CONSTRUCTIBILITY
IN IMPREDICATIVE SET THEORY

by

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ABSTRACT

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In the first chapter of the paper a semantical characterization of the sets $M(\lambda)$ is given, and several elementary results about the impredicative extension (VBI) of the von Neumann-Bernays-Godel set theory are established.

In the second chapter the consistency of VBI and the axiom of constructibility is demonstrated by constructing an inner model $\mathcal{K}$. This inner model is defined by considering classes which are well-orderings and hence may be longer than all ordinals.

In the last chapter it is shown that the inner model $\mathcal{K}$ is a model for a set theory (VBC) with a strong axiom of constructibility. It is then proven that the consistency of VBC (and hence that of VBI) is equivalent to that of a set theory ZF* which is a natural extension of ZF. Applying the Cohen constructions to ZF* shows (constructively) that if VBI is consistent then VBI and the negation of choice are consistent; the desired independence results for the continuum hypothesis and the axiom of constructibility are obtained by the same method.

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INTRODUCTION

We are concerned in this paper with questions of the consistency and independence of the axiom of choice, the general continuum hypothesis, and the axiom of constructibility for the set theory (which we call VBI) which results by adding an impredicative comprehension schema to the Von Neumann-Bernays-Godel set theory (VB). This set theory, which has received little attention in the literature, is set forth in the appendix of Kelly's <u>General Topology</u>, but was previously discussed by Hao Wang (who called it NQ) in Wang [1]. Mostowski [1] implicitly reveals some interesting facts about VBI in his study of the truth definition of Zermelo-Fraenkel set theory in VB. It is our thesis that beneath the surface, VBI possesses certain natural structures and that Godel's notion of constructibility and Cohen's methods can also be applied fruitfully in this new context.

In the first of the three chapters of this paper a number of results are established, some of an introductory nature and others needed as a basis for later work; to some extent we have restated known results in a convenient form, and some of the other observations are probably new. In particular, a method of characterizing the $M(\lambda)$ (see pp. 12) which was suggested by Hilary Putnam has been found useful for summarizing known constructions and suggesting new applications.

In the second chapter we give a constructive proof of the relative consistency of VBI and the axiom of constructibility by constructing a suitable inner model $\mathcal{K}$. In order to define $\mathcal{K}$, it is necessary to consider a certain "outer model" which in a sense goes beyond VBI.

In Mendelson [1], pp. 206, it is conjectured that the Godel consistency proof cannot be carried out for VBI. After the present proof was established, the author learned that Dr. Robert Solovay had previously obtained a proof of consistency. However, his proof was not published, and the present author has not seen his methods of proof.

In the last chapter a strengthening of the consistency result is gotten by showing that the inner model $\mathcal{K}$ satisfies a strong axiom of constructibility which implies that there is (essentially) a well-ordering of all classes. On the basis of this stronger set theory, the outer model may be shown to be a model for a certain modification of Zermelo-Fraenkel set theory. The Cohen constructions may be applied to this latter set theory, and in this indirect manner the desired independence results for VBI may be inferred.

CHAPTER I

PRELIMINARY RESULTS

1. The axiom of constructibility in VB.

Our notation for the von Neumann-Bernays-Godel set theory (VB) is similar to that used in Mendelson [1] and Godel [2]. The predicate calculus with identity is taken as the basic language, and the notion of set ($\mathcal{M}(X) \overset{df}{=} (\exists Y)(X \varepsilon Y)$) and the two-sorted variables are introduced by definition. The axioms are extensionality, pairing, sum set, power set, infinity, replacement, foundation, and comprehension. Specifically, we take the following forms of the axioms:

1. AxExt is $(X)(Y)\ [(u)(u \varepsilon X \Leftrightarrow u \varepsilon Y) \Rightarrow X = Y]$.
2. AxPair is $(x)(y)(\exists z)(u)\ [u \varepsilon z \Leftrightarrow u = x \vee u = y]$.
3. AxSum is $(x)(\exists y)(u)\ [u \varepsilon y \Leftrightarrow (\exists v)[u \varepsilon v \wedge v \varepsilon x]]$ $(y = \mathcal{S}(x))$.
4. AxPower is $(x)(\exists y)(u)[u \varepsilon y \Leftrightarrow u \subseteq x]$ $(y = \mathcal{P}(x))$.
5. AxInf is $(\exists x)\ [x \neq \emptyset \wedge (u)[u \varepsilon x \Rightarrow (\exists y)\ [y \varepsilon x \wedge u \subsetneq y]]]$.
6. AxRep is $(F)(x)\ [Fnc(F) \Rightarrow (\exists y)(v)\ [v \varepsilon y \Leftrightarrow (\exists u)[\langle u,v \rangle \varepsilon F \wedge u \varepsilon x]]]$ $(y = F''x)$.
7. AxFound is $(X)\ [X \neq \emptyset \Rightarrow (\exists u)(u \varepsilon X \wedge u \cap x = \emptyset)]$.
8. AxComp is the schema
   $(X_1) \ldots (X_n)(\exists A)(u)\ [u \varepsilon A \Leftrightarrow \varphi(u, X_1, \ldots, X_n)]$
   where φ has the free variables shown and all bound variables of φ are set variables.

As is well known (see Godel [2], pp. 8) the comprehension schema can be replaced by a finite list of axioms. A strong choice principle for VB and VBI is UC (universal choice): $(\exists F)\,[\mathrm{Fnc}(F) \wedge (x)\,[x \neq \emptyset \Rightarrow F(x) \varepsilon x]]$. This principle, which is easily seen to be equivalent to the statement that the universe has a well-ordering, is not taken as an axiom, since we are interested in its consistency. In the set theory ZF, UC cannot be stated, and a weaker statement $CH \overset{DF}{=} (x)(\exists f)(u)\,[u \subseteq x \wedge u \neq \emptyset \Rightarrow f(u) \varepsilon u]$, which is equivalent to the assertion that each set has a well-ordering, is appropriate.

We shall proceed informally in much of what follows, and many of the common notions will not be explicitly defined except where confusion might arise. See Mendelson [1] for any omitted definitions. However, for convenience, some of the most heavily used notions are listed below.

Definition 1.1

a. $\mathrm{Trans}(X) \overset{DF}{=} (u)\,[u \varepsilon X \Rightarrow u \subseteq X]$.

b. $\mathrm{Ord}(x) \overset{DF}{=} \mathrm{Trans}(x) \wedge (u)\,[u \varepsilon x \Rightarrow \mathrm{Trans}(u)]$

   $On \overset{DF}{=} \{x / \mathrm{Ord}(x)\}$; if $\mathrm{Ord}(x)$ let $x' \overset{DF}{=} x \cup \{x\}$;

   $\mathrm{Suc}(x) \overset{DF}{=} \mathrm{Ord}(x) \wedge (\exists y)\,[x = y']$;

   $\mathrm{Lim}(x) \overset{DF}{=} \mathrm{Ord}(x) \wedge \sim \mathrm{Suc}(x) \wedge x \neq \emptyset$;

   $\mathrm{Fino}(x) \overset{DF}{=} \mathrm{Ord}(x) \wedge \sim \mathrm{Lim}(x) \wedge (u)\,[u \varepsilon x \Rightarrow \sim \mathrm{Lim}(u)]$;

   $\omega \overset{DF}{=} \{x / \mathrm{Fino}(x)\}$.

c. $\mathrm{Rel}(R) \overset{DF}{=} R \subseteq V \times V$; we will often write uRv for $\langle u,v \rangle \varepsilon R$

d. If $\mathrm{Rel}(R)$ let $|R| \overset{DF}{=} \mathcal{D}(R) \cup \mathcal{R}(R)$, that is $|R|$ is the <u>field</u> of R or the union of the domain and range of R.

e. If Rel(R) let $R|A \overset{DF}{=} R \cap (A \times A)$; let $R \restriction A \overset{DF}{=} R \cap (A \times V)$.

f. If Rel(R) let $WF(R) \overset{DF}{=} (A)[(A \subseteq |R| \wedge A \neq \emptyset) \Rightarrow (\exists u)(u \varepsilon A \wedge (v) \sim (vRu \wedge v \varepsilon A))]$.

g. $F{:}X \simeq Y \overset{DF}{=}$ F is a 1:1 function, $X = \mathfrak{D}(F)$ and $Y = \mathfrak{R}(F)$; $X \simeq Y \overset{DF}{=} (\exists F)[F{:}X \simeq Y]$.

h. $F{:}R \approx S \overset{DF}{=} Rel(R) \wedge Rel(S) \wedge F{:}|R| \simeq |S| \wedge (u)(v)[uRv \Leftrightarrow F(u)SF(v)]$; $R \approx S \overset{DF}{=} (\exists F)[F{:}R \approx S]$.

i. Let $E \overset{DF}{=} \{\langle x,y\rangle \mid x \varepsilon y\}$; let $I \overset{DF}{=} \{\langle x,y\rangle \mid x = y\}$.

It is well known that for any set x, if Rel(x) and WF(x) and $x \models Ext$ where Ext is $(u)(v)[(z)(z \varepsilon u \Leftrightarrow z \varepsilon v) \Rightarrow u = v]$, then there is a transitive set t such that $x \approx E|t$. Namely, one defines $g(a) = \emptyset$ where a is the least element of $|x|$, and $g(v) = \{g(u) \mid \langle u,v\rangle \varepsilon x\}$ for other $v \varepsilon |x|$. It is easily verified that $g{:}x \approx E|t$ where t is $g''|x|$. The preceding construction suggests the following useful lemma:

Lemma 1.2

a. If Trans(t) and $f{:}x \approx E|t$, then $f(v) = \{f(u) \mid \langle u,v\rangle \varepsilon x\}$ for all $v \varepsilon |x|$.

b. If Trans(t_1) and Trans(t_2) and $f{:}t_1 \approx t_2$ then $f = I \restriction t_1$ and $t_1 = t_2$. (We shall often write $t_1 \approx t_2$ for $E|t_1 \approx E|t_2$, etc., when dealing with the relation E.)

Proof:

a. If $y \varepsilon f(v)$ then $y = f(u)$ and $\langle u,v\rangle \varepsilon x$ since f is an isomorphism. Conversely, if $\langle u,v\rangle \varepsilon x$ then $f(u) \varepsilon f(v)$, so $f(v) = \{f(u) | \langle u,v\rangle \varepsilon x\}$.

b. By Part (a), $f(v) = \{f(u) | u \varepsilon v\}$. Since any (non-empty) transitive set contains $\emptyset$, $f(\emptyset) = \emptyset$. Take an ε-least v in t_1 such that $f(v) \neq v$. Then $u \varepsilon v \Rightarrow f(u) = u$ by minimality of v, so $f(v) = \{u | u \varepsilon v\} = v$.

AxFound of course guarantees ε-least elements, and in general this proof is illustrative of the use of well-founded classes. Related to AxFound is the useful notion of rank.

Definition 1.3

Let $R(o) = \emptyset$, $R(\alpha+1) = \mathcal{P}(R(\alpha))$, $R(\lambda) = \bigcup_{\beta<\lambda} R(\beta)$ if Lim (λ). Then if $x \varepsilon \bigcup R(\alpha)$, let $\rho(x)$ (the rank of x) be $\mu\alpha[x \varepsilon R(\alpha+1)]$.

It is easily seen that each $R(\alpha)$ is transitive, $\alpha<\beta \Rightarrow R(\alpha) \varepsilon R(\beta)$, and $\alpha \leq \beta \Rightarrow R(\alpha) \subseteq R(\beta)$. Furthermore, it is easy to verify that AxFound is equivalent*, on the basis of the other axioms, to the sentence $\bigcup R(\alpha) = V$. Hence $\rho(x)$ is defined for each set x, and gives a natural partial ordering of the universe with the property that $x \varepsilon y \Rightarrow \rho(x) < \rho(y)$.

The constructible sets, which we shall be concerned with throughout this paper, may be viewed as arising from sort of a "constructive" analog of the preceding construction.

*Of course in the absence of AxFound a different definition of ordinal must be used.

Definition 1.4

a. Fodo(a,b) (a is first order definable over b)

$\leftrightarrow a = \{u | u \varepsilon b \wedge E|b \models \varphi(u, c_1, \ldots, c_n)\}$

where φ has n + 1 free variables, the symbols ε and $=$, and where the constants $c_1, \ldots, c_n$ are in b.

b. Let $M(0) = \emptyset$, $M(\alpha+1) = \{u | \mathrm{Fodo}(u, M(\alpha))\}$, and $M(\lambda) = \bigcup_{\beta<\lambda} M(\beta)$ for $\mathrm{Lim}(\lambda)$. Let $L = \bigcup M(\alpha)$, and for $x \varepsilon L$, let $\delta(x) = \mu\alpha[x \varepsilon M(\alpha+1)]$.

As before, a number of elegant relationships hold. Each $M(\alpha)$ is transitive, $\alpha < \beta \Rightarrow M(\alpha) \varepsilon M(\beta)$, $\alpha \leq \beta \Rightarrow M(\alpha) \subseteq M(\beta)$, $x \varepsilon y \varepsilon L \Rightarrow \delta(x) < \delta(y)$, and $\delta(M(\alpha)) = \alpha$. Also $On \subseteq L$, and in fact $\delta(\alpha) = \alpha$.

The "constructibility" of L is made somewhat clearer by the observation that there is a definable well-ordering of L. We mention specifically the following approach, since it will be of use later.

Lemma 1.5

Let variables x^α range over $M(\alpha)$. Then if $a \varepsilon L$, a can be defined by an expression $\hat{x}^\alpha (Q_1 x_1^{\alpha_1}) \ldots (Q_n x_n^{\alpha_n}) \pi(x^\alpha, x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})$ where $\alpha = \delta(a)$, $\alpha_1, \ldots, \alpha_n \leq \alpha$ and the propositional matrix π contains <u>no constants.</u>

Proof:

By definition $a = \hat{x}^\alpha (Q_1 x_1^\alpha) \ldots (Q_j x_j^\alpha) \varphi(x^\alpha, x_1^\alpha, \ldots, x_j^\alpha, c_1, \ldots, c_k)$, where $\alpha = \delta(a)$.

Each constant c_i, by induction hypothesis, has the desired representation. Transform φ as follows: Replace $t_1 = t_2$ by $(x^\alpha)[x^\alpha \varepsilon t_1 \Longleftrightarrow x^\alpha \varepsilon t_2]$ for any terms t_1 and t_2. Then consider the remaining atoms $b \varepsilon c$, $c \varepsilon v$, and $v \varepsilon c$ where v is a variable and b and c are constants, represented, respectively, by $\hat{x}^\beta \psi(x^\beta)$ and $\hat{x}^\gamma \theta(x^\gamma)$ where $\beta = \delta(b)$ and $\gamma = \delta(c)$, and β and γ are less than α. Replace $b \varepsilon c$ by $(\exists x^{\beta+1})[(y^\beta)[y^\beta \varepsilon x^{\beta+1} \Longleftrightarrow \psi(y^\beta)] \wedge (\exists x^\gamma)[x^{\beta+1} = x^\gamma \wedge \theta(x^\gamma)]]$, $v \varepsilon c$ by $(\exists x^\gamma)[v = x^\gamma \wedge \theta(x^\gamma)]$, and $c \varepsilon v$ by $(\exists x^{\gamma+1})[(y^\gamma)[y^\gamma \varepsilon x^{\gamma+1} \Longleftrightarrow \theta(y^\gamma)] \wedge x^{\gamma+1} \varepsilon v]$. This transformation clearly yields an expression of the desired form, which also defines the set a.

Corollary 1.6

a. L can be well-ordered by assigning to each $a \varepsilon L$ a least finite sequence $\langle \alpha, p, \alpha_1, \ldots, \alpha_n \rangle$ where $\delta(a) = \alpha$, p is the Godel number of a formula of the appropriate form, and $\alpha_1, \ldots, \alpha_n$ are the bounds of the variables.

b. For $\alpha \geq \omega$, $\overline{\overline{M(\alpha)}} = \overline{\overline{\alpha}}$, because, by a well known result, $\overline{\overline{F(\alpha)}} = \overline{\overline{\alpha}}$ where $F(\alpha)$ is the set of all finite sequences of ordinals less than α.

Our chief goal in this section is to prove that there is a sentence σ which characterizes the $M(\lambda)$ in the sense that if $WF(x)$ then $x \models \sigma \Longleftrightarrow (\exists \lambda)[Lim(\lambda) \wedge x \approx E|M(\lambda)]$. We have repeatedly used semantical notions in the foregoing, and in fact Fodo and M were defined in these terms. In the next section it will be necessary to consider a more general truth definition (see Mostowski [1] , pp. 114), which can easily be modified to define semantical concepts for sets in VB. However, it will be convenient for the proof of the above theorem to identify Fodo with the concise formal expression given below. Since this expression coincides in VB with the semantical definition, we henceforth make this identification.

Definition 1.7 (i, j, k, m and n range over integers.)

a. $b^j \overset{DF}{=} \{f \mid f: j \longrightarrow b\}$.

b. $E(b,j) \overset{DF}{=} \{f \mid f: j+2 \longrightarrow b \wedge f(0) \varepsilon f(1)\}$.

c. $I(b,j) \overset{DF}{=} \{f \mid f: j+2 \longrightarrow b \wedge f(0) = f(1)\}$.

d. $\mathrm{Perm}(r_1, h, r_2) \overset{DF}{=} (\exists b)(\exists j)[r_2 \subseteq b^j$
$\wedge h: j \simeq j \wedge (x)[x \varepsilon r_1 \Longleftrightarrow (\exists y)[(y \circ h = x) \wedge y \varepsilon r_2]]]$
(r_1 results by applying the permutation h to r_2).

e. $\mathrm{Exist}(r_1, r_2) \overset{DF}{=} (\exists j)(\exists b)[r_2 \subseteq b^{j+1}$
$\wedge (x)[x \varepsilon r_1 \Longleftrightarrow x: j \longrightarrow b \wedge (\exists u)(x \cup \{\langle j,u \rangle\} \varepsilon r_2)]]$
(r_1 is the existential projection of r_2).

f. $\mathrm{Sub}(r_1, u, r_2) \overset{DF}{=} (\exists j)(\exists b)[r_2 \subseteq b^{j+1}$
$\wedge (x)[x \varepsilon r_1 \Longleftrightarrow (x: j \longrightarrow b \wedge x \cup \{\langle j,u \rangle\} \varepsilon r_2)]]$
(r_1 results by substituting u in r_2).

Definition 1.8

$\mathrm{Fodo}(a,b) \overset{DF}{=} (\exists g)(\exists n)[\mathrm{Fnc}(g)$ and $\mathcal{D}(g) = n+1$

and $g(0) = E(b,j)$
and $g(1) = I(b,j)$ } some $j \geq 0$.

and for $i > 1$

$$
\left.\begin{array}{ll} \text{either} & g(i) = g(j) \cup g(k) \\ \text{or} & g(i) = g(j) - g(k) \end{array}\right\} \quad \begin{array}{l} \text{some } j,k < i, \text{ where for} \\ \text{some } m \\ g(j) \subseteq b^m \text{ and } g(k) \subseteq b^m \end{array}
$$

$$
\begin{array}{ll} \text{or} & g(i) = b^j \end{array} \quad \text{some } j \geq 0
$$

$$
\left.\begin{array}{ll} \text{or} & \text{Perm } (g(i),h,g(k)) \\ \text{or} & \text{Exist } (g(i),g(k)) \\ \text{or} & \text{Sub } (g(i),u,g(k)) \end{array}\right\} \quad \begin{array}{l} \text{some } k < i, \text{ some } h, \\ \text{and some } u \end{array}
$$

and $g(n) \subseteq b^1$ and $a = \{ v \mid \{\langle 0,v\rangle\} \varepsilon\, g(n)\} = \mathcal{R}\mathcal{S}(g(n))]$.

<u>Definition 1.9</u>

Closed $(t) \overset{\text{DF}}{=}$ Trans (t) and $t \neq \emptyset$ and if

$x \varepsilon t$, $y \varepsilon t$ and $j \varepsilon \omega$

(i) $\{x,y\} \varepsilon\ t$

(ii) $x \cup y \varepsilon\ t$

(iii) $x - y \varepsilon\ t$

(iv) $\mathcal{R}\mathcal{S}(x) \varepsilon\ t$ $\quad (\mathcal{R}\mathcal{S}(x) = \{v \mid (\exists u)(\exists z)[\langle u,v\rangle \varepsilon\ z \wedge z\ \varepsilon\ x]\})$.

(v) $x^j\ \varepsilon\ t$

(vi) $E(x,j)\ \varepsilon\ t$

(vii) $I(x,j)\ \varepsilon\ t$

(viii) $y \subseteq x^{j+1} \Rightarrow$ any permutation of y is in t.

(ix) $y \subseteq x^{j+1} \Rightarrow$ the existential projection of y is in t.

(x) $y \subseteq x^{j+1} \wedge u \varepsilon x \Rightarrow$ the substitution of u in y is in t.

We remark that if Trans (t) and $t \neq \emptyset$ then $\emptyset\ \varepsilon\ t$; hence clauses (i) and (ii) imply that $\omega \subseteq t$ and that if $f: j \rightarrow b$ and $b \varepsilon t$, then $f \varepsilon t$.

Lemma 1.10

For $Lim(\lambda)$ each $M(\lambda)$ is closed, and L is closed.

Proof: - Straightforward

Many results in set theory depend on some form of the concept of the absoluteness or invariance of formulas. In particular, Godel [2] pp. 42 formally defines absoluteness and bases much of the proof on this notion. For our purposes the concept of invariance is convenient: Let φ have n free variables and the symbols ε and $=$. Then φ is invariant with respect to a definable collection of sets C if VB $\vdash$ $(x)(a_1) \ldots (a_n)[x \varepsilon C \wedge a_1 \varepsilon x \wedge \ldots \wedge a_n \varepsilon x \Rightarrow [\varphi(a_1, \ldots, a_n) \Leftrightarrow E|x \models \varphi(a_1, \ldots, a_n)]]$.
This, of course, is a metamathematical definition, and within VB we may show certain φ to be invariant with respect to certain C. Especially important is the fact (which is easy to verify) that many formulas are invariant with respect to all transitive sets, for example, Ord(u), Lim(u), Suc(u), Fino(u), Fnc(u), etc.

Lemma 1.11

If Closed(t) then

a. Fodo is invariant with respect to t,

i. e. $a,b \varepsilon t \Rightarrow [Fodo(a,b) \Leftrightarrow t \models Fodo(a,b)]$.

b. $Fodo(a,b) \wedge b \varepsilon t \Rightarrow a \varepsilon t$

Proof:

a. Since t is closed, it is easy to see that there are enough objects in t so that the quantifiers in Fodo can be restricted to t without changing the truth of Fodo(a,b).

b. If $b \in t$ and Fodo(a,b) and g is the finite function in Def. 1.8 then $g(0) = E(b,j) \in t$, $g(1) = I(b,j) \in t$, and indeed each g(i) is in t. Hence $a \in t$.

Definition 1.12

Let Close be the formal sentence consisting of the conjunction of the ten clauses $C_1, C_2, \ldots, C_{10}$ stating, respectively, that the objects mentioned in Def. 1.9 exist. For example, C_1 is $(x)(y)(\exists z)(u)[u \in z \Leftrightarrow u = x \vee u = y]$, C_2 is $(x)(y)(\exists z)(u)[u \in z \Leftrightarrow u \in x \vee u \in y]$, and C_5 is $(x)(u)[\mathrm{Fino}(u) \Rightarrow (\exists y)(z)[z \in y \Leftrightarrow$ $\mathrm{Fnc}(z) \wedge \mathcal{D}(z) = u \wedge \mathcal{R}(z) \subseteq x]]$ and similarly for the other clauses.

Lemma 1.13

Let t be transitive and nonempty. Then Closed(t) $\Leftrightarrow t \models$ Close.

Proof:

($\Leftarrow$) First, if $t \models C_1$, then t is closed under $\{x,y\}$, and if $t \models C_2$ then t is closed under $x \cup y$, as is easy to verify directly. Hence as remarked before, $\omega \subseteq t$ and $x \in t$ $\wedge f: j \rightarrow x \Rightarrow f \in t$ for any $j \in \omega$. It is easy to verify that the other clauses yield the desired closure. For example, regarding C_5, if $t \models (z)[z \in y \Leftrightarrow z: j \rightarrow b]$ for j,b and y in t then $y = b^j$; however without the above observations we could only conclude that $y \subseteq b^j$.

($\Longrightarrow$) is straightforward.

Definition 1.14

$$\mathrm{Cond}(f) \overset{\mathrm{DF}}{\equiv} \mathrm{Fnc}(f) \wedge (\exists u)[\mathrm{Ord}(u) \wedge \mathfrak{D}(f) = u \wedge f(0) = \emptyset \wedge (v)[v' \varepsilon u \Rightarrow f(v') = \{z/\mathrm{Fodo}(z,f(v))\}] \wedge (v)[\mathrm{Lim}(v) \wedge v \varepsilon u \Rightarrow f(v) = \bigcup_{z \varepsilon v} f(z)]] .$$

Lemma 1.15

If Closed(t) and $f \varepsilon t$ then $[\, t \models \mathrm{Cond}(f) \Longleftrightarrow (\exists \gamma)(f = M \restriction \gamma)]$.

Proof:

($\Longrightarrow$) The only problem is the condition $f(\beta+1) = \{z | \mathrm{Fodo}(z,f(\beta))\}$.
Suppose $t \models (z)[z \varepsilon f(\beta+1) \Longleftrightarrow \mathrm{Fodo}(z,f(\beta))]$.
Then by Lemma 1.11, if $t \models \mathrm{Fodo}(a,b)$ for $a,b \varepsilon t$, then Fodo(a,b), so we have $f(\beta+1) \subseteq \{z | \mathrm{Fodo}(z,f(\beta))\}$. Also by Lemma 1.11, if $\mathrm{Fodo}(z,f(\beta))$ then $z \varepsilon t$, and by the absoluteness of Fodo it follows that $t \models \mathrm{Fodo}(z,f(\beta))$. Therefore $\{z | \mathrm{Fodo}(z,f(\beta))\} \subseteq f(\beta+1)$, so $f(\beta+1) = \{z | \mathrm{Fodo}(z,f(\beta))\}$ and, finally, $f = M \restriction \gamma$ where $\gamma = \mathfrak{D}(f)$.

($\Longleftarrow$) This can be checked as above by using Lemma 1.11.

Definition 1.16

Let σ be the sentence

$Ext \wedge Close \wedge (x)(\exists f)\ [Cond(f) \wedge (\exists u)\ (x \in f(u))]$

Lemma 1.17

$$(\alpha)\ [M \restriction \alpha \in M(\alpha+3)]$$

Proof

a. $M \restriction 0 = 0 \in M(1) \subseteq M(3)$.

b. $M \restriction (\beta+1) = (M \restriction \beta) \cup (\{\langle \beta, M(\beta) \rangle\})$ so if $M \restriction \beta \in M(\beta+3)$, clearly $M \restriction (\beta+1) \in M(\beta+4)$.

c. If $Lim(\lambda)$ and $M \restriction \beta \in M(\beta+3)$ for $\beta < \lambda$, then $M \restriction \beta \in M(\lambda)$ for $\beta < \lambda$.
Let $x = \hat{u}^{\lambda}(\exists f^{\lambda})\ [\overline{Fnc}^{\lambda}(f^{\lambda}) \wedge \overline{Cond}^{\lambda}(f^{\lambda}) \wedge u^{\lambda} \in f^{\lambda}]$ (where all quantifiers are restricted to $M(\lambda)$).
Clearly $Fodo(x, M(\lambda))$ and $x = \bigcup_{\beta<\lambda} M \restriction \beta = M \restriction \lambda$, by Lemma 1.15. Hence $M \restriction \lambda \in M(\lambda+1) \subseteq M(\lambda+3)$.

Theorem 1.18

$$[WF(x) \wedge x \models \sigma \iff (\exists \lambda)\ [Lim(\lambda) \wedge x \approx E|M(\lambda)]].$$

Proof

($\Leftarrow$) $M(\lambda)$ for $Lim(\lambda)$ is transitive, closed, and since $M \restriction \beta \in M(\lambda)$ for all $\beta < \lambda$, satisfies $(x)(\exists f)$ $[Cond(f) \wedge (\exists u)\ (x \in f(u))]$.
Hence $E|M(\lambda) \models \sigma$.

($\Rightarrow$) $WF(x) \wedge x \models Ext \Rightarrow x \approx E|t$ for some transitive t. $t \models Close$, so t is closed. Also $t \models (x)(\exists f)$ $[Cond(f) \wedge (\exists u)\ (x \in f(u))]$. By Lemma 1.15, each such f is an $M \upharpoonright \gamma$ for some γ. Hence if $x \varepsilon t$ then $x \varepsilon M(\beta)$ for some β, and $M(\beta) \varepsilon t$ and $M(\beta) \subseteq t$. Therefore t is the union of these $M(\beta)$ and $t = M(\lambda)$ where λ is the least upper bound of these β, and λ is clearly a limit ordinal because $x \varepsilon t \Rightarrow \{x\} \varepsilon t$.

Corollary 1.19

σ holds in L, that is, $VB \vdash \overline{\sigma}$, where the quantifiers of σ range over L. This follows by the method of the first half of 1.18.

This method of analyzing the constructible sets seems to us to give conceptual proofs of the Gödel theorem and related results. Notice, for example, the proof of Gödel's main lemma, namely, $a \varepsilon L \wedge a \subseteq M(\omega_\alpha) \Rightarrow a \varepsilon M(\omega_{\alpha+1})$: Suppose $a \varepsilon M(\lambda)$ for some limit ordinal $\lambda > \omega_\alpha$. Then $M(\lambda) \models \sigma$, and we define N as the closure of $M(\omega_\alpha) \cup \{a\}$ under the Skolem functions of σ in $M(\lambda)$. Then $N \approx M(\beta)$ and $\overline{\overline{M(\beta)}} = \omega_\alpha$ for some β. If $f : N \approx M(\beta)$ then clearly $f''M(\omega_\alpha)$ is transitive, $f \upharpoonright M(\omega_\alpha) = I \upharpoonright M(\omega_\alpha)$, and since $f(a) = f''a$ (see Lemma 1.2), $f(a) = a \varepsilon M(\beta) \varepsilon M(\omega_{\alpha+1})$.

To summarize the Gödel proof for VB, we have shown $VB \vdash$ [L has a well-ordering], so $VB \& V = L \vdash UC$. By the above lemma, $VB \& V = L \vdash GCH$ (where $GCH \overset{DF}{=} (\alpha)[\mathcal{P}(\omega_\alpha) \simeq \omega_{\alpha+1}]$) since $VB \& V = L \vdash [\mathcal{P}(M(\omega_\alpha)) \subseteq M(\omega_{\alpha+1})]$. It is easy to check that $VB \vdash \overline{VB}$ where the classes of the inner model are those X such that $X \subseteq L \wedge (u)[u \varepsilon L \Rightarrow X \cap u \varepsilon L]$.

By Corollary 1.19, $VB \vdash \overline{V = L}$, so finally the two constructions $VB \vdash \overline{VB} \& \overline{V = L}$ and $VB \& V = L \vdash UC \wedge GCH$ conclude the relative consistency proof. This proof can be easily modified for ZF, since it can be shown constructively that for each axiom A_i of ZF, $ZF \vdash \overline{A}_i$.

It is also clear that if there are well-founded models for ZF, then there are minimal models (see Cohen [1]). For if $x \models ZF$, then by the Godel proof x has a submodel y such that $y \models ZF, \sigma$ so $y \approx E|M(\alpha)$ for some α. If $\gamma_o = \mu\beta[M(\beta) \models ZF]$ (hence γ_o is countable), then clearly $M(\gamma_o)$ is minimal in the sense that any well-founded model of ZF contains a submodel isomorphic to $M(\gamma_o)$.

Another useful application (see Cohen [3] , pp. 110) is the proof that for each n $ZF \vdash (\exists\alpha)(\overline{\overline{\alpha}} = \omega \wedge M(\alpha) \models ZF_n)$ where ZF_n is the conjunction of the first n axioms of ZF. By the Godel proof, one can show within ZF that $(ZF_n \wedge \sigma)$ holds in L, that is, $ZF \vdash \overline{(ZF_n \wedge \sigma)}$. Taking the prenex of the formula $(ZF_n \wedge \sigma)$ one can explicitly define the Skolem "functionals" of $(ZF_n \wedge \sigma)$ over L. By closing $\{\emptyset\}$ under these functionals one obtains a countable set x such that $E|x \models (ZF_n \wedge \sigma)$, and hence $(\exists\alpha)(\overline{\overline{\alpha}} = \omega \wedge M(\alpha) \models ZF_n)$. An immediate consequence is that if ZF is consistent, it is not finitely axiomatizable, since for each n, $ZF \vdash (ZF_n$ is consistent).

We note finally that an analog of Theorem 1.18 does not hold for the sets $R(\alpha)$ which are also a natural category of models for set theory. This is because for any $\alpha > \omega$, $R(\alpha) \models (\exists x)\mathrm{Lim}(x)$, and hence there is a countable set s which is elementarily equivalent to $R(\alpha)$; but s cannot be isomorphic to an $R(\gamma)$ because if $R(\gamma)$ is countable, then $\gamma \leq \omega$, and $R(\gamma) \models \sim(\exists x)\mathrm{Lim}(x)$.

2. The impredicative extension of VB.

Let VBI be the system obtained from VB by replacing AxComp by the impredicative comprehension schema

$$(X_1) \ldots (X_n)(\exists A)(u)\,[u \in A \Longleftrightarrow \varphi(u, X_1, \ldots, X_n)] \qquad \text{(ImComp)}$$

where φ may have bound class variables. This system was first considered in the literature by Hao Wang in Wang [1] .

Clearly VBI has no more expressive power than VB, but it is much stronger because within VBI a truth definition for ZF can be given and so the consistency of ZF (and hence that of VB) is provable in VBI. Of course the truth definition for ZF can also be stated in VB, so necessarily some of the basic properties of the truth definition cannot be established in VB. This state of affairs is examined in detail in Mostowski [1] to which we now refer the reader. We also point out that our study of constructibility in VBI will rely largely on this truth definition which can easily be extended to arbitrary relations R. By <u>formula</u> we shall always mean a formula with the predicates ε and $=$, to be interpreted in the obvious way. It is not necessary to repeat the definition of satisfaction in detail, but we indicate the method used.

A "finite sequence of classes" need not exist in the usual sense, but the notion can be reinterpreted.

Definition 2.1

a. If Fin(c)(i.e. $(\exists i)[i \varepsilon \omega \wedge i \simeq c]$) then SC(X,c) (X is a sequence of classes over c) $(x)[x \varepsilon X \Longrightarrow x:c \longrightarrow V]$. That is, a sequence of classes over c is a class of sequences over c.

b. FSC(X) (X is a finite sequence of classes) $(\exists i)[i \varepsilon \omega \wedge SC(X,i)]$

c. If SC(X,i) let X_j (the j-th term of X) for $0 \leq j < i$ be $\{u \mid (\exists f)[\langle j,u \rangle \varepsilon f \wedge f \varepsilon X]\}$.

In brief, we may define satisfaction of a formula (or its Godel number p) in R by: $R \models \varphi(f(0), \ldots, f(j))$ (where $f:j \longrightarrow |R|$ and φ has j free variables) if and only if $(\exists X)(\exists n+1)[SC(X,n+1) \wedge P(X,p) \wedge f \varepsilon X_n]$ where P(x,p) says, roughly, that the structure of the finite sequence of classes corresponds to the structure of the formula coded into p. P contains no bound <u>class</u> variables, so there is only one bound class variable in the satisfaction predicate. <u>Truth</u> and <u>validity</u> may then be defined.

In VBI one can show that if $R \models \varphi$ and $\varphi \vdash \psi$, then $R \models \psi$. One can also prove the general sentence that all axioms of ZF are true in V, and thus the sentence that all theorems of ZF are true in V. Since there are sentences of ZF which are demonstrably false in V, the consistency of ZF is provable in VBI. Within VB one can prove $\Theta(0) \wedge (n)\,[\Theta(n) \Rightarrow \Theta(n+1)]$ where $\Theta(n)$ says, roughly, that all formulas of ZF with a proof of length less than n are valid in V, but, of course, one cannot prove $(n)\,\Theta(n)$ in VB. By adding to VB the induction schema $\Theta(0) \wedge (n)\,[\Theta(n) \Rightarrow \Theta(n+1)] \Rightarrow (n)\,\Theta(n)$ for all Θ, one obtains a system VB + Ind which is intermediate in strength between VB and VBI. In VBI (as opposed to VB) the comprehension axioms give a set x such that $(n)\,[n \in x \Leftrightarrow \sim\Theta(n)]$ for impredicative Θ, so the induction schema is provable in VBI.

Below several general observations are sketched concerning the relationship of VBI to other set theories.

<u>Lemma 2.2</u>

$$\mathrm{VBI} \vdash (\exists \alpha)\,[\overline{\alpha} = \omega \wedge M(\alpha) \models ZF].$$

<u>Proof</u>

From the Godel proof, $ZF \vdash \overline{ZF}, \overline{\sigma}$.
Since $V \models ZF$, $V \models \overline{ZF}, \overline{\sigma}$ so $L \models ZF, \sigma$.

One could define the Skolem functions for σ in L (which, of course, are proper classes), or one could use the following method of Tarski-Vaught [1] for defining elementary submodels. For any set x, $x \subseteq L$, let
$g(x) = \{u \mid (\exists \varphi)(\exists f)[\,u = \mu v\,[L \models \varphi(v, f(0), \ldots, f(k))]$

$\wedge f{:}k \rightarrow x \wedge \varphi$ has $k+1$ free variables$]\}$.
Clearly $x \subseteq g(x) \subseteq L$. Let $h(0) = \{\emptyset\}$, $h(i+1) = g(h(i))$ and $d = \bigcup h(i)$. It is easy to see that for any $a_1, \ldots, a_j \in d$, $d \models \psi(a_1, \ldots, a_j)$ if and only if $L \models \psi(a_1, \ldots, a_j)$, and that d is countable. Since $L \models \sigma$, $d \models \sigma$ and $d \approx M(\alpha)$ for some countable α. Thus $M(\alpha) \models ZF$.

From the above it follows that $M(\alpha+1) \models$ VB, since in general if t is a transitive model of ZF, then $\{u \mid Fodo(u,t)\}$ is a transitive model of VB. If UC is added to VBI, then one can infer at once that there is an $R(\alpha)$ such that $R(\alpha) \models ZF$: Choose a well-ordering of V and modify the preceding construction by defining

$g(x) = \{u \mid (\exists \varphi)(\exists f)\ [u = \mu v\, [V \models \varphi(v, f(0), \ldots, f(k))]$
$\wedge f{:}k \rightarrow x \wedge \varphi$ has $k+1$ free variables$]\}$, and
$h(0) = \{\emptyset\}$, $h(2i+1) = g(h(2i))$ and $h(2i+2) = R(\alpha_i)$
where $\alpha_i = \mu\beta[h(2i+1) \subseteq R(\beta)]$. Clearly $\bigcup h(i) = R(\alpha)$ where $\alpha = \mathcal{S}(\{\alpha_i\})$, and $R(\alpha) \models ZF$. (It is possible to dispense with UC, for example by using the construction of Montague-Vaught [1].) One cannot, however, show within VBI that there is a β such that $R(\beta) \models$ VB because one could immediately conclude that $R(\beta) \models$ VBI.

It is now obvious that the set theory VB+Ind is much weaker than VBI, since if $M(\alpha+1) \models$ VB then $M(\alpha+1) \models$ VB+Ind. This fact would hold for any transitive model t, since the finite ordinals are absolute with respect to t, and so if $t \models \Theta(0) \wedge (n)[\Theta(n) \Rightarrow \Theta(n+1)]$ then $t \models (n)\Theta(n)$, by induction on n.

In order to give models for ZF or VB it is natural to introduce the notion of inaccessible ordinal. Call α regular (Reg(α)) if $(\beta)(f)[\beta < \alpha \wedge f: \beta \to \alpha \Rightarrow \delta(f''\beta) < \alpha]$. $\mathrm{In}(\alpha) \overset{\mathrm{Df}}{\equiv} \mathrm{Reg}(\alpha) \wedge (\exists \lambda)[\mathrm{Lim}(\lambda) \wedge \alpha = \omega_\lambda]$. This particular definition is sometimes called <u>weakly</u> inaccessible; see Montague-Vaught [1] for a general discussion of different definitions of inaccessibility.

Let ZF^ι be the set theory formed by adding the axiom $(\exists\alpha)\mathrm{In}(\alpha)$ to ZF. If one were to add CH and GCH to ZF^ι it could be shown that $R(\iota + 1) \models$ VBI where $\iota = \mu\alpha \mathrm{In}(\alpha)$. However, there is a more interesting model which can be shown to exist in ZF^ι without any additional axioms, namely, let $\mathfrak{Q} = \mathcal{P}(M(\omega_\iota)) \cap M(\omega_{\iota+1})$ (recall that $\omega_\iota = \iota$). That is, the "classes" are those subsets of $M(\iota)$ which are constructible, and the "sets" are just the members of $M(\iota)$. Using the Godel lemma, it is clear that $\mathfrak{Q} \models$ AxPower, and the only problematical axioms are AxRep and ImComp. AxRep holds because if $x \in M(\iota)$, then $x \in M(\beta)$ for some $\beta < \iota$. Therefore $\overline{\overline{x}} \leq \overline{\overline{\beta}}$ and if $f: x \to M(\iota)$, then $f''x \subseteq M(\gamma)$ for some $\gamma < \iota$ since $\mathrm{In}(\iota)$. Thus if f is a constructible function $f''x \in M(\iota)$. For ImComp, let $A = \{x | x \in M(\iota) \wedge \overline{\varphi}(x, \overline{X}_1, \ldots, \overline{X}_n)\}$ where quantifications are restricted to $\mathcal{P}(M(\omega_\iota)) \cap M(\omega_{\iota+1})$. Clearly Fodo$(A, M(\omega_{\iota+1}))$ so A is constructible, and $A \subseteq M(\omega_\iota)$, so by the Godel lemma $\delta(A) < \omega_{\iota+1}$. That is to say, $A \varepsilon \mathcal{P}(M(\omega_\iota)) \cap M(\omega_{\iota+1})$ and $\mathfrak{A} \models$ ImComp. It is also obvious from previous considerations that $\mathfrak{A} \models$ VBI&V$=$L, so this gives a non-constructive proof of the consistency of VBI&V$=$L.

As regards constructive relative consistency proofs, it is clear that Von Neumann's proof of the consistency of AxFound by using the inner model $\mathcal{N}(X) \overset{\text{Df}}{\equiv} X \subseteq \bigcup R(\alpha)$ may be carried out with no change in the impredicative set theory.

The situation, however, is quite different for the Godel proof. Specifically, one needs to define a collection of subclasses $\overline{X}$ of L such that they are closed under comprehension, that is, $(\exists \overline{A})(\bar{u}) [\bar{u} \varepsilon \overline{A} \Longleftrightarrow \overline{\varphi}(\bar{u}, \overline{X}_1, \ldots, \overline{X}_n)]$ for arbitrary φ. The "sets" would be just the members of L, and the classes $\overline{X}$ would have to have the property that $u \varepsilon L \Longrightarrow (\overline{X} \cap u) \varepsilon L$. This property would also be sufficient to establish $\overline{\text{AxRep}}$, which is the only non-trivial axiom besides $\overline{\text{ImComp}}$. This follows because if $\overline{\text{Fnc}}(\overline{F})$ then $\text{Fnc}(\overline{F})$ and if $x \varepsilon L$ then $\overline{F}''x \subseteq M(\alpha)$ for some α by AxRep. Let $f = F \cap (x \times M(\alpha))$. Then $f''x = \overline{F}''x$ and $(x \times M(\alpha)) \varepsilon L$, so the property would give $f \varepsilon L$ and hence $\overline{F}''x \varepsilon L$.

In the proof for VB, one simply defined $\mathcal{L}(X) \overset{\text{Df}}{\equiv} X \subseteq L \wedge (u)[u \varepsilon L \Longrightarrow u \cap X \varepsilon L]$. Without using the finite axiomitizability of VB, it is easily seen that $\overline{\text{AxComp}}$ holds. Namely, it will suffice to show that if $A = \{\bar{u} \mid \overline{\varphi}(\bar{u}, \overline{X}_1, \ldots, \overline{X}_n)\}$ for predicative φ, then $A \cap M(\alpha) \varepsilon L$ for all α. But $A \cap M(\alpha) = \hat{x}^\alpha \overline{\varphi}(x^\alpha, \overline{X}_1, \ldots, \overline{X}_n)$

and by a repeated application of AxRep (as in Cohen [3], pp. 106) one gets $A \cap M(\alpha) = \hat{x}^{\alpha}\psi(x^{\alpha}, c_1, \ldots, c_n)$ where ψ has only bounded quantifiers, and $c_1, \ldots, c_n \in L$. Thus $A \cap M(\alpha) \in L$.

This method, however, does not appear to work for the inner model proof for VBI. The natural model $\mathfrak{Q} = \mathcal{P}(M(\omega_\iota)) \cap M(\omega_{\iota+1})$ suggests that the construction of L be in some sense extended. Intuitively we may think of On as ω_ι, the first inaccessible. Although there are no ordinals longer than ω_ι, one can consider classes which are well-orderings, and these, intuitively, might be of length β for each $\beta < \omega_{\iota+1}$. This is reminiscent of the fact that each countable ordinal can be represented by a relation on the integers. Indeed there will be other analogies between the present project of constructing classes which are essentially $M(\beta)$ for $\beta > \omega_\iota$, and certain methods of hierarchy theory.

CHAPTER II

THE AXIOM OF CONSTRUCTIBILITY IN VBI

1. Complete classes.

In order to define the inner model it will be necessary to define what might be called an <u>outer model</u>. That is, we shall define certain objects called <u>structures</u>; for the structures a membership relation and an equivalence relation may then be defined, and the resulting construction will give a "model" for a certain natural extension of ZF.

We remark that unless otherwise noted the theorems and lemmas are statements provable in VBI, and schemata are, as usual, infinite collections of individual statements.

Well-founded classes with various additional properties are used throughout, and it is convenient to abbreviate $(u)\ [uRx \Rightarrow uRy]$ by $x \subseteq_R y$. Then clearly $R \models \mathrm{Ext}$ if and only if $x \subseteq_R y \wedge y \subseteq_R x \Rightarrow x = y$.

<u>Definition 1.1</u>

$\mathrm{Comp}(R) \overset{\mathrm{DF}}{\equiv} \mathrm{WF}(R) \wedge R \models \mathrm{Ext}$ (R is <u>complete</u>).

Comp(R) is the analog of transitivity for classes which may be too long to be isomorphic to transitive classes. Clearly a <u>set</u> x is complete if and only if it is isomorphic to a transitive set.

Definition 1.2

If Comp(R), call A transitive in R if $A \subseteq |R| \wedge (u)(x)\,[x \in A \wedge uRx \Rightarrow u \in A]$.

Theorem 1.3

If Comp(R) and A and B are transitive in R, and $F{:}R|A \approx R|B$ then $F = I|A$ (and $A = B$).

Proof:

Consider an R- least element a of A such that $F(a) \neq a$. If uRa then $u \in A$ by the transitivity of A, and $F(u) = u$ by the minimality of a, and F(u)RF(a), so $a \subsetneq_R F(a)$. Conversely, if xRF(a), then $x \in B$, by the transitivity of B, so $x = F(u)$, and uRa since F is an isomorphism. But then $F(u) = u = x$, so xRa and $F(a) \subsetneq_R a$. Since $R \models Ext$, $a = F(a)$, which is a contradiction.

Corollary 1.4

a. If Comp(R) then R has only the trivial automorphism since $|R|$ is transitive in R.

b. $Comp(R) \wedge Comp(S) \wedge F{:}R \approx S \wedge G{:}R \approx S \Rightarrow F = G$ because $G^{-1} \circ F{:}R \approx R$ and $G^{-1} \circ F = I\,|\,|R|$ by part (a).

Definition 1.5

Suppose Comp(R) and Comp(S).

a. $F{:}R \leq S \overset{\text{df}}{\equiv} (\exists A)\,[A \subseteq |S| \wedge A$ is transitive in S $\wedge F{:}R \approx S|A]$.

b. $R \leq S \overset{\text{df}}{=} (\exists F)\,[F{:}R \leq S]$.

Lemma 1.6

Suppose R, S, and T are complete.

a. $F{:}R \leq R \Longleftrightarrow F = I \mid |R|$.

b. $F{:}R \leq S \wedge G{:}S \leq T \Longrightarrow G \circ F{:}R \leq T$.

c. $F{:}R \leq S \wedge G{:}S \leq R \Longrightarrow F{:}R \approx S$.

Proof:

(a) is immediate from Theorem 1.3. As for (b), if $F{:}R \approx S \mid A$ and $G{:}S \approx T \mid B$ then it is easy to check that $(G \circ F)\text{"}\,|R|$ is transitive in T, and hence $G \circ F{:}R \leq T$. Part (c) follows from parts (a) and (b).

Lemma 1.7

If R and S are well-orderings then $R \leq S$ or $S \leq R$.

Proof:

It is easy to see that if R is a well-ordering then Comp(R); also, A is transitive in R if an only if $A = |R|$ or A is a segment of the form $R_a = \{u \mid uRa\}$ for some a. By Corollary 1.4 there can be at most one isomorphism mapping a segment R_a onto a segment S_b.
Let $G = \{x \mid (\exists F)(\exists a)(\exists b)\,[F{:}R_a \approx S_b \wedge x \in F]\}$.
Thus G is the union of the partial isomorphism, and it is clear that either $\mathcal{D}(G) = |R|$ or $\mathcal{R}(G) = |S|$, or else G could be extended by adding $\langle r,s \rangle$ where r and s are, respectively, the least elements of $|R|$ and $|S|$ not covered

by G . Hence either $G:R \leq S$ or $G^{-1}:S \leq R$.

We remark that an impredicative comprehension axiom is used here, and appears to be essential. Thus the preceding lemma is an example of a natural and intuitively true sentence which (apparently) cannot be proven in VB.

Definition 1.8

If Comp(R), let $W(R) \overset{DF}{=} \{x \mid x \in |R| \wedge R \models Ord(x)\}$.

Lemma 1.9

W(R) is transitive in R.

Proof:

W(R) is transitive in R just in case whenever $R \models Ord(y)$ and xRy then $R \models Ord(x)$. $Ord(y) \overset{DF}{=} Trans(y) \wedge (v)(v \in y \Rightarrow Trans(v))$, so if $R \models Ord(y)$ and xRy then $R \models Trans(x)$. If uRx then uRy (because $R \models Trans(y)$) so $R \models Trans(u)$. Hence $R \models [Trans(x) \wedge (u)(u \in x \Rightarrow Trans(u))]$ and $x \in W(R)$.

Lemma 1.10

$R \mid W(R)$ is a well-ordering.

Proof:

$R \mid W(R)$ is well-founded and so if $x \in W(R)$ then $\sim xRx$. For $x,y,z \in W(R)$, if xRy and yRz then xRz because $R \models Trans(z)$. Hence we have only to show comparability, namely, if $x,y \in W(R)$ then xRy or yRx or $x = y$. Let x_0 be a least member x of W(R) such that there is a y in W(R) and $x \neq y$ and $\sim xRy$ and $\sim yRx$, and for x_o let y_o be a least such y. If uRx_o then $u \in W(R)$ by Lemma 1.9 so uRy_o or $u = y_o$ or y_oRu, by the minimality of x_o. Clearly uRy_o and hence $x_o \subsetneqq_R y_o$. Similarly $y_o \subsetneqq_R x_o$, and so $x_o = y_o$.

Lemma 1.11

If Comp(R) and $R \models Fino(r)$ then there are finitely many $u \in |R|$ such that uRr.

Proof:

We may assume $\omega+1 \leq W(R)$, because otherwise $W(R) \leq \omega$ and the statement is trivial. If $f: \omega+1 \leq W(R)$ then $r = f(i)$ for some $i \in \omega$, because otherwise either $r = f(\omega)$ or $f(\omega)Rr$; but $R \models Lim(f(\omega))$ which would contradict $R \models Fino(r)$. Since $r = f(i)$ it is clear that if uRr then $u = f(j)$ for some $j < i$, so the lemma is proved.

Definition 1.12

If Comp(R) and $x \in |R|$, define Cl(x,R) (the closure of x in R) by:

$$v \in Cl(x,R) \Longleftrightarrow (\exists F)(\exists n)\ [\ SC(F,n+1) \wedge F_o = \{x\}$$
$$\wedge\ (i)(0 \leq i < n \Longrightarrow F_{i+1} = \{u \mid (\exists y)(uRy \wedge y \in F_i)\}\ )$$
$$\wedge\ (\exists i)(0 \leq i \leq n \wedge v \in F_i)].$$

Lemma 1.13

a. Cl(x,R) is transitive in R.

b. If A is transitive in R and $x \varepsilon A$ then $Cl(x,R) \subseteq A$.

Proof:

a. If $v \varepsilon Cl(x,R)$ and zRv, then there is an F and an n such that $SC(F,n+1)$ and $v \varepsilon F_i$ for some $0 \leq i \leq n$. If $i < n$ then $z \varepsilon F_{i+1}$ and $z \varepsilon Cl(x,R)$; if $i = n$ then extend F to G, where $G_{n+1} = \{u | (\exists y)[uRy \wedge y \varepsilon F_n]\}$. Clearly $z \varepsilon G_{n+1}$ and so $z \varepsilon Cl(x,R)$.

b. Suppose A is transitive in R and $x \varepsilon A$. If $v \varepsilon Cl(x,R)$, then there is a finite sequence of classes F such that $v \varepsilon F_n$. $F_o = \{x\} \subseteq A$. If $F_i \subseteq A$ then F_{i+1} $\{u | (\exists y)(uRy \wedge y \varepsilon F_i)\} \subseteq A$ because A is transitive. Hence $Cl(x,R) \subseteq A$.

Definition 1.14

$Norm(R) \overset{DE}{=} WF(R) \wedge R \models \sigma$ (R is normal).

Trivially Norm(R) implies Comp(R). The normal classes are just the "long" $M(\alpha)$ needed to define the inner model, and the normal sets are just the sets isomorphic to $M(\lambda)$ for $Lim(\lambda)$.

Definition 1.15

$\hat{R}$ is a structure if $\hat{R} = \langle R, x \rangle$ where Norm(R) and $x \varepsilon |R|$.

The structures will be the points of the outer model mentioned earlier. In the remainder of this section we define a membership relation η and an equivalence relation $\equiv$ between structures; the lemmas, however, use only the fact that R is complete.

Definition 1.16

$$\langle R,x\rangle \ \eta \ \langle S,y\rangle \overset{\mathrm{DF}}{=} (\exists z)\,[\mathrm{Cl}(x,R) \approx \mathrm{Cl}(z,S) \wedge zSy] \,.$$

Definition 1.17

$$\hat{R} \equiv \hat{S} \overset{\mathrm{DF}}{=} (\hat{T})\,[\hat{T}\,\eta\,\hat{R} \Leftrightarrow \hat{T}\,\eta\,\hat{S}] \,.$$

Lemma 1.18

$\langle R,x\rangle \equiv \langle S,y\rangle$ if and only if $\mathrm{Cl}(x,R) \approx \mathrm{Cl}(y,S)$.

Proof:

If $F:\mathrm{Cl}(x,R) \approx \mathrm{Cl}(y,S)$ and $\langle T,t\rangle \ \eta \ \langle R,x\rangle$ then suppose $G:\mathrm{Cl}(t,T) \approx \mathrm{Cl}(r,R) \wedge rRx$.
Clearly $F\circ G:\mathrm{Cl}(t,T) \approx \mathrm{Cl}(F(r),S)$ and $F(r)Sy$, and so $\langle T,t\rangle \ \eta \ \langle S,y\rangle$. Thus $\langle R,x\rangle \equiv \langle S,y\rangle$. Conversely, suppose $\langle R,x\rangle \equiv \langle S,y\rangle$. It is immediate from the definition of $\mathrm{Cl}(x,R)$ that $v \in \mathrm{Cl}(x,R) \Leftrightarrow$ $[v = x \vee (\exists z)(zRx \wedge v \in \mathrm{Cl}(z,R))]$. It is also clear that $\langle R,r\rangle \ \eta \ \langle R,x\rangle$ if and only if rRx, so if rRx then $(\exists s)\,[sSy \wedge \mathrm{Cl}(r,R) \approx \mathrm{Cl}(s,S)]$. Similarly if sSy then $(\exists r)\,[rRx \wedge \mathrm{Cl}(s,S) \approx \mathrm{Cl}(r,R)]$.
Let $G = \{u \mid (\exists F)(\exists r)(\exists s)\,[rRx \wedge sSy \wedge F:\mathrm{Cl}(r,R) \approx \mathrm{Cl}(s,S) \wedge u \in F]\}$, and let $H = G \cup \{\langle x,y\rangle\}$.
It is easy to see that $H:\mathrm{Cl}(x,R) \approx \mathrm{Cl}(y,S)$.

Lemma 1.19

If $\hat{R} \equiv \hat{S}$ and $\hat{S} \eta \hat{T}$ then $\hat{R} \eta \hat{T}$.

Proof:

Let $\hat{R} = \langle R,x \rangle$, $\hat{S} = \langle S,y \rangle$ and $\hat{T} = \langle T,z \rangle$. By the preceding lemma, if $\hat{R} \equiv \hat{S}$ then $Cl(x,R) \approx Cl(y,S)$. So if $\hat{S} \eta \hat{T}$, then $(\exists t)\,[Cl(y,S) \approx Cl(t,T) \wedge tTz]$ and $Cl(x,R) \approx Cl(t,T)$ for the same t, and hence $\hat{R} \eta \hat{T}$.

Theorem 1.20 (Schema)

If $\hat{R} \equiv \hat{S}$ and $\hat{\Phi}(\hat{S})$ then $\hat{\Phi}(\hat{R})$, where $\hat{\Phi}$ is any formula involving only quantification over structures and the relations η and $\equiv$.

Proof:

This is immediate from the fact that if $\hat{R} \equiv \hat{S}$ then for any $\hat{U}$, $\hat{U} \eta \hat{R} \iff \hat{U} \eta \hat{S}$ (by definition of $\equiv$) and $\hat{R} \eta \hat{U} \iff \hat{S} \eta \hat{U}$ (by Lemma 1.19).

2. Analysis of normal classes

It will be shown in this section that any normal class is built up from its "ordinals" in a manner similar to an $M(\lambda)$. Hence a normal class R will be essentially determined by W(R). The normal classes will be comparable under $\preceq$, and each normal class will have a natural well-ordering.

The predicate Fodo was originally defined for sets.

However in VBI the definition can be extended to arbitrary classes. Fodo(A,B,R), "A is first order definable over B with respect to R", $\overset{DF}{\equiv}$ $Rel(R) \wedge B \subseteq |R| \wedge (\exists G)(\exists n)[SC(G,n+1) \wedge G_0 = R(B,j) \wedge G_1 = I(B,j) \wedge \ldots \wedge G_n \subseteq B^1 \wedge A = \{v | \{\langle 0,v \rangle\} \varepsilon G_n\}$, where $R(B,j) \overset{DF}{\equiv} \{f | f : j+2 \rightarrow B \wedge f(0)Rf(1)$. That is, $I(B,j)$, B^j, and the various operations, such as union and permutation, can be extended to classes in the obvious way, and to define Fodo(A,B,R), Definition 1.8 of Chapter I is modified by using a finite sequence of classes G. It is clear that Fodo(a,b) $\Longleftrightarrow$ Fodo(a,b,E), so the notation is consistent.

Lemma 2.1

Fodo(A,B,R) if and only if there is a formula $\varphi(u,x_1, \ldots, x_n)$ and $b_1, \ldots, b_n \varepsilon B$ such that $A = \{u | (R|B) \models \varphi(u,b_1, \ldots, b_n)\}$.

Proof:

This is immediate because if Fodo(A,B,R) then to the sequence of classes G there corresponds a formula φ such that $A = \{u | (R|B) \models \varphi(u,b_1, \ldots, b_n)\}$. Conversely, for any formula φ, put it in prenex normal form Ψ, and then the class $A = \{u | (R|B) \models \Psi(u,b_1, \ldots, b_n)\}$ can be defined by an appropriate sequence G.

Definition 2.2

Suppose R is normal.

a. If $a \varepsilon |R|$ let $a^* = \{u | uRa\}$.

b. If $R \models \text{Fino}(d)$ let $d\# = \text{cardinal}(d^*)$.

c. Let r_j be that unique element of $|R|$ such that
$R \models [(u) \sim (u \varepsilon r_o) \wedge r_1 = \{r_o\}$
$\wedge r_2 = \{r_o, r_1\} \wedge . \ . \ . \ . \wedge r_j = \{r_o, \ . \ . \ . \ , r_{j-1}\}]$.

d. If $R \models h: r_j \simeq r_j$ let
$h'' = \{ \langle i,k \rangle | R \models \langle r_i, r_k \rangle \varepsilon h\}$.

e. If $R \models [f: r_j \rightarrow b]$ let $f' = \{ \langle i,u \rangle | R \models [\langle r_i, u \rangle \varepsilon f]\}$.

f. If $R \models x \subseteq b^{r_j}$ let $x! = \{f' | R \models f \varepsilon x\}$.

(Although this notation is ambiguous, it will be clear from context which R is meant.)

The preceding definitions are made with a view to coding the "finite functions" and "sets of finite functions" of R into genuine finite functions and classes of finite functions. In connection with the definitions we point out several facts: If $a^* = b^*$ then $a = b$ because $R \models \text{Ext}$; if $R \models \text{Fino}(d)$ then $R \models d = r_j$ where $j = d\#$; if $R \models h: r_j \simeq r_j$ then $h'': j \simeq j$; if $R \models f: r_j \rightarrow b$ then $f': j \rightarrow b^*$; if $R \models x \subseteq b^{r_j}$ then $x! \subseteq (b^*)^j$. Other facts needed for the basic Lemma 2.4 are listed below.

Lemma 2.3

Suppose R is normal.

a. If $R \models x = b^{r_j}$ then $x! = (b^*)^j$.

b. If $R \models [x \subseteq y \subseteq b^{r_j}]$ then $x! \subseteq y!$.

c. If $R \models [z = x \cup y \wedge x \subseteq b^{r_j} \wedge y \subseteq b^{r_j}]$ then $z! = x! \cup y!$.

d. Similarly for $z = x - y$.

e. If $R \models x = E(b, r_j)$ then $x! = R(b^*, j)$.

f. If $R \models x = I(b, r_j)$ then $x! = I(b^*, j)$.

g. If $R \models \mathrm{Perm}(x, h, y)$ then $\mathrm{Perm}(x!, h'', y!)$.

h. If $R \models \mathrm{Exist}(x, y)$ then $\mathrm{Exist}(x!, y!)$.

i. If $R \models \mathrm{Sub}(x, u, y)$ then $\mathrm{Sub}(x!, u, y!)$.

j. If $R \models [a = \mathcal{R}\mathcal{J}(x) \wedge x \subseteq b^{r_1}]$ then $a^* = \mathcal{R}\mathcal{J}(x!)$.

Proof:

These proofs are straightforward verifications.

Lemma 2.4

Suppose R is normal. If $R \models \mathrm{Fodo}(a, b)$ then $\mathrm{Fodo}(a^*, b^*, R)$; if $\mathrm{Fodo}(A, b^*, R)$ for some A, then there is an $a \in |R|$ such that $a^* = A$ and $R \models \mathrm{Fodo}(a, b)$.

Proof:

Suppose $R \models \mathrm{Fodo}(a, b)$. Then there is a $g \in |R|$ such that $R \models \mathrm{Fnc}(g)$ and $R \models g(r_o) = E(b, r_j)$ and $R \models g(r_1) = I(b, r_j)$ and, . . . , and $R \models g(r_n) \subseteq b^{r_1}$ and $R \models a = \mathcal{R}\mathcal{J}(g(r_n))$. There is a finite sequence of classes G such that $G_o = g(r_o)!$, $G_1 = g(r_1)!$, . . . , $G_n = g(r_n)!$. By Lemma 2.3 we have $a^* = \mathcal{R}\mathcal{J}(G_n)$, and $\mathrm{Fodo}(a^*, b^*, R)$, since the sequence of classes G satisfies the required condition.

Conversely, if $\mathrm{Fodo}(A, b^*, R)$ then there is a finite sequence $G_o = R(b^*, j)$, $G_1 = I(b^*, j)$, . . . , $G_n \subseteq (b^*)^1$ and $A = \mathcal{R}\mathcal{J}(G_n)$. Since $R \models \sigma$ there are sets $x_o, x_1, \ldots, x_n$ and a such that $R \models x_o = E(b, r_j)$,

$R \models x_1 = I(b,r_j)$, . . . , $R \models x_n \subseteq b^{r,}$ and $R \models a = \mathcal{RS}(x_n)$.
Also there is a g such that $R \models g = \{\langle r_o, x_o \rangle, \ldots, \langle r_n x_n \rangle\}$.
Thus $R \models$ Fodo(a,b). By Lemma 2.3 $x_i! = G_i$, so
$a^* = \mathcal{RS}(x_n!) = \mathcal{RS}(G_n) = A$.

Lemma 2.5

If Norm(R) and $R \models [\mathrm{Cond}(f) \wedge \mathrm{Cond}(g)]$ then f and g "agree on their common domain":
$R \models (u)(v)(w)\ [f(w) = u \wedge g(w) = v \Rightarrow u = v]$.

Proof:

Let w be the least member of W(R) such that $f(w) \neq g(w)$. w cannot be r_o because $f(r_o) = r_o = g(r_o)$. If $R \models \mathrm{Lim}(w)$ then $R \models f(w) = \{u \mid (\exists z)[z \in w \wedge f(z) = u]\}$ so if $R \models f(z) = g(z)$ for z such that zRw, then $R \models f(w) = g(w)$. If there is a z such that $R \models w = z \cup \{z\}$, then $R \models (u)\ [u \in f(w) \Leftrightarrow \mathrm{Fodo}(u, f(z))]$ so again $R \models f(w) = g(w)$ because $R \models f(z) = g(z)$.

Definition 2.6

If Norm(R), let $M(w,R) = \{x \mid (\exists f)[R \models \mathrm{Cond}(f) \wedge x \in f(w)]\}$ for $w \in W(R)$.

It is easy to check that for each $w \in W(R)$ there is an $f \in |R|$ and $R \models [\mathrm{Cond}(f) \wedge (\exists u)(\langle w,u \rangle \in f)]$; furthermore, from the preceding lemma, $M(w,R) = f(w)^*$ for any such f. It is also clear that $|R|$ is the union of all M(w,R).

Lemma 2.7

a. $M(w,R)$ is transitive in R for all $w \in W(R)$.

b. If w_1Rw_2 then $M(w_1,R) \subseteq M(w_2,R)$.

Proof:

a. It is clear that $M(r_o,R) = \emptyset$ and $M(r,R) = \{r_o\}$. Also if $R \models Lim(w)$ then $M(w,R) = \bigcup_{zRw} M(z,R)$. Thus if there is a w such that $M(w,R)$ is not transitive in R, then the least such w, call it w_1, must be a successor, $R \models [w_1 = z \cup \{z\}]$ for a certain $z \in W(R)$.
If $x \in M(w_1,R)$ then $R \models Fodo(x,f(z))$ and $R \models x \subseteq f(z)$. If vRx then $vRf(z)$, and since $M(z,R)$ is transitive in R, $R \models (u)[u \in v \Longleftrightarrow u \in f(z) \wedge u \in v]$. Hence $R \models Fodo(v,f(z))$ and $v \in M(w_1,R)$, so $M(w_1,R)$ is transitive in R.

b. By the preceding proof, if $R \models [w = z \cup \{z\}]$, then $M(z,R) \subseteq M(w,R)$. Hence it follows by induction that w_1Rw_2 implies $M(w_1,R) \subseteq M(w_2,R)$.

Theorem 2.8

If $Norm(R)$ and $Norm(S)$ and $W(R) \preceq W(S)$ then $R \preceq S$.

Proof:

Suppose $F:W(R) \preceq W(S)$. We will extend F to a function $G:R \preceq S$, by showing that there exist isomorphisms $G_w:M(w,R) \approx M(F(w),S)$. It will suffice to show that if $G_z:M(z,R) \approx M(F(z),S)$ then, if $R \models [r = z \cup \{z\}]$ and $R \models [s = F(z) \cup \{F(z)\}]$ (and hence $s = F(r)$), there is an isomorphism $G_r: M(r,R) \approx M(s,S)$. If $x \varepsilon M(r,R)$ then $R \models Fodo(x,f(z))$ and hence $Fodo(x^*,M(z,R),R)$ by Lemma 2.4. Since $G_z: M(z,R) \approx M(F(z),S)$ clearly $Fodo(Y,M(F(z),S),S)$ where $Y = G_z''x^*$. Hence by Lemma 2.4 there is a $y \varepsilon |S|$ such that $y^* = Y$. Let $G_r(x)$ be that unique y. Clearly G_r is the desired isomorphism $M(r,R) \approx M(s,S)$.

Corollary 2.9

a. If Norm(R) and Norm(S) then $R \trianglelefteq S$ or $S \trianglelefteq R$.

b. If $W(R) \approx W(S)$ then $R \approx S$.

c. If $W(R) \approx \{z | z \varepsilon W(S) \wedge zSy\}$ then $R \approx S|M(y,S)$.

Proof:

(a) and (b) are immediate from Lemma 1.7 and Lemma 1.6. (c) follows from the proof of 2.8.

Lemma 2.10

If W is a well-ordering then $\{f | (\exists j)[f:j \to |W|]\}$ has a well-ordering which we call FS(W).

Proof:

Let $m(f)$ = the maximum of $\{f(i)\}$ in $|W|$.

Let $\langle f,g \rangle \varepsilon FS(W) \overset{DF}{\equiv} [m(f) < m(g)]$
$\vee [m(f) = m(g) \wedge \mathcal{D}(f) < \mathcal{D}(g)]$
$\vee [m(f) = m(g) \wedge \mathcal{D}(f) = \mathcal{D}(g) \wedge (\exists i)(\langle f(i),g(i) \rangle \varepsilon W$
$\wedge (k)(k < i \Rightarrow f(k) = g(k)))]$. It is easy to check that FS(W) is a well-ordering.

Theorem 2.11

Every normal class has a well-ordering.

Proof:

For every $x \in |R|$ define its order: $\partial(x) = w$ if $x \in M(z,R) - M(w,R)$ where $R \models z = w \cup \{w\}$. If $\partial(x) = w$ then $R \models \mathrm{Fodo}(x, f(w))$ so $\mathrm{Fodo}(x^*, M(w,R), R)$. Thus there are formulas φ such that $x^* = \{u \mid (R|M(w,R)) \models \varphi(u, c_1, \ldots, c_k)\}$. Just as in the case of the $M(\alpha)$ (see Lemma 1.5, Chapter I) one may introduce bounded variables x^z for arbitrary $z \in W(R)$ which are understood to range over $M(z,R)$. Then a completely analogous argument shows that there corresponds to each x finite sequences $\langle w, r_p, w_1, \ldots, w_n \rangle$ where $w = \partial(x)$, p is the Godel number of a formula defining x^* and $w_1, \ldots, w_n$ are the quantifier bounds.
Let $\xi : |R| \longrightarrow \{f \mid (\exists j)(f : j \rightarrow W(R))\}$ where $\xi(x)$ is the least f in the well-ordering $FS(R \mid W(R))$ which corresponds to x. Then we define a well-ordering of $|R|$ by $x \underset{R}{\leq} y$ if and only if $\langle \xi(x), \xi(y) \rangle \in FS(R|W(R))$. We point out that this well-ordering has the further property that if $F : R \preceq S$, then $[x \underset{R}{\leq} y \Longleftrightarrow F(x) \underset{S}{\leq} F(y)]$.

Definition 2.12

Let $\Gamma(\hat{X}, \hat{Y}) \overset{\mathrm{DF}}{=} (\exists R)(\exists a)(\exists b)[\hat{X} \equiv \langle R, a \rangle \wedge \hat{Y} \equiv \langle R, b \rangle \wedge a <_R b]$.

Theorem 2.13

Γ is well-defined with respect to $\equiv$ - types, and has the properties

a. $(\hat{X}) \sim \Gamma(\hat{X},\hat{X})$

b. $(\hat{X})(\hat{Y})\,[\hat{X} \equiv \hat{Y} \vee \Gamma(\hat{X},\hat{Y}) \vee \Gamma(\hat{Y},\hat{X})]$

c. $(\hat{X})(\hat{Y})(\hat{Z})[\Gamma(\hat{X},\hat{Y}) \wedge \Gamma(\hat{Y},\hat{Z}) \Rightarrow \Gamma(\hat{X},\hat{Z})]$

d. (Schema) For any formula $\Pi(\hat{X}_o, \ldots, \hat{X}_n)$ where Π is well defined with respect to $\equiv$ - types,

$$(\hat{X}_1) \ldots (\hat{X}_n)\,[(\exists \hat{R})\Pi(\hat{R},\hat{X}_1, \ldots, \hat{X}_n)$$
$$\Rightarrow (\exists \hat{S})(\Pi(\hat{S},\hat{X}_1, \ldots, \hat{X}_n) \wedge (\hat{T})(\Gamma(\hat{T},\hat{S})$$
$$\Rightarrow \sim \Pi(\hat{T},\hat{X}_1, \ldots, \hat{X}_n)))]$$

Proof:

a. If $\hat{X} \equiv \langle R,a\rangle \equiv \langle R,b\rangle$ for some R, then $Cl(a,R) \approx Cl(b,R)$ and $a = b$.

b. Suppose $\hat{X} = \langle X,x\rangle$ and $\hat{Y} = \langle Y,y\rangle$. Then by comparability we may assume $X \leq Y$ and $\hat{X} \equiv \langle Y,z\rangle$. Then according as $y = z$, $y <_Y z$, or $z <_Y y$ it follows that $\hat{X} \equiv \hat{Y}, \Gamma(\hat{Y},\hat{X})$ or $\Gamma(\hat{X},\hat{Y})$.

c. In the definition of Γ, if for some R $\hat{X} \equiv \langle R,a\rangle$ and $\hat{Y} \equiv \langle R,b\rangle$ and $a <_R b$, then for any other normal class S, if $\hat{X} \equiv \langle S,c\rangle$ and $\hat{Y} \equiv \langle S,d\rangle$, then $c <_S d$, by the properties of the well-orderings $<_R$ and $<_S$ (see Theorem 2.11). Therefore given $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ there is an R such that $\hat{X} \equiv \langle R,a\rangle$, $\hat{Y} \equiv \langle R,b\rangle$ and $\hat{Z} \equiv \langle R,c\rangle$ and $a <_R b$ and $b <_R c$. Hence $a <_R c$ and $\Gamma(\hat{X},\hat{Z})$.

d. For the given formula Π, suppose $\Pi(\hat{R},\hat{X}_1, \ldots, \hat{X}_n)$, where $\hat{R} \equiv \langle R,a\rangle$. Let b be the least (relative to $<_R$) u such that $\Pi(\langle R,u\rangle,\hat{X}_1, \ldots, \hat{X}_n)$, and let $\hat{S} \equiv \langle R,b\rangle$. Then clearly $\Pi(\hat{S},\hat{X}_1, \ldots, \hat{X}_n)$ and $\Gamma(\hat{T},\hat{S}) \Rightarrow \sim\Pi(\hat{T},\hat{X}_1, \ldots, \hat{X}_n)$ because if $\Gamma(\hat{T},\hat{S})$ then there is an element $c \in |R|$ such that $c <_R b$ and $\hat{T} \equiv \langle R,c\rangle$.

It is clear that Γ is essentially a well-ordering of the ($\equiv$ - types of) structures. For any formula Π which is well-defined with respect to $\equiv$ - types, it will be convenient to let $\Pi_\mu(\hat{X}_o,\hat{X}_1, \ldots, \hat{X}_n)$

$$\overset{\text{Df}}{\equiv} \Pi(\hat{X}_o,\hat{X}_1, \ldots, \hat{X}_n) \wedge (\hat{U})(\Gamma(\hat{U},\hat{X}_o) \Rightarrow \sim\Pi(\hat{U},\hat{X}_1, \ldots, \hat{X}_n)).$$

By the preceding theorem Π_μ is well-defined with respect to $\equiv$ - types and has the properties:

a. $\Pi_\mu(\hat{X}_o, \ldots, \hat{X}_n) \Rightarrow \Pi(\hat{X}_o, \ldots, \hat{X}_n)$

b. $(\exists \hat{R})\,\Pi(\hat{R},\hat{X}_1, \ldots, \hat{X}_n) \Rightarrow (\exists \hat{R})\,\Pi_\mu(\hat{R},\hat{X}_1, \ldots, \hat{X}_n)$

c. $\Pi_\mu(\hat{R},\hat{X}_1, \ldots, \hat{X}_n) \wedge \Pi_\mu(\hat{S},\hat{X}_1, \ldots, \hat{X}_n) \Rightarrow \hat{R} \equiv \hat{S}$.

<u>Theorem 2.14</u>

$$\text{Norm}(R) \Rightarrow (\exists S)(\text{Norm}(S) \wedge \sim S \leq R)$$

(That is, every normal class can be properly extended.)

<u>Proof:</u>

Intuitively the normal classes are $M(\lambda)$ for $\text{Lim}(\lambda)$ and the method of extending $M(\lambda)$ to $M(\lambda+\omega)$ can be modified and applied to normal classes. We define a predicate

Extend(X,Y) which holds for a given $X = \langle A_1, W_1 \rangle$ and a certain $Y = \langle A_2, W_2 \rangle$, where A_1 is a relation such that all elements of $|A_1|$ are of the form $\langle i, x \rangle$ for i less than some fixed k, and W_1 is a well-ordering of $|A_1|$:

Given any $\langle A_1, W_1 \rangle$ of the described form, let j be the least k such that $\langle i,x \rangle \in |A_1| \Rightarrow i < k$. Let C be the class of all finite sequences $\langle j, p, a_1, \ldots, a_n \rangle$ such that

(i) p is the Godel number of a formula φ with $n+1$ free variables.

(ii) There is no $a \in |A_1|$ such that $A_1 \models (u)\,[u \in a \Leftrightarrow \varphi(u, a_1, \ldots, a_n)]$

(iii) There is no $\langle q, b_1, \ldots, b_m \rangle$ preceding $\langle p, a_1, \ldots, a_n \rangle$ (in the natural well-ordering induced by W_1) such that $A_1 \models (u)\,[\psi(u, b_1, \ldots, b_m) \Leftrightarrow \varphi(u, a_1, \ldots, a_n)]$ where p codes φ and q codes ψ.

Define $|A_2| = |A_1| \cup C$ and

$A_2 = A_1 \cup \{\langle x,y \rangle \mid x \in |A_1| \wedge y \in C \wedge A_1 \models \varphi(x)\}$

(where $y = \langle p, a_1, \ldots, a_n \rangle$ determines the formula $\varphi(u, a_1, \ldots, a_n)$). $|A_2|$ can be given the obvious well-ordering. Under these conditions we say Extend($\langle A_1, W_1 \rangle$, $\langle A_2, W_2 \rangle$).

Given a normal class R code it into

$R_0 = \{\langle\langle 0,x \rangle, \langle 0,y \rangle\rangle \mid \langle x,y \rangle \in R\}$. Let $W_0 = <_{R_0}$.

Then let $S = \{y \mid (\exists G)(\exists n)\,[SC(G, n+1) \wedge G_0 = \langle R_0, W_0 \rangle$

$\wedge (i)(i < n \Rightarrow \text{Extend}(G_i, G_{i+1}))$

$\wedge (\exists B)(\exists W)(G_n = \langle B, W \rangle \wedge y \in B)]\}$.

By the method of extension it is straightforward to check that S is normal.

3. The inner model $\mathcal{K}$.

For the universe of the inner model $\mathcal{K}$, we shall single out certain subclasses of L which are in a natural sense projections of structures on L.

Definition 3.1

a. $\mathrm{Proj}(\hat{R}) \overset{DF}{=} \{x \mid \langle L,x \rangle \eta \hat{R}\}$.

b. $\mathcal{K}(A) \overset{DF}{=} (\exists \hat{R})[A = \mathrm{Proj}(\hat{R})]$.

It is clear that $\langle L,x \rangle$ is a structure for each $x \in L$ and that $x = \mathrm{Proj}(\langle L,x \rangle)$. Therefore $x \in L \Rightarrow \mathcal{K}(x)$, and the "sets" of $\mathcal{K}$ are exactly the members of L. We shall often use $\bar{x}$ and $\bar{X}$ to denote sets and classes of $\mathcal{K}$.

In order to prove the relativized axioms it will suffice to show

$x \in L \wedge \mathcal{K}(A) \Rightarrow x \cap A \in L$ (for $\overline{\text{AxRep}}$) and

$\mathcal{K}(\{\bar{x} \mid \Phi(\bar{x},\bar{X}_1, \ldots, \bar{X}_n)\})$ (for $\overline{\text{ImComp}}$).

Theorem 3.2

If $x \in L$ and $\mathcal{K}(A)$ then $x \cap A \in L$.

Proof:

It will suffice to show that if $\mathcal{K}(A)$ then $M(\alpha) \cap A \in L$ for every α. Suppose $A = \mathrm{Proj}(\hat{R})$ and $\hat{R} = \langle R,a \rangle$. By comparability we may assume $(\exists f)(f:M(\alpha+\omega) \trianglelefteq R)$ (otherwise $R \trianglelefteq M(\alpha+\omega)$ and $A \in L$). Let $m = f(M(\alpha))$, and let $g = f \restriction M(\alpha)$. It is obvious that m^* is transitive in R. Let $r \in |R|$ be such that $R \models r = m \cap a$. Let N be the closure of $m^* \cup \{r\}$ under the Skolem functions of σ in R. Since m^* is a set, N is a set and $R|N \models \sigma$. Hence there is an h, $h:R|N \approx E|M(\beta)$ for some β. Consider $h \circ g: M(\alpha) \longrightarrow M(\beta)$. It is clear that $(h \circ g)''M(\alpha)$ is transitive, and that $(h \circ g) \restriction M(\alpha) = I \restriction M(\alpha)$. We claim that $h(r) = A \cap M(\alpha)$, and that $A \cap M(\alpha) \in M(\beta)$. If $x \in A \cap M(\alpha)$ then $g(x)Rr$ and $g(x) \in N$ so $h(g(x)) \in h(r)$ and $A \cap M(\alpha) \subseteq h(r)$. Conversely, if $x \in h(r)$, then $h^{-1}(x)Rr$ so $h^{-1}(x)Rm$ and $h^{-1}(x)Ra$, so $g^{-1}(h^{-1}(x)) = x \in A \cap M(\alpha)$. Therefore $h(r) = A \cap M(\alpha)$.

Definition 3.3

a. Let $\check{L}$ be a fixed structure such that $(\hat{U})[\hat{U} \eta \check{L} \iff (\exists x)(\hat{U} \equiv \langle L, x \rangle)]$ (by the proof of Theorem 2.14 there is such an $\check{L} = \langle L^+, \ell \rangle$ where L^+ is a shortest proper extension of L).

b. Let variables $\check{R}, \check{S}, \check{X}$, etc., range over structures $\hat{R}$ such that $(\hat{U})[\hat{U} \eta \hat{R} \Longrightarrow \hat{U} \eta \check{L}]$.

Lemma 3.4

$$(A)[\mathcal{H}(A) \Longrightarrow (\exists \check{R})(A = \mathrm{Proj}(\check{R}))].$$

Proof:

Suppose $A = \mathrm{Proj}(\hat{R})$. We may assume $L^+ \trianglelefteq R$ because otherwise $R \trianglelefteq L$ and $\hat{R}$ is already an $\check{R}$. If $F:L^+ \trianglelefteq R$, let a be such that $R \models a = F(\ell) \cap r$ where $\hat{R} = \langle R, r\rangle$. Clearly $\hat{U}\, \eta \langle R, a\rangle \Rightarrow \hat{U}\, \eta\, \breve{L}$ so $\langle R, a\rangle$ is an $\breve{R}$, and also $\mathrm{Proj}(\langle R, a\rangle) = \mathrm{Proj}(\hat{R}) = A$.

By the definition of projection it is clear that $\hat{R} \equiv \hat{S} \Rightarrow \mathrm{Proj}(\hat{R}) = \mathrm{Proj}(\hat{S})$. We have singled out the structures $\breve{R}$ because, as is easy to check, they have the further properties that
$\breve{R} \equiv \breve{S} \Leftrightarrow \mathrm{Proj}(\breve{R}) = \mathrm{Proj}(\breve{S})$ and $\breve{R}\, \eta\, \breve{S} \Leftrightarrow \mathrm{Proj}(\breve{R}) \in \mathrm{Proj}(\breve{S})$.
More generally, schema 3.6 holds.

<u>Definition 3.5</u>

Let Φ, Ψ, and Θ from now on be formulas with variables X_i, and predicate letters $\in$ and $=$.
Let $\bar{\Phi}$, $\hat{\Phi}$, and $\breve{\Phi}$ be defined by the replacements:

$\bar{\Phi}$:replace X_i by $\bar{X}_i$.

$\hat{\Phi}$:replace X_i by $\hat{X}_i$, $\in$ by η and $=$ by $\equiv$.

$\breve{\Phi}$:replace X_i by $\breve{X}_i$, $\in$ by η and $=$ by $\equiv$.

<u>Lemma 3.6(Schema)</u>

For any Φ, $(\breve{X}_1) \ldots (\breve{X}_n)(\bar{X}_1) \ldots (\bar{X}_n)$
$[\mathrm{Proj}(\breve{X}_1) = \bar{X}_1 \wedge \ldots \wedge \mathrm{Proj}(\breve{X}_n) = \bar{X}_n \Rightarrow (\breve{\Phi}(\breve{X}_1, \ldots, \breve{X}_n) \Leftrightarrow \bar{\Phi}(\bar{X}_1, \ldots, \bar{X}_n))]$.

<u>Proof:</u>

This is a straightforward induction using Lemma 3.4 since $\breve{X}_i\, \eta\, \breve{X}_j \Leftrightarrow \bar{X}_i \in \bar{X}_j$ and $\breve{X}_i \equiv \breve{X}_j \Leftrightarrow \bar{X}_i = \bar{X}_j$.

Definition 3.7

a. Let ImComp($\bar{\Phi}$) be the sentence

$$(\bar{X}_1)\ldots(\bar{X}_n)(\exists\bar{A})(\bar{x})[\bar{x}\,\varepsilon\,\bar{A}\Leftrightarrow\bar{\Phi}(\bar{x},\bar{X}_1,\ldots,\bar{X}_n)].$$

b. Let Aus(Ψ) be the sentence

$$(\hat{X}_1)\ldots(\hat{X}_n)(\hat{R})(\exists\hat{S})(\hat{U})[\hat{U}\eta\hat{S}\Leftrightarrow \hat{\Psi}(\hat{U},\hat{X}_1,\ldots,\hat{X}_n)\wedge\hat{U}\eta\hat{R}].$$

Lemma 3.8(Schema)

For any Φ there is a Ψ such that Aus(Ψ) implies ImComp(Φ).

Proof:

The formula $\check{\Phi}$ can be expressed by a formula $\hat{\Psi}$ since $(\exists\check{X})\Pi(\check{X})\Leftrightarrow(\exists\hat{X})[\Pi(\hat{X})\wedge(\hat{U})(\hat{U}\eta\hat{X}\Rightarrow\hat{U}\eta\check{L})]$. Then Aus($\Psi$) gives $(\exists\hat{S})(\hat{U})[\hat{U}\eta\hat{S}\Leftrightarrow\hat{\Psi}(\hat{U},\check{X}_1,\ldots,\check{X}_n)\wedge\hat{U}\eta\check{L}]$ where $\mathrm{Proj}(\check{X}_i)=\bar{X}_i$. By Lemma 3.6, $\mathrm{Proj}(\hat{S})=\bar{A}$ and ImComp(Φ) follows.

It follows from the preceding that the relativized comprehension axioms will be provable in VBI if the sentences Aus(Ψ) (which, of course, comprise the Aussonderung schema for the outer model) are provable in VBI. This demonstration, which uses a Skolem construction, will occupy the remainder of this section.

Definition 3.9

Let $\hat{\sigma}$ be the sentence which results from σ by replacing ε by η and $=$ by $\equiv$ and variables x_i by $\hat{X}_i$.

Theorem 3.10

$\hat{\sigma}$ (that is, VBI $\vdash \hat{\sigma}$).

Proof:

The clause Ext becomes
$(\hat{X})(\hat{Y})[(\hat{U})(\hat{U}\eta\hat{X} \Longleftrightarrow \hat{U}\eta\hat{Y}) \Longrightarrow \hat{X}\equiv\hat{Y}]$ and is trivially true. Consider the other components of $\hat{\sigma}$; for example, $(\hat{X})(\hat{Y})(\exists\hat{Z})(\hat{U})[\hat{U}\eta\hat{Z} \Longleftrightarrow \hat{U}\equiv\hat{X} \vee \hat{U}\equiv\hat{Y}]$. For any given $\hat{X} = \langle X,x\rangle$ and $\hat{Y} = \langle Y,y\rangle$ we may assume there is an $F, F:X \preceq Y$, by the comparability of normal classes. Let z be that member of $|Y|$ such that $Y \models z = \{F(x),y\}$. Then clearly $(\hat{U})[\hat{U}\eta\langle Y,z\rangle \Longleftrightarrow \hat{U}\equiv\hat{X}\vee\hat{U}\equiv\hat{Y}]$ and $\langle Y,z\rangle$ is the desired $\hat{Z}$. This same method may be used to establish the other clauses.

It will be necessary to define for formulas $\hat{\Phi}$ certain Skolem functionals. Suppose $\hat{\Phi}$ is a sentence (in prenex normal form) of the form $(\hat{X})(\exists\hat{Y})\hat{\Psi}(\hat{X},\hat{Y})$. Then let $\mathfrak{F}(\hat{X},\hat{Y}) \overset{DF}{\equiv} \hat{\Theta}_\mu(\hat{Y},\hat{X})$ where $\hat{\Theta}(\hat{Y},\hat{X}) \overset{DF}{\equiv} \hat{\Phi}(\hat{X},\hat{Y})$. By Lemma 2.13, $\mathfrak{F}(\hat{X},\hat{Y}_1) \wedge \mathfrak{F}(\hat{X},\hat{Y}_2) \Longrightarrow \hat{Y}_1 \equiv \hat{Y}_2$ so $\mathfrak{F}$ is essentially a function of the variable $\hat{X}$. Clearly to each existential variable $\hat{Y}_i$ corresponds a Skolem functional $\mathfrak{F}_i$ defined in terms of the functionals $\mathfrak{F}_k$ for $k < i$, which is (essentially) a function of the preceding universal variables.

For formulas φ with free variables the usual definition of Skolem functions may be modified so as to give the necessary functions for forming elementary submodels. The following method seems to be applicable to all situations which arise, and it makes possible a clear inductive proof that for arguments in the submodel, a formula holds in the submodel if and only if it holds in the original model. (This property is also clear for situations such as Lemma 3.13 where individual formulas are dealt with and the "models" are too large to be models in the usual sense.) Given a formula $\varphi(u_1, \ldots, u_n)$ with $n \geq 0$ free variables, put it in prenex normal form using only the quantifier $\exists$ (hence (v) becomes $\sim(\exists v)\sim$). To each subformula of the form $(\exists v_i)\,\theta(u_1, \ldots, u_n, v_1, \ldots, v_{i-1}, v_i)$ there corresponds a function $f_i(u_1, \ldots, u_n, v_1, \ldots, v_{i-1})$ which equals the least v_i such that $\theta(u_1, \ldots, u_n, v_1, \ldots, v_i)$ holds in the original model, if such a v_i exists, otherwise it is some chosen constant. Thus to each bound variable v_i there corresponds a function f_i which is a function of the n free variables plus i-1 other variables. It is clear that this method may be extended to define Skolem functionals of formulas $\hat{\Phi}(\hat{U}_0, \ldots, \hat{U}_n)$.

Consider Aus(Ψ) for a given $\hat{\Psi}(\hat{U}_0, \ldots, \hat{U}_n)$. Select $\hat{R}, \hat{B}_1, \ldots, \hat{B}_n$. In order to prove that there is an $\hat{S}$ such that $(\hat{U})\,[\hat{U} \eta \hat{S} \Longleftrightarrow \hat{\Psi}(\hat{U}, \hat{B}_1, \ldots, \hat{B}_n) \wedge \hat{U} \eta \hat{R}\,]$, it is convenient to define the Skolem hull $\mathcal{H}$ of $\hat{R}$ with respect to $\hat{\Psi}$ and $\hat{B}_1, \ldots, \hat{B}_n$.

Definition 3.11(Schema)

$\mathcal{H}(\hat{X}) \overset{DF}{\equiv} (\exists G)(\exists m)[SC(G,m+1)$ and for $i \leq m$

$[(\exists u)(G_i \equiv \langle R,u\rangle)$

or $G_i \equiv \hat{B}_1, \ldots, $ or $G_i \equiv \hat{B}_n$

or $\mathcal{F}(G_{i_1}, \ldots, G_{i_k}, G_i)]$ for some $i_1, \ldots, i_k < i$ where $\mathcal{F}$ is a Skolem functional of $\hat{\Psi}(\hat{U})$ or $\hat{\sigma}$

and $G_m \equiv \hat{X}]$.

Definition 3.12(Schema)

a. Let $\sigma^{\mathcal{H}}$ be the sentence resulting by restricting the quantifiers of $\hat{\sigma}$ to $\mathcal{H}$.

b. Let $\Psi^{\mathcal{H}}(\hat{U})$ be the formula resulting from $\hat{\Psi}(\hat{U})$ by restricting the quantifiers to $\mathcal{H}$.

Lemma 3.13(Schema)

a. $\sigma^{\mathcal{H}}$

b. If $\mathcal{H}(\hat{U})$ then
$\hat{\Psi}(\hat{U}) \Longrightarrow \Psi^{\mathcal{H}}(\hat{U})$ and $\sim\hat{\Psi}(\hat{U}) \Longrightarrow \sim \Psi^{\mathcal{H}}(\hat{U})$.

Proof:

This follows immediately from the fact that $\mathcal{H}$ is "closed" under the necessary Skolem functionals.

If $\mathcal{H}(\hat{X})$, then each derivation, or sequence of classes G, such that $G_m \equiv \hat{X}$ may be coded into an integer p such that p describes how the last term is derived from the structures $\hat{B}_1, \ldots, \hat{B}_n$ using j structures $\langle R,u_1\rangle, \ldots, \langle R,u_j\rangle$. If code names for $\langle R,u_1\rangle, \ldots, \langle R,u_j\rangle$ are supplied then a complete description of G is given up to the $\equiv$ - type of its terms, because the Skolem functionals are well-defined with respect to $\equiv$ - types and give essentially unique values.

Definition 3.14(Schema)

$$\mathrm{Code}(\hat{X},x) \overset{\mathrm{df}}{\equiv} \mathcal{H}(\hat{X}) \wedge x = \langle p, \xi(u_1), \ldots, \xi(u_j)\rangle$$

where p is the Godel number of a sequence of classes G yielding $\hat{X}$, and $\langle R,u_1\rangle, \ldots, \langle R,u_j\rangle$ are the structures used in the derivation G, as described above. (See Theorem 2.11 for the definition of ξ.)

Remark: a. A fixed well-ordering of the sequences $\langle p, \xi(u_1), \ldots, \xi(u_j)\rangle$ may be defined in the obvious way.

b. If $\mathrm{Code}(\hat{X},x)$ and $\mathrm{Code}(\hat{Y},x)$ then $\hat{X} \equiv \hat{Y}$ because, as noted before, the terms of G are determined by x up to $\equiv$ - type.

Definition 3.15(Schema)

$$\mathrm{Map}(\hat{X},x) \overset{\mathrm{df}}{\equiv} \mathcal{H}(\hat{X}) \wedge x = \mu y\, [(\exists \hat{Y})(\hat{Y} \equiv \hat{X} \wedge \mathrm{Code}(\hat{Y},y))] .$$

Lemma 3.16(Schema)

$$\mathrm{Map}(\hat{X},x) \wedge \mathrm{Map}(\hat{Y},y) \Rightarrow [x = y \Leftrightarrow \hat{X} \equiv \hat{Y}]$$

Proof:

If $\hat{X} \equiv \hat{Y}$ then $x = y$ by the definition of Map. Conversely, if $x = y$ then there are $\hat{X}_1$ and $\hat{Y}_1$ such that $\hat{X} \equiv \hat{X}_1$ and $\text{Code}(\hat{X}_1, x)$ and $\hat{Y} \equiv \hat{Y}_1$ and $\text{Code}(\hat{Y}_1, x)$. By Part (b) of the above remark, $\hat{Y}_1 \equiv \hat{X}_1$ so $\hat{X} \equiv \hat{Y}$.

From the preceding it is clear that Map is essentially a 1:1 function onto its range.

Definition 3.17(Schema)

Let $H \overset{DF}{=} \{\langle x,y\rangle \mid (\exists \hat{X})(\exists \hat{Y})(\text{Map}(\hat{X},x) \wedge \text{Map}(\hat{Y},y) \wedge \hat{X} \eta \hat{Y})\}$.

Lemma 3.18(Schema)

For any formula Φ ,

$$(X_1) \ldots (X_m)(x_1) \ldots (x_m)\, [\text{Map}(\hat{X}_1,x_1) \wedge \ldots \wedge \text{Map}(\hat{X}_m,x_m) \Longrightarrow [\Phi^{\mathcal{H}}(\hat{X}_1, \ldots, \hat{X}_m) \Longleftrightarrow H \models \Phi(x_1, \ldots, x_m)]]$$

where $\Phi^{\mathcal{H}}$ is the restriction of Φ to $\mathcal{H}$.

Proof:

This schema follows immediately from Lemma 3.16 and Definition 3.17.

Lemma 3.19(Schema)

H is normal.

Proof:

$H \models \sigma$ by Lemma 3.18, and it will thus suffice to show that H is well-founded. Let $A \subseteq |H|$. Take any $a \in A$ and any $\hat{X}$ such that $\mathrm{Map}(\hat{X},a)$. Let $D = \{x | x \in |X| \wedge (\exists b)[\mathrm{Map}(\langle X,x\rangle ,b) \wedge b \in A]\}$. Let d_o be an X- least element of D. Then $\mathrm{Map}(\langle X,d_o\rangle ,a_o)$ for a certain a_o and $a_o \in A$. We claim that a_o is an H- least element of A. If $a_1 H a_o$ and $a_1 \in A$, then consider any $\hat{Y}$ such that $\mathrm{Map}(\hat{Y},a_1)$. Since $a_1 H a_o$, $\hat{Y}\ \eta \langle X,d_o\rangle$ and so there is a d_1 such that $\hat{Y} \equiv \langle X,d_1\rangle$ and $d_1 X d_o$, which contradicts the minimality of d_o.

Several elementary observations will be essential for the proof of the main theorem. First, $F_1 : R \trianglelefteq H$ where $F_1(u) = v \Longleftrightarrow \mathrm{Map}(\langle R,u\rangle ,v)$: It is clear that F_1 is a 1:1 function; also the range of F_1 is transitive because if $F_1(u) = v$ and xHv, then by definition, there is an $\hat{X}$ such that $\mathrm{Map}(\hat{X},x)$ and $\hat{X}\ \eta \langle R,u\rangle$ so that $\hat{X} \equiv \langle R,y\rangle$ for some y, and x is in the range of F_1. The second point is that if for some u $\langle H,t\rangle \equiv \langle R,u\rangle$ then $\mathrm{Map}(\langle H,t\rangle ,t)$: If $\langle H,t\rangle \equiv \langle R,u\rangle$ then there is an isomorphism $F_2 : \mathrm{Cl}(u,R) \approx \mathrm{Cl}(t,H)$. $F_2 = F_1 \restriction \mathrm{Cl}(u,R)$ because $\mathrm{Cl}(u,R)$ is transitive in R. Since $F_2(u) = t$, $t = F_1(u)$ and $\mathrm{Map}(\langle R,u\rangle ,t)$, so also $\mathrm{Map}(\langle H,t\rangle ,t)$. Finally, observe that if $\mathrm{Map}(\langle R,u\rangle ,t)$ then $\langle R,u\rangle \equiv \langle H,t\rangle$: $F_1 : R \trianglelefteq H$ so $\mathrm{Cl}(u,R) \approx \mathrm{Cl}(F_1(u),H)$, but if $\mathrm{Map}(\langle R,u\rangle ,t)$ then $F_1(u) = t$ so $\langle R,u\rangle \equiv \langle H,t\rangle$.

<u>Theorem 3.20(Schema)</u>

$$(\exists \hat{S})(\hat{U})[\hat{U}\ \eta\ \hat{S} \Longleftrightarrow \hat{\Psi}(\hat{U},\hat{B}_1, \ldots , \hat{B}_n) \wedge \hat{U}\ \eta\ \hat{R}]$$

Proof:

Let S be a proper extension of H and let $G:H \leq S$. Let H_1 be the image of H in S (that is, $H_1 = S|(G''|H|)$). There is a point $x \varepsilon |S|$ such that $\hat{R} \equiv \langle S,x \rangle$, and for each formula Φ, a point y, such that $(u)\,[uSy \Longleftrightarrow H_1 \models \Phi(u)]$. Let $Map(\hat{B}_i, b_i)$ for $i = 1, \ldots, n$, and let $c_1, \ldots, c_n$ be the images of $b_1, \ldots, b_n$ in $|H_1|$ under G. Then let $z \varepsilon |S|$ be such that $(u)\,[uSz \Longleftrightarrow H_1 \models \Psi(u, c_1, \ldots, c_n)]$ and let $s \varepsilon |S|$ be such that $S \models s = x \wedge z$.

We claim then that $\hat{S} = \langle S,s \rangle$ is the desired structure. If $\hat{U} \eta \hat{S}$ then $\hat{U} \eta \langle S,x \rangle$ and so $\hat{U} \eta \hat{R}$. Also $\hat{U} \eta \langle S,z \rangle$ so there is a $u \varepsilon |H_1|$ such that $\hat{U} \equiv \langle S,u \rangle$ and $H_1 \models \Psi(u, c_1, \ldots, c_n)$. Let $t \varepsilon |H|$ be that element such that $G(t) = u$. Then $H \models \Psi(t, b_1, \ldots, b_n)$ and by Lemma 3.18, $\Psi^{*}(\hat{T}, \hat{B}_1, \ldots, \hat{B}_n)$ where $Map(\hat{T}, t)$. By Lemma 3.13 $\hat{\Psi}(\hat{T}, \hat{B}_1, \ldots, \hat{B}_n)$. Finally $\hat{T} \equiv \hat{U}$, because $\hat{U} \eta \hat{R}$ and $\hat{U} \equiv \langle H,t \rangle$ and so $Map(\hat{U}, t)$.

Conversely, if $\hat{\Psi}(\hat{U}, \hat{B}_1, \ldots, \hat{B}_n) \wedge \hat{U} \eta \hat{R}$ then it is straightforward to verify that $\hat{U} \eta \hat{S}$.

CHAPTER III

INDEPENDENCE RESULTS FOR VBI

1. The strong axiom of constructibility.

In the preceding chapter it was shown that $\mathcal{K}$ was an inner model for VBI plus the usual axiom of constructibility which states that each set is constructible. However in VBI one can consider a stronger statement which asserts that each class is constructible:

$(A)(\exists \hat{R})[A = \mathrm{Proj}(\hat{R})]$.

Let VBC be the set theory formed by adding $(A)(\exists \hat{R})[A = \mathrm{Proj}(\hat{R})]$ to VBI. Within VBC it can be shown that a very strong choice principle holds; namely, there is a definable linear ordering Δ of all classes which is essentially a well-ordering.

Definition 1.1

Let $\Delta(X,Y)$ be the sentence

$(\exists \hat{R})[X = \mathrm{Proj}(\hat{R}) \wedge (\hat{S})(Y = \mathrm{Proj}(\hat{S}) \Rightarrow \Gamma(\hat{R},\hat{S}))]$

Theorem 1.2

VBC $\vdash$

$[(X)\sim\Delta(X,X) \wedge (X)(Y)(\Delta(X,Y) \vee X = Y \vee \Delta(Y,X)) \wedge$
$(X)(Y)(Z)(\Delta(X,Y) \wedge \Delta(Y,Z) \Rightarrow \Delta(X,Z))]$

and for each $\Psi(Y,X_1, \ldots, X_n)$,

VBC $\vdash (X_1) \ldots (X_n)[(\exists Y)\Psi(Y,X_1, \ldots, X_n)$

$$\Longrightarrow (\exists Y)[\Psi(Y,X_1, \ldots, X_n) \wedge (Z)(\Delta(Z,Y) \Rightarrow \sim\Psi(Z,X_1, \ldots, X_n))]]$$

Proof:

These properties of Δ follow at once from the corresponding properties of Γ.

The two main points of this section are first to show that within VBI the inner model $\mathcal{K}$ is a model for VBC, and then to examine the outer model constructed within VBC(see Theorem 3.7) which will suggest a means for obtaining independence results for VBI. In order to establish the first point the essential fact needed is that every constructible class is constructed by a constructible class.

Lemma 1.3

$$(A)[\mathcal{K}(A) \Longrightarrow (\exists T)(\exists t)(A = \mathrm{Proj}(\langle T,t\rangle) \wedge \mathcal{K}(T))]$$

Proof:

We may assume $A = \mathrm{Proj}(\breve{R})$ where $\breve{R} = \langle R,a\rangle$. Consider the structure $\langle L^+, \ell\rangle$ mentioned in section 3 of Chapter II. We may assume that there is a function F, $F:L^+ \leq R$ because otherwise $R \leq L$ and the theorem would follow. By the definition of ℓ, $F(\ell)^*$ is the isomorphic image of L in R, that is, $E|L \approx R|F(\ell)^*$. Let X be the closure of $F(\ell)^* \cup \{a\}$ under the Skolem functions of σ in R. Thus $R|X$ is normal and $\mathrm{Proj}(\langle R|X,a\rangle) = A$. It is also clear that $R|X$ can be coded into a relation $T \subseteq L$: First, one can let members of $F(\mathcal{L})^*$ be represented by finite sequences of ordinals, and members of the Skolem closure of $F(\mathcal{L})^* \cup \{a\}$ can be mapped into finite sequences in a manner analogous to the definition of the normal class H from the Skolem hull $\mathcal{H}$.

However, the fact that T is normal and $T \subseteq L$ does not by itself give $\mathcal{H}(T)$. It is necessary to show that T is itself the projection of a structure $\langle B,b \rangle$. This follows in a straightforward manner from the following facts: (i) The natural function G which maps L into finite sequences of ordinals is constructible, and (ii) the natural well-ordering W of finite sequences $\langle p, f_1, \ldots, f_m \rangle$ where $p \in \omega$ and the f_i are finite sequences of ordinals is constructible since $\mathcal{H}$ is a model of VBI. Also (iii) the well-ordering of R is definable over R in the sense that there is a sentence $\varphi(x,y)$ such that $x <_R y$ if and only if $R \models \varphi(x,y)$. Thus (iv) the Skolem functions of σ are definable over R in the sense that for each Skolem function F_i there is a sentence θ_i such that $F_i(x_1, \ldots, x_{n_i}) = y$ if and only if $R \models \theta_i(x_1, \ldots, x_{n_i}, y)$. Clearly R can be extended to a structure S such that S has points representing the images of the function G and the well-ordering W, and the Skolem functions F_i. Hence T', the isomorphic image of T in R, can be defined by a sentence $\Psi(u)$ such that $u \in T'$ if and only if $S \models \Psi(u)$. Finally, S may be extended to a structure B, and ψ represented by a point b, so that $\mathrm{Proj}(\langle B,b\rangle) = T$.

Theorem 1.4

$$\mathrm{VBI} \vdash \overline{(A)(\exists \hat{R})[A = \mathrm{Proj}(\hat{R})]}$$

Proof:

From Lemma 3.3, for every constructible A there is a constructible R such that $A = \mathrm{Proj}(\langle R,a\rangle)$. It is clear that if R is well-founded then R must be well-founded in $\mathcal{H}$; also if $R \models \sigma$ then $R \models \sigma$ in $\mathcal{H}$ because R is constructible and hence only constructible classes are needed in the truth definition. Finally, observe that $A = \{x \mid \langle L,x\rangle \,\eta\, \langle R,a\rangle\}$ is true in $\mathcal{H}$ because if $\langle L,x\rangle \,\eta\, \langle R,a\rangle$ then there is a function $f{:}\mathrm{Cl}(x,L) \approx \mathrm{Cl}(r,R)$. $R \mid \mathrm{Cl}(r,R)$ is well-founded in $\mathcal{H}$ and $\mathcal{H}$ is a model for VBI, so there is a constructible function $g{:}E \mid t \approx R \mid \mathrm{Cl}(r,R)$ for some transitive t. But then $t = \mathrm{Cl}(x,L)$ and $f = g$.

In VBI it was possible to show that the outer model satisfied Aussonderung. In VBC one has the ordering Δ and it is possible to show that the outer model satisfies the replacement schema. In order to do this we shall need the following lemma.

Lemma 1.5

If Z is a well-ordering with no largest element then there is a normal class R such that $R \mid W(R) \approx Z$.

Proof:

Let Z be a shortest well-ordering such that it has no largest element and there is no R such that $R|W(R) \approx Z$. Then either Z has a largest limit element or not. In the first case, if z is the largest limit element then $Z|z^* \approx R|W(R)$ for some R, and since R can be extended, clearly $Z \approx S|W(S)$ for some S. If Z has no largest limit element then one can construct the desired R by essentially taking the union of the normal classes for segments of Z. That is, if z is a limit element of Z, then let R_z $= \{ \langle f_1, f_2 \rangle | (\exists R)(\exists F) [F: R|W(R) \approx Z|z^* \wedge (\exists x_1)(\exists x_2)(\langle x_1, x_2 \rangle \varepsilon R \wedge f_1 = F o \xi(x_1) \wedge f_2 = F o \xi(x_2))] \}$. If $R = \bigcup R_z$ then R is normal and $R|W(R) \approx Z$.

Definition 1.6

For each formula $\Psi(X_1, X_2, \ldots, X_{n+2})$ let Rep(Ψ) be the sentence:

$$(\hat{R})(\hat{X}_1) \ldots (\hat{X}_n) [(\hat{A}) [\hat{A} \eta \hat{R} \Rightarrow (\exists \hat{B}) [\hat{\Psi}(\hat{A}, \hat{B}) \wedge (\hat{C})(\hat{\Psi}(A, C) \Rightarrow \hat{B} \equiv \hat{C})]] \Rightarrow (\exists \hat{S})(\hat{B}) [\hat{B} \eta \hat{S} \Longleftrightarrow (\exists \hat{A})(\hat{A} \eta \hat{R} \wedge \hat{\Psi}(\hat{A}, \hat{B}))]]$$

Theorem 1.7

VBC $\vdash$ Rep(Ψ) for each Ψ.

Proof:

Consider a given $\hat{R} = \langle R, r \rangle$ and $\hat{X}_1, \ldots, \hat{X}_n$, and suppose that $\widehat{\Psi}$ satisfies the hypothesis of being a "function" on $\widehat{R}$. For each a such that aRr let $\widehat{Ba}$ be the Δ-least structure such that $\widehat{\Psi}(\langle R,a\rangle,\widehat{Ba})$. Define a well-ordering W of pairs $\langle a,b\rangle$ such that aRr and $b \in Ba$ by $\langle\langle a_1,b_1\rangle,\langle a_2,b_2\rangle\rangle \in W$

$$\Longleftrightarrow [a_1 <_R a_2] \vee [a_1 = a_2 \wedge b_1 <_{Ba_1} b_2].$$

This well-ordering is the well-ordering of a normal class U by the preceding lemma, and U is longer than each Ba. Hence if $\widehat{\Psi}(\widehat{A},\widehat{B})$ and $\widehat{A}\,\gamma\,\widehat{R}$ then $\widehat{B} \equiv \langle U,u\rangle$ for a certain $u \in |U|$.

Using an inductive procedure as in Cohen [3] pp. 106, one can by this method get bounds on all of the quantifiers of $\widehat{\Psi}$; that is, if $\widehat{A}\,\gamma\,\widehat{R}$ then $\widehat{\Psi}(\widehat{A},\widehat{B}) \Longleftrightarrow \Pi(\widehat{A},\widehat{B})$ where Π is formed from $\widehat{\Psi}$ by restricting each quantifier Q_i to range over structures of the form $\langle T_i,x\rangle$ for a certain T_i. If S is a normal class longer than R,U, and the T_i, then there is a point $s \in |S|$ such that $\widehat{S} = \langle S,s\rangle$ is the desired "range" of $\widehat{\Psi}$ on $\widehat{R}$.

2. Outer models and Cohen models

It is clear that the outer models which have arisen satisfy (in the obvious sense) the axioms of certain set theories related to ZF.

<u>Definition 2.1</u>

a. Let ZF^- be the set theory formed by omitting the power set axiom from ZF.

b. Let ZF* be the set theory formed by adding to ZF^- the formal sentences

(i) $(\exists\alpha)\mathrm{In}(\alpha)$

(ii) σ

(iii) $(x)(\delta(x) < \mu\alpha\mathrm{In}(\alpha) \Rightarrow (\exists y)$
$(u)(u \in y \Leftrightarrow u \subseteq x))$.

In condition (i) the predicate $\mathrm{In}(\alpha)$ can be taken as $\mathrm{Reg}(\alpha) \wedge (\beta)[\beta < \alpha \Rightarrow (\exists\gamma)(\beta < \overline{\overline{\gamma}} < \alpha)]$ which avoids reference to the function (actually a definable predicate in ZF) $f: \beta \rightarrow \omega_\beta$; notice further that without the full power set axiom, it cannot be shown that the cardinals do not form a set, so in general the function mentioned above is not defined on all ordinals. Condition (ii) states that every set is constructible, and (iii) stipulates that every set constructed by an ordinal less than the first inaccessible has a power set. The set of axioms as stated is highly redundant, but this is of no concern for the present purposes. Despite the lack of syntactical elegance ZF* has a natural model. If ω_ι is the first inaccessible then just as $M(\omega_\iota)$ may be considered as the intended model of ZF, $M(\omega_{\iota+1})$ is the intended model of ZF*. As stated in the lemma below, the consistency of VBI is equivalent to that of ZF*, which is related to the original motivation of considering within VBI classes which are intuitively isomorphic to $M(\beta)$ for $\beta < \omega_{\iota+1}$. Obviously if the full power set axiom were added to ZF* then the resulting system would be ZF^{ι}&$V = L$, within which the consistency of VBI is provable.

Lemma 2.2

It can be shown constructively that VBI is consistent if and only if ZF* is consistent.

Proof:

If VBI is consistent then the inner model $\mathcal{M}$ yields the consistency of VBC. Within VBC the outer model clearly satisfies ZF*, the only problematical axioms being the replacement schema. Conversely within ZF* there is an inner model for VBI, namely, $\mathcal{V}(x) \overset{df}{=} x \subseteq M(\omega_\iota)$. The comprehension axioms of VBI hold in $\mathcal{V}$ since the Aussonderung schema (which is implied by the replacement schema) holds in ZF*, and the axiom of replacement holds in $\mathcal{V}$ because in ZF* all sets are constructible and ω_ι is inaccessible.

In order to apply the Cohen construction to ZF* and thus obtain the desired independence results for VBI, it is convenient to consider another modification of ZF.

Definition 2.3

a. Let Mod be the sentence

$$(\exists f)(\exists \alpha)\,[\mathrm{Lim}(\alpha) \wedge \alpha > \omega \wedge \mathrm{Fnc}(f) \wedge \mathfrak{D}(f) = \alpha+1$$
$$\wedge\, f(0) = 0 \wedge (\beta)\,[\beta < \alpha \Rightarrow (u)(u \in f(\beta+1) \Leftrightarrow u \subseteq f(\beta))]$$
$$\wedge\,(\lambda)\,[\mathrm{Lim}(\lambda) \wedge \lambda < \alpha+1 \Rightarrow f(\lambda) = \bigcup_{\beta<\lambda} f(\beta)] \wedge$$

$\bigwedge(x)(g)\,[x \in f(\alpha) \wedge g \subseteq f(\alpha) \wedge \mathrm{Fnc}(g) \Longrightarrow g''x \in f(\alpha)]]$

b. Let ZF^+ be the set theory formed by adding Mod to ZF^-.

The reason for considering ZF^+ is that the axiom Mod guarantees the existence of an inner model for VBI since it says that $R(\alpha)$ exists for a limit ordinal α and that replacement holds in the inner model $\mathcal{U}(x) \overset{\mathrm{DF}}{\equiv} x \subseteq R(\alpha)$. A condition stating that there is an inaccessible β and that $R(\beta)$ exists would not by itself guarantee that $\mathcal{U}'(x) \overset{\mathrm{DF}}{\equiv} x \subseteq R(\beta)$ is an inner model for VBI since GCH and CH are needed to show that replacement holds in $\mathcal{U}'$, and we are here concerned with Cohen models within which GCH and CH may not hold. It is not difficult to verify that ZF* is equivalent to $ZF^+ \& \sigma$.

<u>Definition 2.4</u>

Let Σ_1, Σ_2, and Σ_3 be the formal statements:

Σ_1: There is a non-constructible set of integers $\wedge$ CH $\wedge$ GCH.

Σ_2: $2^{\aleph_0} = \aleph_2 \wedge$ CH

Σ_3: $\mathcal{P}(\omega)$ has no well-ordering.

<u>Lemma 2.5</u>

For $i = 1, 2$, and 3, if $ZF^+ \& \Sigma_i$ is consistent, then VBI& Σ_i is consistent.

Proof:

The inner model $\mathcal{U}$ immediately yields the consistency of VBI& Σ_1 because its universe is of the form $R(\alpha)$ and hence includes all sets of integers.

Lemma 2.6

If $\mathfrak{M} = E|M(\alpha)$ is a countable model of ZF* then the Cohen constructions yield countable transitive models $\mathfrak{N}_i$ of $ZF^+ \& \Sigma_1$.

Proof:

The only new point involved is to show that Mod holds in $\mathfrak{N}_i$. Let ι be the first inaccessible in $\mathfrak{M}$. It is convenient to prove first the sublemma that if $\beta < \iota$ (and hence $F_\beta \varepsilon F_\iota$) and $F_\xi \subseteq F_\iota$ and F_ξ is a function, then $F_\xi{}''F_\beta \varepsilon F_\iota$ (which immediately shows that ι is also inaccessible in $\mathfrak{N}_i$): In $\mathfrak{M}$, for each $\alpha < \beta$, let $h_\alpha(P) = \mu\gamma[P \Vdash F_\xi(F_\alpha) = F_\gamma \wedge \gamma < \iota]$ if such a γ exists and $h_\alpha(P) = 0$ otherwise. Since ι is inaccessible in $\mathfrak{M}$, the range of each h_α is bounded by some $\bar{\alpha} < \iota$. If $g(\beta)$ is the least upper bound of $\{\bar{\alpha} \mid \alpha < \beta\}$ then $g(\beta) < \iota$. Clearly in $\mathfrak{N}_i$ if $F_\xi(F_\alpha) = u$ for any $F_\alpha \varepsilon F_\beta$ then $u = F_\gamma$ for some $\gamma \leq g(\beta)$ and so $F_\xi{}''F_\beta \varepsilon F_\iota$. Returning to the main lemma, it can be shown by induction, using the sublemma, that for each $\beta < \iota$ there is a function f_β in F_ι satisfying the recursion condition in Mod. Hence there is a function f defined on $\iota + 1$ satisfying the recursion condition, that is, $f(\alpha)$ is $R(\alpha)$ in $\mathcal{N}_i$ for each $\alpha \leq \iota$, and $f(\iota) \subseteq F_\iota$. It is clear conversely that $F_\iota \subseteq f(\iota)$, by the definition of the function F. But then by the sublemma, $x \in f(\iota) \wedge g \subseteq f(\iota) \wedge \mathrm{Fnc}(g)$ $\Longrightarrow g''x \in f(\iota)$ so Mod holds in $\mathcal{N}_i$.

Theorem 2.7

It can be shown constructively that the consistency of VBI implies the consistency of VBI& Σ_i for $i = 1,2,$ and 3.

Proof:

The theorem follows from the observation that the same method mentioned in Cohen [3] pp. 110 can also be used here, because for each n, $ZF^* \vdash (\exists \alpha)[\bar{\bar{\alpha}} = \omega \wedge M(\alpha) \models ZF_n{}^*]$ where $ZF_n{}^*$ is the conjunction of the first n axioms of ZF*, since the same Skolem submodel argument may be applied. Thus the Cohen construction may be carried out piece-wise, inferring $\mathcal{N}_i \models ZF_p^+ \& \Sigma_i$ by considering an $\mathcal{M}$ such that $\mathcal{M} \models ZF_n{}^*$ for a suitable n effectively calculable from p.

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