basis becomes 1, $\Phi'(\{h_i^r = h_{i+1}^l = g_i\}) = 1$ which can be equivalently written as a product of dimers between neighboring sites $|\Phi\rangle' = \prod_i \sum_{g_i} |h_i^r = g_i, h_{i+1}^l = g_i\rangle$. In this way we have mapped each degree of freedom g_i into a dimer and the total wave-function takes the same form as Eq.(4).

It is also important to check that time reversal symmetry acts as in Eq.(1) on the edge degree of freedom in the ground state. To see this, note that after the duality transformation, time reversal symmetry acts separately on each site and is composed of complex conjugation K together with unitary transformation on the basis states

$$
\begin{aligned}
M_{lr}(g)|h_i^l, h_i^r\rangle' &= \frac{\nu_2^{s(g)}(h_i^l, h_i^r, g^*)}{\nu_2(gh_i^l, gh_i^r, g^*)}|gh_i^l, gh_i^r\rangle' \\
&= \frac{\nu_2^{s(g)}(h_i^l, g^{-1}g^*, g^*)}{\nu_2^{s(g)}(h_i^r, g^{-1}g^*, g^*)}|gh_i^l, gh_i^r\rangle'
\end{aligned}
$$

which factorizes on the two degrees of freedom and acts in the same way as in Eq.(1) on the edge. Therefore, our quantized topological term Eq.(6) provides a proper description of the SPT order in the Haldane chain.

The advantage of reformulating the Haldane chain in the path integral language is that it can be easily generalized to all spatial dimensions d and all symmetry groups G. The two cocycles $\nu_2(g_0, g_1, g_2)$ used in the construction have higher dimensional analogues – the $(d+1)$-cocycles $\nu_{d+1}(g_0, ..., g_{d+1})$, which are maps from $d+2$ group elements to a $U(1)$ phase factor and satisfy

$$\nu_{d+1}^{s(g)}(g_0, g_1, ..., g_{d+1}) = \nu_{d+1}(gg_0, gg_1, ..., gg_{d+1}), g \in G$$

$$\text{and} \quad \prod_{i=0}^{d+2} \nu_{d+1}^{(-1)^i}(g_0, .., g_{i-1}, g_{i+1}, ..., g_{d+2}) = 1$$

A quantized topological term in the path integral of a $(d+1)D$ system can be written in terms of the $(d+1)$-cocycles by (1) discretizing the $(d+1)D$ space time with triangularization (triangle in $(1+1)D$ and tetrahedron in $(2+1)D$ etc); (2) putting group element labeled degrees of freedom onto the vertices (3) assigning action amplitude to each simplex with the corresponding cocycle. The path integral then takes the form

$$Z = |G|^{-N_v} \sum_{\{g_i\}} \prod_{\{ij...k\}} \nu_{d+1}^{s_{ij...k}}(g_i, g_j, ..., g_k) \quad (8)$$

where $s_{ij...k} = \pm 1$ depending on the orientation of the simplex $ij...k$. Similar to the $(1+1)D$ case, it can be shown that the path integral is symmetric under symmetries in group G, the action amplitude $e^{-S(\{g_i\})} = \prod_{\{ij...k\}} \nu_{d+1}^{s_{ij...k}}(g_i, g_j, ..., g_k)$ is in a fixed point form and is quantized to 1 on a closed manifold.

The ground state wave-function can be obtained from the action amplitude on a open geometry as discussed before $\Psi(\{g_i\}_M) = \prod_{\{i...j*\}} \nu_2(g_i, ..., g_j, g^*)$, where $\{g_i\}_M$ is on M and g^* is inside M_{ext}. $\prod_{\{i...j*\}}$ is the product over all simplices. An exactly soluble Hamiltonian can be constructed to realize this state as the gapped ground state. To see this, note that Ψ can be mapped through a local unitary transformation $U = \prod_{\{i...j*\}} \nu_2^{-1}(g_i, ..., g_j, g^*)|g_i...g_j\rangle\langle g_i...g_j|$ to a total product state $\Psi_0(\{g_i\}_M) = 1$. As Ψ_0 is the gapped ground state of $H_0 = -\sum_k |\phi_k\rangle\langle\phi_k|$, $|\phi_k\rangle = \sum_{g_i \in G} |g_i\rangle$, Ψ is the gapped ground state of $H = U^\dagger H_0 U = \sum_k -U^\dagger|\phi_k\rangle\langle\phi_k|U$, which is local and has symmetry G.

The nontrivial SPT order of the system can be seen explicitly from its boundary. The path integral of degrees of freedom on the boundary can be obtained by putting the topological term on an open geometry as shown in Fig.2 (c) for $(1+1)D$. Note that the manifold M now corresponds to the space time manifold of the boundary degrees of freedom. The path integral for the boundary then reads

$$Z_b = |G|^{-N_v^M} \sum_{\{g_i\}_M} \prod_{\{i...j*\}} \nu_{d+1}^{s_{i...j*}}(g_i, .., g_j, g^*) \quad (9)$$

where $\{g_i\}_M$ is the field on M and g^* is inside M_{ext}. The action amplitude for each $\{g_i\}_M$ configuration does not depend on the choice of g^*.

This term can be thought of as a discretized version of the Wess-Zumino-Witten(WZW) term[37,38] in nonlinear σ-models because 1. it is a path integral of $(d-1)+1$ dimensional systems written on an extended $(d+1)D$ manifold with a boundary 2. the action amplitude does not depend on how the extended field in the interior of the $(d+1)D$ manifold is chosen 3. its field takes value in a group G and the path integral is invariant under the action of $g \in G$. On the other hand, this term is more general than the original continuous WZW term because it applies to discrete groups like Z_2^T while the continuous WZW term only works for continuous groups. We expect that the boundary theory described by such a discretized WZW term will be gapless as long as symmetry is not broken, similar to systems described by continuous WZW terms. Actually this has been firmly established in $(1+1)D$ and $(2+1)D$. In $(1+1)D$, as we have discussed with the example of time reversal symmetry, symmetry action on the edge degree of freedom does not have a one-dimensional representation, therefore the edge state will always be degenerate. In $(2+1)D$, we have proved rigorously using the tool of matrix product unitary operator that the symmetry action on the boundary requires it to be gapless as long as symmetry is not broken.[39,40] Therefore, the boundary of the systems we constructed carries gapless excitations protected by certain symmetry, which reflects the nontrivial SPT order of the system.

In conclusion, we have presented a systematic construction of symmetry protected topological phases in bosonic systems in any spatial dimension d and with any

symmetry of group G. In particular, we wrote down a quantized topological term in the path integral of the system based on nontrivial $(d+1)$-cocycles of the symmetry group. From the quantized topological term we found that the ground state wave-function is short range entangled and the boundary of the systems carries gapless excitations protected by the symmetry of the system, which clearly demonstrates the nontrivial SPT order of the system. The number of nontrivial cocycles, hence the number of nontrivial SPT phases, for some simple symmetry groups are summarized in table I. Our construction is non-perturbative and works for strongly interacting bosonic systems.

Appendix A: Projective Representation

Matrices $u(g)$ form a projective representation of symmetry group G if

$$u(g_1)u(g_2) = \omega(g_1,g_2)u(g_1g_2), \qquad g_1,g_2 \in G. \quad \text{(A1)}$$

Here $\omega(g_1,g_2) \in U(1)$ and $\omega(g_1,g_2) \neq 1$, which is called the factor system of the projective representation. The factor system satisfies

$$\omega^{s(g_1)}(g_2,g_3)\omega(g_1,g_2g_3) = \omega(g_1,g_2)\omega(g_1g_2,g_3), \quad \text{(A2)}$$

for all $g_1,g_2,g_3 \in G$, where $s(g_1) = 1$ if g_1 is unitary and $s(g_1) = -1$ if g_1 is anti-unitary. If $\omega(g_1,g_2) = 1$, $\forall g_1,g_2$, this reduces to the usual linear representation of G.

A different choice of pre-factor for the representation matrices $u'(g) = \beta(g)u(g)$ will lead to a different factor system $\omega'(g_1,g_2)$:

$$\omega'(g_1,g_2) = \frac{\beta(g_1)\beta^{s(g_1)}(g_2)}{\beta(g_1g_2)}\omega(g_1,g_2). \quad \text{(A3)}$$

We regard $u'(g)$ and $u(g)$ that differ only by a pre-factor as equivalent projective representations and the corresponding factor systems $\omega'(g_1,g_2)$ and $\omega(g_1,g_2)$ as belonging to the same class ω.

Suppose that we have one projective representation $u_1(g)$ with factor system $\omega_1(g_1,g_2)$ of class ω_1 and another $u_2(g)$ with factor system $\omega_2(g_1,g_2)$ of class ω_2, obviously $u_1(g) \otimes u_2(g)$ is a projective presentation with factor system $\omega_1(g_1,g_2)\omega_2(g_1,g_2)$. The corresponding class ω can be written as a sum $\omega_1 + \omega_2$. Under such an addition rule, the equivalence classes of factor systems form an Abelian group, which is called the second cohomology group of G and is denoted as $\mathcal{H}^2[G,U(1)]$. The identity element $1 \in \mathcal{H}^2[G,U(1)]$ is the class that corresponds to the linear representation of the group.

$\omega(g_1,g_2)$ can be equivalently expressed as $\nu_2(\tilde{g}_0,\tilde{g}_1,\tilde{g}_2)$ with correspondence

$$\omega(g_1,g_2) = \nu_2(E,g_1,g_1g_2) \quad \text{(A4)}$$

where E is the identity element in the symmetry group G. $\nu_2(g_0,g_1,g_2)$ satisfies

$$\nu_2^{s(g)}(g_0,g_1,g_2) = \nu_2(gg_0,gg_1,gg_2), \ g \in G \quad \text{(A5)}$$

and the condition Eq.(A2) satisfied by $\omega(g_1,g_2)$ becomes

$$\frac{\nu_2(g_1,g_2,g_3)\nu_2(g_0,g_1,g_3)}{\nu_2(g_0,g_2,g_3)\nu_2(g_0,g_1,g_2)} = 1 \quad \text{(A6)}$$

Similarly, $\beta(g_1)$ can be equivalently written as $\nu_1(E,g_1)$ where

$$\nu_1^{s(g)}(g_0,g_1) = \nu_1(gg_0,gg_1) \quad \text{(A7)}$$

The equivalence relation Eq.(A3) between $\omega(g_1,g_2)$'s then becomes

$$\nu_2'(g_0,g_1,g_2) = \frac{\nu_1(g_0,g_1)\nu_1(g_1,g_2)}{\nu_1(g_0,g_2)}\nu(g_0,g_1,g_2). \quad \text{(A8)}$$

In the following we will show $M(g)$ defined in Eq.(1) forms a projective representation of symmetry group G. It is easy to check that:

$$\begin{aligned}
&M(g_1)M(g_2)|g_0\rangle \\
=&M(g_1)\nu_2^{s(g_2)}(g_0,g_2^{-1}g^*,g^*)|g_2g_0\rangle \\
=&\nu_2^{s(g_1)}(g_2g_0,g_1^{-1}g^*,g^*)\nu_2^{s(g_1g_2)}(g_0,g_2^{-1}g^*,g^*)|g_1g_2g_0\rangle
\end{aligned} \quad \text{(A9)}$$

with $g^* = E$. We note that in the last line, the sign factor $s(g_1g_2)$ arises because the complex conjugate K could have a nontrivial action on $\nu_2^{s(g_2)}(g_0,g_2^{-1}g^*,g^*)$ if g_1 is an antiunitary. According to the 2-cocycle condition Eq. (A6), we have:

$$\begin{aligned}
&M(g_1)M(g_2)|g_0\rangle \\
=&\nu_2(g^*,g_1g^*,g_1g_2g^*)\nu_2(g_1g_2g_0,g^*,g_1g_2g^*)|g_1g_2g_0\rangle \\
=&\nu_2(g^*,g_1g^*,g_1g_2g^*)\nu_2^{s(g_1g_2)}(g_0,(g_1g_2)^{-1}g^*,g^*)|g_1g_2g_0\rangle \\
=&\nu_2(E,g_1,g_1g_2)M(g_1g_2)|g_0\rangle \\
=&\omega_2(g_1,g_2)M(g_1g_2)|g_0\rangle
\end{aligned} \quad \text{(A10)}$$

Now it is clear that $M(g)$ defined in Eq.(1) forms a projective representation of the symmetry group G. However, $M(g)$ is usually reducible. It can be reduced to a direct sum of several irreducible projective representations which belong to the same class. Only irreducible projective representations describe the edge states of general 1D SPT phases.

Appendix B: Group cohomology

The above discussion on the factor system of a projective representation can be generalized which gives rise to a cohomology theory of groups. In this section, we will briefly describe the group cohomology theory.

For a group G, let M be a G-module, which is an abelian group (with multiplication operation) on which G acts compatibly with the multiplication operation (*ie* the abelian group structure) on M:

$$g \cdot (ab) = (g \cdot a)(g \cdot b), \quad g \in G, \quad a, b \in M. \quad (B1)$$

For the cases studied in this paper, M is simply the $U(1)$ group and a a $U(1)$ phase. The multiplication operation ab is the usual multiplication of the $U(1)$ phases. The group action is trivial $g \cdot a = a$ ($g \in G$, $a \in U(1)$) if g is unitary and $g \cdot a = a^*$ if g is anti-unitary.

Let $\nu_n(g_0, ..., g_n)$ be a function of $(n+1)$ group elements whose value is in the G-module $U(1)$. In other words, $\nu_n : G^{(n+1)} \to U(1)$. ν_n satisfies

$$\nu_n^{s(g)}(g_0, g_1, ..., g_n) = \nu_n(gg_0, gg_1, ..., gg_n), g \in G \quad (B2)$$

We will call such a map ν_n an n-cochain: Let $\mathcal{C}^n(G, U(1)) = \{\nu_n\}$ be the space of all n-cochains. Note that $\mathcal{C}^n(G, U(1))$ is an Abelian group under the function multiplication $\nu_n''(g_0, ..., g_n) = \nu_n(g_0, ..., g_n)\nu_n'(g_0, ..., g_n)$. We define a map d_n from $\mathcal{C}^n[G, U(1)]$ to $\mathcal{C}^{n+1}[G, U(1)]$:

$$(d_n \nu_n)(g_0, g_1, ..., g_{n+1})$$
$$= \prod_{i=0}^{n+1} \nu_n^{(-1)^i}(g_0, .., g_{i-1}, g_{i+1}, ..., g_{n+1}) \quad (B3)$$

Let

$$\mathcal{B}^n(G, U(1)) = \{\nu_n | \nu_n = d_{n-1}\nu_{n-1}, \nu_{n-1} \in \mathcal{C}^{n-1}(G, U(1))\}$$

and

$$\mathcal{Z}^n(G, U(1)) = \{\nu_n | d_n \nu_n = 1, \nu_n \in \mathcal{C}^n(G, U(1))\}$$

$\mathcal{B}^n(G, U(1))$ and $\mathcal{Z}^n(G, U(1))$ are also Abelian groups which satisfy $\mathcal{B}^n(G, U(1)) \subset \mathcal{Z}^n(G, U(1))$ where $\mathcal{B}^1(G, U(1)) \equiv \{1\}$. The n-cocycle of G is defined as

$$\mathcal{H}^n(G, U(1)) = \mathcal{Z}^n(G, U(1))/\mathcal{B}^n(G, U(1)) \quad (B4)$$

Let us discuss some simple cases. From

$$(d_1 \nu_1)(g_0, g_1, g_2) = \nu_1(g_0, g_1)\nu_1(g_1, g_2)/\nu_1(g_0, g_2)$$

we see that

$$\mathcal{Z}^1(G, U(1)) = \{\nu_1 | \nu_1(g_0, g_1)\nu_1(g_1, g_2) = \nu_1(g_0, g_2)\}.$$

Since $\mathcal{B}^1(G, U(1)) \equiv \{1\}$ is trivial, $\mathcal{H}^1(G, U(1)) = \mathcal{Z}^1(G, U(1))$. If we define $\alpha(g) = \nu_1(E, g)$, it is easy to see that the first-cocycle is related to the one-dimensional representations of the group G.

From

$$(d_2 \nu_2)(g_0, g_1, g_2, g_3)$$
$$= \nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)/\nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)$$

we see that

$$\mathcal{Z}^2(G, U(1)) = \{\nu_2 |$$
$$\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3) = \nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)\}.$$

and

$$\mathcal{B}^2(G, U(1)) = \{\nu_2 |$$
$$\nu_2(g_0, g_1, g_2) = \nu_1(g_0, g_1)\nu_1(g_1, g_2)/\nu_1(g_0, g_2)\}.$$

The 2-cocycles $\mathcal{H}^2(G, U(1)) = \mathcal{Z}^2(G, U(1))/\mathcal{B}^2(G, U(1))$ classify the projective representations discussed in section A.

From

$$(d_3 \nu_3)(g_0, g_1, g_2, g_3, g_4)$$
$$= \frac{\nu_3(g_1, g_2, g_3, g_4)\nu_3(g_0, g_1, g_3, g_4)\nu_3(g_0, g_1, g_2, g_3)}{\nu_3(g_0, g_2, g_3, g_4)\nu_3(g_0, g_1, g_2, g_4)} \quad (B5)$$

we see that

$$\mathcal{Z}^3(G, U(1)) = \{\nu_3 |$$
$$\frac{\nu_3(g_1, g_2, g_3, g_4)\nu_3(g_0, g_1, g_3, g_4)\nu_3(g_0, g_1, g_2, g_3)}{\nu_3(g_0, g_2, g_3, g_4)\nu_3(g_0, g_1, g_2, g_4)} = 1\}.$$

and

$$\mathcal{B}^3(G, U(1)) = \{\nu_3 |$$
$$\nu_3(g_0, g_1, g_2, g_3) = \frac{\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)}\}$$

which give us the 3-cocycle $\mathcal{H}^3(G, U(1)) = \mathcal{Z}^3(G, U(1))/\mathcal{B}^3(G, U(1))$.

Appendix C: Relationship with the topological-θ term of the $O(3)$ non-linear sigma model

In the section, we will show the amplitudes Eq.(6) can be formally regarded as the discrete group and discrete space time generalization of topological θ term of non-linear sigma model. We will first review that the Haldane phase (a non-trivial 1D SPT phase) is described by a 2π-quantized topological term in continuous non-linear σ-model.[41] However, such kind of 2π-quantized topological terms cannot describe more general 1D SPT phases. We argue that to describe SPT phases correctly, we must generalize the 2π-quantized topological terms to discrete space-time.

Before considering a spin-1 chain, let us first consider a $(0+1)$D non-linear σ-model that describes a single spin, whose imaginary-time action is given by

$$S = \oint dt \frac{1}{2g}(\partial_t \boldsymbol{n}(t))^2$$
$$+ is \int_{D^2} dt \, d\xi \, \boldsymbol{n}(t, \xi) \cdot [\partial_t \boldsymbol{n}(t, \xi) \times \partial_\xi \boldsymbol{n}(t, \xi)] \quad (C1)$$

where $\boldsymbol{n}(t)$ is an unit 3d vector and we have assumed that the time direction form a circle. The second term is the Wess-Zumino-Witten (WZW) term.[37,38] We note that the WZW term cannot be calculated from the field $\boldsymbol{n}(t)$ on the time-circle. We have to extend $\boldsymbol{n}(t)$ to a disk D^2 bounded by the time-circle: $\boldsymbol{n}(t) \rightarrow \boldsymbol{n}(t,\xi)$ (see Fig. 4). Then the WZW term can be calculated from $\boldsymbol{n}(t,\xi)$. When $2s$ is an integer, WZW terms from different extensions only differ by a multiple of $2\mathrm{i}\pi$. So e^{-S} is determined by $\boldsymbol{n}(t)$ and is independent of how we extend $\boldsymbol{n}(t)$ to the disk D^2.
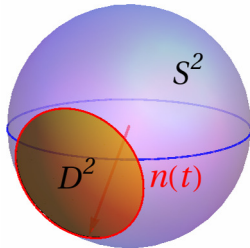

FIG. 4: If we extend $\boldsymbol{n}(t)$ that traces out a loop to $\boldsymbol{n}(t,\xi)$ that covers the shaded disk, then the WZW term $\int_{D^2} \mathrm{d}t\,\mathrm{d}\xi\, \boldsymbol{n}(t,\xi) \cdot [\partial_t \boldsymbol{n}(t,\xi) \times \partial_\xi \boldsymbol{n}(t,\xi)]$ corresponds to the area of the disk.

The ground states of the above non-linear σ-model have $2s+1$ fold degeneracy, which form the spin-s representation of $SO(3)$. The energy gap above the ground state approaches to infinite as $g \rightarrow \infty$. Thus a pure WZW term describes a pure spin-s spin.
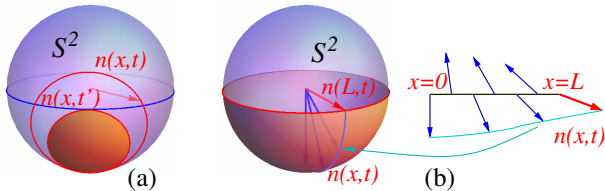

FIG. 5: (a) The topological term W describes the number of times that $\boldsymbol{n}(x,t)$ wraps around the sphere (as we change t). (b) On an open chain $x \in [0,L]$, the topological term W in the (1+1)D bulk becomes the WZW term for the end spin $\boldsymbol{n}_L(t) = \boldsymbol{n}(L,t)$ (where the end spin at $x=0$ is hold fixed).

To obtain the action for the $SO(3)$ symmetric antiferromagnetic spin-1 chain, we can assume that the spins $\boldsymbol{S}_i$ are described by a smooth unit vector field $\boldsymbol{n}(x,t)$: $\boldsymbol{S}_i = (-)^i \boldsymbol{n}(ia,t)$ (see Fig. 5b). Putting the above single-spin action for different spins together, we obtain the following (1+1)D non-linear σ-model[42]

$$S = \int \mathrm{d}x\,\mathrm{d}t\, \frac{1}{2g}(\partial \boldsymbol{n}(x,t))^2 + \mathrm{i}\theta W, \quad \theta = 2\pi, \quad (\text{C2})$$

where $W = (4\pi)^{-1} \int \mathrm{d}t\,\mathrm{d}x\, \boldsymbol{n}(t,x) \cdot [\partial_t \boldsymbol{n}(t,x) \times \partial_x \boldsymbol{n}(t,x)]$ and $\mathrm{i}\theta W$ is the topological term.[42] If the space-time manifold has no boundary, then $\mathrm{e}^{-\mathrm{i}\theta W} = 1$ when $\theta =$

0 mod 2π. We will call such a topological term – a 2π-quantized topological term. The above non-linear σ-model describes a gapped phase with short range correlation and the $SO(3)$ symmetry, which is the Haldane phase.[13] In the low energy limit, g flows to infinity and the fixed-point action contains only the 2π-quantized topological term. Such a non-linear σ-model will be called topological non-linear σ-model.

It appears that the 2π-quantized topological term has no contribution to the path integral and can be dropped. In fact, the 2π-quantized topological term has physical effects and cannot be dropped. On an open chain, the 2π-quantized topological term $2\pi\mathrm{i}W$ becomes a WZW term for the boundary spin $\boldsymbol{n}_L(t) \equiv \boldsymbol{n}(x=L,t)$ (see Fig. 5).[41] The motion of $\boldsymbol{n}_L$ is described by Eq.(C1) with $s=1/2$. So the Haldane phase of spin-1 chain has a spin-1/2 boundary spin at each chain end![15,16,41]

We see that the Haldane phase is described by a fixed point action which is a topological non-linear σ-model containing only the 2π-quantized topological term. *The non-trivialness of the Haldane phase is encoded in the non trivially quantized topological term.*[41,43] From this example, one might guess that various SPT phases can be classified by various topological non-linear σ-models, and thus by various 2π-quantized topological terms. But such a guess is not correct.

This is because the fixed-point action (the topological non-linear σ-model) describes a short-range-correlated state. Since the renormalized cut-off length scale of the fixed-point action is always larger than the correlation length, the field $\boldsymbol{n}(x,t)$ fluctuates strongly even at the cut-off length scale. Thus, the fixed-point action has no continuum limit, and must be defined on discrete space-time. On the other hand, in our fixed-point action, $\boldsymbol{n}(x,t)$ is assumed to be a continuous field in space-time. The very existence of the continuum 2π-quantized topological term depends on the non-trivial mapping classes from the *continuous* space-time manifold T^2 to the *continuous* target space S^2. It is not self consistent to use such a continuum topological term to describe the fixed-point action for the Haldane phase.

As a result, the continuum 2π-quantized topological terms fail to classify bosonic SPT phases. For example, different possible continuum 2π-quantized topological terms in Eq.(C2) are labeled by integers, while the integer spin chain has only two gapped phases protected by spin rotation symmetry: all even-integer topological terms give rise to the trivial phase and all odd-integer topological terms give rise to the Haldane phase. Also non-trivial SPT phases may exist even when there is no continuum 2π-quantized topological terms (such as when the symmetry G is discrete).

However, the general idea of using fixed-point actions to classify SPT phases is still correct. But, to use 2π-quantized topological terms to describe bosonic SPT phases, we need to generalize them to discrete space-time. In the following, we will show that this indeed can be done, using the (1+1)D model (C2) as an example.

A discrete (1+1)D space-time is given by a branched triangularization (see Fig. 2 (a)). Since $S = \int \mathrm{d}x \, \mathrm{d}t L$, on triangularized space-time, we can rewrite

$$e^{-S} = \prod \nu^{s(i,j,k)}(\boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_k),$$

$$\nu^{s(i,j,k)}(\boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_k) = e^{-\int_\triangle \mathrm{d}x \, \mathrm{d}t \, L} \in U(1), \qquad \text{(C3)}$$

where $\int_\triangle \mathrm{d}x \, \mathrm{d}t \, L$ is the action on a single triangle. We see that, on discrete space-time, the action and the path integral are described by a 3-variable function $\nu(\boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_k)$, which is called action amplitude. The $SO(3)$ symmetry requires that

$$\nu(g\boldsymbol{n}_i, g\boldsymbol{n}_j, g\boldsymbol{n}_k) = \nu(\boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_k), \quad g \in SO(3). \quad \text{(C4)}$$

In order to use the action amplitude $\nu^{s(i,j,k)}(\boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_k)$ to describe a 2π-quantized topological term, we must have $\prod \nu^{s(i,j,k)}(\boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_k) = 1$ on any sphere. This can be satisfied iff $\prod \nu^{s(i,j,k)}(\boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_k) = 1$ on a tetrahedron – the simplest discrete sphere (See Fig. 2(b)):

$$\frac{\nu(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)\nu(\boldsymbol{n}_0, \boldsymbol{n}_1, \boldsymbol{n}_3)}{\nu(\boldsymbol{n}_0, \boldsymbol{n}_2, \boldsymbol{n}_3)\nu(\boldsymbol{n}_0, \boldsymbol{n}_1, \boldsymbol{n}_2)} = 1. \qquad \text{(C5)}$$

(Another way to define topological term on discretized space-time can be found in Ref. 44.)

A $\nu(\boldsymbol{n}_0, \boldsymbol{n}_1, \boldsymbol{n}_2)$ that satisfies Eq.(C4) and Eq.(C5) is called a 2-cocycle. If $\nu(\boldsymbol{n}_0, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is a 2-cocycle, then

$$\nu'(\boldsymbol{n}_0, \boldsymbol{n}_1, \boldsymbol{n}_2) = \nu(\boldsymbol{n}_0, \boldsymbol{n}_1, \boldsymbol{n}_2)\frac{\mu(\boldsymbol{n}_1, \boldsymbol{n}_2)\mu(\boldsymbol{n}_0, \boldsymbol{n}_1)}{\mu(\boldsymbol{n}_0, \boldsymbol{n}_2)} \quad \text{(C6)}$$

is also a 2-cocycle, for any $\mu(\boldsymbol{n}_0, \boldsymbol{n}_1)$ satisfying $\mu(g\boldsymbol{n}_0, g\boldsymbol{n}_1) = \mu(\boldsymbol{n}_0, \boldsymbol{n}_1)$, $g \in SO(3)$. Since $\nu(\boldsymbol{n}_0, \boldsymbol{n}_2, \boldsymbol{n}_3)$ and $\nu'(\boldsymbol{n}_0, \boldsymbol{n}_2, \boldsymbol{n}_3)$ can continuously deform into each other, they correspond to the same kind of 2π-quantized topological term. So we say that $\nu(\boldsymbol{n}_0, \boldsymbol{n}_2, \boldsymbol{n}_3)$ and $\nu'(\boldsymbol{n}_0, \boldsymbol{n}_2, \boldsymbol{n}_3)$ are equivalent. The equivalent classes of the 2-cocycles $\nu(\boldsymbol{n}_0, \boldsymbol{n}_2, \boldsymbol{n}_3)$ give us $\mathcal{H}^2[S^2, U(1)]$ – the 2-cohomology group of sphere S^2 with $U(1)$ coefficient. $\mathcal{H}^2(S^2, U(1))$ classifies the 2π-quantized topological term in *discrete* space-time and with S^2 as the target space.

Does $\mathcal{H}^2[S^2, U(1)]$ classify the SPT phases with $SO(3)$ symmetry? The answer is no. We know that S^2 is just one of the symmetric spaces of $SO(3)$. To classify the SPT phases, we need to replace the target space S^2 by the maximal symmetric space, which is the group itself $SO(3)$ (see Ref. 45 for more discussions). So we need to consider discrete non-linear σ-model described by $\nu(g_i, g_j, g_k)$, $g_i, g_j, g_k \in SO(3)$. Now the 2-cocycle conditions becomes

$$\nu(gg_i, gg_j, gg_k) = \nu(g_i, g_j, g_k) \in U(1),$$
$$\frac{\nu(g_1, g_2, g_3)\nu(g_0, g_1, g_3)}{\nu(g_0, g_2, g_3)\nu(g_0, g_1, g_2)} = 1, \qquad \text{(C7)}$$

which defines a "group cohomology" $\mathcal{H}^2[SO(3), U(1)]$. It classifies the 2π-quantized topological term for the maximal symmetric space. It also classifies the SPT phases with $SO(3)$ symmetry in (1+1)D.

Appendix D: Branched triangulation and topological invariant amplitudes

As we have shown, the bosonic symmetry protected order in $d + 1$ D can be described by the amplitude:

$$Z = \frac{\sum_{\{g_i\}}}{|G|^{N_v}} \prod \nu_{d+1}^{s_{ij\ldots k}}(g_i, g_j, \cdots, g_k), \qquad \text{(D1)}$$

where $g_i, g_j, \cdots, g_k \in G$ are group elements of the symmetry group G and $|G|$ is the order of G. The $d + 1$-cocycle $\nu_{d+1}^{s_{ij\ldots k}}(g_i, g_j, \cdots, g_k)$ is defined on the branched $d + 1$-simplex, with $s_{ij\ldots k} = \pm 1$ uniquely determined by the orientation of the corresponding $d + 1$-simplex. In the following, we will show they are topological invariant amplitudes. Hence, they represent a class of fixed point amplitudes of (symmetry protected) topologically ordered phases.

First, we need to give a branching structure to the discretized space-time. A branching is a choice of an orientation of each edge of an n-simplex such that there is no oriented loop on any triangle. For example, Fig. 6 (a) is a branched 2-simplex and (c) is a branched 3-simplex. However, (b) is not an allowed branching because all its three edges contain the same orientations and thus form an oriented loop. (d) is also not allowed because one of its triangle contains an oriented loop. It is easy to check that any consistent branched structure can induce a canonical ordering for the vertices of n-simplex. Indeed, the branching structure will also induce an canonical orientation of the n-simplex, see Fig. 7.
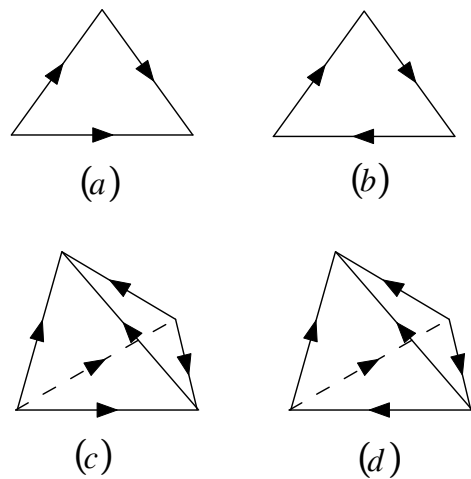

FIG. 6: Examples of allowed((a),(c)) and unallowed((b),(d)) branching for 2-simplex and 3-simplex.

To understand the geometric meaning of the topological invariance of the above amplitude, let us first discuss (1+1)D fixed-point action-amplitude with a symmetry group G. For a $(1 + 1)$D system on a complex with a branching structure, a fixed-point action-amplitude (see
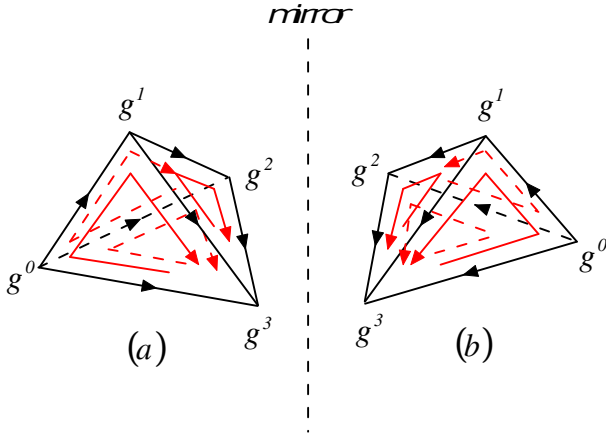
FIG. 7: (Color online)(a): A branching structure of a n-simplex will induce a canonical ordering for its vertices. For example, if a 3-simplex contains a vertex with no incoming edge, then we can label this vertex as v^0 and canonically label the vertices of the rest 2-simplex as v^1, v^2, v^3. Such a scheme can be applied for arbitrary n-simplex if n-simplex has a canonical label. (b): A branching structure will also induce a canonical orientation for the n-simplex. For example, we can use the right hand rule to determine the orientations of surfaces of the tetrahedron, then the orientation of the tetrahedron can be uniquely determined by the surface opposite to g^0
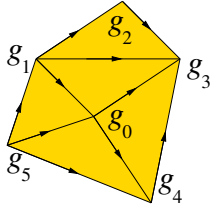
FIG. 8: (Color online) The graphic representation of the action-amplitude $e^{-S(\{g_i\})}$ on a complex with a branching structure represented by the arrows on the edge. The vertices of the complex are labeled by i. Note that the arrows never for a loop on any triangle.

Fig. 8)

$$e^{-S(\{g_i\})} = \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k)$$
$$= \nu_2^{-1}(g_1, g_2, g_3)\nu_2(g_0, g_4, g_3)\nu_2^{-1}(g_5, g_1, g_0) \times$$
$$\nu_2(g_1, g_0, g_3)\nu_2^{-1}(g_5, g_0, g_4) \qquad (D2)$$

where each triangle contribute to a phase factor $\nu_2^{s_{ijk}}(g_i, g_j, g_k)$, $\prod_{\{ijk\}}$ multiply over all the triangles in the complex Fig. 8. Note that the first variable g_i in $\nu_2(g_i, g_j, g_k)$ correspond to the vertex with two out going edges, the second variable g_j to the vertex with one out going edges, and the third variable g_k to the vertex with no out going edges. $s_{ijk} = \pm 1$ depending on the orienta-

tion of $i \to j \to k$ to be anti-clock-wise or clock-wise.

In order for the action-amplitude to represent a quantized topological θ-term, we must choose $\nu_2(g_i, g_j, g_k)$ such that

$$e^{-S(\{g_i\})} = \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k) = 1 \qquad (D3)$$

on closed space-time complex without boundary, in particular, on a tetrahedron with four triangles (see Fig. 2(b)):

$$e^{-S(\{g_i\})} = \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k)$$
$$= \frac{\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_1, g_2)\nu_2(g_0, g_2, g_3)} = 1 \qquad (D4)$$

Also, in order for our system to have the symmetry generated by the group G, its action-amplitude must satisfy

$$e^{-S(\{g_i\})} = e^{-S(\{gg_i\})}, \text{ if } g \text{ contains no T}$$
$$\left(e^{-S(\{g_i\})}\right)^\dagger = e^{-S(\{gg_i\})}, \text{ if } g \text{ contains one T} \qquad (D5)$$

where T is the time-reversal transformation. This requires

$$\nu_2^{s(g)}(g_i, g_j, g_k) = \nu_2(gg_i, gg_j, gg_k). \qquad (D6)$$

Eq.(D5) and Eq.(D6) happen to be the conditions of 2-cocycles $\nu_2(g_0, g_1, g_2)$ of G. Thus the action-amplitude Eq.(D2) constructed from a 2-cocycle $\nu_2(g_0, g_1, g_2)$ is a quantized topological θ-term.

If $\nu_2(g_0, g_2, g_3)$ satisfy Eq.(D5) and Eq.(D6), then

$$\nu_2'(g_0, g_2, g_3) = \nu_2(g_0, g_2, g_3)\frac{\mu_1(g_1, g_2)\mu_1(g_0, g_1)}{\mu_1(g_0, g_2)} \qquad (D7)$$

also satisfy Eq.(D5) and Eq.(D6), for any $\mu_1(g_0, g_1)$ satisfying $\mu_1(gg_0, gg_1) = \mu_1(g_0, g_1)$, $g \in G$. So $\nu_2'(g_0, g_2, g_3)$ also gives rise to a quantized topological θ-term. As we continuously deform $\mu_1(g_0, g_1)$, the two quantized topological θ-terms can be smoothly connected. So we say that the two quantized topological θ-terms obtained from $\nu_2(g_0, g_2, g_3)$ and $\nu_2'(g_0, g_2, g_3)$ are equivalent. We note that $\nu_2(g_0, g_2, g_3)$ and $\nu_2'(g_0, g_2, g_3)$ differ by a 2-coboundary $\frac{\mu_1(g_1, g_2)\mu_1(g_0, g_1)}{\mu_1(g_0, g_2)}$. So the equivalence classes of $\nu_2(g_0, g_2, g_3)$ is nothing but the cohomology group $\mathcal{H}^2[G, U_T(1)]$. Therefore, the quantized topological θ-terms are classified by $\mathcal{H}^2[G, U_T(1)]$.

We can also show that Eq.(D2) is a fixed-point action-amplitude from the cocycle conditions on $\nu_2(g_i, g_j, g_k)$. The cocycle condition Eq.(D4) induces two renormalization moves in the discretized manifold. Fig. 9 represents a $2 \leftrightarrow 2$ moves:

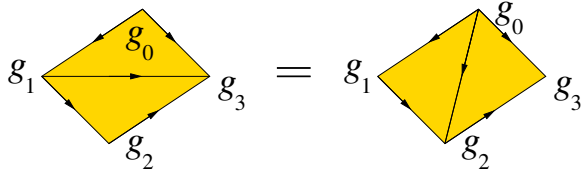$$\nu_2(g_0, g_1, g_3)\nu_2(g_1, g_2, g_3) = \nu_2(g_0, g_1, g_2)\nu_2(g_0, g_2, g_3) \qquad (D8)$$

FIG. 9: (Color online) Graphic representation of $\nu_2(g_0, g_1, g_2)\nu_2(g_0, g_2, g_3) = \nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)$ The arrows on the edges represent the branching structure.
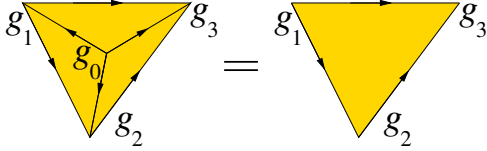

FIG. 10: (Color online) Graphic representation of $\nu_2(g_1, g_2, g_3) = \nu_2(g_0, g_1, g_2)\nu_2(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_3)$. The arrows on the edges represent the branching structure.

and Fig. 10 represents a $1 \leftrightarrow 3$ move:

$$\nu_2(g_1, g_2, g_3) = \nu_2(g_0, g_1, g_2)\nu_2(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_3) \tag{D9}$$

By using these two moves, different triangularization of the space-time can be mapped into each other without changing the action amlitude Eq.(D1). Therefore, the action-amplitude is a fixed point form.

Geometrically, these two moves can be obtained by projecting a 3 simplex onto two two-dimensional surfaces.

Fig.11 (a) gives the $2 \leftrightarrow 2$ move and Fig.11 (b) gives the $1 \leftrightarrow 3$ move. Note that the projection from opposite directions will induce opposite chiralities for the boundary of the tetrahedron, that's why we need to change the
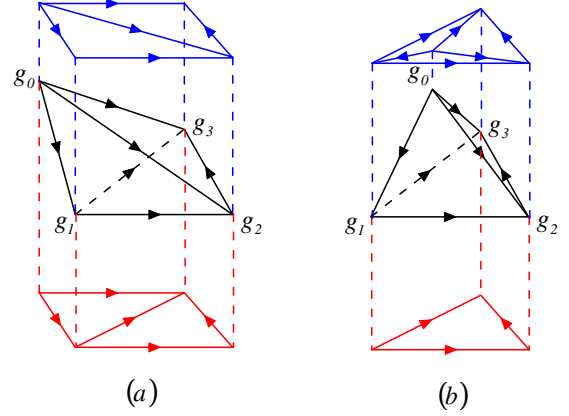

(a)   (b)

FIG. 11: $(d_2\nu_2)(g_0, g_1, g_2, g_3)$ can be represented as the boundary of a 3-simplex. (a) and (b) correspond to two different basic moves of 2-simplexes.

chiralities of the triangular in one side of basic moves. Such a change correspond to inverse ν_2 when we move it from left side to right side of the above equation, which is consistent with our rules of algebra. Similar process also works in higher dimensions. Different moves in d-dimension can be obtained from projecting a $d + 1$-simplex in different ways. Such moves induces a renormalization flow in the discrete space-time under which the action-amplitude Eq.(D1) is a fixed point.

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