The State-Variable Approach to Continuous Estimation

by

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B.S., University of Southern California
(1961)

S.M., Massachusetts Institute of Technology
(1963)

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philiosophy

at the

Massachusetts Institute of Technology
February, 1966

Signature of Author__
       Department of Electrical Engineering, February 7, 1966

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Abstract

A new approach is presented for the continuous, nonlinear filtering problem. The approach is based on the state-variable representation for random processes and makes use of the Markovian nature of the associated state vector. An equation is derived for the probability density of the instantaneous value of the desired process given the past values of the observed process in which it is imbedded. Equations for the estimate of the desired process are obtained by employing the fact that the minimum-mean-square-error estimate is the conditional mean.

The nonlinear filtering problem has applications in several disciplines among which are the theories of optimal control, radar, sonar, seismic estimation, and communication. Only applications to communication theory are presented in the thesis. These cover a variety of commonly occurring, linear and nonlinear, analog modulation schemes and continuous channels. The approach can be used to treat the estimation of Markovian messages observed, after a nonlinear modulator, in a channel introducing Markovian disturbances; Gaussian messages and channel disturbances are a special case. Demodulator structures are specified which are physically realizable and whose output is a close approximation to the minimum-mean-square-error estimate of the message and, if desired, the channel disturbances. Particular emphasis is given to phase and frequency modulation. The channels considered include Rayleigh channels and fixed channels with memory.

An analysis of quasi-optimum phase and frequency demodulators is presented. An exact analysis of a quasi-optimum PM demodulator for a first-order Gaussian message process is given; an equation is derived for the stationary probability density of the estimation error. The above threshold performance of quasi-optimum FM demodulators for the Butterworth class of Gaussian-message spectra is given. Simulation results are presented for an FM demodulator and a first-order Gaussian-message spectrum.

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Acknowledgement

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Notation

$\underline{v}(t)$ Lower-case, underscored letters denote column vectors.

$v_i(t)$ The i-th component of the vector $\underline{v}(t)$

$\dfrac{d}{dt}\underline{v}(t)$ A vector whose i-th component is $\dfrac{d}{dt}v_i(t)$

$M(t)$ Capital letters denote matrices.

$M'(t)$ Transpose of $M(t)$

$M^{-1}(t)$ Inverse of $M(t)$

$\underline{f}[t:\underline{v}(t)]$ A column vector whose components are nonlinear, no-memory, time-varying transformations of the vector $\underline{v}(t)$

$D[\underline{f}(t:\underline{v})]$ The Jacobian matrix associated with $\underline{f}[t:\underline{v}(t)]$ -- the (i-row, j-column)-element of the matrix is $\dfrac{\partial}{\partial v_i}f_j[t:\underline{v}(t)]$.

$\hat{\underline{v}}(t)$ Circumflex denotes the exact minimum-mean-square error estimate

$\underline{v}^*(t)$ Asterisk denotes the approximate minimum-mean-square estimate

$\underline{v}_{t_0}t$ Denotes the set of waveforms $\{\underline{v}(\tau): t_0 \leq \tau \leq t\}$

I. Introduction

1.1 Description of the Problem

In this study, we shall present an approach to the problem of continuously estimating the components of a vector random process, $\underline{x}(t)$, based upon the past values of a noisy observed vector, $\underline{r}(t)$, in which it is imbedded. For reasons which will become apparent, we call our approach the "state-variable" approach.

The components of $\underline{x}(t)$ and $\underline{r}(t)$ are assumed to be continuous processes which may or may not be stationary. $\underline{r}(t)$ is continuously observed over the interval (t_0, t), where t_0 is the initial observation time and t is the final observation time for which the estimate is desired.

We shall consider estimates of $\underline{x}(t)$ which are optimum in the sense that the mean-square error in estimating each component of $\underline{x}(t)$ is minimum.

The problem to be examined is commonly referred to as the filtering problem. It arises in several disciplines, among which are the theories of optimal control, radar, sonar, seismic estimation, and communication. The state-variable approach is applicable to all these disciplines. However, we shall apply it only in the analog communication theory context. The applications given provide an interpretation of the approach as well as illustrate its broad scope.

1.2 Description of the Approach Developed

We shall assume that $\underline{x}(t)$ is a continuous vector Markov process. The conditional probability density of $\underline{x}(t_i) = \underline{x}_i$, given the values of $\underline{x}(t)$ at the preceding, ordered times, $t_{i-1} > t_{i-2} > \cdots > t_0$, then satisfies:

$$p(\underline{x}_i \mid \underline{x}_{i-1}, \underline{x}_{i-2}, \cdots, \underline{x}_0) = p(\underline{x}_i \mid \underline{x}_{i-1}) \qquad (1.1)$$

Thus, $\underline{x}_i$ is independent of $\underline{x}_{i-2}, \underline{x}_{i-3}, \cdots, \underline{x}_0$ when $\underline{x}_{i-1}$ is known.

Our reason for considering continuous vector Markov processes is that they can be used to characterize the response of a wide class of linear and nonlinear systems to excitation functions which are white Gaussian processes. Markov processes arise very naturally as a characterization when systems are represented by equations of state. These equations are of the form:*

$$\frac{d}{dt}\underline{x}(t) = \underline{f}[t:\underline{x}(t)] + \underline{\xi}(t) \qquad (1.2)$$

where the components of $\underline{x}(t)$ are a set of system state variables and the components of $\underline{f}[t:\underline{x}(t)]$ are suitably restricted, nonlinear transformations of $\underline{x}(t)$. By the definition of state, the value of $\underline{x}(t_i)$ is determined by the value of $\underline{x}(t_{i-1})$, for $t_i > t_{i-1}$, and the values of $\underline{\xi}(t)$ in the interval (t_{i-1}, t_i); it does not depend on values of $\underline{x}(t)$ for $t < t_{i-1}$. If the components of $\underline{\xi}(t)$ are processes with independent increments, then the behavior of $\underline{\xi}(t)$ in the interval (t_{i-1}, t_i) is independent of its behavior outside the interval. It follows, intuitively, that the probability density of the state vector, $\underline{x}(t)$, satisfies Eq. 1.1 and, therefore, that $\underline{x}(t)$ is a vector Markov process. It will be continuous if $\underline{f}[t:\cdot]$ is suitably restricted and if $\underline{\xi}(t)$ has components which are white Gaussian processes.

A conditional probability density functional for $\underline{x}(t)$, given the past values of $\underline{r}(t)$, is derived. Use is then made of the fact that the

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* An interpretation of the state equation is provided in Ch. 5 by several examples.

least-squares estimate of $\underline{x}$(t) is the conditional mean. The exact least-squares estimate, unfortunately, cannot be realized practically except under very special conditions.* Nevertheless, the equations specifying the exact estimate lead naturally to the consideration of an approximate estimate . The approximate estimate closely matches the exact estimate when the disturbances processes introduce small perturbations in the observed processes. Furthermore, the equations for the approximate estimate can be readily implemented in the form of a physically realizable processor.

We shall refer to the approximate least-squares estimate as the "quasi-optimum" estimate.

1.3 The Relationship of the Approach Developed to Alternate Approaches

The state-variable approach to continuous estimation provides a logical addition to existing approaches and, moreover, expands the scope of the problems which can be treated.

That a logical addition is provided can be seen by examining Fig. 1.1 where where we have classified existing approaches according to whether or not they are structured and according to the manner in which the random processes are represented. The principle structured approaches are those of Wiener[1] and Kalman and Bucy.[2] For these, the estimate of $\underline{x}$(t) is restricted to being a linear transformation of $\underline{r}$(t). The principle nonstructured approach is that originating with Lehan and Parks[3] and extended by Youla[4] and Van Trees[5-8] in which

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* The exact estimate is "physically" realizable since it depends only on the past of $\underline{r}$(t).

	STRUCTURED	NONSTRUCTURED
SPECTRAL REPRESENTATION	Wiener [1]	Lehan and Parks [3] Youla [4] Van Trees [5-8]
STATE REPRESENTATION	Kalman and Bucy [2]	

Figure 1.1 Approaches to Continuous Estimation

no restriction is made on the form of the transformation of $\underline{r}(t)$.* With regard to the manner of representing the random processes, we note that both Wiener and Lehan and Parks use a spectral or, equivalently, a correlation-function representation while, on the other hand, a state representation is used by Kalman and Bucy. A logical addition to these existing approaches is provided by the technique to be presented because it is nonstructured and uses a state representation for the random processes.

The scope of the estimation problems which can be treated is expanded by the state-variable approach. The components of $\underline{x}(t)$ are no longer restricted to being sample functions from Gaussian processes as they are with the MAP approach. The broader class of continuous Markov processes, of which Gaussian processes are a special case, can now be treated.

1.4 Previous Related Studies

The equation for the conditional probability density of $\underline{x}(t)$ which is developed in Ch. III was first derived correctly by Kushner[9] whose derivation is closely followed here. A more recent derivation has been given by Bucy.[10] Incorrect versions of the equation appear in the literature[11-14] and some of these have been noted and discussed by Kushner,[9,15] Wonham,[16] and Mortensen.[17] We shall indicate in Sec. 3.2.1 how the incorrect equation can arise by neglecting a significant term in our derivation.

Discrete counterparts to the estimation model presented in Ch. II, or to special cases of it, have been studied by Wonham,[16,18] Weaver,[19] and Cox.[20-22]

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* A maximum a posteriori probability criterion is used with this approach. For this reason, we shall refer to it as the "MAP" approach.

Special cases of the general continuous estimation model of Ch. II have been examined in the past. Snyder [23] studied the estimation of one-dimensional Gaussian processes contained as modulation in a signal observed in white Gaussian noise. Bucy [10] studied the estimation of one-dimensional Markov processes, but he did not consider the application or implementation of his results. Several related, not widely-known studies have been made in the USSR. [24-32] However, the Russian studies are often based upon an incorrect equation for the conditional probability density functional and some caution must be exercised in the use of the stated results.

1.5 Organization of the Thesis

Chapters 2,3, and 4 are devoted to the mathematical development of the state-variable approach to continuous estimation. The Estimation Model is defined in Chapter 2. An equation for the conditional probability density for $x(t)$, given the observations accumulated up to time t, is derived in Chapter 3. The equations for the optimum and quasi-optimum estimate of $x(t)$ are derived in Chapter 4.

The first part of Chapter 5 contains the definition of the Communication Model and a discussion relating this model to the Estimation Model. The remainder of the chapter contains a variety of examples which serve to illustrate the broad scope of the approach.

Chapter 6 contains the results of an analysis of the quasi-optimum PM and FM demodulators derived in the examples of Chapter 5.

II. The Estimation Model

The Estimation Model is shown in Fig. 2.1. Quantities appearing in the model will now be defined. The interpretation of the model is deferred to Ch. IV where we shall define a related Communication Model which will be studied in detail.

Let $\underline{x}(t)$ be an m-dimensional vector Markov process described by the stochastic differential equation:

$$d \underline{x}(t) = \underline{f}[t: \underline{x}(t)] dt + d \underline{\chi}(t) \tag{2.1}$$

where $\underline{f}[t: \underline{x}(t)]$ is an m-dimensional vector whose components are memoryless, nonlinear transformations of $\underline{x}(t)$ and $\underline{\chi}(t)$ is an m-dimensional vector whose components are Wiener processes. Let the covariance matrix associated with $\underline{\chi}(t)$ be:

$$E[\underline{\chi}(t) \, \underline{\chi}'(u)] = X \min(t,u) \tag{2.2}$$

where X is a symmetric, non-negative definite, m x m matrix. The elements of X, denoted by X_{ij}, may be time-varying.

Eq. 2.1 defines a continuous vector Markov process provided the components of $\underline{f}[t: \cdot]$ are suitably restricted. The restrictions are discussed by Doob,[33] Khazen;[34] a heuristic discussion is given by Wishner.[35] The principle restriction, which insures the continuity of $\underline{x}(t)$, is that each component of $\underline{f}[t: \cdot]$ must satisfy the Lipschitz condition.* We shall hereafter assume that $\underline{f}[t: \cdot]$ meets all the necessary requirements so that $\underline{x}(t)$ is a continuous vector Markov process.

Observe that more than one vector process can be represented by

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* If $||f_i[t: \underline{x}_1] - f[t: \underline{x}_2]|| < k ||\underline{x}_1 - \underline{x}_2||$, where k is a constant, then $f_i[t: \cdot]$ satisfies the Lipschitz condition. The vertical bars are defined by: $||\underline{x}|| = \max(x_1, x_2, \cdots, x_m)$.
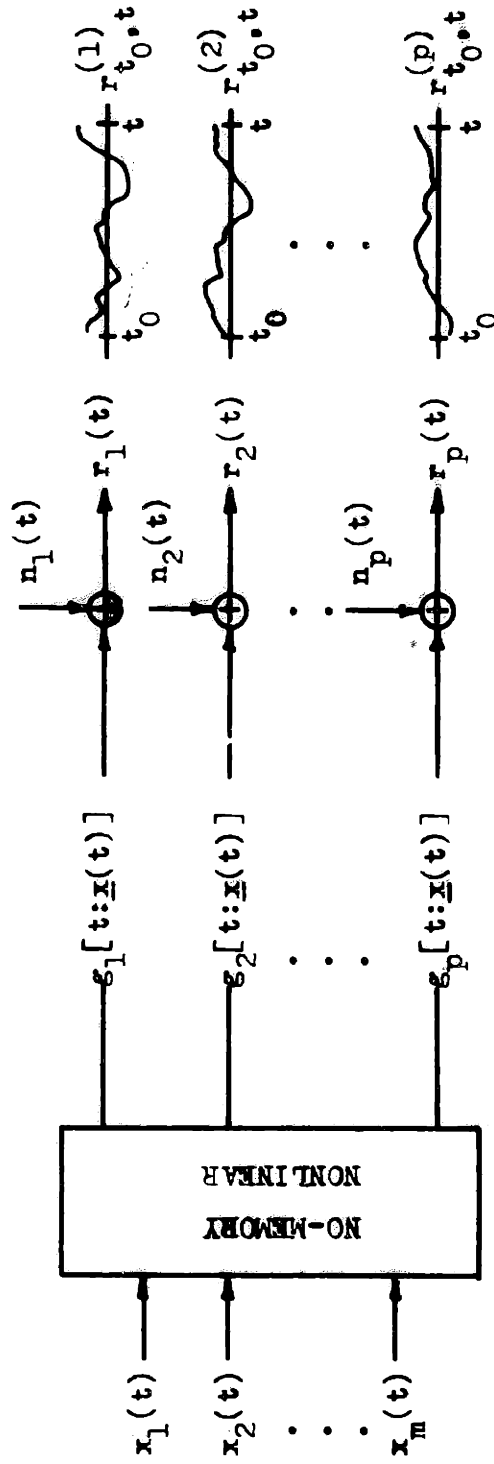
Figure 2.1 The Estimation Model

simply adjoining the individual vectors to form $\underline{x}(t)$. Observe also that $\underline{x}(t)$ can have deterministic components in which case the corresponding elements of X are zero.

In later sections, we shall be interested in estimating scalar Gaussian processes which are stationary and have rational spectra. These processes can be represented in the form of Eq. 2.1 by letting $\underline{f}[t:\underline{x}(t)] = F \underline{x}(t)$, where F is a time-invariant, $m \times m$ matrix. X must also be time invariant. By carefully choosing such a state representation, one component of $\underline{x}(t)$ can be made to correspond directly to the scalar process. Moreover, when several scalar processes are represented by adjoining their individual state vectors, each will correspond directly to one of the components of $\underline{x}(t)$. A particularly convenient state representation for scalar Gaussian processes is presented in Appendix A1. We shall use this representation exclusively in the applications to follow.

In a fairly straightforward way, it can be demonstrated that the amplitude probability density, $p(\underline{x}; t)$, associated with the Markov process, $\underline{x}(t)$, satisfies the Fokker-Planck equation:

$$\frac{\partial}{\partial t}p(\underline{x}; t) = -\sum_{i=1}^{m} \frac{\partial}{\partial x_i}[f_i(t:\underline{x})p(\underline{x}; t)] + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}X_{ij}\frac{\partial^2}{\partial x_i \partial x_j}p(\underline{x}; t) \qquad (2.3)$$

with appropriate boundary conditions. The derivation of Eq. 2.3 is given, for example, by Doob,[33] Wishner,[35] Bharucha-Reid,[36] and Stratonovich.[37]

We shall now define the noisy observed process. Let $\underline{\dot{y}}(t)$ be a p-dimensional vector random process described by the stochastic differential equation:

$$d\,\underline{y}(t) = \underline{g}[t:\underline{x}(t)]\,dt + d\,\underline{\eta}(t) \qquad (2.4)$$

where $\underline{g}[t:\underline{x}(t)]$ is a p-dimensional vector whose components are

memoryless, nonlinear transformations of $\underline{x}(t)$ and $\underline{\eta}(t)$ is a

p-dimensional vector whose components are Wiener processes. Simply

for the convenience of notation, we shall assume that $\underline{x}(t)$ and $\underline{\eta}(t)$ are

statistically independent. Let the covariance matrix associated with

$\underline{\eta}(t)$ be given by:

$$E[\underline{\eta}(t) \ \underline{\eta}'(u)] = N \min(t,u) \tag{2.5}$$

where N is a symmetric, positive-definite, p x p matrix. It is assumed

that N^{-1} exists; this requires that none of the components of $\underline{\eta}(t)$ be

zero. The elements of N may be time varying.

Some of the statistics of $d\underline{y} = \underline{y}(t+dt) - \underline{y}(t)$ will be required in

later sections. We shall cite them here for convenience. The first

property to be noted is that when $\underline{x}(t)$ is known, $d\underline{y}$ has a normal

distribution:

$$p(d\underline{y} | \underline{x}; t) \sim \exp - \frac{1}{2dt} \Big[d\underline{y} - \underline{g}[t:\underline{x}]dt \Big]' N^{-1} \Big[d\underline{y} - \underline{g}[t:\underline{x}]dt \Big] \tag{2.6}$$

Observe, secondly, that to terms of order dt:

$$E[d\underline{y} \ d\underline{y}'] = N \ dt \tag{2.7}$$

as can be demonstrated by using Eqn's. 2.4 and 2.5. Furthermore, all

higher-order moments of $d\underline{y} \ d\underline{y}'$ are of order greater than dt. This

implies that $d\underline{y} \ d\underline{y}' / dt$ is essentially deterministic and equal to N for

dt vanishingly small. Thus, to terms of order dt:

$$d\underline{y} \ d\underline{y}' = E[d\underline{y} \ d\underline{y}'] = N dt \quad \text{(dt infinitesimal)} \tag{2.8}$$

A more rigorous discussion justifying Eq. 2.8 is given by Kushner.[9] An

excellent discussion is also given by Gray and Caughey.[45]

Eqn's. 2.1 and 2.4 jointly define a continuous, (m+p)-dimensional,

vector Markov process whose components are the combined components of $\underline{x}(t)$ and $\underline{y}(t)$.

Formally dividing Eqn's. 2.1 and 2.4 by dt results in the more familiar looking expressions:

$$\frac{d}{dt}\underline{x}(t) = \underline{f}[t:\underline{x}(t)] + \underline{\xi}(t) \tag{2.9}$$

and

$$\frac{d}{dt}\underline{y}(t) \equiv \underline{r}(t) = \underline{g}[t:\underline{x}(t)] + \underline{n}(t) \tag{2.10}$$

where $\underline{\xi}(t) = d\underline{\chi}(t)/dt$ and $\underline{n}(t) = d\underline{\eta}(t)/dt$ are m-and p-dimensional vectors whose components are white Gaussian processes. The associated covariance matrices are $X\delta(t-u)$ and $N\delta(t-u)$, respectively.

We shall assume that the actually observed process, $\underline{r}(t) \equiv d\underline{y}(t)/dt$, is available from an initial observation time, t_0, until the present time, t. The entire observed waveform, $\{\underline{r}(\tau): t_0 \leq \tau \leq t\}$, will be denoted by $\underline{r}_{t_0,t}$. Similarly, the entire waveform, $\{\underline{y}(\tau): t_0 \leq \tau \leq t\}$, will be denoted by $\underline{y}_{t_0,t}$.

The Estimation Model as shown in Fig. 2.1 is now defined. At this point, we turn our attention to obtaining a sufficiently detailed description of the model so that the filtering problem can be studied. It is known[38] that the minimum-mean-square-error estimate* of $\underline{x}(t)$, given $\underline{r}_{t_0,t}$, is specified by the conditional mean:

$$\hat{\underline{x}}(t) = \int_{m\text{-fold}} \underline{x}\, p(\underline{x}; t \,|\, \underline{r}_{t_0,t})\, d\underline{x} \tag{2.11}$$

The statistical description we require is, therefore, the conditional probability density functional, $p(\underline{x}; t \,|\, \underline{r}_{t_0,t})$, for which an equation is derived in the next chapter.

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* The least-squares estimate of $\underline{x}$ is a vector whose i-th component is the least-squares estimate of x_i.

III. Derivation of an Equation for the Conditional

Probability Density Functional, $p(\underline{x}; t \,|\, \underline{r}_{t_0}, t)$

3.1 Summary of Results

In this chapter, we shall derive the following differential equation

for $p(\underline{x}; t \,|\, \underline{y}_{t_0}, t) = p(\underline{x}; t \,|\, \underline{r}_{t_0} t)$:

$$p(\underline{x}; t+dt \,|\, \underline{y}_{t_0}, t+dt) - p(\underline{x}; t \,|\, \underline{y}_{t_0}, t) = - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [f_i(t:\underline{x}) p(\underline{x}; t \,|\, \underline{y}_{t_0}, t)] \, dt +$$

$$\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} X_{ij} \frac{\partial^2}{\partial x_i \partial x_j} p(\underline{x}; t \,|\, \underline{y}_{t_0}, t) \, dt + \tag{3.1}$$

$$p(\underline{x}; t \,|\, \underline{y}_{t_0}, t) [\underline{g}(t:\underline{x}) - E\underline{g}(t:\underline{x})]' N^{-1} [d\underline{y}(t) - E\underline{g}(t:\underline{x}) \, dt]$$

where E indicates expectation with respect to $p(\underline{x}; t \,|\, \underline{y}_{t_0}, t)$. The left

side along with the first two terms of the right side of Eq. 3.1 are

recognized as the Fokker-Planck equation associated with $\underline{x}(t)$, as

given by Eq. 2.3. The last term on the right represents the modification

to the Fokker-Planck equation resulting from the observation of $\underline{r}(t)$.

When $\underline{g}[t:\underline{x}(t)]$, and hence $\underline{r}(t)$, does not depend on $\underline{x}(t)$, then the last

term is zero and the equation reduces the the original Fokker-Planck

equation as expected.

3.2 Derivation

We shall follow Kushner's[9] derivation of Eq. 3.1. Two steps are

taken: (i) changes in $p(\underline{x}; t \,|\, \underline{y}_{t_0}, t)$ resulting from an incremental

observation, $d\underline{y}(t) = \underline{y}(t+dt) - \underline{y}(t)$, with $\underline{x}(t)$ fixed are accounted for in

step one; (ii) changes resulting from incremental changes in $\underline{x}(t)$ are

then accounted for in step two.

3.2.1 Derivation: Step One

Consider the effect on $p(\underline{x}; t \mid \underline{y}_{t_0}, t)$ of an incremental change, $d\underline{y}(t)$, in the observation vector. Clearly:

$$p(\underline{x}; t \mid \underline{y}_{t_0}, t+dt) = p(\underline{x}; t \mid \underline{y}_{t_0}, t, d\underline{y})$$

$$= \frac{p(d\underline{y} \mid \underline{x}; t, \underline{y}_{t_0}, t) \, p(\underline{x}; t \mid \underline{y}_{t_0}, t)}{p(d\underline{y} \mid \underline{y}_{t_0}, t)}$$

$$= \frac{p(d\underline{y} \mid \underline{x}; t, \underline{y}_{t_0}, t) \, p(\underline{x}; t \mid \underline{y}_{t_0}, t)}{\int p(d\underline{y} \mid \underline{x}; t, \underline{y}_{t_0}, t) \, p(\underline{x}; t \mid \underline{y}_{t_0}, t) \, d\underline{x}} \qquad (3.2)$$

It is seen upon examining Eqn's. 2.4 and 2.6 that the probability density of $d\underline{y}$ is determined when $\underline{x}$ is known and that the density is normal. Eq. 3.2 then becomes:

$$p(\underline{x}; t \mid \underline{y}_{t_0}, t+dt) =$$

$$\frac{p(\underline{x}; t \mid \underline{y}_{t_0}, t) \, \exp - \dfrac{1}{2dt} [d\underline{y} - \underline{g}(t:\underline{x})dt]' \, N^{-1} [d\underline{y} - \underline{g}(t:\underline{x})dt]}{\int p(\underline{x}; t \mid \underline{y}_{t_0}, t) \, \exp - \dfrac{1}{2dt} [d\underline{y} - \underline{g}(t:\underline{x})dt]' \, N^{-1} [d\underline{y} - \underline{g}(t:\underline{x})dt] \, d\underline{x}} \qquad (3.3)$$

After cancelling terms common to the numerator and denominator and defining the scalar, $z(d\underline{y}, dt)$, we obtain:

$$z(d\underline{y}, dt) \equiv \frac{p(\underline{x}; t \,|\underline{y}_{t_0},t+dt)}{p(\underline{x}; t\,|\underline{y}_{t_0},t)}$$

(3.4)

$$= \frac{\exp\{d\underline{y}'\,N^{-1}\underline{g}(t:\underline{x}) - \tfrac{1}{2}\,\underline{g}'(t:\underline{x})N^{-1}\underline{g}(t:\underline{x})dt\}}{\int p(\underline{x}; t\,|\underline{y}_{t_0},t)\,\exp\{d\underline{y}'\,N^{-1}\underline{g}(t:\underline{x}) - \tfrac{1}{2}\,\underline{g}'(t:\underline{x})N^{-1}\underline{g}(t:\underline{x})dt\}\,d\underline{x}}$$

An expression for $p(\underline{x}; t\,|\underline{y}_{t_0},t+dt)$ in terms of $p(\underline{x}; t\,|\underline{y}_{t_0},t)$ can be obtained from Eq. 3.4 by expanding $z(d\underline{y}, dt)$ in a multidimensional Taylor series and keeping terms up to the order of dt. Observe from Eq. 2.7 that terms up to the second order in $d\underline{y}$ must be retained since they are of order dt in the mean. The expansion of $z(d\underline{y}, dt)$ with terms up to the order dt is:

$$z(d\underline{y}, dt) = z(0,0) + dt\,\frac{\partial}{\partial(dt)}\,z(d\underline{y}, dt)\Big|_{0,0} + \sum_{i=1}^{r} dy_i\,\frac{\partial}{\partial y_i}z(d\underline{y}, dt)\Big|_{0,0} +$$

$$\tfrac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}dy_i dy_j\,\frac{\partial^2}{\partial y_i \partial y_j}\,z(d\underline{y}, dt)\Big|_{0,0}$$

(3.5)

$$= z(0,0) + dt\,\frac{\partial}{\partial(dt)}z(d\underline{y}, dt)\Big|_{0,0} + d\underline{y}'D[z(d\underline{y}, dt)]_{0,0} +$$

$$\tfrac{1}{2}\,d\underline{y}'D\Big[D[z(d\underline{y}, dt)]\Big]_{0,0}d\underline{y}$$

where $D[z(d\underline{y}, dt)]$ is a column vector or first derivatives and $D\Big[D[z(d\underline{y}, dt)]\Big]$ is a matrix of second derivatives, both with respect to the components of $d\underline{y}$. The individual terms required for the evaluation of Eq. 3.5 can be obtained by manipulation of Eq. 3.4. The results we obtain are:

$$z(0,0) = 1$$

$$\frac{\partial}{\partial (dt)} z(d\underline{y}, dt)\Big|_{0,0} = -\tfrac{1}{2} \underline{g}'(t:\underline{x})N^{-1}\underline{g}(t:\underline{x}) + \tfrac{1}{2} E\underline{g}'(t:\underline{x})N^{-1}\underline{g}(t:\underline{x})$$

$$D[z(d\underline{y}, dt)]_{0,0} = N^{-1}\underline{g}(t:\underline{x}) - N^{-1}E\underline{g}(t:\underline{x})$$

$$D\Big[D[z(d\underline{y}, dt)]\Big]_{0,0} = [N^{-1}\underline{g}(t:\underline{x})][N^{-1}\underline{g}(t:\underline{x})]' -$$

$$2[N^{-1}\underline{g}(t:\underline{x})][N^{-1}E\underline{g}(t:\underline{x})]' +$$

$$2[N^{-1}E\underline{g}(t:\underline{x})][N^{-1}E\underline{g}(t:\underline{x})]' -$$

$$E[N^{-1}\underline{g}(t:\underline{x})][N^{-1}\underline{g}(t:\underline{x})]'$$

where E indicates expectation with respect to $p(\underline{x}; t\,|\underline{y}_{t_0,t})$. A typical

term resulting from the substitution of $D\Big[D[z(d\underline{y}, dt)]\Big]$ into Eq. 3.5 is:

$$\tfrac{1}{2} d\underline{y}' N^{-1}\underline{g}(t:\underline{x})\Big[N^{-1}\underline{g}(t:\underline{x})\Big]' d\underline{y} = \tfrac{1}{2} d\underline{y}' N^{-1}\underline{g}(t:\underline{x})\underline{g}'(t:\underline{x})N^{-1}d\underline{y}$$

$$= \tfrac{1}{2} \underline{g}'(t:\underline{x})N^{-1}d\underline{y}\, d\underline{y}' N^{-1}\underline{g}(t:\underline{x})$$

where we have used the fact that $d\underline{y}' N^{-1}\underline{g}(t:\underline{x})$ is a scalar. Since we

are retaining only terms up to the order of dt and since dt is

infinitesimal, $d\underline{y}\, d\underline{y}'$ may be replaced by Ndt, as indicated by Eq. 2.8.

Thus, the typical term becomes $\tfrac{1}{2} \underline{g}'(t:\underline{x})N^{-1}\underline{g}(t:\underline{x})dt$. This procedure

can be repeated for other terms associated with $D\Big[D[z(d\underline{y}, dt)]\Big]_{0,0}$.

The result of substituting the individual terms into Eq. 3.5 is:

$$z(d\underline{y}, dt) = 1 + d\underline{y}' N^{-1}\underline{g}(t:\underline{x}) - d\underline{y}' N^{-1}\Big[E\underline{g}(t:\underline{x})\Big] -$$

$$\underline{g}'(t:\underline{x})N^{-1}\Big[E\underline{g}(t:\underline{x})\Big]dt + \Big[E\underline{g}(t:\underline{x})\Big]' N^{-1}\Big[E\underline{g}(t:\underline{x})\Big]dt$$

Hence:

$$z(d\underline{y}, dt) = 1 + [d\underline{y} - E\underline{g}(t:\underline{x})dt]' N^{-1}[\underline{g}(t:\underline{x}) - E\underline{g}(t:\underline{x})] \qquad (3.6)$$

where E indicates expectation with respect to $p(\underline{x}; t | \underline{y}_{t_0}, t)$.

By using the definition of $z(d\underline{y}, dt)$ from Eq. 3.4, we conclude that the effect of an incremental observation, $d\underline{y}$, is, to terms of order dt, an incremental change in $p(\underline{x}; t | \underline{y}_{t_0}, t)$ given by:

$$p(\underline{x}; t | \underline{y}_{t_0}, t+dt) - p(\underline{x}; t | \underline{y}_{t_0}, t) =$$

$$p(\underline{x}; t | \underline{y}_{t_0}, t) [\underline{g}(t:\underline{x}) - E\underline{g}(t:\underline{x})]' N^{-1}[d\underline{y} - E\underline{g}(t:\underline{x})dt] \qquad (3.7)$$

As an aside, it is of interest to investigate the effect of overlooking the second order terms of $d\underline{y}$ in the expansion of $z(d\underline{y}, dt)$ given in Eq. 3.5. The erroneous expansion:

$$z(d\underline{y}, dt) = z(0,0) + dt \frac{\partial}{\partial(dt)} z(d\underline{y}, dt)\Big|_{0,0} + d\underline{y}' D[z(d\underline{y}, dt)]_{0,0}$$

leads eventually to:

$$p(\underline{x}; t | \underline{y}_{t_0}, t+dt) - p(\underline{x}; t | \underline{y}_{t_0}, t) =$$

$$-\tfrac{1}{2} p(\underline{x}; t | \underline{y}_{t_0}, t)\left[[\tfrac{d}{dt}\underline{y} - \underline{g}(t:\underline{x})]' N^{-1}[\tfrac{d}{dt}\underline{y} - \underline{g}(t:\underline{x})] - \right.$$

$$\left. E[\tfrac{d}{dt}\underline{y} - \underline{g}(t:\underline{x})]' N^{-1}[\tfrac{d}{dt}\underline{y} - \underline{g}(t:\underline{x})] \right] dt$$

as can be easily verified. When this expression is used in Part Two of the derivation, rather than Eq. 3.7, an incorrect equation for $p(\underline{x}; t | \underline{y}_{t_0}, t)$ results. This incorrect equation has appeared frequently in the literature.[11-14,24-28,32]

3.2.2 Derivation: Step Two

For convenience, let the right side of Eq. 3.7 be abbreviated by $dq(t:\underline{x})$. Then:

$$p(\underline{x};\, t\,|\underline{y}_{t_0},t+dt) \;=\; p(\underline{x};\, t\,|\underline{y}_{t_0},t,d\underline{y}) \;=\; p(\underline{x};\, t\,|\underline{y}_{t_0},t) \;+\; dq(t:\underline{x}) \qquad (3.8)$$

The effect on the conditional probability density of an incremental change in $\underline{x}(t)$ will now be examined. The derivation closely parallels that usually given for the Fokker-Planck equation, Eq. 2.3.

Observe initially that:

$$p(\underline{x};\, t+dt\,|\underline{y}_{t_0},t+dt) \;=\; \int p(\underline{x};\, t+dt\,|\underline{u};\, t,\, \underline{y}_{t_0},t,\, d\underline{y})\; p(\underline{u};\, t\,|\underline{y}_{t_0},t+dt)\; d\underline{u}$$

$$=\; \int p(\underline{x};\, t+dt\,|\underline{u};\, t)\; p(\underline{u};\, t\,|\underline{y}_{t_0},t+dt)\; d\underline{u} \qquad (3.9)$$

where $p(\underline{x};\, t+dt\,|\underline{u};\, t)$ is the transition probability associated with the Markov process, $\underline{x}(t)$; $\underline{x}$ and $\underline{u}$ are the realizations of the process at times $t+dt$ and t, respectively. Use has been made of the fact that when $\underline{x}(t)=\underline{u}$ is known, no information about $\underline{x}(t+dt)=\underline{x}$ is provided by either $\underline{y}_{t_0},t$ or $d\underline{y}$. $\underline{y}_{t_0},t$, which depends only on the past of $\underline{x}(t)$ before time t, provides no information because of the Markovian nature of $\underline{x}(t)$. $d\underline{y} = \underline{g}[t:\underline{x}(t)] + \underline{\eta}(t)$ provides no additional information because of the assumed independence of $\underline{\chi}(t)$ and $\underline{\eta}(t)$.

Let $h(\underline{x})$ be an arbitrary function possessing a multidimensional Taylor expansion:

$$h(\underline{x}) = h(\underline{u}) \;+\; \sum_{i=1}^{m}(x_i - u_i)\,\frac{\partial h(\underline{x})}{\partial x_i}\bigg|_{\underline{u}} \;+\; \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(x_i - u_i)(x_j - u_j)\,\frac{\partial^2 h(\underline{x})}{\partial x_i \partial x_j}\bigg|_{\underline{u}} \;+\; \cdots$$

and satisfying Eq. 3.14. Since $x_i - u_i = dx_i = f_i[t:\underline{u}]dt + d\chi_i$, we have:

$$E[x_i - u_i] = f_i(t: \underline{u}) \, dt$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (3.11)

$$E[x_i - u_i][x_j - u_j] = X_{ij} \, dt$$

where E indicates expectation with respect to $p(\underline{x}; t+dt \,|\, \underline{u}; t)$. All

higher order moments of $(x_i - u_i)$ are of order greater than dt. They

can, therefore, be neglected.

We now multiply both sides of Eq. 3.9 by $h(\underline{x})$ and integrate. The

result is:

$$\int h(\underline{x}) \, p(\underline{x}; t+dt \,|\, \underline{y}_{t_0}, t+dt) \, d\underline{x} = \iint h(\underline{x}) p(\underline{x}; t+dt \,|\, \underline{u}; t) \, p(\underline{u}; t \,|\, \underline{y}_{t_0}, t+dt) d\underline{u} d\underline{x}$$

$$(3.12)$$

Substituting the expansion for $h(\underline{x})$ into the right side of Eq. 3.12,

integrating with respect to $\underline{x}$, using Eq. 3.11, and keeping terms to the

order of dt, results in:

$$\int h(\underline{x}) \, p(\underline{x}; t+dt \,|\, \underline{y}_{t_0}, t+dt) \, d\underline{x} = \int \left[h(\underline{u}) + \sum_{i=1}^{m} f_i(t: \underline{u}) \left. \frac{\partial h(\underline{x})}{\partial x_i} \right|_{\underline{u}} dt + \right.$$

$$(3.13)$$

$$\left. \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} X_{ij} \left. \frac{\partial^2 h(\underline{x})}{\partial x_i \partial x_j} \right|_{\underline{u}} dt \right] p(\underline{u}; t \,|\, \underline{y}_{t_0}, t+dt) \, d\underline{u}$$

We obtain the final result by integrating the last two terms on the right

by parts. Provided $h(\underline{x})$ satisfies the limiting conditions (suppressing

arguments):

$$\left. f_i \, p \, h \right|_{-\infty}^{+\infty} = \left. X_{ij} \, p \, \frac{\partial h}{\partial x_i} \right|_{-\infty}^{+\infty} = \left. X_{ij} \, h \, \frac{\partial p}{\partial x_j} \right|_{-\infty}^{+\infty} = 0 \qquad (3.14)$$

$$\text{for } i, j = 1, 2, \cdots, m$$

the result we obtain, after changing integration variables from $\underline{u}$ to $\underline{x}$, is:

$$\int h(\underline{x})\, p(\underline{x};\, t+dt\, |\underline{y}_{t_0},t+dt)\, d\underline{x} \;=\; \int h(\underline{x})\Big[p(\underline{x};\, t\, |\underline{y}_{t_0},t+dt) \;-$$

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i}[f_i(t:\underline{x})\, p(\underline{x};\, t\, |\underline{y}_{t_0},t+dt)]dt \;+ \qquad\qquad (3.15)$$

$$\tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} X_{ij}\, \frac{\partial^2}{\partial x_i \partial x_j}\, p(\underline{x};\, t\, |\underline{y}_{t_0},t+dt)\, dt\Big]\, d\underline{x}$$

Finally, we use the arbitrariness of $h(\underline{x})$ and the expression for $p(\underline{x};\, t\, |\underline{y}_{t_0},t+dt)$ of Eq. 3.8 to conclude:

$$p(\underline{x};\, t+dt\, |\underline{y}_{t_0},t+dt) - p(\underline{x};\, t\, |\underline{y}_{t_0},t) \;=\; dq(t:\underline{x}) \;-$$

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i}[f_i(t:\underline{x})\, p(\underline{x};\, t\, |\underline{y}_{t_0},t)]dt \;+ \qquad\qquad (3.16)$$

$$\tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} X_{ij}\, \frac{\partial^2}{\partial x_i \partial x_j}\, p(\underline{x};\, t\, |\underline{y}_{t_0},t)\, dt$$

where only terms to the order of dt have been retained. Eq. 3.1 follows easily by substituting the definition of $dq(t:\underline{x})$ into Eq. 3.16.

IV. Derivation of an Equation for the Approximate
Least-Squares Estimate of $\underline{x}(t)$

In this chapter, we shall derive the equations for the estimate of $\underline{x}(t)$. The interpretation of the equations is deferred to the following chapter.

4.1 Summary of Results

Two equations specify the approximate minimum-mean-square-error estimate, $\underline{x}^*(t)$. The first equation is:

$$\frac{d}{dt}\underline{x}^*(t) = \underline{f}[t:\underline{x}^*(t)] + V^*(t) \, D[\underline{g}(t:\underline{x}^*)]N^{-1}\{\underline{r}(t) - \underline{g}[t:\underline{x}^*(t)]\} \tag{4.1}$$

$D[\underline{g}(t:\underline{x}^*)]$ is an m x m Jacobian matrix whose (i-row, j-column)-element is $\dfrac{\partial}{\partial x_i^*}g_j[t:\underline{x}^*(t)]$. $V^*(t)$ is a non-negative definite, symmetric, m x m error-covariance matrix which is specified by the second equation:

$$\frac{d}{dt}V^*(t) = D'[\underline{f}(t:\underline{x}^*)]V^*(t) + V^*(t)D[\underline{f}(t:\underline{x}^*)] + X(t) + $$
$$V^*(t) \, D\left[D[\underline{g}(t:\underline{x}^*)]N^{-1}\{\underline{r}(t) - \underline{g}(t:\underline{x}^*)\}\right] V^*(t) \tag{4.2}$$

where $D[\cdot]$ is the Jacobian matrix associated with the vector enclosed within its square brackets.

We shall refer to Eq. 4.1 as the "processor equation" and to Eq. 4.2 as the "variance equation." Observe that in general the processor and variance equations are coupled and that both depend on the observations.

The initial conditions associated with the two equations are determined as follows. If $\underline{x}(t_0)$ is known, then $\underline{x}^*(t_0) = \underline{x}(t_0)$ and

$V^*(t_0) = 0$. On the other hand, if only the initial distribution, $p(\underline{x}; t)$, is known, then:

$$\underline{x}^*(t_0) = \int \underline{x}\, p(\underline{x}; t_0)\, d\underline{x}$$

and

$$V^*(t_0) = \int [\underline{x} - \underline{x}^*(t_0)][\underline{x} - \underline{x}^*(t_0)]' p(\underline{x}; t_0)\, d\underline{x}$$

Some special cases of Eqn's. 4.1 and 4.2 are of interest so we shall list them here.

4.1.1 Special Case: $\underline{f}$ and $\underline{g}$ linear transformations of $\underline{x}(t)$

When $\underline{x}(t)$ and $\underline{y}(t)$ are defined by:

$$d\underline{x}(t) = F(t)\underline{x}(t)\, dt + d\underline{\chi}(t)$$

and

$$d\underline{y}(t) = G(t)\underline{x}(t)\, dt + d\underline{\eta}(t)$$

both $\underline{x}(t)$ and $\underline{y}(t)$ are vector Gaussian processes. The exact and approximate estimates are equal under these circumstances and Eqn's. 4.1 and 4.2 become:

$$\frac{d}{dt}\hat{\underline{x}}(t) = F(t)\, \hat{\underline{x}}(t) + V(t)G'(t)N^{-1}\{\underline{r}(t) - G(t)\, \hat{\underline{x}}(t)\} \tag{4.3}$$

and

$$\frac{d}{dt}V(t) = F(t)V(t) + V(t)F'(t) + X(t) - V(t)G'(t)N^{-1}G(t)V(t) \tag{4.4}$$

Eqn's. 4.3 and 4.4 are entirely equivalent to the results on Kalman and Bucy.[2] As noted by them, the variance equation for this case is a matrix Riccati equation whose properties have been studied extensively. The following properties hold under suitable conditions:

(P1) An analytic solution exists. (see Ref. 2)

(P2) The solution is unique and determined by the specification

of an initial, non-negative definite matrix, $V(t_0)$.

(P3) A unique steady-state solution exists and V(t) converges to

this solution for any initial, non-negative matrix, $V(t_0)$.

The conditions under which P1, P2, and P3 hold are given by Kalman

and Bucy.[2] A sufficient condition for P3 is that the components of

$\underline{x}(t)$ and $\underline{y}(t)$ be stationary.

We observe that the variance equation in this instance does not

depend on either the observed processes or the estimate of $\underline{x}(t)$.

Consequently, V(t) can be determined before observations are taken.

Solving Eq. 4.4 for V(t) involves solving $\frac{1}{2}m(m+1)$ nonlinear differential

equations. A salient feature of the Kalman-Bucy approach is that

the equations can be solved numerically by computer thereby completely

determining the structure of the processor whose output is the estimate.

When steady-state conditions exist, $dV(t)/dt = 0$ and Eq. 4.4 becomes

$\frac{1}{2}m(m+1)$ quadratic algebraic equations whose solution is non-negative

definite. The algebraic equations can be solved numerically but for

even moderate values of m a large number of decisions are required

to determine the non-negative definite solution. An alternate technique

is to allow the numerical solution to the differential equations to approach

the unique steady-state value guaranteed by P3.

4.1.2 Special Case: f a linear transformation of $\underline{x}(t)$

When $\underline{x}(t)$ and $\underline{y}(t)$ are defined by:

$$d\underline{x}(t) = F(t) \underline{x}(t) dt + d\underline{x}(t)$$

and

$$d\underline{y}(t) = \underline{g}[t: \underline{x}(t)] dt + d\underline{\eta}(t)$$

$\underline{x}(t)$ is a vector Gaussian process. Most of the applications presented

in Ch. 5 fall within this case. Eqn's. 4.1 and 4.2 become:

$$\frac{d}{dt} \underline{x}^*(t) = F(t)\,\underline{x}^*(t) + V^*(t)D[\underline{g}(t:\underline{x}^*)]N^{-1}\{\underline{r}(t) - \underline{g}[t:\underline{x}^*(t)]\} \qquad (4.5)$$

and

$$\frac{d}{dt} V^*(t) = F(t)V^*(t) + V^*(t)F(t) + X(t) +$$

$$V^*(t)D\left[D[\underline{g}(t:\underline{x}^*)]N^{-1}\{\underline{r}(t) - \underline{g}(t:\underline{x}^*)\}\right]V^*(t) \qquad (4.6)$$

4.2 Derivations

4.2.1 Derivation of the Processor Equation

An equation for the exact minimum-mean-square-error estimate, $\hat{\underline{x}}(t)$, can be obtained in a straightforward way from the equation for $p(\underline{x};\,t\,|\underline{y}_{t_0},t)$ derived in Ch. III by using the fact that $\hat{\underline{x}}(t)$ is the conditional mean; that is:

$$\hat{\underline{x}}(t) = \int \underline{x}\,p(\underline{x};\,t\,|\underline{y}_{t_0},t)\,d\underline{x}$$

Multiplying both sides of Eq. 3.1 by $\underline{x}$ and integrating results in:

$$\hat{\underline{x}}(t+dt) - \hat{\underline{x}}(t) \equiv d\hat{\underline{x}}(t) = E\underline{f}(t:\underline{x})\,dt + E\left[\{\underline{x} - \hat{\underline{x}}(t)\}\,\underline{g}'(t:\underline{x})\right]N^{-1}[d\underline{y} - E\underline{g}(t:\underline{x})dt]$$

$$(4.7)$$

where integration by parts has been used and the following boundary conditions have been assumed to exist (supressing arguments):

$$p\Big|_{-\infty}^{+\infty} = \frac{\partial p}{\partial x_i}\Big|_{-\infty}^{+\infty} = x_i\,\frac{\partial p}{\partial x_i}\Big|_{-\infty}^{+\infty} = f_i p\Big|_{-\infty}^{+\infty} = x_i f_i p\Big|_{-\infty}^{+\infty} = 0 \qquad (4.8)$$

$$\text{for } i = 1, 2, \cdots, m$$

The expectations in Eq. 4.7 are with respect to $p(\underline{x}; t | \underline{y}_{t_0}, t)$.

We now assume that the following Taylor expansions for $\underline{f}[t; \underline{x}(t)]$ and $\underline{g}[t; \underline{x}(t)]$ exist:

$$\underline{f}(t; \underline{x}) = \underline{f}(t; \hat{\underline{x}}) + \sum_{i=1}^{m}(x_i - \hat{x}_i) \frac{\partial}{\partial x_i} \underline{f}(t; \underline{x}) \Big|_{\hat{\underline{x}}} +$$

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(x_i - \hat{x}_i)(x_j - \hat{x}_j) \frac{\partial^2}{\partial x_i \partial x_j} \underline{f}(t; \underline{x}) \Big|_{\hat{\underline{x}}} + \cdots$$

$$(4.9)$$

$$\underline{g}(t; \underline{x}) = \underline{g}(t; \hat{\underline{x}}) + \sum_{i=1}^{m}(x_i - \hat{x}_i) \frac{\partial}{\partial x_i} \underline{g}(t; \underline{x}) \Big|_{\hat{\underline{x}}} +$$

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(x_i - \hat{x}_i)(x_j - \hat{x}_j) \frac{\partial^2}{\partial x_i \partial x_j} \underline{g}(t; \underline{x}) \Big|_{\hat{\underline{x}}} + \cdots$$

The second term in each expansion may be written as $D'[\underline{f}(t; \hat{\underline{x}})](\underline{x} - \hat{\underline{x}})$ and $D'[\underline{g}(t; \hat{\underline{x}})](\underline{x} - \hat{\underline{x}})$, respectively.

The equation for the exact estimate can be obtained by substituting these expansions into Eq. 4.7. The resulting expression can neither be solved nor readily implemented because of the general existence of an infinite number of terms in the expansions of Eq. 4.9. It is natural, therefore, to consider the truncation of the expansions on the assumption that the components of the error vector, $\underline{x}(t) - \hat{\underline{x}}(t)$, are small. This assumption can be expected to be valid when the disturbance processes introduce only small perturbations in the observed processes.

Let $\underline{x}^*(t)$ be the approximate minimum-mean-square-error estimate of $\underline{x}(t)$ which is specified by the substitution of the expansions for $\underline{f}[t; \underline{x}(t)]$ and $\underline{g}[t; \underline{x}(t)]$ into Eq. 4.7 and the retention of the most

significant terms. Whenever $\underline{f}[t:\underline{x}(t)]$ and $\underline{g}[t:\underline{x}(t)]$ are linear functions of $\underline{x}(t)$, no approximation is involved and the exact and approximate estimates are identical. The equation we obtain for $\underline{x}*(t)$ is:

$$d\underline{x}*(t) = \underline{f}[t:\underline{x}*(t)]dt + V*(t)D[\underline{g}(t:\underline{x}*)]N^{-1}\{d\underline{y} - \underline{g}[t:\underline{x}*(t)]dt\} \qquad (4.10)$$

where $V*(t)$ is a symmetric, non-negative definite, m x m error covariance matrix defined by $V*(t) = E[\underline{x} - \underline{x}*(t)][\underline{x} - \underline{x}*(t)]'$. The processor equation is obtained by the formal division of Eq. 4.10 by dt.

We note that the terms of Eq. 4.9 having the most significant effect on the processor equation are the first two of each expansion. Consequently, the approximation is, in effect, a linearization about the estimate. This implies that to within the approximation, $p(\underline{x}; t|\underline{y}_{t_0}, t)$ is normal with mean $\underline{x}*(t)$.

4.2.2 Derivation of the Variance Equation

We now turn to the derivation of an equation for $V*(t)$. An equation for $v*_{k\ell}(t)$, the (k,ℓ)-element of $V*(t)$, is first obtained by the following procedure: (i) multiply the equation for $p(\underline{x}; t|\underline{y}_{t_0}, t)$ by $[x_k - \hat{x}_k(t)][x_\ell - \hat{x}_\ell(t)]$; (ii) integrate to obtain an equation for the (k,ℓ)-element of the exact error-covariance matrix; (iii) use the expansions for $\underline{f}[t:\underline{x}(t)]$ and $\underline{g}[t:\underline{x}(t)]$ and keep only the most significant terms.

Proceeding with steps i and ii, we use:

$$[x_k - \hat{x}_k(t)][x_\ell - \hat{x}_\ell(t)] = [x_k - \hat{x}_k(t+dt)][x_\ell - \hat{x}_\ell(t+dt)] + d\hat{x}_k(t)d\hat{x}_\ell(t) -$$
$$[x_k - \hat{x}_k(t+dt)]d\hat{x}_\ell(t) - [x_\ell - \hat{x}_\ell(t+dt)]d\hat{x}_k(t) \qquad (4.11)$$

to obtain:

$$dv_{k\ell}(t) + d\hat{x}_k(t)d\hat{x}_\ell(t) =$$

$$E\left\{ \underline{f}(t:\underline{x})[\underline{x} - \hat{\underline{x}}(t)]' + [\underline{x} - \hat{\underline{x}}(t)]\underline{f}'(t:\underline{x}) \right\}_{k\ell} dt + X_{k\ell}(t) dt +$$

$$\hspace{8cm} (4.12)$$

$$E\{x_k - \hat{x}_k(t)\}\{x_\ell - \hat{x}_\ell(t)\}[\underline{g}(t:\underline{x}) - E\underline{g}(t:\underline{x})]' N^{-1}[d\underline{y} - E\underline{g}(t:\underline{x}) dt]$$

where integration by parts has been used to obtain the first three terms on the right. We now substitute the expansions for $\underline{f}[t:\underline{x}(t)]$ and $\underline{g}[t:\underline{x}(t)]$ given in Eq. 4.9 into Eq. 4.12 and keep only the most significant terms. We also use the fact that to within the approximation, $p(x; t|\underline{y}_{t_0}, t)$ is normal with mean $\underline{x}^*(t)$; consequently, odd moments of the components of the error vector, $\underline{x} - \underline{x}^*(t)$, are zero and even moments factor into products of second moments. The equation we obtain for $v_{k\ell}^*(t)$ is:

$$dv_{k\ell}^*(t) + dx_k^*(t)dx_\ell^*(t) =$$

$$\left\{ D'[\underline{f}(t:\underline{x}^*)]V^*(t) + V^*(t)D[\underline{f}(t:\underline{x}^*)] + X(t) \right\}_{k\ell} dt +$$

$$\hspace{8cm} (4.13)$$

$$\left[\sum_{i=1}^{m} \sum_{j=1}^{n} v_{ki}^*(t)v_{\ell j}^*(t) \frac{\partial^2}{\partial x_i^* \partial x_j^*} \underline{g}'[t:\underline{x}^*(t)] \right] N^{-1}\{d\underline{y} - \underline{g}[t:\underline{x}^*(t)] dt\}$$

The second term on the left, $dx_k^* dx_\ell^* = [(d\underline{x}^*)(d\underline{x}^*)']_{k\ell}$, remains to be examined. Using Eq. 4.10 and keeping terms to the order of dt, we have:

$$(d\underline{x}^*)(d\underline{x}^*)' = V^* D[\underline{g}(t:\underline{x}^*)]N^{-1}d\underline{y}\,d\underline{y}'\,N^{-1}D'[\underline{g}(t:\underline{x}^*)]V^* \hspace{2cm} (4.14)$$

Since we are retaining only those terms of order dt and since dt is infinitesimal, $d\underline{y}\,d\underline{y}'$ may be replaced by Ndt, as indicated by Eq. 2.8. Hence, to terms of order dt:

$$(d\underline{x}^*)(d\underline{x}^*)' = V^*(t)D[\underline{g}(t:\underline{x}^*)]N^{-1}D'[\underline{g}(t:\underline{x}^*)]V^*(t)\,dt$$

Substituting this result into Eq. 4.13, we have:

$$dv^*_{k\ell}(t) = \Big\{D'[\underline{f}(t:\underline{x}^*)]V^*(t) + V^*(t)D[\underline{f}(t:\underline{x}^*)] + X(t) +$$

$$V^*(t)D[\underline{g}(t:\underline{x}^*)]N^{-1}D'[\underline{g}(t:\underline{x}^*)]V^*(t)\Big\}_{k\ell}\,dt +$$

(4.15a)

$$\left[\sum_{i=1}^{m}\sum_{j=1}^{m}v^*_{ki}(t)v^*_{\ell j}(t)\,\frac{\partial^2}{\partial x^*_i \partial x^*_j}\,\underline{g}'[t:\underline{x}^*(t)]\right]N^{-1}\{d\underline{y} - \underline{g}[t:\underline{x}^*(t)]dt\}$$

$$= \Big\{D'[\underline{f}(t:\underline{x}^*)]V^*(t) + V^*(t)D[f(t:\underline{x}^*)] +$$

(4.15b)

$$V^*(t)D\Big[D[\underline{g}(t:\underline{x}^*)]N^{-1}\{\underline{r}(t) - \underline{g}(t:\underline{x}^*)\}\Big]V^*(t)\Big\}_{k\ell}\,dt$$

That Eqn's. 4.15a and 4.15b are equal may be demonstrated by expanding the matrix expressions. The variance equation, Eq. 4.2, follows by the formal division of Eq. 4.15b by dt.

V. Applications to Communication Theory

Several applications of the state-variable approach to continuous estimation will be given in the following sections. These applications serve the purpose of providing an interpretation of the Estimation Model of Ch. 2 as well as the processor and variance equations of Ch. 4. Moreover, the broad scope of the problems which can be treated in a uniform manner is illustrated.

We shall begin by defining a general Communication Model which is directly related to the Estimation Model. Special cases will then be investigated in detail. These may be divided into three categories: (i) $x(t)$ and $r(t)$ are Gaussian processes; (ii) $x(t)$ is a Gaussian process and $r(t)$ is not; (iii) neither $x(t)$ nor $r(t)$ are Gaussian processes. In the communication theory context, i includes Gaussian message-- linear modulation schemes, ii includes Gaussian message-- Gaussian channel disturbance-- nonlinear modulation schemes, and iii includes Markovian message-- Markovian channel disturbance-- nonlinear modulation schemes. Categories i and ii can be treated by the alternate , MAP approach to continuous estimation. Category iii contains cases which cannot be treated by any alternate approach.

Van Trees[7,8] and Thomas and Wong[39] have used the MAP approach to study communication models which are equivalent to special cases of our model. We shall indicate the relationship between their demodulators and ours. Recall that the MAP approach leads to an integral equation for the estimate and that the equation corresponds to a physically unrealizable demodulator. An approximation to the unrealizable demodulator is made for the purpose of implementation. The approximation holds closely when the disturbance processes introduce only small

perturbations in the observed processes and consists of a cascade of a nonlinear, physically-realizable demodulator and a linear, physically-unrealizable filter. The nonlinear, physically-realizable portion of the cascade approximation is identical to our demodulator.

5.1 The Communication Model

The Communication Model is shown in Fig. 5.1.

Let $\underline{a}(t)$ be an n-dimensional state vector representing the output of an analog message source. $\underline{a}(t)$ is a continuous vector Markov process defined by the stochastic differential equation:

$$d\underline{a}(t) = \underline{f}_a[t:\underline{a}(t)] \, dt + d\underline{\alpha}(t) \tag{5.1}$$

where $\underline{\alpha}(t)$ is an n-dimensional vector whose components are Wiener processes. Let the covariance matrix associated with $\underline{\alpha}(t)$ be given by:

$$E[\underline{\alpha}(t) \, \underline{\alpha}'(u)] = A \min(t,u) \tag{5.2}$$

where A is a non-negative definite, n x n matrix which may be time-varying. More than one message can be represented by simply adjoining their individual state vectors in the formation of $\underline{a}(t)$. Of course, Gaussian messages with rational spectra are a special case of Eq. 5.1 with $\underline{f}_a[t:\underline{a}(t)] = F_a \, \underline{a}(t)$.

$\underline{a}(t)$ is transformed by a modulator into c signals, represented by the vector $\underline{s}[t:\underline{u}(t)]$, appropriate for transmission over the channel. The modulator consists of linear filtering followed by a memoryless, nonlinear modulator. The linear filtering may be time-varying and is described by the state equation:

$$d\underline{u}(t) = F_u(t) \, \underline{u}(t) \, dt + L_a(t) \, \underline{a}(t) \, dt \tag{5.3}$$

where $\underline{u}(t)$ is an ℓ-dimensional vector.
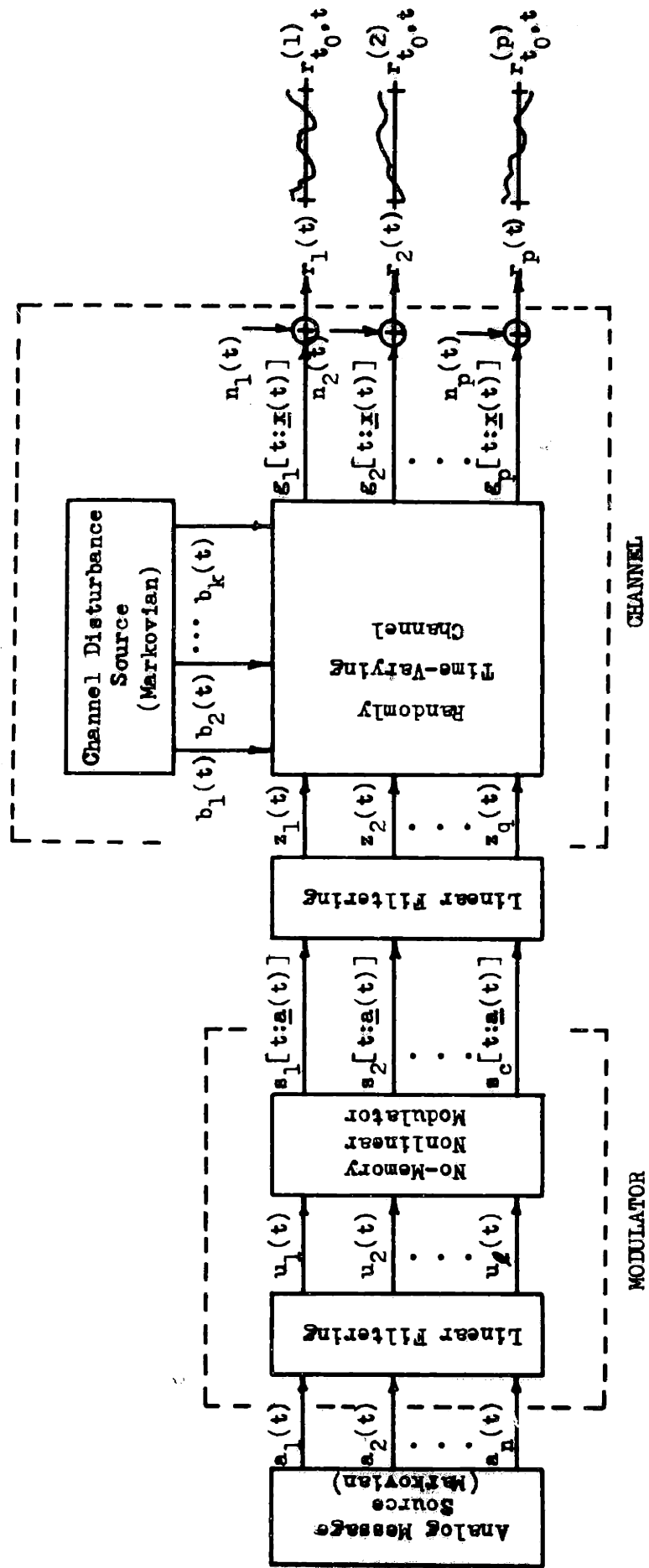
Figure 5.1   The Communication Model

A second linear-filtering operation follows the modulator. It is described by the state equation:

$$d\underline{z}(t) = F_z(t) \, \underline{z}(t) \, dt + L_s(t) \, \underline{s}[t:\underline{u}(t)] \, dt \tag{5.4}$$

where $\underline{z}(t)$ is a q-dimensional vector. We shall allow this filtering to be associated with either the modulator or the channel depending upon the particular application.

The modulator, including possible linear filtering at its output, contains as special cases: linear-modulation schemes, such as AM, AM-DSB/SC, AM-SSB, etc.; nonlinear-modulation schemes, such as PM, FM, preemphasized FM, etc.; deversity-modulation schemes, such as frequency-diversity PM and FM; and multilevel-modulation schemes, such as PM_n/PM and FM_n/FM.

The channel inputs are transformed into p signals which are represented by the vector, $\underline{g}[t:\underline{x}(t)]$. Each component of $\underline{g}[t:\underline{x}(t)]$ is observed in additive white Gaussian noise. The observed process can be described by the stochastic differential equation:

$$d\underline{y}(t) = \underline{g}[t:\underline{x}(t)] \, dt + d\underline{\eta}(t) \tag{5.5}$$

where $\underline{\eta}(t)$ is a p-dimensional vector whose components are Wiener processes. Let the covariance matrix associated with $\underline{\eta}(t)$ be given by:

$$E[\underline{\eta}(t) \, \underline{\eta}'(u)] = N \min(t,u) \tag{5.6}$$

where N is a symmetric, positive-definite, p x p matrix which may be time varying. We assume N^{-1} exists; this requires that none of the components of $\underline{\eta}(t)$ be zero. The actually observed process is $\underline{r}(t) = d\underline{y}(t)/dt$. Note that we have defined $\underline{y}(t)$ for the Communication Model in exactly the same way as $\underline{y}(t)$ for the Estimation Model of Ch. 2 (compare Eqn's. 2.4 and 5.5.)

Disturbance processes, such as additive and multiplicative processes, are introduced in the randomly time-varying portion of the channel. These processes can be Markovian in general and are described by the stochastic differential equation:

$$d\underline{b}(t) = \underline{f}_b[t:\underline{b}(t)]\,dt + d\underline{\beta}(t) \tag{5.7}$$

where $\underline{b}(t)$ and $\underline{\beta}(t)$ are k-dimensional vectors. The components of $\underline{\beta}(t)$ are Wiener processes and the associated covariance matrix is given by:

$$E[\underline{\beta}(t)\,\underline{\beta}'(u)] = B\,\min(t,u) \tag{5.8}$$

where B is a symmetric, non-negative definite, k x k matrix which may be time varying. Of course, as a special case, the disturbance processes can be Gaussian processes with rational spectra.

The channel, including possible linear filtering at its input, contains as special cases: simple additive channels; Gaussian multiplicative channels, such as Rayleigh channels, Rician channels, etc.; fixed channels with memory; multilink channels; and other commonly occurring channels. The Markovian disturbance processes which we include in the model cannot be treated with any alternate approach.

Collecting all the equations describing the Communication Model, we have:

$$d\underline{a}(t) = \underline{f}_a[t:\underline{a}(t)]\,dt + d\underline{\alpha}(t) \tag{5.1}$$

$$d\underline{u}(t) = F_u(t)\,\underline{u}(t)\,dt + E_a(t)\,\underline{a}(t)\,dt \tag{5.3}$$

$$d\underline{z}(t) = F_z(t)\,\underline{z}(t)\,dt + L_s(t)\,\underline{s}[t:\underline{u}(t)]\,dt \tag{5.4}$$

$$d\underline{b}(t) = \underline{f}_b[t:\underline{b}(t)]\,dt + d\underline{\beta}(t) \tag{5.7}$$

and

$$d\underline{y}(t) = \underline{g}[t:\underline{x}(t)]\,dt + d\underline{\eta}(t) \tag{5.5}$$

The relationship between the Communication Model and the Estimation Model is completed when we define the vector $\underline{x}(t)$ which is obtained by adjoining $\underline{a}(t)$, $\underline{u}(t)$, $\underline{z}(t)$, and $\underline{b}(t)$. Let:

$$\underline{x}(t) = \begin{bmatrix} \underline{a}(t) \\ \hline \underline{u}(t) \\ \hline \underline{z}(t) \\ \hline \underline{b}(t) \end{bmatrix} \qquad \underline{f}[t:\underline{x}(t)] = \begin{bmatrix} \underline{f}_a[t:\underline{a}(t)] \\ \hline F_u(t)\,\underline{u}(t) + L_a(t)\,\underline{a}(t) \\ \hline F_z(t)\underline{z}(t) + L_s(t)\,\underline{s}[t:\underline{u}(t)] \\ \hline \underline{f}_b[t:\underline{b}(t)] \end{bmatrix}$$

and   (5.9)

$$\underline{X}(t) = \begin{bmatrix} \underline{\alpha}(t) \\ \hline 0 \\ \hline 0 \\ \hline \underline{\beta}(t) \end{bmatrix}$$

Then:

$$d\underline{x}(t) = \underline{f}[t:\underline{x}(t)]\,dt + d\underline{X}(t) \qquad\qquad (5.10)$$

Equation 5.10 describes $\underline{x}(t)$ for the Communication Model and is identical to Eq. 2.1 describing $\underline{x}(t)$ for the Estimation Model. The order in which $\underline{a}(t)$, $\underline{u}(t)$, $\underline{z}(t)$, and $\underline{b}(t)$ are placed in forming $\underline{x}(t)$ is arbitrary.

With the definition of the Communication Model now completed, we turn our attention to the consideration of applications. The procedure we shall follow for each of the applications presented in the following sections is: (i) specify the particular communication model for the application; (ii) identify $\underline{x}(t)$, $\underline{f}[t:\underline{x}(t)]$, $\underline{\xi}(t) = d\underline{X}(t)/dt$, $X(t)$, $\underline{r}(t) = d\underline{y}(t)/dt$, $\underline{g}[t:\underline{x}(t)]$, $\underline{n}(t) = d\underline{\eta}(t)/dt$, and $N(t)$; (iii) use the processor and variance equations of Ch. 4 to determine the structure of the demodulator. The state representation indicated in App. A1 will be used for all applications involving Gaussian processes.

5.2 Applications when $\underline{x}(t)$ and $\underline{r}(t)$ are Gaussian

For Examples 5.2.1 through 5.2.3, $\underline{x}(t)$ and $\underline{r}(t)$ are described by:

$$\frac{d}{dt}\underline{x}(t) \; = \; F(t)\,\underline{x}(t) \; + \; \underline{\xi}(t) \tag{5.11}$$

and

$$\frac{d}{dt}\underline{y}(t) \; = \; \underline{r}(t) \; = \; G(t)\,\underline{x}(t) \; + \; \underline{n}(t) \tag{5.12}$$

Eqn's. 4.3 and 4.4 are the processor and variance equations.

Example 5.2.1 Single Message, No Modulation, Additive White Noise Channel

Consider the communication model shown in Fig. 5.2a. a(t) is a nonstationary Gaussian message and n(t) is a white Gaussian process whose correlation function is $N_1(t)\,\delta(t-u)$. a(t) and n(t) are uncorrelated. The equations describing the model are:

$$\frac{d}{dt}\underline{x}(t) \; = \; F(t)\,\underline{x}(t) \; + \; \underline{\xi}(t) \tag{5.13}$$

and

$$\frac{d}{dt}y(t) \; = \; r(t) \; = \; x_1(t) \; + \; n(t) \tag{5.14}$$

where $\underline{x}(t)$ is an m-dimensional vector with $x_1(t) = a(t)$. $F(t)$ and $\underline{\xi}(t)$ are given by:

$$F(t) \; = \; \begin{bmatrix} -\psi_1(t) & 1 & 0 & \cdots & \\ -\psi_2(t) & 0 & 1 & & \\ -\psi_3(t) & 0 & 0 & & \\ \cdot & & & \ddots & \\ \cdot & & & & 1 \\ \cdot & & & & \\ -\psi_m(t) & 0 & 0 & & 0 \end{bmatrix} \; ; \; \underline{\xi}(t) \; = \; \begin{bmatrix} \lambda_1(t)\,\xi(t) \\ \lambda_2(t)\,\xi(t) \\ \lambda_3(t)\,\xi(t) \\ \cdot \\ \cdot \\ \cdot \\ \lambda_m(t)\,\xi(t) \end{bmatrix}$$
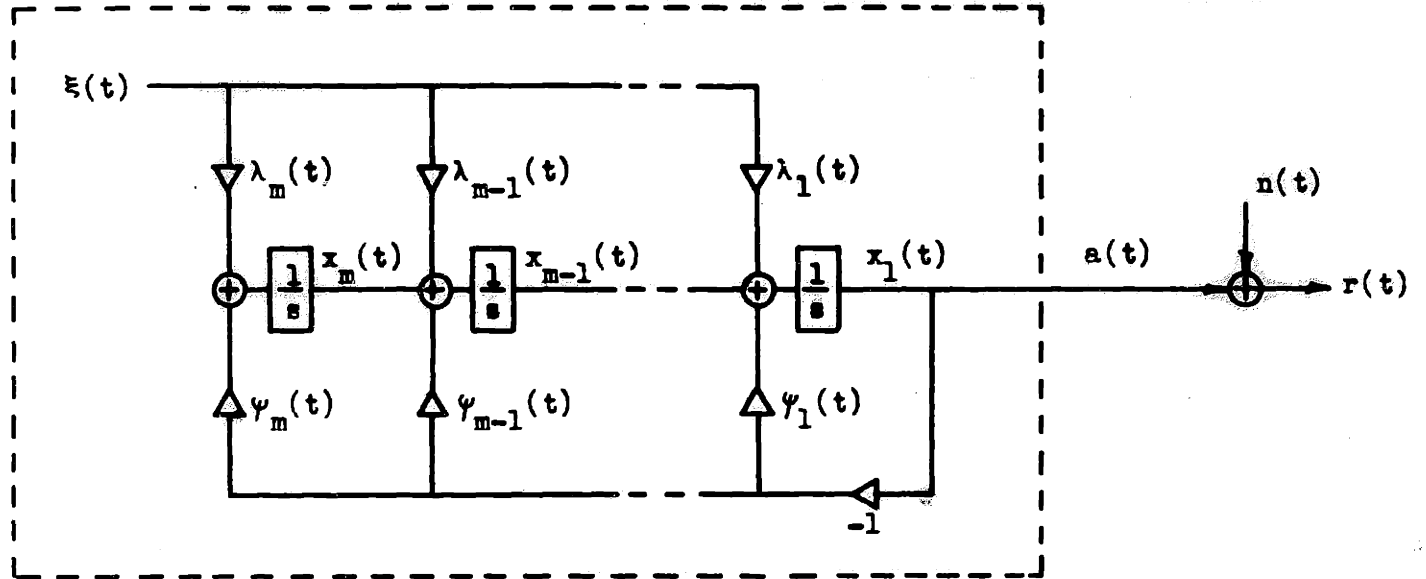
Analog Message Source

(a)
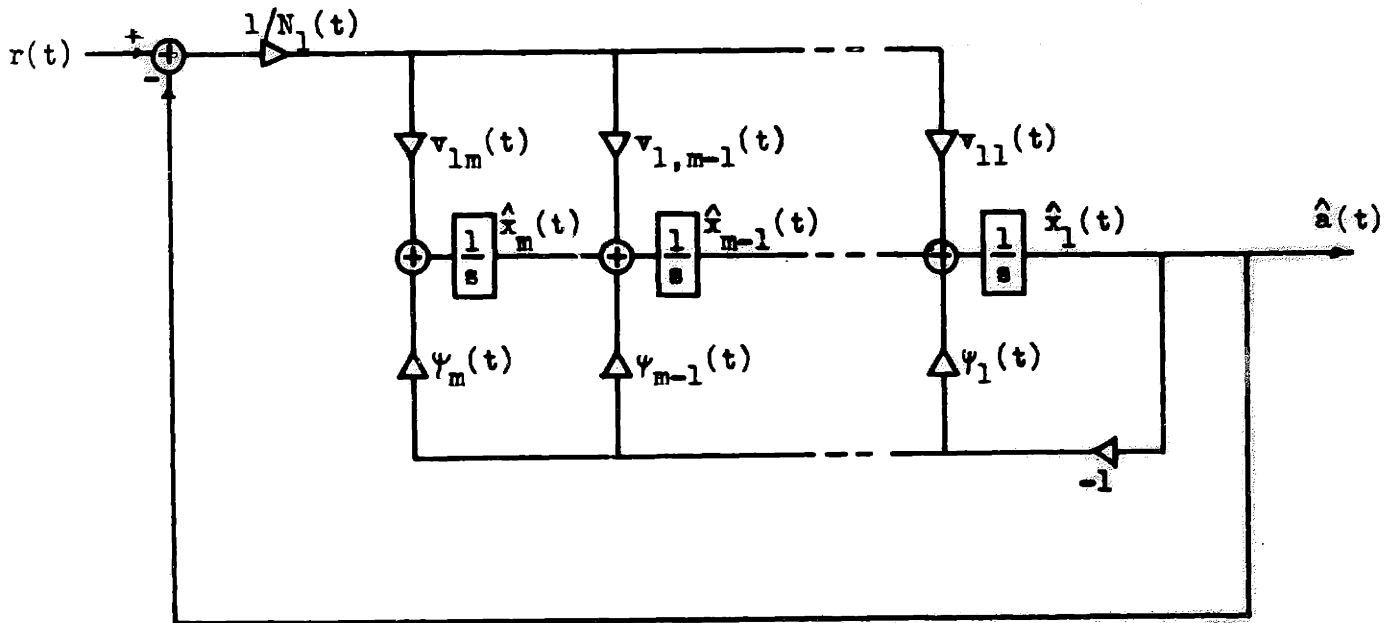
(b)

Figure 5.2 (a) Nonstationary Gaussian Message in an Additive White Noise Channel (b) Optimum Filter for Estimating the Message

We assume that $E[\underline{\xi}(t)\,\underline{\xi}'(u)] = X(t)\,\delta(t-u)$ is known. From Eq. 5.14, we observe that $G(t) = [1\ 0\ 0\ \cdots\ 0]$.

The processor and variance equations, Eqn's. 4.3 and 4.4, become:

$$\frac{d}{dt}\,\hat{\underline{x}}(t) = F(t)\,\hat{\underline{x}}(t) + \frac{1}{N_1(t)}\begin{bmatrix} v_{11}(t) \\ v_{12}(t) \\ \cdot \\ \cdot \\ \cdot \\ v_{1m}(t) \end{bmatrix}\{r(t) - \hat{x}_1(t)\} \tag{5.15}$$

and

$$\frac{d}{dt}\,V(t) = F(t)\,V(t) + V(t)\,F'(t) + X(t) -$$

$$\frac{1}{N_1(t)}\begin{bmatrix} v_{11}^2(t) & v_{11}(t)v_{12}(t) & \cdots & v_{11}(t)v_{1m}(t) \\ v_{12}(t)v_{11}(t) & v_{12}^2 & \cdots & v_{12}(t)v_{1m}(t) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ v_{1m}(t)v_{11}(t) & \cdots & & v_{1m}^2(t) \end{bmatrix} \tag{5.16}$$

By comparing Eqn's. 5.13 and 5.15, we obtain the optimum processor shown in Fig. 5.2b. We observe that the processor depends only on the first column of $V(t)$.

Components of $V(t)$ can be determined numerically or can be generated as the output of a system specified by Eq. 5.16. The components of $V(t)$ are of interest for two reasons: first, they complete the structure of the processor; and second, they describe the estimation performance of the processor. Except for some simple cases considered below, we shall not give solutions to the variance equation. Rather, we shall be interested only in obtaining the general structure of the optimum processor.

A special case arises when $a(t)$ is a stationary process and $t_0 = -\infty$ so

that steady-state conditions exist. In this instance, $a(t)$ has a rational spectrum and the communication model has the alternate form shown in Fig. 5.3a. Correspondingly, the optimum filter has the alternate form of Fig. 5.3b. Let $G_{opt}(s)$ denote the filter appearing in the forward path. Several interesting properties of $G_{opt}(s)$ are evident by inspection. These are:

(1) The poles of $G_{opt}(s)$ coincide with those of the message shaping filter.

(2) The order of the numerator of $G_{opt}(s)$ is exactly one less than the denominator since v_{11}, the mean-square estimation error, is nonzero.

(3) The zeros of $G_{opt}(s)$ depend only on the first row, or column, of the error-covariance matrix, V.

(4) The coefficients of the numerator polynomial are error covariances associated with state variables.

(5) $v_{11} = N_1 \lim_{s \to \infty} s G_{opt}(s) = N_1 g_{opt}(0^+)$, where $g_{opt}(t)$ is the impulse response corresponding to $G_{opt}(s)$.

We have obtained properties 1, 2, and 5 directly from the solution to the Wiener-Hopf equation.[41] (5) gives an expression for the mean-square error in terms of the optimum loop filter. An alternate expression, not requiring a determination of $G_{opt}(s)$, is:

$$v_{11} = N_1 \int_{-\infty}^{\infty} \log\left[1 + \frac{S_a(\omega^2)}{N_1}\right] \frac{d\omega}{2\pi} \tag{5.17}$$

where $S_a(\omega^2)$ watts/cps is the power density spectrum of $a(t)$. We shall see later that this expression is useful in the evaluation of quasi-optimum phase demodulators. The expression was originally derived by Yovits and Jackson.[42] We have given a simplified derivation which begins with the solution to the Wiener-Hopf equation.[40] Alternate

$$\frac{\lambda_1 s^{m-1} + \cdots + \lambda_m}{s^m + \psi_1 s^{m-1} + \cdots + \psi_m}$$

Analog Message Source

(a)

$$\frac{v_{11} s^{m-1} + v_{12} s^{m-2} + \cdots + v_{1m}}{s^m + \psi_1 s^{m-1} + \cdots + \psi_m}$$
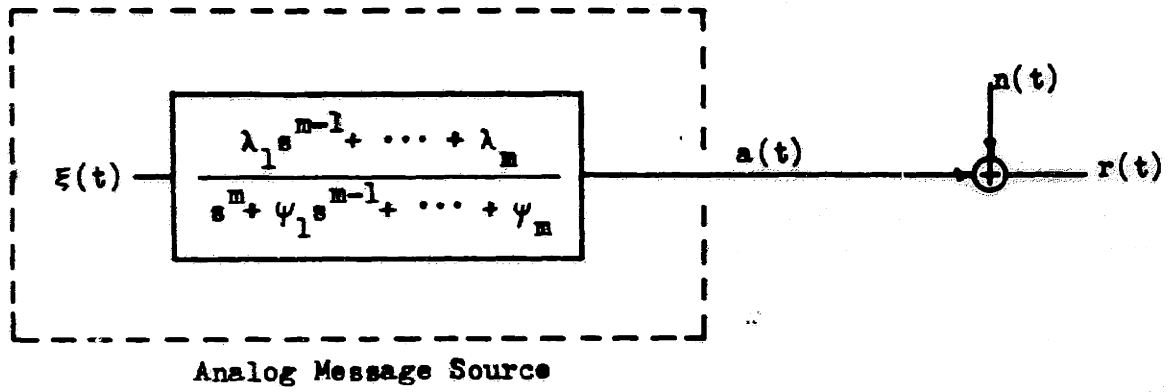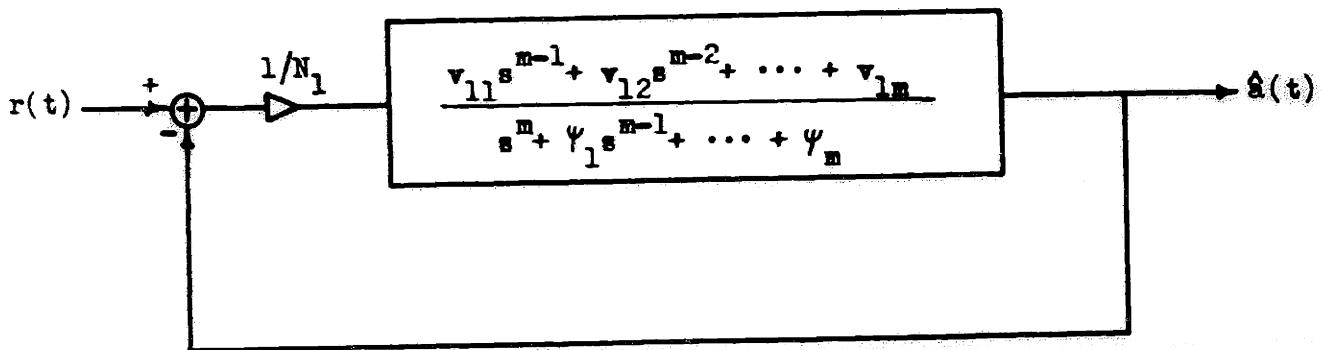
(b)

Figure 5.3 (a) Stationary Gaussian Message in Additive White Noise Channel (b) Optimum Filter for Estimating the Message

derivations are given by Viterbi and Cahn[43] and Viterbi.[44]

5.2.1.1 Special Case of a One-Dimensional Message: No Modulation

As a simple example, consider the one-dimensional communication model of Fig. 5.4a. For this case, the spectrum of a(t) is $\dfrac{2Pk}{\omega^2 + k^2}$ and

$F = -k$, $X = 2Pk$. The processor and variance equations are:

$$\frac{d}{dt}\hat{x}_1(t) = -k\,\hat{x}_1(t) + \frac{1}{N_1}v_{11}\{r(t) - \hat{x}_1(t)\} \qquad (5.18)$$

and

$$0 = -2k\,v_{11} + 2Pk - \frac{1}{N_1}v_{11}^2 \qquad (5.19)$$

The solution to the variance equation is:

$$v_{11} = \frac{2P}{1 + \sqrt{1 + \Lambda}} \qquad (5.20)$$

where $\Lambda = 2P/kN_1$ is the signal-to-noise ratio in the message bandwidth.* The optimum filter is shown in Fig. 5.4b. A closed-loop version of the filter is shown in Fig. 5.4c; this latter realization would arise most naturally with the Wiener approach.

Example 5.2.2 Single Message, Integrator, White Noise Channel

Consider the communication model shown in Fig. 5.5a. a(t) is a stationary Gaussian message which is integrated in a modulator before transmission through the channel. n(t) is a white Gaussian process of

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* The message bandwidth is defined to be the width of a rectangular spectrum of height $S_a(0)$ and same total area as $S_a(\omega^2)$.

(a)
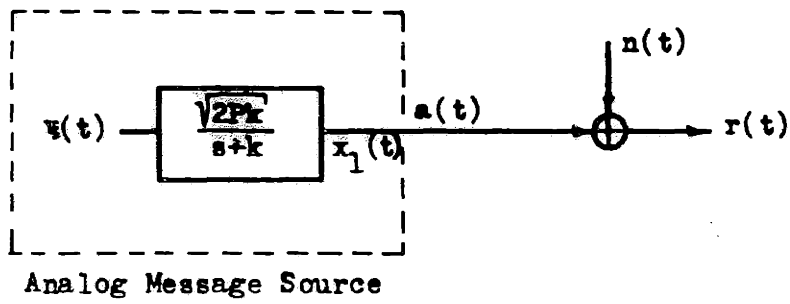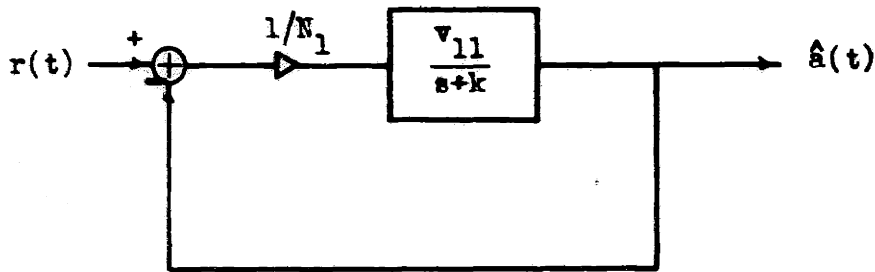
(b)
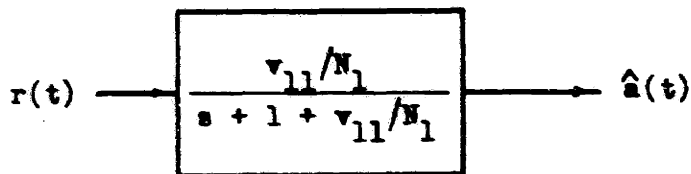
(c)

Figure 5.4 (a) One-Dimensional Gaussian Message Observed in White Gaussian Noise (b,c) Two Realizations for the Optimum Filter

of spectral height N_0 watts/cps. a(t) and n(t) are uncorrelated. The integration occurring in the modulator is a particular example of linear filtering which might arise. It will occur again when we consider frequency modulation.

The state vector associated with the analog message source satisfies:

$$da(t) = F_a \underline{a}(t) dt + d\underline{\alpha}(t) \tag{5.21}$$

where

$$\underline{a}(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ \cdot \\ \cdot \\ \cdot \\ a_n(t) \end{bmatrix} \qquad F_a = \begin{bmatrix} -\psi_1 & 1 & 0 & & & \\ -\psi_2 & 0 & 1 & & & \\ -\psi_3 & 0 & 0 & 1 & & \\ \cdot & & & & & \\ \cdot & & & & & 1 \\ \cdot & & & & & \\ -\psi_n & 0 & 0 & & & 0 \end{bmatrix} \tag{5.22}$$

and

$$\frac{d}{dt} \underline{\alpha}(t) = \begin{bmatrix} \lambda_1 \xi(t) \\ \lambda_2 \xi(t) \\ \lambda_3 \xi(t) \\ \cdot \\ \cdot \\ \cdot \\ \lambda_n \xi(t) \end{bmatrix}$$

We assume that $E[\underline{\alpha}(t) \underline{\alpha}'(u)] = A \min(t,u)$ is known. Note that $a(t) = a_1(t)$.

The equation describing the modulator is:

$$du(t) = a(t) dt = a_1(t) dt \tag{5.23}$$

$$\xi(t) \rightarrow \boxed{\frac{\lambda_1 s^{n-1} + \cdots + \lambda_n}{s^n + \psi_1 s^{n-1} + \cdots + \psi_n}} \xrightarrow[x_1(t)]{a(t)} \boxed{\frac{1}{s}} \xrightarrow[x_0(t)]{u(t)} \oplus \xrightarrow{} r(t)$$

with $n(t)$ entering at the summing junction.

Analog Message Source

(a)

$$r(t) \xrightarrow{+} \oplus \xrightarrow{} \triangleright \frac{1}{N_0} \rightarrow \boxed{\frac{v_{01} s^{n-1} + v_{02} s^{n-2} + \cdots + v_{0n}}{s^n + \psi_1 s^{n-1} + \cdots + \psi_n}} \xrightarrow{\hat{x}_1(t)} \hat{a}(t)$$

$\hat{u}(t)$

$\triangledown v_{00}$

$\hat{x}_0(t) \quad \boxed{\frac{1}{s}} \quad \oplus$
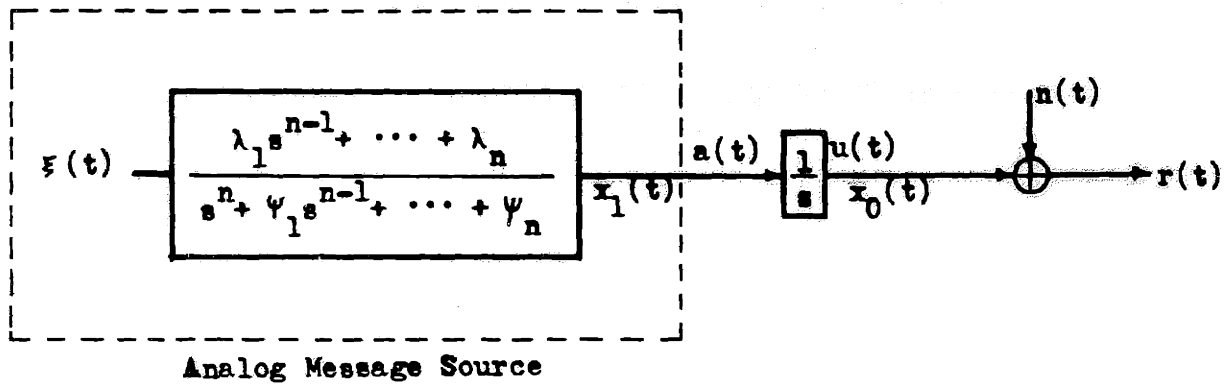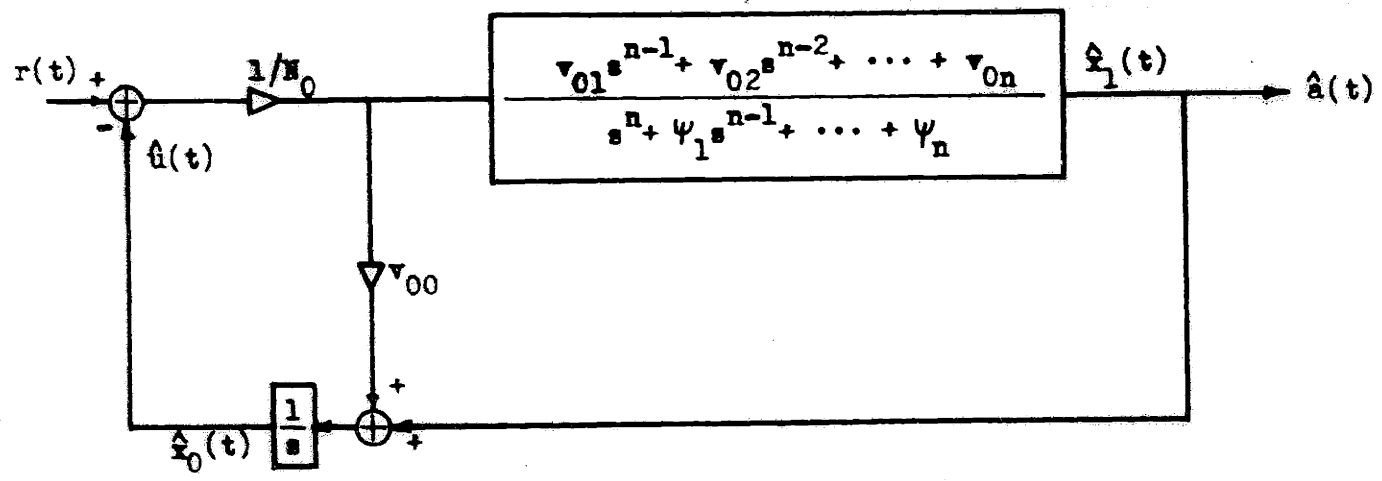
(b)

Figure 5.5 (a) Stationary Gaussian Message with an Integrator and an Additive White Noise Channel (b) The Optimum Filter

Define $\underline{x}(t)$ by:

$$\underline{x}(t) \;=\; \begin{bmatrix} u(t) \\ ---- \\ \underline{a}(t) \end{bmatrix} \;=\; \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{bmatrix} \tag{5.24}$$

Then $\underline{x}(t)$ satisfies:

$$\frac{d}{dt}\underline{x}(t) \;=\; F\,\underline{x}(t) \;+\; \underline{\xi}(t) \tag{5.25}$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ 0 & -\psi_1 & 1 & 0 & \\ 0 & -\psi_2 & 0 & 1 & \\ & . & & & \\ & . & & & 1 \\ & . & & & \\ 0 & -\psi_n & 0 & 0 & 0 \end{bmatrix} ; \quad \underline{\xi}(t) = \begin{bmatrix} 0 \\ \lambda_1\xi(t) \\ \lambda_2\xi(t) \\ . \\ . \\ . \\ \lambda_n\xi(t) \end{bmatrix} ; \quad X = \begin{bmatrix} 0 & | & 0 \\ --- & | & --- \\ 0 & | & A \end{bmatrix}$$

Note that $u(t) = x_0(t)$ and $a(t) = x_1(t)$.

The received signal is described by:

$$r(t) \;=\; u(t) \;+\; n(t) \;=\; x_0(t) \;+\; n(t) \tag{5.26}$$

Thus, $G(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$.

Under steady-state conditions, the processor and variance equations, Eqn's. 4.3 and 4.4, become:

$$\frac{d}{dt}\hat{\underline{x}}(t) \;=\; F\hat{\underline{x}}(t) \;+\; \frac{1}{N_0} \begin{bmatrix} v_{00} \\ v_{01} \\ v_{02} \\ . \\ . \\ . \\ v_{0n} \end{bmatrix} \{r(t) - \hat{x}_0(t)\} \tag{5.27}$$

and

$$0 = FV + VF' + X - \frac{1}{N_0} \begin{bmatrix} v_{00}^2 & v_{00}v_{01} & & & v_{00}v_{0n} \\ v_{01}v_{00} & v_{01}^2 & & & v_{01}v_{0n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ v_{0n}v_{00} & v_{0n}v_{01} & & & v_{0n}^2 \end{bmatrix}$$ (5.28)

The processor equation leads to the realization shown in Fig. 5.5b. Once again, we observe that the optimum filter depends only on the first row, or column, of V.

Expressions can be given for v_{00} and v_{11}, the error variances associated with estimating the message, a(t), and the integrated message, u(t). The expressions are convenient because they do not require a determination of the optimum filter or a solution of the variance equation for their evaluation. We shall see later that they are useful in the evaluation of the performance of optimum FM demodulators. The expressions, which are derived in Appendix A2, are:

$$v_{00} = N_0 f(0) \tag{5.29}$$

and

$$v_{11} = \frac{N_0}{3} f^3(0) + F(0) \tag{5.30}$$

where $f(0)$ and $F(0)$ are related to $S_a(\omega^2)$, the spectrum of a(t), by:

$$f(0) = \int_{-\infty}^{\infty} \log \left[1 + \frac{S_a(\omega^2)}{\omega^2 N_0} \right] \frac{d\omega}{2\Pi} \tag{5.31}$$

and

$$F(0) = \int_{-\infty}^{\infty} \omega^2 N_0 \log\left[1 + \frac{S_a(\omega^2)}{\omega^2 N_0}\right] \frac{d\omega}{2\Pi} \qquad (5.32)$$

5.2.2.1 Special Case of a One-Dimensional Message: Integral Modulation

As a simple example, consider the one-dimensional message used in the communication model of Fig. 5.6a. For this case, the spectrum of $a(t)$ is $\dfrac{2Pk}{\omega^2 + k^2}$ watts/cps. We have:

$$\underline{x}(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} \; ; \quad F = \begin{bmatrix} 0 & 1 \\ 0 & -k \end{bmatrix} \; ; \quad X = \begin{bmatrix} 0 & 0 \\ 0 & 2Pk \end{bmatrix}$$

The optimum filter is shown in Fig. 5.6b. After some straightforward manipulation, the variance equation leads to three equations for the components of V:

$$\frac{d}{dt} v_{00} = 2v_{01} - \frac{1}{N_0} v_{00}^2 = 0$$

$$\frac{d}{dt} v_{01} = v_{11} - kv_{01} - \frac{1}{N_0} v_{00} v_{01} = 0 \qquad (5.33)$$

$$\frac{d}{dt} v_{11} = -2kv_{11} + 2kP - \frac{1}{N_0} v_{01}^2 = 0$$

An equation for v_{00} is obtained by eliminating v_{01} and v_{11} from the second equation:

$$\frac{1}{4k^2} \Lambda = \left[\frac{v_{00}}{2kN_0}\right]^4 + 2\left[\frac{v_{00}}{2kN_0}\right]^3 + \left[\frac{v_{00}}{2kN_0}\right]^2$$

(a)
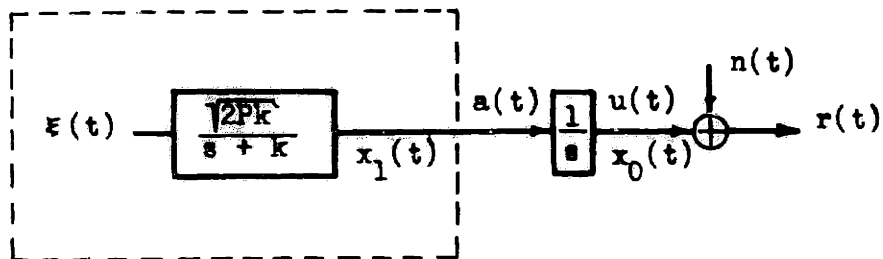
$$E\xi(t)\xi(u) = \delta(t-u)$$
$$En(t)n(u) = N_0\delta(t-u)$$
$$En(t)\xi(u) = 0$$
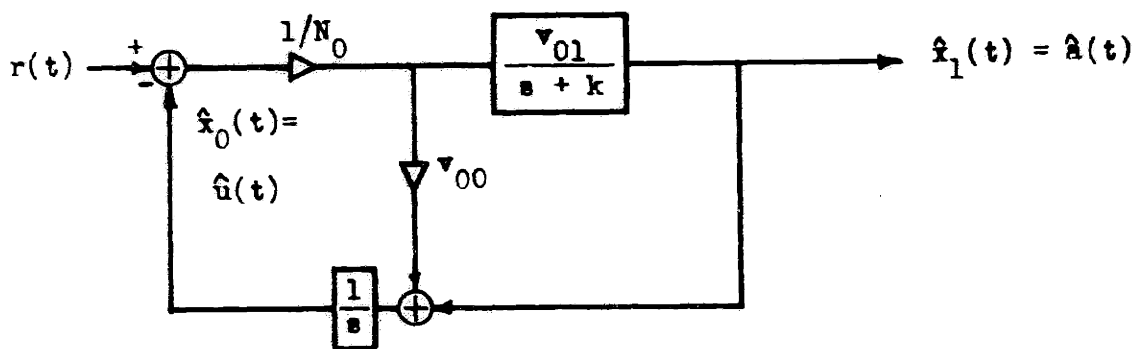
(b)

Figure 5.6 (a) One-Dimensional Message with Integral Modulation and a White Noise Channel (b) The Optimum Filter

This equation becomes:

$$\frac{\Lambda}{4k^2} = \left\{ \left[\frac{v_{00}}{2kN_0} \right]^2 + \left[\frac{v_{00}}{2kN_0} \right] \right\}^2 \tag{5.34}$$

where $\Lambda = 2P/kN_0$ is the signal-to-noise ratio in the message bandwidth. Solving for v_{00}, we obtain:

$$v_{00} = \frac{2N_0 \Lambda^{\frac{1}{2}}}{1 + \sqrt{1 + \frac{2}{k} \Lambda^{\frac{1}{2}}}} \tag{5.35}$$

v_{01} and v_{11} are then easily found to be:

$$v_{01} = \frac{2N_0 \Lambda}{\left\{ 1 + \sqrt{1 + \frac{2}{k} \Lambda^{\frac{1}{2}}} \right\}^2} \tag{5.36}$$

$$v_{11} = P - \frac{2N_0 \Lambda^2}{\left\{ 1 + \sqrt{1 + \frac{2}{k} \Lambda^{\frac{1}{2}}} \right\}^4 k} \tag{5.37}$$

We have assumed that $a(t)$ is stationary and that $t_0 = -\infty$ so that steady-state conditions exist. If t_0 is finite, then the only modification to the optimum filter is that v_{00} and v_{01} are time-varying gains rather than constants. We shall see in Ch. 6 that the approach to steady state is rapid compared to the message correlation time. For this reason, the steady-state assumption is usually valid in practice.

Example 5.2.3 Single Message, AM-DSB/SC, Additive White Noise Channel

Consider the communication model of Fig. 5.7. $a(t)$ is a stationary Gaussian message and $n(t)$ is a white Gaussian noise of spectral height N_1 watts/cps. $a(t)$ and $n(t)$ are uncorrelated. $a(t)$ amplitude modulates a sinusoidal carrier whose frequency is large compared to significant frequencies of $a(t)$. This is a typical example in which the signal component of $r(t)$ has a bandpass spectrum which is essentially disjoint from that of $a(t)$. The results obtained are similar to those obtained for other linear modulation procedures.

The state vector associated with the message source satisfies:

$$\frac{d}{dt} \underline{x}(t) = F \underline{x}(t) + \underline{\xi}(t) , \quad \text{where } a(t) = x_1(t) \tag{5.38}$$

F and $\underline{\xi}(t)$ are given by:

$$F = \begin{bmatrix} -\psi_1 & 1 & 0 & 0 & \cdots & 0 \\ -\psi_2 & 0 & 1 & 0 & & \\ -\psi_3 & 0 & 0 & 1 & & \\ \cdot & & & & & \\ \cdot & & & & & 1 \\ \cdot & & & & & \\ -\psi_m & 0 & 0 & \cdots & & 0 \end{bmatrix} \qquad \underline{\xi}(t) = \begin{bmatrix} \lambda_1 \xi(t) \\ \lambda_2 \xi(t) \\ \lambda_3 \xi(t) \\ \cdot \\ \cdot \\ \cdot \\ \lambda_m \xi(t) \end{bmatrix}$$

We assume that $E[\underline{\xi}(t) \underline{\xi}'(u)] = X \delta(t-u)$ is known.

The received signal is described by:

$$r(t) = a(t)\sqrt{2} \sin \omega_0 t + n(t)$$
$$= x_1(t)\sqrt{2} \sin \omega_0 t + n(t) \tag{5.39}$$

so that $G(t) = [1 \ 0 \ 0 \ \cdots \ 0]\sqrt{2} \sin \omega_0 t$.
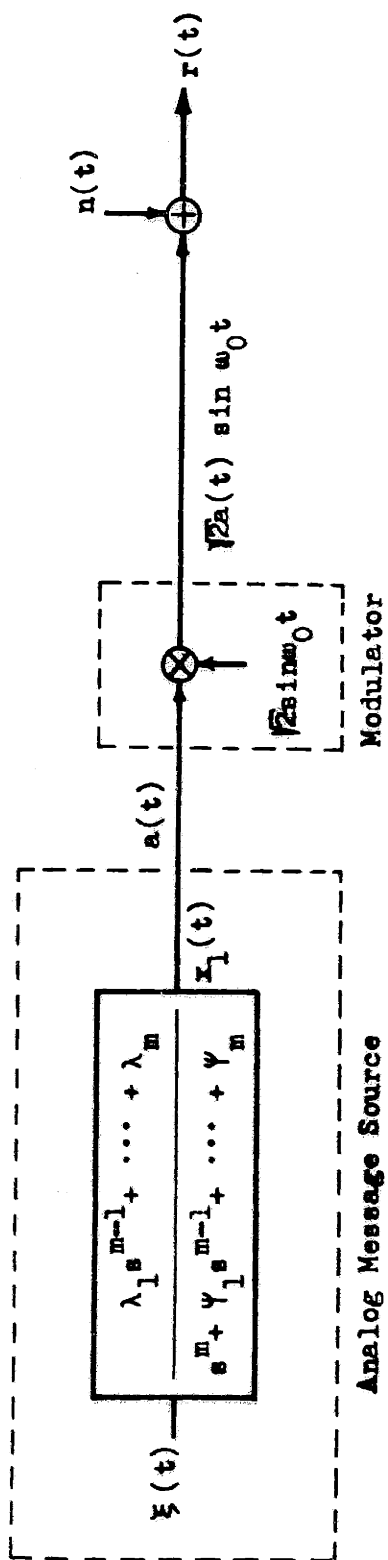
Figure 5.7 Single Message Transmitted by AM-DSB/SC
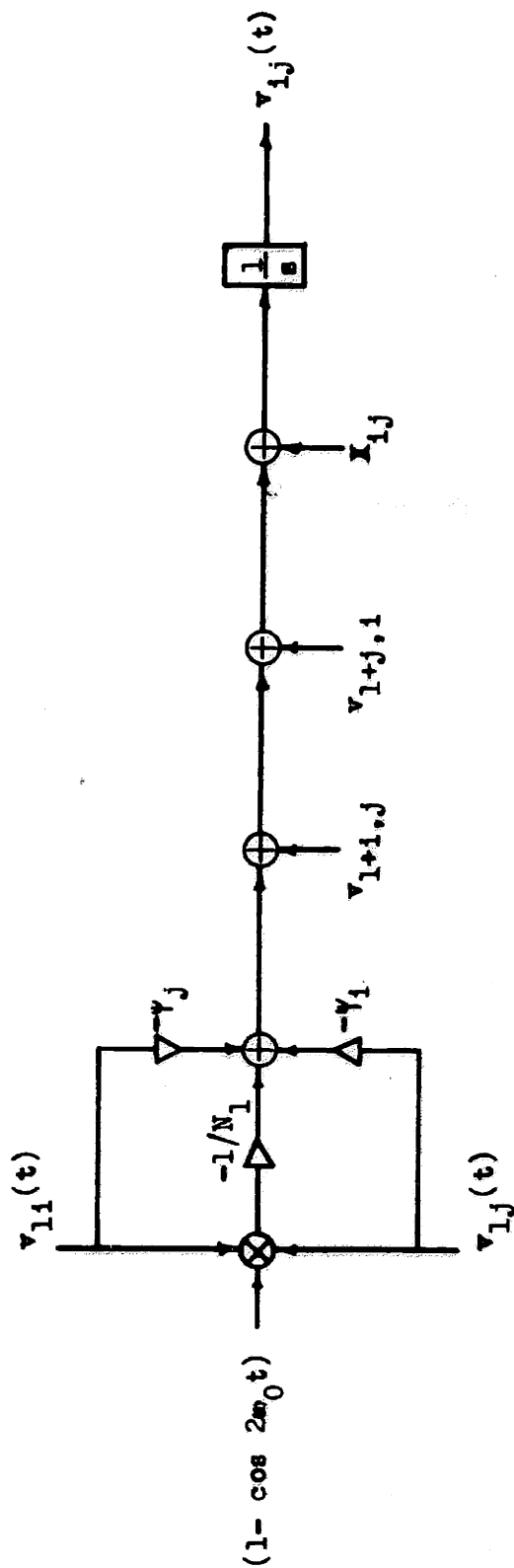
Figure 5.8 A Realization for the (1,j)-Element of V(t)

First, we shall examine the variance equation, which is:

$$\frac{d}{dt} V(t) = FV + VF' + X - \frac{2}{N_1} \begin{bmatrix} v_{11}^2(t) & v_{11}(t)v_{12}(t) & \cdots & v_{11}(t)v_{1m}(t) \\ v_{12}(t)v_{11}(t) & v_{12}^2(t) & & v_{12}(t)v_{1m}(t) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ v_{1m}(t)v_{11}(t) & \cdots & & v_{1m}^2(t) \end{bmatrix} \sin^2 \omega_0 t \qquad (5.40)$$

From this equation, it is found that the (i,j)-element of $V(t)$ satisfies:

$$\frac{d}{dt} v_{ij}(t) = - \psi_i v_{1j}(t) - \psi_j v_{1i} + v_{i+1,j}(t) + v_{j+1,i}(t) + X_{ij} -$$

$$(5.41)$$

$$\frac{1}{N_1} v_{1i}(t)v_{1j}(t)\{1 - \cos 2\omega_0 t\}$$

$v_{ij}(t)$ can be realized as the output of the system diagrammed in Fig. 5.8. Inspection of the figure indicates that the components of $V(t)$ are slowly varying and that the double-frequency signals associated with $\cos 2\omega_0 t$ have no effect because they will not propagate through the low-pass filter. Consequently, the variance equation can be rewritten as Eq. 5.16, the variance equation associated with the no-modulation case of Example 5.2.1. It follows that $V(t)$ and, therefore, the estimation performance, are the same for both examples.

We now examine the processor equation under the assumption that $t_0 = -\infty$ so that steady-state conditions exist. The processor equation is:

$$\frac{d}{dt} \hat{\underline{x}}(t) = F \hat{\underline{x}}(t) + \frac{1}{N_1} \begin{bmatrix} v_{11} \\ v_{12} \\ \cdot \\ \cdot \\ v_{1m} \end{bmatrix} \sqrt{2} \sin \omega_0 t \{r(t) - \hat{x}_1(t)\sqrt{2}\sin \omega_0 t\} \qquad (5.42)$$

The optimum filter is shown in Fig. 5.9a. The signal at the output of the upper multiplier is $r(t)\sqrt{2}\sin\omega_0 t - \hat{x}_1(t) + \hat{x}_1(t)\cos 2\omega_0 t$. We observe that $\hat{x}_1(t)$ is slowly varying so that the double-frequency term will not propagate through the low-pass filter. This implies that the processor equation can be rewritten as:

$$\frac{d}{dt}\hat{\underline{x}}(t) = F\hat{\underline{x}}(t) + \frac{1}{N_1}\begin{bmatrix} v_{11} \\ v_{12} \\ . \\ . \\ . \\ v_{1m} \end{bmatrix}\{r(t)\sqrt{2}\sin\omega_0 t - \hat{x}_1(t)\} \qquad (5.43)$$

The modified demodulator is shown in Fig. 5.9b.

5.3 Applications when $\underline{x}(t)$ is Gaussian and $\underline{r}(t)$ is not

For examples 5.3.1 through 5.3.5, $\underline{x}(t)$ and $\underline{r}(t)$ are described by:

$$\frac{d}{dt}\underline{x}(t) = F(t)\underline{x}(t) + \underline{\xi}(t) \qquad (5.44)$$

and

$$\frac{d}{dt}\underline{y}(t) = \underline{r}(t) = \underline{g}[t:\underline{x}(t)] + \underline{n}(t) \qquad (5.45)$$

In general, we shall assume that F and X are constant matrices so that $\underline{x}(t)$ is stationary. In this way, the rational polynomial realization of Appendix A1 can be used. The nonstationary case can be treated by the simple modification of employing the alternate realization of Appendix A1. Eqn's. 4.5 and 4.6 are the processor and variance equations.
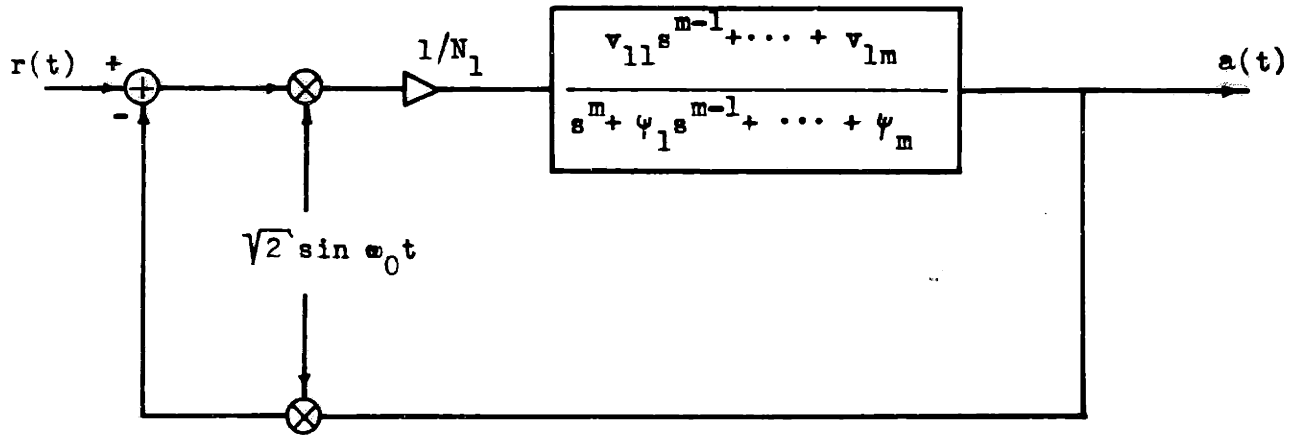
(a)
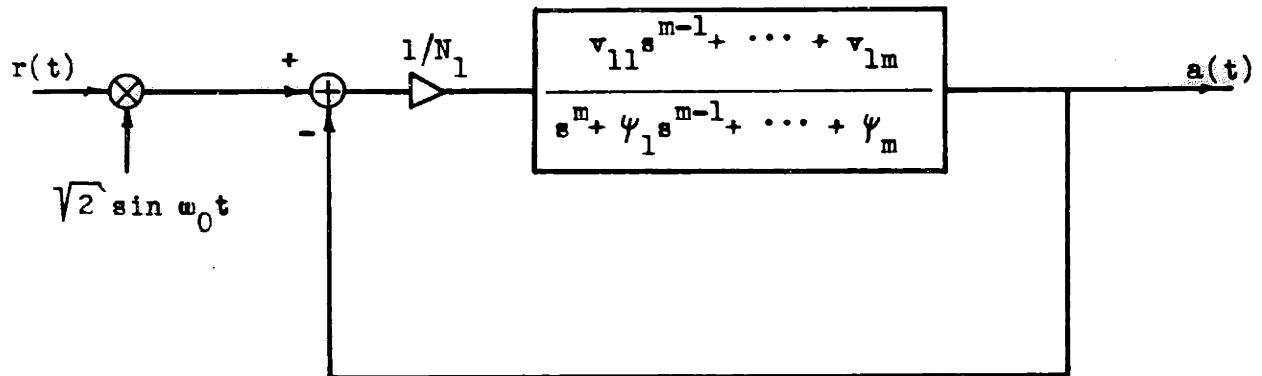
(b)

Figure 5.9 (a,b)   Two Realization for the Optimum
AM-DSB/SC Demodulator

Example 5.3.1 Single Message, PM, Additive White Noise Channel

Consider the communication model shown in Fig. 5.10. a(t) is a stationary Gaussian message and n(t) is a white Gaussian process of spectral height N_1 watts/cps. a(t) phase modulates a sinusoidal carrier whose nominal frequency is large compared to significant frequencies of a(t). We shall assume that the variance of a(t) is unity so that β can be interpreted as the modulation index.

The results obtained in this example are typical of those obtained for other memoryless, nonlinear modulation schemes in which signals in r(t) associated with the message vary rapidly. A close resemblance exists between the results obtained here and those obtained in the linear modulation cases of Ex's. 5.2.1 and 5.2.3.

The equations describing the communication model are:

$$\frac{d}{dt}\underline{x}(t) = F\,\underline{x}(t) + \underline{\xi}(t) \ , \qquad a(t) = x_1(t) \tag{5.46}$$

and

$$\frac{d}{dt}y(t) = r(t) = C\sin[\omega_0 t + \beta a(t)] + n(t) \tag{5.47}$$

$\underline{x}(t)$ is an m-dimensional vector and F and $\underline{\xi}(t)$ are the same as defined for Eq. 5.38. We assume that $E[\underline{\xi}(t)\,\underline{\xi}'(u)] = X\delta(t-u)$ is known. We observe that $\underline{g}[t:\underline{x}(t)] = C\sin[\omega_0 t + \beta x_1(t)]$, a scalar. Hence:

$$D[\underline{g}(t:\underline{x})] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \beta C \cos[\omega_0 t + \beta x_1(t)] \tag{5.48}$$

After some manipulation, the processor and variance equations,
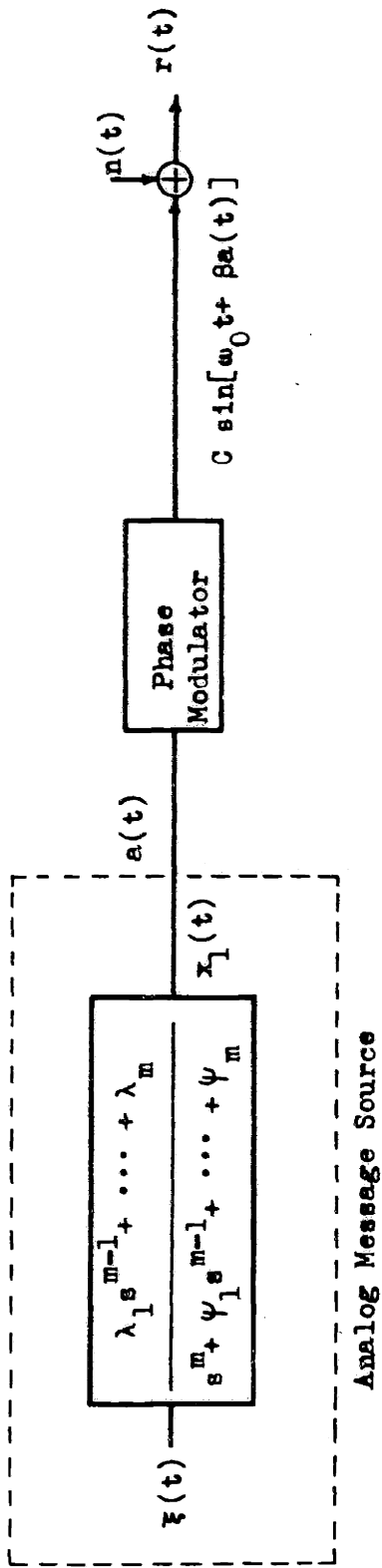
Figure 5.10 A Single Message Transmitted by Phase Modulation
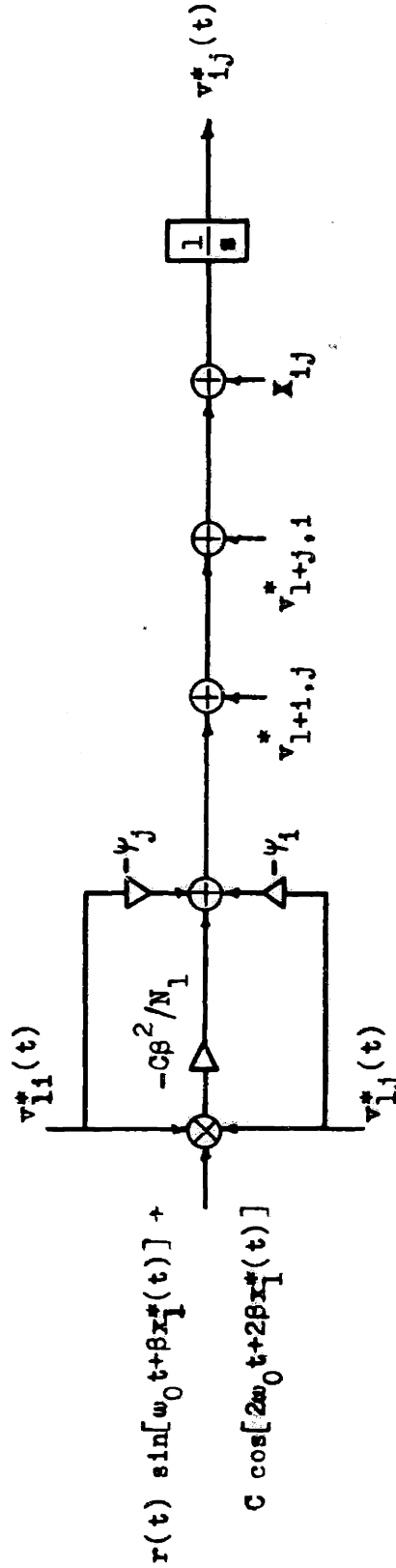
Figure 5.11 A Realization for the $(1,j)$-Element of $V*(t)$

Eqn's. 4.5 and 4.6, become:

$$\frac{d}{dt}\underline{x}^*(t) = F\underline{x}^*(t) + \frac{1}{N_1}\begin{bmatrix} v_{11}^*(t) \\ v_{12}^*(t) \\ \cdot \\ \cdot \\ \cdot \\ v_{1m}^*(t) \end{bmatrix}\beta C\cos[\omega_0 t + \beta x_1^*(t)]\{r(t) - C\sin[\omega_0 t + \beta x_1^*(t)]\}$$

(5.49)

and

$$\frac{d}{dt}V^*(t) = FV^*(t) + V^*(t)F' + X -$$

$$\frac{1}{N_1}\begin{bmatrix} v_{11}^*(t)v_{11}^*(t) & v_{11}^*(t)v_{12}^*(t) & \cdots & v_{11}^*(t)v_{1m}^*(t) \\ v_{12}^*(t)v_{11}^*(t) & v_{12}^*(t)v_{12}^*(t) & & v_{12}^*(t)v_{1m}^*(t) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ v_{1m}^*(t)v_{11}^*(t) & & & v_{1m}^*(t)v_{1m}^*(t) \end{bmatrix}\beta^2 C\{r(t)\sin[\omega_0 t + \beta x_1^*(t)] + C\cos[2\omega_0 t + 2\beta x_1^*(t)]\}$$

(5.50)

We shall examine the variance equation first. From Eq. 5.50, the (i,j)-element of $V^*(t)$ satisfies:

$$\frac{d}{dt}v_{ij}^*(t) = -\psi_i v_{ij}^*(t) - \psi_j v_{1i}^*(t) + v_{i+1,j}^*(t) + v_{j+1,i}^*(t) + X_{ij} -$$

$$\frac{1}{N_1}\beta^2 C\, v_{1i}^*(t)v_{1j}^*(t)\{r(t)\sin[\omega_0 t + \beta x_1^*(t)] + C\cos[2\omega_0 t + 2\beta x_1^*(t)]\}$$

(5.51)

$v_{ij}^*(t)$ can be realized as the output of the system diagrammed in Fig. 5.11. Let us now conjecture that the components of $V^*(t)$ are slowly varying. We shall find that to a close approximation this is,

in fact, true. Then the double-frequency terms associated with $\cos[2\omega_0 t + 2\beta x_1^*(t)]$ will not propagate through the low-pass filtering. Consequently, $\cos[2\omega_0 t + 2\beta x_1^*(t)]$ has negligible effect and can be dropped. The input to the multiplier is then $r(t)\sin[\omega_0 t + \beta x_1^*(t)]$. It is through this term that the variance equation is coupled to both $r(t)$ and $\underline{x}^*(t)$. This coupling is a great disadvantage practically because $V^*(t)$ and, therefore, the structure of the demodulator, cannot be determined prior to making observations. For this reason, it is worthwhile to examine $r(t)\sin[\omega_0 t + \beta x_1^*(t)]$ critically so as to obtain any possible simplification. We shall find that a significant simplification is possible.

Observe that the coupling term may be rewritten as:

$$r(t)\sin[\omega_0 t + \beta x_1^*(t)] = n(t)\sin[\omega_0 t + \beta x_1^*(t)] +$$

$$C\sin[\omega_0 t + \beta x_1(t)]\sin[\omega_0 t + \beta x_1^*(t)]$$

$$= n(t)\sin[\omega_0 t + \beta x_1^*(t)] +$$

$$\tfrac{1}{2}C\cos\beta[x_1(t) - x_1^*(t)] - \qquad (5.52)$$

$$\tfrac{1}{2}C\cos[2\omega_0 t + \beta x_1(t) + \beta x_1^*(t)]$$

Again, the double-frequency term can be disregarded. The second term on the right can be expanded as:

$$\tfrac{1}{2}C\cos\beta[x_1(t) - x_1^*(t)] = \tfrac{1}{2}C - \tfrac{1}{4}C\beta^2[x_1(t) - x_1^*(t)]^2 + \cdots \qquad (5.53)$$

Within the approximation for which the demodulator is optimum, all terms of the expansion except the first can be neglected -- the others lead to terms of the order of the sixth moment of the error at the

output of the multiplier. Thus, to a good approximation for small error, we have:

$$r(t) \sin[\omega_0 t + \beta x_1^*(t)] \approx \tfrac{1}{2} C \{1 + \frac{2}{C} n(t) \sin[\omega_0 t + \beta x_1^*(t)]\} \qquad (5.54)$$

$n(t)$ is a white process, by which we mean that it has a flat spectrum at least over the frequency range where it has effect. In reality, $n(t)$ has a finite variance given by $N_1 W_c$ where W_c is the channel or receiver input bandwidth. By increasing the channel signal-to-noise ratio, $C^2 / 2N_1 W_c$, it is possible to make the probability of excursions of $2 n(t)/C$ outside a range around its mean, zero, as small as desired. since the magnitude of $\sin(\cdot)$ is bounded by unity, this implies:

$$\tfrac{1}{2} C \{1 + \frac{2}{C} n(t) \sin[\omega_0 t + \beta x_1^*(t)]\} \approx \tfrac{1}{2} C$$

almost always when the sinal-to-noise ratio is sufficiently large. We conclude that for large channel signal-to-noise ratio:

$$r(t) \sin[\omega_0 t + \beta x_1^*(t)] \approx \tfrac{1}{2} C \qquad (5.55)$$

The approximations have effected an uncoupling of the variance equation from $r(t)$ and $\underline{x}^*(t)$ thereby making a practical simplification of importance. The variance equation becomes:

$$\frac{d}{dt} V^*(t) = FV^* + V^*F' + X - \frac{\beta^2 C^2}{2N_1} \begin{bmatrix} v_{11}^*(t)v_{11}^*(t) & v_{11}^*(t)v_{12}^*(t) & \cdots & v_{11}^*(t)v_{1m}^*(t) \\ v_{12}^*(t)v_{11}^*(t) & v_{12}^*(t)v_{12}^*(t) & & v_{12}^*(t)v_{1m}^*(t) \\ \vdots & & & \\ v_{1m}^*(t)v_{11}^*(t) & \cdots & & v_{1m}^*(t)v_{1m}^*(t) \end{bmatrix}$$

$$(5.56)$$

This equation is nearly identical to the variance equation associated

with the no-modulation case of Ex. 5.2.1 (see Eq. 5.16.) Only the noise level must be modified. V*(t) can be determined prior to making any observations, just as in the no-modulation case. It follows from Eq. 5.17 that the mean-square estimation error, v_{11}^* , is given by:

$$v_{11}^* = \frac{2N_1}{\beta^2 C^2} \int_{-\infty}^{\infty} \log\left[1 + \frac{\beta^2 C^2}{2N_1} S_a(\omega^2)\right] \frac{d\omega}{2\Pi} \tag{5.57}$$

We observe that a useful measure for evaluating the performance of the processor is $\beta^2 v_{11}^*$, the mean-square error in estimating the total phase, $\beta a(t) = \beta x_1(t)$.

In the steady-state, the processor equation, Eq. 5.49, leads to the quasi-optimum PM demodulator of Fig. 5.12a. It is seen that the subtractive sinusoidal signal results only in double-frequency terms at the output of the multiplier. Since these will not propagate through the filter, the subtractive branch can be discarded. The simplified demodulator, a phase-locked loop, is shown in Fig. 5.12b. Several interesting properties of the loop filter can be deduced by inspection. These are similar to the properties observed in the linear modulation case of Example 5.2.1. The optimum PM demodulator bears a distinct resemblance to the linear, no-modulation processor of Fig. 5.3b.

5.3.1.1 Special Case of a One-Dimensional Message: PM

As a simple example, consider the one-dimensional message in the communication model of Fig. 5.13a. For this case, the spectrum of a(t) is $\dfrac{2k}{\omega^2 + k^2}$ watts/cps so that a(t) has a variance of unity. Eqn's. 5.46 and 5.47 become:

$$\frac{d}{dt} x_1(t) = -k x_1(t) + \sqrt{2k}\, \xi(t) \tag{5.58}$$
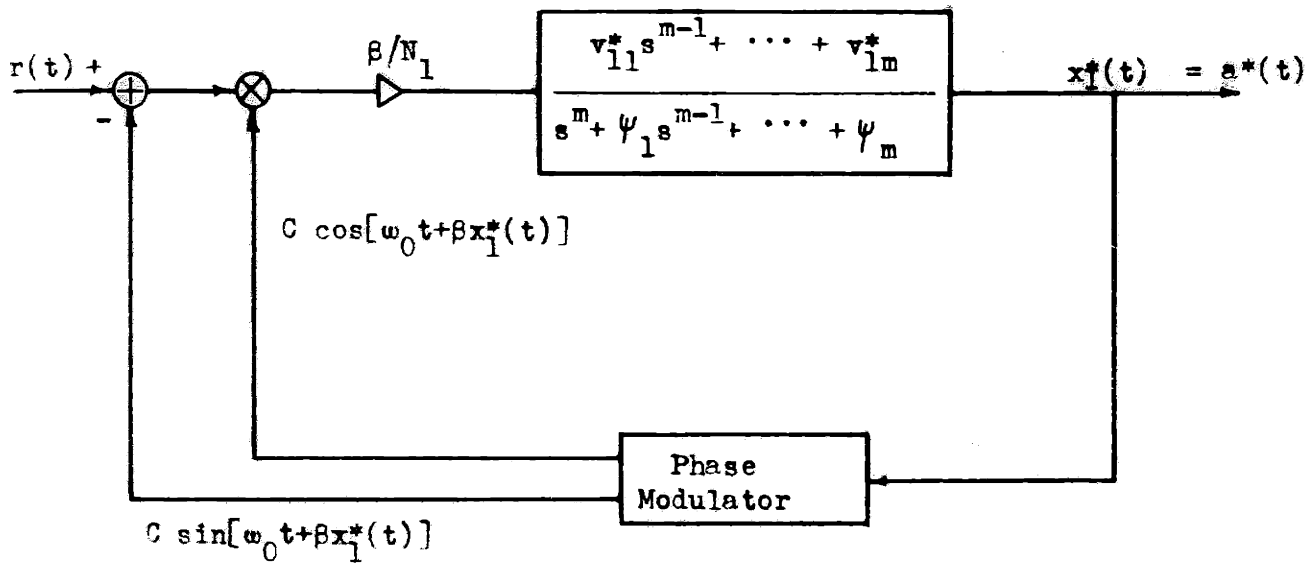
(a)
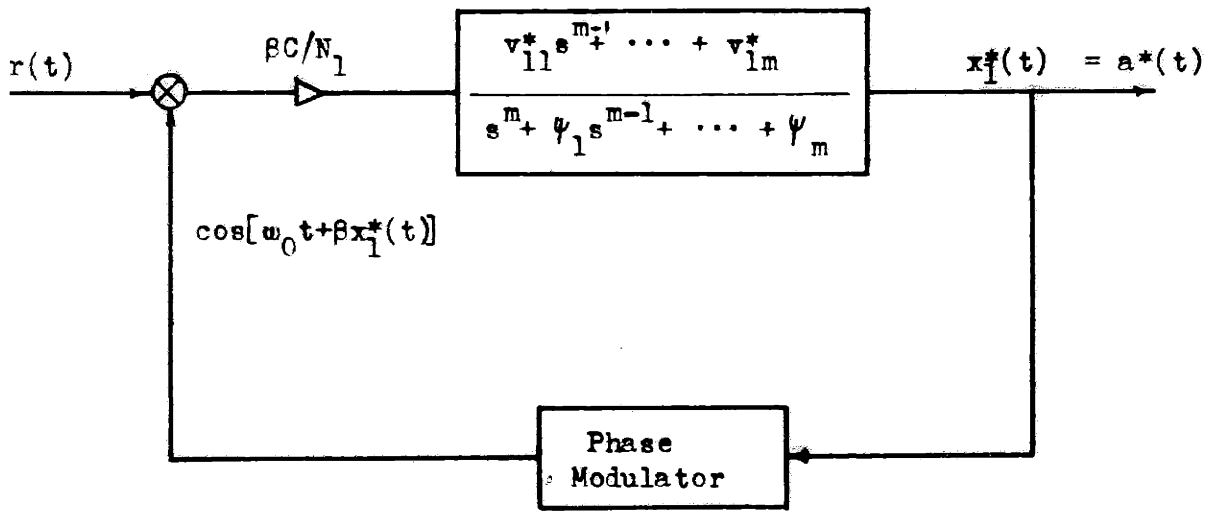
(b)

Figure 5.12 (a,b)   Two Realizations for a Quasi-Optimum
PM Demodulator
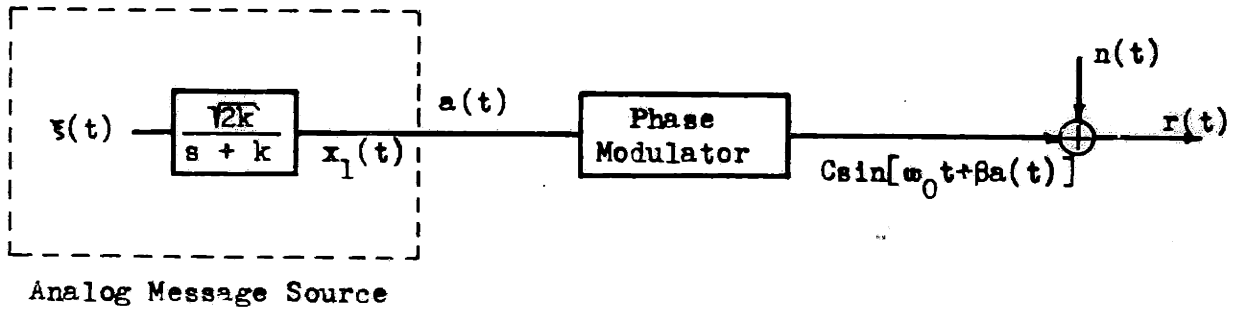
(a)
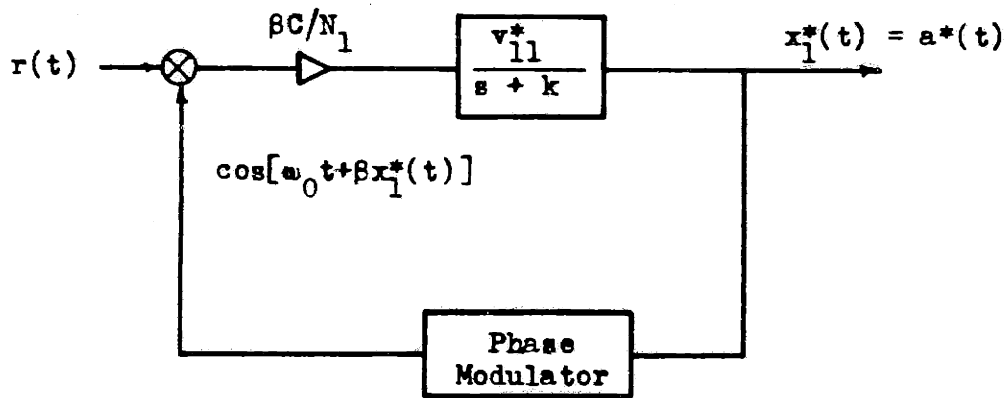
(b)

Figure 5.13 (a) PM with a One-Dimensional Message
(b) Quasi-Optimum Demodulator for a One-Dimensional
Message

and

$$\frac{d}{dt} y(t) = r(t) = C \sin[\omega_0 t + \beta x_1(t)] + n(t) \qquad (5.59)$$

The processor and variance equations, Eqn's. 5.49 and 5.56, become:

$$\frac{d}{dt} x_1^*(t) = -k x_1^*(t) + \frac{\beta C}{N_1} v_{11}(t) r(t) \cos[\omega_0 t + \beta x_1^*(t)] \qquad (5.60)$$

and

$$\frac{d}{dt} v_{11}^*(t) = -2k v_{11}^*(t) + 2k - \frac{\beta^2 C^2}{2 N_1} v_{11}^{*2}(t) \qquad (5.61)$$

where double-frequency terms have been neglected.

The steady-state solution to the variance equation is:

$$v_{11}^* = \frac{2}{1 + \sqrt{1 + \beta^2 \Lambda}} \qquad (5.62)$$

where $\Lambda = C^2 / k N_1$ is the signal-to-noise ratio in the message bandwidth. The optimum demodulator is shown in Fig. 5.13b. In Ch. 6, we shall analyze the performance of this demodulator. An explicit expression for the probability density of the steady-state estimation error, $\beta[x_1(t) - x_1^*(t)]$, is derived. The expression is valid in all regions of operation including threshold and below.

Example 5.3.2 Single Message, FM, Additive White Noise Channel

Consider the communication model shown in Fig. 5.14a. a(t) is a stationary Gaussian message and n(t) is a White Gaussian process of spectral height N_0 watts/cps. a(t), which is uncorrelated with n(t), frequency modulates a sinusoidal carrier whose nominal frequency is large compared to significant frequencies of a(t). We shall assume that the variance of a(t) is unity. d_f is then the standard
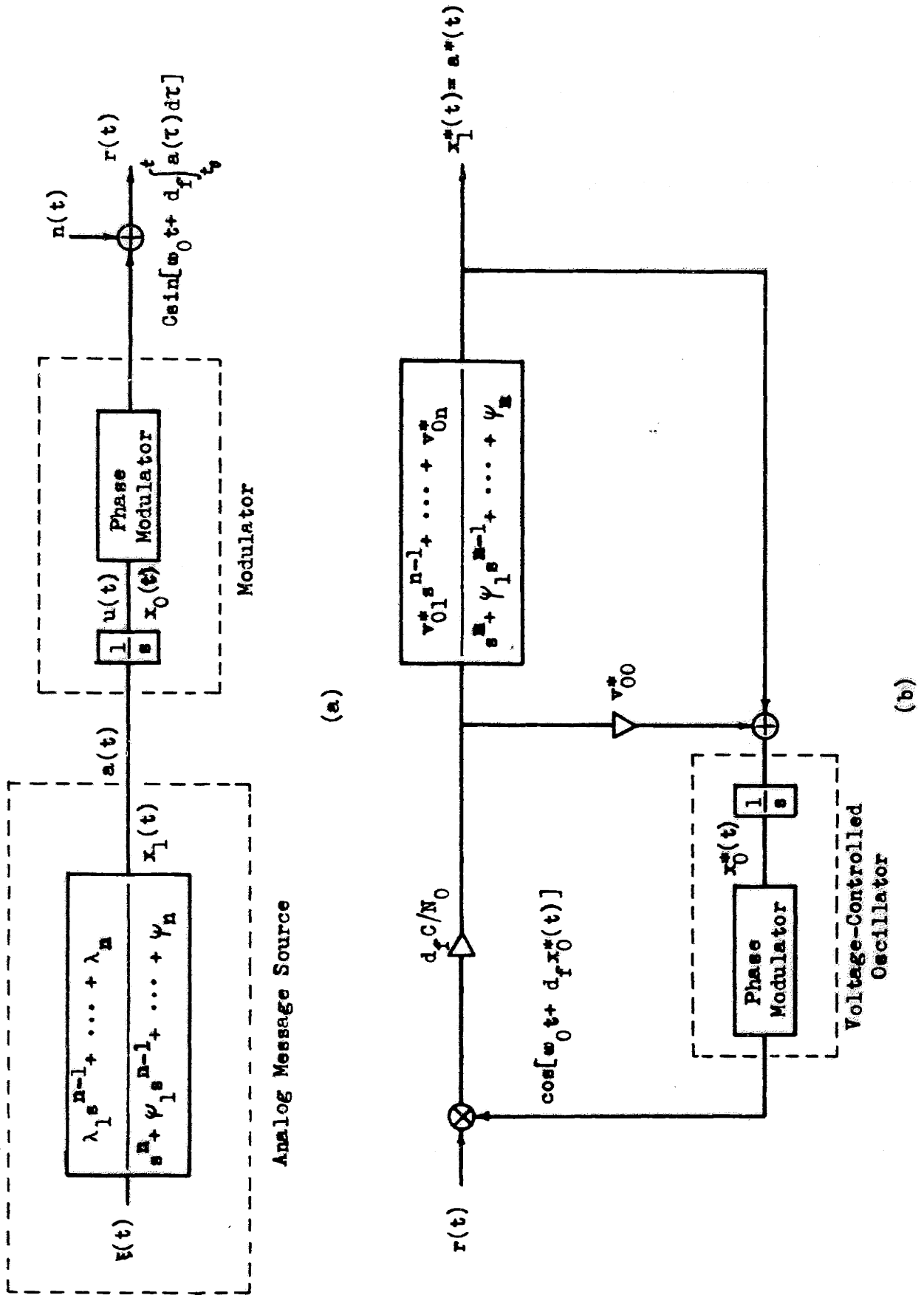
Figure 5.14 (a) A Single Message Transmitted by Frequency Modulation
(b) Quasi-Optimum FM Demodulator

deviation of the frequency from ω_0.

The results obtained for this example are typical of those obtained for other nonlinear modulation schemes with memory. They bear a close resemblance to the results of the linear, integral modulation case of Ex. 5.2.2.

The state vector associated with the analog message source satisfies:

$$d\underline{a}(t) = F_a \, \underline{a}(t) \, dt + d\underline{\alpha}(t) \tag{5.63}$$

where:

$$\underline{a}(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ \cdot \\ \cdot \\ \cdot \\ a_n(t) \end{bmatrix} \; ; \quad F_a = \begin{bmatrix} -\psi_1 & 1 & 0 & 0 \\ -\psi_2 & 0 & 1 & 0 \\ -\psi_3 & 0 & 0 & 1 \\ \cdot & & & \\ \cdot & & & 1 \\ \cdot & & & \\ -\psi_n & 0 & \cdots & 0 \end{bmatrix} \; ;$$

$$\frac{d}{dt} \underline{\alpha}(t) = \begin{bmatrix} \lambda_1 \dot{\xi}(t) \\ \lambda_2 \dot{\xi}(t) \\ \lambda_3 \dot{\xi}(t) \\ \cdot \\ \cdot \\ \cdot \\ \lambda_n \dot{\xi}(t) \end{bmatrix}$$

We assume that $E[\underline{\alpha}(t) \, \underline{\alpha}(u)] = A \min(t,u)$ is known. Note that $a(t) = a_1(t)$. $u(t)$ is defined by:

$$du(t) = a(t) \, dt \tag{5.64}$$

Define $\underline{x}(t)$ by:

$$\underline{x}(t) \quad = \begin{bmatrix} u(t) \\ \underline{a}(t) \end{bmatrix} = \begin{bmatrix} x_0(t) \\ x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix} \tag{5.65}$$

Then $\underline{x}(t)$ satisfies:

$$\frac{d}{dt} \underline{x}(t) = F \underline{x}(t) + \underline{\xi}(t) \tag{5.66}$$

where F, $\underline{\xi}(t)$, and X are the same as defined for Eq. 5.25 of the linear, integral-modulation case. Note that $u(t) = x_0(t)$ and $a(t) = x_1(t)$.

The received signal is described by:

$$r(t) = C \sin[\omega_0 t + d_f \int_{t_0}^{t} a(\tau) d\tau] + n(t)$$

$$= C \sin[\omega_0 t + d_f x_0(t)] + n(t) \tag{5.67}$$

so that $\underline{g}[t:\underline{x}(t)] = C \sin[\omega_0 t + d_f x_0(t)]$ and

$$D[\underline{g}(t:\underline{x})] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} d_f C \cos[\omega_0 t + d_f x_0(t)] \tag{5.68}$$

After some manipulation, the variance and processor equations, Eqn's. 4.5 and 4.6, become:

$$\frac{d}{dt} \underline{x}^*(t) = F\underline{x}^*(t) + \frac{1}{N_0} \begin{bmatrix} v_{00}^*(t) \\ v_{01}^*(t) \\ \cdot \\ \cdot \\ \cdot \\ v_{0n}^*(t) \end{bmatrix} d_f C\cos[\omega_0 t + d_f x_0^*(t)]\{r(t) - C\sin[\omega_0 t + d_f x_0^*(t)]\}$$

(5.69)

and

$$\frac{d}{dt} V^*(t) = FV^*(t) + V^*(t)F' + X -$$

(5.70)

$$\begin{bmatrix} v_{00}^*(t)v_{00}^*(t) & v_{00}^*(t)v_{01}^*(t) & \cdots & v_{00}^*(t)v_{0n}^*(t) \\ v_{01}^*(t)v_{00}^*(t) & v_{01}^*(t)v_{01}^*(t) & & v_{01}^*(t)v_{0n}^*(t) \\ \cdot \\ \cdot \\ \cdot \\ v_{0n}^*(t)v_{00}^*(t) & \cdots & & v_{0n}^*(t)v_{0n}^*(t) \end{bmatrix} d_f^2 C \{r(t)\sin[\omega_0 t + d_f x_0^*(t)] + C\cos[2\omega_0 t + 2d_f x_0^*(t)]\}$$

We observe that Eq. 5.70 is equivalent to Eq. 5.50, the variance equation for the PM case. Therefore, the arguments leading to the simplified variance equation, Eq. 5.56, carry over and Eq. 5.70 becomes:

$$\frac{d}{dt} V^*(t) = FV^*(t) + V^*(t)F' + X -$$

$$\frac{d_f^2 C^2}{2N_0} \begin{bmatrix} v_{00}^*(t)v_{00}^*(t) & v_{00}^*(t)v_{01}^*(t) & \cdots & v_{00}^*(t)v_{0n}^*(t) \\ v_{01}^*(t)v_{00}^*(t) & v_{01}^*(t)v_{01}^*(t) & & v_{01}^*(t)v_{0n}^*(t) \\ \cdot \\ \cdot \\ \cdot \\ v_{0n}^*(t)v_{00}^*(t) & \cdots & & v_{0n}^*(t)v_{0n}^*(t) \end{bmatrix}$$

(5.71)

With a change in noise level, Eq. 5.71 is identical to the variance equation for the linear, integral-modulation case, Eq. 5.28. By using

Eqn's. 5.29 and 5.30, we see that the steady-state mean-square errors in estimating $x_0(t)$, the integrated message, and $x_1(t)$, the message, are given by:

$$v_{00}^* = \frac{2N_0}{d_f^2 C^2} f(0) \qquad (5.72)$$

and

$$v_{11}^* = \frac{2N_0}{3d_f^2 C^2} f^3(0) + F(0) \qquad (5.73)$$

where

$$f(0) = \int_{-\infty}^{\infty} \log\left[1 + \frac{d_f^2 C^2}{2N_0} \frac{S_a(\omega^2)}{\omega^2}\right] \frac{d\omega}{2\Pi}$$

and

$$F(0) = \int_{-\infty}^{\infty} \frac{2N_0}{d_f^2 C^2} \omega^2 \log\left[1 + \frac{d_f^2 C^2}{2N_0} \frac{S_a(\omega^2)}{\omega^2}\right] \frac{d\omega}{2\Pi}$$

A useful measure for evaluating the performance of the FM demodulator is $d_f^2 v_{00}^*$, the mean-square error in estimating the total phase, $d_f x_0(t)$. Eqn's. 5.72 and 5.73 are important because with them, the performance can be studied without determining the structure of the demodulator. The equations have been evaluated numerically for the Butterworth class of message spectra. The results are presented in Ch. 6.

In the steady state, the processor equation, Eq. 5.69, leads to the quasi-optimum FM demodulator of Fig. 5.14b. Double-frequency terms have been omitted. The FM demodulator bears a distinct resemblance to the linear, integral-modulation processor of Fig. 5.5b.

5.3.2.1 Special Case of a One-Dimensional Message: FM

As a simple example, consider the one-dimensional message in the communication model of Fig. 5.15a. For this case, the spectrum of a(t) is $\dfrac{2k}{\omega^2 + k^2}$ watts/cps so that a(t) has a variance of unity.

Eqn's. 5.66 and 5.77 become:

$$\frac{d}{dt}\,\underline{x}(t) \;=\; F\,\underline{x}(t) \;+\; \underline{\xi}(t) \tag{5.74}$$

and

$$r(t) \;=\; C\sin[\omega_0 t + d_f\,x_0(t)] \;+\; n(t) \tag{5.75}$$

where

$$\underline{x}(t) \;=\; \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} \;;\quad F \;=\; \begin{bmatrix} 0 & 1 \\ 0 & -k \end{bmatrix} \;;\quad X \;=\; \begin{bmatrix} 0 & 0 \\ 0 & 2k \end{bmatrix}$$

The processor and variance equations, Eqn's. 5.69 and 5.70, become:

$$\frac{d}{dt}\underline{x}^*(t) \;=\; F\underline{x}^*(t) \;+\; \frac{1}{N_0}\begin{bmatrix} v^*_{00}(t) \\ v^*_{01}(t) \end{bmatrix} d_f C\, r(t)\cos[\omega_0 t + d_f\, x^*_0(t)] \tag{5.76}$$

and

$$\frac{d}{dt} V^*(t) \;=\; F V^*(t) \;+\; V^*(t) F' \;+\; X \;-$$

$$\frac{1}{N_0}\begin{bmatrix} v^*_{00}v^*_{00} & v^*_{00}v^*_{01} \\ v^*_{01}v^*_{00} & v^*_{01}v^*_{01} \end{bmatrix} d_f^2 C\, r(t)\sin[\omega_0 t + d_f\, x^*_0(t)] \tag{5.77}$$

where double-frequency terms have been neglected. The simplified variance equation, Eq. 5.71, becomes:
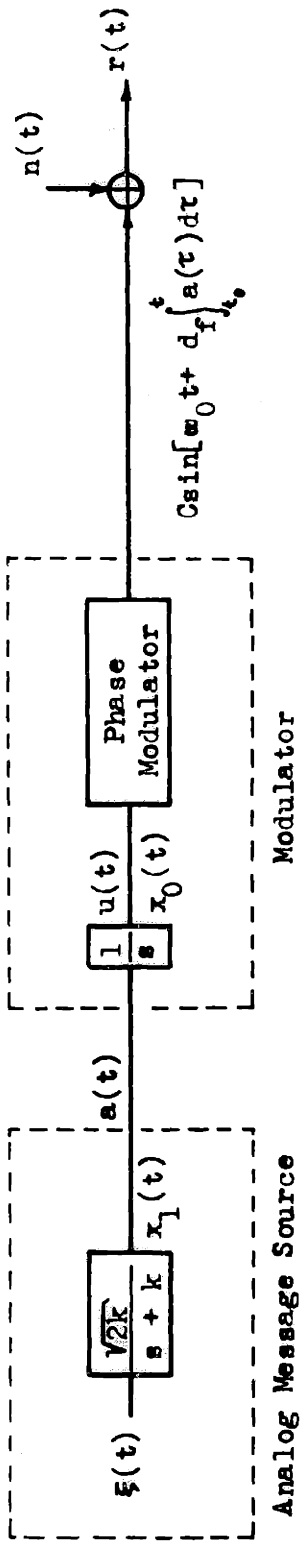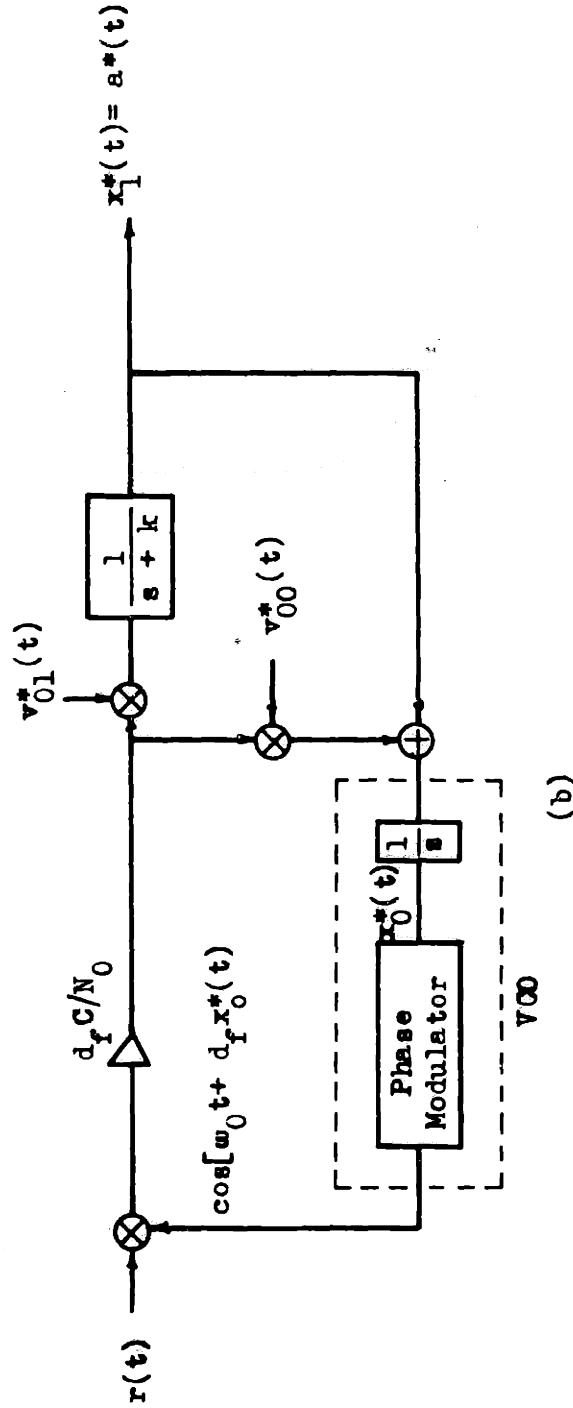
Figure 5.15 (a) FM with a One-Dimensional Message
(b) Quasi-Optimum FM Demodulator in Transient Case

$$\frac{d}{dt} V*(t) = FV*(t) + V*(t)F' + X -$$

(5.78)

$$\frac{d_f^2 C^2}{2N_0} \begin{bmatrix} v^*_{00}v^*_{00} & v^*_{00}v^*_{01} \\ v^*_{01}v^*_{00} & v^*_{01}v^*_{01} \end{bmatrix}$$

The quasi-optimum FM demodulator for the one-dimensional message is shown in Fig. 5.15b. In Ch. 6, we shall present the results of a computer simulation of this demodulator. For the simulation, $v^*_{00}(t)$ and $v^*_{01}(t)$ were generated by both Eqn's. 5.77 and 5.78. The results of the simulation indicate that the variance equations are equivalent and that the approach to steady state is rapid compared to the message correlation time. The steady-state values for the components of V* can be obtained from Eqn's. 5.35-5.37 by substituting $2N_0/d_f^2 C^2$ for N_0. Let $\Lambda = C^2/kN_0$, the signal-to-noise ratio in the message bandwidth, and let $\beta = d_f/k$, the modulation index. After some manipulation, we obtain:

$$d_f^2 v^*_{00} = \frac{4\beta \Lambda^{-\frac{1}{2}}}{1 + \sqrt{1 + 2\beta\Lambda^{\frac{1}{2}}}}$$

(5.79)

$$d_f v^*_{01} = \frac{4\beta}{\left\{ 1 + \sqrt{1 + 2\beta\Lambda^{\frac{1}{2}}} \right\}^2}$$

(5.80)

$$v^*_{11} = 1 - \frac{4\beta^2\Lambda}{\left\{ 1 + \sqrt{1 + 2\beta\Lambda^{\frac{1}{2}}} \right\}^4}$$

(5.81)

Example 5.3.3 Single Message,FM, c Diversity Channels

The diversity communication system of Fig. 5.16 consists of a single, stationary Gaussian message transmitted by c frequency-modulated signals, each differing only in amplitude, over c links. Each link has additive observation noise. The model can also be interpreted as representing a fixed, known, multipath channel with different gains associated with each path. Other diversity modulation schemes, such as frequency-diversity FM, are treated in a fashion parallel to this example.

We shall use the representation for FM of Ex. 5.3.2 so that $\underline{x}(t)$ is given by Eq. 5.66.

Let the additive disturbances be independent and white Gaussian. N is then of the form:

$$N = \begin{bmatrix} N_1 & & & \\ & N_2 & & 0 \\ & & \cdot & \\ 0 & & \cdot & \\ & & & N_c \end{bmatrix} \tag{5.82}$$

The received signal is:

$$\underline{r}(t) = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \cdot \\ \cdot \\ \cdot \\ C_c \end{bmatrix} \sin[\omega_0 t + d_f x_0(t)] + \underline{n}(t) \tag{5.83}$$

$D[\underline{g}(t:\underline{x})]$ is given by:
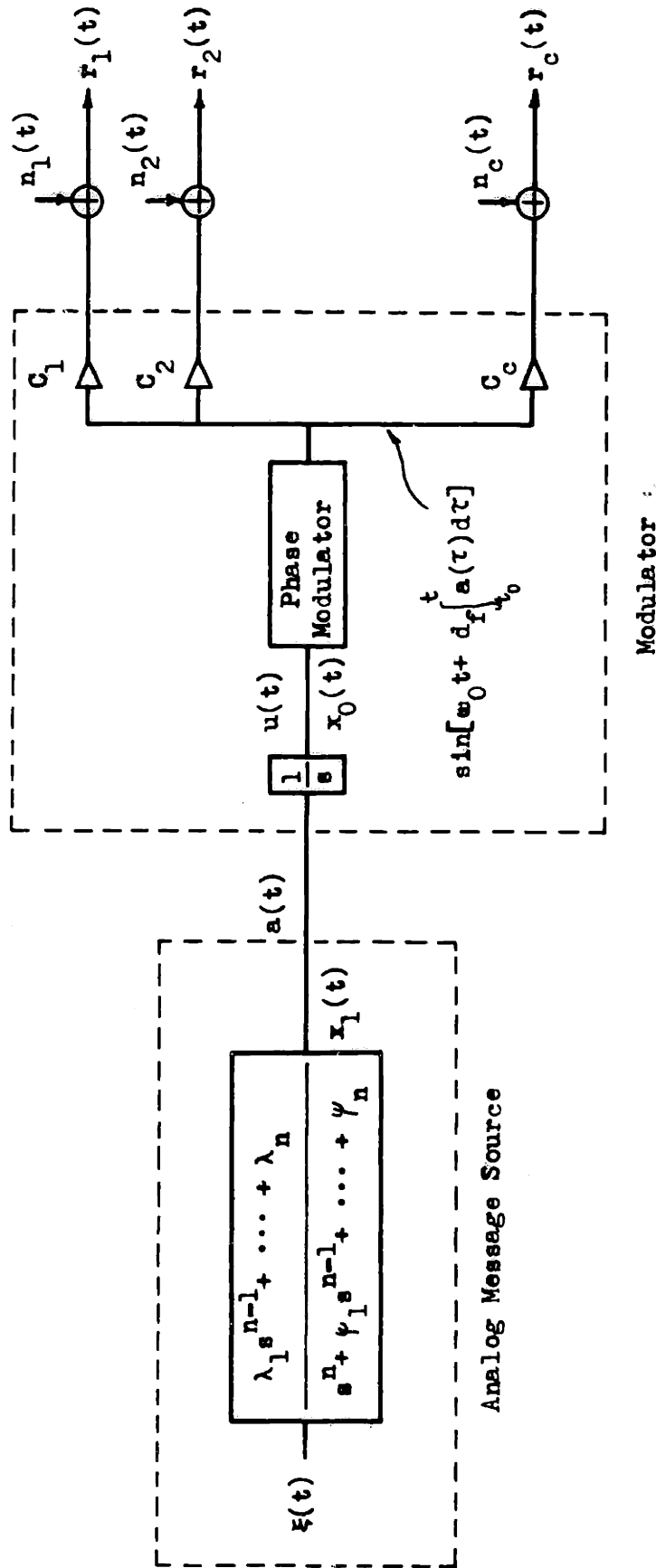
Figure 5.16 FM Diversity Communication System

$$D[\underline{g}(t:\underline{x})] = \begin{bmatrix} C_1 & C_2 & C_3 & \cdots & C_c \\ 0 & 0 & 0 & & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & & 0 \end{bmatrix} d_f \cos[\omega_0 t + d_f x_0(t)] \qquad (5.84)$$

After some manipulation, the processor equation, Eq. 4.5, becomes:

$$\frac{d}{dt} \underline{x}^*(t) = F\underline{x}^*(t) + \begin{bmatrix} v_{00}(t) \\ v_{01}(t) \\ v_{02}(t) \\ \cdot \\ \cdot \\ \cdot \\ v_{0n}(t) \end{bmatrix} \left\{ \sum_{i=1}^{c} \frac{C_i}{N_i} r_i(t) \right\} d_f \cos[\omega_0 t + d_f x_0^*(t)] \qquad (5.85)$$

when double-frequency terms are neglected.

The variance equation reduces to a linear equivalent variance equation just as in the PM and FM examples considered previously.

The quasi-optimum, diversity FM demodulator, in the steady state, is shown in Fig. 5.17. It consists of a maximal-ratio combiner followed by a scalar FM demodulator.

Example 5.3.4 Single Message, FM, Simple Multiplicative Channel

For the communication model shown in Fig. 5.18, the received signal is:

$$r(t) = C b(t) \sin[\omega_0 t + d_f \int_{t_0}^{t} a(\tau) d\tau] + n(t) \qquad (5.86)$$

a(t), the message, b(t), the multiplicative disturbance, and n(t) are uncorrelated, stationary, Gaussian processes. The nonstationary case
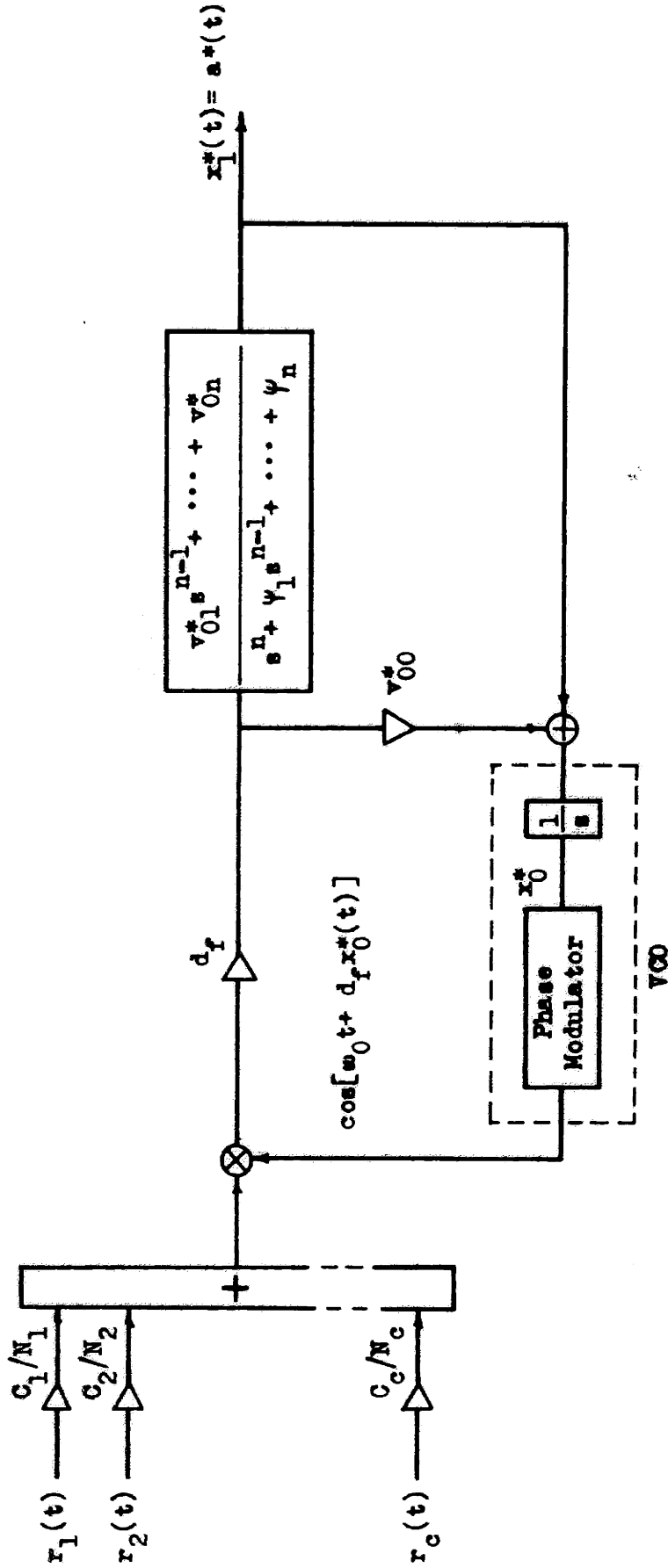
Figure 5.17 Quasi-Optimum Demodulator for a FM Diversity
Communication System

can be treated in a similar manner.

The simple multiplicative channel does not occur naturally in practice, but the procedure used here is identical to that use for Rayleigh channels, Rician channels, and other channels composed of combinations óf simple multiplicative channels.

The algebraic manipulations for this example are lengthy so we shall present only the broader aspects of the derivation.

The (m+1+k)-dimensional vector, $\underline{x}(t)$, obtained by adjoining the (m+1)-dimensional state vector associated with a(t) and u(t) with the k-dimensional vector associated with b(t), satisfies:

$$\frac{d}{dt}\underline{x}(t) = F\,\underline{x}(t) + \underline{\xi}(t) \tag{5.87}$$

where

$$\underline{x}(t) = \begin{bmatrix} a(t) \\ \hline b(t) \end{bmatrix} = \begin{bmatrix} x_0(t) \\ x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_m(t) \\ x_{m+1}(t) \\ \cdot \\ \cdot \\ \cdot \\ x_{m+k}(t) \end{bmatrix} \quad ; \quad \underline{\xi}(t) = \begin{bmatrix} 0 \\ \lambda_1 \xi_1(t) \\ \lambda_2 \xi_1(t) \\ \cdot \\ \cdot \\ \lambda_m \xi_1(t) \\ \lambda_{m+1} \xi_{m+1}(t) \\ \lambda_{m+2} \xi_{m+1}(t) \\ \cdot \\ \cdot \\ \lambda_{m+k} \xi_{m+1}(t) \end{bmatrix}$$

where $u(t) = x_0(t)$, $a(t) = x_1(t)$, and $b(t) = x_{m+1}(t)$. F is defined by:
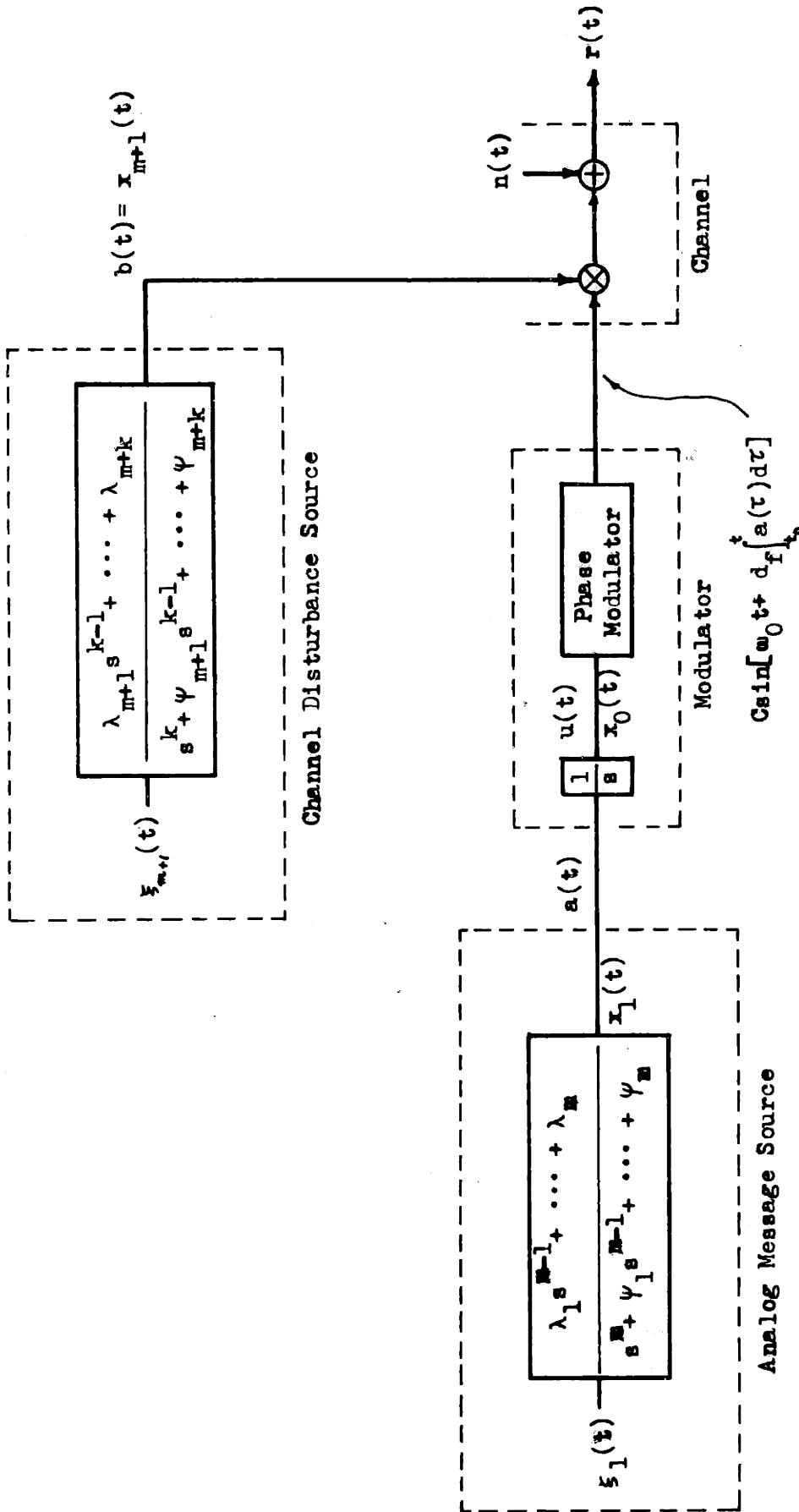
Figure 5.18 FM with a Simple Multiplicative Channel

$$
F = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & & & & & & \\
0 & -\psi_1 & 1 & 0 & & & & & & & & \\
0 & -\psi_2 & 0 & 1 & & & & & 0 & & & \\
& \vdots & & & & 1 & & & & & & \\
0 & -\psi_m & 0 & & & 0 & & & & & & \\
\hline
& & & & & & -\psi_{m+1} & 1 & 0 & 0 & & \\
& & & & & & -\psi_{m\ast 2} & 0 & 1 & 0 & & \\
& & 0 & & & & -\psi_{m+3} & 0 & 0 & 1 & & \\
& & & & & & \vdots & & & & & 1 \\
& & & & & & -\psi_{m+k} & 0 & & & & 0
\end{bmatrix}
= \begin{bmatrix} F_a & 0 \\ \hline 0 & F_b \end{bmatrix}
$$

n(t) is a white Gaussian disturbance of spectral height N_0 watts/cps. We shall assume that:

$$
E[\,\underline{\xi}(t)\,\underline{\xi}'(u)\,] = X\,\delta(t-u) = \begin{bmatrix} X_a & 0 \\ \hline 0 & X_b \end{bmatrix}\delta(t-u)
$$

is known.

From Eq. 5.86 and Fig. 5.18, it is seen that:

$$
\underline{g}[t:\underline{x}(t)] = C\,x_{m+1}(t)\,\sin[\omega_0 t + d_f x_0(t)]
$$

so that:

$$
D[\underline{g}(t:\underline{x})] = C\begin{bmatrix}
x_{m+1}(t)\,d_f\cos[\omega_0 t + d_f x_0(t)] \\
0 \\
\vdots \\
0 \\
\sin[\omega_0 t + d_f x_0(t) \\
0 \\
\vdots \\
0
\end{bmatrix}
\begin{array}{l}
\text{—— (row 0)} \\
\\
\\
\\
\text{—— (row m+1)} \\
\\
\text{(5.88)} \\
\\
\end{array}
$$

Let V*(t) be given by:

$$V*(t) = \begin{bmatrix} V^*_{aa}(t) & \vline & V^*_{ab}(t) \\ \hline V^*_{ba}(t) & \vline & V^*_{bb}(t) \end{bmatrix}$$

Since a(t) and b(t) are uncorrelated, $V^*_{ab}(t_0) = V^*_{ba}(t_0) = 0$. Further, by a straightforward, but tedious, manipulation of the variance equation, Eq. 4.6, it can be demonstrated that $V^*_{ab}(t) = V^*_{ba}(t) = 0$ for all time given the zero initial condition. Moreover, it can be shown that:

$$\frac{d}{dt} V^*_{aa}(t) = F_a V^*_{aa}(t) + V^*_{aa}(t) F'_a + X_a -$$

$$(5.89)$$

$$\frac{C^2 d_f^2 P_b}{2N_0} \begin{bmatrix} v^*_{00}(t)v^*_{00}(t) & v^*_{00}(t)v^*_{01}(t) & \cdots & v^*_{00}(t)v^*_{0m}(t) \\ v^*_{01}(t)v^*_{00}(t) & v^*_{01}(t)v^*_{01}(t) & & v^*_{01}(t)v^*_{0m}(t) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ v^*_{0m}(t)v^*_{00}(t) & \cdots & & v^*_{0m}(t)v^*_{0m}(t) \end{bmatrix}$$

and

$$\frac{d}{dt} V^*_{bb}(t) = F_b V^*_{bb}(t) + V^*_{bb}(t) F'_b + X_b -$$

$$(5.90)$$

$$\frac{C^2}{2N_0} \begin{bmatrix} \text{matrix of the same form} \\ \text{as for Eq. 5.89 but with} \\ \text{elements:} \\ v^*_{m+1,i}(t)v^*_{m+1,j}(t) \quad \text{for} \\ i,j = m+1, m+2, \cdots, m+k \end{bmatrix}$$

where double-frequency terms have been omitted and where an argument

parallel to that used for eliminating coupling terms in the PM case of Ex. 5.3.1 has been used. P_b is the power in the multiplicative disturbance, $b(t)$.

In the steady-state, the processor equation becomes:

$$\frac{d}{dt} \underline{x}^*(t) = F \underline{x}^*(t) +$$

$$\frac{C}{N_0} \begin{bmatrix} v_{00}^* \, d_f \, x_{m+1}^*(t) \cos[\omega_0 t + d_f x_0^*(t)] \\ v_{01}^* \, d_f \, x_{m+1}^*(t) \cos[\omega_0 t + d_f x_0^*(t)] \\ \cdot \\ \cdot \\ v_{0m}^* d_f x_{m+1}^*(t) \cos[\omega_0 t + d_f x_0^*(t)] \\ v_{m+1,m+1}^* \sin[\omega_0 t + d_f x_0^*(t)] \\ v_{m+1,m+2}^* \sin[\omega_0 t + d_f x_0^*(t)] \\ \cdot \\ \cdot \\ v_{m+1,m+k}^* \sin[\omega_0 t + d_f x_0^*(t)] \end{bmatrix} \{r(t) - C x_{m+1}^*(t)\sin[\omega_0 t + d_f x_0^*(t)]\}$$

$$(5.91)$$

Eq. 5.91 leads to the following two equations for $\underline{a}^*(t)$ and $\underline{b}^*(t)$:

$$\frac{d}{dt} \underline{a}^*(t) = F_a \underline{a}^*(t) + \frac{C}{N_0} \begin{bmatrix} v_{00}^* \\ v_{01}^* \\ \cdot \\ \cdot \\ \cdot \\ v_{0m}^* \end{bmatrix} d_f b^*(t) \, r(t) \cos[\omega_0 t + d_f u^*(t)] \qquad (5.92)$$

$$(\text{where } a_0^* = u^* = x_0^*)$$

and

$$\frac{d}{dt}\underline{b}^*(t) = F_b \underline{b}^*(t) +$$

$$\frac{C}{N_0}\begin{bmatrix} v^*_{m+1,m+1} \\ v^*_{m+1,m+2} \\ \cdot \\ \cdot \\ \cdot \\ v^*_{m+1,m+k} \end{bmatrix} \{r(t)\sin[\omega_0 t + d_f u^*(t)] - \frac{C}{2}\underline{b}^*(t)\} \qquad (5.93)$$

$$(\text{where } b^* = b^*_1)$$

where double-frequency terms have been neglected. These equation bear a close resemblance to the processor equations for the FM and AM-DSB/SC cases of Ex's. 5.2.3 and 5.3.2. The realization of the quasi-optimum demodulator is shown in Fig. 5.19. It has an intuitively appealing interpretation. The structure of the upper branch corresponds to the quasi-optimum FM demodulator when $b(t)$ is known and the lower branch corresponds to the optimum AM-DSB/SC demodulator when $u(t)$ is known. Since $u(t)$ and $b(t)$ are not known, the demodulator substitutes their best estimates.

Example 5.3.5 Single Message, FM, Rayleigh Channel

A communication model for a frequency-modulation scheme and a Rayleigh Channel is shown in Fig. 5.20. The transmitted signal is:

$$s[t:\underline{a}(t)] = C\sin[\omega_0 t + d_f \int_{t_0}^{t} a(\tau)\,d\tau]$$

$$= C\sin[\omega_0 t + d_f u(t)] \qquad (5.94)$$

The received signal is:

$$r(t) = b_1(t)C\sin[\omega_0 t + d_f u(t)] + b_2(t)C\cos[\omega_0 t + d_f u(t)] + n(t)$$
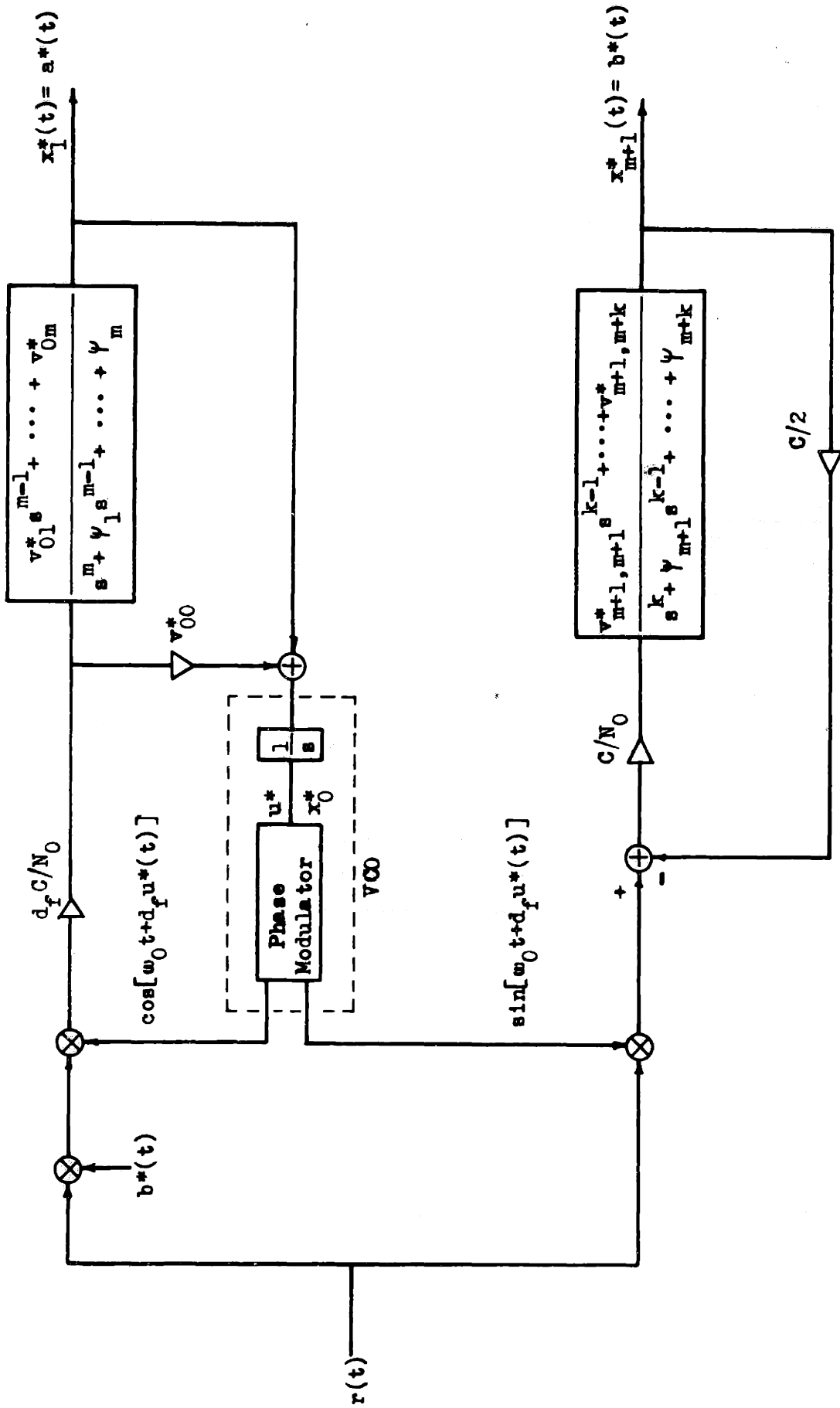
$$(5.95)$$

Figure 5.19 Quasi-Optimum Demodulator for FM with a Multiplicative Channel

where $b_1(t)$, $b_2(t)$, $a(t)$, and $n(t)$ are uncorrelated Gaussian processes. $b_1(t)$ and $b_2(t)$ have identical power density spectra and $n(t)$ is white with a spectral height N_0 watts/cps.

Let $\rho(t)$ and $\theta(t)$ be defined by:

$$\rho^2(t) = b_1^2(t) + b_2^2(t) \tag{5.96}$$

and

$$\theta(t) = \arctan[b_2(t)/b_1(t)] \qquad 0 \leq \theta < 2\Pi \tag{5.97}$$

Then $\rho(t)$ is Rayleigh distributed and $\theta(t)$ is uniformly distributed between 0 and 2Π. The received signal can now be expressed as:

$$r(t) = \rho(t) \, C \, \sin[\omega_0 t + d_f \int_{t_0}^{t} a(\tau) d\tau + \theta(t)] + n(t) \tag{5.98}$$

We shall not present the detailed manipulations required to determine the processor and variance equations since they are tedious but straightforward. The derivations closely parallel those for the simple multiplicative channel.

The $(m+1+2k)$-dimensional vector, $\underline{x}(t)$, satisfies:

$$\frac{d}{dt}\underline{x}(t) = F \, \underline{x}(t) + \underline{\xi}(t) \tag{5.99}$$

where

$$\underline{x}(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \\ \cdot \\ \cdot \\ x_m(t) \\ x_{m+1}(t) \\ \cdot \\ \cdot \\ x_{m+k+1} \\ \cdot \\ \cdot \\ x_{m+2k} \end{bmatrix} \quad ; \quad \underline{\xi}(t) = \begin{bmatrix} 0 \\ \lambda_1 \xi_1(t) \\ \lambda_m \xi_1(t) \\ \lambda_{m+1} \xi_{m+1}(t) \\ \lambda_{m+1} \xi_{m+k+1}(t) \\ \lambda_{m+k} \xi_{m+k+1}(t) \end{bmatrix}$$
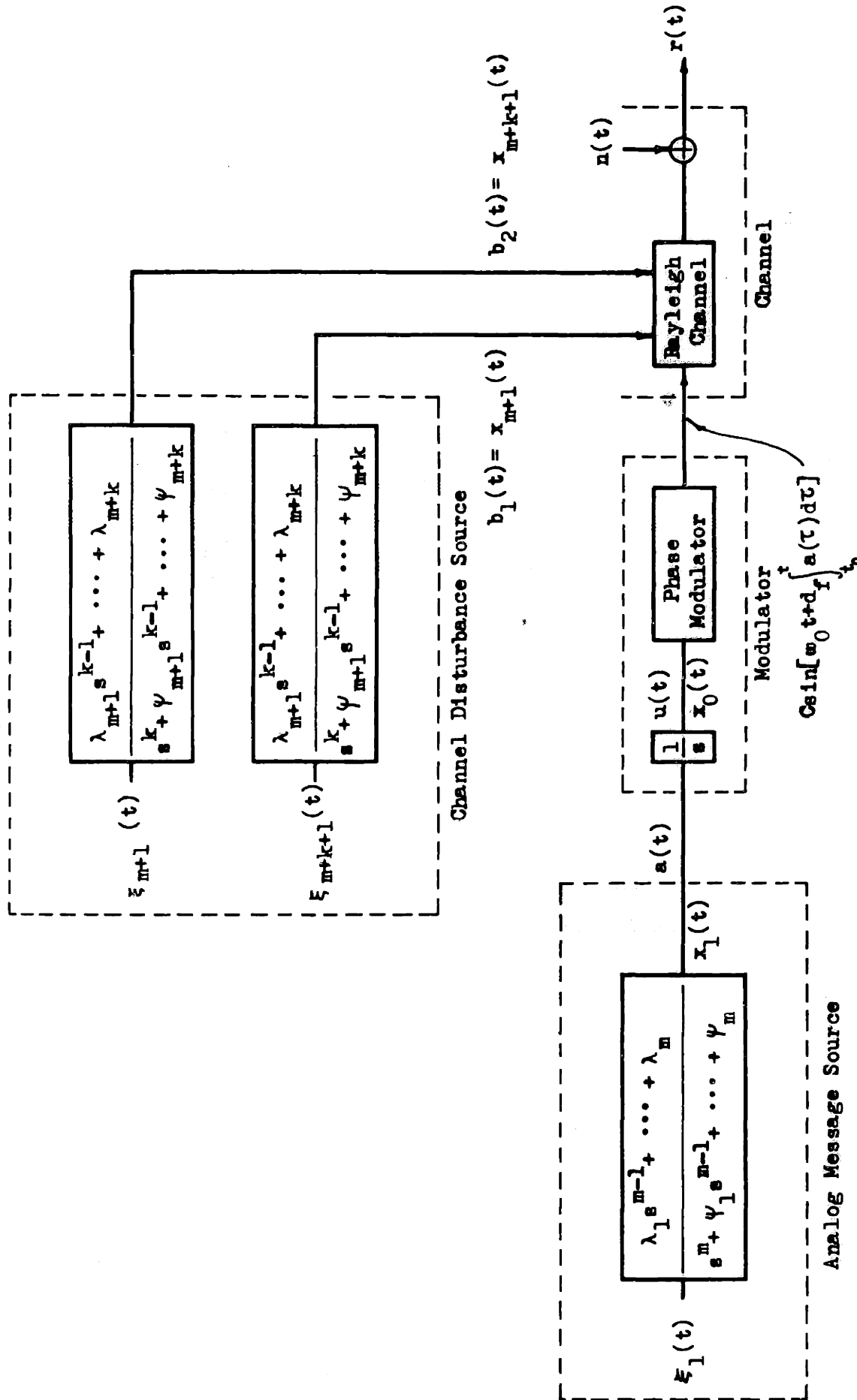
Figure 5.20 A Single Frequency-Modulated Signal Transmitted via a Rayleigh Channel

Note that $u(t) = x_0(t)$, $a(t) = x_1(t)$, $b_1(t) = x_{m+1}(t)$, and $b_2(t) = x_{m+k+1}(t)$. The matrix, F, is defined by:

$$F = \begin{bmatrix} F_a & 0 & 0 \\ 0 & F_b & 0 \\ 0 & 0 & F_b \end{bmatrix}$$

where F_a and F_b are the same as for Eq. 5.87 of the simple multiplicative channel case.

From Eq. 5.95 and Fig. 5.20, we see that $g[t: \underline{x}(t)]$ satisfies:

$$g[t: \underline{x}(t)] = x_{m+1}(t) C \sin[\omega_0 t + d_f x_0(t)] \;+$$

(5.100)

$$x_{m+k+1}(t) C \cos[\omega_0 t + d_f x_0(t)]$$

Thus:

$$D[g(t: \underline{x})] =$$

$$C \begin{bmatrix} x_{m+1}(t) d_f \cos[\omega_0 t + d_f x_0(t)] - x_{m+k+1}(t) d_f \sin[\omega_0 t + d_f x_0(t)] \\ 0 \\ \vdots \\ \sin[\omega_0 t + d_f x_0(t)] \\ 0 \\ \vdots \\ \cos[\omega_0 t + d_f x_0(t)] \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \text{(row 0)} \\ \\ \\ \\ \\ \text{(row m+1)} \\ \\ \\ \\ \\ \text{(row m+1+k)} \\ \\ \\ \\ \\ \end{matrix}$$

It can be demonstrated that $V^*(t)$ has the form:

$$V^*(t) = \begin{bmatrix} V^*_{aa}(t) & 0 & 0 \\ 0 & V^*_{bb}(t) & 0 \\ 0 & 0 & V^*_{bb}(t) \end{bmatrix}$$

where the equation for $V^*_{aa}(t)$ is identical to Eq. 5.89 and the equation for $V^*_{bb}(t)$ is identical to Eq. 5.90.

The steady-state processor equation, Eq. 4.5, leads to the realization shown in Fig. 5.21 when only significant terms are retained. The two lower branches correspond to the optimum AM-DSB/SC demodulators for estimating $b_1(t)$ and $b_2(t)$ when $u(t)$ is known. The upper branch corresponds to the quasi-optimum FM demodulator for estimating $a(t)$ when $b_1(t)$ and $b_2(t)$ are known. Since these signals are not known, their best estimates are used.

Let $\rho^*(t)$ and $\theta^*(t)$ be defined by:

$$[\rho^*(t)]^2 = [b_1^*(t)]^2 + [b_2^*(t)]^2 \tag{5.101}$$

and

$$\theta^*(t) = \arctan[b_2^*(t)/b_1^*(t)] \ , \quad 0 \le \theta^* < 2\Pi \tag{5.102}$$

Then the upper branch of the quasi-optimum demodulator has the alternate form shown in Fig. 5.22. This realization bears a resemblance to the demodulator for the simple multiplicative channel.

Example 5.3.6   Single Message, PM, Random Phase Channel

The purpose of this example is to indicate that nonmultiplicative
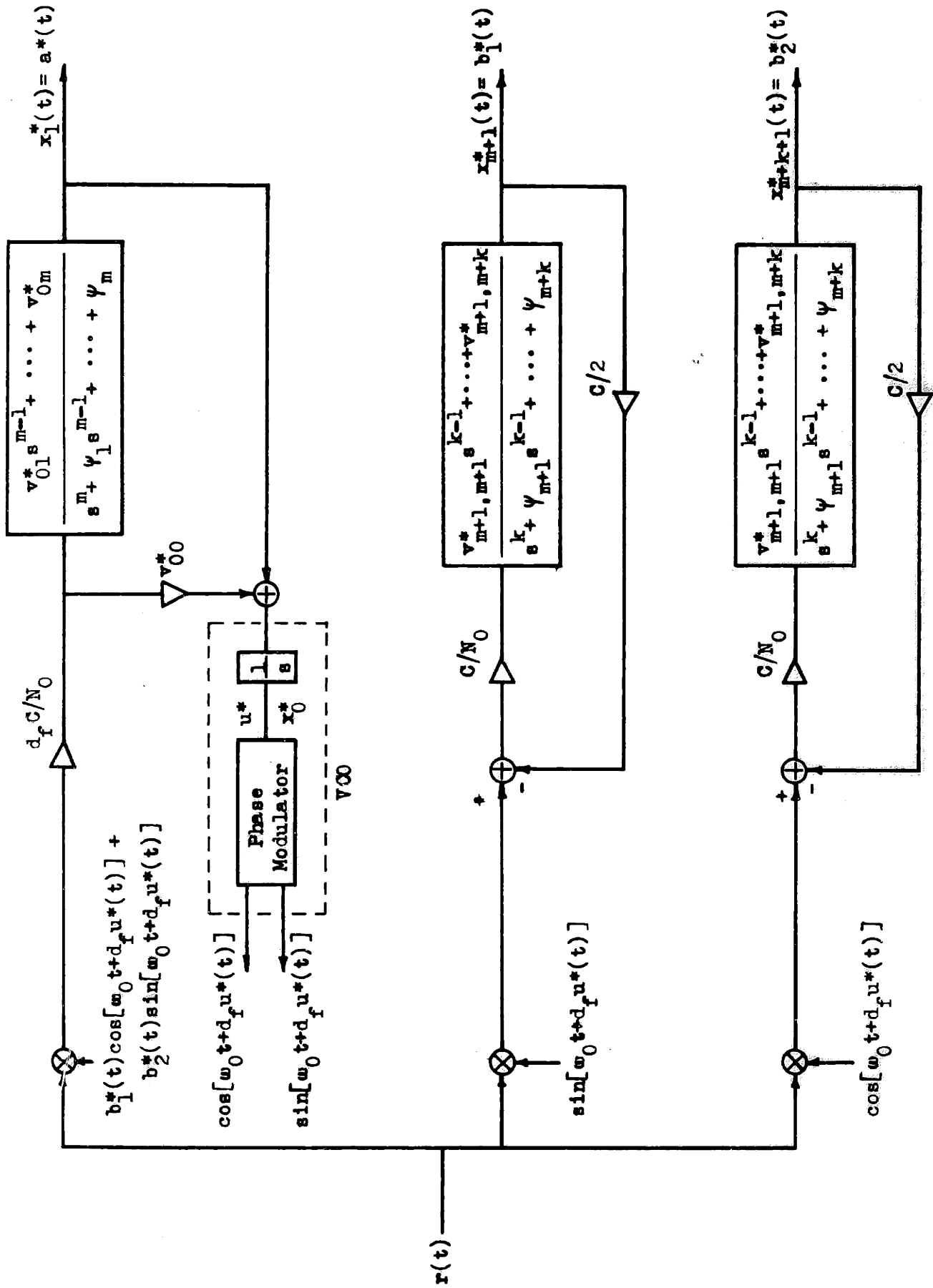
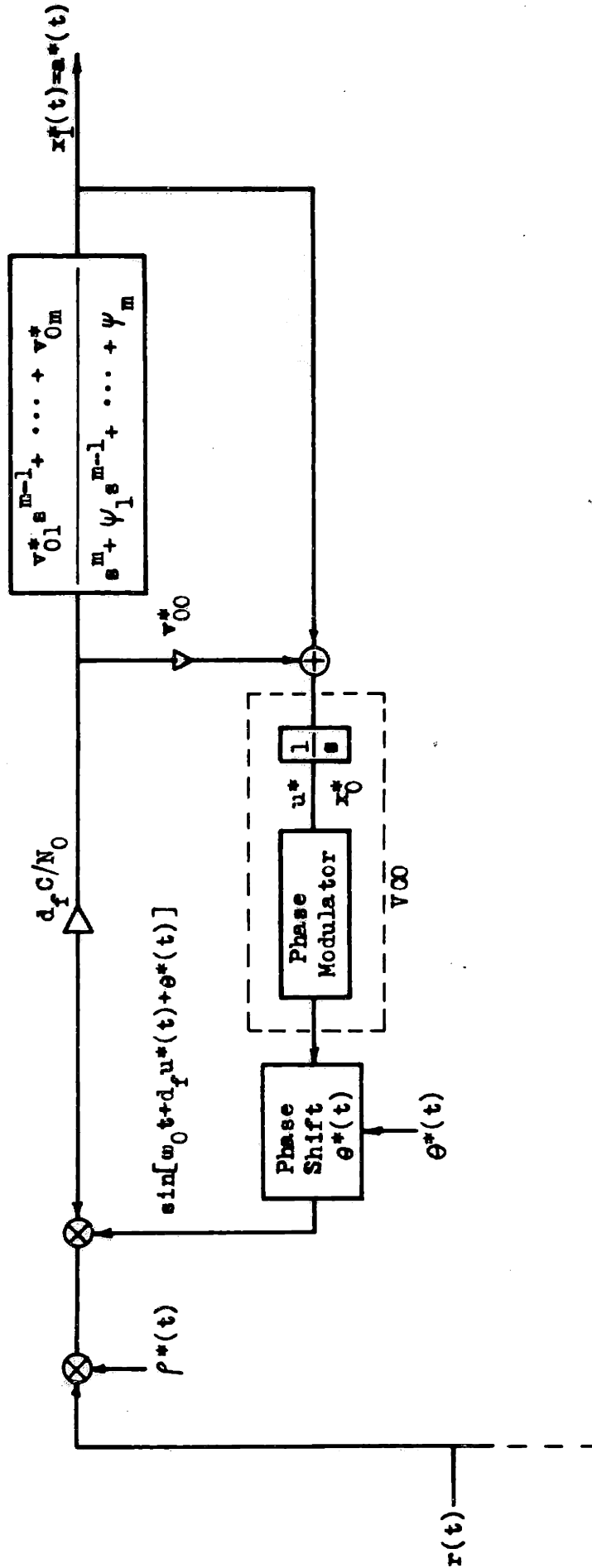Figure 5.21 Quasi-Optimum Demodulator for FM and a Rayleigh Channel

Figure 5.22 Alternate Realization for the Upper Branch of the
Quasi-Optimum Demodulator for FM and a Rayleigh Channel

channel disturbances fall within the scope of the approach and the procedure for treating a specific instance of such a disturbance.

The communication model of Fig. 5.23 represents the transmission of a phase-modulated signal over a channel which introduces a random phase disturbance. The model can be interpreted as representing a phase-modulation system with an unstable oscillator. $\varphi(t)$ has the spectrum $1/\tau\omega^2$ watts/cps which is identical to that used by Edson[46] and Develet[47] to characterize oscillator instabilities. $\xi_0(t)$, $\xi_1(t)$, and $n(t)$ are white Gaussian processes; $\xi_0(t)$ and $n(t)$ have spectral heights of unity and N_0 watts/cps, respectively. τ is the time required for an accumulation of one radian (rms) of phase drift.

Let $\underline{x}(t)$ be an $(m+1)$-dimensional vector defined by:

$$\frac{d}{dt}\,\underline{x}(t) \;=\; F\,\underline{x}(t) \;+\; \underline{\xi}(t) \tag{5.103}$$

where

$$\underline{x}(t) \;=\; \begin{bmatrix} \varphi(t) \\ \text{----} \\ \underline{a}(t) \end{bmatrix} \;=\; \begin{bmatrix} x_0(t) \\ x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_{m+1}(t) \end{bmatrix} \;;\quad \underline{\xi}(t) \;=\; \begin{bmatrix} \tau^{-\frac{1}{2}}\xi_0(t) \\ \lambda_1\xi_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \lambda_m\xi_1(t) \end{bmatrix}$$

and F is defined by:
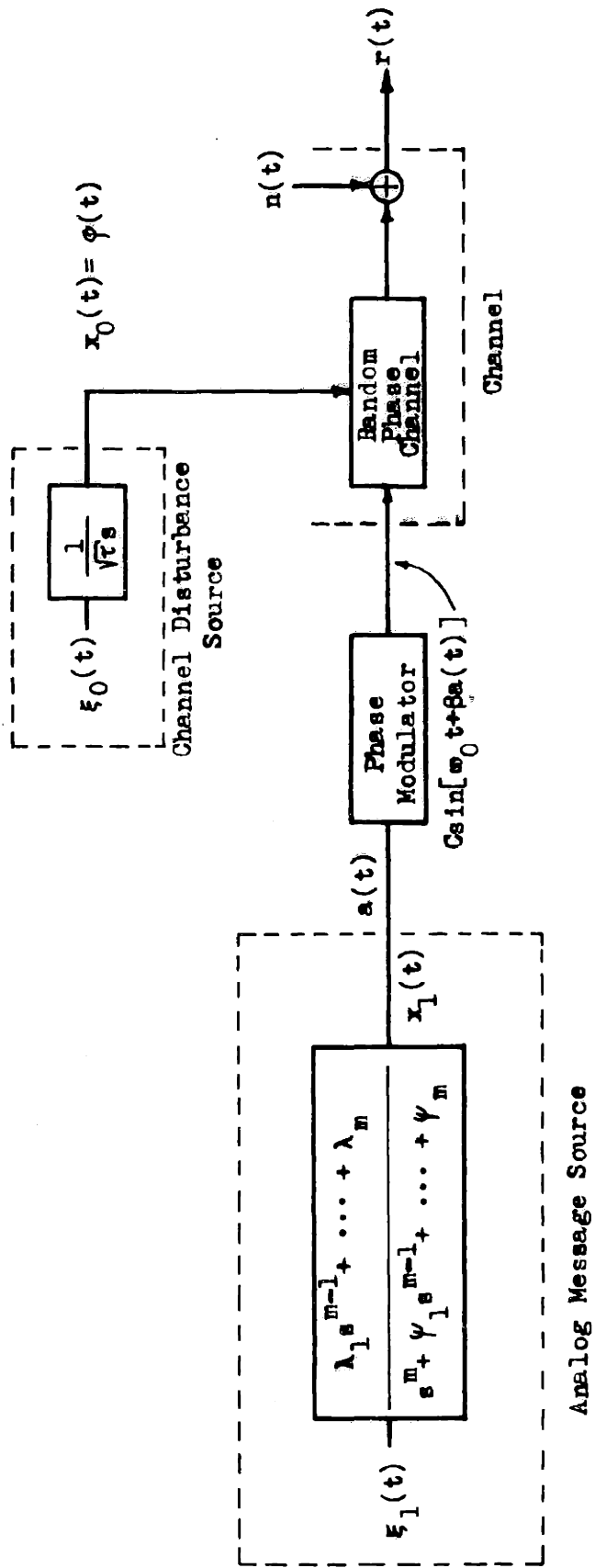
Figure 5.23 Phase Modulation and a Random Phase Channel (Oscillator Instability)

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & \\ 0 & -\psi_1 & 1 & 0 & \\ 0 & -\psi_2 & 0 & 1 & \\ & \cdot & & & \\ & \cdot & & & 1 \\ & \cdot & & & \\ 0 & -\psi_m & 0 & & 0 \end{bmatrix}$$

Note that $\varphi(t) = x_0(t)$ and $a(t) = x_1(t)$.

The received signal is given by:

$$r(t) = C \sin[\omega_0 t + \beta a(t) + \varphi(t)] + n(t) \qquad (5.104)$$

sо that

$$g[t: \underline{x}(t)] = C \sin[\omega_0 t + \beta x_1(t) + x_0(t)] \qquad (5.105)$$

Then

$$D[g(t: \underline{x})] = \begin{bmatrix} 1 \\ \beta \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} C \cos[\omega_0 t + \beta x_1(t) + x_0(t)] \qquad (5.106)$$

As in the case of the pure, phase-modulation example, Ex. 5.3.1, the variance equation, Eq. 4.6, becomes uncoupled from r(t) and $\underline{x}^*(t)$ when only significant terms are retained. From the processor equation, Eq. 4.5, we obtain:

$$\frac{d}{dt} \underline{x}^*(t) = F \underline{x}^*(t) +$$

$$\frac{1}{N_0} \begin{bmatrix} v_{00}^*(t) + \beta v_{10}^*(t) \\ v_{01}^*(t) + \beta v_{11}^*(t) \\ \cdot \\ \cdot \\ \cdot \\ v_{0m}^*(t) + \beta v_{1m}^*(t) \end{bmatrix} C \, r(t) \cos[\omega_0 t + \beta x_1^*(t) + x_0^*(t)] \qquad (5.107)$$

where double-frequency terms have been neglected. It is observed that the equation depends only on two columns of the error-covariance matrix, V*(t). The quasi-optimum demodulator under steady-state conditions is shown in Fig. 5.24.

5.4   Applications when Neither $\underline{x}(t)$ nor $\underline{r}(t)$ are Gaussian

For Examples 5.4.1 and 5.4.2, $\underline{x}(t)$ and $\underline{r}(t)$ are described by:

$$\frac{d}{dt} \underline{x}(t) = \underline{f}[t: \underline{x}(t)] + \underline{\xi}(t) \qquad (5.108)$$

and

$$\frac{d}{dt} \underline{y}(t) = \underline{r}(t) = \underline{g}[t: \underline{x}(t)] + \underline{n}(t) \qquad (5.109)$$

Equations 4.1 and 4.2 are the variance and processor equations.

Example 5.4.1   Single Message, FM, Fixed Channel with Memory

The purpose of this example is to indicate how fixed channels with memory are treated. The same procedure is also followed for modulation schemes with linear filtering after a nonlinear transformation.
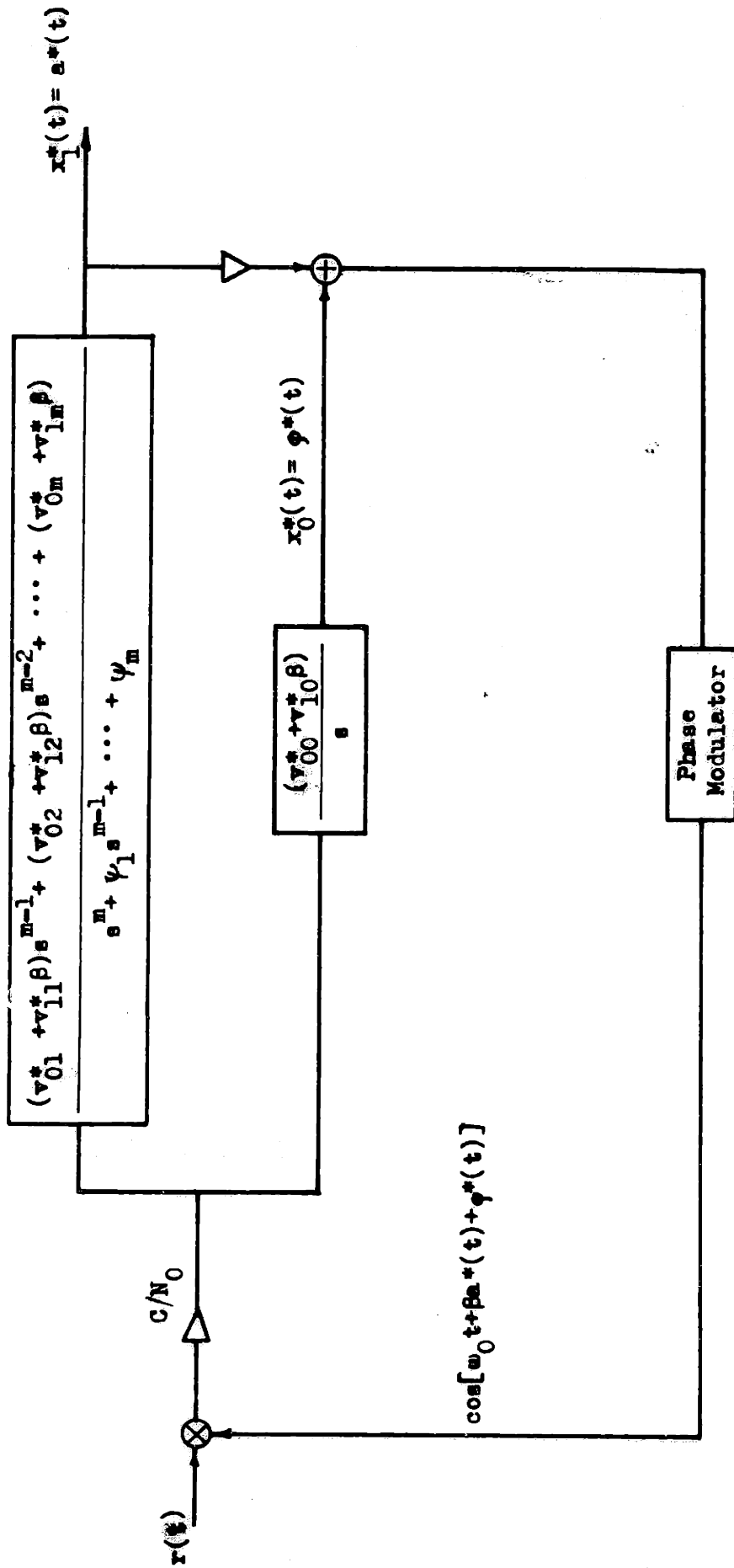
Figure 5.24 Quasi-Optimum Demodulator for Estimating a Phase Modulated Signal in a Random-Phase Channel

The communication model of Fig. 5.25 represents the transmission of a frequency-modulated signal over a channel having a bandpass transmission characteristic which is known and fixed. The frequency response of the channel is normalized to unity at the nominal frequency of the transmitted signal, ω_0. Let the channel filter be described by the state equation:

$$\frac{d}{dt}\begin{bmatrix} x_{m+1}(t) \\ x_{m+2}(t) \end{bmatrix} = \begin{bmatrix} -2\alpha & 1 \\ -\omega_0^2 & 0 \end{bmatrix}\begin{bmatrix} x_{m+1}(t) \\ x_{m+2}(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 2\alpha C \sin[\omega_0 t + d_f x_0(t)] \qquad (5.110)$$

where $z(t) = x_{m+1}(t)$ is the response of the filter to the channel input, $C \sin[\omega_0 t + d_f x_0(t)]$. A time-variant channel response can be treated by simply modifying the filter state equation.

As in the pure FM example, Ex. 5.3.2, a(t) and u(t) are described by:

$$\frac{d}{dt}\begin{bmatrix} x_0(t) \\ x_1(t) \\ . \\ . \\ . \\ x_m(t) \end{bmatrix} = F \begin{bmatrix} x_0(t) \\ x_1(t) \\ . \\ . \\ . \\ x_m(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda_1 \xi(t) \\ . \\ . \\ . \\ \lambda_m \xi(t) \end{bmatrix} \qquad (5.111)$$

where F is the same as defined for Eq. 5.25 of the linear, integral-modulation case.

The (m+3) dimensional vector, $\underline{x}(t)$, obtained by adjoining the 2-dimensional channel state vector with the (m+1)-dimensional message and modulator state vector, satisfies:

$$\frac{d}{dt}\underline{x}(t) = \underline{f}[t : \underline{x}(t)] + \underline{\xi}(t) \qquad (5.112)$$
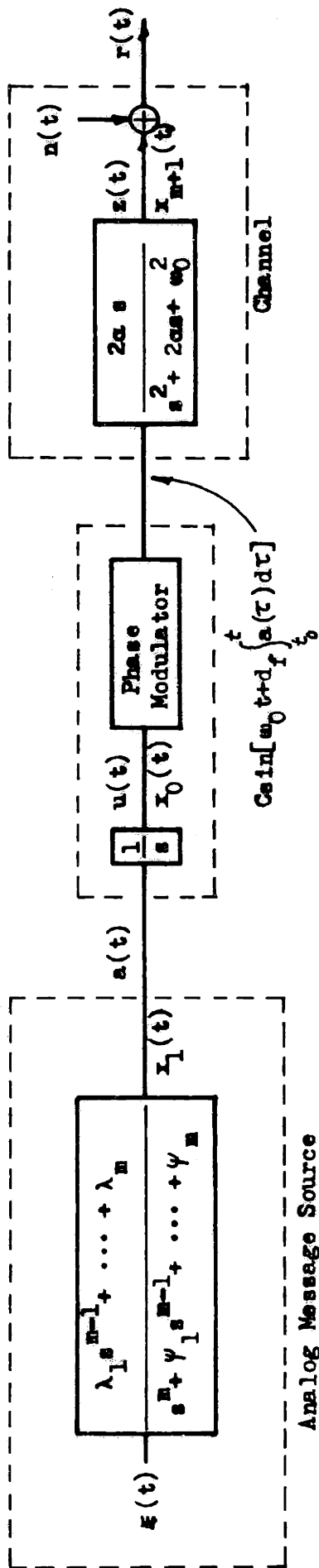
Figure 5.25 FM Transmitted through a Fixed Channel with Memory

where:

$$\underline{x}(t) \;=\; \begin{bmatrix} x_0(t) \\ x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_m(t) \\ x_{m+1}(t) \\ x_{m+2}(t) \end{bmatrix}; \qquad \underline{\xi}(t) \;=\; \begin{bmatrix} 0 \\ \lambda_1 \xi(t) \\ \cdot \\ \cdot \\ \cdot \\ \lambda_m \xi(t) \\ 0 \\ 0 \end{bmatrix}$$

Note that $u(t) = x_0(t)$, $a(t) = x_1(t)$, and $s(t) = x_{m+1}(t)$. The vector, $\underline{f}[t:\underline{x}(t)]$ is given by:

$$\underline{f}[t:\underline{x}(t)] \;=\; \left[\begin{array}{c|cc} F & & 0 \\ \hline & -2\alpha & 1 \\ 0 & & \\ & -\omega_0^2 & 0 \end{array} \right] \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \end{bmatrix} 2\alpha C \sin[\omega_0 t + d_f x_0(t)]$$

It is observed that $\underline{f}[t:\underline{x}(t)]$ is composed of a linear and a nonlinear transformation of $\underline{x}(t)$. Let F_ℓ denote the matrix associated with the linear transformation. Then:

$$D[\underline{f}(t:\underline{x})] \;=\; F_\ell' + \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & & & & & 0 \end{bmatrix} 2\alpha C\, d_f \cos[\omega_0 t + d_f x_0(t)]$$

$$(5.113)$$

The $(1, m+2)$-element is the only nonzero element in the second matrix on the right side.

The received signal is:

$$r(t) = z(t) + n(t) = x_{m+1}(t) + n(t) \qquad (5.114)$$

where $n(t)$ is a white Gaussian process of spectral height N_0 watts/cps. We see that $g[t: \underline{x}(t)] = x_{m+1}(t)$ and, therefore, that:

$$D[g(t: \underline{x})] = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad (5.115)$$

The variance equation is coupled to the estimate of $\underline{x}(t)$ and there is no apparent way to simplify the equation. Since it is quite long, but easily derived, we shall not include it here.

The processor equation, Eq. 4.1, becomes:

$$\frac{d}{dt} \underline{x}^*(t) = \underline{f}[t: \underline{x}(t)] + \frac{1}{N_0} \begin{bmatrix} v^*_{0,m+1}(t) \\ v^*_{1,m+1}(t) \\ \cdot \\ \cdot \\ \cdot \\ v^*_{m+2,m+1}(t) \end{bmatrix} \{r(t) - x^*_{m+1}(t)\} \qquad (5.116)$$

When the definition for $\underline{f}[t: \underline{x}(t)]$, Eq. 5.116 becomes:

$$\frac{d}{dt}\underline{x}^*(t) = F_\ell\,\underline{x}^*(t) + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \end{bmatrix} 2\alpha C \sin[\omega_0 t + d_f x_0^*(t)] +$$

$$\text{(5.117)}$$

$$\frac{1}{N_0}\begin{bmatrix} v_{0,m+1}^*(t) \\ v_{1,m+1}^*(t) \\ \cdot \\ \cdot \\ \cdot \\ v_{m+2,m+1}^*(t) \end{bmatrix} \{r(t) - x_{m+1}^*(t)\}$$

The following two equations are implied by Eq. 5.117:

$$\frac{d}{dt}\begin{bmatrix} x_0^*(t) \\ x_1^*(t) \\ \cdot \\ \cdot \\ \cdot \\ x_m^*(t) \end{bmatrix} = F\begin{bmatrix} x_0^*(t) \\ x_1^*(t) \\ \cdot \\ \cdot \\ \cdot \\ x_m^*(t) \end{bmatrix} + \frac{1}{N_0}\begin{bmatrix} v_{0,m+1}^*(t) \\ v_{1,m+1}^*(t) \\ \cdot \\ \cdot \\ \cdot \\ v_{m,m+1}^*(t) \end{bmatrix} \{r(t) - x_{m+1}^*(t)\} \qquad \text{(5.118)}$$

and

$$\frac{d}{dt}\begin{bmatrix} x_{m+1}^*(t) \\ x_{m+2}^*(t) \end{bmatrix} = \begin{bmatrix} -2\alpha & 1 \\ -\omega_0^2 & 0 \end{bmatrix}\begin{bmatrix} x_{m+1}^*(t) \\ x_{m+2}^*(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 2\alpha C \sin[\omega_0 t + d_f x_0^*(t)] +$$

$$\text{(5.119)}$$

$$\frac{1}{N_0}\begin{bmatrix} v_{m+1,m+1}^*(t) \\ v_{m+2,m+1}^*(t) \end{bmatrix} \{r(t) - x_{m+1}^*(t)\}$$

Eqn's. 5.118 and 5.119 lead to the quasi-optimum demodulator shown in Fig. 5.26.
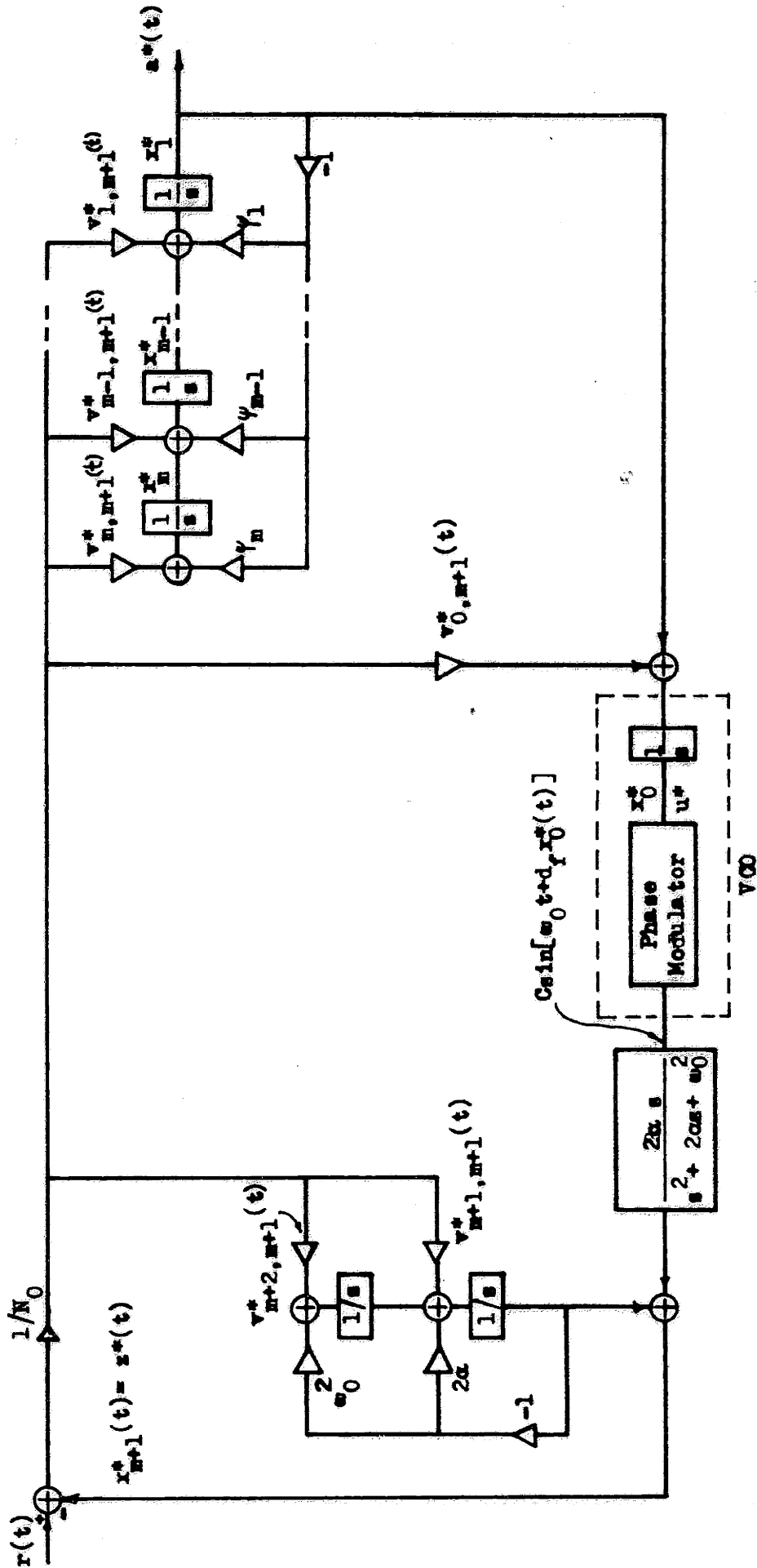
Figure 5.26 Quasi-Optimum Demodulator for FM Transmitted via a Fixed Channel with Memory

Example 5.4.2 Markovian Message, No Modulation, Additive White
Noise Channel

In this final example, we shall consider the simple model of
Fig. 5.27a which cannot be treated by the alternate MAP approach
to continuous estimation. Attention is restricted to the consideration
of a one-dimensional Markovian message, $x(t)$, observed without
modulation in white Gaussian noise. The equation describing $x(t)$ is:

$$\frac{d}{dt} x(t) = f[x(t)] + \xi(t) \tag{5.120}$$

where $\xi(t)$ is a white Gaussian process of spectral height X watts/cps.
If $f[x(t)] = -k\, x(t)$, then $x(t)$ is Gaussian and the model reduces to
the one-dimensional case of Example 5.2.1. Otherwise, $x(t)$ is
non-Gaussian and has a stationary amplitude probability density
given by:

$$p(x) = C \exp\left\{ \frac{2}{X} \int^{x} f(u)\, du \right\} \tag{5.121}$$

where C is a normalization constant and $f(x)$ is assumed to be
negative for large positive values of x and positive for large negative
values of x. Eq. 5.121 can be derived by use of the Fokker-Planck
equation, Eq. 2.3. This has been accomplished by Andronov, Pontryagin,
and Witt[48] and Barrett.[49]

The observed signal is:

$$r(t) = x(t) + n(t) \tag{5.122}$$

where $n(t)$ is a white Gaussian process of spectral height N_0 watts/cps.
Thus, $g[t:x(t)] = x(t)$ and $D[g(t:x)] = 1$.

The variance and processor equations, Eqn's. 4.1 and 4.2, become:
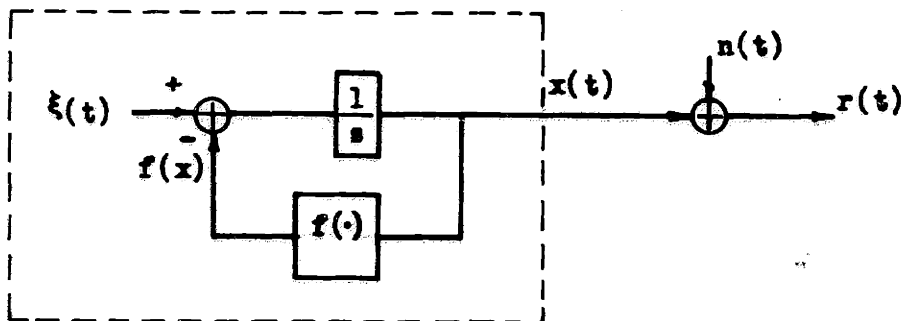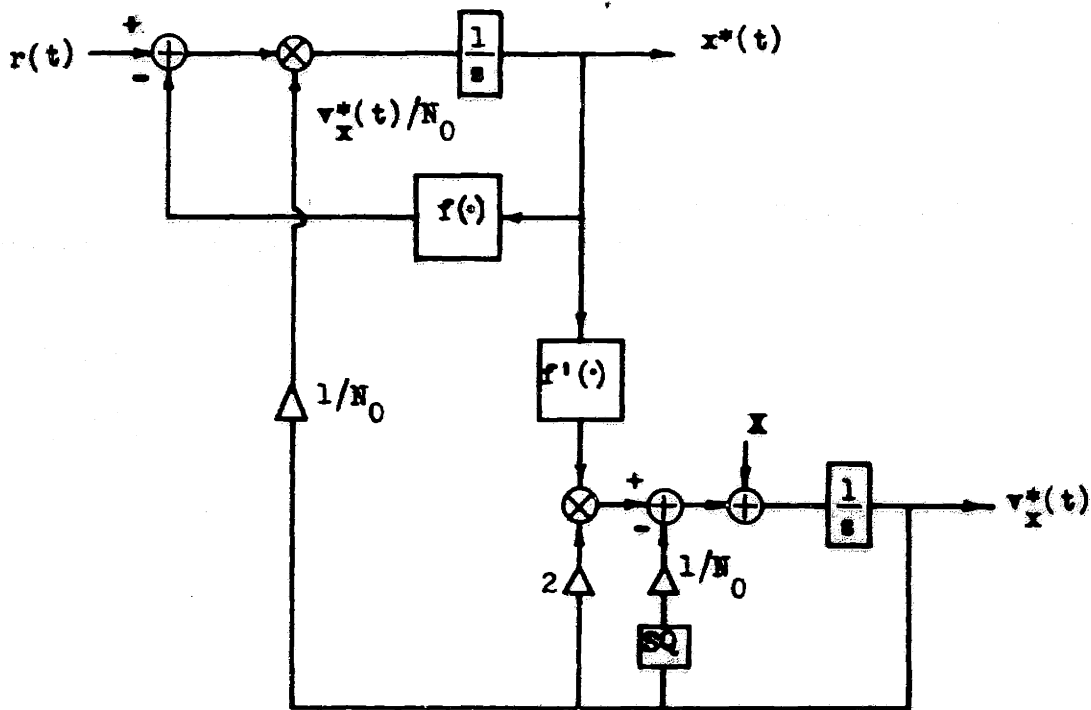
Figure 5.27 (a) Model for a One-Dimensional Markov Process Observed in White Noise (b) Quasi-Optimum Estimator

$$\frac{d}{dt} x^*(t) = f[x^*(t)] + \frac{1}{N_0} v_x^* \{r(t) - x^*(t)\} \qquad (5.123)$$

and

$$\frac{d}{dt} v_x^*(t) = 2 v_x^*(t) \frac{\partial}{\partial x^*} f[x^*(t)] + X - \frac{1}{N_0} v_x^{*2}(t) \qquad (5.124)$$

The system for simultaneously generating the quasi-optimum estimate, $x^*(t)$, and the error variance, $v_x^*(t)$, is shown in Fig. 5.27b.

5.5 Critique of the State-Variable Approach

The usefulness of the state-variable approach to continuous estimation has been illustrated by several examples, most of which can also be treated by the MAP approach as well. We note here several advantages and disadvantages of the state-variable approach compared to the MAP approach.

Some advantages are:

(1) Insight into the structure of the quasi-optimum demodulator is provided.

(2) An important advantage of Kalman-Bucy filtering over Wiener filtering is its amenability to numerical solutions. Similarly, in those instances where the variance equation has a linear equivalent (e.g., PM, FM), this is an advantage of the state-variable approach over the MAP approach.

(3) Realizable demodulators result directly.

(4) A class of non-Gaussian message and channel disturbances can be treated. In the communication theory context, it is not yet clear what usefulness this has. However, applications in control theory can be given. These arise when it is desired to estimate the state variables of a nonlinear, dynamic system based on noisy observations of the state variables.

Some disadvantages are:

(1) It is necessary that random processes and linear filtering be representable by equations of state. Thus, Gaussian processes with nonrational spectra cannot be treated. A particular linear operation which arises, for example, in array problems and cannot be treated directly is that of pure delay.

(2) In many instances, the algebraic manipulations are tedious, particularly when working with the variance equation.

(3) Some useful results can be more readily obtained from the

solution to the Wiener-Hopf equation (e.g., Eqn's. 5.17, 5.29, and 5.30).

(4)   The unrealizable filtering problem cannot be treated easily.

The state-variable and MAP approaches complement each other in the sense that from different criteria of optimality and different mathematical techniques, identical estimators are obtained.

VI. Analysis of the Performance of Quasi-Optimum

PM and FM Demodulators

In this chapter, we shall examine the performance of the quasi-optimum PM and FM demodulators derived in the examples of the preceding chapter. The PM case will be presented first. An exact analysis of a quasi-optimum phase estimator for a one-dimensional Gaussian message process is given; the probability density of the estimation error is derived. In the FM case, we first consider the performance for a class of message spectra -- the Butterworth class. A comparison is made between actual performance and information theoretical bounds on the performance. Curves indicating the performance under threshold and bandwidth limitations are given. We then present the results of a computer simulation of a quasi-optimum FM demodulator for estimating a one-dimensional Gaussian message.

The procedure for analyzing quasi-optimum PM and FM demodulators has been given by Van Trees[5,6] and Viterbi and Cahn.[45] We assume a familiarity with these procedures, particularly those discussed by Van Trees. These authors consider two types of quasi-optimum PM and FM demodulators. The first corresponds to zero-delay estimation of the message; our demodulators fall into this category. The second corresponds to infinite-delay estimation. The modification we require for the infinite-delay case is that of post-cascading our demodulators with unrealizable, linear filters. These filters can be realized approximately by using delays.

6.1 Analysis of Quasi-Optimum PM Demodulators

The quasi-optimum PM demodulator is shown in Fig. 5.12b. It has the base-band equivalent shown in Fig. 6.1. This equivalent has been derived by Van Trees[5,6,50] and Viterbi.[51] $n'(t)$ is uncorrelated with $a(t)$ and is a white Gaussian process of spectral height $2N_1/C^2$ watts/cps. The estimation error, $e_a = a(t) - a^*(t)$, has been minimized by the choice of the loop filter. The minimum-mean-square estimation error, σ_a^2, is given by Eq. 5.57 provided the conditions under which the demodulator was derived are satisfied. These conditions require that the channel
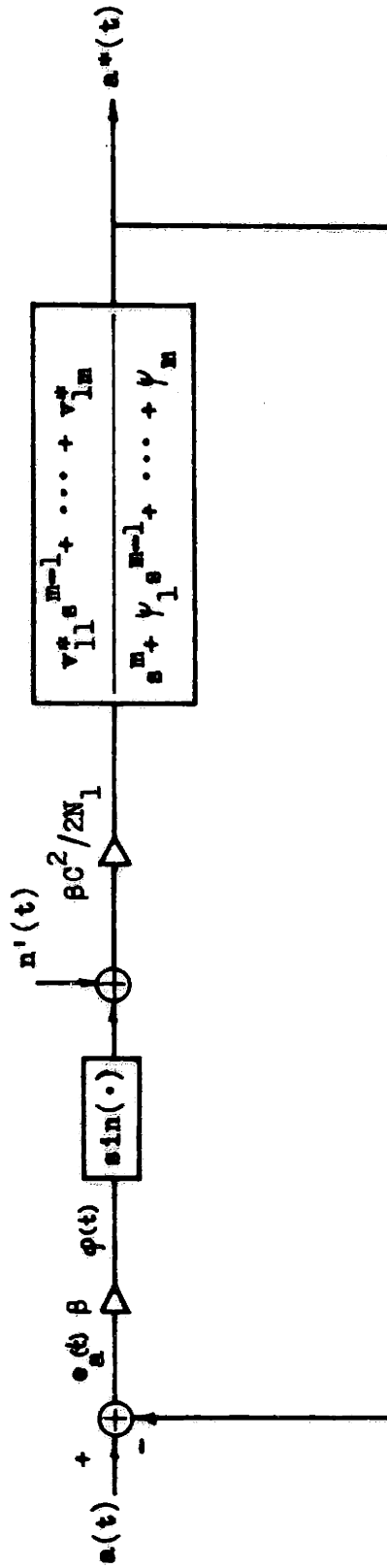
Figure 6.1 Baseband Equivalent of the Quasi-Optimum
PM Demodulator Shown in Figure 5.12b

signal-to-noise ratio, $C^2/2N_1W_c$, be large; W_c is the channel, or receiver-input, bandwidth. So long as this condition holds, the total phase error, φ, is small and, consequently, $\sin \varphi \approx \varphi$. As the channel signal-to-noise ratio decreases, the mean-square-error increases. Finally, threshold occurs at which point $\sin \varphi$ cannot be approximated by φ and Eq. 5.57 no longer describes the performance. Thus, σ_φ^2 is given by:

$$\sigma_\varphi^2 = \frac{2N_1}{C^2} \int_{-\infty}^{\infty} \log\left[1 + \frac{\beta^2 C^2}{2N_1} S_a(\omega^2)\right] \frac{d\omega}{2\Pi} \quad \text{provided } \sigma_\varphi^2 < \sigma_{cr}^2$$

(6.1)

σ_{cr}^2 is the critical value of σ_φ^2 where threshold occurs; it is roughly 0.5 rad.^{2}

Eq. 6.1 has been used by Viterbi and Cahn[45] to analyze the performance of quasi-optimum, zero-delay PM demodulators operating above threshold. The Butterworth class of message spectra was used.

We shall concentrate on the analysis of the PM demodulator for a first-order Butterworth message spectrum. This case was examined in Ex. 5.3.1.1. The demodulator is shown in Fig. 5.13b and its base-band equivalent in Fig. 6.2. $\xi(t)$ and $n'(t)$ are white Gaussian processes of spectral heights 1 and $2N_1/C^2$ watts/cps, respectively. The spectrum of $a(t)$ is $\dfrac{2k}{\omega^2+k^2}$ watts/cps. $a(t)$ has a normalized variance of unity.

The performance above threshold can be studied by evaluating Eq. 6.1 or by using Eq. 5.62. The result is:

$$\sigma_\varphi^2 = \frac{2\beta^2}{1 + \sqrt{1 + \beta^2 \Lambda}} \quad \text{for } \sigma_\varphi^2 < \sigma_{cr}^2$$

$$\Lambda = C^2/kN_1$$

(6.2)

Eq. 6.2 does not provide an accurate description of the performance in the vicinity of threshold and below. This region of operation has been studied by means of computer simulation by Zaorski.[51] We shall present an exact analysis of the performance. The analysis is valid in all regions
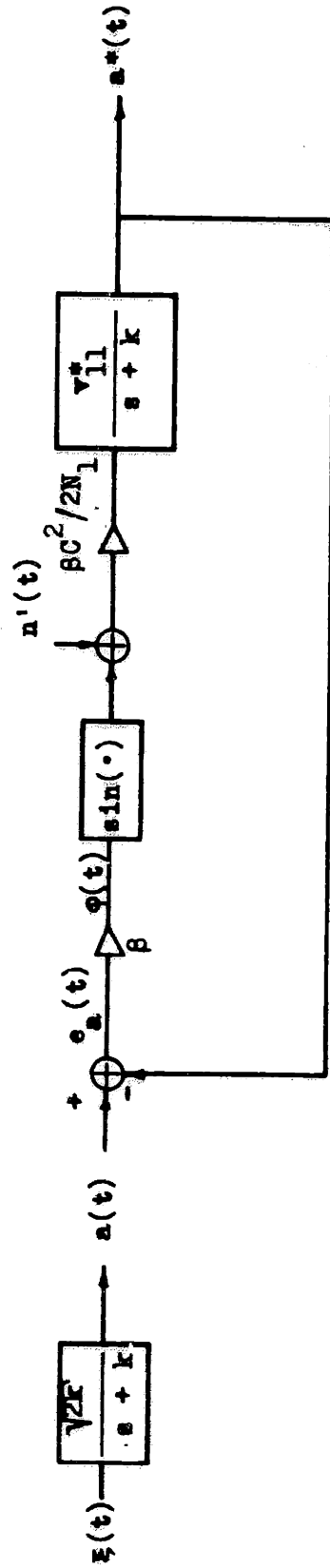
Figure 6.2 Baseband Equivalent of a Quasi-Optimum FM Demodulator
for a First-Order Butterworth Message Spectrum

$$v_{11}^* = \frac{2}{1+r}$$

$$r = \sqrt{1+\beta^2\Lambda} \quad , \quad \Lambda = C^2/kN_1$$

of operation including threshold and below.

The exact analysis is possible because $\varphi(t)$ is a one-dimensional Markov process for which the associated Fokker-Planck equation can be solved. Use of the Fokker-Planck equation in the study of phase demodulators originated with Tikhonov[53,54] in the USSR. His work has been discussed by Viterbi.[51] Tikhonov examined cases where there was no modulation (i.e., $\beta = 0$) and where the loop was not optimum. Our analysis extends their results to include modulation and optimum loops.

Referring to Fig. 6.2, we observe that the differential equation describing $a*(t)$ is:

$$\frac{d}{dt}a*(t) + k a*(t) = \frac{\beta C^2 v_{11}^*}{2N_1} [\sin \varphi + n'(t)] \tag{6.3}$$

where

$$v_{11}^* = \frac{2}{1+\gamma} = \frac{2}{1 + \sqrt{1 + \beta^2 \Lambda}} , \quad \Lambda = C^2/kN_1$$

In terms of the total phase error, $\varphi(t)$, we have $a*(t) = a(t) - \frac{1}{\beta} \varphi(t)$.

Substituting this expression into Eq. 6.3 and using:

$$\frac{d}{dt}a(t) + k a(t) = \sqrt{2k}\, \xi(t) \tag{6.4}$$

we obtain:

$$\frac{d}{dt}\varphi(t) = - k \varphi(t) - \frac{\beta^2 C^2 v_{11}^*}{2N_1} \sin \varphi(t) + \lambda(t) \tag{6.5}$$

where

$$\lambda(t) = \beta\sqrt{2k}\, \xi(t) - \frac{\beta^2 C^2 v_{11}^*}{2N_1} n'(t)$$

is a white Gaussian process having a spectral height $\dfrac{4\beta^2 k \gamma}{1+\gamma}$ watts/cps where $\gamma = \sqrt{1 + \beta^2 \Lambda}$. As described by Eq. 6.5, $\varphi(t)$ is seen to be a

one-dimensional Markov process. Consequently, $p(\varphi)$, the steady-state probability density of the total phase error, satisfies the Fokker-Planck equation, Eq. 2.3:

$$\frac{d}{d\varphi}\left[k\varphi + \frac{k\beta^2\Lambda}{1+\gamma}\sin\varphi\right]p(\varphi) + \frac{2\beta^2 k\gamma}{1+\gamma}\frac{d^2}{d\varphi^2}p(\varphi) = 0 \qquad (6.6)$$

with the boundary conditions, $p(\pm\infty) = 0$, and the normalization requirement, $\int_{-\infty}^{\infty}p(\varphi)\,d\varphi = 1$. Integrating and using the boundary conditions, we obtain:

$$p(\varphi) = C\,\exp\left\{-\frac{1+\gamma}{4\beta^2\gamma}\varphi^2 + \frac{\Lambda}{2\gamma}\cos\varphi\right\} \qquad (6.7)$$

The constant, C, can be determined by using the normalization requirement and the expansion:[38]

$$\exp\left\{\frac{\Lambda}{2\gamma}\cos\varphi\right\} = \sum_{\nu=-\infty}^{\infty}I_\nu\left(\frac{\Lambda}{2\gamma}\right)\cos\nu\varphi \qquad (6.8)$$

where $I_\nu(\cdot)$ is a Bessel function of order ν. The final result is:

$$p(\varphi) = \frac{\exp\left\{-\frac{1+\gamma}{4\beta^2\gamma}\varphi^2 + \frac{\Lambda}{2\gamma}\cos\varphi\right\}}{2\beta\sqrt{\frac{\Pi\gamma}{1+\gamma}}\sum_{\nu=-\infty}^{\infty}I_\nu\left(\frac{\Lambda}{2\gamma}\right)\exp\left\{-\frac{\beta^2\gamma}{1+\gamma}\nu^2\right\}} \qquad (6.9)$$

Some plots of $p(\varphi)$ for $\varphi \geq 0$ and different values of β and $\Lambda = C^2/kN_1$ are given in Fig. 6.3. The following observations can be made:

(1) $p(\varphi)$ is not periodic. Consequently, measurement of φ modulo 2Π is not meaningful.

(2) The central lobe of $p(\varphi)$ is always larger than the side lobes. This implies that the error has a tendency to return to zero when cycles are skipped.
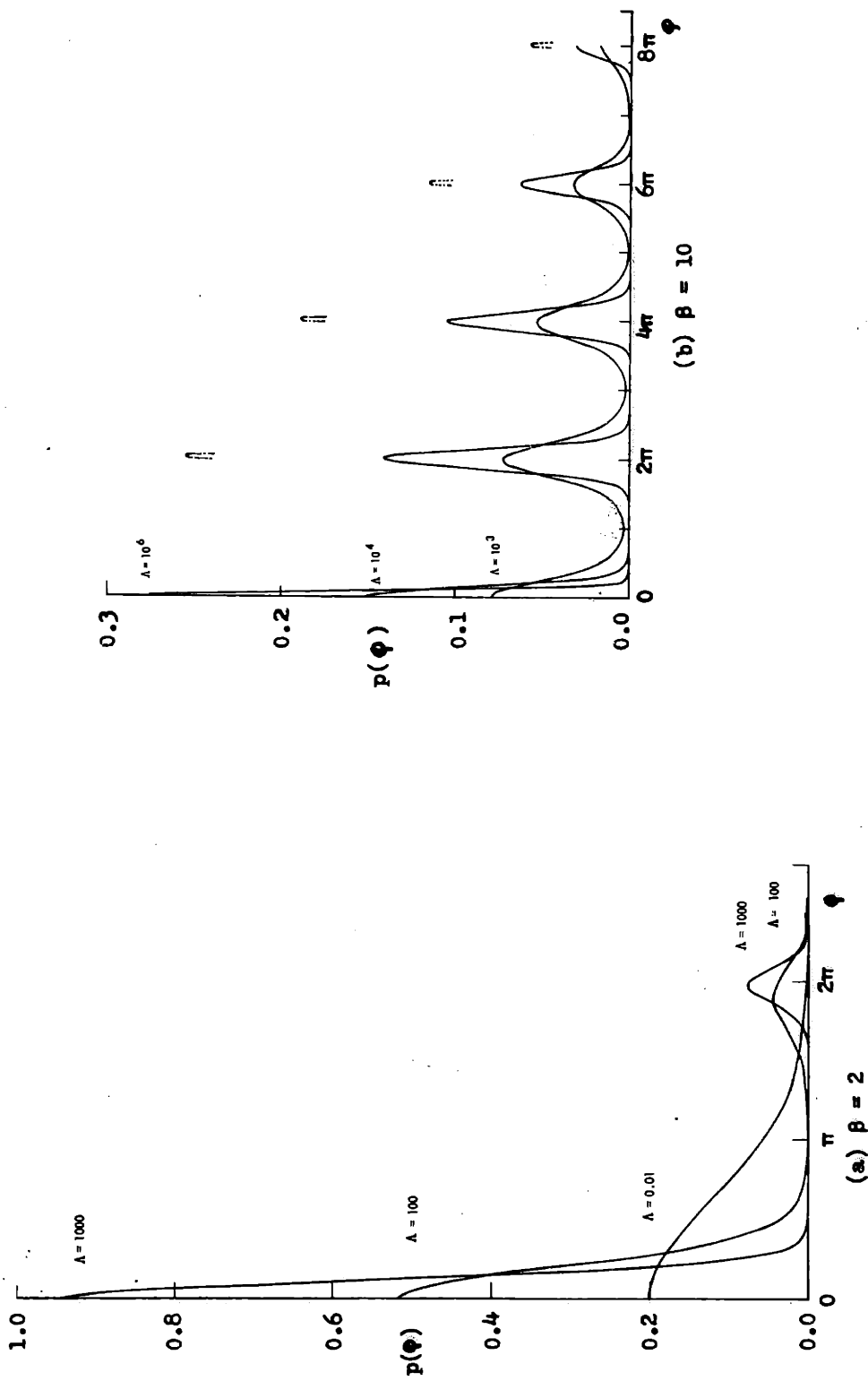
Figure 6.3 Probability Density of the Total-Phase Estimation Error for an Optimum Phase Demodulator and a First-Order Butterworth Message Spectrum

(3) $p(\varphi)$ has a Gaussian envelope with variance $\dfrac{2\beta^2\gamma}{1+\gamma}$. Therefore, $p(\varphi)$ has only a central lobe for small index (β small) PM. This implies that no cycle skipping or threshold behavior will be exhibited for small index PM.

(4) For large signal-to-noise ratio, the central lobe of $p(\varphi)$ is Gaussian with variance $2\beta^2/(1+\gamma) = v_{11}^* \beta^2$.

6.2 Analysis of Quasi-Optimum FM Demodulators

The quasi-optimum FM demodulator is shown in Fig. 5.14b. It has the base-band equivalent shown in Fig. 6.4. The equivalent is derived in a fashion parallel to that used in the PM case. $\xi(t)$ and $n'(t)$ are uncorrelated white Gaussian processes. The spectral height of $n'(t)$ is $2N_0/C^2$ watts/cps.

Two errors are of interest: (i) $e_a = a(t) - a^*(t)$, the error in the estimation of the message; (ii) $\varphi(t) = d_f e_u(t) = d_f[u(t) - u^*(t)]$, the error in estimating the total phase. The mean-square values of $e_a(t)$ and $\varphi(t)$ are given by Eqn's 5.73 and 5.72, respectively, provided the conditions under which the demodulator was derived are satisfied. These require that the channel signal-to-noise ratio be large. As in the phase-modulation case, threshold occurs when the signal-to-noise ratio decreases and the mean-square value of $\varphi(t)$, σ_φ^2, reaches some critical value, σ_{cr}^2, which is roughly $1/4$ rad.^{2} (this value is based on experimental results presented in Sec. 6.2.2.1.) Eqn's. 5.72 and 5.73 no longer describe the performance below threshold. Rewriting the equations, we have:

$$\sigma_\varphi^2 = \frac{2N_0}{C^2} f(0) \qquad\qquad \text{provided } \sigma_\varphi^2 \leq \sigma_{cr}^2 \approx 1/4 \ \text{rad.}^2 \qquad (6.10)$$

and

$$\sigma_a^2 = \frac{2N_0}{3d_f^2 C^2} f^3(0) + F(0) \qquad \text{provided } \sigma_\varphi^2 \leq \sigma_{cr}^2 \approx 1/4 \ \text{rad.}^2 (6.11)$$
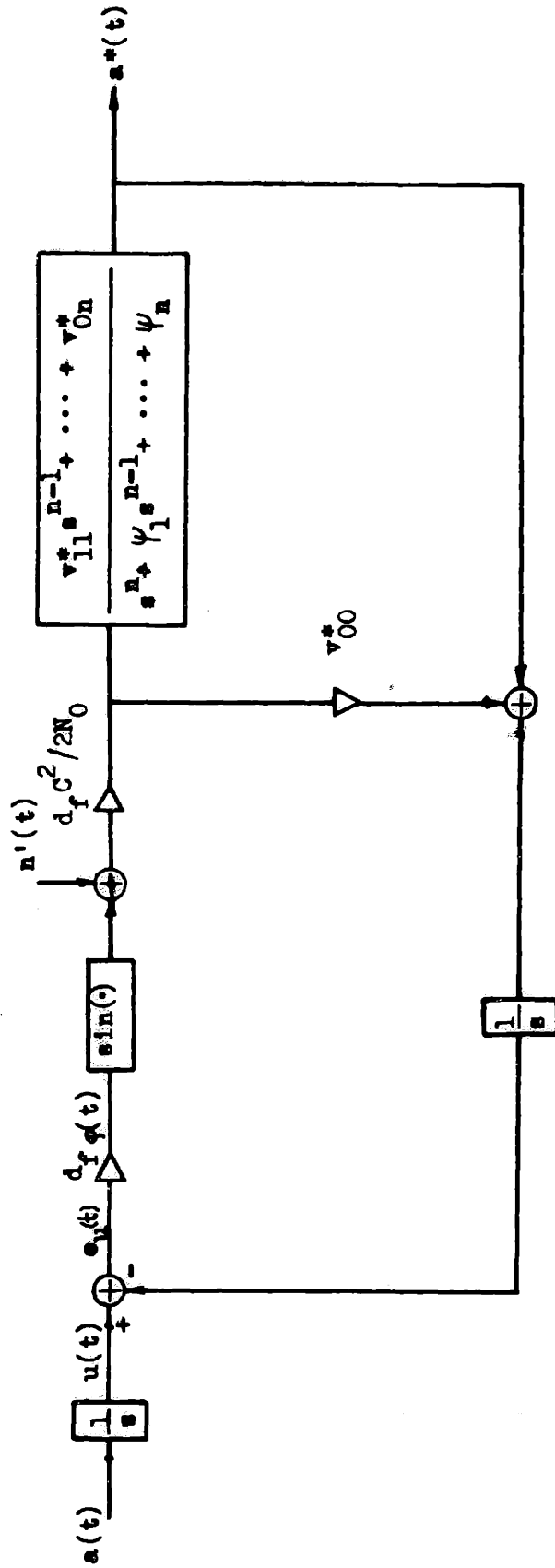
where

Figure 6.4 Baseband Equivalent of the Quasi-Optimum FM
Demodulator Shown in Figure 5.14b

$$f(0) = \int_{-\infty}^{\infty} \log \left[1 + \frac{d_f^2 C^2}{2N_0} \frac{S_a(\omega^2)}{\omega^2} \right] \frac{d\omega}{2\Pi}$$

and

$$F(0) = \int_{-\infty}^{\infty} \frac{2N_0}{d_f^2 C^2} \omega^2 \log \left[1 + \frac{d_f^2 C^2}{2N_0} \frac{S_a(\omega^2)}{\omega^2} \right] \frac{d\omega}{2\Pi}$$

Eq. 6.11 is the mean-square estimation-error for the zero-delay, quasi-optimum FM demodulator we have derived. An equation for the mean-square error of the corresponding infinite-delay demodulator has been given by Van Trees.[5,6] It is:

$$\sigma_a^2 \bigg|_{\substack{\text{inf.} \\ \text{del.}}} = \int_{-\infty}^{\infty} \frac{S_a(\omega^2) \dfrac{2N_0}{d_f^2 C^2} \omega^2}{S_a(\omega^2) + \dfrac{2N_0}{d_f^2 C^2} \omega^2} \frac{d\omega}{2\Pi} \qquad \text{provided } \sigma_\varphi^2 \le \sigma_{cr}^2 \approx 1/4 \text{ rad.}^2$$

$$(6.12)$$

6.2.1 Performance of Quasi-Optimum FM Demodulators for a Class of Message Spectra

We have evaluated Eqn's. 6.10 and 6.11 numerically for the Butterworth class of message spectra. For this class:

$$S_a(\omega^2) = \frac{1/W_n}{\left(\dfrac{\omega}{k}\right)^{2n} + 1} \qquad \text{watts/cps} \qquad (6.13)$$

where

$$W_n = \frac{k}{2n \sin(\Pi/2n)} \qquad \text{cps.} \qquad (6.14)$$

is the equivalent rectangular bandwidth of the message. Eqn's. 6.10 and

6.11 now become:

$$\sigma^2_{\varphi,n} = \frac{1}{\Lambda} \frac{\sin(\Pi/2n)}{(\Pi/2n)} \int_0^\infty \log\left[1 + \frac{\Lambda\beta^2}{x^2(x^{2n}+1)}\right] dx \qquad \sigma^2_{\varphi,n} \lesssim \sigma^2_{cr} \approx 1/4 \text{ rad.}^2$$

(6.15)

and

$$\sigma^2_{a,n} = \frac{1}{\Lambda\beta^2} \frac{\sin(\Pi/2n)}{(\Pi/2n)} \frac{1}{3\Pi^2} \left[\int_0^\infty \log\left[1 + \frac{\Lambda\beta^2}{x^2(x^{2n}+1)}\right] dx\right]^3 +$$

(6.16)

$$\frac{1}{\Lambda\beta^2} \frac{\sin(\Pi/2n)}{(\Pi/2n)} \int_0^\infty x^2 \log\left[1 + \frac{\Lambda\beta^2}{x^2(x^{2n}+1)}\right] dx$$

$$\sigma^2_{\varphi,n} \lesssim \sigma^2_{cr} \approx 1/4 \text{ rad.}^2$$

where $x = \omega/k$, $\Lambda = C^2/2N_0 W_n$, and $\beta = d_f/k$. Λ is the signal-to-noise ratio in the message bandwidth (equivalent rectangular) and β is the modulation index. Eq. 6.15 can be evaluated for $n = 1$ and $n = \infty$. The results are:

$$\sigma^2_{\varphi,1} = \frac{4\beta\Lambda^{-\frac{1}{2}}}{1 + \sqrt{1 + 2\beta\Lambda^{\frac{1}{2}}}} \qquad \text{(from Eq. 5.79)} \qquad (6.17)$$

$$\sigma^2_{\varphi,\infty} = \frac{1}{\Lambda}\left[\log(1 + \Lambda\beta^2) + 2\beta\Lambda^{\frac{1}{2}} \tan^{-1}\frac{1}{\beta\Lambda^{\frac{1}{2}}}\right] \qquad \text{(Ref. 43)} \quad (6.18)$$

qWe have plotted $1/\sigma^2_{\varphi,n}$ for $n = 1, 2, 5,$ and ∞ in Fig's. 6.5a - 6.8a.

A threshold constraint of one-forth is indicated. The curves accurately describe the performance when $1/\sigma^2_{\varphi,n}$ is above the constraint level.

We have plotted $1/\sigma^2_{a,n}$ for $n = 1, 2, 5,$ and ∞ in Fig's. 6.5b - 6.8b.

Figure 6.5a   Inverse Mean-Square Error in Estimation of Phase for
a Quasi-Optimum FM Demodulator (First-Order Butterworth Message
Spectrum)

$$\Lambda = \frac{c^2}{2N_0 W_1} = \frac{c^2}{N_0 k} \quad \text{(SNR in Message Bandwidth)}$$

Figure 6.5b A Comparison of Inverse Mean-Square Error in Estimation
of Frequency for Zero- and Infinite-Delay, Quasi-Optimum FM
Demodulators and the Information Theoretical Bound on the Inverse
Mean-Square Error Using Any Modem. The above threshold performance
for fixed β and zero delay is also indicated. (First-Order
Butterworth Message Spectrum)

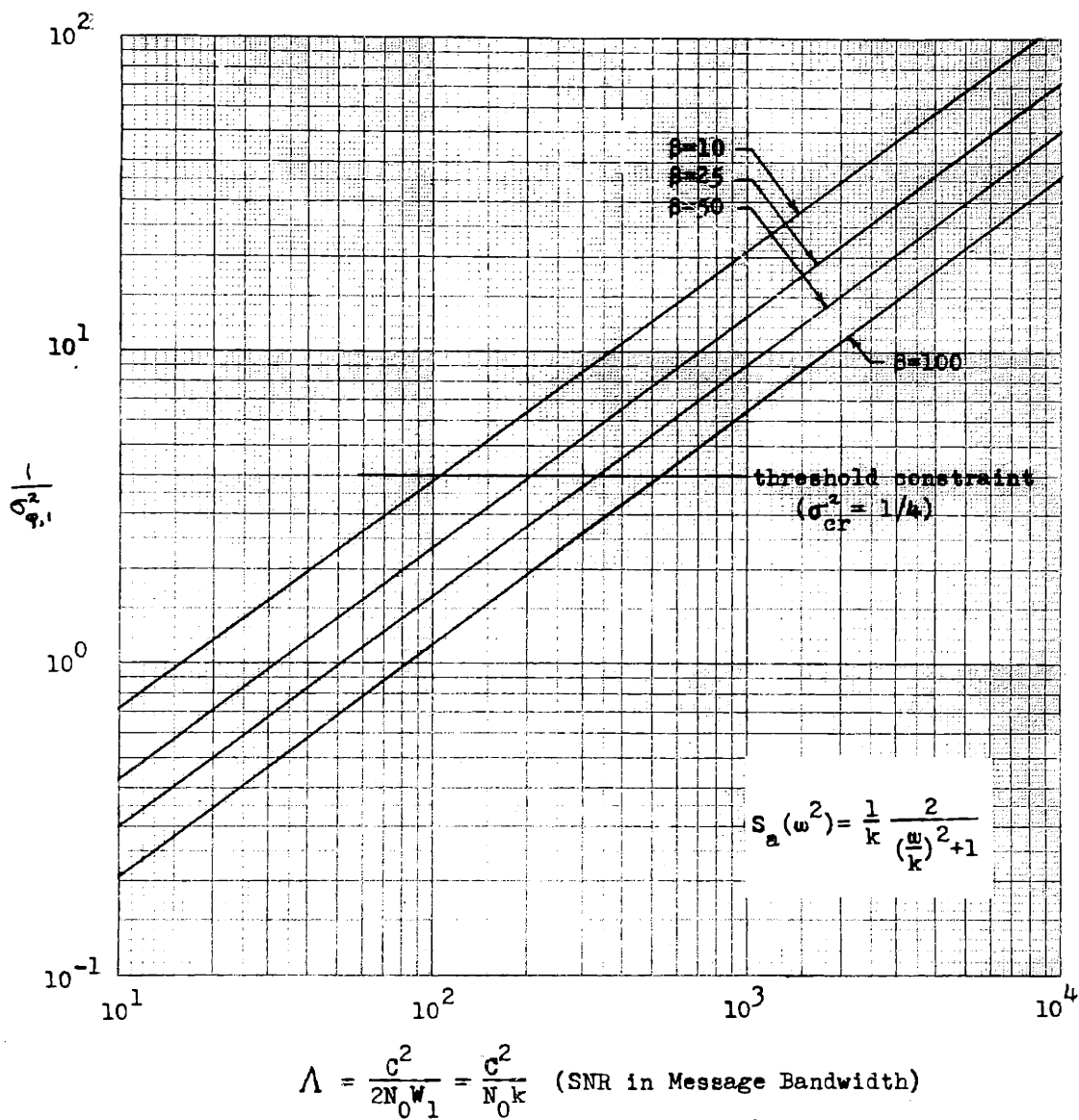$$\Lambda = \frac{c^2}{2N_0W_2} = \frac{c^2\sqrt{2}}{N_0k} \quad \text{(SNR in Message Bandwidth)}$$

Figure 6.6a   Inverse Mean-Square Error in Estimation of Phase for a Quasi-Optimum FM Demodulator (Second-Order Butterworth Message Spectrum)

$$\Lambda = \frac{c^2}{2N_0W_2} = \frac{c^2\sqrt{2}}{N_0k} \quad \text{(SNR in Message Bandwidth)}$$
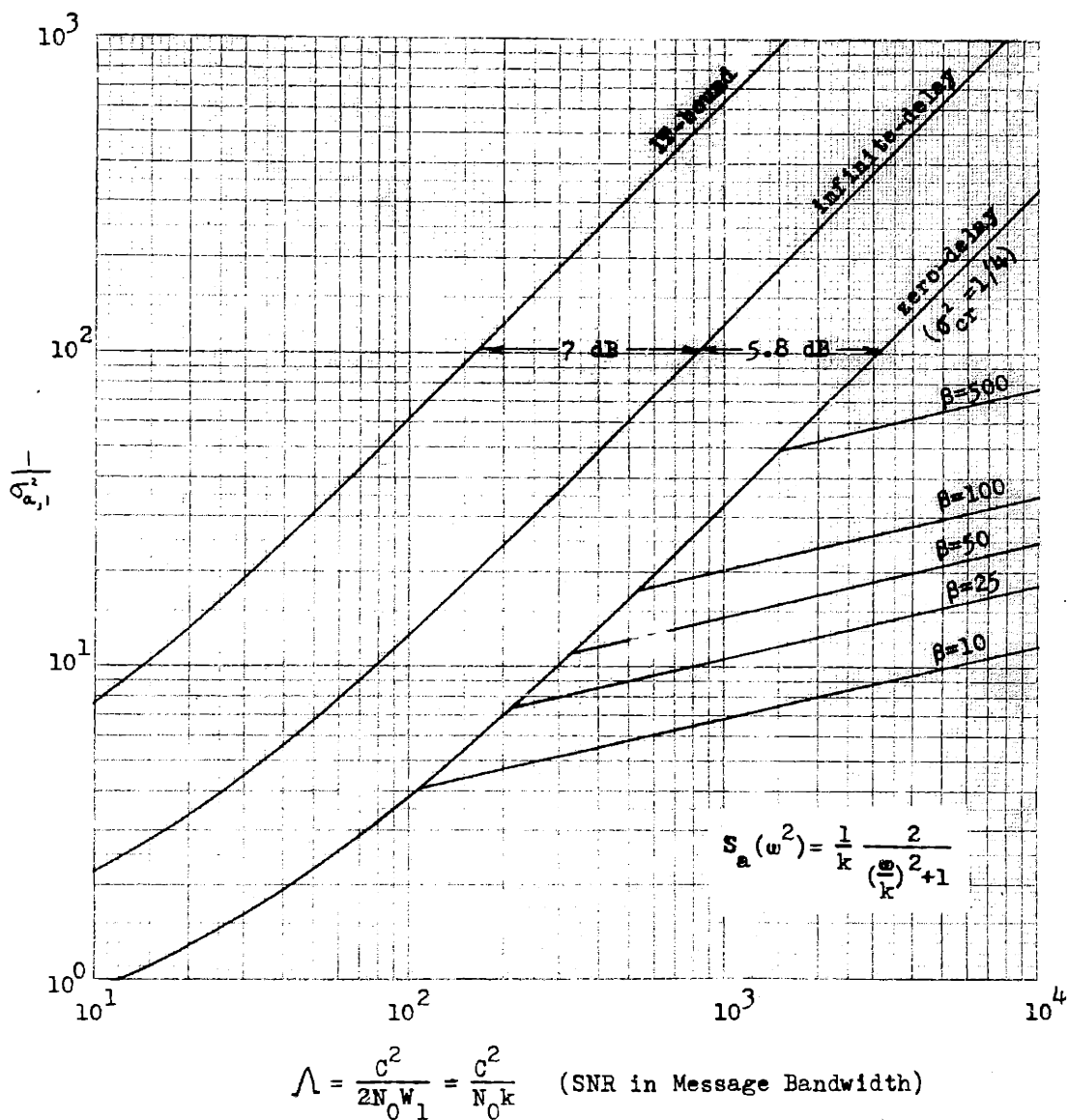
Figure 6.6b A Comparison of Inverse Mean-Square Error in Estimation of Frequency for Zero- and Infinite-Delay, Quasi-Optimum FM Demodulators and the Information Theoretical Bound on the Inverse Mean-Square Error Using Any Modem. The above threshold performance for fixed β and zero delay is also indicated. (Second-Order Butterworth Message Spectrum)

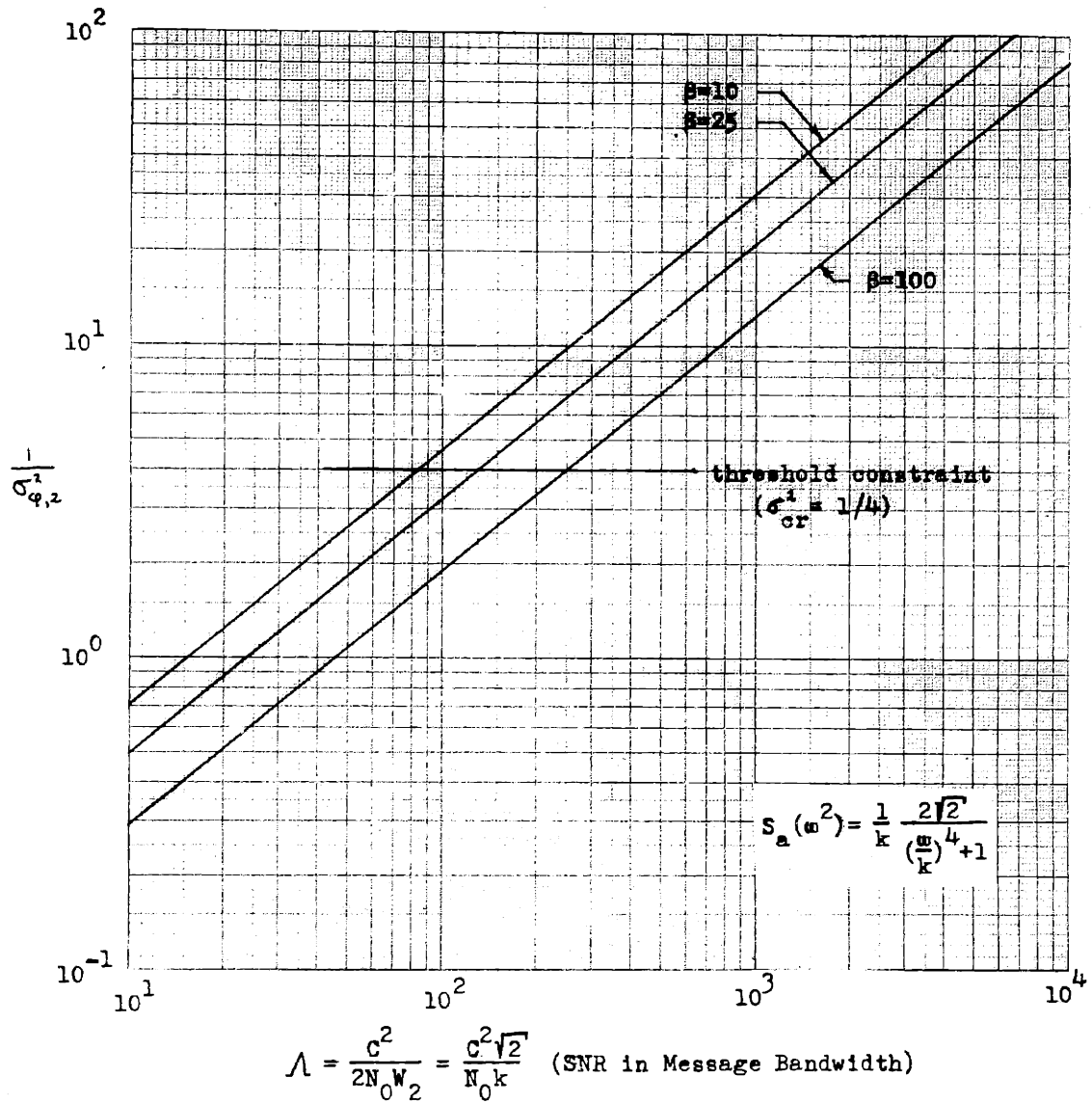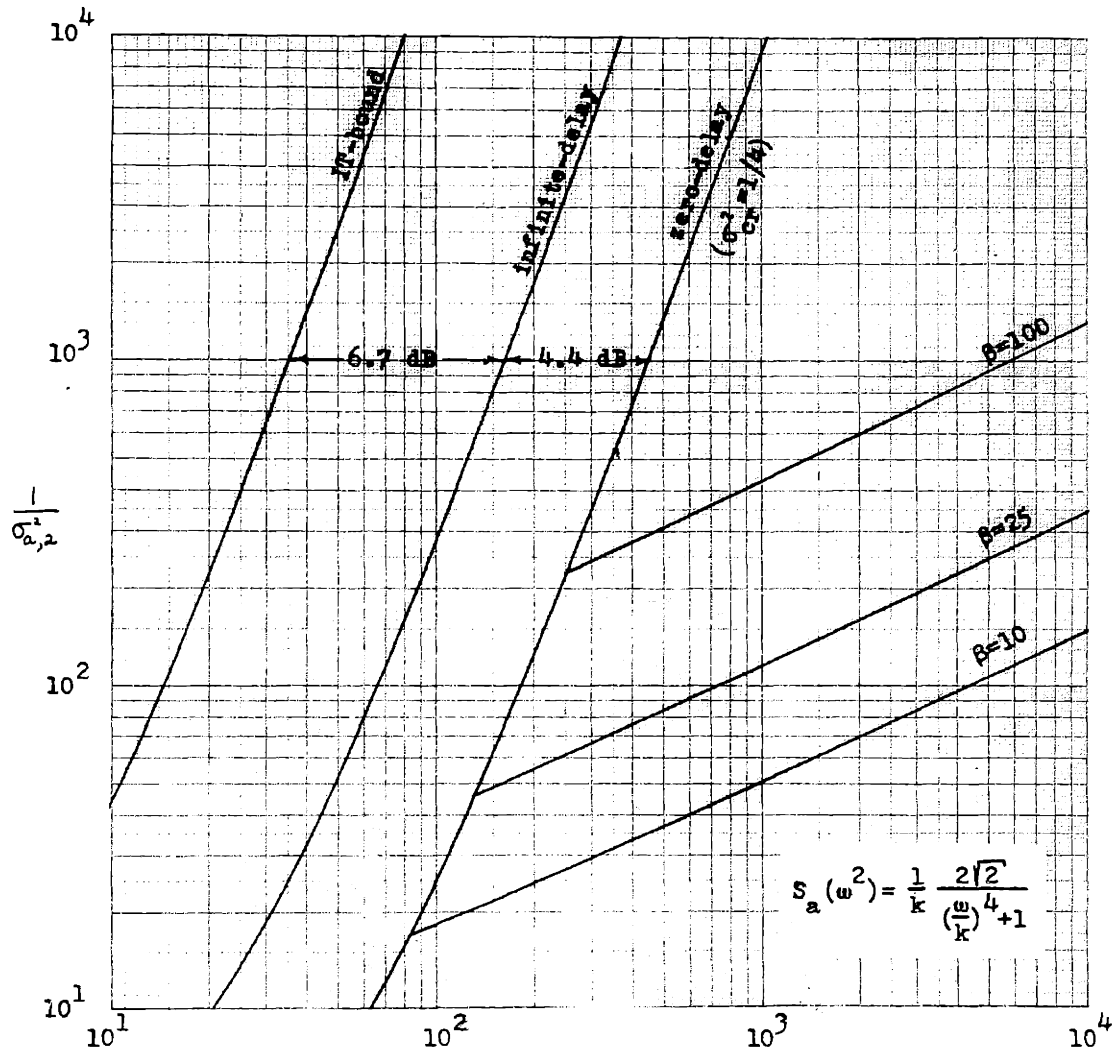$$\Lambda = \frac{c^2}{2N_0W_5} = \frac{c^2(1.55)}{N_0k} \quad \text{(SNR in Message Bandwidth)}$$

Figure 6.7a Inverse Mean-Square Error in Estimation of Phase for a Quasi-Optimum FM Demodulator (Fifth-Order Butterworth Message Spectrum)

$$\Lambda = \frac{c^2}{2N_0 W_5} = \frac{c^2(1.55)}{N_0 k} \quad \text{(SNR in Message Bandwidth)}$$

Figure 6.7b A Comparison of Inverse Mean-Square Error in Estimation of Frequency for Zero- and Infinite-Delay, Quasi-Optimum FM Demodulators and the Information Theoretical Bound on the Inverse Mean-Square Error Using Any Modem. The above threshold performance for fixed β and zero delay is also indicated. (Fifth-Order Butterworth Message Spectrum)

$$\Lambda = \frac{c^2}{2N_0 W_\infty} = \frac{c^2 \pi}{2N_0 k} \quad \text{(SNR in Message Bandwidth)}$$
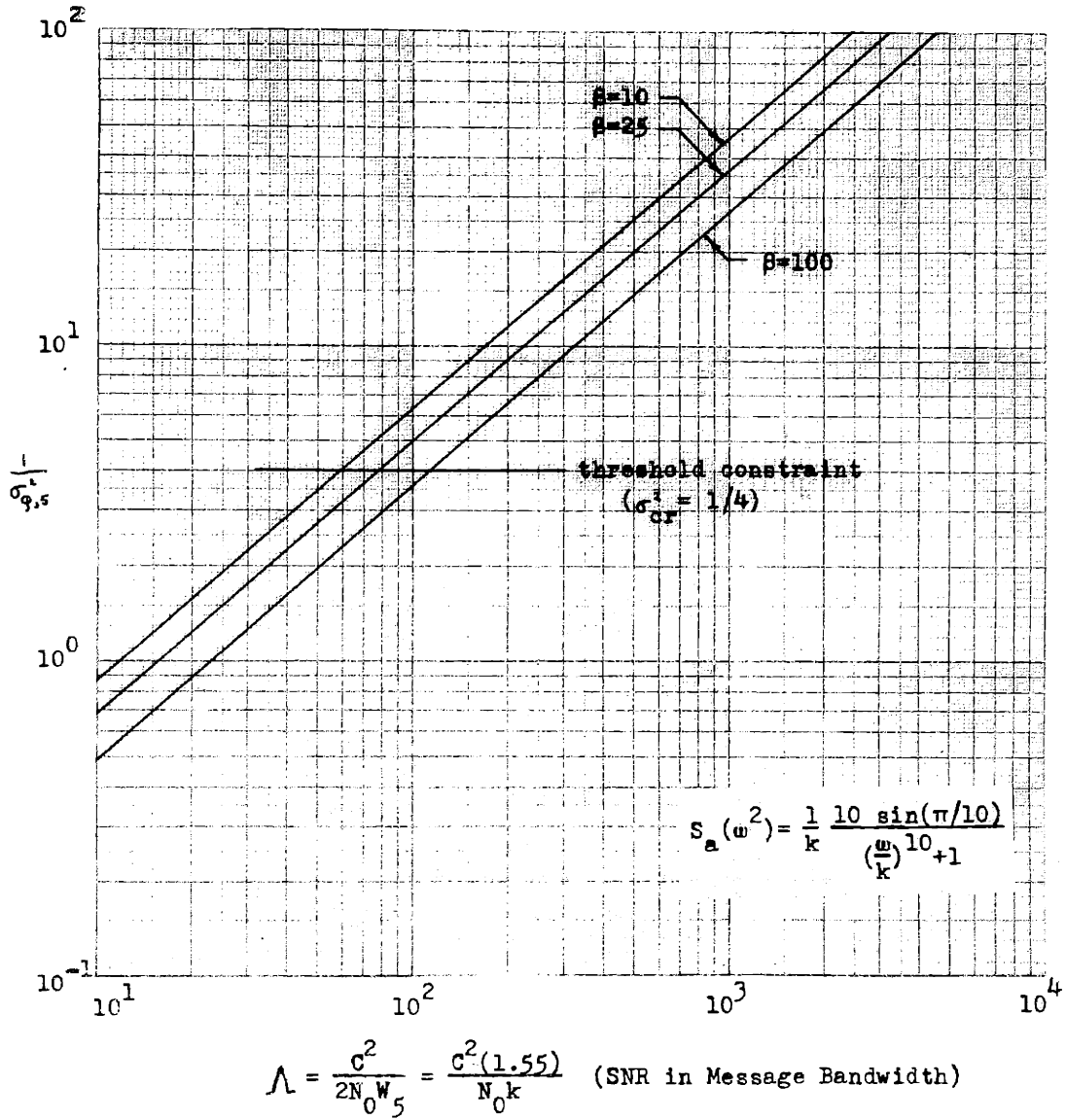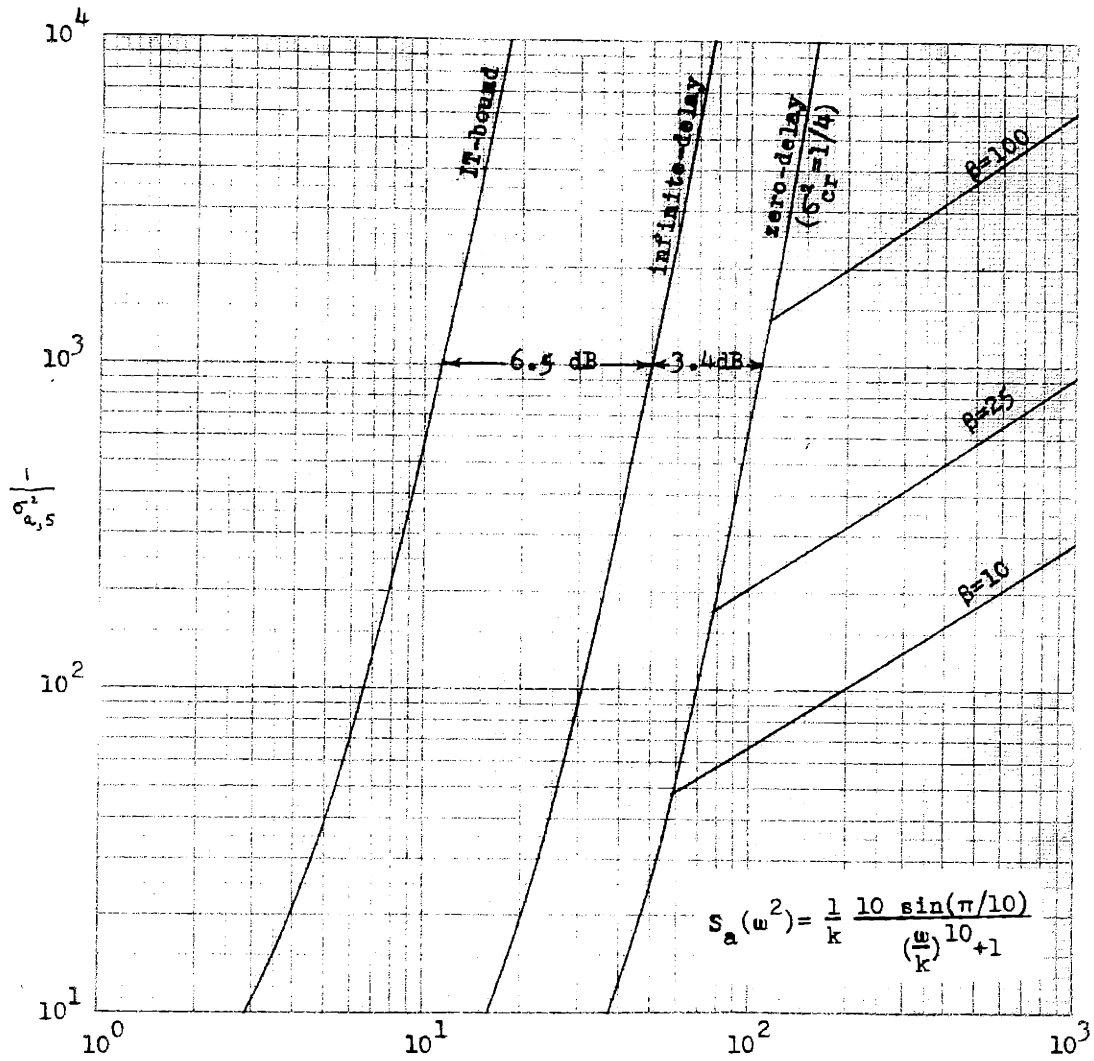
Figure 6.8a   Inverse Mean-Square Error in Estimation of Phase for a Quasi-Optimum FM Demodulator (Infinite-Order Butterworth Message Spectrum)

$$\Lambda = \frac{C^2}{2N_0 W_\infty} = \frac{C^2 \pi}{2N_0 k} \quad \text{(SNR in Message Bandwidth)}$$

Figure 6.8b A Comparison of Inverse Mean-Square Error in Estimation of Frequency for Zero- and Infinite-Delay, Quasi-Optimum FM Demodulators and the Information Theoretical Bound on the Inverse Mean Square-Error Using Any Modem. The above threshold performance for fixed β and zero delay is also indicated. (Infinite-Order Butterworth Message Spectrum)
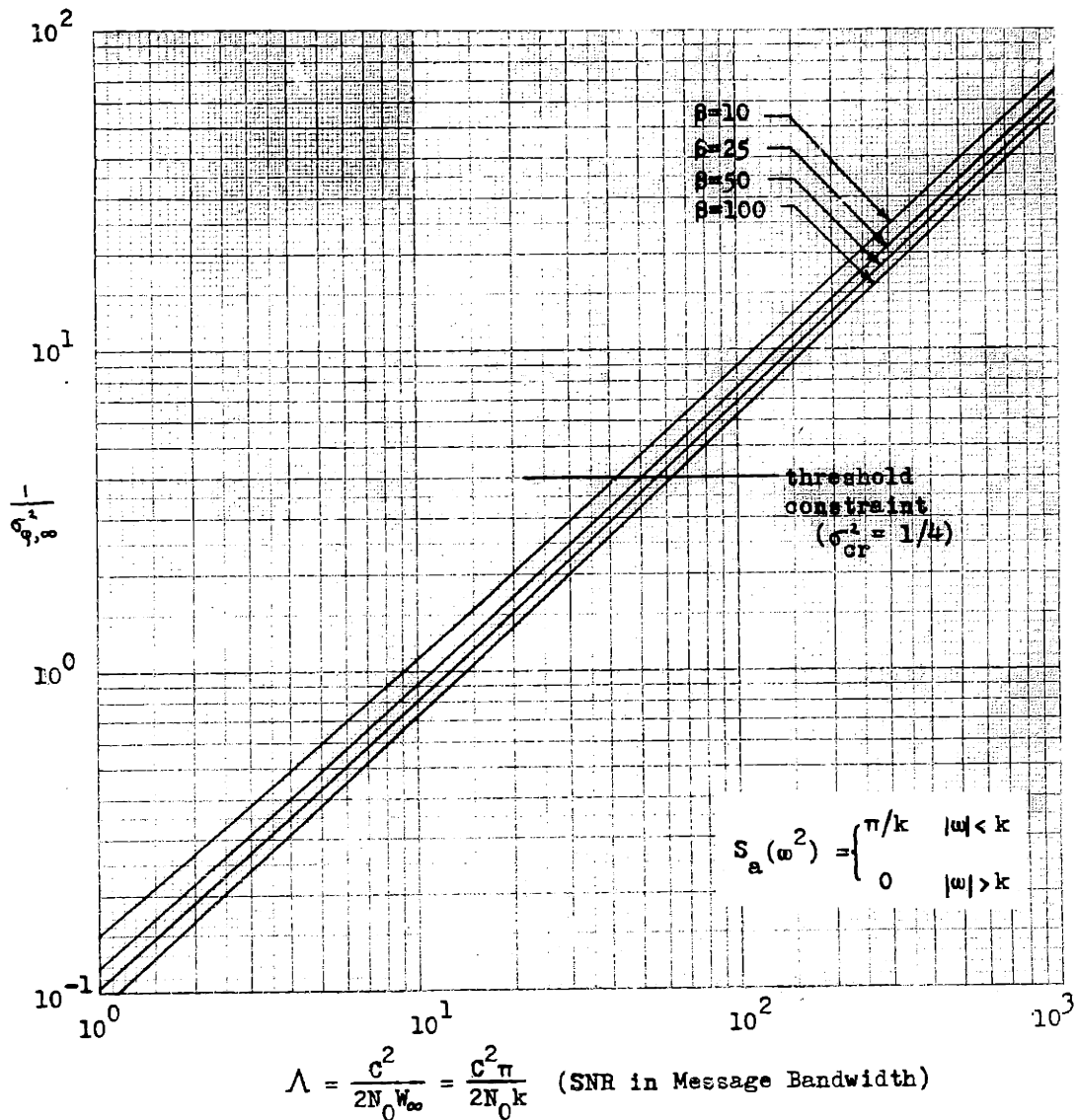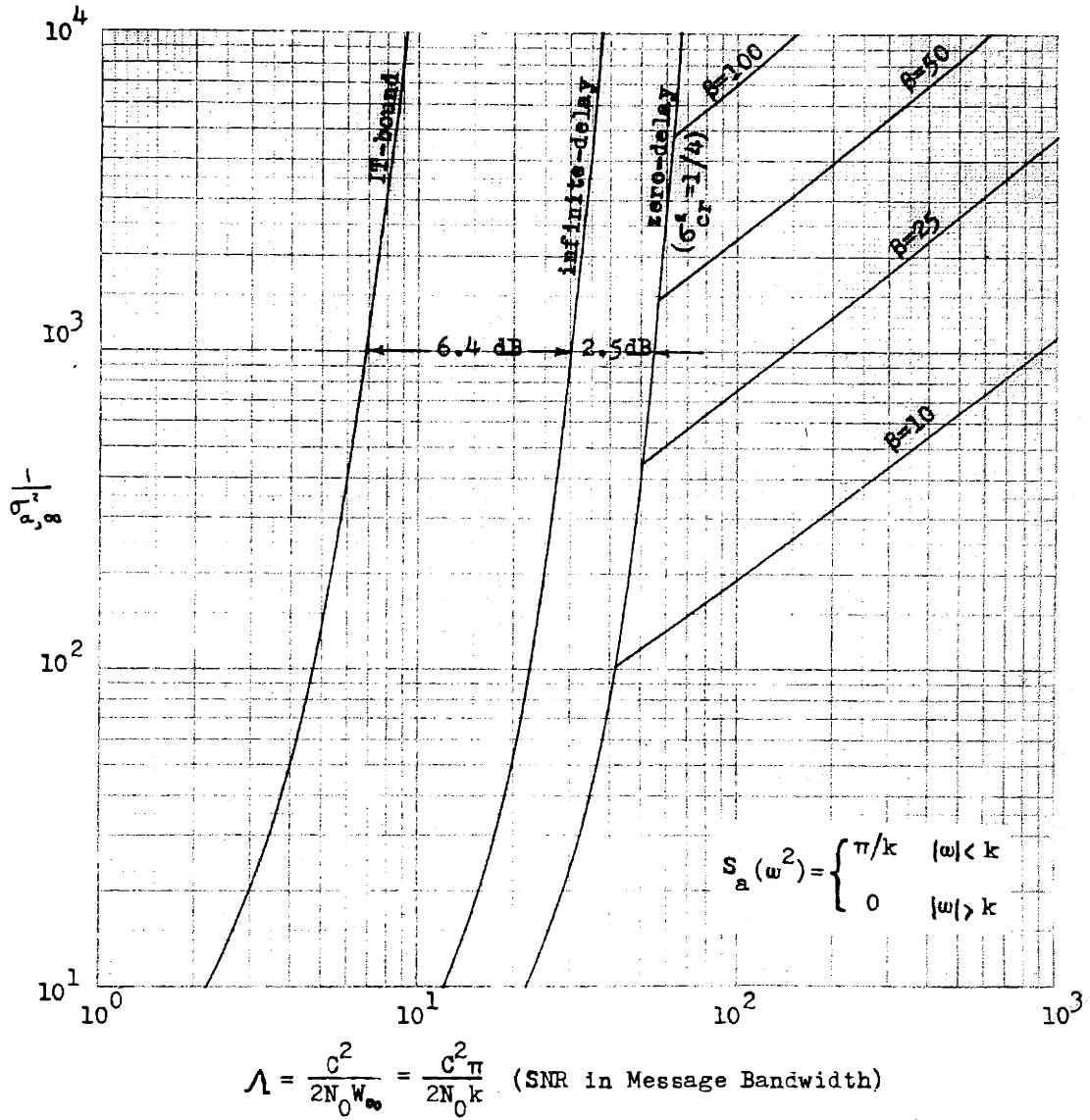
(The appropriate curve is the one appearing on the extreme right in each figure; the other curves will be explained below.) $1/\sigma_{a,n}^2$, for fixed β, is shown only for values of Λ corresponding to above threshold performance. The critical value of Λ for each β is determined from the $1/\sigma_{\varphi,n}^2$ curves; these values are connected to form the threshold constraint line labeled "zero-delay." The following observations can be made:

(i) From the $1/\sigma_{a,n}^2$ curves, the slope of the fixed β lines increases from about 0.25 for n= 1 to about 0.77 for n= ∞. The slope of the fixed β lines determines the rate at which increasing Λ will improve the performance when operating under a fixed bandwidth constraint.

(ii) From the $1/\sigma_{a,n}^2$ curves, the slope of the threshold constraint line increases from about 1 for n= 1 to about 12 for n=∞. The slope of the threshold constraint line determines the rate at which increasing Λ will improve the performance when no bandwidth constraint exists.

(iv) For any given value of β and Λ, the performance improves as n increases.

For the Butterworth class of message spectra, the equation for the mean-square error in the infinite-delay case, Eq. 6.12, becomes:

$$\sigma_{a,n}^2\bigg|_{\substack{inf.\\del.}} = \frac{\sin(\Pi/2n)}{(\Pi/2n)} \int_0^\infty \frac{x^2}{x^{2(n+1)} + x^2 + \Lambda\beta^2} \, dx \qquad (6.19)$$

$$\sigma_{\varphi,n}^2 \leq \sigma_{cr}^2 \approx 1/4 \ \text{rad.}^2$$

where x, Λ, and β are as defined previously. Eqn. 6.19 can be evaluated for n =1 and n= ∞. The results are:

$$\sigma_{a,1}^2\bigg|_{\substack{inf.\\del.}} = \frac{1}{\sqrt{1 + 2\beta\Lambda^{\frac{1}{2}}}} \qquad \sigma_{\varphi,1}^2 \leq \sigma_{cr}^2 \approx 1/4 \ \text{rad.}^2 \qquad (6.20)$$

and

$$\sigma_{a,\infty}^2 \bigg|_{\substack{\text{inf.} \\ \text{del.}}} = 1 - \beta\Lambda^{\frac{1}{2}}\tan^{-1}\frac{1}{\beta\Lambda^{\frac{1}{2}}} \qquad \sigma_{\varphi,\infty}^2 \leq \sigma_{cr}^2 \approx 1/4 \text{ rad.}^2$$

(6.21)

We have plotted $1/\sigma_{a,n}^2$ for $n = 1, 2, 5,$ and ∞ for the infinite-delay case in Fig's. 6.5b- 6.8b. For clarity, only the threshold performance line, labeled "infinite-delay," is shown. The amount of threshold improvement which can be attained by adding delay is evident; it ranges from 6 dB. for $n = 1$ to 2.5 dB. for $n = \infty$.

Goblick[55] has presented information-theoretical bounds on the performance attainable with any modem used for communicating Gaussian messages whose spectra are of the Butterworth class. Van Trees[56] has discussed the use of these bounds for evaluating the performance of angle-modulation schemes. We have included the bounds in Fig's. 6.5b-6.8b. It is seen that the actual performance of a zero-delay, quasi-optimum FM demodulator ranges from about 13 dB., for $n = 1$, to about 9 dB., for $n = \infty$, away from the theoretical bound. These values are based on the particular threshold constraint level of $1/4$ rad.^{2} which we have choosen to match experimental results mentioned in Sec. 6.2.2.1. The values increase as the constraint level is made smaller.

6.2.2 Performance of a Quasi-Optimum FM Demodulator for a First-Order Butterworth Message Spectrum: Simulation Results

In this section, we shall present the results of a computer simulation of the FM demodulator derived in Ex. 5.3.2.1 and shown in Fig. 5.15b. The baseband equivalent for the demodulator is shown in Fig. 6.9. $\xi(t)$ and n'(t) are uncorrelated white Gaussian processes with spectral heights of unity and $2N_0/C^2$ watts/cps, respectively. As mentioned in the discussion of the example, the two time-varying gains, $v_{00}^*(t)$ and $v_{01}^*(t)$, were generated in two ways. The first way was by simulating Eqn. 5.77, the coupled variance equation. The equation is shown in block diagram form in Fig. 6.10a; the baseband equivalent is shown in Fig. 6.10b. n''(t)
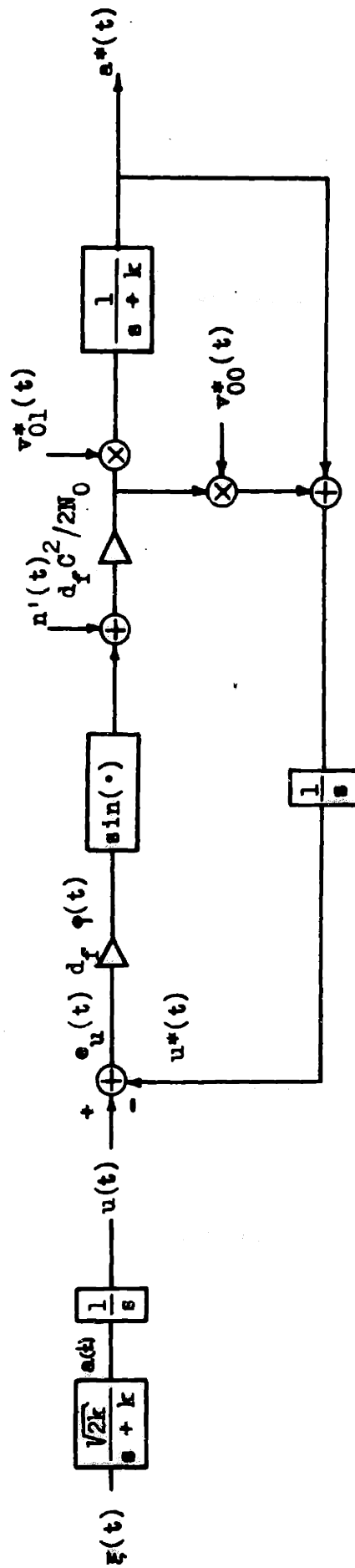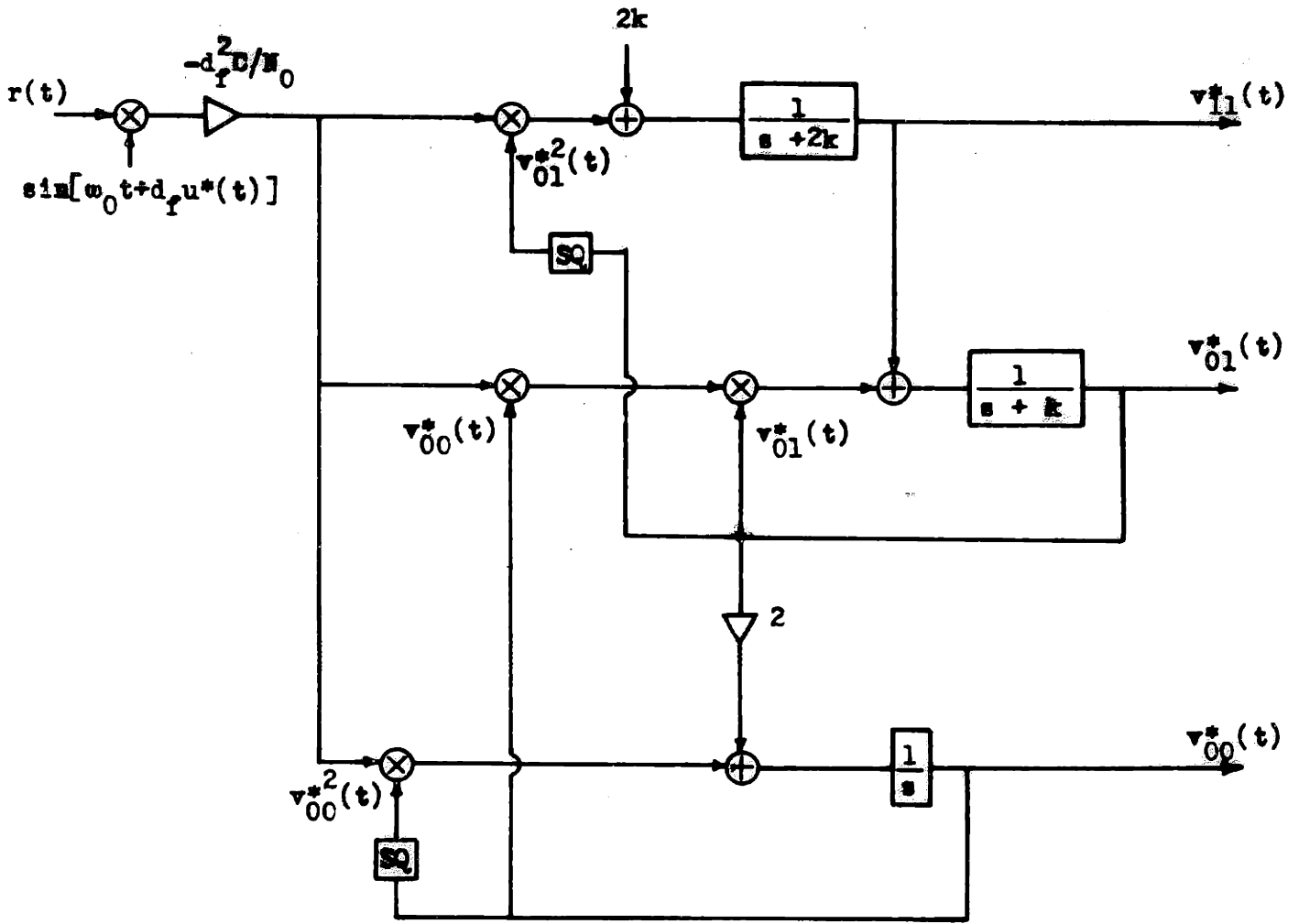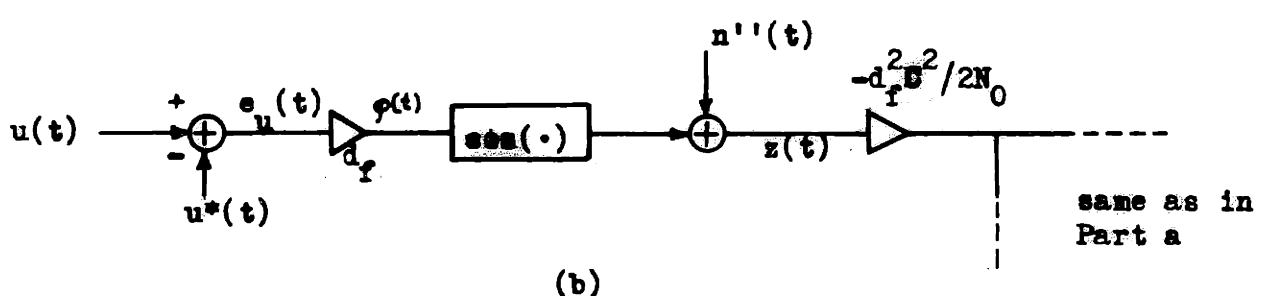
Figure 6.9 Quasi-Optimum FM Demodulator used in Simulation Study

(a)

(b)

Figure 6.10 (a) Processor for Generating Solution to the
Coupled Variance Equation   (b) Its Baseband Equivalent

is a white Gaussian process which is uncorrelated with $\xi(t)$ and $n'(t)$; it has a spectral height of $2N_0/C^2$ watts/cps. The baseband equivalent of the variance equation is coupled to that of the demodulator by $\varphi(t)$, the error in estimating the total phase, $d_f u(t)$. The second way of generating the two time-varying gains was by simulating Eqn. 5.78, the uncoupled variance equation. The solution to the equation can be produced by setting $z(t)$, of Fig. 6.10b, equal to unity, in which case the generation of the gains is uncoupled from the demodulator.

Three cases are of interest:

(i) The first is that of studying the performance of the quasi-optimum FM demodulator when the uncoupled variance equation is used and steady-state conditions exist.

(ii) The second is that of studying the performance when the uncoupled variance equation is used and transient conditions exist. In this instance, the variance equation can be simulated either simultaneously with the demodulator or in advance.

(iii) The third is that of studying the performance when the coupled variance equation is used and transient conditions exist. The variance equation must be simulated simultaneously with the demodulator.

6.2.2.1 Performance with the Uncoupled Variance Equation: Steady-State

In this case, the gains, $v_{00}^*(t)$ and $v_{01}^*(t)$, of Fig. 6.9 are constants which are given by Eqn's. 5.79 and 5.80.

It can be demonstrated that the error in estimating the message, $e_a(t) = a(t) - a^*(t)$, and the error in estimating the total phase, $\varphi(t) = d_f[u(t) - u^*(t)]$, form a two-dimensional Markov process. For this purpose, we derive the differential equations describing the two errors. Following the procedure used to derive Eq. 6.5, we obtain:

$$\frac{d}{dt} e_a(t) = -k\, e_a(t) - \frac{v_{01}^* d_f C^2}{2N_0} \sin \varphi(t) + \left[\sqrt{2k}\, \xi(t) - \frac{v_{01}^* d_f D^2}{2N_0} n'(t) \right]$$

$$\frac{d}{dt} \varphi(t) = d_f e_a(t) - \frac{v_{00}^* d_f^2 C^2}{2N_0} \sin \varphi(t) - \left[\frac{v_{00}^* d_f^2 C^2}{2N_0} n'(t) \right]$$

The bracketed expression in each equation is a white Gaussian process. The two-dimensional vector with $e_a(t)$ and $\varphi(t)$ as its components satisfies an equation of the form of Eq. 2.1 and is, therefore, a two-dimensional Markov process. Unfortunately, the corresponding Fokker-Planck equation for the joint probability density, $p(e_a, \varphi)$, appears to be analytically intractable.

Zaorski[52] has simulated the demodulator for the steady-state case and we have reproduced his results in Fig's. 6.11a and 6.11b. We have also superimposed the theoretically derived performance curves of Fig's. 6.5a and 6.5b on his results. The theoretical curves match the actual performance curves very well above threshold.

6.2.2.2 Performance with the Uncoupled Variance Equation: Transient Case

We have simultaneously simulated the demodulator and the uncoupled variance equation in the transient case. We assumed that at the initial observation time, $t_0 = 0$, the message was known to be zero, $a(0) = 0$. The appropriate initial condition for the variance equation is then $V^*(0) = 0$. The transient solution for two components of the uncoupled variance equation, $v_{00}^*(t)$ and $v_{11}^*(t)$, are shown in Fig's. 6.12-6.15* as the smooth curves. It is seen that the steady-state solution is reached in about one-forth the message correlation time. This accounts for the fact that the long-term ($\sim$150 message correlation times) performance we observed was identical to that observed by Zaorski.

6.2.2.3 Performance with the Coupled Variance Equation: Transient Case

We have also simultaneously simulated the demodulator and the coupled variance equation in the transient case. The same initial conditions were used. The transient solution for $v_{00}^*(t)$ and $v_{11}^*(t)$ is shown as the rapidly varying curves in Fig's. 6.12-6.15. The

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* The small discontinuities in the curves are due to truncation errors.
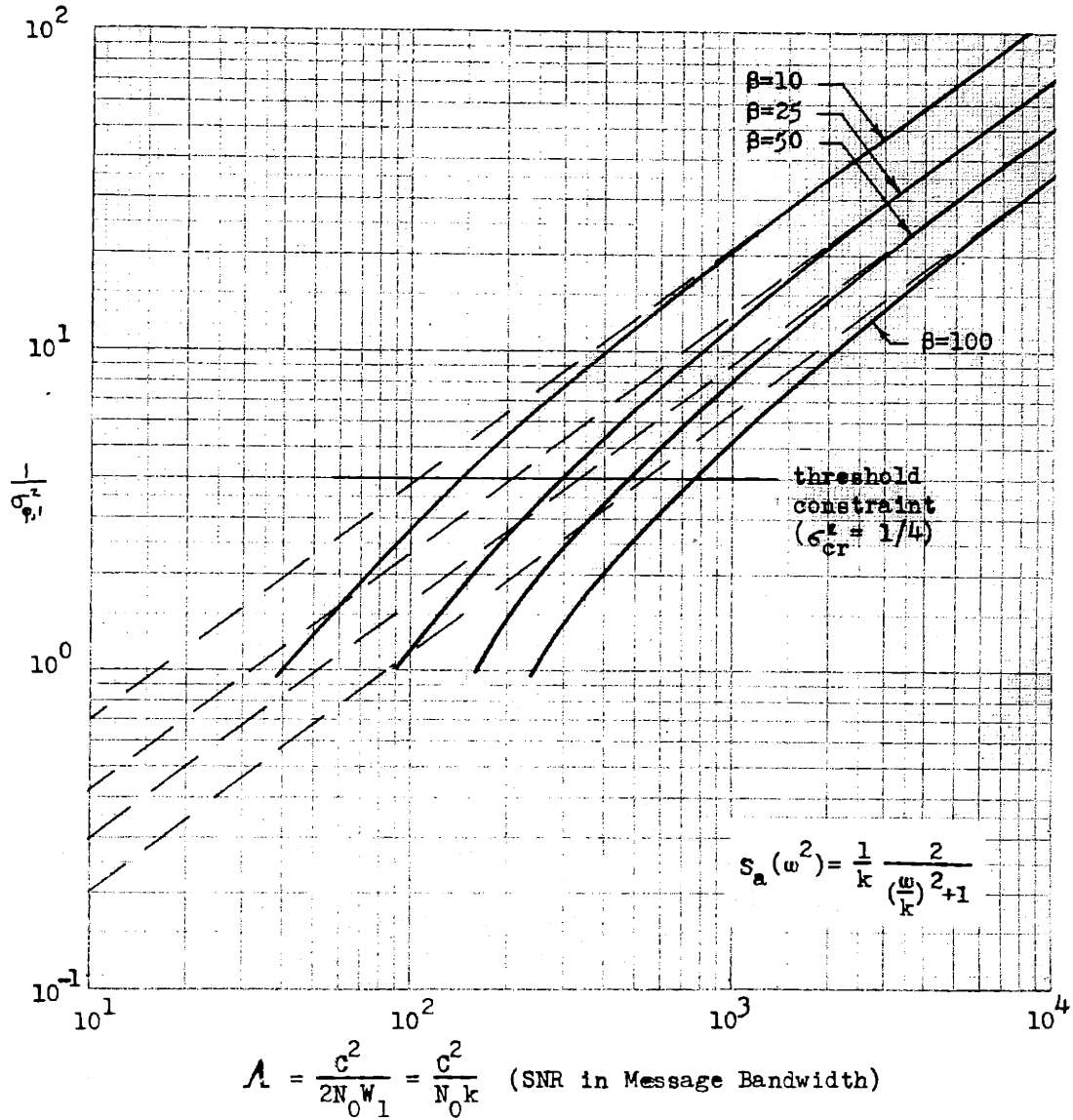
Figure 6.11a **Solid Lines:** Experimental Performance of a Quasi-Optimum FM Demodulator in Steady-State (from Zaorski[52])
Dashed Lines: Theoretical Performance (from Fig. 6.5a)
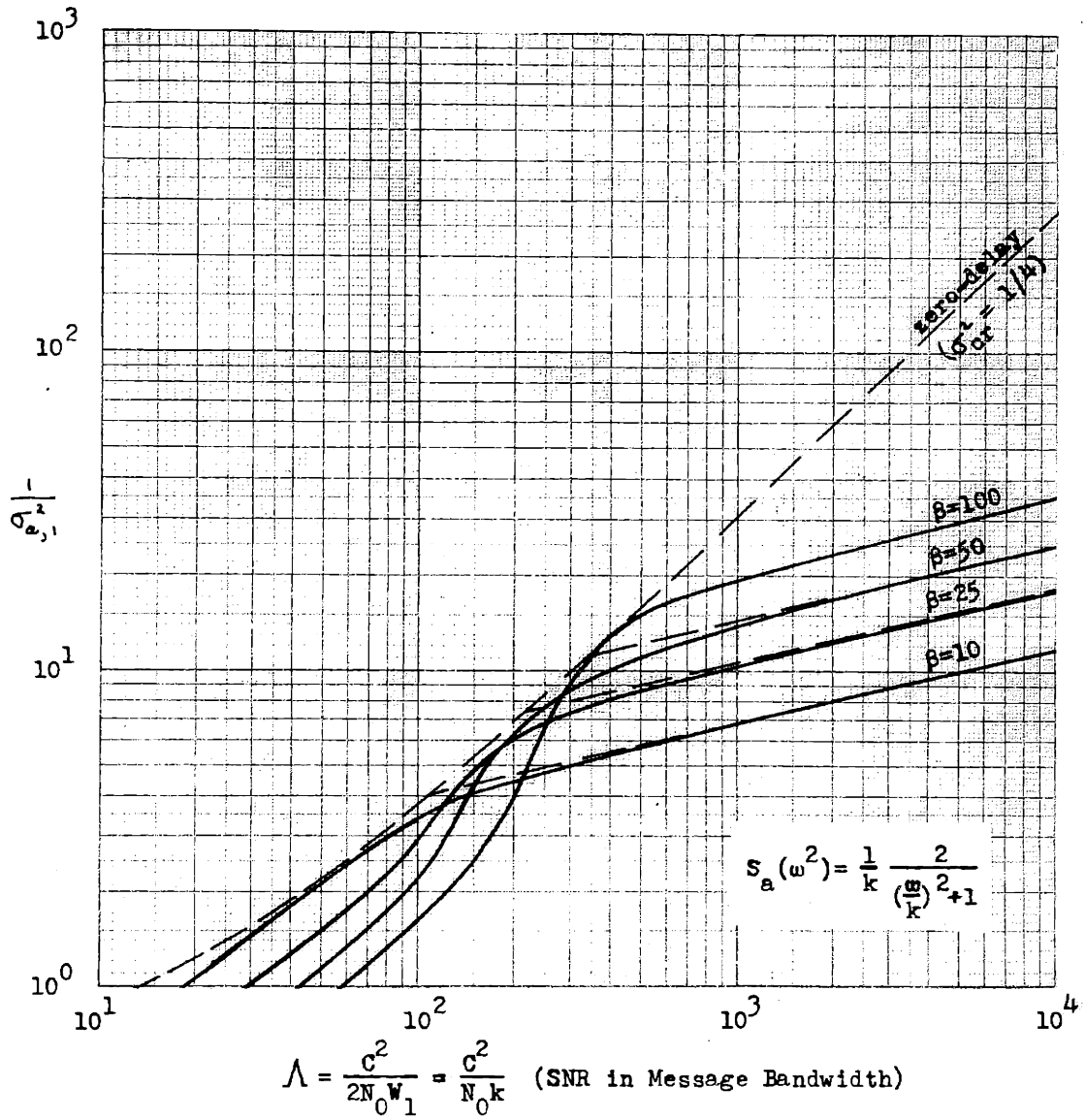
Figure 6.11b Solid Lines: Experimental Performance of a Quasi-
Optimum FM Demodulator in Steady-State (from Zaorski[52])
Dashed Lines: Theoretical Performance (from Fig. 6.5b)
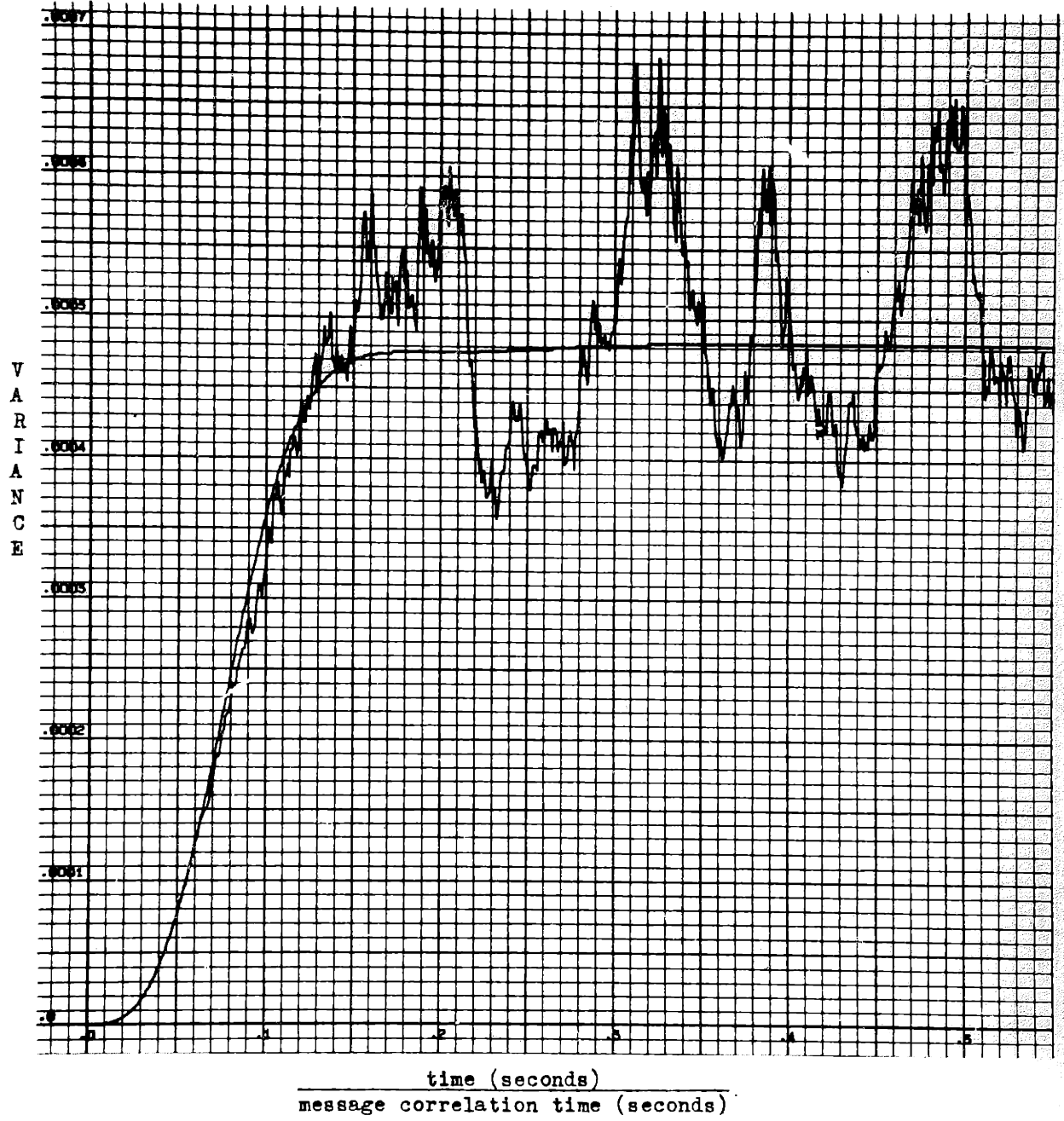
Figure 6.12 $v_{00}^{*}(t)$ for $\beta = 10$ and $\Lambda = 1000$
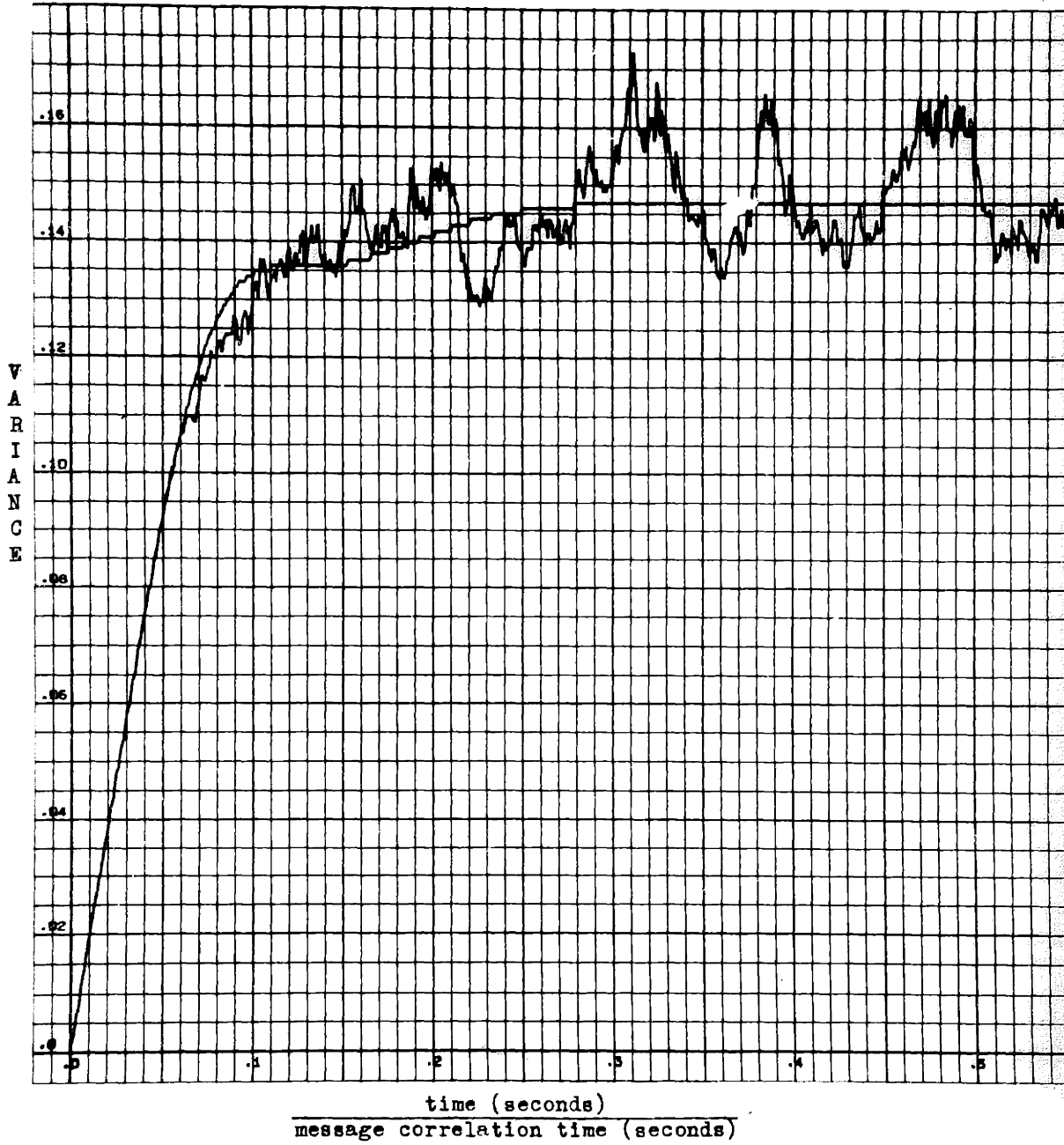
Figure 6.13 $v_{11}^{*}(t)$ for $\beta = 10$ and $\Lambda = 1000$

$$\frac{\text{time (seconds)}}{\text{message correlation time (seconds)}}$$
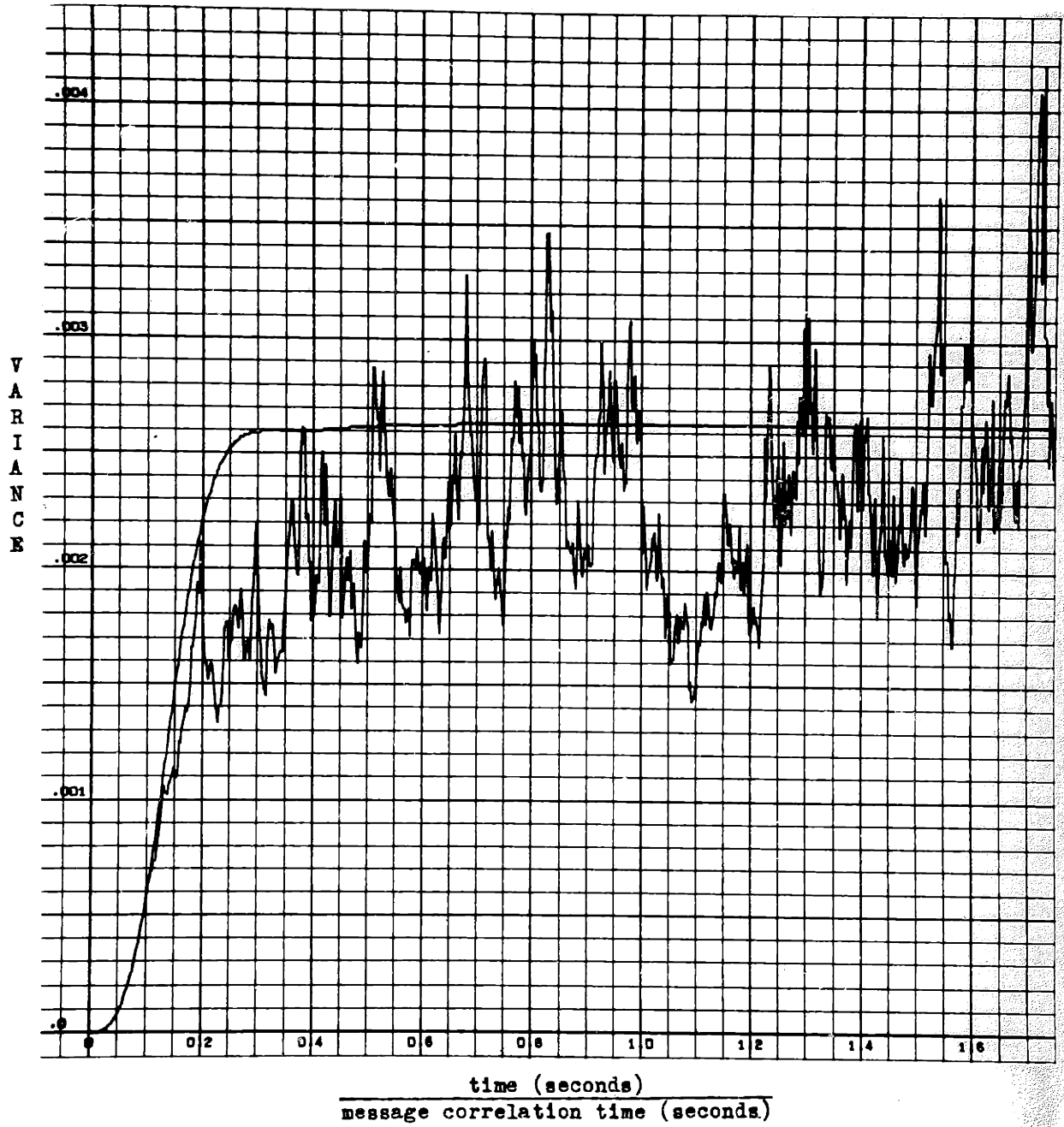
Figure 6.14 $v_{00}^*(t)$ for $\beta = 10$ and $\Lambda = 100$
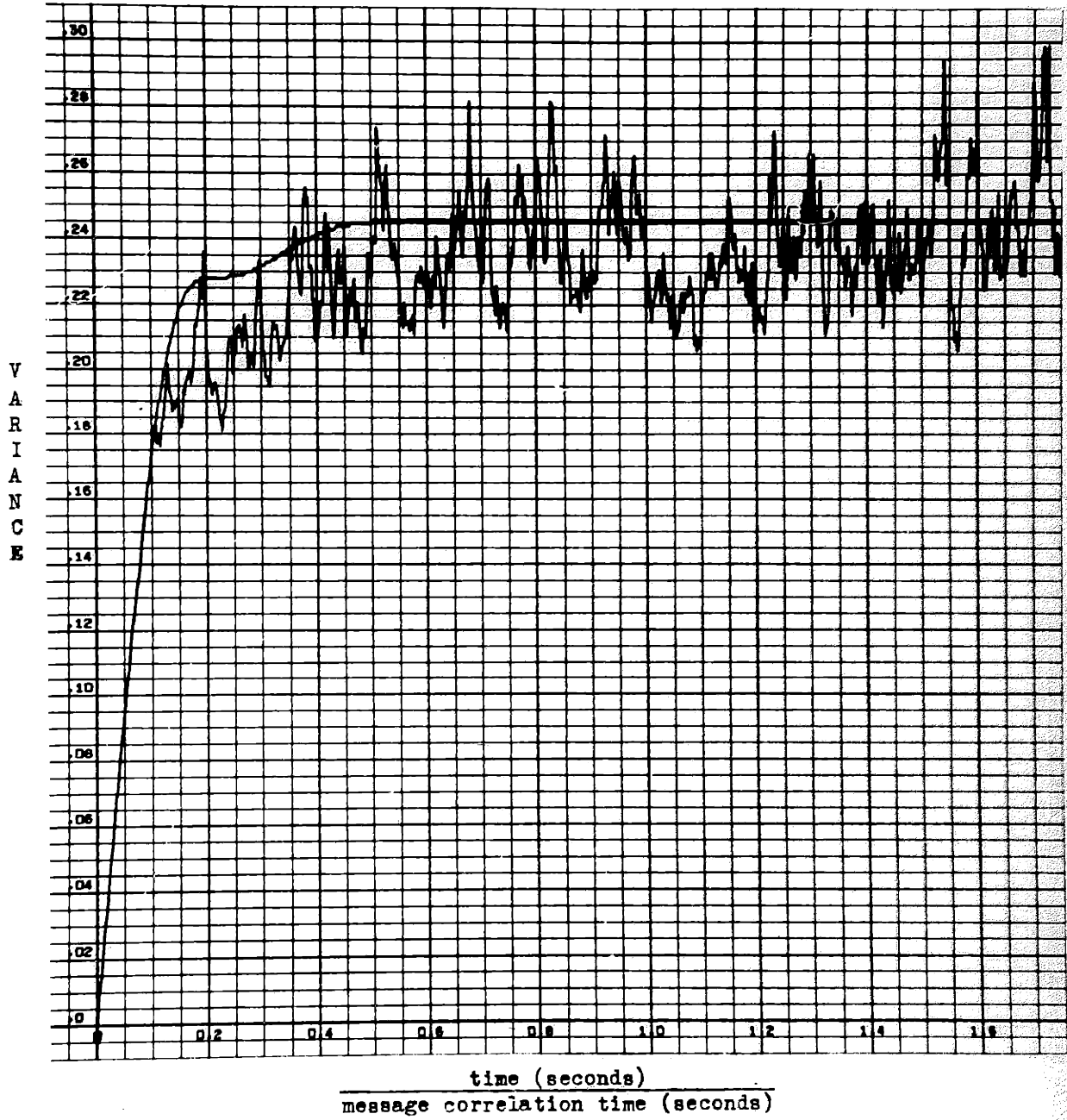
Figure 6.15 $v_{11}^*(t)$ for $\beta = 10$ and $\Lambda = 100$

solutions are seen to vary rapidly around the solutions for the corresponding uncoupled variance equation case. They reach a stationary behavior in about one-forth a message correlation time. In Table 6.1, we have indicated the observed performance for a limited number of values of β and Λ. Zaorski's results are also listed. It is observed that the performance is nearly the same for both cases, even when the demodulator operates below threshold. We can account for this by observing that the rapid fluctuations in the gains, $v^*_{00}(t)$ and $v^*_{01}(t)$, will not propagate through the low-pass filters of the demodulator (see Fig. 6.9). Consequently, the gains can be replaced by their short-term time averages. The result is the same as generating the gains with the uncoupled variance equation.

β	$\Lambda = c^2/N_0 k$	Length of Simulation (Mes. Corr. Times)	Coupled Variance Equation		Uncoupled Variance Equation	
			$1/\sigma_{\varphi,1}^2$	$1/\sigma_{a,1}^2$	$1/\sigma_{\varphi,1}^2$	$1/\sigma_{a,1}^2$
10	1000	170	20.4	6.7	20.0	6.8
10	100	183	2.9	3.5	2.8	3.8
10	40	46	0.9	2.8		
50	1000	9	8.2	13.5	7.8	13.8

Table 6.1

Appendix A1.

State Representation for Gaussian Processes

Any stationary, scalar Gaussian process, $x(t)$, with a rational spectrum approaching zero for high frequencies can be represented by the differential equation:

$$\frac{d^m}{dt^m} x(t) + \psi_1 \frac{d^{m-1}}{dt^{m-1}} x(t) + \cdots + \psi_m x(t) =$$

$$\lambda_1 \frac{d^{m-1}}{dt^{m-1}} \xi(t) + \lambda_2 \frac{d^{m-2}}{dt^{m-2}} + \cdots + \lambda_m \xi(t) \tag{A1.1}$$

where $\psi_1, \cdots, \psi_m$ and $\lambda_1, \cdots, \lambda_m$ are constants and $\xi(t)$ is a white Gaussian process. As is well-known, $x(t)$ can be realized by the passage of $\xi(t)$ through the filter shown in Fig. A1.1a. Alternate realizations can be obtained by representing $x(t)$ by one of several possible equations of state. A particular state-representation we shall use, of which a detailed account is given by Zadeh and Desoer,[57] is:

$$\frac{d}{dt} x_1(t) \quad = \quad - \psi_1 x_1(t) \quad + \quad x_2(t) \quad + \quad \lambda_1 \xi(t)$$

$$\frac{d}{dt} x_2(t) \quad = \quad - \psi_2 x_1(t) \quad + \quad x_3(t) \quad + \quad \lambda_2 \xi(t)$$

$$\frac{d}{dt} x_3(t) \quad = \quad - \psi_3 x_1(t) \quad + \quad x_4(t) \quad + \quad \lambda_3 \xi(t)$$

$$\vdots \tag{A1.2}$$

$$\frac{d}{dt} x_{m-1}(t) \quad = \quad - \psi_{m-1} x_1(t) \quad + \quad x_m(t) \quad + \quad \lambda_{m-1} \xi(t)$$

$$\frac{d}{dt} x_m(t) \quad = \quad - \psi_m x_1(t) \quad \quad \quad + \quad \lambda_m \xi(t)$$

where:

$$x(t) = x_1(t)$$

$$\xi(t) \longrightarrow \boxed{\dfrac{\lambda_1 s^{m-1} + \lambda_2 s^{m-2} + \cdots + \lambda_m}{s^m + \psi_1 s^{m-1} + \psi_2 s^{m-2} + \cdots + \psi_m}} \longrightarrow x(t)$$
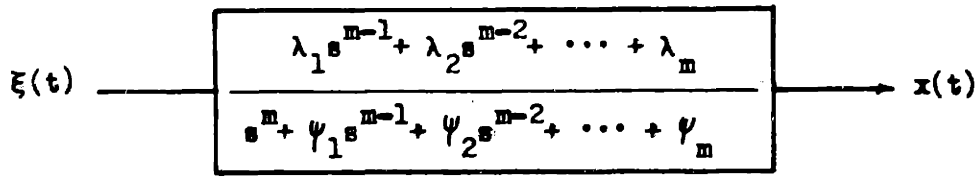
(a)
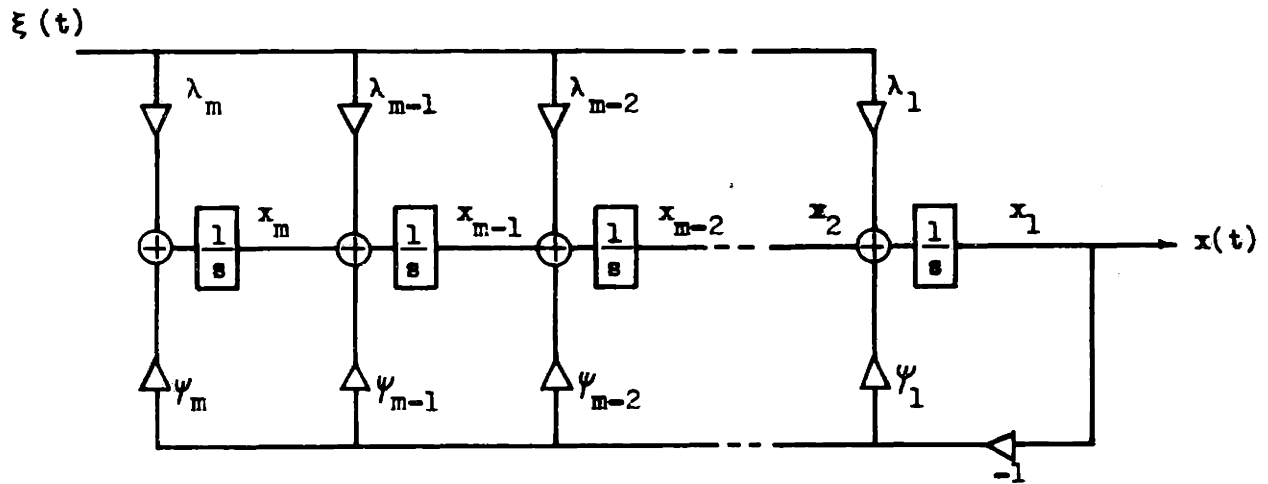
(b)

Figure A1.1 Two Realizations for any Gaussian
Process with a Rational Spectrum Approaching
Zero for High Frequencies

Eq. A1.2 leads to the alternate realization shown in Fig. A1.1b. We shall represent the equation in matrix notation as:

$$\frac{d}{dt} \underline{x}(t) = F \underline{x}(t) + \underline{\xi}(t) \tag{A1.3}$$

where

$$F = \begin{bmatrix} -\psi_1 & 1 & 0 & 0 & \cdots \\ -\psi_2 & 0 & 1 & 0 & \cdots \\ -\psi_3 & 0 & 0 & 1 & \\ \cdot & & & & \ddots \\ \cdot & & & & 1 \\ \cdot & & & & \\ -\psi_m & 0 & \cdots & & 0 \end{bmatrix}, \qquad \underline{\xi}(t) = \begin{bmatrix} \lambda_1 \xi(t) \\ \lambda_2 \xi(t) \\ \lambda_3 \xi(t) \\ \cdot \\ \cdot \\ \cdot \\ \lambda_m \xi(t) \end{bmatrix} \tag{A1.4}$$

Observe that F contains all the denominator coefficients associated with the rational polynomial realization and, correspondingly, $\underline{\xi}(t)$ contains all the numerator coefficients. Because of this feature, the rational polynomial representation can be obtained by inspection from the state representation, and vice-versa. Also observe that the scalar process, $x(t)$, corresponds directly to one of the components of $\underline{x}(t)$.

Any nonstationary scalar Gaussian process which can be represented by Eq. A1.1 with time-varying coefficients, $\psi_1(t), \cdots, \psi_m(t)$ and $\lambda_1(t)$, $\lambda_2(t), \cdots, \lambda_m(t)$, can equally well be represented by Eq. A1.2 with the corresponding coefficients now varying. The filter of Fig. A1.1b with varying gains can be used to realize $x(t)$.

Appendix A2.

Optimum Linear Filtering of an Integrated Signal in White Noise

In this appendix, we shall derive Eqn's. 5.29 and 5.30.

Let an analog message, $a(t)$, be passed through an ideal integrator to produce a signal, $u(t)$, which is observed in additive white noise, $n(t)$. Let the observed signal, $r(t)$, defined by:

$$r(t) = u(t) + n(t) = \int_{-\infty}^{t} a(\tau) \, d\tau + n(t)$$

be available over the interval $(-\infty, t)$. The spectra associated with $a(t)$,

$u(t)$, and $n(t)$ are rational and will be denoted by $S_a(\omega^2)$, $S_u(\omega^2) = S_a(\omega^2)/\omega^2$, and N_0 watts/cps, respectively.

Beginning with the solution to the Wiener-Hopf equation, we shall demonstrate:

(i) The optimum linear filter for estimating $a(t)$ without delay is given by:

$$H_{opt}(\omega) = j\omega \sqrt{N_0} \left[1 - \frac{j\omega + f(0)}{\left[S_a(\omega^2) + \omega^2 N_0 \right]^+} \right] \qquad (A2.1)$$

(ii) The minimum-mean-square error, v_{aa}, in estimating $a(t)$ without
delay is:

$$v_{aa} = \frac{N_0}{3} f^3(0) + F(0) \qquad (A2.2)$$

where

$$f(0) = \int_{-\infty}^{\infty} \log \left[1 + \frac{S_a(\omega^2)}{\omega^2 N_0} \right] \frac{d\omega}{2\Pi} \qquad (A2.3)$$

and

$$F(0) = \int_{-\infty}^{\infty} \omega^2 N_0 \log \left[1 + \frac{S_a(\omega^2)}{\omega^2 N_0} \right] \frac{d\omega}{2\Pi} \qquad (A2.4)$$

We also note that as a consequence of the results of Yovits and Jackson[42] or Snyder,[40] the minimum-mean-square error in estimating $u(t)$ without

delay is given by $N_0 f(0)$. This is just a simple application of Eq. 5.17.

Eq. A2.1 is an expression for the closed-loop version of the optimum filter shown in Fig. 5.5b.

The expression for v_{aa} is significant because it involves only the known input spectra and does not require a determination of $H_{opt}(\omega)$ for its evaluation. An identical expression has been given by Becker, Chang, and Lawton[58] whose derivation is considerably more involved than that presented here. For the following derivations, we closely parallel Snyder.[40]

A2.1 Derivation of the Expression for $H_{opt}(\omega)$

From the solution to the Wiener-Hopf equation, we have:

$$H_{opt}(\omega) = \frac{1}{\left[S_u(\omega^2) + N_0\right]^+} \left[\frac{j\omega S_u(\omega^2)}{\left[S_u(\omega^2) + N_0\right]^-}\right]_+$$

$$= j\omega - \frac{1}{\left[S_u(\omega^2) + N_0\right]^+} \left[\frac{j\omega N_0}{\left[S_u(\omega^2) + N_0\right]^-}\right]_+ \qquad (A2.5)$$

where the superscripts "+" and "-" indicate spectral factorization and the subscript "+" indicates taking the realizable part of a partial fraction expansion. The bracketed expression in Eq. A2.5 with the subscript "+" is a rational function whose numerator is of degree exactly one greater than its denominator. This expression has the form $j\omega k_1 + k_0 +$ [unrealizable terms] when expanded in a partial fraction. We obtain $k_1 = \sqrt{N_0}$ provided $\lim_{\omega \to \infty} S_u(\omega^2) = 0$. Consequently:

$$H_{opt}(\omega) = j\omega - \frac{j\omega\sqrt{N_0} + k_0}{\left[S_u(\omega^2) + N_0\right]^+} \qquad (A2.6)$$

We shall prove below that $k_0^2 = N_0 f^2(0)$ and that $\left|H_{opt}(\omega)/j\omega\right| \le 1$ at $\omega = 0$ so that k_0 is positive. Eq. A2.1 then follows from Eq. A2.6 by

using the fact that $S_u(\omega^2) = S_a(\omega^2)/\omega^2$.

A2.2 Derivation of the Expression for v_{aa}

The minimum-mean-square error is given by:

$$v_{aa} = \int_{-\infty}^{\infty} \left| 1 - \frac{1}{j\omega} H_{opt}(\omega) \right|^2 S_a(\omega^2) \frac{d\omega}{2\Pi} + \int_{-\infty}^{\infty} \omega^2 N_0 \left| \frac{1}{j\omega} H_{opt}(\omega) \right|^2 \frac{d\omega}{2\Pi} \qquad (A2.7)$$

From Eq. A2.1, we have:

$$\left| 1 - \frac{1}{j\omega} H_{opt}(\omega) \right|^2 = \frac{\omega^2 N_0 + k_0^2}{S_a(\omega^2) + \omega^2 N_0} \qquad (A2.8)$$

Let

$$\frac{1}{j\omega} H_{opt}(\omega) = \left| \frac{1}{j\omega} H_{opt}(\omega) \right| e^{j\varphi(\omega)}$$

where $\left| \frac{1}{j\omega} H_{opt}(\omega) \right|$ is an even function of ω and $\varphi(\omega)$ is an odd function of ω. Then:

$$\left| 1 - \frac{1}{j\omega} H_{opt}(\omega) \right|^2 = 1 + \left| \frac{1}{j\omega} H_{opt}(\omega) \right|^2 - 2 \left| \frac{1}{j\omega} H_{opt}(\omega) \right| \cos \varphi(\omega)$$

Using Eq. A2.8, we easily obtain:

$$\left| \frac{1}{j\omega} H_{opt}(\omega) \right|^2 = \frac{k_0^2 - S_a(\omega^2)}{S_a(\omega^2) + \omega^2 N_0} + 2 \left| \frac{1}{j\omega} H_{opt}(\omega) \right| \cos \varphi(\omega) \qquad (A2.9)$$

Substituting Eqn's. A2.8 and A2.9 into A2.7, we obtain:

$$v_{aa} = \int_{-\infty}^{\infty} \left\{ k_0^2 + 2\omega^2 N_0 \left| \frac{1}{j\omega} H_{opt}(\omega) \right| \cos \varphi(\omega) \right\} \frac{d\omega}{2\Pi}$$

$$\leq \int_{-\infty}^{\infty} \left\{ k_0^2 + 2\omega^2 N_0 \frac{1}{j\omega} H_{opt}(\omega) \right\} \frac{d\omega}{2\Pi} \qquad (A2.10)$$

We now make the following observations:

(i) $\left| \frac{1}{j\omega} H_{opt}(\omega) \right| \leq 1$. To prove this assume that $\left| \frac{1}{j\omega} H_{opt}(\omega) \right| > 1$ over some range of frequencies and examine Eq. A2.7. Replacing $\left| \frac{1}{j\omega} H_{opt}(\omega) \right|$ by 1 at these frequencies reduces the mean-square error resulting in a contradiction since v_{aa} is already minimum.

(ii) $H_{opt}(\omega) \approx k_0^2 / 2j\omega N_0$ for large ω for otherwise, from Eq. A2.10, v_{aa} diverges. That $H_{opt}(\omega)$ behaves as $1/j\omega$ for ω large can also be deduced from Fig. 5.5b.

(iii) $\int_{-\infty}^{\infty} 2\omega^2 N_0 \frac{1}{n} \left\{ \frac{1}{j\omega} H_{opt}(\omega) \right\}^n \frac{d\omega}{2\Pi} = 0$ for $n = 2, 3, 4, \cdots$. A simple application of contour integration shows that the integral is zero for $n = 2, 3, \cdots$ since the integrand is right-half plane analytic and behaves as $1/\omega^{2n-2}$ for ω large.

(iv) $\int_{-\infty}^{\infty} \frac{1}{n} \left\{ \frac{1}{j\omega} H_{opt}(\omega) \right\}^n \frac{d\omega}{2\Pi} = 0$ for $n = 1, 2, 3, \cdots$. The proof is identical to that of (iii).

Using these observations and the logarithmic expansion:

$$- \log (1-x) = x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \cdots \qquad \text{for } |x| < 1$$

Eq. A2.10 becomes:

$$v_{aa} = \int_{-\infty}^{\infty} \left\{ k_0^2 - 2\omega^2 N_0 \log \left(1 - \frac{1}{j\omega} H_{opt}(\omega) \right) \right\} \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \left\{ k_0^2 - \omega^2 N_0 \log \left| 1 - \frac{1}{j\omega} H_{opt}(\omega) \right|^2 \right\} \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \left\{ k_0^2 - \omega^2 N_0 \log \frac{\omega^2 N_0 + k_0^2}{S_a(\omega^2) + \omega^2 N_0} \right\} \frac{d\omega}{2\pi} \qquad (A2.11)$$

Also, (iv) leads to:

$$0 = \int_{-\infty}^{\infty} \log \left| 1 - \frac{1}{j\omega} H_{opt}(\omega) \right|^2 \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \log \frac{\omega^2 N_0 + k_0^2}{S_a(\omega^2) + \omega^2 N_0} \frac{d\omega}{2\pi} \qquad (A2.12)$$

Eqn's. A2.11 and A2.12 can be obtained from the colored-noise results of Yovits and Jackson.[42] In this form, however, v_{aa} is difficult to evaluate since k_0^2 must first be determined from the integral equation, Eq. A2.12. We shall now perform some manipulations which lead to the convenient expression for v_{aa} given above.

Let:

$$f(\lambda) = -\int_{-\infty}^{\infty} \log \frac{\omega^2 N_0 + \lambda}{S_a(\omega^2) + \omega^2 N_0} \frac{d\omega}{2\pi} \qquad (A2.13)$$

We seek λ such that $f(\lambda = k_0^2) = 0$. Differentiating Eq. A2.13:

$$\frac{d}{d\lambda} f(\lambda) = -\int_{-\infty}^{\infty} \frac{1}{\omega^2 N_0 + \lambda} \frac{d\omega}{2\pi} = -\frac{1}{2\sqrt{N_0 \lambda}}$$

Integrating and introducing the appropriate boundary condition, we then obtain:

$$f(\lambda) = -\sqrt{\frac{\lambda}{N_0}} + f(0)$$

$$(A2.14)$$

where $f(0)$ is defined by Eq. A2.3. Setting $\lambda = k_0^2$ we then obtain

$$k_0^2 = N_0 f^2(0)$$

$$(A2.15)$$

Following the same procedure, let:

$$F(\lambda) = \int_{-\infty}^{\infty} \left\{ \lambda - \omega^2 N_0 \log \frac{\omega^2 N_0 + \lambda}{S_a(\omega^2) + \omega^2 N_0} \right\} \frac{d\omega}{2\Pi}$$

Then $F(\lambda = k_0^2) = v_{aa}$. Differentiating and integrating as before, we obtain:

$$F(\lambda) = \frac{\lambda^{3/2}}{3\sqrt{N_0}} + F(0)$$

$$(A2.16)$$

where $F(0)$ is defined by Eq. A2.4. Letting $\lambda = k_0^2$ and using Eq. A2.15, we then obtain from Eq. A2.16:

$$v_{aa} = \frac{N_0}{3} f^3(0) + F(0)$$

$$(A2.17)$$

which is the desired result.

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