THE ONE-DIMENSIONAL INVERSE SCATTERING PROBLEM

by

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ABSTRACT

The Inverse Problem of Scattering Theory (designated hereafter by IPST) consists of deriving information about the scatterer from the knowledge of the scattered data. This problem has been extensively developed for the radial Schrodinger equation, chiefly in view of nuclear physics.

The theory of the IPST provides an exceedingly powerful "theoretical tool" which can be successfully applied to some problems of interest in the electromagnetic synthesis of inhomogeneous media, viz., synthesis of non-uniform transmission lines, dielectric filter design for optical applications, ionospheric diagnosis, to name but a few.

We strongly believe that the IPST unfolds a promising and entirely new approach to electromagnetic problems which, up to the present time, have mostly been treated using more or less trial and error numerical procedures. Still its potentiality for Electrical Engineering seems to have remained unexplored, a part from a few isolated investigations.

We would like to consider this thesis as an introduction to a systematic exploitation of the IPST's results in view of electromagnetic synthesis problems. Accordingly, we have tried to give a complete review of what has been done on the IPST both in its one-dimensional and radial formulations. We have limited our consideration to one-channel, non-relativistic scattering of S-waves by local, real, velocity-independent potentials without bound states, planning to undertake these generalizations in further research.

The interesting scatterers for practical applications are obviously those which occupy a finite region of space. Consequently, our focal point of interest was the problem of finding conditions on the chosen scattered data in order that the corresponding scatterer be indeed finite. Although this problem was solved for the radial case (where one is concerned with only one "cutoff"), its solution for the one-dimensional case (where two "cutoffs" are required) was unknown up to the present time.

We have been able to give a complete solution of such a problem, viz., the problem of synthesizing a finite range one-dimensional potential producing a prescribed phase-delay as a function of frequency. Conditions ensuring that a function of frequency is the phase-delay produced by some finite range one-dimensional potential were also found.

Thesis Supervisor: William P. Allis
Title: Professor of Physics

ACKNOWLEDGEMENT

I owe a very great debt of gratitude to Professor W. P. Allis for his patient and generous supervision of this work. To know him was one of the most rewarding and rich experiences during these years at the Institute. Being unable to adequately express my feeling, I should like to use Knecht's * words:

..." Und erlaube mir noch ein Wort ", fing der Glasperlenspielmeister wieder mit leiser Stimme an. " Ich möchte dir gerne noch etwas über die Heiterkeit sagen, über die der Sterne und die des Geistes, und auch über unsre kastalische Art von Heiterkeit. Du hast eine Abneigung gegen die Heiterkeit, vermutlich weil du einen Weg der Traurigkeit hast gehen müssen, und nun scheint dir alle Helligkeit und gute Laune, und namentlich unsre kastalische, seicht und kindlich, auch feige, eine Flucht vor den Schrecken und Abgründen der Wirklichkeit in eine klare, wohl-geordnete Welt bloßer Formen und Formeln, bloßer Abstraktionen und Abgeschliffenheiten. Aber, mein lieber Trauriger, mag es diese Flucht auch geben, mag es an feigen, furchtsamen, mit bloßen Formeln spielenden Kastaliern nicht fehlen, ja sollten sie bei uns sogar in der Mehrzahl sein - dies nimmt der echten Heiterkeit, der des Himmels und der des Geistes, nichts von ihrem Wert und Glanz. Den Leichtzufriedenen und Scheinheiteren unter uns stehen andere gegenüber, Menschen und Generationem von Menschen, deren Heiterkeit nicht Spiel und Oberfläche, sondern Ernst und Tiefe ist. Einen habe ich gekannt, es war unser ehemaliger Musikmeister, den du einst in Waldzell auch je und je gesehen hast; dieser Mann hat in seinen letzten Lebensjahren die Tugend der Heiterkeit in solchem Maße besessen, daß sie von ihm ausstrahlte wie das Licht von einer Sonne, daß sie als Wohlwollen, als Lebenslust, als gute Laune, als Vertrauen und Zuversicht auf alle überging und in allen weiterstrahlte, die ihren Glanz ernstlich aufgenommen und in sich eingelassen hatten. "...

I am also deeply indebted to Doctor H. E. Moses, who introduced me to this area of research, and provided invaluable support throughout the course of this work.

I wish to express my gratitude to Professor S. J. Mason for his suggestions and constant encouragement, and also for his kindliness and human understanding as my Graduate Counselor.

My appreciation also goes to the "Conselho Nacional de Pesquisas", Brazil, whose fellowship made this research possible.

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* - from Hermann Hesse's "Das Glasperlenspiel" (Magister Ludi)

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..." Thinking that it was time to bring down the Monarch from his raptures to the level of common sense, I determined to endeavour to open up to him some glimpses of the truth, that is to say of the nature of things in Flatland. So I began thus: " How does your Royal Highness distinguish the shapes and positions of his subjects ? I for my part noticed by the sense of sight, before I entered your Kingdom, that some of your people are Lines and others Points, and that some of the lines are larger - " You speak of an impossibility " , interrupted the King; " you must have seen a vision; for to detect the difference between a Line and a Point by the sense of sight is, as every one knows, in the nature of things, impossible; but it can be detected by the sense of hearing, and by the same means my shape can be exactly ascertained. Behold me - I am a Line, the longest in Lineland, over six inches of Space - "
" Of length" , I ventured to suggest. " Fool " , said he, " Space is Length. Interrupt me again, and I have done " ...

(excerpt from " Flatland " , by E. Abbot)

CHAPTER I

INTRODUCTION

1.1 General Background

The theory of potential scattering has evolved during the last two decades to a highly sophisticated level[(1)*].

The vast majority of its results concern the well-known "radial equation" :

$$\frac{d^2\psi}{dr^2} + k^2\psi - \frac{\ell(\ell+1)}{r^2}\psi - V(r)\psi = 0 \qquad (1)$$

$$0 \le r < \infty \qquad (1a)$$

which is called for in the quantum-mechanical description of certain three-dimensional physical systems.

On the other hand, the so-called "one-dimensional equation":

$$\frac{d^2\psi}{dx^2} + k^2\psi - V(x)\psi = 0 \qquad (2)$$

$$-\infty < x < \infty \qquad (2a)$$

has received relatively little consideration, the reason being probably that one seldom meets "one - dimensional potentials V(x)" in quantum scattering theory. We shall see, however, that this equation assumes an important role in connection with electromagnetic problems.

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* The superscripted parentheses will here and henceforward indicate the number of a specific reference contained in the Bibliography.

It is somewhat surprising that Electrical Engineering has not yet taken substantial advantage from such a rich field which has been instrumental to quantum scattering theorists for more than twenty years. Let us examine briefly where could it extract its greatest dividends and speculate on why such a possibility has not been generally acknowledged.

A large class of electromagnetic problems can be reduced to the one-dimensional steady-state* wave equation:

$$\boxed{\frac{d^2E}{dx^2} + k^2 U(x) E = 0} \tag{3}$$

A typical set of "scattering" boundary conditions is expressed by the asymptotic forms:

$$E \sim e^{ikx} + R(k) e^{-ikx} \qquad x \rightarrow -\infty \tag{3a}$$

$$E \sim T(k) e^{ikx} \qquad x \rightarrow +\infty \tag{3b}$$

We shall describe in the next section the distinction between "direct" and "inverse" problems. For the moment let us simply indicate that in the direct problem the scatterer U(x) is given, and one seeks to determine R(k) and T(k).

For such a problem it seems that there is not much to gain from the results of potential scattering theory. After all, direct problems for the wave equation were under the keen scrutiny of J.C. Maxwell and his

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* A time factor $e^{-i\omega t}$ is assumed throughout this thesis.

successors before Schrodinger came into the world - and it was not until 1947 that the first really relevant investigation [2] on potential scattering was published.

The question then arises - is there anything to be gained from the "inverse" problem formulation ?

The situation here is entirely different and the answer is an emphatic yes. Till the present time, except for a few isolated investigations to be discussed subsequently, the electromagnetic synthesis of inhomogeneous media [as for example to construct a $U(x)$ which would produce a prescribed $R(k)$] has been treated using numerical methods of trial and error, with little or no attention given to the analytic properties of the quantities involved.

We would like to quote here from a very recent article [3] by prominent authorities in optical research:

..." Depending on the way in which the rates are varied, different refractive index profiles $n = n(x)$ as a function of the coordinate x along the normal of the film surface are obtained. As in the case of multiple homogeneous films, the problem is to find a function $n(x)$ which gives a desired reflectance $R = R(\lambda, \theta)$ as a function of wave-length λ and angle of incidence θ . *Unfortunately, there exists no general solution of this problem and one has to use more or less trial and error methods in the design process* " ... *

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* Italics are ours.

On the other hand, the inverse problem for the "quantum scattering equations (1) and (2) has been solved more than a decade ago. We shall see, and this shall be indeed the focal point of our thesis, that its results bring invaluable assistance to certain electromagnetic synthesis problems.

As a matter of fact, the general solution of the problem stated in Reference (3) does exist (in its "diagnosis" formulation*) and was pointed out by I. Kay[(4)] as far back as 1955, although in a quite succint description (as an application of the inverse one-dimensional problem rather than as a central subject).

His solution rests chiefly upon two important results:

(i) the existence of an algorithm for solving the inverse problem for equation (2). This is the main subject of his paper.

(ii) the existence of a change of variables which transforms equation (3) into an equation of type (2). Such is the well known Liouville's transformation[(5)].

Notice that (ii) is an indispensable step in order to use (i). It is so because the algorithm derived in (i) requires velocity-independent (independent of k) potentials and, comparing equations (2) and (3), one sees that the coefficient of the wave-function is respectively $k^2 - V(x)$ and $k^2 . U(x)$. An important exception to this is

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* See next section

the case of a cold collisionless plasma, for which U(x) depends on k in such a way that $k^2 . U(x, k)$ has the desired form of equation (2), without any transformation.

These results show that an entirely new approach to electromagnetic synthesis has been disclosed by the advent of a successfull solution to the inverse problem of scattering theory.

It is therefore legitimate to ask why are there so few papers on electromagnetic synthesis using such an approach. We can only speculate on the various reasons for this fact.

It seems to us that the complexity of the algorithm is not in cause, since it only involves a quadrature followed by the solution of a linear Fredholm integral equation which lends itself naturally to the usual approximation techniques. Furthermore, it constitutes indeed a powerful "theoretical tool" through which one is able to obtain realizability conditions which clarify the physics of the problem.

We rather believe that the main reasons for such an "ostracism" are twofold:

(i) Electrical Engineers are generally unaware of the existence of a solution to the quantum scattering inverse problem for the one-dimensional Schrodinger equation, although the mapping of this equation into the wave equation by means of the Liouville transformation is widely used (in connection with the various approximation techniques for solving the wave equation).

(ii) Physicists are interested primarily in the radial equation for quan*um scattering applications - this fact has hindered the development of the one-dimensional inverse problem as compared to the radial inverse problem.

1.2 Direct and Inverse Problems

In order to develop further the "vocabulary" with which we shall describe our objective in this work, let us determine in detail what is meant by direct and inverse problems, and how these problems may be interpreted according to their formulation.

Roughly speaking, the problems characterized by equations (1) and (2) can be classified in two broad classes:

1.2.1 Direct Problems

This is the familiar class of problems where the potential V is given and, by solving the corresponding equation, all the desired information about the physical system is obtained.

We shall assume throughout this thesis that the potentials are such that the system does not have any bound states - this assumption does not represent a very serious restriction, and most of our results can be modified to account for possible discrete points in the spectrum.

These problems have been extensively studied (references 6 to ~~104~~ 62 in the Bibliography) and consequently we turn now to our real concern : the inverse problem.

1.2.2 Inverse Problems

This class is less widely known, and is much more recent. The potential is unknown and one seeks to reconstruct it from the knowledge of the scattered data. By "scattered data" we mean quantities which can be measured "far away" from the scatterer, plus whatever additional data as might turn out to be necessary in order to determine the potential (as for example binding energies and related normalization constants, if the potential supports bound states).

It is important to understand that the inverse problem may be interpreted in two different ways:

i) "diagnosis" problem - the scattered data is given and one seeks for an algorithm determining the corresponding potential.

ii) "synthesis" problem - one wishes to construct a potential which would produce a prescribed set of scattered data but, in addition, seeks for "realizability conditions", viz., which classes of potentials correspond to which classes of scattered data.

Let us emphasize the distinction between these two formulations by giving examples drawn from electromagnetic situations:

i) to determine the ionization density of the ionosphere from the time delay of a pulse radio wave which has been transmitted from the earth and reflected back by the ionosphere. This is a "diagnosis" problem - one does not have to worry about the realizability of the <u>measured</u> time delay.

ii) to construct a piece of inhomogeneous dielectric such that, when an electromagnetic wave is reflected from it, the reflection coefficient has a prescribed variation with frequency. This is a "synthesis" problem - one has to be sure that the prescribed variation of the reflection coefficient with frequency is indeed realizable with a physical piece of dielectric. Among other requirements, the piece should not have infinite length !

1.3 Objective of this Thesis

1.3.1 General Purpose - Finite Range Potentials

We would like to consider this thesis as an introduction to the application of the quantum inverse scattering problem to electromagnetic synthesis problems.

This being so, much of our work has been bestowed to the study of the quantum inverse scattering problem in itself, since it belongs to a highly specialized branch of physics involving rather complex analytical techniques.

This thesis can be mentally divided in two parts:

i) the description of both the historical development and the main results of the theory of the inverse scattering problem, in order to have a solid foundation upon which the applications and further results shall rest. A few original results, in connection with the one-dimensional inverse problem, shall be included in this account.

ii) the complete solution (realizability conditions and algorithm) of of a specific electromagnetic synthesis problem which, to the best of our knowledge, has remained unsolved up to the present time.

The interesting scatterers for practical applications are obviously those which occupy a finite region of space. Throughout this work we shall therefore be concerned with the synthesis of FINITE RANGE one-dimensional potentials, i.e., potentials which are identically zero outside a finite interval (also called : of compact support).

A twofold task is implied in such an objective : once a physically interesting set is chosen as scattered data, we must be able

i) to find an algorithm through which the potential can be found

ii) to give sufficient conditions ensuring that the corresponding potential has indeed a finite range.*

To our knowledge, no solution for one-dimensional inverse scattering problems has been found which satisfies (ii). We have been able (chapter V) to find such a complete solution for a one-dimensional inverse problem where the total wave-function phase-delay is chosen as scattered data.

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* Besides the conditions ensuring that the set is indeed a set of scattered data.

1.3.2 Brief Description of Contents

In Chapter II we have tried to give a detailed account on what has been done on the Inverse Problem up to the present time, including formulations which in a strict sense cannot be considered to correspond to "scattering" problems, being rather of the type "natural modes". We felt that it would be of some interest to describe in more detail certain important papers which were apparently forgotten in the existing literature on the problem, as for example, the "inverse geophysical problem" treated by R. E. Langer in 1933. In general, emphasis was given to works directly relevant to our objectives in this thesis.

Chapter III deals with the Inverse Problem both in its Radial and One-Dimensional formulations. In the former case, we merely summarize the important results which shall be used in connection with our main present interest, viz., chapter V. For the one-dimensional problem, most results derived are new and hopefully shall contribute to further development of different formulations of the problem (different sets of scattered data as input).

In Chapter IV we list some examples to show how certain electromagnetic problems can be reduced to Sturm-Liouville equations which are ready to be processed according to the theory of the Inverse Quantum Scattering Problem. This list is by no means exhaustive, and it might be interesting indeed to proceed to a systematic investigation in order to examine problems of interest which can be reduced to Inverse Sturm-Liouville problems; of course we need not be restricted to electromagnetic problems - Acoustics, Hydrodynamics, Heat Conduction, Elasticity, etc..., might benefit as well.

Chapter V contains the major practical results in this work, by means of which a solution to an unsolved problem in electromagnetic synthesis is derived. Essentially, we obtain a complete solution (algorithm and sufficient conditions) for a one-dimensional problem involving a FINITE RANGE potential (dielectric slab), where the input data consists of the total wave-function (electric field) phase-delay as a function of energy (frequency). Sections 5.3 and 5.4 examine the possibility of solving similar problems for different input data.

In Chapter VI we have listed what we believe to be the most promising directions for further development of this work. We also state our feeling that the subject remains vastly unexplored and that this thesis can serve only as an introduction to its applications, although we were fortunate enough to obtain the results of chapter V.

Finally, an Appendix gives an example of the method derived in chapter V.

CHAPTER II

HISTORICAL DEVELOPMENT OF THE INVERSE PROBLEM

2.1 Introduction

In 1894, Lord Rayleigh (63) considers the problem of determining the partial tones of a stretched string of variable density $\rho_0 + \Delta\rho$ where $\Delta\rho(x)$ is relatively small. The approximate formula is given:

$$T_n^2 = \frac{4\ell^2 \rho_0}{\tau . n^2} (1 + \alpha_n)$$

where T_n is the period of the n^{th} component vibration, ℓ is the string's length, τ its tension and

$$\alpha_n = \frac{2}{\ell} \int_0^{\frac{\ell}{2}} \frac{\Delta\rho}{\rho_0} \left[1 - \cos \frac{2\pi n x}{\ell} \right] dx$$

By expanding $\Delta\rho$ from 0 to $\ell/2$ in the Fourier series

$$\frac{\Delta\rho}{\rho_0} = A_0 + A_1 \cos \frac{2\pi x}{\ell} + \ldots\ldots\ldots + A_n \cos \frac{2\pi n x}{\ell} + \ldots\ldots\ldots$$

Lord Rayleigh obtains the following expression for α_n ::

$$\alpha_n = A_0 - \frac{1}{2} A_n$$

where

$$A_0 = \frac{2}{\ell} \int_0^{\frac{\ell}{2}} \frac{\Delta\rho}{\rho_0} dx$$

$$A_n = \frac{4}{\ell} \int_0^{\frac{\ell}{2}} \frac{\Delta\rho}{\rho_0} \cos \frac{2\pi n x}{\ell} dx$$

Lord Rayleigh detects an essential feature in these results - we quote:

..." This equation, as it stands, gives the changes in period in terms of the changes of density supposed to be known. *And it shows conversely that a variation of density may always be found which will give prescribed arbitrary displacements to all periods* *. This is a point of some interest."...

As far as we know this is the first solution, although approximate, to an inverse problem for the wave equation. We notice however that this is not an inverse "scattering" problem, since here the continuous spectrum of the wave operator is empty. The input data consists of a sequence of numbers; accordingly, we shall designate such problems by the generic term "discrete inverse problems", as opposed to inverse "scattering" problems where, even

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* Italics are ours. As an example, Lord Rayleigh solves the problem of prescribing $\alpha_n = 0$ for $n \neq 1$, viz., the pitch of the fundamental tone is displaced by α_1, all other tones remainig unaltered. The corresponding density variation is easily derived:

$$A_n = 0 \quad (n \neq 1) \qquad\qquad A_1 = -2\alpha_1$$

Hence:

$$\frac{\Delta \rho}{\rho_0} = -2\alpha_1 \cos \frac{2\pi x}{\ell}$$

though the spectrum may have a discrete part, there is always a continuous part which correspond to the "scattering" solutions. For the sake of completeness we shall first give a brief account on the "discrete inverse problem", which initiated the development of the whole theory.

2.2 The Discrete Inverse Problem

The vast majority of investigations deal with eigenvalue problems for the well-known [64, 65, 66] Sturm-Liouville operator. (Notice that the string problem undertaken by Lord Rayleigh is an exception to this statement, although it can be reduced to a Sturm-Liouville problem, as shall be seen).

The first rigourous result in this field seems to have been given by V. A. Ambartsumyan [67] in 1929. Considering the eigenvalue problem:

$$L\psi_n = \lambda_n \psi_n \qquad (n = 0, 1, 2, \ldots\ldots)$$

$$\psi_n'(0) = \psi_n'(\pi) = 0$$

where

$$L \equiv -\frac{d^2}{dx^2} + V(x)$$

and V(x) is a real continuous function for $x \in [0, \pi]$, he proved the following theorem [*] :

" If $\lambda_n = n^2$ then $V(x) \equiv 0$."

Next comes an important paper by G, Borg [(68)], published in 1946. Roughly, his results can be sketched as follows; consider the two boundary-value problems for the same operator L:

$$L\Psi = \lambda\Psi \quad (\lambda = \lambda_0, \lambda_1, \ldots) \quad \begin{cases} \Psi'(0) - h\Psi(0) = 0 \\ \Psi'(\pi) + H\Psi(\pi) = 0 \end{cases} \qquad (I)$$

$$L\Psi = \mu\Psi \quad (\mu = \mu_0, \mu_1, \ldots) \quad \left\{ \quad \Psi'(0) - h_1\Psi(0) = 0 \right. \qquad (II)$$

Borg shows that the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ determine V(x), h, h_1, and H uniquely. He also shows that in general (Ambartsumyan's result is an obvious exception) one spectrum is not sufficient to determine V(x) ; he attempts to reconstruct V(x) from the knowledge of two spectra, obtaining a partial answer [conditions ensuring that there exists a V(x) corresponding to prescribed λ_n's and μ_n's are not derived].

Apparently unaware of Lord Rayleigh's results on the variable density stretched string, Borg tackles the general problem using the full machinery developed for the inverse Sturm-Liouville problem. In order to do so, he transforms the wave equation:

$$\frac{d^2 z}{dt^2} + k^2 P(t) z(t) = 0$$

[*] We have here a simple example of a set of "scattered data" which determines the potential uniquely.

where:

$$k = \frac{\omega}{\sqrt{c}} \qquad t \in [0, \ell] \qquad z(0) = z(\ell) = 0$$

into the Sturm-Liouville eqation:

$$\frac{d^2 y}{dx^2} + \left[\ell_0^2 k^2 + V(x) \right] y = 0$$

by means of the so-called Liouville transformation [5] :

$$x = \frac{1}{\ell} \int_0^t \sqrt{P(\xi)}\, d\xi$$

$$\ell_0 = \frac{1}{\pi} \int_0^\ell \sqrt{P(\xi)}\, d\xi = \frac{\ell}{\pi} x(\ell)$$

$$y(x) = \sqrt[4]{P(t)}\, z(t)$$

$$V(x) = \frac{1}{\sqrt[4]{P(t)}} \frac{d^2}{dx^2} \left[\sqrt[4]{P(t)} \right]$$

This transformation constitutes the "bridge" through which all the results derived for the inverse quantum scattering problem shall be applied to electromagnetic inverse problems. To our knowledge, it has been given for the first time in 1837 [5] as an approximate method for solving the wave equation which has evolved into what is called today the WKBJ approximation.

A decisive advance in the whole theory was obtained by V. A. Marchenko in 1950 [69], who showed that V(x) is determined uniquely by the so-called spectral function.

Let us say a few words about the spectral function of a regular Sturm-Liouville problem. Define $\varphi(x,\lambda)$ to be the solution of the equation:

$$L\varphi - \lambda\varphi = 0 \qquad x \in [0,\pi]$$

which satisfies the initial conditions:

$$\varphi(0,\lambda) = 1$$
$$\varphi'(0,\lambda) = h$$

and let $\{\lambda_n\}$ be the spectrum. Define the normalization constants

$$\alpha_n = \int_0^\pi \varphi^2(x,\lambda_n)\,dx$$

the spectral function is then determined by:

$$\rho(\lambda) = \sum_{\lambda_n < \lambda} \frac{1}{\alpha_n}$$

For example, the simple problem $V(x) \equiv 0$, $h = 0$, has the spectral function:

$$\rho_0(\lambda) = \frac{2}{\pi}\sqrt{\lambda}$$

as it can be easily seen from the definition.

A fundamental property of $\rho(\lambda)$ is that for any function $f(x) \in L^2[0,\pi]$ there exists a function $E(\lambda)$ implicitly defined by:

$$\lim_{t \to \pi} \int_{-\infty}^{+\infty} \left[E(\lambda) - \int_0^t f(x)\varphi(x,\lambda)dx \right] d\rho(\lambda) = 0$$

and the Parseval equation:

$$\int_0^{\pi} f^2(t)dt = \int_{-\infty}^{+\infty} E^2(\lambda)d\rho(\lambda)$$

holds.

Marchenko shows that $\rho(\lambda)$ completely determines V(x) and the number h.

At almost the same time, M. G. Krein [70] obtains a complete, although somewhat indirect, solution of the "two spectra" inverse problem attempted by Borg.

In 1951 appears the fundamental paper [71] of I. M. Gelfand and B. M. Levitan [actually a series of two papers]. An algorithm for determining V(x) from $\rho(\lambda)$ is given, as well as necessary and sufficient conditions for a function of λ to be the spectral function of some Sturm-Liouville operator. Let us briefly describe their algorithm:

a) Define $\sigma(\lambda)$ by the relations:

$$\rho(\lambda) = \begin{cases} \frac{2}{\pi}\sqrt{\lambda} & \lambda \geq 0 \\ \sigma(\lambda) & \lambda < 0 \end{cases}$$

b) obtain f(x, y) through:

$$f(x,y) = \int_{-\infty}^{+\infty} \cos\sqrt{\lambda}\,x \, . \cos\sqrt{\lambda}\,y \, . \, d\sigma(\lambda)$$

c) solve the linear Fredholm (x is a parameter) integral equation:

$$K(x,y) = -f(x,y) - \int_0^x f(y,z) K(x,z) dz \qquad y < x$$

d) obtain V(x) and h from K(x, y) using:

$$V(x) = 2 \frac{d}{dx} K(x,x)$$

$$h = K(0,0)$$

e) $\varphi(x,\lambda)$ has the integral representation:

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}\,x + \int_0^x K(x,y) \cos\sqrt{\lambda}\,y \, dy$$

It shall be seen in the next section that Gelfand-Levitan's results originated the whole theory of the inverse quantum scattering problem, although earlier attempts existed which made use of iteration schemes formally derived.

We shall end here our discussion on discrete inverse problems by saying that they have generally evolved through generalizations of the earlier problems, as for example, to various cases of singular Sturm-Liouville operators. More recent literature relevant to some of these investigations is listed in the Bibliography from (72) to (76) [(72) and (74) contain extensive list of respectively 78

2.3 The Radial Inverse Scattering Problem

In 1947, C. Froberg [77], under a suggestion from W. Pauli, examines the problem of determining the potential from the phase-shifts. His results, as we shall see, triggered a rapid succession of papers on this subject - Heisenberg had suggested that the S-matrix contains enough physical information to overtake the fundamental role played by the Hamiltonian (or the equations of motion) in atomic problems, conjecturing that the binding energies could be obtained from the analytic continuation of the S-matrix into the complex energy plane - this question was one of lively interest among physicists and, since the S-matrix is closely connected with the phase-shifts, it was of some importance to determine if the potential could be reconstructed knowing the phase.

But before going any further, let us proceed chronologically and describe first a continuous inverse problem treated by R. E. Langer in 1933 [78]. This isolated work seems to have passed unnoticed to most workers in the field - as a matter of fact the only mention we know of Langer's research is given in Borg's paper [68].

The mathematical problem treated by Langer originates from a geophysical problem formulated by L. B. Slichter [79]. The problem is to determine the conductivity of the earth in a certain region as a function of depth. The experimental procedure consists essentially on measuring the surface electrical potential distribution created by the supply of a direct electric current through a small electrode to the surface of the earth. The issue is: can one find the conductivity knowing the surface potential distribution ?

Under certain idealizations, Maxwell's equations for this time-independent problem

$$\nabla . \vec{J} = 0 \qquad \vec{J} = \sigma \vec{E} \qquad \vec{E} = -\nabla \Phi$$

can easily be reduced to the partial differential equation :

$$\sigma(x) \left\{ \frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Phi}{\partial \rho} + \frac{\partial^2 \Phi}{\partial x^2} \right\} + \frac{d\sigma}{dx} \cdot \frac{\partial \Phi}{\partial x} = 0$$

where x is the depth, ρ the horizontal distance from the electrode, Φ (x, ρ) the electrical potential and σ(x) the conductivity.

Separating variables :

$$\Phi(x, \rho) = \sum_\lambda U_\lambda(\rho)\, u_\lambda(x)$$

Langer obtains the two ordinary differential equations:

$$U_\lambda'' + \frac{1}{\rho} U_\lambda' + \lambda^2 U_\lambda = 0$$

$$(\sigma u_\lambda')' - \lambda^2 \sigma u_\lambda = 0$$

$U_\lambda(\rho)$ satisfies a Bessel equation, and must remain finite and vanish at infinity as $(\rho)^{-1}$, therefore:

$$U_\lambda(\rho) \equiv J_0[\lambda\rho]$$

$u_\lambda(x)$ satisfies a Sturm-Liouville equation, and must be of exponential form, since $\sigma(x)$ is positive. Denote by $u_1(x, \lambda)$ a solution which is positive and monotonically decreasing with x.

Since $\frac{\partial\Phi}{\partial x}$ must vanish everywhere at the surface, except at the electrode $[\rho = a]$, the actually distribution is given by:

$$\Phi(\rho, x) = \frac{-c}{2\pi a\sigma(0)} \int_0^\infty \frac{u_1(x,\lambda)}{u_1'(0,\lambda)} J_0(\lambda\rho)\sin\lambda a \, d\lambda$$

where c is the current and a the radius of the electrode. The surface potentials are:

$$\Phi(\rho, 0) = \frac{-c}{2\pi\sigma(0)} \int_0^\infty \Omega(\lambda)\frac{\sin\lambda a}{\lambda a} J_0(\lambda\rho)\, d\lambda$$

where:
$$\Omega(\lambda) = -\lambda\frac{u_1(0,\lambda)}{u_1'(0,\lambda)}$$

Now this is a Fourier-Bessel integral transform which can be inverted into:

$$\Omega(\lambda)\left\{\frac{c\sin\lambda a}{2\pi a\lambda^2\sigma(0)}\right\} = \int_0^\infty \Phi(\rho, 0) J_0(\lambda\rho)\, d\rho$$

It follows that the surface data determine $\Omega(\lambda)$ uniquely. Langer shows that the knowledge of $\Omega(\lambda)$ for $\lambda \in (0, \infty)$ determines the conductivity $\sigma(x)$. Let us very briefly sketch his solution. Defining

$$v(x, \lambda) = - \frac{u_1(x,\lambda)}{u_1'(x,\lambda)}$$

he shows that $v(x, \lambda)$ and $\Omega(\lambda)$ have the asymptotic representations for large λ :

$$v(x,\lambda) \sim 1 + \sum_{n=1}^{\infty} \frac{V_n(x)}{\lambda^n}$$

$$\Omega(\lambda) \sim 1 + \sum_{n=1}^{\infty} \frac{\omega_n}{\lambda^n}$$

Proving that

$$\frac{\sigma'(x)}{\sigma(x)} \equiv -2V_1(x)$$

he thus reduces the problem to the determination of $v_1(x)$ from the ω_n's . Deriving recurrence relations for $\left\{ \frac{d^n}{dx^n} V_1 \right\}_{x=0}$ in terms of the ω_n's , $v_1(x)$ is then obtained through its MacLaurin expansion.

It might be interesting to examine if we can apply the Gelfand-Levitan formalism to this problem and, in case we can, to compare its results with Langer's.

Let us return now to the "mainstream" of investigations on the inverse quantum scattering problem. As we mentioned at the beginning of this section, it was on a suggestion of W. Pauli that C. Froberg considered the problem of determining the the potential from the phase-shift. Froberg's results were published in a series of three papers (77, 80, 81).

In the first paper, Froberg starts from the radial equation (1) and, defining the asymptotic phase $\delta(k)$ as usual

$$\psi \sim \sin\left(kr - \tfrac{1}{2}\pi\ell + \delta\right) \qquad r \to +\infty$$

he bases his method upon the equation, obtained by L. Hulthen (82)

$$k \sin\delta(k) = -\int_0^\infty u_\ell V(r)\psi\, dr$$

where u_l is the $V \equiv 0$ solution:

$$u_\ell = \sqrt{\frac{\pi k r}{2}}\, J_{\ell+\frac{1}{2}}(kr)$$

Approximating ψ by the first term u_l of its Picard's expansion

$$k \sin\delta(k) \simeq -\int_0^\infty u_\ell^2 V(r)\, dr$$

and differentiating with respect to k, he obtains the integral equation for V(r):

$$f_0(k) = \frac{d}{dk}\left\{k\sin\delta(k)\right\} \simeq -2\int_0^\infty u_\ell u_\ell' \, r V(r)\, dr \tag{4}$$

for which he gives, without proof, the solution:

$$V(r) \simeq \frac{-8}{\pi \ell!} \int_0^\infty k f_0(k) dk \int_0^1 (ktr)^\ell (1-t^2)^\ell v_\ell(2ktr) dt \tag{5}$$

where:

$$v_\ell(z) = (-1)^\ell \sqrt{\frac{\pi z}{2}}\, J_{-\ell-\frac{1}{2}}(z)$$

for s-waves, he obtains thus the simple result:

$$V(r) \simeq -\frac{4}{\pi} \int_0^\infty f_0(k) \sin \frac{2kr}{r}\, dk$$

His next paper (80) supplies proofs for these results. For the solution of the integral equation (4) he refers to N. Zeilon (83) and shows that the integral equation of the first kind for W:

$$f(x) = \int_0^\infty W(xt).h(t)\, dt$$

has the solution:

$$W(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \frac{\int_0^\infty f(v) v^{-i\xi-1} dv}{\int_0^\infty h(v) v^{i\xi} dv} \right\} x^{i\xi} d\xi$$

and from this expression he proceeds to get Eq.(5), adding in proof that (5) can be simplified as to give:

$$V(r) \simeq \frac{-8}{\pi r} \int_0^\infty f_0(k) u_\ell(kr) v_\ell(kr) dk$$

the method is then applied to the s- and p-waves of the Yukawa potential:

$$V(r) = -b\alpha \frac{e^{-\alpha r}}{r}$$

finding that his first approximation yields the exact result for both cases S and P. Finally, a Note is given based upon R. Jost's suggestions, which simplifies the derivation of (5) and dispenses with Zeilon's formula.

In the meantime, while Froberg was preparing his third paper he had several discussions with E. Hylleraas *. Thereupon Hylleraas undertook investigations on the same subject, publishing an article [84] in which he criticizes Froberg's results on grounds of the lack of convergence proofs as well as of a clearer formulation of the method for higher approximations. He proceeds to give a different treatment to the problem, starting with validity conditions for the Fourier transformations:

$$F(r) = \frac{4}{\pi}\int_0^\infty G(k)\sin 2kr\,dk$$

$$G(k) = \int_0^\infty F(r)\sin 2kr\,dr$$

in the whole region $0 \leq r < \infty$, $0 \leq k < \infty$. He points out that one has to be careful to examine the convergence of each integral separately in order to eliminate the risk of having δ-functions arising from improper product integrals, viz. :

* When both were at Princeton (as stated by Froberg in his third paper [81], p. 2)

$$\lim_{N\to\infty} \frac{4}{\pi}\int_0^N \sin 2kr \sin 2kr' dk = \delta(r-r') - \delta(r+r')$$

He proceeds to prove the validity of the pair of integral transforms:

$$k \sin(\delta_V - \delta_U) = -\int_0^\infty (V-U)\, Y_k(r)\, dr$$

$$V - U = \frac{4}{\pi}\int_0^\infty \sin(\delta_V - \delta_U)\, Z_k'(r)\, dk \qquad (6)$$

where U is an auxiliary potential defining the solutions of the corresponding radial equation having the aymptotic forms:

$$\Psi_{1U} \backsim \sin\left(kr - \tfrac{1}{2}\pi\ell + \delta_U\right)$$

$$\Psi_{2U} \backsim \cos\left(kr - \tfrac{1}{2}\pi\ell + \delta_U\right)$$

and:

$$Y_k(r) = \Psi_{1V} \cdot \Psi_{1U}$$

$$Z_k(r) = \Psi_{1U}\Psi_{2V} + \Psi_{2U}\Psi_{1V}$$

δ_U, δ_V are the phase-shifts corresponding respectively to U(r) and V(r). Based upon Eq. (6), he constructs an iteration scheme in the following way:

a) Start from a good guess U(r) and, by solving the radial equation, obtain Ψ_{1U}, Ψ_{2U} and δ_U.

b) Replace the unkown functions Ψ_{1V} and Ψ_{2V} in Eq.(6) by

ψ_{1U} and ψ_{2U}, obtaining thus an approximate value of the difference, say $V_1 - U$, and consequently a first approximation $V_1(r)$ to the potential $V(r)$.

c) Put $U \equiv V_1$ and repeat the same process.

Applying this scheme to an inverse square potential, Hylleraas is led to the conclusion that if a U(r) can be found such that for all k the difference

$$|\delta_U - \delta_V|$$

is smaller than some given number, his method is convergent. He ends with the statement: ..." Thus it has been proved that a perturbing central field of force can be uniquely determined from the observed scattering phase-shift. "...

A few months later, Froberg's third paper [81] is published, in which he simplifies the first approximation and indicates how the higher order approximations are obtained. The simplification is the one already mentioned in the "add in proof" of his second paper, and higher approximations are examined in a substantial way only for S-waves, although a rather complicated formula for the second approximation is given for all l. Froberg uses known potentials as first approximation (for S-waves only) and proceeds to examine the corresponding second approximation using both his and Hylleraas method (the potentials used are Yukawa's, Gaussian and Well). Hylleraas' proof of convergence is criticized on grounds that improper integrals arise from the constant difference of phase-shifts produced by the inverse square potentials used in his proof. The paper ends with a procedure for obtaining the phase-shifts from the differential cross-section. In an Appendix, formulae in connection with Hadamard's "partie finie" of a divergent integral are given for several cases.

From this examination of Froberg's and Hylleraas' results, we feel that both investigations are somewhat inconclusive, inasmuch as they depend on iteration schemes whose convergence is not rigourously examined. Furthermore, the problem of giving conditions on a function of k in order that it be the phase-shift corresponding to some potential V(r) is left untouched.

Four months later, V. Bargmann[85] gives an example of two potentials having the same phase-shift, and points out an error in Hylleraas' argument. Bargmann shows that the potentials:

$$V_1(r) = \frac{-6\lambda^2 e^{-\lambda r}}{(1+e^{-\lambda r})^2}$$

$$V_2(r) = \frac{-24\lambda^2 e^{-2\lambda r}}{(1+3e^{-\lambda r})^2}$$

yield the same Jost function and, consequently, the same phase-shift (for S-waves only). He mentions, according to Moller[86], that infinitely many different Hamiltonians can be found which yield the same phase-shifts. However, he remarks that we cannot be sure to find among them one which corresponds to an ordinary central field of force. Commenting on Hylleraas' results, he first modifies Eq. (6) by interchanging differentiation with respect to r and integration with respect k, obtaining

$$V - U = \frac{4}{\pi} \frac{d}{dr} \int_0^\infty \sin [\delta_V - \delta_U] \chi_k(r)\, dk$$

By doing so he eliminates the appearance of the divergent integrals objected to by Froberg. Nevertheless, the incorrectness in Hylleraas argument does not lie here but rather in his assuming two equations (used in his proof of validity of equation 6) to be equivalent - Bargmann shows that in fact, they are independent since one corresponds to a completeness relation, while the other represents an orthogonality relation ; moreover, the former does not seem to hold when either V or U support bound states, because it is well-known that in this case the "improper eigenfunctions" alone do not form a complete system. Bargmann ends his comments by conjecturing that Eq. (6) may be generally valid if the potentials V <u>and</u> U do not support bound states. An "add in proof" is given, in which Bargmann mentions that potentials having same phase-shifts (S-waves) but different discrete energy levels, as well as a non-vanishing potential which causes no S-scattering, can be constructed.

A few months later, a second paper by Bargmann [87] is published, in which he develops in a more systematic way the construction of (S-wave) phase-equivalent potentials. He also gives N. Levinson's results (communicated to him prior to publication):

" two potentials decreasing sufficiently fast at infinity are identical if they have i) same phase-shifts for all l [88] ii) same phase-shift for $l = 0$ and neither supports bound states [89] ".

Bargmann points out, according to Kramers, Ma and Jost, that the S-matrix may have "redundant" zeros, i.e., zeros which do not correspond to bound states, and therefore in general the phase-shifts do not determine the energy of bound states. However, he proves that if

$$\int_0^\infty r\,|V(r)|\,dr < \infty \qquad (7)$$

two potentials having the same Jost function are phase-equivalent and have same bound states, which are given by the zeros of the Jost function on the negative * imaginary axis. In addition, he proves that a zero-energy bound state cannot occur if (7) holds. He also states, according to a remark by Levinson that if $f(k)$ is differentiable (including the origin) and

---

* The Jost solution is defined using $f(r,k) \to e^{-ikr}$ when $r \to \infty$

$$\int_0^\infty r^2 |V(r)| dr < \infty$$

it can be shown that the number m of bound states is finite and

$$\eta(0) = \begin{cases} -m\pi & f(0) \neq 0 \\ -(m+\frac{1}{2})\pi & f(0) = 0 \end{cases}$$

(assuming $\eta(k)$ to be normalized by $\eta(\infty) = 0$)

A systematic construction of phase-equivalent potentials (S-waves) is then undertaken, by defining:

$$\chi(r,k) = e^{ikr} \varphi(r,k)$$

where $\varphi(r, k)$ is the regular solution of Eq. (1) which vanishes at the origin. $\chi(r, k)$ obviously satisfies the equation:

$$\chi'' - 2ik\chi' - V(r)\chi = 0$$

assuming χ's such that $\lim_{r \to \infty} \chi(r, k) = \chi(\infty, k) \neq 0$, the Jost function is obtained in terms of χ :

$$f(k) = \frac{\chi(0,k)}{\chi(\infty,k)}$$

once $\chi(r, k)$ is known, V(r) is immediately obtained :

$$V(r) = \frac{\chi'' - 2ik\chi'}{\chi}$$

Bargmann comprehensively treats the two cases:

a) $\chi(r, k) = 2k + i\, a(r)$

b) $\chi(r, k) = 4k^2 + 2i\, a(r)\, k + b(r)$

where the undetermined coefficients a(r) and b(r) are obtained through differential equations using the fact that V(r) is independent of k. These choices for $\chi(r, k)$ produce the potentials:

1. $$V_1(r) = \frac{\rho\sigma \{ 4\sigma\rho + (\rho-\sigma)^2 \cosh[(\rho+\sigma)\kappa - 2\theta] - (\rho+\sigma)^2 \cosh(\rho-\sigma)\kappa \}}{\{ \sigma \sinh(\rho\kappa - \theta) - \rho \sinh(\sigma\kappa - \theta) \}^2}$$

for which the Jost function is:

$$f_I(k) = \frac{2k + i(\rho+\sigma)}{2k - i(\rho-\sigma)}$$

$f_I(k)$ is independent of θ, therefore $V_1(r)$ forms a continuous family of phase-equivalent potentials, all having a bound state of energy

$$E_I = -\frac{1}{4}(\rho+\sigma)^2$$

2. $$V_2(r) = \frac{\rho\sigma \{ 4\rho\sigma + (\rho-\sigma)^2 \cosh(\rho+\sigma)\kappa - (\rho+\sigma)^2 \cosh[(\rho-\sigma)\kappa + 2\theta] \}}{\{ \sigma \sinh(\rho\kappa + \theta) - \rho \sinh(\sigma\kappa - \theta) \}^2}$$

$$V_3(r) = -2\frac{\rho}{\sigma} \cdot \frac{(\rho+\sigma)^2 e^{-(\rho+\sigma)\kappa}}{\left[1 + \frac{\rho}{\sigma} e^{-(\rho+\sigma)\kappa} \right]^2}$$

$$V_4(r) = \frac{-\rho\sigma\left\{4\rho\sigma + (\rho-\sigma)^2\cosh(\rho+\sigma)r + (\rho+\sigma)^2\cosh\left[(\rho-\sigma)r - 2\theta\right]\right\}}{\left\{\sigma\cosh(\rho r-\theta) + \rho\cosh(\sigma r+\theta)\right\}^2}$$

all of which have the same Jost function:

$$f_{II}(k) = \frac{2k + i(\rho-\sigma)}{2k - i(\rho+\sigma)}$$

V_2, V_3 and V_4 are phase-equivalent, having one boud state with energy

$$E_{II} = -\frac{1}{4}(\rho-\sigma)^2$$

Bargmann remarks that V_1, V_2, V_3 and V_4 are all phase-equivalent, since:

$$\frac{f_I(k)}{f_I(-k)} = \frac{f_{II}(k)}{f_{II}(-k)} = e^{2i\eta(k)}$$

3. $$V_5(r) = 2\rho^2 \cdot \frac{\rho r\sinh\rho r - 2(\cosh\rho r - 1) - \delta}{(\sinh\rho r - \rho r + \delta)^2}$$

for which the Jost function is

$$f_{III}(k) = \frac{2k + i\rho}{2k - i\rho}$$

$f_{III}(k)$ is independent of δ, hence V_5 forms a continuous family of phase-equivalent potentials all having one bound state with energy

$$E_{III} = -\frac{1}{4}\rho^2$$

4. $$V_6(r) = \frac{6(r-\alpha)\left[(r-\alpha)^3 - 2\gamma^3\right]}{\left[(r-\alpha)^3+\gamma^3\right]^2} \qquad (\gamma > \alpha \geq 0)$$

for which the Jost function is

$$f_{IV}(k) = 1 + \frac{3\alpha\left[1-i\alpha k\right]}{(\gamma^3-\alpha^3)k^2}$$

Bargmann remarks that for $\alpha = 0$, $f(k) \equiv 1$, and therefore we have here a continuous (in γ) family of zero-phase equivalent potentials, having a zero-energy bound state. In addition, if $\alpha > 0$ by a suitable choice of α and γ, V_6 can be made phase-equivalent to an Eckart potential which has no bound state (although V_6 supports a bound state).

Bargmann points out that these anomalies are related to the slow decrease [$V_6 \sim \frac{6}{r^2}$ $r \to \infty$] of the potentials considered, and ends with the remarks:

i) if $f(r, k)$ is entire in the complex k-plane, the results of the S-matrix theory hold. He proves that $f(r, k)$ is entire if

$$\int_0^\infty e^{\alpha r} |V(r)| \, dr < \infty$$

for all $\alpha > 0$. [Bargmann was apparently unaware of Stone's results (18) which can be easily shown to be equivalent to his].

ii) Levinson's result (already mentioned) can be put under the form

if

$$V_i(r) + \frac{\ell(\ell+1)}{r^2} \geq 0 \qquad \text{and} \qquad \int_0^\infty r\,|V_i(r)|\,dr < \infty \quad (i=1,2)$$

then

$$\eta_l^{(1)}(k) = \eta_l^{(2)}(k) \quad \text{for all } l \text{ implies } V_1(r) \equiv V_2(r) .$$

Almost at the same time, an important work by N. Levinson (89) comes out, in which he proves the uniqueness of the potential for a given S-wave phase-shift, under certain conditions. His results are supported by rigorous mathematical proofs. We shall limit ourselves to sketch his main points:

I. If V(r) is piecewise continuous, non-negative, and $rV(r) \in L_1(0, \infty)$,

i) $\eta(k)$ determines V(r) uniquely

ii) A(k) and $\eta(k)$ are continuous functions of k, where

$$f(k) = A(k)\, e^{i\,\eta(k)}$$

is the Jost function*.

iii) $\eta(k)$ determines A(k) uniquely and conversely

II. If V(r) is measurable and real and

$$\int_0^1 r|V(r)|\,dr + \int_1^\infty r^2|V(r)|\,dr < \infty$$

i) V(r) is uniquely determined by $\eta(k)$ if there are no eigenvalues.

ii) the result on $\eta(0)$ already mentioned by Bargmann holds, and consequently, if $\eta(\infty) - \eta(0) < \pi$

there are no bound states

iii) $\eta(k)$ determines A(k) uniquely and conversely in this case.

III. If $u(r) \in L_2(0, \infty)$ the Parseval equation holds:

$$\int_0^\infty |u(r)|^2 dr = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{k^2}{[A(k)]^2} dk \left| \int_0^\infty \varphi(r,k) u(r) dr \right|^2$$

IV. If $rV(r) \in L_1(o, \infty)$, $\varphi(r, k)$ is an entire function of

$p = \sigma + i\tau$ which for all p satisfies:

$$|\varphi(r,p)| \leq \frac{K r e^{|\tau| r}}{1 + |p| r}$$

and as $|p| \to \infty$

$$\varphi(r,p) = \frac{\sin pr}{p} + o\left(\frac{e^{|\tau| r}}{|p|} \right)$$

V. The following bound is derived for $f(r, p)$:

$$\left| f(r,p) - e^{ipr} \right| \leq \frac{K e^{-\tau r}}{|p|} \int_x^\infty |V(\xi)| d\xi \qquad (\tau \geq 0)$$

VI. If V(r) is non-negative, f(p) is analytic for $\tau > 0$ and continuous for $\tau \geq 0$. For $\tau \geq 0$, f(p) vanishes only for p on the imaginary axis. If $k_n = i\tau_n$ ($\tau_n > 0$) is such a zero of f(p), then $\varphi(r, i\tau_n)$ is a proper eigenfunction of Eq. (1) satisfying :

$$\varphi(r, i\tau_n) = C_n f(r, i\tau_n) \to 0 \qquad r \to \infty$$

$$C_n \neq 0$$

for large $|p|$:

$$f(p) = 1 + o(1)$$

uniformly for $0 \leq \arg p \leq \pi$.

We notice that, besides Bargmann, also Levinson seems to have been unaware of Stone's results [18] which are somewhat overlaped here.

VI. For $\varepsilon > 0$:

$$\log f(p) = \frac{2p}{\pi i} \int_0^\infty \frac{\log A(k)}{k^2 - p^2} dk$$

and

$$\eta(k) = \lim_{\varepsilon \to +0} \operatorname{Im} \log f(p)$$

Levinson ends by adding in proof (March, 1949) a generalization of his method for higher angular momenta, as already mentioned in Bargmann's paper.

Approximately at the same time, G. Borg [90] obtains similar results, basing his investigation on the Weyl-Titchmarsh function m(k). Since only the formalism differs, we shall not pursue a detailed description of his method.

In 1950, V. A. Marchenko [69] shows that the spectral function determines V(r) uniquely (see Sec. 2.2).

The issue was then : what must be prescribed, in addition to the phase-shift, in order to determine the spectral function ? Levinson and Borg had already proved that, if there is no discrete spectrum, V(r) is uniquely determined by the phase-shift. The lack of uniqueness was therefore related to the existence of bound states. On the other hand, Bargmann (87) had given examples where same phase-shift and same bound states where produced by different potentials.

In 1951, the fundamental papers of Gelfand and Levitan (71) provided an effective way of reconstructing V(r) from the spectral function (see Sec. 2.2) - no application to scattering theory was mentioned.

In 1952, Marchenko (91) shows that the phase-shift determines only the continuous part of the spectral function ; he shows that the additional data required when there is a discrete spectrum is formed by the eigenvalues plus related normalization constants.

At approximately the same time, R. Jost and W. Kohn (92), unaware of the recent Russian developments, and based upon Bargmann's and Levinson's results, describe two methods of constructing V(r) from $\eta(k)$ (S-waves) in the form of series. They point out that Levinson's first result can also be used by choosing a phase-shift corresponding to an l so high as to exclude bound states, thereby determinig V(r) uniquely. A convergence proof for the first method is given, and it is shown that the convergence conditions are not very restrictive.

Comments are offered on the limited usefulness of these procedures to the analysis of two-nucleon systems.

The series solution is rather involved. Furthermore, if there are bound states, the method yields only one potential out of the corresponding family. Their results are applied to Eckart and exponential potentials - convergence is found to be quite rapid in both cases.

The second series method is based upon the equation:

$$-k \tan \eta(k) = \int_0^{\infty} \sin k\kappa \, V(\kappa)\, \varphi(\kappa,k)\, d\kappa$$

They conclude by surveying possible generalizations of the method to $l > 0$, tensor forces, Coulomb fields and relativistic scattering.

In an Appendix, a generalization of Levinson's theorem is given, according to which if there are m bound states, in order to determine the potential uniquely one must give m additional positive parameters C_i , related to the normalization of the proper eigenfunctions. We remark that this result was already implicit in Marchenko's analysis [(91)].

An add in proof mentions their recent acquaintance with Gelfand-Levitan's results [71]. Also, the discovery of a complete set (in C_i) of phase equivalent potentials having the same binding energies is reported. - they refer to B. Holmberg [93] who independently finds some of their results.

B. Holmberg in July 1952 [93] examines the nonuniqueness of the potential when there are bound states. He seems to be unaware of Levinson's results, as well as of the recent developments made by Marchenko, Borg and Gelfand-Levitan, and bases his analysis upon Froberg's and specially Hylleraas' results. Using Hylleraas' functions $Z_k(r)$ he gives an explicit formula for the difference between two phase-equivalent potentials having one bound state with same binding energy, if one special potential and its corresponding continuous eigenfunction is known.

Immediately after, comes out a beautiful paper by R. Jost and W. Kohn [94], where an explicit method for the construction of an entire class of potentials having (for a fixed l) same phase-shift and energy spectrum, is given. The analysis carries over to potentials with purely discrete spectra. Although the authors are aware of the Gelfand-Levitan formalism, they do not use it in this investigation. They start by assuming an infinitesimal change $\delta V(r)$ of the potential and giving the corresponding changes of the phase-shift and binding energies:

$$\delta E_n = \int_0^\infty \delta V(r) \, [\varphi_n(r,k)]^2 \, dr$$

$$\delta \eta(k) = -\frac{1}{k} \int_0^\infty \delta V(r) \, [\varphi(r,k)]^2 \, dr$$

where $\varphi_n(r, k)$ is the proper eigenfunction corresponding to E_n, and $\varphi(r, k)$ is the regular solution normalized to

$$\sin [\, kr + \eta(k) \,]$$

at infinity.

Consequently, if $\delta V(r)$ can be chosen to be orthogonal to the square of all eigenfunctions, the E_n's and $\eta(k)$ will remain unaltered. The authors proceed by proving that the functions :

$$\varphi_n'(r,k)\varphi_n(r,k)$$

have this property, and by judiciously integrating the increments $\delta V(r)$ they obtain the family of potentials

$$V(\lambda, r) \equiv V(r) + \frac{\lambda}{N} f_1(r) f_1'(r) + \frac{\lambda^2}{8N^2} [\, f_1(r) \,]^4$$

where:

$$N = 1 + \frac{\lambda}{4} \int_r^\infty f_1^2(r') \, dr'$$

$$f_1(r) = f(r, -i\zeta_1)$$

$$E_1 = -\zeta_1^2 \qquad \text{is the binding energy and}$$

$$f(r, k) \longrightarrow \exp[-ikr] \qquad \text{when } r \longrightarrow \infty$$

They use the fact that this family is complete to implement their series construction of $V(r)$ - all potentials can now be obtained by conveniently varying λ.

A proof is given that if the discrete spectrum and the phase-shift are given for two angular momenta, the potential is uniquely determined.

A potential equivalent to the deuteron square well is shown. The same method is similarly applied to potentials with purely discrete spectra. Potentials equivalent to respectively the one-dimensional box and the harmonic oscillator are shown as examples.

It is only in 1953 that, almost simultaneously and independently, N. Levinson and Jost-Kohn take advantage of the Gelfand-Levitan formalism. We shall first discuss Jost-Kohn's paper [95], which was received slightly before Levinson's (respectively September and October 1952).

Gelfand-Levitan's work [71] was brought to the attention of Jost-Kohn by Lars Gårding. They immediately adapted it to the which incorporates the phase-shift, the binding energies and m additional parameters, where m is the number of bound states.

Unaware of the second paper by Gelfand-Levitan (71), the authors first adapt their results to the boundary condition $\varphi(o, k) = 0$, not included in Gelfand-Levitan's first results.

They assume $rV(r)$ and $r^2V(r)$ to be absolutely integrable and proceed to define the spectral function $\rho(E)$ $[E = k^2]$ in terms of familiar "quantum scattering" quantities, by the relations:

i) $$\rho(-\infty) = 0$$

ii) $$\frac{d\rho}{dE} = \begin{cases} \sum_{n=1}^{m} C_n \, \delta(E - E_n) & E < 0 \\ \frac{1}{\pi} \frac{\sqrt{E}}{|f(\sqrt{E})|^2} & E \geq 0 \end{cases}$$

iii) $$C_n = \frac{1}{\int_0^\infty [\varphi(\kappa, E_n)]^2 \, d\kappa} = \frac{2i\tau_n f'(0, i\tau_n)}{\dot{f}(-i\tau_n)}$$

iv) $$\dot{f}(-i\tau_n) = \left\{ \frac{d}{dk} f(k) \right\}_{k = -i\tau_n}$$

We have already discussed some properties of the spectral function, as well as the Gelfand-Levitan algorithm (see Sec. 2.2), therefore we shall only briefly sketch the two applications given by Jost and Kohn:

a) The dependence on the m parameters C_n for a family of phase-equivalent potentials having same binding energies is found to be:

$$V(r) = V_1(r) - 2 \frac{d^2}{dx^2} \log \mathrm{Det} \; \| M_{ij}(r) \|$$

where $V_1(r)$ is the particular potential corresponding to C_{1n} and

$$M_{ij}(r) = -\delta_{ij} - (C_i - C_{1i}) \int_0^x \varphi_{1i}(t)\varphi_{1j}(t)dt$$

$$\varphi_{1j}(t) = \varphi_1(x, E_j)$$

b) The independence of the binding energies from the phase-shift is examined and, as an example, the authors explicitly construct all potentials corresponding to the phase shift $\eta(k)$ given by

$$k \cot g\, \eta(k) = -\alpha + \frac{1}{2} x_0 k^2$$

$$\alpha > 0$$

$$x_0 > 0$$

$$2\alpha x_0 < 1$$

and a bound state located at an arbitrary negative energy E_1. They find:

$$V(r) = V_1(r) - 2C \frac{\partial}{\partial r} \frac{\{\varphi_1(r,E_1)\}^2}{1 + C\int_0^r \{\varphi_1(t,E_1)\}^2 dt}$$

where $V_1(r)$ is one of Bargmann's [87] potentials.

As we mentioned before, N. Levinson independently applied [96] the Gelfand-Levitan formalism to the inverse scattering problem, arriving at some of the conclusions of Jost-Kohn and giving the approximate formula for the variation of $V(r)$ with $\rho(\lambda)$:

$$\delta V(r) \sim -4\int_{-\infty}^{+\infty} \varphi(r,k)\varphi'(r,k)\, d[\delta\rho(\lambda)]$$

We hope to have given a rather detailed description of the earlier investigations on the inverse scattering problem, which indeed laid the foundations for its future development. From this point up to 1959, the literature on the subject is covered in a comprehensive survey article by L. D. Faddeyev [97] which appeared in Russia in 1959 and was published in English in 1963.

Consequently, from now on only papers bearing a direct relevance to our objectives shall be focused upon and we shall describe them accordingly in the corresponding sections where they belong. For the remaining literature, we refer to the following sources, besides Faddeyev's review :

In chapter I, section G of their book [7], Wu and Ohmura give 20 references on the inverse scattering problem, and describe in an Appendix the generalized formulations of I. Kay and H. E. Moses.

Another important review article by R. Newton [8] came out in 1960, in which proofs of the main results in both direct and inverse radial problems can be found.

De Alfaro and Regge [1], in chapter 12 of their book, derive the main results relevant to the radial inverse problem.

A rigorous, pure-mathematical treatment of the radial inverse problem is given by Agranovich and Marchenko [98] in a book which was translated into English in 1963.

In Dunford and Scwartz's [14] treatise, part II, p. 1622, we find a discussion on the inverse problem presented in a pure-mathematical form.

Finally, a very recent article by P. Swan and W. A. Pearce [99] came out in April 1966 in which a critical review is made, a new approximation method is derived, and an extensive bibliography is given.

Not included in neither of these sources are the following works:

A. A. Kostarev, in 1964 (100) and 1965 (101) discusses a perturbation theory of the inverse problem for a certain class of S-matrices, the convergence of the series obtained being proved for this class. He bases his method upon Marchenko's formulation for the radial inverse problem.

G. Burdet and M. Giffon, in 1964 (102, 103) and 1965 (104) discuss the inverse problem for fixed energy and varying angular momentum. Defining a spectral function for this problem, they prove that $r^{-1} . f(l, k, r)$ form a complete system in the complex l-plane, where f is the Jost solution, defined by $f(l, k, r) \rightarrow \exp[-ikr]$.

Further papers which have only an indirect relevance to our problem and are not mentioned in this literature are L. D. Faddeyev's (105) and M. M. Lavrent'ev's (106, 107).

2.4 The One-Dimensional Inverse Problem

As we mentioned earlier, the bulk of the results on the inverse scattering problem concern the radial equation. To our knowledge, only four authors have focused their attention on the one-dimensional case on a theoretical basis.

The first investigation seems to have been made by A. Sh. Bloch[119] which reconstructs the potential from the spectral matrix function.

H. E. Moses and I. Kay[108, 109, 110, 111] have comprehensively treated in a series of papers, the inverse problem in a more general formulation which not only clarified many of the ideas involved in the Gelfand-Levitan solution, but also allowed for degeneracies thus containing the one-dimensional case as a special application.

In several other publications[112, 113, 114, 115, 116, 117] they have developed their theory for the one-dimensional problem and pointed out its applications to some electromagnetic problems. We shall describe later their main results in this direction.

In 1958, Faddeyev[118] gives a realizability condition for potentials such that :

$$\int_{-\infty}^{+\infty} [1 + |x|] \, |V(x)| \, dx < \infty$$

in terms of the reflection coefficients "from both sides" [wave from $-\infty$ and wave from $+\infty$].

However, his condition seems to be unpractical, since it involves both reflection coefficients - one of which is unknown . One should like to have a condition on the quantity chosen as datum of the problem, as for example on a reflection coefficient alone (from either side, but just one). This point will appear in all its importance if it is understood that both the algorithm for determining the potential and the "realizability condition" must apply to the same set of quantities chosen as scattered data.

2.5 Applications to Electromagnetic Problems

We have already mentioned that this constitutes a vastly unexplored subject - we only know of two authors which have focused their attention in such problems:

C. B. Sharpe, in a series of two papers (120, 121) is concerned with the synthesis of non-uniform lines having a rational input admittance. Necessary and sufficient conditions are given for a rational function to be realizable as the input admittance of an infinite line. He gives a closed-form expression of the characteristic impedance $Z_0[x]$ of a line in this class as a functional of its input admittance. Uniqueness of the solution is also examined.

However, he fails to "terminate the line" , viz., to give conditions in order that $Z_0[x]$ be a constant for x greater than some x_0 . As a matter of fact, his choice of rational input admittances automatically excludes this possibility, as indeed he proves in his second paper [121].

G. L. Brown, in a recent (1965) Ph. D. thesis [122], studies the class of non-uniform transmission lines which are non-uniform in a finite interval, say $0 \leq r \leq D$.

Starting from the usual transmission lines equations:

$$C(x)\frac{\partial}{\partial t}V(x,t) + \frac{\partial}{\partial x}I(x,t) = 0$$

$$L(x)\frac{\partial}{\partial t}I(x,t) + \frac{\partial}{\partial x}V(x,t) = 0$$

and assuming $C(r) \equiv C_0$ and $L(r) \equiv L_0$ for $r > D$, he studies the initial boundary value problem for I and V when $I(0,t) \equiv 0$ and the initial state $[\, I(r,0) \, ; \, V(r,0) \,]$ is required to be in the Hilbert space defined by the energy integral:

$$\int_0^\infty \left\{ L(x)\left|I(x,0)\right|^2 + C(x)\left|V(x,0)\right|^2 \right\} dx$$

Defining the reflection coefficient $R(\omega)$ as usual, he is concerned in the first and second parts of the thesis with the direct problem, both in its steady-state and transient formulation. The last part focuses on the inverse problem of determining $L(r)$ and $C(r)$, given the reflection coefficient $R(\omega)$ and the function:

$$j(x) = \int_0^x \left[C(\xi)L(\xi)\right]^{\frac{1}{2}} d\xi$$

The procedure is analogous to the Gelfand-Levitan method for the radial inverse scattering problem.

However, also he fails to "terminate the line". We quote:

..." Unsolved is the problem of finding conditions on $R(\omega)$ to ensure that it corresponds to a set [C, L] from a given class. "...

He puts forward a very valuable suggestion (p. 15) :

... " A different approach to the inverse problem of quantum scattering theory due to Agranovich and Marchenko leads to simpler sufficient conditions for the solvability of the inverse problem. It is probable that a similar approach can be developed for the transmission line problem "...

As a matter of fact, it is indeed such an approach that we shall use in our main problem (chapter V).

CHAPTER III

THE INVERSE PROBLEM

3.1 Radial Case

We shall give in this section the main results in which we shall be interested concerning the radial problem, restricting ourselves to the zero-angular momentum case.

3.1.1 The Jost function and its Analytic Properties

We shall make the following assumptions on the potential $V(r)$:

$$V(r) \equiv 0 \qquad \text{for } r > R \tag{8}$$

$$V(r) \text{ real and piecewise continuous} \tag{8a}$$

These conditions cover all cases in which we shall presently focus our attention.

There exists a solution $f(r, k)$ of equation (1) defined by the boundary condition :

$$f(r, k) \equiv e^{ikr} \qquad \text{for } r \geqslant R \tag{9}$$

It can be easily shown that $f(r, k)$ satisfies the Volterra linear integral equation:

$$f(r,k) = e^{ikr} - \int_r^\infty \frac{\sin k(r-r')}{k} V(r') f(r',k)\, dr' \tag{10}$$

For real values of k, it follows from (1) and (8a) :

$$f(r, -k) \equiv f^{*}(r, k) \qquad (11)$$

Since we shall extend the definition of f(r, k) to complex values of k (hereafter designated by $p = \sigma + i\tau$), Schwartz's reflection principle shows that in any region of analyticity connected with the real axis, we must have:

$$f(r, -p^{*}) \equiv f^{*}(r, p) \qquad (11a)$$

Equation (10) can be solved by iteration, and the series of iterations converges uniformly for any finite p. Consequently, f(r, p) is an entire function of p.

The function :

$$f(p) \equiv f(0, p)$$

is the Jost function [2] for the potential V(r).

Two very important results to our objective concerning potentials satisfying (8) were proved respectively by T. Regge [28] and A. G. Ramm [123].

Regge, in 1958, proves that f(p) is an entire function of order one and type 2R. He gives the following properties of f(p) :

i) $\lim_{|p| \to \infty} f(p) = 1$ on the real axis and upper half plane (12)

(this result can be found in Levinson's paper [89] in a more complete form)

ii) f(p) has an infinite number of zeros, but only a finite number of them lies on the imaginary axis. (13)

Assuming that V(r) can be developed near $r = R$ in an asymptotic series whose principal term is

$$C\,[\,R - x\,]^{\lambda}$$

he proceeds to show that all iterated but the first Born approximation give a negligible contribution for large $|p|$ in the lower half-plane, and gives the principal term in the asymptotic series for f(p) :

$$f(p) \sim C\,\frac{e^{2ikx}}{k^{\lambda+2}} \tag{14}$$

inducing from (14) that: f(p) is of order one and type 2R. (15)

and that on any ray in the lower half-plane there is at most a finite number of zeros of f(p). He also gives the Hadamard's expansion of f(p) :

$$f(p) = f(0)\,e^{ikR} \prod_{n=1}^{\infty} \left(1 - \frac{p}{p_n}\right) \tag{17}$$

In a later paper (124), Regge simplifies Gelfand-Levitan's procedure by examining the physical aspect of the problem. We quote:

..." The extreme tail of the potential has usually no bearing on physically measurable quantities. If we truncate a Gaussian potential, by letting it vanish identically, farther than several times its range, we shall cause a little change in the phase-shifts but drastic changes in the position of the large zeros of the Jost function. The small zeros will remain almost unaffected."...

He proceeds to examine the Gelfand-Levitan algorithm, making the approximation that $f(k)$ is a polynomial, based on the remarks quoted above. $f(k)$ has then a finite number of zeros k_n ; he considers the solutions $\varphi(r, k_n)$ such that :

$$\varphi(0, k_n) = 0$$
$$\varphi'(0, k_n) = 1$$

and shows that they form a complete set in the interval $[0, 2R]$, by means of the inverse Paley-Wiener's theorem. He uses this fact to prove that the set:

$$\varphi_0(r, k_n) = \frac{\sin k_n \kappa}{k_n}$$

is also complete in $[0, 2R]$. The mathematical machinery thus developed is then used to simplify Gelfand-Levitan's algorithm.

A. G. Ramm, in 1965, proves that a necessary and sufficient condition for the potential to be of finite range is that $f(p)$ is an entire function with an order of growth not higher than the first, such that $f(p) - 1$ is square-integrable.

Combining Ramm's and Regge's results, we can state:

A sufficient condition for $V(r) \equiv 0$ for $r > R$ is that $f(p)$ be an entire function of order one and type 2R, such that $f(p) - 1$ is square integrable.

This result shall find application in section 5.3.

To end this discussion on the Jost function, let us now show the connection between $f(k)$ and the scattering phase-shift and amplitude. Consider the solution of equation (1) defined by the initial values :

$$\begin{aligned} \varphi(0, k) &= 0 \\ \varphi'(0, k) &= 1 \end{aligned} \qquad (18)$$

It can be shown [96] that for large r :

$$\varphi(r, k) \sim \frac{\alpha(k)}{k} \sin\left[kr - \eta(k)\right] \qquad (19)$$

$\alpha(k)$ is the asymptotic amplitude and $\eta(k)$ the phase-shift.

It can be easily shown that :

$$f(k) = \alpha(k) e^{i\eta(k)} \qquad (20)$$

Quantities of interest in the scattering problem are also:

$$S(k) = e^{-2i\eta(k)} = \frac{f(-k)}{f(k)} \qquad (21)$$

designated by "scattering function" and the "weight function" :

$$W(k) = \left\{ \frac{1}{\alpha(k)} \right\}^2 = \frac{1}{f(k)f(-k)}$$

3.1.2 The Radial Inverse Problem Algorithms

There are three main approaches to the radial inverse problem, which are closely related:

i) Gelfand and Levitan

ii) Agranovich and Marchenko

iii) Krein

Historically, the Gelfand-Levitan algorithm was the first real break-through in the radial inverse problem. However, we shall focus our attention in Agranovich-Marchenko's approach, which we find more suitable to our purpose.

Agranovich and Marchenko start with the scattering function $S(k)$ and prove the following theorem[97]:

" Any function $S(k)$ with the properties :

i) $\quad |S(k)| = S(\infty) = S(0) = 1 \qquad (23)$

ii) $\quad S(-k) = S^*(k) = S^{-1}(k) \qquad (24)$

iii) $\quad S(k) = 1 + \int_{-\infty}^{+\infty} F(t) \exp[-ikt]\, dt$ where $F(t)$ is absolutely integrable in $[-\infty, +\infty]$. $\qquad (25)$

iv) $\quad \left\{ \arg S(k) \right\}_{-\infty}^{+\infty} = -4im\pi \qquad (m \geq 0) \qquad (26)$

is the scattering function corresponding to a potential $V(r)$ having m bound states and a continuous spectrum along $[0, \infty]$. $V(r)$ may be a generalized function such as the derivative of a locally summable function. A necessary and sufficient condition for $rV(r)$ to be absolutely integrable in $[0, \infty]$ is that $tF'(t)$ be absolutely integrable in the same interval. "

In this thesis we shall assume $m = 0$, restricting ourselves to potentials which do not support bound states, unless explicitly stated.

The following inequalities are derived:

$$\left| F'(2r) + \frac{1}{4} V(r) \right| \leq C \left\{ \int_r^{\infty} |V(r')| \, dr' \right\}^2 \tag{27}$$

$$\left| F'(2r) + \frac{1}{4} V(r) \right| \leq C \left\{ \int_{2r}^{\infty} |F'(r')| \, dr' \right\}^2 \tag{28}$$

and the integral representation for $f(r, k)$ is obtained:

$$f(r,k) = e^{ikr} + \int_r^{\infty} A(r,r') e^{ikr'} \, dr' \tag{29}$$

where $A(r, r')$ is square integrable in r' [123].

From the linear Fredholm integral equation (the so-called Marchenko's equation):

$$A(r,r') = F(r+r') + \int_{r}^{\infty} A(r,\rho)\, F(\rho+r')\, d\rho \qquad r < r' \tag{30}$$

$A(r, r')$ is obtained, and the potential is simply:

$$V(r) = -2 \frac{d}{dr} A(r, r) \tag{31}$$

It is also proved that $f(k)$ can be reconstructed from $S(k)$. Normalizing the phase-shift to be zero at the origin and at infinity (no bound states) :

$$\eta(k) = \frac{i}{2} \log S(k)$$

the Wiener-Levi theorem shows that

$$\eta(k) = -\int_0^{\infty} \gamma(t) \sin kt\, dt \tag{32}$$

where $\gamma(t)$ is absolutely integrable in $[0, \infty]$. $f(k)$ is then obtained through:

$$f(k) = \exp \int_0^{\infty} \gamma(t) e^{ikt} dt \tag{33}$$

Before examining Gelfand-Levitan's and Krein's formalisms let us state the following result:

" A necessary and sufficient condition for $V(r) \equiv 0$ for $r > R$ is that :

$$F'(t) \equiv 0 \quad \text{for} \quad t > 2R \quad " \tag{34}$$

which follows immediately from the inequalities (27) and (28).

ii) Gelfand and Levitan

Starting with the weight function $W(k)$, satisfying

1. $W(k) > 0 \qquad W(-k) \equiv W(k)$
2. $W(k) = 1 + \int_{-\infty}^{+\infty} H(t)\exp[ikt]\,dt$ where $H(t)$ is absolutely integrable in $[-\infty, +\infty]$

they construct the funtion :

$$\Omega(r, r') = \frac{2}{\pi}\int_0^{\infty} \sin kr \,\sin kr' \,[W(k) - 1]\,dk \tag{35}$$

and solving the linear Fredholm integral equation [Gelfand-Levitan's equation] :

$$K(r, r') = -\Omega(r, r') - \int_0^{r} K(r, \rho)\,.\,\Omega(\rho, r')\,d\rho \qquad r' < r \tag{36}$$

they obtain the potential:

$$V(r) = 2\frac{d}{dr} K(r, r) \tag{37}$$

$f(k)$ can be reconstructed from $W(k)$ in the same way as from $S(k)$.

iii) Krein's approach

from the Gelfand-Levitan analysis it is clear that

$$\Omega(\kappa,\kappa') = H(\kappa-\kappa') - H(\kappa+\kappa')$$

where:

$$H(t) = \frac{1}{\pi}\int_0^\infty [W(k)-1]\cos kt\, dk$$

Looking for solutions of the Gelfand-Levitan equation under the form:

$$K(\kappa,\kappa') = \Gamma_{2\kappa}(\kappa-\kappa') - \Gamma_{2\kappa}(\kappa+\kappa')$$

one obtains for $\Gamma_{2k}(t)$ the equation (Krein's equation) :

$$\Gamma_{2\kappa}(t) + H(t) + \int_0^{2\kappa} \Gamma_{2\kappa}(s)\, H(s-t)\, ds = 0 \qquad (38)$$

which of course is equivalent to the Gelfand-Levitan equation.

3.2 One-Dimensional Case

We would like first to give some results concerning the solutions of Equation (2), where, unless otherwise stated, V(x) is assumed to be real, bounded, piecewise continuous and absolutely integrable in $[-\infty, +\infty]$.

3.2.1 Spectrum of the Operator $L \equiv -\frac{d^2}{dx^2} + V(x)$

For the classes of potentials assumed here, L has a spectral representation with a spectrum composed of a discrete and a continuous part. The eigenfunctions corresponding to values of k in the discrete spectrum are usually called "proper eigenfunctions", whereas the "improper eigenfunctions" correspond to values of k in the continuum. The proper eigenfunctions are those solutions of equation (2) which are square-integrable in the real line. These solutions must vanish exponentially at infinity. From their asymptotic forms:

$$\psi(x,k) \sim A_+ e^{ikx} + B_+ e^{-ikx} \qquad x \to +\infty \tag{39}$$

$$\psi(x,k) \sim A_- e^{ikx} + B_- e^{-ikx} \qquad x \to -\infty \tag{40}$$

it follows that k must be complex [complex values of k shall be alternatively designated hereafter by $p = \sigma + i\tau$] with $\tau \neq 0$. On the other hand, the eigenvalues are p^2, and these must be real since L is Hermitian - therefore $\sigma = 0$ and the discrete spectrum is composed of points $p = i\tau$ on the imaginary axis, for which the eigenvalues of L are negative numbers.

We recall our previous assumption that the potentials considered in this thesis <u>do not</u> support bound states - there exists no negative number E such that the equation $Lu = Eu$ has a square-integrable solution.

It might be of some interest to mention here a property of the spectrum of L which is particular to the one-dimensional range[125], i.e., for $x \in [-\infty, +\infty]$, and does not hold in more than one dimensional. Suppose $V(x) \not\equiv 0$ everywhere except for an interval (a, b) in which it assumes only non-positive values ; then, for any (a, b) and whatever non-positive values V(x) assumes in (a, b), L has at least one discrete eigenvalue - "there is a bound state for any attractive potential, no matter how weak"[9]. It can also be proved that of all potentials V(x) with the same value of

$$\int_{-\infty}^{+\infty} V(x)\,dx$$

the δ-function potential :

$$V_\delta(x) = \delta(x-x_0)\int_{-\infty}^{+\infty} V(x')\,dx'$$

has the lowest energy (in units $\hbar = 2m = 1$):

$$E_\delta = -\left(\int_{-\infty}^{+\infty} V(x)\,dx\right)^2$$

If in addition to $V(x) \in L_1$ we assume that $x\,V(x) \in L_1$ and also that $k = 0$ is not in the discrete spectrum, then it can be shown that L has only a finite number of eigenvalues[126].

3.2.2 The Scattering Matrix

In the asymptotic forms (39) and (40) the coefficients $A_\pm$ and $B_\pm$ are independent of x but may depend on k.

We have assumed throughout this thesis a time factor $\exp[-i\omega t]$ where $\omega = kv$ and v is the velocity of the wave front in the medium of propagation - the potential only causes dispersion, leaving unaltered the value of v.

The asymptotic forms (39) and (40) are linear combinations of factors which can be physically interpreted as simple progressive waves moving to $+\infty$ or to $-\infty$ in the region where V(x) vanishes. A "left (to $-\infty$) moving wave" is then proportional to exp[-ikx] and a "right (to $+\infty$) moving wave" is proportional to exp[ikx].

It is of interest to consider as an "outgoing wave" a wave moving away from the scattering potential - it can be either a left moving wave at $-\infty$ or a right moving wave at $+\infty$. Accordingly, an "incoming wave" can be defined as a right moving wave at $-\infty$ or as a left moving wave at $+\infty$.

The set of incoming waves can be considered as a two-dimensional vector space Ω_i in which each vector Λ_i is a column where the upper component is the complex amplitude of the right moving wave and the lower component is the complex amplitude of the left moving wave. In a similar way, the set of all outgoing waves form a vector space Ω_o of vectors Λ_o.

It is easy to show that there is a linear relation between Λ_o and Λ_i. One way to do so is to use the properties of the Wronskian[126] of two solutions of Eq. (2). Let us briefly recall the definition of the Wronskian, which we shall use on several occasions:

$$W\{\varphi_1(x,k);\varphi_2(x,k)\} \equiv \varphi_1\frac{\partial\varphi_2}{\partial x} - \varphi_2\frac{\partial\varphi_1}{\partial x} \tag{41}$$

here, the important property of W is that if φ_1 and φ_2 are solutions of Eq. (2), then:

$$\frac{\partial}{\partial x} W\{\varphi_1 ; \varphi_2\} \equiv 0 \tag{42}$$

Using the asymptotic forms of an arbitrary pair of solutions of Eq. (2), we write:

$$W(-\infty) = W(+\infty)$$

and we find that there exists a two by two matrix S, such that:

$$\Lambda_o = S \Lambda_i \tag{43}$$

S is called the scattering matrix. It transforms each vector in the vector space of incoming waves into the corresponding vector in the vector space of outgoing waves.

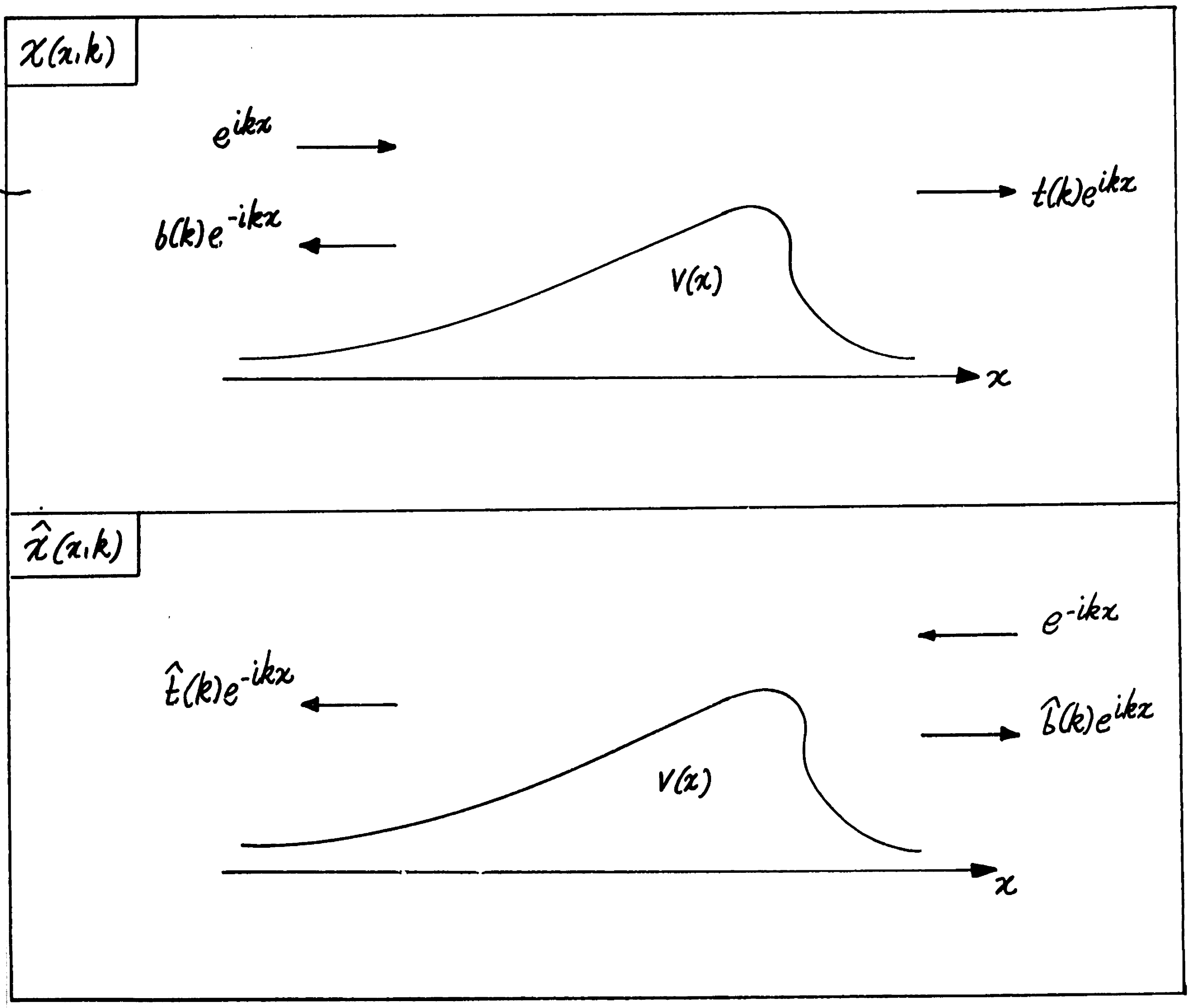
It is important to realize that S is independent of the particular solutions used to equate the values of the Wronskian at infinity - it is the same for all solutions of Eq. (2), depending uniquely upon V(x).

Let us then find the elements of S by considering two particular solutions $\chi(x, k)$ and $\hat{\chi}(x, k)$ defined by their asymptotic forms:

$$\chi(x,k) \sim \begin{cases} e^{ikx} + b(k)e^{-ikx} & x \to -\infty \\ t(k)e^{ikx} & x \to +\infty \end{cases}$$

$$\hat{\chi}(x,k) \sim \begin{cases} \hat{t}(k)e^{-ikx} & x \to -\infty \\ e^{-ikx} + \hat{b}(k)e^{ikx} & x \to +\infty \end{cases}$$

These solutions are physically described as sketched below:



This interpretation suggests the nomenclature:

$b(k)$: reflection coefficient "from the left"

$t(k)$: transmission coefficient "from the left".

$\hat{b}(k)$: reflection coefficient "from the right".

$\hat{t}(k)$: transmission coefficient "from the right".

Using these solutions in the definition of the scattering matrix:

$$\begin{bmatrix} t \\ b \end{bmatrix} = S \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \hat{b} \\ \hat{t} \end{bmatrix} = S \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

it follows from these relations that :

$$S = \begin{bmatrix} t(k) & \hat{b}(k) \\ b(k) & \hat{t}(k) \end{bmatrix} \tag{44}$$

Since V(x) is real, and Eq.(2) involves only k^2 , it follows that, for real values of k, if u(x, k) is a solution, u(x, -k) is also a solution, and we must have:

$$u(x, -k) \equiv u^{*}(x, k) \tag{45}$$

we deduce from this property that:

$$S(-k) = S^{*}(k) \tag{46}$$

Using again the Wronskian of two solutions of Eq.(2), now chosen as u(x, k) and u(x, -k) , it can be easily shown that, for real values of k:

reciprocity law $$t(k) = \hat{t}(k) \tag{47}$$

energy conservation law $$1 - |b(k)|^2 = 1 - |\hat{b}(k)|^2 = |t(k)|^2 \tag{48}$$

phase law $$b(-k)\,t(k) + \hat{b}(k)\,t(-k) = 0 \tag{49}$$

which express that , for real values of k, S is a unitary matrix.

We shall be interested in the analytic continuation of the quantities involved in our problem from real values of k to the complex plane $p = \sigma + i\tau$. Schwartz's reflection principle shows, in conjonction with (45), that in any region connected with the real axis (region of analyticity, of course) :

$$b(-p^{*}) = b^{*}(p) \tag{50}$$

$$\hat{b}(-p^{*}) = \hat{b}^{*}(p) \tag{51}$$

$$t(-p^{*}) = t^{*}(p) \tag{52}$$

3.2.3 Analytic Properties of S - Solutions φ and $\hat{\varphi}$

If the discrete spectrum of L is empty and V(x) decreases sufficiently fast at infinity (in any case, our interest lies on finite range potentials, so we shall not precise this point further), it can be proved[110] that the elements of S are regular in the upper half-plane $\mathrm{Im}\,p > 0$ and are continuous down to the real axis. In this region they have the asymptotic restrictions, for large $|p|$:

$$b(p) = O\left[\frac{1}{|p|}\right] \tag{53}$$

$$\hat{b}(p) = O\left[\frac{1}{|p|}\right] \tag{54}$$

$$t(p) = 1 + O\left[\frac{1}{|p|}\right] \tag{55}$$

A theorem due to M.H. Stone[18] considerably clarifies these statements. It is curious to remark that his important analysis, although published 40 years ago, seems to have passed unnoticed in this field. Stone shows that if V(x) is absolutely integrable, then Eq.(2) has two linearly independent solutions $\varphi(x, p)$ and $\hat{\varphi}(x, p)$ which, as functions of p, are analytic in the upper half-plane $\mathrm{Im}\,p > 0$ and continuous in $\mathrm{Im}\,p \geqslant 0$, except possibly at $p = 0$, having the asymptotic forms:

$$\varphi(x,p) = e^{-ipx}\left[\,1 + \frac{m(x,p)}{p}\,\right] \tag{56}$$

$$\hat{\varphi}(x,p) = e^{ipx}\left[\,1 + \frac{\hat{m}(x,p)}{p}\,\right] \tag{57}$$

where the m's are uniformly bounded, $-\infty < x < +\infty$, $\mathrm{Im}\,p \geq 0$, for large values of $|p|$.

In addition, he proves that if there is a positive number H such that:

$$\int_{-\infty}^{+\infty} e^{2H|x|}\,|V(x)|\,dx < \infty \tag{58}$$

then $\varphi(x, p)$ and $\hat{\varphi}(x, p)$ are regular in the half-plane $\mathrm{Im}\,p > -H$, except possibly for $p = 0$.

This theorem is proved using the following integral equations for φ and $\hat{\varphi}$:

$$\varphi(x,k) = e^{-ikx} + \int_{-\infty}^{x} \frac{\sin k(x-x')}{k}\, V(x')\,\varphi(x',k)\,dx' \tag{59}$$

$$\hat{\varphi}(x,k) = e^{ikx} - \int_{x}^{\infty} \frac{\sin k(x-x')}{k}\, V(x')\,\hat{\varphi}(x',k)\,dx' \tag{60}$$

For finite range potentials such as we are considering, condition (58) is satisfied for <u>any</u> positive number H . Consequently, we infer from Stone's theorem that φ and $\hat{\varphi}$ are regular functions of p in the entire p-plane, except possibly for p = 0.

Let us complete our knowledge of the asymptotic behaviour of φ and $\hat{\varphi}$; we already know that :

$$\varphi(x,p) \sim e^{-ipx} \qquad x \to -\infty \tag{61}$$

$$\hat{\varphi}(x,p) \sim e^{ipx} \qquad x \to +\infty \tag{62}$$

$\chi(x,k)$ and $\chi(x,-k)$ constitutes a fundamental set for Eq.(2). Consequently, $\hat{\varphi}(x,k)$ can be expressed as:

$$\hat{\varphi}(x,k) = \hat{\ell}_1(k)\chi(x,k) + \hat{\ell}_2(k)\chi(x,-k) \tag{63}$$

when $x \to +\infty$, we have:

$$e^{ikx} = \hat{\ell}_1(k)t(k)e^{ikx} + \hat{\ell}_2(k)t(-k)e^{-ikx}$$

it follows that :

$$\hat{\varphi}(x,k) = \frac{\chi(x,k)}{t(k)} \tag{64}$$

and, from the asymptotic form of $\mathcal{X}(x, k)$ for $x \to -\infty$:

$$\hat{\varphi}(x,k) \sim \frac{e^{ikx} + b(k)e^{-ikx}}{t(k)} \qquad x \to -\infty \tag{65}$$

Similarly:

$$\varphi(x,k) = \frac{\hat{\mathcal{X}}(x,k)}{t(k)} \tag{66}$$

hence:

$$\varphi(x,k) \sim \frac{e^{-ikx} + \hat{b}(k)e^{ikx}}{t(k)} \qquad x \to +\infty \tag{67}$$

From Stone's theorem and the asymptotic form (65) it follows that for any potential which decreases faster than any exponential at infinity :

$\frac{1}{t(p)}$ and $\frac{b(p)}{t(p)}$ are regular functions of p in the entire complex p-plane, except possibly at the origin.

The following results shall be useful later:

i) it can be proved[118] that if $|b(0)| = 1$, then

$b(0) = \hat{b}(0) = -1$, and obviously $t(0) = 0$.

ii) the Wronskian of φ and $\hat{\varphi}$ is easily obtained:

$$W\{\hat{\varphi},\varphi\} = \frac{2ik}{t(k)} \tag{68}$$

if $xV(x)$ is absolutely integrable, Friedrichs (126) proves that φ and $\hat{\varphi}$ are continuous in k at the point $k = 0$, and therefore W is continuous at $k = 0$, where it assumes the value $W[\hat{\varphi}_0 ; \varphi_0]$, $\hat{\varphi}_0$ and φ_0 being defined by the integral equations:

$$\hat{\varphi}_0(x) = 1 - \int_x^{\infty} (x - x') V(x') \hat{\varphi}_0(x') dx' \tag{69}$$

$$\varphi_0(x) = 1 + \int_{-\infty}^{x} (x - x') V(x') \varphi_0(x') dx' \tag{70}$$

iii) Friedrichs also proves that all the poles of t (p) are contained in a circle centered at the origin, with radius 12M, where:

$$M = \int_{-\infty}^{+\infty} \left| V(x) \right| . \ dx \tag{71}$$

We shall postpone further consideration on the analytic properties of the elements of the scattering matrix till we dispose of the results of the inverse problem.

3.2.4 A General Integral Representation for the Solutions of Eq. (2)

In this section our endeavour is to obtain an integral representation for the solutions of Eq. (2). Such a representation shall be of paramount importance in the solution of the inverse problem, since it is through its kernel that the potential is obtained.

It is of some interest to conserve as much generality as possible and compatible with simplicity, in order to be able to take advantage of it in further applications.

Consistently with these remarks, let us make the Ansatz:

$$\varphi(x,k) = \Psi(x,k) + \int_{a(x)}^{b(x)} \Lambda(x,y)\Psi(y,k)\,dy \tag{72}$$

where:

$$L_x^V[\varphi] \equiv \frac{d^2\varphi}{dx^2} + k^2\varphi - V(x)\varphi = 0 \tag{73}$$

and $\Psi(x,k)$ is left arbitrary for the moment. $a(x)$ and $b(x)$ are such that $a(x) \leqslant b(x)$ for all x.

Applying the operator L_x^V to both sides of Eq. (72) :

$$L_x^V[\varphi(x,k)] = L_x^V[\Psi(x,k)] + \frac{d^2}{dx^2}\int_{a(x)}^{b(x)} \Lambda\Psi\,dy + \left[k^2 - V(x)\right]\int_{a(x)}^{b(x)} \Lambda\Psi\,dy \tag{74}$$

Straightforward computation gives:

$$\frac{d^2}{dx^2}\int_{a(x)}^{b(x)} \Lambda(x,y)\Psi(y,k)\,dy = b'(x)\Lambda_x[x,b(x)]\Psi[b(x),k] - a'(x)\Lambda_x[x,a(x)]\Psi[a(x),k]$$

$$+ b''(x)\Lambda[x,b(x)]\Psi[b(x),k] + [b'(x)]^2\Lambda_y[x,b(x)]\Psi[b(x),k] + b'(x)\Lambda_x[x,b(x)]\Psi[b(x),k]$$

$$+ [b'(x)]^2\Lambda[x,b(x)]\Psi'[b(x),k] - a''(x)\Lambda[x,a(x)]\Psi[a(x),k] - a'(x)\Lambda_x[x,a(x)]\Psi[a(x),k]$$

$$- [a'(x)]^2\Lambda_y[x,a(x)]\Psi[a(x),k] - [a'(x)]^2\Lambda[x,a(x)]\Psi'[a(x),k] + \int_{a(x)}^{b(x)}\Lambda_{xx}(x,y)\Psi(y,k)\,dy$$

where:

$$\Lambda_x(x,y) = \frac{\partial}{\partial x}\Lambda(x,y)$$

$$\Psi'[a(x),k] = \left\{\frac{d}{dy}\Psi(y,k)\right\}_{y=a(x)}$$

Substituting this expression into (74), it follows:

$$L_x^V[\varphi(x,k)] = L_x^V[\Psi(x,k)] + \int_{a(x)}^{b(x)} dy\,\Psi(y,k)\,L_x^V[\Lambda(x,y)]$$

$$+ \left\{2b'(x)\Lambda_x[x,b(x)] + b''(x)\Lambda[x,b(x)] + [b'(x)]^2\Lambda_y[x,b(x)]\right\}\Psi[b(x),k]$$

$$+ [b'(x)]^2\Lambda[x,b(x)]\Psi'[b(x),k]$$

$$- \left\{2a'(x)\Lambda_x[x,a(x)] + a''(x)\Lambda[x,a(x)] + a'(x)\Lambda_y[x,a(x)]\right\}\Psi[a(x),k]$$

$$- [a'(x)]^2\Lambda[x,a(x)]\Psi'[a(x),k]$$

Now let us focus our attention upon the term:

$$\int_{a(x)}^{b(x)} dy\, \Psi(y,k)\, L_x^V \left[\Lambda(x,y) \right] \tag{75}$$

Assume that we can find an operator M_y such that $\Lambda(x, y)$ satisfies the partial differential equation:

$$L_x^V \left[\Lambda(x,y) \right] = M_y \left[\Lambda(x,y) \right] \tag{76}$$

and let us make the further Ansatz that:

$$M_y \equiv L_y^{V_a} \equiv \frac{d^2}{dy^2} + k^2 - V_a(y) \tag{77}$$

where $V_a(y)$ is an auxiliary potential left arbitrary for the moment. $\Lambda(x, y)$ satisfies the equation:

$$\frac{\partial^2 \Lambda}{\partial x^2} - \frac{\partial^2 \Lambda}{\partial y^2} = \left[V(x) - V_a(y) \right] \Lambda \tag{78}$$

From Lagrange's identity, it follows that:

$$\Psi(y,k)\, L_y^{V_a} \left[\Lambda(x,y) \right] - \Lambda(x,y)\, \widetilde{L_y^{V_a}} \left[\Psi(y,k) \right] \equiv \frac{d}{dy} P\left[\Lambda, \Psi \right] \tag{79}$$

where $P[\Lambda, \Psi]$ is the usual bilinear concomittant. On the other hand, $L_y^{V_a}$ is obviously self-adjoint, therefore:

$$\Psi(y,k)\left\{\frac{\partial^2\Lambda}{\partial y^2}+k^2\Lambda-V_a(y)\Lambda\right\}-\Lambda(x,y)\left\{\frac{\partial^2\Psi}{\partial y^2}+k^2\Psi-V_a(y)\Psi\right\}$$

$$\equiv \Psi\frac{\partial^2\Lambda}{\partial y^2}-\Lambda\frac{\partial^2\Psi}{\partial y^2} \equiv -\frac{\partial}{\partial y}\left\{\Lambda\frac{\partial\Psi}{\partial y}-\Psi\frac{\partial\Lambda}{\partial y}\right\}$$

Hence the bilinear concomittant is computed:

$$P[\Lambda,\Psi] = \Psi\frac{\partial\Lambda}{\partial y}-\Lambda\frac{\partial\Psi}{\partial y}$$

We are ready now to compute the term (75) :

$$\int_{a(x)}^{b(x)} dy\,\Psi(y,k)\,L_x^V[\Lambda(x,y)] = \int_{a(x)}^{b(x)} dy\,\Psi(y,k)\,M_y[\Lambda(x,y)] = \int_{a(x)}^{b(x)} dy\,\Lambda(x,y)\,\tilde{M}_y[\Psi(y,k)]$$

$$+\int_{a(x)}^{b(x)} \frac{d}{dy}P[\Lambda,\Psi]\,dy = \int_{a(x)}^{b(x)} dy\,\Lambda(x,y)\,L_y^{V_a}[\Psi(y,k)] + \Big\{P[\Lambda,\Psi]\Big\}_{a(x)}^{b(x)}$$

Let us select $\Psi(x, k)$ such that : $L_y^{V_a}[\Psi(y, k)] = 0$ (80)

Then, using the following expression for the bilinear concomittant:

$$\Big\{P[\Lambda,\Psi]\Big\}_{a(x)}^{b(x)} = \Lambda_y[x,b(x)]\,\Psi[b(x),k]-\Lambda[x,b(x)]\,\Psi'[b(x),k]$$

$$-\Lambda_y[x,a(x)]\,\Psi[a(x),k]+\Lambda[x,a(x)]\,\Psi'[a(x),k]$$

we obtain:

$$L_x^V\,[\varphi(x,k)] = L_x^V\,[\Psi(x,k)] + \{[b'(x)]^2-1\}\,\Lambda[x,b(x)]\,\Psi'[b(x),k]$$

$$-\{[a'(x)]^2-1\}\,\Lambda[x,a(x)]\,\Psi'[a(x),k]$$

$$+\left\{2b'(x)\Lambda_x[x,b(x)] + b''(x)\Lambda[x,b(x)] + \left([b'(x)]^2+1\right)\Lambda_y[x,b(x)]\right\}\Psi[b(x),k]$$

$$-\left\{2a'(x)\Lambda_x[x,a(x)] + a''(x)\Lambda[x,a(x)] + \left([a'(x)]^2+1\right)\Lambda_y[x,a(x)]\right\}\Psi[a(x),k]$$

and finally, since

$$L_x^V\,[\Psi(x,k)] \equiv [V_a(x)-V(x)]\,\Psi(x,k)$$

$$L_x^V\,[\varphi(x,k)] \equiv 0$$

we obtain our condition, which must hold for all x and k :

$$[V_a(x)-V(x)]\,\Psi(x,k) + \{[b'(x)]^2-1\}\,\Lambda[x,b(x)]\,\Psi'[b(x),k]$$

$$-\{[a'(x)]^2-1\}\,\Lambda[x,a(x)]\,\Psi'[a(x),k]$$

$$+\left\{b''(x)\Lambda[x,b(x)] + 2b'(x)\Lambda_x[x,b(x)] + \left([b'(x)]^2+1\right)\Lambda_y[x,b(x)]\right\}\Psi[b(x),k]$$

$$-\left\{a''(x)\Lambda[x,a(x)] + 2a'(x)\Lambda_x[x,a(x)] + \left([a'(x)]^2+1\right)\Lambda_y[x,a(x)]\right\}\Psi[a(x),k]$$

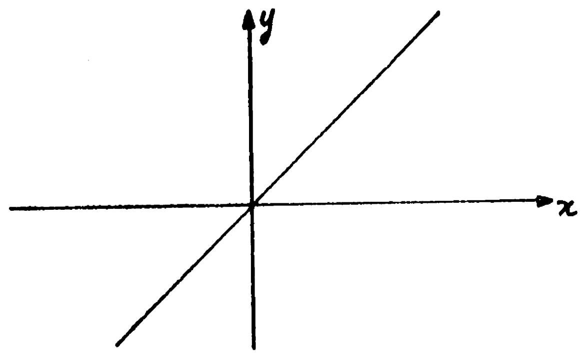
$$\equiv 0 \tag{81}$$

3.2.5 Kernels $K(x, y)$ and $\hat{K}(x, y)$

Let us particularize the results in the preceding section in order to obtain our previously defined solutions φ and $\hat{\varphi}$.

i) Kernel $K(x, y)$. Obtained in the special case where:

$$V_0(x) \equiv 0$$
$$\psi(x,k) \equiv e^{-ikx}$$
$$a(x) = -\infty$$
$$b(x) = x$$



we shall define $K(x, y) \equiv 0$ in the patched region $(y > x)$.

identity (81) is satisfied by requiring:

$$2 \frac{d}{dx} \Lambda(x,x) \equiv V(x)$$
$$\Lambda(x,-\infty) \equiv 0$$
$$\Lambda_y(x,-\infty) \equiv 0$$

The theory of partial differential equations shows that there exists a solution to the characteristic problem:

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial y^2} \equiv V(x) K \qquad (82)$$

$$2 \frac{d}{dx} K(x,x) \equiv V(x) \qquad (83)$$
$$K(x,-\infty) \equiv 0$$
$$K_y(x,-\infty) \equiv 0 \qquad (84)$$

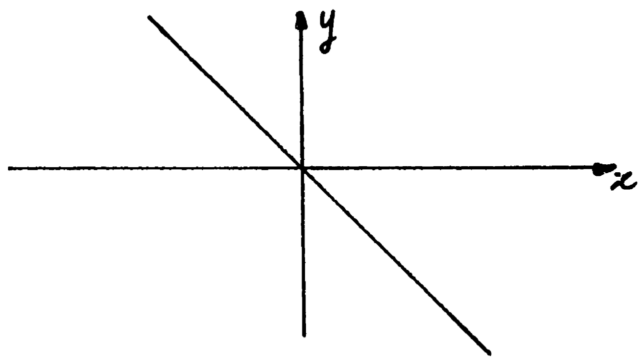
$$K_y(x, -\infty) \equiv 0 \tag{85}$$

This kernel K(x, y) produces the previously defined solution $\varphi(x, k)$:

$$\varphi(x,k) = e^{-ikx} + \int_{-\infty}^{x} K(x,y)\, e^{-iky}\, dy \tag{86}$$

ii) Kernel $\hat{K}(x, y)$. Obtained in the special case where:

$$V_a(x) \equiv 0$$
$$\psi(x,k) \equiv e^{-ikx}$$
$$a(x) \equiv x$$
$$b(x) \equiv +\infty$$



similarly, defining $\hat{K}(x, y)$ to be zero in the patched region ($y < x$), the characteristic problem giving $\hat{K}(x, y)$ is:

$$\frac{\partial^2 \hat{K}}{\partial x^2} - \frac{\partial^2 \hat{K}}{\partial y^2} \equiv V(x)\, K \tag{87}$$

$$2 \frac{d}{dx} \hat{K}(x,x) \equiv -V(x) \tag{88}$$

$$\hat{K}(x, \infty) \equiv 0 \tag{89}$$

$$\hat{K}_x(x, \infty) \equiv 0 \tag{90}$$

and $\hat{K}(x, y)$ defines then the solution:

$$\hat{\varphi}(x,k) = e^{ikx} + \int_x^{\infty} \hat{K}(x,y)\, e^{iky}\, dy \tag{91}$$

3.2.6 Completeness Relations for φ and $\hat{\varphi}$.

It can be shown (110) that Eq. (2) has the solutions:

$$\psi_{\mp}(x,k) = e^{ikx} \mp \sqrt{\frac{\pi}{2}}\, \frac{i}{|k|} \int_{-\infty}^{+\infty} e^{\pm i|k||x-x'|}\, V(x')\, \psi_{\mp}(x',k)\, dx' \tag{92}$$

these solutions have the well-known completeness relations (we are always assuming an empty discrete spectrum) :

$$\int_{-\infty}^{+\infty} \psi_{\pm}^{*}(x,k)\, \psi_{\pm}(x',k)\, dk = 2\pi\, \delta(x-x') \tag{93}$$

Comparing the respective integral equations, we infer the following relations:

$$\psi_-(x,k) \equiv t(k)\hat{\varphi}(x,k) \tag{94}$$

$k > 0$

$$\psi_+(x,k) \equiv t(-k)\varphi(x,-k) \tag{95}$$

$$\psi_-(x,k) \equiv t(-k)\varphi(x,-k) \tag{96}$$

$k < 0$

$$\psi_+(x,k) \equiv t(k)\hat{\varphi}(x,k) \tag{97}$$

and the hermiticity law:

$$\psi_{\mp}(x,-k) \equiv \psi_{\pm}^*(x,k) \tag{98}$$

Using the completeness relation (93) for, say $\psi_+(x,k)$:

$$\int_{-\infty}^{0} \psi_+^*(x,k)\psi_+(x',k)dk + \int_0^{\infty} \psi_+^*(x,k)\psi_+(x',k)dk = 2\pi\delta(x-x')$$

from (97) and (95), it follows :

$$2\pi\delta(x-x') = \int_{-\infty}^{0} t(-k)\hat{\varphi}(x,-k)t(k)\hat{\varphi}(x',k)dk + \int_0^{\infty} t(k)\varphi(x,k)t(-k)\varphi(x',-k)dk$$

and expressing $\hat{\varphi}(x,k)$ in terms of $\varphi(x,k)$ and $\varphi(x,-k)$, we obtain the completeness relation for $\varphi(x,k)$:

$$2\pi\delta(x-x') = \int_{-\infty}^{+\infty}\varphi(x,k)\varphi(x',-k)\,dk + \int_{-\infty}^{+\infty} b(k)\varphi(x,k)\varphi(x',k)\,dk \tag{99}$$

proceeding in a similar way, we obtain for $\hat{\varphi}(x, k)$:

$$2\pi\delta(x-x') = \int_{-\infty}^{+\infty}\hat{\varphi}(x,k)\hat{\varphi}(x',-k)\,dk + \int_{-\infty}^{+\infty}\hat{b}(k)\hat{\varphi}(x,k)\hat{\varphi}(x',k)\,dk \tag{100}$$

3.2.7 The Inverse One-Dimensional Algorithm

In what follows, we shall borrow heavily from Faddeyev [118]. Let us first recall the relations between the solutions φ and $\hat{\varphi}$:

$$t(k)\varphi(x,k) = \hat{\varphi}(x,-k) + \hat{b}(k)\hat{\varphi}(x,k) \tag{101}$$

$$t(k)\hat{\varphi}(x,k) = \varphi(x,-k) + b(k)\varphi(x,k) \tag{102}$$

and their integral representations:

$$\varphi(x,k) = e^{-ikx} + \int_{-\infty}^{x} K(x,y)e^{-iky}\,dy$$

$$\hat{\varphi}(x,k) = e^{ikx} + \int_{x}^{\infty} \hat{K}(x,y)e^{iky}\,dy$$

B. Levin proves (127) that if V(x) satisfies:

$$\int_{-\infty}^{+\infty} \left(1 + |x|\right) |V(x)|\, dx < \infty \tag{103}$$

then:

$$\int_{-\infty}^{a} dx \int_{-\infty}^{x} dy \, |K(x,y)|^2 \leq C_a \qquad a < \infty \tag{104}$$

$$\int_{\hat{a}}^{\infty} dx \int_{x}^{\infty} dy \, |\hat{K}(x,y)|^2 \leq C_{\hat{a}} \qquad \hat{a} > -\infty \tag{105}$$

Our aim is to take the Fourier transformation of equations (101) and (102) and to express the new equations thus obtained in terms of K, $\hat{K}$ and the Fourier transforms of b, $\hat{b}$ and t. Consequently, let us introduce the Fourier transforms:

$$R(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} b(k) e^{-ikt} dk \tag{106}$$

$$\hat{R}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{b}(k) e^{ikt} dk \tag{107}$$

$$\Gamma(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[t(k) - 1 \right] e^{-ikt} dk \tag{108}$$

It can be proved (127) that $R(t)$, $\hat{R}(t)$ and $\Gamma(t)$ are quadratically integrable functions of t over the entire axis and, since we are assuming no bound states,

$$\Gamma(t) \equiv 0 \quad \text{for} \quad t < 0 \tag{109}$$

[t(p) is regular on the upper half plane, and there (55) holds].

From the convolution theorem for Fourier transforms, it follows from these considerations that equations (101) and (102) are equivalent to the equations:

$$\int_{-\infty}^{+\infty} K(x,z)\Gamma(z-y)dz + K(x,y) + \Gamma(x-y) = \int_{-\infty}^{+\infty} \hat{K}(x,z)\hat{R}(y+z)dz + \hat{K}(x,y) + \hat{R}(x+y) \tag{110}$$

$$\int_{-\infty}^{+\infty} \hat{K}(x,z)\Gamma(y-z)dz + \hat{K}(x,y) + \Gamma(y-x) = \int_{-\infty}^{+\infty} K(x,z)R(y+z)dz + K(x,y) + R(x+y) \tag{111}$$

Now, from (109) we obtain Gelfand-Levitan-Kay-Moses' equations:

$$K(x,y) + R(x+y) + \int_{-\infty}^{x} R(y+z)\,K(x,z)dz = 0 \qquad y < x \tag{112}$$

$$\hat{K}(x,y) + \hat{R}(x+y) + \int_{x}^{\infty} \hat{R}(y+z)\,\hat{K}(x,z)dz = 0 \qquad y > x \tag{113}$$

It is also proved (118) that $R(t)$ and $\hat{R}(t)$ can be differentiated once and, from (103) :

$$\int_{\hat{a}}^{\infty} (1+|t|)\, |\hat{R}'(t)|\, dt < C_{\hat{a}} \qquad \hat{a} > -\infty \tag{114}$$

$$\int_{-\infty}^{a} (1+|t|)\, |R'(t)|\, dt < C_{a} \qquad a < \infty \tag{115}$$

We shall prove in the next section that necessary and sufficient conditions for $V(x) \equiv 0$ for $x < 0$ and $x > \delta$ are:

$$R(t) \equiv 0 \quad \text{for} \quad t < 0 \tag{116}$$

$$\hat{R}(t) \equiv 0 \quad \text{for} \quad t > 2\delta \tag{117}$$

Combining this result with Faddeyev's [118], it follows that, in order that the symmetric unitary matrix $S(k)$ be the S-matrix of the operator:

$$L \equiv -\frac{d^2}{dx^2} + V(x)$$

such that $V(x)$ has the finite range $[o, \delta]$, it is necessary and sufficient that $b(k)$, $\hat{b}(k)$ and $t(k)$ be such that:

i) for all real k : (except possibly $k = 0$)

$$|b(k)| < 1 \quad \text{and} \quad |\hat{b}(k)| < 1 \tag{118}$$

$$\text{if } |b(0)| = 1, \text{ or } |\hat{b}(0)| = 1, \text{ then } b(0) = \hat{b}(0) = -1 \tag{119}$$

ii) (53), (54) and (55) hold

iii) S is regular in the upper half-plane

iv) b and $\hat{b}$ are such that (116) and (117) hold.

The potential V(x) is then determined uniquely. One can use either equation (112) or (113), depending on the given scattered data, together with the conditions

$$V(x) \equiv 2 \frac{d}{dx} K(x, x) \tag{120}$$

$$V(x) \equiv -2 \frac{d}{dx} \hat{K}(x, x) \tag{121}$$

3.2.8 Conditions for finite range potentials using the algorithm

1. A necessary and sufficient condition for $K(x, y) \equiv 0$ for $x < 0$ is that $R(x) \equiv 0$ for $x < 0$. Similarly, a necessary and sufficient condition for $K(x, y) \equiv 0$ for $x > \delta$ is that $R(x) \equiv 0$ for $x > 2\delta$.

Necessity: Eq.(112) shows that $R(x + y) \equiv 0$. Since $y < x < 0$ $x + y < 2x < 0$, therefore $R(x) \equiv 0$ for $x < 0$.

Sufficiency: when $x < 0$ and $y < x$, $x + y < 0$. Eq.(112) becomes

$$K(x,y) + \int_{-\infty}^{x} R(y+z) K(x,z) dz \equiv 0$$

since $z \leq x$, $y < x$, it follows that $y + z < 0$, hence the only remaining term is $K(x, y) \equiv 0$

the procedure is the same for $\hat{K}(x, y)$.

$K(x, y) \equiv 0$ for $x < 0$ and $\hat{K}(x, y) \equiv 0$ for $x > \delta$ are obviously sufficient conditions for $V(x)$ to have a finite range $[0, \delta]$. Let us show that they are also necessary.

Since for $x < 0$ $V(x) \equiv 0$, Eq. (86) becomes:

$$\int_{-\infty}^{x} K(x,y) e^{-iky} dy \equiv 0 \tag{122}$$

similarly, for $x > \delta$, Eq. (91) becomes:

$$\int_{x}^{\infty} \hat{K}(x,y) e^{iky} dy \equiv 0 \tag{123}$$

Equations (112) and (113) can be written under the form:

$$K(x,y) + R(x+y) + \frac{1}{2\pi}\int_{-\infty}^{x} K(x,z)dz \int_{-\infty}^{+\infty} b(k) e^{-ik(y+z)} dk = 0$$

$$\hat{K}(x,y) + \hat{R}(x+y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{K}(x,z)dz \int_{-\infty}^{+\infty} \hat{b}(k) e^{ik(y+z)} dk = 0$$

and interchanging the order of integration, it follows:

$$K(x,y) + R(x+y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(k) e^{-iky} dk \int_{-\infty}^{x} K(x,z) e^{-ikz} dz = 0 \tag{124}$$

$$\hat{K}(x,y) + \hat{R}(x+y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{b}(k) e^{iky} dk \int_{x}^{\infty} \hat{K}(x,z) e^{ikz} dz = 0 \tag{125}$$

using (122) and (123), we obtain:

$$K(x,y) \equiv -R(x+y) \qquad x<0 \tag{126}$$

$$\hat{K}(x,y) \equiv -\hat{R}(x+y) \qquad x>\delta \tag{127}$$

Therefore, making $x \equiv y$:

$$K(x,x) = -R(2x) \equiv -\frac{1}{2\pi}\int_{-\infty}^{+\infty} b(k)e^{-2ikx}dk \qquad x<0 \tag{128}$$

$$\hat{K}(x,x) = -\hat{R}(2x) \equiv -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{b}(k)e^{2ikx}dk \qquad x>\delta \tag{129}$$

Since $V(x) \equiv 0$ in this range of values of x, we must have, according to (120) and (121) :

$$R'(2x) \equiv 0 \qquad x < 0$$
$$\hat{R}'(2x) \equiv 0 \qquad x > 2\delta$$

Hence $R(x)$ and $\hat{R}(x)$ must be equal to constants in this range. Now, since $b(k)$ and $\hat{b}(k)$ are absolutely integrable, the Riemann-Lebesgue theorem show that these constants must be equal to zero, which proves our assertion that $K(x, y) \equiv 0$ for $x < 0$ and $\hat{K}(x, y) \equiv 0$ for $x > \delta$ are not only sufficient, but also necessary conditions for $V(x)$ to have the finite range $[0, \delta\ ]$.

We have proved the theorem:

A necessary and sufficient condition for $V(x) \equiv 0$ outside the finite interval $[0, \delta]$ is that:

$$\int_{-\infty}^{+\infty} b(k) e^{-ikx} dk \equiv 0 \qquad x < 0 \tag{130}$$

$$\int_{-\infty}^{+\infty} \hat{b}(k) e^{ikx} dk \equiv 0 \qquad x > 2\delta \tag{131}$$

the first relation ensuring cut-off for $x < 0$, the second for $x > \delta$.

Using the phase-law (49) we can bring condition (131) to the form:

$$\int_{-\infty}^{+\infty} \frac{b(k)\, t(-k)}{t(k)} e^{-ikx} dk \equiv 0 \qquad x > 2\delta \tag{132}$$

However, we are unable to impose a cutoff "on the right" by the sole use of the reflection coefficient on the left explicitly.

2. A necessary and "heuristically sufficient" (this shall be explained later) condition for $V(x)$ to have the finite range $[0, \delta]$ is that:

$$\int_{-\infty}^{+\infty} k \frac{b(k)}{t(k)} e^{-ikx} dk \equiv 0 \qquad x \notin [0, 2\delta] \tag{133}$$

Assume $V(x) \equiv 0$ for $x \notin [0, \delta]$. From (65) and (67) it follows that:

$$\varphi(x,k) \equiv \frac{e^{-ikx} + \hat{b}(k)\, e^{ikx}}{t(k)} \qquad x < 0 \tag{134}$$

$$\hat{\varphi}(x,k) \equiv \frac{e^{ikx} + b(k)\, e^{-ikx}}{t(k)} \qquad x > \delta \tag{135}$$

Using the integral representations (86) and (91):

$$\frac{e^{-ikx} + \hat{b}(k) e^{ikx}}{t(k)} = e^{-ikx} + \int_{-\infty}^{x} K(x,y) e^{-iky}\, dy$$

$$\frac{e^{ikx} + b(k) e^{-ikx}}{t(k)} = e^{ikx} + \int_{x}^{\infty} \hat{K}(x,y) e^{iky}\, dy$$

Let us first focus our attention on $x < 0$, using $K(x, y)$:

$$\int_{-\infty}^{x} K(x,y) e^{-iky}\, dy = \left[\frac{1}{t(k)} - 1\right] e^{-ikx} + \frac{\hat{b}(k)}{t(k)} e^{ikx}$$

from the phase law and Eq. (124) it follows that:

$$0 = K(x,y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(k) e^{-ik(x+y)} dk + \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(k) e^{-iky} \left\{ \left[\frac{1}{t(k)} - 1\right] e^{-ikx} - \frac{b(-k)}{t(-k)} e^{ikx} \right\}$$

or:

$$0 = K(x,y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{b(k)}{t(k)} e^{-ik(x+y)} dk - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|b(k)|^2}{t(-k)} e^{ik(x-y)} dk$$

Make $x = y$ and differentiate with respect to x:

$$0 \equiv \frac{d}{dx} K(x,x) - \frac{i}{\pi}\int_{-\infty}^{+\infty} k \frac{b(k)}{t(k)} e^{-2ikx} dk$$

Since $V(x) \equiv 0$ for $x < 0$, using (120) we are left with:

$$\int_{-\infty}^{+\infty} k \frac{b(k)}{t(k)} e^{-ikx} dk \equiv 0 \qquad x<0$$

and the necessity is proved for cutoff "on the left". Let us undertake the same procedure for the other side $x > \delta$, using $\hat{K}(x, y)$; using (91) we get:

$$\int_{x}^{\infty} \hat{K}(x,y) e^{iky} dy \equiv \left[\frac{1}{t(k)} - 1\right] e^{ikx} + \frac{b(k)}{t(k)} e^{-ikx}$$

from (125) it follows that:

$$0 \equiv \hat{K}(x,y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{b}(k) e^{ik(x+y)} dk + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{b}(k) e^{iky} dk \left\{\left[\frac{1}{t(k)} - 1\right] e^{ikx} + \frac{b(k)}{t(k)} e^{-ikx}\right\}$$

or:

$$0 \equiv \hat{K}(x,y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\hat{b}(k)}{t(k)} e^{ik(x+y)} dk + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{b(k)\hat{b}(k)}{t(k)} e^{ik(y-x)} dk$$

making $x = y$ and differentiating with respect to x:

$$0 \equiv \frac{d}{dx}\hat{K}(x,x) + \frac{i}{\pi}\int_{-\infty}^{+\infty} k \frac{\hat{b}(k)}{t(k)} e^{2ikx} dk$$

and since $V(x) \equiv 0$ for $x > \delta$, it follows from (121) :

$$\int_{-\infty}^{+\infty} k \frac{\hat{b}(k)}{t(k)} e^{ikx} dk \equiv 0 \qquad x > 2\delta$$

now using the phase law this expression is brought to the form:

$$\int_{-\infty}^{+\infty} k \frac{b(k)}{t(k)} e^{-ikx} dk \equiv 0 \qquad x > 2\delta$$

and the necessity is proved for a cutoff "on the right.

We see that the great advantage of formulating our conditions through b/t is that we are able to express the condition for <u>cutoff on both sides</u> using b/t alone. Such was not possible with the reflection coefficients - the reason for this possibility comes from the unitarity of the S-matrix which, through the phase law enables to express b/t in terms of its analogue on the right $\hat{b}/t$.

We have been led to the conclusion that the function

$$C(x) = \frac{2i}{\pi} \int_{-\infty}^{+\infty} k \frac{b(k)}{t(k)} e^{-2ikx} dk \tag{136}$$

plays an important role in the formulation of our finite range potentials problem :

A necessary condition for $V(x)$ to have finite range $[0, \delta]$ is that the function $C(x)$ have same finite range.

It is important to realize that this is an exceedingly strong condition, what suggests that it might be a sufficient condition as well. Let us make this possibility at least heuristically plausible through the following considerations:

We shall prove in the next section that

$$k\,\frac{b(k)}{t(k)} = -\frac{i}{2}\int_{-\infty}^{+\infty} e^{ikx'}\,V(x')\,\hat{\varphi}(x';k)\,dx' \tag{137}$$

hence, using the definition of C(x) :

$$C(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-2ikx}\,dk \int_{-\infty}^{+\infty} e^{ikx'}\,V(x')\,\hat{\varphi}(x';k)\,dx'$$

Substituting $\hat{\varphi}(x', k)$ by its integral representation (91) :

$$C(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-2ikx}\,dk \left\{ \int_{-\infty}^{+\infty} e^{2ikx'}\,V(x')\,dx' + \int_{-\infty}^{+\infty} e^{ikx'}\,V(x')\,dx' \int_{x'}^{\infty} \hat{K}(x';z)\,e^{ikz}\,dz \right\}$$

and interchanging the order of integration:

$$C(x) \equiv V(x) + 2\int_{-\infty}^{x} V(x')\,\hat{K}(x', 2x-x')\,dx' \tag{138}$$

Proceeding analogously for $\varphi(x, k)$ with (86) :

$$C(x) \equiv V(x) + 2\int_{x}^{\infty} V(x')\,K(x', 2x-x')\,dx' \tag{139}$$

Assuming that C(x) and $\hat{K}(x, y)$ are given functions, (12)
Eq. (138) is a Volterra equation for V(x). It is well-known that if the kernel $\hat{K}(x, y)$ is bounded, such equations have no eigenvalues and their solution is unique. Let us make the Ansatz that $\hat{K}$ is indeed bounded. Therefore, if $C(x) \equiv 0$ in a certain interval (a, b), the equation has the trivial solution $V(x) \equiv 0$ in this

interval - from the uniqueness theorem we infer that this is the only possible solution in (a, b), and the sufficiency of condition (133) is made somewhat plausible. We are well aware of the "hand-waving" character of this argument, specially because we did not ascertain whether condition (133) ensures a bounded kernel. As an example of what happens if the kernel is unbounded consider:

$$K(x,y) \equiv -\frac{e^y}{\cosh x}$$

in this case, equation:

$$0 \equiv V(x) + 2\int_x^\infty K(x', 2x-x')V(x')dx'$$

is satisfied not only for $V \equiv 0$, but also by:

$$V(x) \equiv -\frac{2}{\cosh^2 x}$$

(this is one of the "reflectionless" potentials derived by I. Kay and H. E. Moses (115) - it follows from the conditions of uniqueness of the inverse problem that all these non-trivial potentials necessarily support bound states).

Let us end by referring to chapter V, where a more convincent argument for the sufficiency of (133) is given.

3.2.9 Scattering Matrix in terms of φ and $\hat{\varphi}$

In this section we shall obtain expressions for b, $\hat{b}$, t and b/t in terms of the functions φ and $\hat{\varphi}$.

From the integral equation (59), which can be expressed under the form:

$$\varphi(x,k) = e^{-ikx}\left\{1 - \frac{1}{2ik}\int_{-\infty}^{x} e^{ikx'}V(x')\varphi(x';k)dx'\right\} + e^{ikx}\frac{1}{2ik}\int_{-\infty}^{+\infty} e^{-ikx'}V(x')\varphi(x';k)dx'$$

and the asymptotic expression (65), we infer:

$$\frac{1}{t(k)} = 1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{ikx'}V(x')\varphi(x';k)dx'$$

$$\frac{\hat{b}(k)}{t(k)} = \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{-ikx'}V(x')\varphi(x';k)dx'$$

from these expressions, we obtain :

$$\hat{b}(k) = \frac{\frac{1}{2ik}\int_{-\infty}^{+\infty} e^{-ikx'}V(x')\varphi(x';k)dx'}{1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{ikx'}V(x')\varphi(x';k)dx'} \tag{140}$$

$$t(k) = \frac{1}{1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{ikx'}V(x')\varphi(x';k)dx'} \tag{141}$$

Proceeding similarly with $\hat{\varphi}$, we obtain successively:

$$\hat{\varphi}(x,k) = e^{ikx}\left\{1 - \frac{1}{2ik}\int_x^{\infty} e^{-ikx'}V(x')\hat{\varphi}(x';k)dx'\right\} + e^{-ikx}\frac{1}{2ik}\int_x^{\infty} e^{ikx'}V(x')\hat{\varphi}(x';k)dx'$$

$$\hat{\varphi}(x,k) \sim \frac{e^{ikx} + b(k)e^{-ikx}}{t(k)}$$

$$\frac{1}{t(k)} = 1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{-ikx'}V(x')\hat{\varphi}(x';k)dx'$$

comparing this last expression with (141) it follows:

$$\int_{-\infty}^{+\infty} e^{-ikx'}V(x')\hat{\varphi}(x';k)dx' \equiv \int_{-\infty}^{+\infty} e^{ikx'}V(x')\varphi(x';k)dx' \tag{142}$$

and:

$$\frac{b(k)}{t(k)} = \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{ikx'}V(x')\hat{\varphi}(x';k)dx' \tag{143}$$

using the phase law and comparing this expression with the analogcus

obtained with :

$$\int_{-\infty}^{+\infty} e^{-ikx'}V(x')\hat{\varphi}(x';-k)dx' = \int_{-\infty}^{+\infty} e^{-ikx'}V(x')\varphi(x';k)dx' \tag{144}$$

also:

$$b(k) = \frac{\frac{1}{2ik}\int_{-\infty}^{+\infty} e^{ikx'}V(x')\hat{\varphi}(x';k)dx'}{1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{-ikx'}V(x')\hat{\varphi}(x';k)dx'} \tag{145}$$

These relations should be helpful in further study of the analytic properties of the scattering matrix. First we need to know more about the solutions φ and $\hat{\varphi}$.

2.3.10 Analytic Properties of φ and $\hat{\varphi}$.

In this section we shall take advantage of a method used by N. Levinson [89] (in connection with the radial equation) in order to study the properties of the solutions φ and $\hat{\varphi}$ as analytic functions of the complex variable p .

In order to study the properties of φ from its integral equation (59) it is convenient to define the sequence $\{\varphi_n\}$ through the relations:

$$\varphi_n(x,p) = e^{-ipx} + \int_{-\infty}^{x} \frac{\sin p(x-x')}{p} V(x') \varphi_{n-1}(x';p) dx' \tag{146}$$

$$\varphi_0(x,p) \equiv 0 \tag{147}$$

we shall assume $0 \leq x \leq \delta$, since V(x) has this finite range. hence:

$$\varphi_n(x,p) = e^{-ipx} + \int_{0}^{x} \frac{\sin p(x-x')}{p} V(x') \varphi_{n-1}(x';p) dx' \tag{148}$$

We have:

$$|\varphi_1(x,p) - \varphi_0(x,p)| \equiv |e^{-ipx}| \equiv e^{\tau x} \tag{149}$$

and :

$$|\varphi_2(x,p) - \varphi_1(x,p)| \leq \int_0^x \left|\frac{\sin p(x-x')}{p}\right| |V(x')| |\varphi_1 - \varphi_0| dx'$$

$$\leq \int_0^x \left|\frac{\sin p(x-x')}{p}\right| |V(x')| e^{\tau x'} dx' \tag{150}$$

It can be easily shown that:

$$\left|\frac{\sin px}{p}\right| \leq |x| e^{|\tau x|} \tag{151}$$

therefore:

$$\left|\frac{\sin p(x-x')}{p}\right| \leq (x-x') e^{|\tau|(x-x')}$$

since $x' \leq x$

1. Let us assume $\operatorname{Im} p \geq 0$. In this case:

$$\left|\frac{\sin p(x-x')}{p}\right| \leq (x-x') e^{\tau(x-x')}$$

Hence:

$$|\varphi_2(x,p) - \varphi_1(x,p)| \leq \int_0^x |V(x')| e^{\tau x'} (x-x') e^{\tau(x-x')} dx' \equiv e^{\tau x} \int_0^x (x-x') |V(x')| dx'$$

Since V(x) is piecewise continuous in $[0, \delta]$, we assume:

$$\left| V(x) \right| \leq M$$

therefore:

$$\left| \varphi_2 - \varphi_1 \right| \leq M e^{\delta x} \int_0^x (x-x')\,dx \equiv M e^{\delta x} \frac{x^2}{2!}$$

$$\left| \varphi_3 - \varphi_2 \right| \leq \int_0^x (x-x') e^{\delta(x-x')} M \frac{x'^2}{2!} e^{\delta x'} M\,dx' = \frac{M^2}{2!} e^{\delta x} \int_0^x x'^2 (x-x')\,dx'$$

$$= \frac{M^2}{2!} e^{\delta x} \frac{x^4}{4.3} = M^2 e^{\delta x} \frac{x^4}{4!}$$

similarly:

$$\left| \varphi_n - \varphi_{n-1} \right| \leq M^{n-1} e^{\delta x} \frac{x^{2n-2}}{(2n-2)!} = \frac{(\sqrt{M}\, x)^{2n-2}}{(2n-2)!} e^{\delta x} \tag{152}$$

Notice that for all finite x [a fortiori for $0 \leq x \leq \delta$], the term:

$$\frac{(\sqrt{M}\, x)^{2n-2}}{(2n-2)!} \longrightarrow 0 \qquad \text{when } n \rightarrow \infty$$

therefore $\{\varphi_n\}$ converges to a function which is analytic on the upper half-plane and continuous down to the real axis. It is obvious from the construction of $\{\varphi_n\}$ that this function is φ.

Let us make an estimate on $\left|\varphi(x,p)\right|$, by considering:

$$\left|\varphi_n(x,p)\right| \leq \left|\varphi_n - \varphi_{n-1} + \varphi_{n-1} - \varphi_{n-2} + \varphi_{n-2} - \cdots\cdots + \varphi_3 - \varphi_2 + \varphi_2 - \varphi_1 + \varphi_1\right.$$

$$\leq \sum_{i=1}^{n} \left|\varphi_i - \varphi_{i-1}\right|$$

$$\leq e^{\epsilon x}\left\{\frac{(\sqrt{M'}\,x)^{2n-2}}{(2n-2)!} + \cdots\cdots + \frac{(\sqrt{M'}\,x)^4}{4!} + \frac{(\sqrt{M'}\,x)^2}{2!} + 1\right\}$$

when $n \longrightarrow \infty$ the right hand side of this inequality represents the infinite series expansion of $\cosh \sqrt{M'}.x$, consequently:

$$\left|\varphi(x,p)\right| \leq e^{\epsilon x}\cosh\sqrt{M'}.x \qquad \left\{\begin{array}{l} 0 \leq x \leq \delta \\ Im\, p = \epsilon \geq 0 \end{array}\right. \qquad (1$$

2. Let us examine now the lower half-plane behaviour ($Im\ p < 0$).

In this case, we shall use:

$$\left|\frac{\sin p(x-x')}{p}\right| \leq (x-x')\, e^{-\epsilon(x-x')}$$

hence:

$$\left|\varphi_2(x,p) - \varphi_1(x,p)\right| \leq \int_0^x |V(x')|\,(x-x')\,e^{-\epsilon(x-x')}\,e^{\epsilon x'}\,dx'$$

$$\leq M e^{-\epsilon x}\int_0^x (x-x')\,e^{2\epsilon x'}\,dx' \equiv M e^{\epsilon x}\int_0^x \xi\, e^{-2\epsilon\xi}\,d\xi$$

Now, we have:

$$\int_0^x \xi e^{-2\epsilon\xi} d\xi \equiv \frac{1}{4\epsilon^2}\left\{ 1 - (1+2\epsilon x) e^{-2\epsilon x} \right\}$$

$$\equiv \frac{1}{4\epsilon^2}\left\{ 1 - 2\epsilon x e^{-2\epsilon x} - e^{-2\epsilon x} \right\} \leq -\frac{x}{2\epsilon} e^{-2\epsilon x}$$

hence, we get a bound for the first difference:

$$\left| \varphi_2 - \varphi_1 \right| \leq -\frac{M}{2\epsilon} e^{-\epsilon x} x$$

and therefore:

$$\left| \varphi_3 - \varphi_2 \right| \leq M \int_0^x (x-x') e^{-\epsilon(x-x')} \left(-\frac{M}{2\epsilon}\right) x' e^{-\epsilon x'} dx' \equiv \frac{-M^2}{2\epsilon} e^{-\epsilon x} \frac{x^3}{3!}$$

$$\left| \varphi_4 - \varphi_3 \right| \leq M \int_0^x (x-x') e^{-\epsilon(x-x')} \left(\frac{-M^2}{2\epsilon}\right) \frac{x'^3}{3!} e^{-\epsilon x'} dx' \equiv \frac{-M^3}{2\epsilon} e^{-\epsilon x} \frac{x^5}{5!}$$

$$\int_0^x (x-x') \frac{x'^3}{3!} dx' \equiv \frac{x^5}{5!}$$

in general:

$$\left| \varphi_n - \varphi_{n-1} \right| \leq \frac{-\sqrt{M}}{2\epsilon} e^{-\epsilon x} \frac{(\sqrt{M}\, x)^{2n-3}}{(2n-3)!} \qquad (1$$

Since the right hand side term tends to zero when $n \to \infty$, we infer that $\{\varphi_n\}$ tends to an analytic function which, by construction of the sequence satisfies the integral equation and therefore is φ. We obtain the following estimate for $|\varphi|$ using the same method of summation of the bounding series:

$$\left| \varphi(x,p) \right| \leq -\frac{\sqrt{M}}{2\tau} e^{-\tau x} \left[1 + \sinh \sqrt{M}\, x \right] \tag{155}$$

$$0 \leq x \leq \delta \qquad Im p = \tau < 0$$

Of course we can proceed in the same manner to obtain analogous results for $\hat{\varphi}$. However, since our objective in this analysis is the study of the analytical properties of the scattering matrix, the relations (142) and (144) dispense with this procedure.

3.2.11 Integral Representations for K and $\hat{K}$

In this section we shall merely use the Gelfand-Levitan equations together with the integral representations of φ and $\hat{\varphi}$ in order to obtain the immediate result:

$$K(x,y) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} b(k)\, \varphi(x,k)\, e^{-iky}\, dk \tag{156}$$

and:

$$\hat{K}(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{b}(k)\,\hat{\varphi}(x,k)\,e^{iky}\,dk \tag{157}$$

These results should be helpful in connection with the sufficiency of condition (133), using the bounds derived for the solutions φ and $\hat{\varphi}$ together with the representation of the scattering matrix in terms of these solutions.

CHAPTER IV

APPLICATION TO ELECTROMAGNETIC PROBLEMS

We shall list here some examples of electromagnetic problems which can be mapped into the inverse scattering problem. This list is by no means exhaustive and we give it here with the hope that it will suggest some ideas for a more systematic search of possible mappings.

In what follows we shall borrow heavily from H. E. Moses (117).

4.1 Cold Collisionless Plasmas

Let us consider the reflection and transmission of a plane pulse of electromagnetic radiation, propagating along a direction z, by an isotropic plasma which is not necessarily homogeneous in the direction z.. Let us consider linear polarization of the electric field $E(z, t)$ in the direction x, and focus our attention on steady-state solutions:

$$E(z,t) = E(z,\omega)\, e^{-i\omega t}$$

The differential equation satisfied by the transverse component $E(z, \omega)$ is:

$$\frac{d^2 E}{dz^2} + k^2 n^2(z,\omega) E = 0 \qquad (158)$$

For a cold collisionless plasma, we have:

$$n^2(z,\omega) = 1 - \frac{4\pi e^2}{m} \frac{N(z)}{\omega^2}$$

where N(z) is the electron density. Substituting this in (158) :

$$\frac{d^2E}{dz^2} + \left\{ k^2 - \frac{4\pi e^2}{mc^2} N(z) \right\} E = 0 \qquad (159)$$

It is easily seen that, by defining a scattering potential (velocity-independent) :

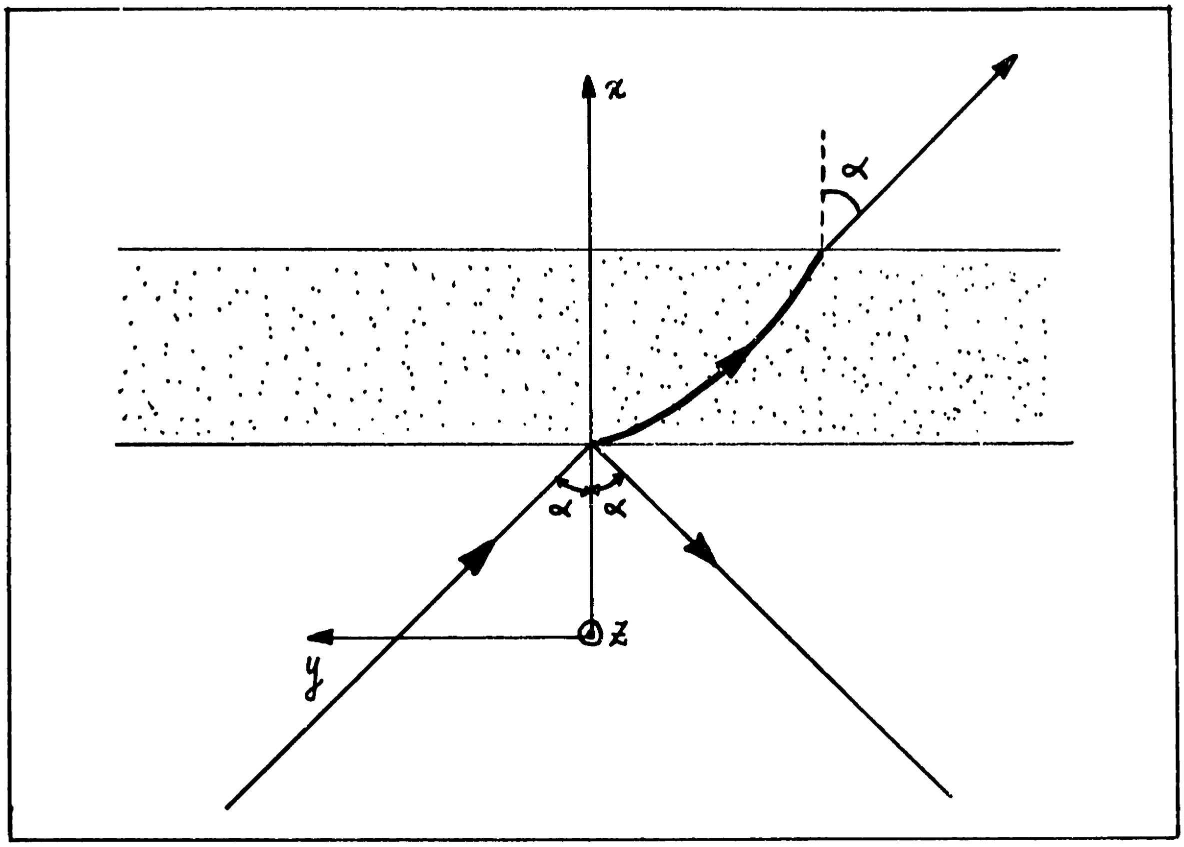
$$V(z) = \frac{4\pi e^2}{mc^2} N(z)$$

we have exactly a one-dimensional Schrodinger problem - all one has to do is to transcribe its results to this case.

4.2 Scattering by a dielectric slab

We shall consider here dielectric media characterized by the electrical properties : dielectric "constant" $\mathcal{E} \equiv \mathcal{E}(x)$ magnetic permeability $\mu = \mu_0$ and zero conductivity. Two cases are of interest: the angle of incidence is variable and the frequency fixed, or the frequency varies at normal incidence. The later case is identical with the transmission line problem which shall be sketched in the next section ; accordingly, we shall focus our attention on the former case.

Let us assume $\mathcal{E}(x) \equiv 1$ for $x < 0$ and $x > \delta$.



Consider a plane electromagnetic wave of frequency ω_0 impinging on the slab, the angle of incidence being α and the plane of incidence is the plane xy.

From Maxwell's equations :

$$\nabla . \mathcal{E}(x) \vec{E} = 0$$

$$\nabla . \vec{H} = 0$$

$$\nabla \times \vec{E} = i\omega_0\mu_0 \vec{H}$$

$$\nabla \times \vec{H} = -i\omega_0 \mathcal{E}(x) \vec{E}$$

we obtain, by eliminating H:

$$\nabla^2 \vec{E} + \mu_0\omega_0^2 \mathcal{E}(x) \vec{E} + \nabla \left[\frac{\vec{E} . \nabla \mathcal{E}}{\mathcal{E}} \right] = 0$$

Let E_z be the component of the elctric field normal to the plane of incidence ; E_z satisfies the equation:

$$\nabla^2 E_z + k_0^2 n^2(x) E_z = 0$$

where:

$$k_0 = \frac{\omega_0}{c}$$

$$n^2(x) = \mathcal{E}(x)/\mathcal{E}_0$$

the boundary conditions on E_z are:

$$E_z = e^{ik_0(x\cos\alpha + y\sin\alpha)} + R\, e^{-ik_0(x\cos\alpha - y\sin\alpha)} \qquad x \leq 0$$

$$E_z = T e^{ik_0(x\cos\alpha + y\sin\alpha)}$$

B is the reflection and T the transmission coefficients.

Separate variables by defining u(x) :

$$E_z = u(x)\, e^{ik_0 \sin\alpha\, y}$$

u(x) satisfies the equation:

$$\frac{d^2u}{dx^2} + \left\{ k_0^2\cos^2\alpha - k_0^2\left[1 - n^2(x)\right]\right\} u = 0$$

Define:

$$p = k_0 \cos\alpha$$

$$V(x) = k_0^2\left[1 - n^2(x)\right] = \frac{\omega_0^2}{\mathcal{E}_0 c^2}\left[\mathcal{E}_0 - \mathcal{E}(x)\right]$$

Notice that for usual materials, $\mathcal{E}(x) \geq \mathcal{E}_0$ thus $V(x) \leq 0$. It is well-known that in the one-dimensional case all potentials which are non-positive support bound states - this property must be kept in mind in this application.

In order that E_z satisfy the prescribed boundary conditions it is necessary and sufficient that we impose on $u(x)$ the boundary conditions:

$$u(x) = e^{ipx} + R\, e^{-ipx} \qquad x < 0$$

$$u(x) = T e^{ipx} \qquad x > \delta$$

and we see that our electromagnetic problem has been reduced to a Schrödinger one-dimensional inverse scattering problem. Notice that, since $\alpha \in [0, \pi]$, $p \in [-k_0, k_0]$; this shall be no problem however because $b(p)$ is an analytic function and giving its values in a continuous part of the region of analyticity will determine its analytic continuation in $[-\infty, +\infty]$.

4.3 Transmission Lines

The transmission lines equations for a line with distributed impedances $L(z)$ and $C(z)$ such that $L(z) \equiv L_0$ and

$$C(z) \equiv \begin{cases} C(z) & 0 < z < \delta \\ C_0 & z < 0 \qquad z > \delta \end{cases}$$

are:

$$\frac{\partial V}{\partial z} = i\omega L_o I \tag{160}$$

$$\frac{\partial I}{\partial z} = i\omega C(z) V \tag{161}$$

assuming steady-state solutions of frequency ω .

Eliminating the current I from (160) and (161), we obtain the wave equation for V:

$$\frac{d^2 V}{dz^2} + \omega^2 L_o C(z) V = 0 \tag{162}$$

Define :

We wish to impose the boundary conditions:

$$V(z) = e^{ikz} + B(k)e^{-ikz} \qquad z<0 \tag{163}$$

$$V(z) = T(k)e^{ikz} \qquad z>\delta \tag{164}$$

In order to reduce (162) to a Sturm-Liouville form we use the Liouville transformation already used by G. Borg (68), defining here:

$$\eta(z) = \left[\frac{C(z)}{C_0} \right]^{\frac{1}{4}}$$

$$\frac{dx}{dz} = \left[\frac{C(z)}{C_0} \right]^{\frac{1}{2}}$$

$$u = \eta V$$

It is readily shown that $u(x, k)$ satisfies:

$$\frac{d^2u}{dx^2} + k^2u - q(x)\,u = 0 \tag{165}$$

where:

$$q(x) = \frac{1}{\eta} \frac{d^2\eta}{dx^2} \tag{166}$$

the $q(x)$ thus obtained is indeed independent of frequency.

The new variable x is defined by:

$$x = \int_0^z \left[\frac{C(\xi)}{C_0} \right]^{\frac{1}{2}} d\xi$$

when z varies from 0 to δ, x varies from 0 to Δ, where:

$$\Delta = \int_0^\delta \left[\frac{C(\xi)}{C_0} \right]^{\frac{1}{2}} d\xi$$

notice also that $q(x) \equiv 0$ for $x < 0$ and $x > \Delta$.

A necessary and sufficient condition for V to satisfy the boundary conditions (163) and (164) is that u satisfies:

$$u(x,k) = e^{ikx} + B(k)e^{-ikx} \qquad x < 0 \tag{167}$$

$$u(x,k) = T(k)e^{ik(\delta-\Delta)}e^{ikx} \qquad x > \Delta \tag{168}$$

Thus the transmission line problem has been reduced to an equivalent Schrodinger problem. It is possible [117] to include the more general transmission line problem where C(z) has different values for $x < 0$ and $x > \Delta$.

CHAPTER V

THE PHASE-DELAY SYNTHESIS PROBLEM

In this chapter we shall give a complete solution to the problem of synthesizing a FINITE RANGE potential producing a prescribed phase-delay between steady-state waves "at the right" and "at the left" of the potential.

We shall proceed by setting up an associated radial problem, and showing that our data is simply related to the scattering function of the associated problem. Agranovich-Marchenko's theory shall then be used in order to solve the inverse radial problem to which our phase-delay synthesis has been reduced.

5.1 Statement of the Problem

Consider a finite range potential $V(x)$ such that $V(x) \equiv 0$ for $x < 0$ and $x > \delta$. We shall assume $V(x)$ to be real and piecewise continuous.

Define the real functions $\mathcal{A}(x,k), \Theta(x,k)$ by:

$$\chi(x,k) = \mathcal{A}(x,k)\, e^{i\,\Theta(x,k)} \tag{169}$$

where $\chi(x, k)$ was previously defined.

From (45) it follows that $\mathcal{A}$ is even and Θ is odd, as functions of k, for real values of k.

Our problem can be stated in two parts:

1. Given

$$\Delta(\delta,k) = \Theta(\delta,k) - \Theta(0,k) \tag{170}$$

(designated hereafter by "phase-delay function"), to find the corresponding scattering potential having finite range δ (δ is also given).

2. To give necessary and sufficient conditions on a function of k in order that it be the phase-delay function of some finite range $[0, \delta]$ one-dimensional potential.

5.2 The Associated Radial Problem

Let us consider the radial equation:

$$\frac{d^2\psi}{dr^2} + k^2\psi - V(r)\psi = 0$$

in which, for $r \geq 0$, V(r) is the same function previously defined for x in the range $-\infty$, $+\infty$.

Let f(r, k) be the solution of this equation defined by the boundary condition:

$$f(r,k) = e^{ikr} \qquad r \geq \delta \tag{171}$$

from the value of the Wronskian:

$$W\left[f(r,k)\,;\,f(r,-k)\right] = 2ik$$

it follows that $f(r,k)$ and $f(r,-k)$ form a fundamental set of solutions of equation (1) for $k \neq 0$.

$\chi(r,k)$ is obviously a solution of Eq.(1) for $r \geq 0$.

Consequently it can be expressed in terms of $f(r,k)$ and $f(r,-k)$:

$$\chi(r,k) = A_1(k)f(r,k) + A_2(k)f(r,-k)$$

recalling that for $r \geq \delta$:

$$\chi(r,k) = t(k)e^{ikr}$$

it follows immediately from (171) that:

$$A_1(k) \equiv t(k)$$

$$A_2(k) \equiv 0$$

hence:

$$\chi(r,k) \equiv t(k)f(r,k) \qquad r \geq 0 \tag{172}$$

In this manner we have connected the radial and one-dimensional problems. Notice that (172) holds irrespective of the behaviour of $V(x)$ for $x < 0$, viz.,

$$\frac{\chi(r,k)}{t(k)}$$

for $r \geq 0$ is independent of the values of $V(r)$ for $r < 0$.

Of course this was to be expected, as seen by integrating Eq. (1) "backwards", i.e., from $r = \delta$ to the left.

On the other hand, the conditions imposed on V(x) ensure that χ and f are continuous functions of the position, with continuous derivatives with respect to the position. Expressing their continuity at the origin:

$$t(k)f(0,k) \equiv \chi(0,k) \equiv 1 + b(k) \tag{173}$$

$$t(k)f'(0,k) \equiv \chi'(0,k) \equiv ik\left[1 - b(k)\right] \tag{174}$$

[using the expression for $\chi(x,k)$ when $x \leq 0$]

These equations are the foundation of our argument - through them we have been able to connect the scattered data b and t of the one-dimensional problem to the Jost function of the radial problem. [Eq. (174) is less important for our present discussion, since it involves f'(o, k) which does not seem to be simply related to f(o, k).].

In terms of the one-dimensional wave function $\chi(x, k)$ the Jost function becomes:

$$f(0,k) \equiv \frac{\chi(0,k)}{\chi(\delta,k)} e^{ik\delta} \tag{175}$$

hence, using the previously defined functions A and η :

$$A(k) = \frac{\mathcal{A}(0,k)}{\mathcal{A}(\delta,k)} \tag{176}$$

$$\eta(k) = \Theta(0,k) - \Theta(\delta,k) + k\delta = k\delta - \Delta(\delta,k) \tag{177}$$

the scattering function of the associated radial problem can be expressed in terms of our one-dimensional data:

$$S(k) \equiv e^{2i\,[\Delta(\delta,k) - k\delta]} \tag{178}$$

5.3 Solution of the Problem

According to 5.1 we shall take up successively each of the two parts into which our problem was divided.

5.3.1 The Algorithm

Based upon Agranovich-Marchenko's results, which were described in chapters II and III, we can set-up a step-by-step procedure to obtain V(x), once $\Delta[\delta, k]$ and δ are given:

i) Compute:

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[e^{2i(\Delta - k\delta)} - 1\right]e^{ikt}\,dk \tag{179}$$

ii) Solve the Fredholm linear integral equation (x is a parameter):

$$A(x,y) = F(x+y) + \int_{x}^{\infty} F(y+z)A(x,z)\,dz \qquad y \geq x \geq 0 \tag{180}$$

iii) Obtain the potential through:

$$V(x) \equiv \begin{cases} -2\frac{d}{dx}A(x,x) & x \geq 0 \quad (181) \\ 0 & x < 0 \quad (182) \end{cases}$$

5.3.2 Necessary and Sufficient Conditions for the Synthesis

Using again Agranovich-Marchenko's results stated in Sec. 2.1 we obtain:

In order that a function $\Delta(k)$ be the phase-delay function corresponding to a finite range potential $V(x) \equiv 0$

for $x \notin [0, \delta]$, it is necessary and sufficient that:

i) $\Delta(k)$ be real

ii) $\Delta(0) = 0$ and $\Delta(k) \sim k\delta \quad k \to \infty$

iii) $\Delta(-k) = -\Delta(k)$

iv)
$$e^{2i[\Delta(k) - k\delta]} = 1 + \int_{-\infty}^{+\infty} F(t) e^{-ikt} dt$$

where $F(t)$ is absolutely integrable

v) $F'(t) \equiv 0$ for $t > 2\delta$

These conditions ensure that a radial potential $V(r)$ exists which vanishes identically for $r > \delta$. Defining the one dimensional potential as:

$$V_1(x) \equiv \begin{cases} 0 & x < 0 \\ V(x) & x \geq 0 \end{cases}$$

we see from 5.1 and 5.2 that $V_1(x)$ is the solution to our problem, viz., it is a finite range $[0, \delta]$ one-dimensional potential whose phase-delay function is indeed $\Delta(k)$.

It is interesting to observe that condition (ii) above ensures that the radial potential $V(r)$ does not support bound states, according to Levinson's theorem, and is uniquely determined from the scattering function alone. However, it does not ensure that $V(x)$ do not support bound states (we have already stressed this fundamental difference between one-dimensional and radial problems) - this is made transparent by considering a radial square well having a very shallow depth, such that it does not support any bound state ; our one-dimensional potential shall have at least one bound state, irrespective of the well's depth.

5.4 Relative Amplitude

For some applications, it is of interest to prescribe the relative amplitude:

$$\alpha(S,k) = \frac{A(0,k)}{A(S,k)}$$

instead of the phase-delay function $\Delta(S, k)$.

We would like to point out here that these formulations are equivalent. It was shown in Sec. 3.1 that the Jost function

$$f(p) = \frac{\chi(0,k)}{\chi(S,k)}\, e^{ikS}$$

must be an entire function of p (finite range potential). Consequently, one cannot prescribe arbitrarily both the relative amplitude and the phase-delay - if either one is chosen, the other is automatically determined.

Adapting the procedure given in Sec. 3.1 to this problem, we obtain :

$$\Delta(\delta,k) = k\delta + \int_0^\infty \gamma(t) \sin kt \, dt$$

$$\alpha(\delta,k) = \exp \int_0^\infty \gamma(t) \cos kt \, dt$$

Notice that we can alternatively proceed directly, by means of the Gelfand-Levitan algorithm, since the weight function W(k) is simply related to the relative amplitude $\alpha(\delta, k)$:

$$W(k) = \frac{1}{\alpha^2}$$

5.5. Ratio reflected/transmitted wave

Another problem of some interest is the synthesis of V(x) when b/t is given.

We have already seen in chapter IV that when V(x) has a finite range, b/t is a regular function of p in the entire plane (except possibly at the origin). Additional insight into the analytic properties of b/t is given by Eq. (173).

The analytic properties of the scattering matrix allow us to use a Wiener-Hopf technique in order to determine b and t separately from the knowledge of b/t on the real axis.

Once b is obtained, we can determine V(x) by means of the Gelfand-Levitan-Kay-Moses' algorithm for the one-dimensional inverse problem.

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