PARALLELISM ON p-SURFACES IN RIEMANNIAN MANIFOLDS

by

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(1958)

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February, 1965

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Department of Mathematics, January 11, 1965

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ABSTRACT

A theory of parallelism along p-surfaces in a Riemannian manifold M is developed. The theory exploits the dual nature of a geodesic as

1. an autoparallel curve

and 2. as a critical point in the calculus of variations problem for minimizing the length of curves connecting a pair of points in M .

A specialized definition of a p-plane field along an embedding $f: N^p \to M$ is given. The differential, df, is a p-plane field. The second fundamental form and mean curvature vector field of a p-plane field are defined. A p-plane field is defined to be parallel when its mean curvature vector field is zero. It is shown that the condition for parallelism is equivalent to the vanishing of a certain p-form on a certain principal bundle associated with each p-plane field. The differential p-plane field, df , of a minimal surface is parallel. The sense in which the theory generalizes the case $p=1$ is discussed.

The Main Theorem, proved only in the real analytic case, using the Cartan-Kahler Theorem, states conditions for the existence and uniqueness of a parallel p-plane field along an embedding, in terms of initial data.

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ACKNOWLEDGEMENT

I wish to thank Professor I. M. Singer for suggesting the problem considered in this thesis. I am grateful for his advice and patience during the many, many hours of discussion when the ideas presented here were developed. I also wish to thank Dr. J. H. Simons for several valuable comments he made.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

§1. Statement of the problem considered in this thesis.

1.1. Geodesics arise in the study of Riemannian Geometry in two ways. On the one hand, the concept of parallel translation of vectors is defined in any affinely connected manifold. In a Riemannian manifold there is a unique connection with zero torsion on the bundle of frames; this connection is called the Riemannian connection (see Milnor [6] or Singer [8]). A geodesic is then defined as a curve whose tangent vector field is its own parallel translate with respect to the Riemannian connection. The geodesics are said to be the autoparallel curves.

1.2. On the other hand, the length of a piecewise smooth curve in a Riemannian manifold is defined using the metric. A basic problem (again, see Milnor [6] or Singer [8]) is to find the curves of minimal length connecting a pair of points in the manifold. The problem is attacked using the Calculus of Variations. A length function is defined on the space of piecewise smooth curves with the same end points. One computes the first variation of a one parameter family of such curves and finds that the geodesics, as defined in §1.1, are the critical points of the length function.

In fact, geodesics were classically defined by this condition.

Thus the curves which are autoparallel are the critical points of the length function.

1.3 The problem considered in §1.2 can be formulated in higher dimensions also. Let $p \leq m$. The p-dimensional volume of a compact p dimensional submanifold of an m dimensional Riemannian manifold is defined using the metric, (see Eisenhart [1]). The problem of finding a p-dimensional submanifold of minimal volume which bounds a fixed, closed, p-1 dimensional submanifold again leads to a problem in the Calculus of Variations. The critical points of the volume function are called minimal surfaces. They are the surfaces of mean curvature zero. (again see Eisenhart [1]).

1.4 Keeping in mind what has been said in §1.1, .2 and .3, the problem considered in this thesis can now be stated:

TO FORMULATE A DEFINITION OF A RIEMANNIAN p-CONNECTION AND/OR AN ASSOCIATED CONCEPT OF PARALLELISM ALONG p-SURFACES, FOR WHICH, THE p-DIMENSIONAL MINIMAL SURFACES ARE THE AUTO-PARALLEL SURFACES.

We attempt to get at a p-connection through parallelism. We must try to find out what kind of geometric objects we can expect to define parallel fields of.

At first glance vectors, p-planes, p-frames or m-frames all appear as equally likely candidates, considering only the one dimensional situation.

1.5. However, there are more signposts. The following are basic statements in Riemannian Geometry. See Milnor [6].

THEOREM A.

Let $\gamma: (a,b) \to M$ be a smoothly embedded curve in the Riemannian manifold M. Let $c \in (a,b)$ and $x \in M_{\gamma(c)}$, $||x|| = 1$. Then there is a unique parallel unit vector field X along γ such that

$$X(\gamma(c)) = x$$

THEOREM B.

Let $n \in M$, a Riemannian manifold. Let $x \in M_n$. Then there is a unique geodesic $\gamma: U \to M$ defined on some (not uniquely determined) neighborhood U of the origin in R^1 such that

$$\gamma(0) = n$$
$$\dot{\gamma}(0) = x$$

In each of these theorems information is given on a $p-1 = 0$ dimensional submanifold and extended along a $p=1$ dimensional manifold. The existence and uniqueness theorem of Ordinary Differential Equations is used decisively in their proofs; that is, the existence and uniqueness theorem of O.D.E. as expressed in the theory of differential ideals by the Frobenius Theorem. We are concerned with p-dimensional manifolds, wherein general $p > 1$. If, by analogy to the above, we are willing, in our theory, to allow information to be given initially along a p-1 dimensional submanifold in order to determine a unique parallel field extending the initial data, then we should take note of the Cartan-Kahler Theorem. See Johnson [2]. This theorem expresses, in the language of differential ideals, the Cauchy-Kowalewski Theorem on the existence and uniqueness of the solutions of real analytic partial differential equations. It states the existence and uniqueness of a p-dimensional integral manifold extending a p-1 dimensional integral submanifold (and perhaps satisfying some additional constraints).

1.6. If we let the one dimensional case, the requirement for minimal surfaces and the Cartan-Kahler Theorem guide us, it turns out that there is exactly one geometric object along an embedded p-dimensional surface for which we can define parallelism. This object, defined, say, along the isometric embedding $f: N^p \to M^m$, is somewhat more specialized than a p-plane field. Roughly, it is a vector bundle isometry, covering f, of the tangent bundle $T(N)$ into the tangent bundle $T(M)$. (See §2.2 for details). We will still call this object a p plane field. Notice that the differential df , is such an object.

The condition of parallelism of a p-plane field is defined by the vanishing of a certain vector field along the embedding, which in the case of the p-plane field df is just the mean curvature vector field (see §4 for details).

It is then immediate that the differential df of a minimal surface is parallel. That is, we may say that minimal surfaces are autoparallel.

In the one dimensional case, it is evident that the arc-length parametrized geodesics (which are autoparallel in the usual sense) an autoparallel in our sense (because they are one-dimensional minimal surfaces). Conversely if an arc-length parametrized curve has parallel differential, it is not hard to show that it is a geodesic. Thus for one dimension, autoparallel means the same thing in either sense.

I have not yet proved an analog of Theorem B (§1.5) which should say something like: given the right initial data along a (p-1) dimensional surface, there exists (at least in the real analytic case) a unique p-dimensional minimal surface containing the (p-1) dimensional surface and extending the data given along it.

1.7 The MAIN THEOREM proved here (see Chapter III) is the natural extension of Theorem A (§1.4). It roughly states that there is a unique parallel p-plane field along an embedding that extends #1. a p-plane field given along a (p-1) dimensional submanifold (see §2.3 for definition)

and #2. a (p-1) plane field given everywhere along the embedding (see §2.2 for definition)

Notice that for $p = 1$, #2 of the main theorem is vacuous and #1 is a statement at a point. The main theorem for the one dimensional case, compared to Theorem A, shows what part of the concept of parallel (in the usual sense) we have been able to generalize. If the metric on M is pulled back via γ then $\gamma: (a,b) \to M$ becomes an isometric embedding. Then the vector field along γ, whose existence is asserted by Theorem A, is the image of the unit tangent vector field which is a positive multiple of $\frac{\partial}{\partial t}$ (not in general a constant multiple) under the 1-plane field whose existence is asserted in the main theorem.

1.8. The main theorem is proved only in the real analytic case. It will be of interest to know what differential equations arise from this geometry, for they will determine the possibility of carrying out the same program in the C^∞ case.

The proof of the main theorem depends heavily on the fact that the condition for parallelism can be stated in terms of the vanishing of a p-form (see §5). This may eventually lead us to a statement of what a p-connection is.

It is also of interest to ask if it is possible to define a p-connection or parallelism along p-surfaces in an affinely connected manifold; we note that the torsion zero property of the Riemannian connection was not used.

§2. Definitions, Notation and General Remarks.

2.1. The concepts that we will now discuss are presented in detail in Singer [8] and Sternberg [9]. Nomizu [7] is much briefer. We will use the abbreviation R. M. for Riemannian manifold and dim for dimension. C^ω means real analytic.

Let X^m denote a R.M. of dim m . We will assume that all manifolds are of class C^ω or C^∞ , which we will call smooth. When a choice of either C^ω or C^∞ must be made it will be explicitly indicated. We will agree that

$$T(X) = \text{tangent bundle}$$
$$S(X) = \text{unit sphere bundle}$$

$$F(X) = \text{bundle of frames}$$

$$\mathcal{G}^p(X) = \text{bundle of p-planes}$$

Unless otherwise stated M^m is a fixed smooth R.M. of dim m and $f: N^p \to M^m$ is a fixed isometric embedding of a fixed smooth R.M. of dim p into M .

2.2. In standard terminology a q-plane field along the embedding f is a lift of the submanifold N^p into the Grassmannmanifold $\mathcal{G}^q(M)$. For our purposes this definition will be inadequate. Let $0 \leq q \leq p$.

Definition: A q-plane field along f is a pair (D^q, G^q) consisting of a q-dim smooth vector subbundle D^q of $T(N)$, and a smooth vector bundle mapping $G^q : D^q \to T(M)$ which carries $D^q(n)$, the fibre of D^q over $n \in N$ isometrically into the fibre over $f(n)$.

This can be restated by saying that the diagram

$$\begin{array}{ccc} D^q & \xrightarrow{G^q} & T(M) \\ \downarrow \pi & & \downarrow \pi \\ N & \xrightarrow{f} & M \end{array}$$

commutes and $G^q|_{\pi^{-1}(n)} : \pi^{-1}(n) \equiv D^q(n) \to \pi^{-1}(f(n))$ preserves the inner product. $G^q(n)$ will denote the element of $\mathcal{G}^p(M)$ given by $G^q(n) = G^q(D^q(n))$. The differential df is an example of a p-plane field along f .

2.3. Let $i: N^{p-1} \to N^p$ be an isometric embedding of smooth R.M. We will generally refer to (N^{p-1}, i) as an initial manifold. Denote by $\delta i(T(N^p))$ the vector bundle over N^{p-1} obtained by pulling back $T(N^p)$ via i. Let $h_i: \delta i(T(N^p)) \to T(N^p)$ be the natural vector bundle isometry which makes the diagram

$$\begin{array}{ccc} \delta i(T(N^p)) & \xrightarrow{h_i} & T(N^p) \\ \downarrow \pi & & \downarrow \pi \\ N^{p-1} & \xrightarrow{i} & N^p \end{array}$$

commute.

Let f and i be as above.

<u>Definition</u>: A <u>p-plane field along foi</u> is a vector bundle mapping $G^p: \delta i(T(N^p)) \to T(M)$ which carries the fibre over $n \in N^{p-1}$ isometrically into the fibre over $foi(n)$. That is, the diagram

$$\begin{array}{ccc} \delta i(T(N^p)) & \xrightarrow{G^p} & T(M) \\ \downarrow \pi & & \downarrow \pi \\ N^{p-1} & \xrightarrow{foi} & M \end{array}$$

commutes and $G^p|_{\pi^{-1}(n)}: \pi^{-1}(n) \equiv h_i^{-1}(N^p_{i(n)}) \to \pi^{-1}(foi(n))$ preserves the inner product.

We will generally refer to $G^p: \delta i(T(N^p)) \to T(M)$ as initial conditions along foi. The reasons for the use of the terms initial manifold and initial conditions will become apparent when the reader sees the role they play in the theory.

2.4. Let $G = (T(N^p), G)$ be a p-plane field along f.

Let (D^{p-1}, G^{p-1}) be a (p-1) plane field along f.

Let $G^p\colon \delta i(T(N^p)) \to T(M)$ be a p-plane field along $f \circ i$.

Then $\delta i(D^{p-1}) \subset \delta i(T(N^p))$ is a vector bundle over N^{p-1} obtained by pulling back D^{p-1} via i.

We need the following

<u>Definition</u>: (D^{p-1}, G^{p-1}) and G^p are <u>compatable</u> if

$$G^p|_{\delta i(D^{p-1})} = G^{p-1} \circ h_i|_{\delta i(D^{p-1})}$$

<u>Definition</u>: G <u>extends</u> G^p if $G^p = G \circ h_i$

<u>Definition</u>: G <u>extends</u> (D^{p-1}, G^{p-1}) if $G|_{D^{p-1}} = G^{p-1}$.

2.5. Let (D^q, G^q) be a q-plane field along f.

Let $F(D^q, G^q) = \{(n, e_1, \ldots e_m) \mid n \in N^p,\ e_1, \ldots e_m$ is an orthonormal frame of $M_{f(n)}$, $e_1, \ldots e_q$ is an orthonormal frame of $G^q(n)\}$

<u>Proposition</u>: $F(D^q, G^q)$ is a principal $O(q) \times O(m-q)$ bundle over N with projection $\pi(n, e_1, \ldots e_m) \to n$.

The proof is similar to a proof in Singer [8] Chapter VII, page 3. We must only add that in this case we use q orthonormal vector fields spanning D^q in a neighborhood of $n \in N$, their images under G^q, and $(m-q)$ more orthonormal vector fields which locally fill out the orthogonal complement to $G^q(D^q)$ in $T(M)$ along f.

Following Singer [8] Chapter VII page 15, let B_1 denote the bundle over N whose fibre over $n \in N$ is the fibre of $F(M)$ over $f(n)$. The group and fibre of both $F(M)$ and B_1 is $O(m)$. Let us make the identification

$$O(q) \times O(m-q) \equiv \left(\begin{array}{c|c} O(q) & 0 \\ \hline 0 & O(m-q) \end{array}\right) \subset O(m)$$

Denote by $j: F(D^q,G^q) \to B_1$, the inclusion map. Let r_g denote right translation by g.

<u>Proposition</u>: $(F(D^q,G^q),j)$ is a submanifold of B_1 and with respect to the above identification of groups we have

$$j \circ r_g = r_g \circ j \quad \text{for each} \quad g \in O(q) \times O(m-q).$$

The proof is a direct verification which we omit.

If $(D^q,G^q) = (T(N^p),df)$ then $F(T(N^p),df)$ is just the bundle of adapted frames of the embedding f. This leads us to the

<u>Definition</u>: $F(D^q,G^q)$ is called the <u>bundle of adapted frames of the q-plane field</u> (D^q,G^q) <u>along f</u>.

§3. Remarks on Vector Bundle-Valued Forms and the Cartan-Kahler Theorem.

3,1. As was indicated in (1.6) and (1.8), we will be concerned with vector valued forms and their integral manifolds. In this section we will review some of the standard material from the theory of differential ideals. At the same time, we will put a basic theorem (the Cartan-Kahler Theorem) in the exact context in which we will later need it. Details and proofs can be found in the following references:

Vector-valued and Vector bundle-valued forms.	Koszul [4]
Cartan-Kahler Theorem	Johnson [2], Kuranishi [5] Kahler [3]

The format that I shall follow most closely, however, is given in Hano-Kobayashi [10]. This unfortunately is not generally available.

3.2. Let M be an m-dim smooth manifold. Let $\Lambda^* M$ be the ring of smooth differential forms on M.

Definition: A subset I of $\Lambda^* M$ is called a differential ideal if

1. it is an ideal in $\Lambda^* M$ (with respect to wedge multiplication)
2. it is homogeneous, (whenever it contains a form that is the sum of forms of several degrees then it contains each of the summands).
3. it is closed under the action of the differential operator d, $dI \subset I$.

We note that given any subset $J \subset \Lambda^* M$, the ideal generated by J and dJ is a differential ideal. We call this ideal the differential ideal generated by J.

<u>Definition</u>: A submanifold (N,f) of M is an <u>integral manifold of the differential ideal</u> I (respectively <u>the subset</u> J) if and only if $\delta f(I) = 0$ (respectively $\delta f(J) = 0$).

It is easy to see that (N,f) is an integral manifold of J if and only if it is an integral manifold of the differential ideal generated by J.

<u>Definition</u>: If $n \in M$ and $E^p(n) \subset M_n$ is a p-plane, then $E^p(n)$ is an <u>integral plane of</u> I (or J) if $\omega|_{E^p(n)} = 0$ for each $\omega \in I$ (or $\omega \in J$).

3.3. Let V be a k-dim vector space over R.

<u>Definition</u>: A <u>p-form</u> ω <u>with values in</u> V is a function which assigns to each $n \in M$ a skew symmetric multilinear map of $M_n \times .. \times M_n$ (p-times) into V. We say that ω is smooth, if, whenever we choose a basis $v_1, .. v_k$ of V and write

$$\omega_m(\quad) = \sum_{i=1}^{k} \omega_m^i(\quad) v_i ,$$

the real valued p-forms, $\omega^1, .. \omega^k$ are smooth. This definition is independent of the choice of basis. Let $I(\omega)$ be the differential ideal generated by $J = \{\omega^1, .. \omega^k\}$. $I(\omega)$ is independent of the choice of basis, as it is easy to see. We say that $I(\omega)$ is the <u>differential ideal associated with</u> ω .

<u>Definition</u>: (N,f) is an <u>integral submanifold</u> of ω if $\delta f(\omega) = 0 \in V$, $E^p(n) \subset M_n$ is an <u>integral p-plane</u> of ω if $\omega|_{E^p(n)}(\quad) = 0 \in V$.

Clearly, (N,f) is an integral submanifold of ω if and only if (N,f) is an integral submanifold of $I(\omega)$. The same remark is true for $E^p(n)$.

Thus we see that the integral manifolds of a vector valued form are found by finding the integral manifolds of its associated differential ideal.

3.4. Let $V(M)$ be a k-dim smooth vector bundle over M .

<u>Definition</u>: <u>A p-form</u> ω <u>on</u> M <u>with values in</u> $V(M)$ is a function which assigns to each $n \in M$ a skew symmetric multilinear map of $M_n x..xM_n$ (p-times) into the fibre of $V(M)$ over n . We define smoothness of ω as follows. Choose k independent smooth vector fields $X_1,..X_k$ which span $V(M)$ over a neighborhood of n , and write

$$\omega_m(\quad) = \sum_{i=1}^{k} \omega_m^i(\quad) X_i(m) .$$

Then ω is smooth in the neighborhood of n if the real valued p-forms $\omega_m^1,..\omega_m^k$ are smooth there. The definition does not depend on the choice of vector fields. Let $I(\omega)$ be the differential ideal generated by $J = \{\omega^1, .. \omega^k\}$. $I(\omega)$ is independent of the choice of vector fields. We call $I(\omega)$ the <u>differential</u> <u>ideal</u> <u>associated</u> <u>with</u> ω. Proceed as in (3.3) for the definitions of integral manifold of ω and integral plane of ω; draw the same conclusions.

3.5. Let I be a differential ideal on M. For each $E^p(n) \in \mathcal{G}^p(M)$ $(n \in M)$, we obtain from I a space of linear functionals on M_n, as follows. Let $v_1, .. v_p$ span $E^p(n)$. Let $\tau^s \in I$, where the degree of τ^s is $s \leq p+1$. For each increasing sequence of s-1 integers $1 \leq i_1 \leq i_2 \leq \cdot \leq i_{s-1} \leq p$ the map

$$v \to \tau^s(v_{i_1} \wedge v_{i_2} \wedge .. v_{i_{s-1}} \wedge v)$$

is a linear functional on M_n.

The space spanned by the linear functionals obtained by

1. varying the increasing sequences of s-1 integers and 2. varying τ^s over all forms in I of degree $\leq p+1$ is called the <u>polar</u> <u>space</u> of $E^p(n)$ and written $J(E^p(n), I)$. It is easily seen to be independent of the choice of $v_1, .. v_p$. We define $t(E^p(n)) = \dim J(E^p(n), I)$ and $t_r(E^p(n)) = \max \{t(E^r(n) | E^r(n) \subset E^p(n)\}$ for each $r = 0, 1, .. p-1$.

Observe that if $E^p(n)$ is integral, we must have $t(E^p(n)) \leq m-p$.

3.6. Suppose that ω is a p-form on M with values in a vector space V . Let us now make an observation which we will use later on. Let $E^{p-1}(n)$ be any $(p-1)$ plane at $n \in M$; let $v_1, \dots v_{p-1}$ be any basis of $E^{p-1}(n)$. We wish to compute $t(E^{p-1}(n)) = \dim (J(E^{p-1}(n), I(\omega))$. Since $I(\omega)$ is given by forms of degree at least p , the polar space of $E^p(n)$ is exactly the span of the linear functionals

$$v \to \omega^i(v_1 \wedge v_2 \wedge \dots \quad v_{p-1} \wedge v)$$

where ω^i are the real valued p-forms obtained from ω , by choosing a basis of V as in (3.3) . Hence $t(E^{p-1}(n))$ is the rank of the linear map of $M_n \to V$ given by

$$v \to \omega(v_1 \wedge \dots \quad v_{p-1} \wedge v)$$

In just the same way we see that if ω is a p-form with values in the vector bundle $V(M)$, then $t(E^{p-1}(n)) = \dim (J(E^{p-1}(n), I(\omega))$ is the rank of the linear map: $M_p \to$ fibre of $V(M)$ over n given by

$$v \to \omega(v_1 \wedge \dots \quad v_{p-1} \wedge v)$$

3.7. <u>Definition</u>: A family $\{f^1, f^2, \dots\}$ of smooth functions defined in a neighborhood of $n \in M$ is called

<u>regular</u> <u>of</u> <u>dimension</u> r <u>at</u> n if

1. $f^i(n) = 0$ for each $i=1,2,3...$
2. There is a coordinate system $y^1,..y^m$ defined in a neighborhood U of n such that the set of common zeroes of $\{f^1,f^2,...\}$ in U coincides with the set of common zeroes of $y^{r+1},..y^m$.
3. The subspace of $(M_n)^*$ spanned by the df^i, $i=1,2,...$ has rank $m-r$.

It is evident that the common zeroes would then form a submanifold of dim r passing through n . Let I be a differential ideal on M .

<u>Definition</u>: An integral point $n_0 \in M$ is said to be <u>regular</u> if the system of functions in I is regular of degree r at n_0 and if there exists a neighborhood U of n_0 such that $t(E^0(n))$ is constant for each integral point $E^0(n) = n$ in U .

For $p \geq 1$:

an integral plane $E^p(n_0)$ is said to be <u>regular</u> if $E^p(n_0)$ contains a regular $E^{p-1}(n_0)$ and if $t(E^p(n))$ is constant for all integral p-planes $E^p(n)$ in some neighborhood of $E^p(n_0)$.

<u>Definition</u>: An integral point $n_0 \in M$ is said to be <u>ordinary</u> if the system of functions in I is regular of dimension r at n_0 .

For $p \geq 1$, an integral plane $E^p(n_0)$ is said to be <u>ordinary</u> if $E^p(n_0)$ contains at least one $E^{p-1}(n_0)$ which is regular.

3.8. Let us assume for the rest of §3 that we are dealing with real analytic forms, ideals, etc.

It can be shown that if $E^q(n)$ is regular and $E^q(n) \subset E^p(n)$ then $t(E^q(n)) = t_q(E^p(n))$.

One shows that the set of regular p-plane and the set of ordinary p-planes are each open in the set of integral planes in $\mathfrak{G}^p(M)$. But more importantly, using a natural description of the integral p-planes as the zeroes of a certain family of functions on $\mathfrak{G}^p(M)$, one shows that at a regular or ordinary plane $E^p(n)$, this family of functions is regular of a particular dimension and hence determines the germ of a submanifold of $\mathfrak{G}^p(M)$ through $E^p(n)$.

The Cartan-Kahler Theorem makes use of this information (together with the Cauchy-Kowalewski Theorem) to find p-dimensional integral manifolds through an ordinary p-plane. There are several ways of prescribing data to obtain a unique integral manifold; the theorem that we now quote leads to the formulation we need later on.

<u>Theorem</u> (Cartan-Kahler). Let I be a differential ideal on M^m . Suppose there are no functions belonging to I . Let $E^p(n)$ be an ordinary integral p-plane of I at n , containing the regular integral (p-1) plane $E^{p-1}(n)$.

Suppose $t_{p-1}(E^p(n)) = m-p$.

Let (V^{p-1},j) be a p-1 dim. integral manifold with $n_0 \in V^{p-1}$, $j(n_0) = n$, $dj(V^{p-1}_{n_0}) = E^{p-1}(n)$.

Then there exists a unique p-dim integral manifold (V^p,k) with $n_1 \in V^p$, $k(n_1) = n$, $dk(V^p_{n_1}) = E^p(n)$ which furthermore "extends" (V^{p-1},j) in a neighborhood of n_0 . (Here, "extends" means that there is a neighborhood U of n_0 in V^{p-1} and a natural embedding $\ell: U \to V^p$ which makes the diagram

$$\begin{array}{ccc} (U,n_0) & \xrightarrow{\ j\ } & (M,n) \\ \ell \downarrow & \nearrow_{k} & \\ (V^p,n_1) & & \end{array}$$

commute.

3.9. We now adapt the Cartan-Kahler Theorem to our specific need.

<u>Theorem</u>: Let $V(M)$ be an $(m-p)$ dim vector bundle over M^m . Let ω be a p-form with values in $V(M)$. Let $E^{p-1}(n) \subset M_n$ and $v_1,..v_{p-1}$ be any basis of $E^{p-1}(n)$. Suppose the linear map $T: M_m \to$ fibre of $V(M)$ over n given by

$$v \to \omega(v_1 \wedge v_2 \wedge .. \ v_{p-1} \wedge v)$$

is surjective. Then given any (p-1) dim integral manifold (V^{p-1},j) with $n_o \in V^{p-1}, j(n_o)=n$, $dj(V^{p-1}_{n_o}) = E^{p-1}(n)$, there exists a unique integral manifold (V^p,k) with $n_1 \in V^p$, $k(n_1) = n$ which "extends" (V^{p-1},j) in a neighborhood of n_0 .

Proof: Take $I = I(\omega)$. By (3.6) $t(E^{p-1}(n)) =$ rank $T = m-p$. The continuity of the determinant function then tells us that $t(E)$ is constant for $(p-1)$ planes E near $E^{p-1}(n)$, because the rank of $T = t(E^{p-1}(n))$ is maximal. There is a unique integral p-plane E^p containing E^{p-1} , in fact, $E^p = \ker T$. Thus E^p is ordinary, because we have shown that $E^{p-1}(n)$ is regular. By the remark at the beginning of (3.8) $t_{p-1}(E^p) = t(E^{p-1}(n)) = m-p$. The theorem now follows directly from the Cartan-Kahler Theorem (3.8). Observe that the choice of $v_1, \dots v_{p-1}$ is immaterial.

CHAPTER II

Parallelism for p-Surfaces

§4. The second fundamental form of a p-plane field and the definition of parallelism.

4.1. Let G be a p-plane field along $f: N^p \to M^m$. Let φ (respectively, φ_{rs}) denote the matrix-valued (respectively, real-valued) one forms of the Riemannian connection on $F(M)$. Let the same symbols denote the pull-back of these forms to B_1 . Let $\overline{\varphi} = \delta j \varphi$ and $\overline{\varphi}_{rs} = \delta j \varphi_{rs}$ denote the further pull-back of these forms to $F(T(N),G) = F(G)$, the bundle of adapted frames of G .

Let $b \in F(G)$. Then b induces a natural identification of the fibre through b with $O(p) \times O(m-p)$; $j(b)$ induces a natural identification of the fibre through $j(b)$ with $O(m)$. Restricted to the fibres through b and $j(b)$, the mapping j corresponds under these natural identifications to the map:

$$O(p) \times O(m-p) \rightarrow \left(\begin{array}{c|c} O(p) & 0 \\ \hline 0 & O(m-p) \end{array}\right) \subset O(m)$$

We also have that vertical subspace of $(F(G))_b$, which we denote by $V(F(G)_b)$ is naturally identified with $o(p) \times o(m-p)$, the Lie Algebra of $O(p) \times O(m-p)$; while the vertical subspace of $(B_1)_{j(b)}$, which we denote $V((B_1)_{j(b)})$ is naturally identified with $o(m)$, the Lie Algebra of $O(m)$. Restricted to the vertical subspaces at b and $j(b)$, the mapping dj corresponds under these natural identifications to the map:

$$\text{"}dj\text{"}: o(p) \times o(m-p) \rightarrow \left(\begin{array}{c|c} o(p) & 0 \\ \hline 0 & o(m-p) \end{array}\right) \subset o(m) .$$

Denote by $\alpha: o(m) \rightarrow o(m)$ the projection

$$\alpha: \left(\begin{array}{c|c} A & C \\ \hline -C^T & B \end{array}\right) \longrightarrow \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

where A is $(p \times p)$ skew symmetric and B is $(m-p) \times (m-p)$

skew symmetric.

Then the matrix valued 1-form $\bar{\varphi} - \alpha\ \bar{\varphi} = (1-\alpha)\bar{\varphi}$ is always zero on the vertical subspace of $(F(G))_b$. Equivalently, the real valued 1-forms $\bar{\varphi}_{rs}$, $p+1 \leq r \leq m$, $1 \leq s \leq p$, are zero on the vertical subspace of $(F(G))_b$.

4.2. We now define real valued 1-forms $\bar{\omega}_\beta$ $\beta = 1,\ldots p$ on $F(G)$ as follows. Let $b \in F(G)$ and suppose $b = (n, e_1, \ldots e_m)$ then

$$(\bar{\omega}_\beta)_b(X) = \langle G\ d\pi X,\ e_\beta \rangle$$

(recall that $\pi\colon F(G) \to N$ is the projection map.) It is easy to see that the forms $\bar{\omega}_\beta$ are smooth: choose coordinates on $F(G)$ and compute as in Singer [8] Chapter IV p. 29. One also can check that the $\{\bar{\omega}_\beta\}$ are an independent set of 1-forms and that they vanish on the vertical.

4.3. By (4.1) we see that the 1-forms $\bar{\varphi}_{rs}$, $p+1 \leq r \leq m$, $1 \leq s \leq p$ are linear combinations of the $\bar{\omega}_\beta$. We define

$$\bar{\varphi}_{rs} = \sum_{\beta=1}^{p} b_{rs\beta}\, \bar{\omega}_\beta ,$$

where the $b_{rs\beta}$ are smooth functions on $F(G)$.

Let us agree that $\perp$ denotes orthogonal complement.

For each $b \in F(G)$ define a linear map

$$S(b): [G(n)]^{\perp} \to \mathrm{Hom}(N_n, N_n) \qquad (\pi(b)=n)$$

as follows. If $b = (n, e_1, .. e_p, e_{p+1}, .. e_m)$ we have $G(n) = \mathrm{span}\,(e_1, .. e_p)$ and $G(n)^{\perp} = \mathrm{span}(e_{p+1}, .. e_m)$. Let $S(b)e_r$ be the element of $\mathrm{Hom}(N_n, N_n)$ whose matrix with respect to $G^{-1}(e_1), .. G^{-1}(e_p)$ is $(b_{r\alpha\beta}(b))$. Because the connection form φ is Ad-equivarient, the linear map $S(b)$ is independent of the choice of $b \in \pi^{-1}(n)$. Thus, for each $n \in N$ we have a linear map S_n: $G(n)^{\perp} \to \mathrm{Hom}(N_n, N_n)$.

Definition: The linear map S_n is called the second fundamental form of the p-plane field G along f (at n).

Suppose we take $G = df$ then it follows that S_n as defined here is the same as the second fundamental form of the embedding f as defined for instance in Singer [8] Chapter VII p. 10 .

4.4. Definition: Let G be a p-plane field along f. The vector field g along f, which is dual (with respect to the inner product) to the linear functional trace $\circ\, S_n$ (on $G(n)^{\perp}$) is called the mean curvature vector field of G.

The definition implies that $g(n) \in G(n)^{\perp}$.

If $G = df$, g is, of course, the mean curvature vector of the embedding f.

4.5. Definition: A p-plane field along f is parallel if and only if its mean curvature vector field is identically zero. The definition says that the p-plane field is parallel if and only if the trace of its second fundamental form is zero.

§5. Parallelism as expressed by the vanishing of a p-form on $F(G)$.

5.1. Let G be a p-plane field along f . We shall define smooth one forms $\psi_1, .. \psi_p$ on $F(G)$ with values in R^{m-p}. Let $r_{p+1}, .. r_m$ be a standard orthonormal basis of R^{m-p} equipped with its usual inner product. Let $b \in F(G)$ and $X \in F(G)_b$, we define

$$\langle(\psi_i)_b(X), r_j\rangle = (\overline{\varphi}_{ji})_b(X) \quad i=1,...p, j=p+1,...m$$

We can always wedge together 1-forms with values in a vector space V over R with real values 1-forms, by using the multi-linearity of the map

$$R \times R .. \times R \text{ (k times)} \to V$$

given by

$$(r_1, r_2, .., r_k, v) \to r_1 \cdot r_2 \ldots r_k \cdot v$$

In this sense define the p-form μ on $F(G)$ with values in R^{m-p} by

$$\mu = \sum_{i=1}^{p} \bar{\omega}_1 \wedge \cdot\cdot \wedge \bar{\omega}_{i-1} \wedge \psi_i \wedge \bar{\omega}_{i+1} \wedge \cdot\cdot \wedge \bar{\omega}_p$$

Let $\mathcal{I}$ denote the differential ideal generated by μ. (see §3.2).

Lemma: The mean curvature vector at n, $g(n)$, is zero, if and only if there is an integral p-plane of $\mathcal{I}$ complementary to the vertical at some point b such that $\pi(b) = n$.

Proof: Suppose $b \in F(G)$ with $\pi(b) = n$. Suppose there exists a p-plane P complementary to the vertical at $F(G)_b$ and P is an integral plane of $\mathcal{I}$. We show that $g(n) = 0$.

$b = (n, e_1, \ldots e_m)$ say. Choose $X_i \in P$ such that

$$Gd\pi X_i = e_i .$$

Then $0 = \mu(X_1 \wedge \ldots \wedge X_p) = \sum_{\alpha=1}^{p} \psi_\alpha(X_\alpha) =$

$$= \sum_{t=p+1}^{m} < \sum_\alpha \psi_\alpha(X_\alpha), r_t > r_t = \sum_{t=p+1}^{m} (\sum_\alpha < \psi_\alpha(X_\alpha), r_t >) r_t$$

$$= \sum_{t=p+1}^{m} (\sum_\alpha \bar{\varphi}_{t\alpha}(X_\alpha)) r_t$$

So $0 = \sum_\alpha \bar{\varphi}_{t\alpha}(X_\alpha)$ for $t = p+1, \ldots m$.

By the choice of the X_α

$$\overline{\varphi}_{t\alpha}(X_\alpha) = \sum_\beta b_{t\alpha\beta}(b)\overline{w}_\beta(X_\alpha) = b_{t\alpha\alpha}(b)$$

Hence $0 = \sum_\alpha b_{t\alpha\alpha}(b) \quad t = p+1, \ldots m$.

Thus the linear functional trace o S_n is identically zero on $G(n)$. Therefore $g(n) = 0$. To demonstrate the converse, pick any b with $\pi(b) = n$, say $b = (n, e_1, \ldots e_m)$. Choose any X_α $\alpha = 1 \ldots p$ such that $G d\pi X_\alpha = e_\alpha$ $\alpha = 1 \ldots p$. Let $P = \text{span}\ (X_1, \ldots X_p)$. Now read the computation given above in the reverse order.

Q.E.D.

THEOREM. The following three statements are equivalent.

1. G is parallel
2. $F(G)$ is an integral manifold of $\mathcal{L}$
3. For each $n \in N^p$, there is a neighborhood U_n and a local cross section $c: U_n \to F(G)$ which is an integral submanifold of $\mathcal{L}$.

Proof: 2. implies 3. is obvious

3. implies 1. by the lemma

We claim that 1. implies 2. By the construction in the second part of the proof of the lemma we see that at every point $b \in F(G)$, all the p-planes complementary to the vertical are integral planes of the form μ . These planes are an open dense subset of all the p-planes, hence μ is identically zero on $F(G)$. Therefore the differential ideal it generates, $\mathcal{I}$, is identically zero on $F(G)$. Q.E.D.

§6. The autoparallel p-surfaces.

6.1. THEOREM: If (N^p, f) is a p-dimensional minimal surface in M^m , then df, the tangent p-plane field along f is parallel.

<u>Proof</u>: This theorem is immediate from our definition of parallel as the vanishing of the mean curvature vector field. Q.E.D.

Generalizing in the sense of (1.1), the minimal surfaces are autoparallel.

For every p-plane field G along an embedding $f: N^p \to M^m$, the image of G defines a p-plane field in the Grassmann sense; a lift of N^p into $\mathcal{G}^p(M)$. It is conceivable that there are embeddings $f: N^p \to M^m$ for which the (Grassmann) p-plane field is the image of a p-plane field $G(\neq df)$ along f (in our sense) and that G is parallel along f .

6.2. Let us compare (1.1) with (6.1, when $p=1$) .

Let γ be an arc length parametrized geodesic. Then γ is autoparallel in the sense of (1.1). But γ is also a one dimensional minimal surface hence γ is autoparallel in the sense of (6.1).

Now let γ be an arc length parametrized curve, which as an isometric embedding is a one dimensional minimal surface. By definition it is autoparallel in the sense of

(6.1). We show that it is autoparallel in the sense of (1.1).

Assume γ is defined on a neighborhood U of the origin in R^1. $F(d\gamma)$ is the bundle of adapted frames of the embedding γ. Recall that B_1 is the full O(m) bundle over U and that $j: F(d\gamma) \to B_1$ is the inclusion map.

From the definition of μ (5.1), for p=1, $\mu = \psi_1$. Now ψ_1 is esentially $\delta j(1-\alpha)\varphi$ where α was defined in (4.1). We are assuming $\mu = 0$ on $F(d\gamma)$ or that $(F(d\gamma), j)$ is an integral manifold of the form $(1-\alpha)\varphi$ on B_1. Let $b = (0, \dot{\gamma}(o), e_1(o), .. e_m(o)) \in F(d\gamma)$. Let $\tilde{\gamma}$ be the unique horizontal lift of γ into $F(d\gamma)$ through b with respect to the connection $\alpha o \overline{\varphi}$ on $F(d\gamma)$. $\tilde{\gamma}$ has the form $\tilde{\gamma}(t) = (t, \dot{\gamma}(t), e_2(t), .. e_m(t))$ (recall γ is parametrized by arc length.). $\tilde{\gamma}$ is the unique integral submanifold of $\alpha o \overline{\varphi}$ on $F(d\gamma)$ passing through b. Thus $j o \tilde{\gamma}$ is an integral submanifold of $\alpha o \varphi$ on B_1 passing through $j(b) = b$. Since $(F(d\gamma), j)$ is an integral manifold of $(1-\alpha)\varphi$ on B_1 so is $j o \tilde{\gamma}$. Thus $j o \tilde{\gamma}$ is an integral manifold of $(1-\alpha)\varphi$ and $\alpha o \varphi$ on B_1; that is $j o \tilde{\gamma}$ is an integral manifold of φ on B_1 passing through b. Hence it must be the unique horizontal lift of γ into B_1 with respect to the Riemannian connection and

$$j \circ \tilde{\gamma}(t) = (t, \dot{\gamma}(t), e_2(t), .. e_m(t))$$

Thus γ is a geodesic and autoparallel in the sense of (§1.1). Q.E.D.

If we drop the assumption about arc length parametrization, I don't believe that we can prove this result. This may be tied up with the difficulty in extending Theorem B (§1.5) (see (§1.7)).

CHAPTER III

THE MAIN THEOREM

§7. Statement of the Main Theorem and Remarks.

7.1. If we work only in the real analytic case, we may prove a generalization of Theorem A (§1.5). That is, we are concerned with the problem of finding a parallel p-plane field along an embedding $f: N^p \to M^m$ which extends initial conditions along an initial manifold (see (§2.3)). It turns out that there are always many p-plane fields with these properties. It is only when we introduce the additional requirement that the p-plane field be known a priori in $(p-1)$ dimensions at every point, that we can begin to ask for a unique parallel p-plane field along f. Since this condition is vacuous for the case $p=1$, the problem is a generalization of the classical problem of finding a parallel 1-plane field along a curve, which extends the choice of a 1-plane at some point on the curve. (see also (§1.7)).

7.2. MAIN THEOREM: Let $f\colon N^p \to M^m$ be a C^ω isometric embedding of R.M. Let $n_o \in N^p$ and U_{n_o}, a neighborhood of n_o.

Let (D^{p-1}, G^{p-1}) be a (p-1) plane field along f defined on U_{n_o}.

Let $i\colon N^{p-1} \to N^p$ be an initial manifold with $i(n_1) = n_o$ and $di(N^{p-1}_{n_1}) = D^{p-1}(n_o)$.

Let $G^p\colon \delta i(T(N^p)) \to T(M)$ be initial conditions along $f{\circ}i$ which are compatable with (D^{p-1}, G^{p-1}).

Then there is a p-plane field G along f defined in a neighborhood $\tilde{U}_{n_o}$ of n_o ($\tilde{U}_{n_o} \subset U_{n_o}$) which satisfies the following three conditions.

1. G extends (D^{p-1}, G^{p-1})
2. G extends G^p
3. G is parallel

Furthermore G is uniquely determined by these three conditions except for the choice of the neighborhood $\tilde{U}_{n_o}$ (that is, G is uniquely determined as the germ of a p-plane field along f).

7.3. A remark on the method of proof. The a priori knowledge of (D^{p-1}, G^{p-1}) enables us to set up a family of bundles and mappings by which, the condition for parallelism (expressed by the differential ideal $\mathcal{I}$) can be carried over to a unit sphere bundle over N . Here we will use the initial conditions to determine a unique integral manifold - hence a unique parallel p-plane field along f .

The existence and uniqueness are proved using the Cartan-Kahler Theorem; thus the first result is local. In §9 we will show that under somewhat stronger hypotheses it is possible to obtain a global solution.

§8. Proof of the Main Theorem.

8.1. For the proof we will take $N^p = U_{n_o}$.

As a first step, we will obtain the following basic commutative diagram:

$$
\begin{array}{ccccc}
F^2(D^{p-1},G^{p-1}) & \xrightarrow{\rho} & F(D^{p-1},G^{p-1}) & \xrightarrow{j} & B_1 \subset F(M) \\
\downarrow \pi_1 & & \downarrow \pi & & \downarrow \pi_o \\
S^2(D^{p-1},G^{p-1}) & \xrightarrow{\kappa} & S(D^{p-1},G^{p-1}) & \xrightarrow{i} & S_1 \\
\downarrow P_1 & & \downarrow P & & \downarrow P_o \\
N & \xleftrightarrow{\text{identity}} & N & \xleftrightarrow{\text{identity}} & N
\end{array}
$$

BASIC DIAGRAM

For future reference the mappings will be:

$$
\begin{array}{ccccc}
(n,e_1,\ldots e_m,G) & \xrightarrow{\rho} & (n,e_1,\ldots e_m) & \xrightarrow{j} & (n,e_1,\ldots e_m) \\
\downarrow{\pi_1} & & \downarrow{\pi} & & \downarrow{\pi_o} \\
(n,e_p,G) & \xrightarrow{\varkappa} & (n,e_p) & \xrightarrow{i} & (n,e_p) \\
\downarrow{P_1} & & \downarrow{P} & & \downarrow{P_o} \\
n & \xleftrightarrow{id} & n & \xleftrightarrow{id} & n
\end{array}
$$

We have already obtained $F(D^{p-1},G^{p-1})$, the bundle of adapted frames of the $(p-1)$ plane field (D^{p-1},G^{p-1}) and its inclusion j into B_1. (see (2.5))

8.2. We also obtain from (D^{p-1},G^{p-1}) a unit $(m-p)$ sphere bundle $S(D^{p-1},G^{p-1})$ over N where

$$S(D^{p-1},G^{p-1}) = \{(n,v) \mid v \text{ is a unit vector in } G^{p-1}(n)^{\perp}, n \in N\}$$

Remark: $S(D^{p-1},G^{p-1})$ is an associated bundle of the principal $O(m-p+1)$ bundle over N whose fibre over n is the set of orthonormal $(m-p+1)$ frames of $G^{p-1}(n)^{\perp}$. We denote by P the projection

$$S(D^{p-1},G^{p-1}) \to N$$

$$P(n,v) = n .$$

8.3. We obtain S_1 in the same way as B_1, that is, it is the bundle over N whose fibre over n is the fibre of $S(M)$ over $f(n)$, which is in turn the unit sphere in $M_{f(n)}$. We denote by P_o the projection $S_1 \to N$. We denote by i the inclusion of $S = S(D^{p-1},G^{p-1})$

into S_1 . Note: where it is clear, we will often drop the (D^{p-1},G^{p-1}), e.g. $S(D^{p-1},G^{p-1}) = S$.

Obviously $P_o oi = P$.

Let $\pi_o: B_1 \to S_1$ be given by

$$\pi_o: (n,e_1,..e_m) \to (n,e_p)$$

<u>Remark:</u> B_1 is a principal $O(m-1)$ bundle over S_1 , with projection π_o .

8.4 Let $\pi_o(b) = (n,e_p)$. Then b induces a natural identification of the fibre of B_1 through b (as a bundle over N) with $O(m)$ (considered here as the orthogonal matrices). See for example, Singer [8] Chapter IV. This induces an identification of the fibre of S_1 through $\pi_o(b)$ with

$$O(m) / O(m-1)$$

where $O(m-1)$ is given as the group of matrices in $O(m)$ of the form:

$$\left(\begin{array}{c|c|c} * & o & * \\ \hline o & 1 & o \\ \hline * & o & * \end{array}\right) \leftarrow \text{row } p$$

$\uparrow$ column p

The vertical subspace of $(B_1)_b \equiv V((B_1)_b)$ (as a bundle over N) is then naturally identified with $o(m)$, the skew symmetric matrices. This induces an identification of the vertical subspace of $(S_1)_{\pi_o(b)}$ (as a bundle over N) with

$$o(m)/_{o(m-1)}$$

where $o(m-1)$ is the subalgebra of skew symmetric matrices of the form:

$$\left(\begin{array}{c|c|c} A & o & C \\ \hline o & o & o \\ \hline -C^T & o & B \end{array}\right) \leftarrow \text{row } p$$

$$\uparrow \text{ column } p$$

where A is $(p-1) \times (p-1)$ skew symmetric and B is $(m-p) \times (m-p)$ skew symmetric.

Restricted to the vertical subspaces at b and $\pi_o(b)$, the mapping $d\pi_o$ corresponds, under these natural identifications to the projection

$$\text{"}d\pi_o\text{"}: o(m) \to o(m)/_{o(m-1)}$$

But the exact sequence:

$$o \to o(m-1) \to o(m) \underset{\alpha}{\overset{\text{"}d\pi_o\text{"}}{\rightleftarrows}} o(m)/_{o(m-1)} \to o$$

splits naturally, where α carries the equivalence class of

$$\begin{pmatrix} A & F & C \\ -F^T & o & E \\ -C^T & -E^T & B \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} o & F & o \\ -F^T & o & E \\ o & -E^T & o \end{pmatrix}$$

Thus we can identify $V((S_1)_{\pi_o(b)})$ with the image of α, which by abuse of notation we will consider as a subspace of $V(B_1)_b$. We then know that $d\pi_o$ maps this subspace isomorphically onto $V(S_1)_{\pi_o(b)}$. The vertical of $(B_1)_b$ (as a bundle over S_1) is just $o(m-1)$.

We carry this one step further: Use b to identify the image of α with $e_p^{\perp}$, the $(m-1)$ plane orthogonal to $e_p \subset M_{f(n)}$. Thus we use b to identify $V(S_1)_{\pi_o(b)=(n,e_p)}$ with $e_p^{\perp}$. A computation, using the equivariance of the identifications: $V(B_1)_{b'} \equiv o(m)$, $b' \in \pi_o^{-1}(n,e_p)$ shows that the identification $V(S_1)_{\pi_o(b)} \longleftrightarrow e_p^{\perp}$ is independent of b.

8.5. Now we define

$$\pi\colon F(D^{p-1},G^{p-1}) \to S(D^{p-1},G^{p-1})$$

by $\pi\colon (n,e_1,..e_m) \to (n,e_p)$

We have $\pi_o o j = i o \pi$.

<u>Remark</u>: $F(D^{p-1},G^{p-1})$ is a principal $O(p-1) \times O(m-p)$ bundle over $S(D^{p-1},G^{p-1})$ with projection π.

Let us continue (see (2.5)) the identification

$$O(p-1) \times O(m-p+1) \equiv \left(\begin{array}{c|c} O(p-1) & o \\ \hline o & O(m-p+1) \end{array}\right) .$$

Thus the vertical subspace of F_b, as a bundle over N, is identified with

$$\left(\begin{array}{c|c} o(p-1) & o \\ \hline o & o(m-p+1) \end{array}\right)$$

8.6 <u>Remark</u>: Using the identifications in (8.4) and (8.5) together with the relation $\pi_o o j = i o \pi$, we have, for $\pi(b) = (n,e_p)$ the following three facts.

1. $V(S_{\pi(b)})$ is identified with the matrices of the form

$$\left(\begin{array}{c|c|c} o & o & o \\ \hline o & o & E \\ \hline o & -E^T & o \end{array}\right) \in \left(\begin{array}{c|c} o(p-1) & o \\ \hline o & o(m-p+1) \end{array}\right)$$

   where E lies in the p^{th} row and $-E^T$ lies in the p^{th} column. Again by abuse of language we will consider these matrices as a subspace of $V(F_b)$ (as a bundle over N). $d\pi$ maps this subspace isomorphically onto $V(S_{\pi(b)})$.

2. The mapping $j: F \to B$, restricted to the fibre through b (over S) corresponds to the map:

$$O(p-1) \times O(m-p) \to \left(\begin{array}{c|c|c} O(p-1) & o & o \\ \hline o & 1 & o \\ \hline o & o & O(m-p) \end{array}\right)$$

This says that the vertical subspace of F_b (as a bundle over S) is identified with

$$\left(\begin{array}{c|c|c} o(p-1) & o & o \\ \hline o & o & o \\ \hline o & o & o(m-p) \end{array}\right)$$

3. $V(S_{\pi(b)})$ is identified with the orthogonal complement to span $(G^{p-1}(n), e_p)$ and this identification does not depend on $b \in \pi^{-1}(n, e_p)$.

8.7. Consider the set $F^2 = F^2(D^{p-1}, G^{p-1}) = \{(n, e_1, \ldots e_m, G') | (n, e_1, \ldots e_m) \in F(D^{p-1}, G^{p-1})$, G' maps N_n into span $(e_1, \ldots e_p) \subset M_{f(n)}$ isometrically and $G'|_{D^{p-1}(n)} = G^{p-1}\}$. It is easy to obtain a manifold structure for F^2 . In fact if $b = (n, e_1, \ldots e_m, G') \in F^2$, let $x \in N_n$ be the unique element such that $G'(x) = e_p$. Let X be the unique unit vector field on N , defined in a neighborhood U of n and normal to $D^{p-1}(n')$ for $n' \in U$ with $X(n) = x$. Then a neighborhood of b is defined to be the set of elements in F^2 of the form $(n', e_1', \ldots e_m', G'')$ where $n' \in U$ and $G''(X(n')) = e_p'$. We can now impose a C^ω structure on the neighborhood by requiring its natural identification with an open subset of F to be a diffeomorphism.

Remark: F^2 is an $O(1)$ principal bundle over F with projection $\rho: (n,e_1,\ldots e_m,G') \to (n,e_1,\ldots e_m)$.

$\rho^{-1}(n,e_1,\ldots e_m)$ is the pair of points $(n,e_1,\ldots e_m,G')$ and $(n,e_1,\ldots e_m,G'')$ where G' and G'' are the isometries that extend $G^{p-1}: D^{p-1}(n) \to$ span $(e_1,\ldots e_{p-1})$ and which send N_n into span $(e_1,\ldots e_p)$. ρ is a local diffeomorphism.

8.8. Similarly consider the set $S^2 = S^2(D^{p-1},G^{p-1}) = \{(n,v,G')|(n,v) \in S$, G' extends G^{p-1} and maps $N_n \to$ span $(G^{p-1}(n),v) \subset M_{f(n)}$ isometrically$\}$. A manifold structure on S^2 is obtained in the same manner as that for F^2.

Remark: S^2 is an $O(1)$ principal bundle over S with projection $\varkappa: (n,v,G') \to (n,v)$.

$\varkappa^{-1}(n,v)$ is the pair of points (n,v,G') and (n,v,G'') where G' and G'' are the two isometries that extend $G^{p-1}: D^{p-1}(n) \to G^{p-1}(n)$ and send N_n into span $(G^{p-1}(n),v)\}$. $\varkappa$ is a local diffeomorphism.

8.9. Let $\pi_1: F^2 \to S^2$ by $\pi_1(n,e_1,\ldots e_m,G') \to (n,e_p,G')$

$$P_1: S^2 \to N \text{ by } P_1(n,v,G') = n$$

Remark: F^2 is an $O(p-1) \times O(m-p)$ principal bundle over S^2 with projection π_1. We have $\varkappa \circ \pi_1 = \pi \circ \rho$,

$P_1 = P \circ \rho$ and

$\rho \circ r_\sigma = r_\sigma \circ \rho$ if

$\sigma \in O(p-1) \times O(m-p)$.

8.10. We now conclude the series of identifications given in (8.4), (8.5) and (8.6).

Remark: By means of the local diffeomorphisms ρ and $\varkappa$ we may identify F^2_b with $F_{\rho(b)}$ and $S^2_{\pi_1(b)}$ with $S_{\pi o \rho(b)} = \varkappa o \pi_1(b)$. We define $(d(P_1 o \pi_1))^{-1}(o)$ to be the vertical part of $F^2{}_b$ over N . (denoted $V(F^2{}_b)$) and identify it with $V(F_{\rho(b)}) =$

$$\left(\begin{array}{c|c} o(p-1) & o \\ \hline o & o(m-p+1) \end{array}\right)$$

We define $(dP_1)^{-1}(o)$ to be the vertical part of $S^2{}_{\pi_1(b)}$ (denoted $V(S^2{}_{\pi_1(b)})$) and identify it with

$$V(S_{\pi o \rho(b)}) = (\text{span } (G^{p-1}(n), e_p))^{\perp}$$

Then $d\pi$ maps the subspace of $V(F^2{}_b)$ consisting of the matrices

$$\left(\begin{array}{c|c|c} o & o & o \\ \hline o & o & E \\ \hline o & -E^T & o \end{array}\right)$$

(where E lies in the p^{th} row and $-E^T$ lies in the p^{th} column.) isomorphically onto $V(S^2{}_{\pi_1(b)})$. We note that the vertical subspace of $F^2{}_b$ (as a bundle over S^2) is

$$\begin{pmatrix} o(p-1) & o & o \\ \hline o & o & o \\ \hline o & o & o(m-p) \end{pmatrix}$$

8.11. On each of the six manifolds S^2, S, S_1, F^2, F and B_1, there is a natural C^ω diffeomorphism of period two which we will call the antipodal map $\mathfrak{a}$

On F^2 $\mathfrak{a}$: $(n,e_1,..e_{p-1},e_p,e_{p+1},..e_m,G) \rightarrow$

$$(n,e_1,..e_{p-1},-e_p,e_{p+1},..e_m,G) .$$

$\mathfrak{a}$ is similarly defined on F and B_1 .

On S^2 $\mathfrak{a}$: $(n,v,G) \rightarrow (n,-v,G)$

$\mathfrak{a}$ is similarly defined on S and S_1 .

The antipodal maps commute with the mappings in the basic diagram. We note that on B_1, $\mathfrak{a} = r_\tau$ where the matrix of τ , (τ_{ij}) is given by $\tau_{ij} = o$ if $i \neq j$, $\tau_{ii} = 1$, $i \neq p$, $\tau_{pp} = -1$)

8.12 We have now concluded the construction of the basic diagram and its important properties.

The second step in the proof proceeds as follows. If G is a p-plane field along f which extends G^{p-1} there is a manifold $R = R(G^{p-1},G)$ which is a submanifold both of F^2 and $F(G)$. From the properties of the basic diagram we can construct on F^2 a differential ideal $\mathfrak{g}$ generated by a vector-valued p-form λ . We can then show that R is an integral submanifold of $\mathfrak{g}$ if and only if R is an integral submanifold of $\mathcal{J}$, the ideal for parallelism given in (5.1). The third step of the proof will then be to construct such a manifold R which is an integral manifold of $\mathcal{J}$.

8.13. On F^2, define 1-forms $\lambda_1,\ldots\lambda_p$ with values in R^{m-p} as follows. Let $r_{p+1},\ldots r_m$ be a standard orthonormal basis of R^{m-p} (recall (5.1)). The $\{\varphi_{ij}|i>j\}$ are the one forms of the Riemannian connection on F(M) and B_1 (recall (4.1)). Let $b \in F^2$ and $X \in F^2{}_b$. Define λ_i by

$$\langle\lambda_i|_b(X), r_j\rangle = \delta(j\circ\rho)\varphi_{ji}(X) \quad \text{for } i=1\ldots p,\ j=p+1,\ldots m .$$

One sees that the forms λ_i are C^ω. On F^2, define real valued 1-forms $\omega_1,\ldots\omega_p$ as follows. Let $b \in F^2$ and $x \in (F^2)_b$ say $b = (n,e_1,\ldots e_m,G)$. Define ω_i by

$$\omega_i|_b(x) = \langle\, Gd(P_1\circ\pi_1)x, e_i\rangle , \quad i=1,\ldots p .$$

One sees that the forms ω_i are C^ω.

On F^2 define the p form λ with values in R^{m-p} by

$$\lambda = \sum_{i=1}^{p} \omega_1\wedge\ldots\wedge\omega_{i-1}\wedge\lambda_i\wedge\omega_{i+1}\wedge\ldots\wedge\omega_p$$

Let $\mathcal{J}$ denote the differential ideal generated by λ.

<u>Remark</u>: Since $\delta\mathbf{a}\lambda_i = \lambda_i \quad i<p$ and $\delta\mathbf{a}\lambda_p = -\lambda_p$

and $\delta\mathbf{a}\omega_i = \omega_i \quad i<p$ and $\delta\mathbf{a}\omega_p = -\omega_p$

we have $\delta \mathfrak{a} \lambda = -\lambda$. Therefore the antipodal map $\mathfrak{a}$ on F^2 carries integral p-planes of λ into the same.

8.14. Suppose that a p-plane field G extends (D^{p-1}, G^{p-1}) . Let $R = R(G^{p-1}, G) = \{(n, e_1, .. e_m) | (n, e_1, .. e_m) \in F$ and span $(e_1, .. e_p) = G(n)\}$.

Remark: R is a principal $O(p-1) \times O(1) \times O(m-p)$ bundle over N . By inclusion it is a submanifold of F and there is a natural lift $L: R \to F^2$ given by

$$L: (n, e_1, .. e_m) \to (n, e_1, .. e_m, G|_{N_n})$$

On the other hand, there is a natural inclusion

$\overline{k}: R \to F(G)$, the bundle of adapted frames of G .

Since $\delta \overline{k} \omega_i = \delta L \omega_i \quad i=1,..p$
and $\delta \overline{k} \psi_i = \delta L \lambda_i$, it follows
that $\delta \overline{k} \mu = \delta L \lambda$ .. Recalling that μ generates $\mathcal{d}$ and λ generates $\mathcal{g}$,

(R,L) is an integral manifold of $\mathcal{g}$ if and only if
$(R, \overline{k})$ is an integral manifold of $\mathcal{d}$.

It follows from the theorem of (5.1) that G is a parallel p-plane field along f if and only if $(R(G^{p-1}, G), L)$ is an integral manifold of $\mathcal{g}$.

The third step in the proof is to construct such an integral manifold (R,L) . We will show that, on a certain submanifold of F^2 , λ induces a vector bundle-valued form on S^2 . Using the Cartan-Kahler Theorem, we will obtain an integral manifold of this form on S^2. The integral manifold will represent a parallel p-plane field G along f. We will then use it be construct an integral manifold (R,L).

8.15. Select orthonormal vector fields $Z_1,..Z_{p-1}$ spanning D^{p-1} in some neighborhood U_{n_o}' of n_o. Consider the set

$$E(Z) = E(Z_1,..Z_{p-1},D^{p-1},G^{p-1}) = \{(n,e_1,..e_m,G) \mid (n,e_1,..e_m,G) \in F^2,\ e_i = G^{p-1}(Z_i(n))\ i=1,..p-1\}$$

Remark: $E(Z)$ is a principal $O(m-p)$ bundle over S^2 (or rather, that part of S^2 that lies over U_{n_o}'). The inclusion map

$$k: E(Z) \to F^2$$

commutes with right translation. On the fibre through $b \in E(Z)$, the mapping k corresponds to the mapping

$$O(m-p) \to \left(\begin{array}{c|c} I & o \\ \hline o & O(m-p) \end{array}\right)$$

under the natural identifications of fibres with the group. A corresponding statement holds for the Lie Algebras; that is, the vertical subspace of $E(Z)_b$ is carried into $V(F^2{}_{k(b)})$ by

$$\text{"}dk\text{"}: o(m-p) \to \left(\begin{array}{c|c} o & o \\ \hline o & o(m-p) \end{array}\right)$$

8.16. Lemma: The form λ restricted to $E(Z)$ is a horizontal equivariant $O(m-p)$ form. That is, it vanishes on a p-vector, one of whose vectors lies in $V(E(Z)_b)$ (as a bundle over S^2) and if $\sigma \in O(m-p)$ we have:

$$\delta r_\sigma \lambda = \sigma^{-1}\lambda$$

where $O(m-p)$ acts on R^{m-p} as isometries.

Proof: The first statement follows from

$$\lambda_i|_{V(E(Z)_b)} = o \quad i=1\ldots p ,$$

while the second statement follows from the invariance of the ω_i and the Ad-equivariance of the λ_i on $E(Z)$. Q.E.D.

Because of the remark in 8.15 there is a C^ω p-form λ_Z induced on S^2 (again, on that part over $U_{n_o}{}'$) with values in the associated vector bundle $C(Z)$ over S^2 (with standard fibre R^{m-p}) . See Koszul [4]. This p-form is obtained as follows.

Let

$$q\colon E(Z) \times R^{m-p} \to C(Z)$$

be the natural map of

(principal bundle) x (fibre) → associated bundle

Let $s \in S^2$ and $X_1,\ldots X_p \in S^2{}_s$. Choose $b \in E(Z)$ so that $\pi_1 o k(b) = s$, and choose $\tilde{X}_1,\ldots\tilde{X}_p \in E(Z)_b$ so that

$d(\pi_1 o k)\tilde{X}_i = X_i$. Then $\lambda_Z(X_1 \wedge \ldots \wedge X_p)$ is given by

$$\lambda_Z(X_1 \wedge \ldots \wedge X_p) = q(b, \lambda(\tilde{X}_1 \wedge .. \wedge \tilde{X}_p))$$

The lemma shows that the choices of b, $\tilde{X}_1, .. \tilde{X}_p$ do not effect the definition of λ_Z .

8.17. Let Z_p be one of the two C^ω unit vector fields defined in some neighborhood $U_{n_o}" \subset U_{n_o}'$ containing n_o and such that $Z_p(n) \perp D^{p-1}(n)$ for $n \in U_{n_o}"$.

Since the initial condition G^p are compatable with (D^{p-1}, G^{p-1}) we have a lifting

$$g: N^{p-1} \to S^2$$

given by

$$g(n) = (n, G^p o(h_i|_{N_n})^{-1}(Z_p(n)),\ G^p o(h_i|_{N_n})^{-1})$$

(recall that h_i is the natural vector bundle isometry given in (2.3)).

Let us denote by P^{p-1} the (p-1) plane $dg(N^{p-1}_{n_1})$.

Since g is a lift, $d\pi_1(P^{p-1}) = di(N^{p-1}_{n_1})$.

By hypothesis $di(N^{p-1}_{n_1}) = D^{p-1}(n_o)$.

Thus $d\pi_1(P^{p-1}) = D^{p-1}(n_o)$.

Choose $X_1, .. X_{p-1}$ in P^{p-1} such that $d\pi_1(X_i) = Z_i(n)$ $i=1..p-1$. We wish to apply the Cartan-Kahler Theorem (3.9) to P^{p-1} for the p-form λ_Z on S^2 with values in the vector bundle $C(Z)$.

Main Lemma: The linear map:

$$(S^2)_{g(n_1)} \to \text{fibre of } C(Z) \text{ over } g(n_1)$$

given by

$$T\colon X \to \lambda_Z(X_1 \wedge \ldots \wedge X_{p-1} \wedge X) \text{ is surjective.}$$

The unique integral plane P^p which contains P^{p-1} is complementary to the vertical of $(S^2)_{g(n_1)}$.

Proof: We show that T is surjective. Let us restrict our attention to those $X = X_p \in (S^2)_{g(n_1)}$ for which $dP_1 X_p = Z_p(n_o)$. Now choose $b \in E(Z)$ with $\pi_1 o k(b) = g(n_1)$ (k, the inclusion map, $E(Z) \to F^2$) . Also choose $\widetilde{X}_1, \ldots \widetilde{X}_{p-1}, \widetilde{X}_p \in E(Z)_b$ where $d(\pi_1 o k)\widetilde{X}_i = X_i$. Then

$$\lambda_Z(X_1 \wedge \ldots \wedge X_{p-1} \wedge X_p) = q(b, \lambda(\widetilde{X}_1 \wedge \ldots \wedge \widetilde{X}_{p-1} \wedge \widetilde{X}_p))$$

Since b acts as an isomorphism of R^{m-p} onto the fibre of C_Z through $g(n_1)$, $\lambda_Z(X_1 \wedge \ldots \wedge X_{p-1} \wedge X_p)$ varies over all the fibre as X_p varies over vectors in $(S^2)_{g(n_1)}$ that project to $Z_p(n_o)$ under P_1 if and only if $\lambda(\widetilde{X}_1 \wedge \ldots \wedge \widetilde{X}_{p-1} \wedge \widetilde{X}_p)$ varies over all R^{m-p} as $\widetilde{X}_p$ varies over vectors in $E(Z)_b$ which project to $Z_p(n_o)$ under $d(P_1 o \pi_1 o k)$. By the choice of the $\widetilde{X}_i$ we have

$$\lambda(\widetilde{X}_1 \wedge \ldots \wedge \widetilde{X}_{p-1} \wedge \widetilde{X}_p) = \sum_{i=1}^{p-1} \lambda_i(\widetilde{X}_i) + \lambda_p(\widetilde{X}_p) \ .$$

Thus it is sufficient to show that $\lambda_p(\tilde{X}_p)$ varies over all R^{m-p} as $\tilde{X}_p$ varies over vectors in $E(Z)_b$ which project to $Z_p(n_o)$ under $d(P_1 o \pi_1 ok)$. Since $E(Z)$ is a bundle over S^2, $\pi_1 ok$ is onto and so is $d(\pi_1 ok)$. Hence $d(\pi_1 ok)$ maps $E(Z)_b$ onto $V(S^2{}_{g(n_1)})$. This says (see (8.10)) that $dk(E(Z)_b)$ contains vectors which project to o under $d(P_1 o \pi_1)$ and have the form

$$\begin{pmatrix} * & o & o \\ o & o & E \\ o & -E^T & * \end{pmatrix}$$

when the entries of the (p^{th}) row vector E (equivalently the (p^{th}) column vector $-E^T$) vary over all of R^{m-p}. From the definition (see (8.13)) we observe that λ_p of such vectors varies over all of R^{m-p}. By adding each of these vectors to a fixed $\tilde{X}_p$ where $d(p_1 o \pi_1 ok)(\tilde{X}_p) = Z_p(n_o)$, we obtain a family of vectors which project to $Z_p(n_o)$ and are sent by λ_p onto R^{m-p}. This concludes the proof that T is surjective.

Note: The use of (8.10) in the above proof explains, in part, why we labored to develop the series of identifications in (8.4), (8.5), (8.6), and (8.10).

As in the proof of the Theorem (3.9), we see that $P^p = \ker T$ is the unique (ordinary) integral p-plane at $(S^2)_{g(n_1)}$ which contains P^{p-1}. It remains to show that P^p is complementary to the vertical. If not choose $X_p \in P^p$ and vertical and $X_1, ..X_{p-1}$ as before.

Now choose $\widetilde{X}_1,..\widetilde{X}_{p-1}, \widetilde{X}_p$ so that $d(\pi_1 o k)(\widetilde{X}_i) = X_i$ $i=1...p$. Then

$$o = \lambda(\widetilde{X}_1 \wedge .. \wedge \widetilde{X}_{p-1} \wedge \widetilde{X}_p) = \lambda_p(\widetilde{X}_p)$$

This says that in the matrix representation of $dk(\widetilde{X}_p)$ (above), $E = -E^T = o$. Hence $d(\pi_1 o k)(\widetilde{X}_p) = o$. Therefore $X_p = o$, which shows P^p cannot intersect the vertical.

Q.E.D.

8.18. We conclude from the Cartan-Kahler Theorem (3.9) that there exists a unique integral manifold (of λ_Z) of p-dimensions, say (V^p,g) which "extends" (N^{p-1},g) in a neighborhood of n_1 and whose tangent plane is carried by dg into a complement to the vertical at $(S^2)_{g(n_1)}$. Suppose $n_2 \in V^p$ with $g(n_2) = g(n_o)$ $(n_o \in N^{p-1})$. Then $dP_1|_{dg(V^p_{n_2})}$ is one to one and this must be true on some neighborhood of $n_2 \in V^p$. Thus we can define a cross section $\mathcal{L}$ mapping some neighborhood $\widetilde{U}_{n_o} \subset U_{n_o}$" of $n_o \in N^p$ into S^2, characterized uniquely by the two properties

1. $\mathcal{L} \circ i = g$ on $N^{p-1} \cap i^{-1}(\widetilde{U}_{n_o})$
2. $(\widetilde{U}_{n_o}, \mathcal{L})$ is an integral manifold of λ_Z.

Of course $(\widetilde{U}_{n_o}, \mathcal{L})$ defines on $\widetilde{U}_{n_o}$ a p-plane field along f which extends (D^{p-1}, G^{p-1}) and the initial conditions. By the way we have constructed λ_Z we will see that $(\widetilde{U}_{n_o}, \mathcal{L})$ defines a parallel p-plane field.

8.19. Let $E'(Z)$ be defined by

$$E'(Z) = \{(n, e_1, .. e_m, G) \mid (n, e_1 .. e_m, G) \in E(Z) \text{ and}$$
$$\pi_1 o k(n, e_1, .. e_m, G) = \mathcal{L}(n) \, , \quad n \in \widetilde{U}_{n_o}\}$$

That is $E'(Z)$ is the subspace of $E(Z)$ which is carried onto $\mathcal{L}(\widetilde{U}_{n_o})$ by $\pi_1 o k$.

Remark: $E'(Z)$ is an $O(m-p)$ bundle over $\widetilde{U}_{n_o}$ with projection $P_1 o \pi_1 o k$. The inclusion $k|_{E'(Z)}: E'(Z) \to F^2$ corresponds on the fibre to the map $O(m-p) \to I \times O(m-p) \subset O(p-1) \times O(m-p)$ and hence $dk|_{E'(Z)_b} : V(E'(Z))_b \to V(F^2{}_{k(b)})$ corresponds to

$$o(m-p) \to \left(\begin{array}{c|c} o & o \\ \hline o & o(m-p) \end{array}\right)$$

Remark: $\delta(k|_{E'(Z)}) \lambda = o$. In fact if P^p is complementary to the vertical, $d(\pi_1 o k) P^p = d(\mathcal{L} o P_1 o \pi_1 o k) P^p$ is an integral plane of λ_Z and such P^p are open and dense in all p-planes in $E'(Z)_b$. Hence $(E'(Z), k)$ is an integral manifold of $\mathcal{J}$, the differential ideal generated by λ .

8.20. Let $\sigma \in O(p-1) \times I \subset O(p-1) \times O(m-p)$. A short computation shows that on F^2 , dr_σ carries integral planes of λ which are complementary to the vertical (over N) into the same. Now right translate the set $k(E'(Z))$ by the elements of $O(p) \times I$ in F^2, and thus generate a principal $O(p-1) \times O(m-p)$ bundle over $\tilde{U}_{n_o}$; call it $E''(Z)$. Thus $(E''(Z), \text{inclusion})$ is an integral manifold of λ and hence of $\mathcal{J}$.

8.21. Suppose we use the other unit normal vector field $-Z_p$ to generate a submanifold (N^{p-1}, g^*) from the initial conditions. We then obtain $(\tilde{U}_{n_o}, \mathcal{L}^*)$ as an integral manifold of λ_Z. If $\mathfrak{a}$ is the antipodal map then $g^* = \mathfrak{a} \circ g$. The commutativity of the antipodal maps (8.11), the remark in (8.13) that $\delta \mathfrak{a} \lambda = -\lambda$, and the uniqueness property of integral manifolds extending (p-1) dim integral manifolds, together show that $\mathcal{L}^* = \mathfrak{a} \circ \mathcal{L}$. Hence $(\tilde{U}_{n_o}, \mathcal{L})$ and $(\tilde{U}_{n_o}, \mathcal{L}^*)$ define the same p-plane field G. We also see that:

$$\mathfrak{a} \circ k(E'(Z)) = k(E'(Z)^*) \quad \text{and} \quad \mathfrak{a} \circ k(E''(Z)) = k(E''(Z)^*)$$

Hence $E''(Z)$ and $E''(Z)^*$ are the two disconnected pieces of $R(G^{p-1}, G)$ (see (8.14)) and G is a parallel p-plane field along f defined on the neighborhood $\tilde{U}_{n_o}$ (by (8.20)).

8.22. The last step in the proof is to show that G is unique in the sense prescribed in the statement of the theorem. To do this it is only necessary to show that G is independent of the choice of $Z_1, \ldots Z_{p-1}$.

Let $Y_1, .. Y_{p-1}$ be another choice of vector fields spanning D^{p-1} on $U_{n_o}{}'$. We obtain $E(Y)$ and λ_Y on S^2. We want to show that for given initial conditions λ_Y and λ_Z determine the same integral manifold (and hence, same p-plane field). What we will show is that if P^p is an integral p-plane of λ_Z at $(S^2)_s$, complementary to the vertical, then P^p is also an integral p-plane of λ_Y.

Let $\Omega: E(Z) \to E(Y)$ be the diffeomorphism

$$\Omega(n, G^{p-1}(Z_1(n)), .. G^{p-1}(Z_{p-1}(n), v, e_{p+1} .. e_m, G)$$

$$= (n, G^{p-1}(Y_1(n), .. G^{p-1}(Y_{p-1}(n)), v, e_{p+1}, .. e_m, G)$$

Let b be in the fibre of $E(Z)$ over s, thus $\Omega(b)$ is in the fibre of $E(Y)$ over s. Let $\sigma \in O(p-1) \times I$ be such that

$$(r_\sigma o k)(b) = (k o \Omega)(b)$$

Let $P^p{}_Z$ be a p-plane at $E(Z)_b$ such that $d(\pi_1 o k)P^p{}_Z = P^p$. By assumption:

$$o = \lambda_Z(P^p) = q(b, \lambda(P^p{}_Z)) .$$

Thus $\lambda(P^p{}_Z) = o$, that is $\lambda(dkP^p{}_Z) = o$.
By the observation in (8.20) it follows that $d(r_\sigma o k)P^p{}_Z$ is an integral p-plane of λ at $(r_\sigma o k)(b) = (k o \Omega)(b)$.

If $d(r_\sigma o k)P^p{}_Z$ is in the image of $dk(E(Y)_{\Omega(b)})$

then $(k^{-1} o r_\sigma o k)P^p{}_Z$ is an integral p-plane of λ at $\Omega(b)$. So:

$$q(f(b), \lambda(d(k^{-1} o r_\sigma o k)P^p{}_Z)) = o = \lambda_Y(d(\pi_1 o k o k^{-1} o r_\sigma o k)P^p{}_Z)$$

or

$$o = \lambda_Y(d(\pi_1 o r_\sigma o k)P^p{}_Z) = \lambda_Y(d(\pi_1 o k)P^p{}_Z) = \lambda_Y(P^p) .$$

However, $d(r_\sigma o k)P^p{}_Z$ is in general not in the image of $dk(E(Y)_{\Omega(b)})$. But as we shall now show, each vector v in $d(r_\sigma o k)P^p{}_Z$ differs from a vector in $dk(E(Y)_{\Omega(b)})$ which has the same projection under $d\pi_1$ and which differs from v by a vector in

$$\left(\begin{array}{c|c|c} o(p-1) & o & o \\ \hline o & o & o \\ \hline o & o & o(m-p) \end{array}\right)$$

In such a case we could write

$$d(r_\sigma o k)P^p{}_Z = \text{span } (u_1, \ldots u_p) \quad \text{where}$$

$$u_i = v_i + w_i,\ v_i \in dk(E(Y)_{\Omega(b)}) ,\quad w_i \text{ a matrix as above.}$$

Then we would have

$$\lambda(dk^{-1}(v_1) \wedge \ldots\ldots \wedge dk^{-1}(v_p)) = \lambda(v_1 \wedge \ldots\ldots \wedge v_p)$$

$$= \lambda((u_1 - w_1) \wedge \ldots\ldots \wedge (u_p - w_p))$$

$$= \lambda(u_1 \wedge u_2 \wedge \ldots\ldots \wedge u_p) + K$$

where K is the sum of terms of the form:

λ (some p-vector with some entry w_j alone)

Hence $K = o$. But $\lambda(u_1 \wedge \dots \wedge u_p) = o$ also since $(d(r_\sigma ok))P^p{}_Z$ is an integral p-plane.

We must therefore demonstrate that given $v \in d(r_\sigma ok)P^p{}_Z$ there is a vector in $dk(E(Y)_{\Omega(b)})$, with the same projection as v, which differs from v by a vector given by a matrix of the above type.

Let $w \in P^p{}_Z$ where $d(r_\sigma ok)\, w = v$.

Let $\gamma: R^1 \to E(Z)$ so that $\dot{\gamma}(o) = w$

$ko\gamma(t) = \gamma(t)$

$= (n(t), G^{p-1}(Z_1(n(t))), \dots G^{p-1}(Z_{p-1}(n(t))), v(t), e_{p+1}(t) .. e_m(t), G(t))$

$r_\sigma oko\gamma(t)$

$= (n(t), \sigma G^{p-1}(Z,(n(t))) \dots \sigma G^{p-1}(Z_{p-1}(n(t)), v(t), e_{p+1}(t), .. e_m(t), G(t))$

while $ko\Omega o\gamma(t)$

$= (n(t), G^{p-1}(Y_1(n(t))), \dots G^{p-1}(Y_{p-1}(n(t))), v(t), e_{p+1}(t), .. e_m(t), G(t))$

We note that the projections are equal:

$$\pi_1 or_\sigma oko\gamma(t) = \pi_1 oko\Omega o\gamma(t) = (n(t), v(t), G(t))$$

and $r_\sigma oko\gamma(o) = ko\Omega o\gamma(o)$.

Since $(r_\sigma \dot{o} ko\gamma)(o)$ and $(k\dot{o}\Omega o\gamma)(o)$ have the same projection under $d\pi_1$, they differ by an element of the vertical of $(F^2)_{ko\Omega(b)}$ (over S^2). That is, they differ by an element of

$$\left(\begin{array}{c|c|c} o(p-1) & o & o \\ \hline o & o & o \\ \hline o & o & o(m-p) \end{array}\right)$$

But $v = (r_\sigma \dot{o} k o \gamma)(o)$ and $(k o \dot{\Omega} o \gamma)(o) \in dk(E(Y)_{\Omega(b)})$

Q.E.D.

§9. Corollaries of the Main Theorem.

9.1. Corollary 1. If $di(N^{p-1}{n_1})$ is sufficiently close to $D^{p-1}(n_o)$ the conclusion of the theorem still holds.

Proof: This is just a rephrasing of the fact that the regular integral planes form an open set.

9.2. In the direction of making a more global statement than is given in the main theorem we have:

Corollary 2. Suppose that the relation $di(N^{p-1}{}_{n_1}) = D^{p-1}(n_o)$ holds not just at the one pair of points n_1 and $i(n_1) = n_o$ but at every pair of points $(n, i(n))$ where $n \in N^{p-1}$. Then there is a unique parallel p-plane field G along f, defined in a neighborhood of $i(N^{p-1})$ which extends (D^{p-1}, G^{p-1}) and the initial conditions.

Proof: Patch together local solutions by means of the uniqueness part of the main theorem.

<u>Remark</u>: If the distribution D^{p-1} is involutive (in the sense that its associated ideal generated by one forms in $\Lambda^1 M$ is already a differential ideal), then the integral manifolds of D^{p-1}, obtained by the Frobenius Theorem, satisfy the hypotheses of Corollary 2.

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