ON TRANSIENT MOTIONS
IN A CONTAINED, ROTATING FLUID

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ABSTRACT

We consider here the manner in which rigid fluid rotation is established from a prescribed initial state of motion. The fluid, viscous and incompressible, wholly fills a uniformly rotating container.

Following the general theory developed by Greenspan (1965), a solution to the linear problem is sought in the form of a superposition of all the natural oscillatory modes of the inviscid problem, each of which is corrected for the effects of viscosity. A few additional aspects of the linear theory are presented for containers of arbitrary shape, and the analysis is then applied to spheroids and cylinders. For these configurations it is possible to determine explicitly the inviscid eigenmodes. Numerical calculations illustrate the effect of viscosity and geometry on these modes.

The modal analysis is also used to determine the response of the fluid when the container is oscillated at a fixed frequency, or when an oscillatory body force is applied. The modal amplitudes satisfy an infinite system of linear algebraic equations with constant coefficients which, for some geometries of practical importance (including spheroids) can be rendered finite. Forced oscillation at a resonant frequency is also dealt with by this general method. This leads to a simple formula for the resonant modal amplitude. Calculations of the induced response in rotating spheroids have been made for a few fundamental modes.

As a final application of the general theory, the O(1) inviscid solution is found for a precessing, fluid-filled rotating spheroid. It is shown that this is the only such solution possible within the framework of the above theory.

Since the general theory is not entirely applicable to containers with <u>vertical</u> sidewalls, a separate boundary layer analysis is made on the nonaxisymmetric geostrophic modes in a cylinder. Mass efflux from the sidewall boundary layers is found to be an order of magnitude larger than that predicted by the Ekman layer theory. (This result is in agreement with that determined by Jacobs (1964) in a related problem.)

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INTRODUCTION

In this report we shall be concerned with fluid motions which are in some sense nearly rigid rotations. Primarily we are interested in the manner in which rigid fluid rotation is established from a prescribed initial state of motion within a uniformly rotating container. The initial motion is subject to two constraints. First, we require that it be a physically acceptable motion. That is, it must satisfy the requirements of mass conversation and zero normal velocity at the boundary. Second, we require that it differ only slightly from the ultimate state of rigid rotation. When this latter stipulation is met, it is then meaningful to speak of nearly rigid rotations: The deviations of the transient motions from the steady state are small enough that they may be adequately described by a linear theory.

Two basic properties of rotating fluids are especially relevant to the present problem. The Taylor-Proudman theorem asserts that in slow steady motion of a uniformly rotating incompressible inviscid fluid (angular velocity $\vec{\Omega}$, say) the fluid velocity $\vec{q}$ does not vary in the direction of $\vec{\Omega}$. That is, $(\vec{\Omega}\cdot\nabla)\vec{q} = 0$. (Such motions are called "geostrophic." More broadly, the adjective is applied to almost-steady motions, in which the time scale of the observed motions is long compared to the basic rotation period of the fluid.) The Taylor-Proudman theorem implies that within a closed container, a column of fluid of height h, measured along $\vec{\Omega}$, must move about the interior as a unit, preserving its length. From this crucial constraint stems the general theory of contained geostrophic motions developed by Greenspan (1965). Further aspects of these motions will be discussed below.

The structure and role of rotating viscous boundary layers is also known to differ markedly from that in non-rotating configurations. In the absence of rotation, laminar boundary layers affect the interior, inviscid portions of the fluid solely through viscous diffusion. When the boundaries are rotating, however, additional effects are present. The solution of the rotating viscous flow equations, the Ekman boundary layer, forms an integral part of the present thesis. We shall derive this solution for containers of arbitrary shape, and thereby show that (1) the Ekman layer is established very quickly, within one or two rotation periods; (2) its structure is nearly uniform over the entire solid surface, the exceptions being those points at which twice the local normal component of rotation equals a free inviscid oscillation frequency; and (3) secondary interior motion is directly induced by a small net mass flux from the Ekman layer.

These two fundamental characteristics of rotating fluids--the Taylor-Proudman theorem and the Ekman boundary layer--underlie all of the analysis presented in this thesis. The interior motion induced by the Ekman layers is instrumental in bringing about the steady state, the transient decay time being of the order $(L^2/\nu\Omega)^{1/2}$. (L is a typical dimension of the container, and ν is the kinematic viscosity of the fluid.) This was first clearly demonstrated by Greenspan and Howard (1963), who analyzed the role of the Ekman layer in the mechanics of transition from one state of rigid rotation to a slightly different, faster spinning state. They showed that the interior fluid attains the new angular velocity and vorticity not primarily by viscous diffusion from the boundaries, but by the transport of angular momentum in a closed circulation. Suction into the Ekman layer stretches the vortex lines of the interior fluid, thereby increasing vorticity. To replace fluid entering the Ekman layer (mass conservation) a slow inward radial flow occurs in the interior. This inward motion in turn increases the angular velocity of the interior fluid, in accordance with the principle of conservation of angular momentum. The circulation is closed via the Ekman layers: Fluid entering the layers is thrown radially outwards from the rotation axis by centrifugal action, acquires increased angular momentum, and is injected back into the fluid in the vicinity of the side boundary region where the top and bottom Ekman layers meet.

Greenspan and Howard chose a rather simple configuration (two parallel infinite disks) to illustrate this spin-up principle, but further analysis by Greenspan (1965) has shown that similar results hold for closed containers in general. This was proved by a modal synthesis. A general linear theory was developed to determine the manner in which any prescribed initial state of motion is distributed among all the natural (free) modes of oscillation within an arbitrary container. The basis of this linear theory is a separation of the flow into geostrophic motion and inertial oscillations. (Geostrophic motion corresponds to the totality of eigenfunctions with zero eigenfrequencies, and therefore changes relatively slowly in time. Inertial oscillations comprise all other eigenfunctions; their frequencies σ are restricted to the range $|\sigma| \leq 2\Omega$.) When these modes are corrected for the effects of viscosity, the result is that they all essentially decay in the same "spin-up" time, $(L^2/\nu\Omega)^{1/2}$. An explicit formulation of the decay time for each mode in an arbitrary container is derived in the present thesis.

In summary, there are three time scales which characterize the transient processes: (1) The formation of the Ekman layers, $T_1 \sim O(\Omega^{-1})$; (2) the viscous decay of the initial state of motion, $T_2 \sim O(L^2/\nu\Omega)^{1/2}$; and (3) the dissipation of small residual oscillations resulting from viscous diffusion, $T_3 \sim O(L^2/\nu)$. Therefore, the important phenomena in the transition from initial to final state occur in the time T_2, i.e., before the boundary layers have been appreciably thickened by diffusion.

In addition to this multiple time-scale structure, spatial nonuniformities also occur at certain critical positions in the boundary layers, as has already been mentioned. Bondi and Lyttleton (1953) and Stewartson and Roberts (1963) have both suggested that free shear layers originate at these critical boundary positions and penetrate the interior fluid regions. These effects, however, are of a lower order than those which we shall consider. This was essentially demonstrated by Roberts and Stewartson (1963) and again by Stewartson and Roberts (1963). Experiments to date support these conclusions: no boundary-layer eruptions from the critical positions have been observed. Nevertheless, the very existence of these boundary-layer-nonuniformities makes a comprehensive asymptotic analysis quite difficult, and for this reason only first-order corrections to the inviscid modal solutions are considered.

One further difficulty exists. When the modal analysis is applied to containers with vertical sidewalls, a separate study of the geostrophic mode is necessary. (No such disparity is found for the inertial oscillations.) On the geostrophic time-scale, the vertical side-wall boundary layer structure is significantly different from that of the Ekman layers. There is an outer boundary layer region of thickness $(\nu L^2/\Omega)^{1/4}$, and an inner one of thickness $(\nu L/\Omega)^{1/3}$, each being terminated at the top and bottom by an Ekman layer. The effect of this multiple structure on axisymmetric geostrophic flow in a cylinder was investigated by Greenspan and Howard, and an extension of this work to include nonaxisymmetric motions is made in this thesis. Additional studies have been made by Stewartson (1957) and Jacobs (1964). These analyses all show that the sharp velocity gradients can only be supported through the interaction of the two layers with each other and with the main body of the fluid. This is true whether the gradients occur in the neighborhood of solid boundaries or across free shear layers in the fluid.

As the above discussion shows, boundary layer methods are applied to determine the essential features of the flow in the general case of arbitrary containers. For any particular body, however, more detailed aspects of the flow depend upon a knowledge of the inertial oscillations (eigenfunctions). These are determined by solving the non-self-adjoint Poincaré problem for the dimensionless pressure $p = \Phi e^{i\lambda t}$,

$$\nabla^2 \Phi - \frac{4}{\lambda^2}(\hat{k}\cdot\nabla)^2 \Phi = 0$$

with $\hat{n}\cdot\nabla\Phi - (2/i\lambda)\hat{n}\cdot\hat{k}\times\nabla\Phi - (4/\lambda^2)(\hat{n}\cdot\hat{k})\hat{k}\cdot\nabla\Phi = 0$ on the boundary. This problem has interesting properties aside from its physical relevance: The eigenvalue λ, a real number, appears both in the equation and boundary condition. It is also clear that the equation is hyperbolic, parabolic, or elliptic according as λ^2 is less than, equal to, or greater than 4, whereas the boundary condition is a relation between Φ and its normal and tengential derivatives on a closed boundary. It can be shown, however, that λ^2 must be restricted to the range $\lambda^2 < 4$, so that the equation must be hyperbolic. This in turn means that rotation endows an incompressible fluid with the ability to support traveling waves, a remarkable property. (It is an open question whether Poincaré's problem is the correct linear representation for the flow when $\lambda^2 \approx 4$ (Morgan, 1956) and further analysis is needed in this area.) Explicit separable solutions for the cylinder and ellipsoid may be found, giving the spatial structure of the inertial oscillations, but no solutions are known for other configurations. Some properties of the eigenfunctions for spheroids and cylinders are presented in this thesis.

CHAPTER 1

FORMULATION; EFFECTS OF VISCOSITY

Let a fluid-filled closed container of arbitrary shape rotate with uniform angular velocity $\vec{\Omega}$ about the fixed vertical axis. We choose a reference frame rotating with the container $(\vec{\Omega} = \Omega\hat{k})$ and measure all quantities with respect to this frame. The equations of motion then are

$$\frac{\partial \vec{q}}{\partial t} + \vec{q}\cdot\nabla\vec{q} + 2\vec{\Omega}\times\vec{q} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = -\frac{1}{\rho}\nabla p + \nu\Delta\vec{q} \,, \qquad \nabla\cdot\vec{q} = 0$$

with

$$\vec{q} = 0 \quad \textit{at solid boundaries,}$$

$$\vec{q}(\vec{r},t) = \vec{q}_*(\vec{r}) \quad at \quad t = 0.$$

The initial velocity distribution $\vec{q}_*(\vec{r})$ is assumed to differ only slightly from rigid rotation $\vec{\Omega}\times\vec{r}$. In addition, $\vec{q}_*$ must satisfy the physical requirements of mass conservation and zero normal velocity at the boundary.

If L, $\varepsilon\Omega L$ and Ω^{-1} characterize the length, initial velocity, and time, respectively, then the equations of motion, under the transformations

$$\vec{q} \longrightarrow \varepsilon\Omega L\vec{q}\,, \qquad \vec{r} \longrightarrow L\vec{r}\,, \qquad t \longrightarrow \Omega^{-1}t$$

become

$$\frac{\partial \vec{q}}{\partial t} + \varepsilon \vec{q}\cdot\nabla\vec{q} + 2\hat{k}\times\vec{q} + \nabla p = R^{-1}\Delta\vec{q}\ , \quad \nabla\cdot\vec{q} = 0$$

where $R \equiv \Omega L^2 \nu^{-1} \gg 1$ and p is the actual pressure less the centrifugal pressure $\frac{1}{2}\rho\Omega^2 L^2(\hat{k}\times\vec{r})^2$. Henceforth, all variables are dimensionless. Moreover, we shall consider only the linear problem; nonlinear effects are assumed to be relatively unimportant compared to viscous effects, and accordingly we put $\varepsilon = 0$. Then

$$\frac{\partial \vec{q}}{\partial t} + 2\hat{k}\times\vec{q} + \nabla p = R^{-1}\Delta\vec{q}\ , \quad \nabla\cdot\vec{q} = 0$$

with $\vec{q} = 0$ *at solid boundaries,*

$$\vec{q}(\vec{r},t) = \vec{q}_*(\vec{r}) \quad at \quad t = 0.$$

Since the linearized equations of motion (both inviscid and viscous) admit the separable solution $\vec{q} = Q(\vec{r})e^{i\sigma t}$, substitution of this expression leads to a condition on σ , as we shall see. For such a modal solution,

$$\left.\begin{aligned} i\sigma\vec{Q} + 2\hat{k}\times\vec{Q} &= -\nabla\Phi + R^{-1}\Delta\vec{Q} \\ \nabla\cdot\vec{Q} &= 0 \\ \text{with} \qquad \vec{Q} &= 0 \quad \text{on the boundary } S. \end{aligned}\right\} \qquad (1.1)$$

Multiplying by $\vec{Q}^*$ (the complex conjugate) and integrating over the volume, we readily find, since $\vec{Q}^*\cdot\nabla\Phi = \nabla\cdot(\Phi\vec{Q}^*)$,

$$i\sigma \iiint \vec{Q}\cdot\vec{Q}^* \, dV + 2\hat{k}\cdot \iiint \vec{Q}\times\vec{Q}^* \, dV = R^{-1} \iiint \vec{Q}^*\cdot\Delta\vec{Q} \, dV \qquad (1.2)$$

Now, from the vector identities

$$\operatorname{div}(\vec{a}\times\vec{b}) = \vec{b}\cdot\operatorname{curl}\vec{a} - \vec{a}\cdot\operatorname{curl}\vec{b}$$

$$\Delta\vec{a} = \operatorname{grad}(\operatorname{div}\vec{a}) - \operatorname{curl}(\operatorname{curl}\vec{a})$$

it follows that

$$\vec{Q}^*\cdot\Delta\vec{Q} = \operatorname{div}(\vec{Q}^*\times\operatorname{curl}\vec{Q}) - \operatorname{curl}\vec{Q}\cdot\operatorname{curl}\vec{Q}^* .$$

Further,

$$\vec{Q}\times\vec{Q}^* = i \operatorname{Im}(\vec{Q}\times\vec{Q}^*) .$$

Therefore, we have from (1.2) that

$$\sigma \iiint |\vec{Q}|^2 dV = -\left(\hat{k}\cdot\operatorname{Im}\iiint \vec{Q}\times\vec{Q}^* dV\right) + i\left(R^{-1}\iiint |\operatorname{curl}\vec{Q}|^2 dV\right) \qquad (1.3)$$

For the inviscid problem $(R = \infty)$ it is clear from (1.3) that σ is real. For R large but finite, σ is of course complex; its imaginary part is a measure of the rate of decay of velocity due to viscous effects. Now it is known that direct viscous action is confined to a thin boundary layer at the container wall of thickness $O(R^{-1/2})$ when R >> 1. Until diffusion has had sufficient time to thicken this boundary region, the flow can therefore be studied by the methods of boundary layer analysis. These methods will now be used to determine the complex correction factor to each inviscid eigenfrequency.

CHAPTER 2

THE BOUNDARY LAYER SOLUTION AND VISCOUS CORRECTION FACTOR $s_m^{(1)}$

Following Greenspan (1965), a solution to the linear problem is sought in the form of a superposition of all the natural modes of the inviscid problem ($R = \infty$), each of which is corrected for the effects of viscosity. In order to include all the important phenomena in the analysis, the solution of the fundamental boundary value problem must be uniformly valid in space and time at least through times of the order $R^{1/2}$, the dimensionless spin-up time.

The typical inertial mode is represented as

$$\left.\begin{aligned} \vec{q} &= \left(\vec{Q}_m e^{s_m t} + R^{-1/2}\vec{q}_{m1} + \dots\right) + \left(\tilde{\vec{q}}_m + R^{-1/2}\tilde{\vec{q}}_{m1} + \dots\right) \\ p &= \left(\Phi_m e^{s_m t} + R^{-1/2}\varphi_{m1} + \dots\right) + \left(\tilde{\varphi}_m + R^{-1/2}\tilde{\varphi}_{m1} + \dots\right) \\ \text{with} \quad s_m &= i\lambda_m + R^{-1/2} s_m^{(1)} + \dots \end{aligned}\right\} \quad (2.1)$$

Here (Φ_m, λ_m) or equivalently $(\vec{Q}_m, \lambda_m)$ represents the mth natural mode, and the tilde symbol denotes a boundary layer function of the (stretched) boundary layer coordinate ζ (see below for details). The parameter $s_m^{(1)}$ is the viscous correction factor to the value $i\lambda_m$ determined by inviscid theory. $s_m^{(1)}$ is chosen so that secular terms possessing unacceptable growth rates are eliminated from the solution.

Substitution of these expressions into the basic equations and boundary conditions (1.1) leads to a sequence of problems for the inviscid and boundary layer flows and their mutual interactions.

To obtain the interior equations we substitute (2.1) into (1.1) and take the limit $R \to \infty$ with the interior variable $\vec{r}$ fixed. By equating like powers of $R^{-1/2}$, we arrive at an asymptotically valid problem sequence. Denoting the limit procedure by Lim_1,

$$Lim_1 \equiv \lim_{\substack{R \to \infty \\ \vec{r}\ \text{FIXED}}}$$

we require

$$Lim_1 \left\{ \tilde{\vec{q}}_m + R^{-1/2} \tilde{\vec{q}}_{m1} + \ldots \right\} = 0,$$

with a similar equation for the pressure.

Then, from the basic relation for the interior equations,

$$Lim_1 \left\{ \left(\frac{\partial}{\partial t} + 2\hat{k}\times - R^{-1}\Delta \right)\left(\vec{Q}_m e^{s_m t} + R^{-1/2}\vec{q}_{m1} + \ldots \right) + \nabla\left(\Phi_m e^{s_m t} + R^{-1/2}\varphi_{m1} + \ldots \right) \right\} = 0,$$

it follows that the O(1) balance is

$$\left.\begin{aligned} i\lambda_m \vec{Q}_m + 2\hat{k}\times\vec{Q}_m + \nabla\Phi_m &= 0 \\ \nabla\cdot\vec{Q}_m &= 0 . \end{aligned}\right\} \qquad (2.2)$$

The corresponding $O(R^{-1/2})$ equations are found by eliminating the O(1) balance from (1.1), dividing the remaining terms by $R^{-1/2}$, and applying the limit procedure

Lim_1 . The result is

$$\left.\begin{aligned} \frac{\partial \vec{q}_{m1}}{\partial t} + 2\hat{k}\times\vec{q}_{m1} + \nabla\varphi_{m1} &= -s_m^{(1)}\vec{Q}_m e^{s_m t} \\ \nabla\cdot\vec{q}_{m1} &= 0. \end{aligned}\right\} \qquad (2.3)$$

We postpone consideration of the boundary conditions (no fluid motions relative to the boundaries) until the remaining equations have been established.

In the boundary layer, $\zeta = R^{1/2}\hat{n}\cdot(\vec{r}_0 - \vec{r})$, and $\hat{n}\cdot\nabla = -R^{1/2}\frac{\partial}{\partial\zeta}$. $\hat{n}$ is the unit outward normal at the surface $\vec{r} = \vec{r}_0$. We substitute (2.1) and take the limit $R \to \infty$ with ζ fixed. This means that the interior functions (those lacking the tilde) are evaluated at the boundary, but these can be eliminated by using the interior equations written above. With

$$Lim_2 \equiv \lim_{\substack{R\to\infty \\ \zeta \text{ FIXED}}} \qquad \text{and} \qquad R^{-1}\Delta = \frac{\partial^2}{\partial\zeta^2} + O(R^{-1/2}),$$

we have

$$Lim_2\left\{\left(\frac{\partial}{\partial t} + 2\hat{k}\times - \frac{\partial^2}{\partial\zeta^2}\right)\tilde{\vec{q}}_m - R^{1/2}\hat{n}\frac{\partial}{\partial\zeta}\left(\tilde{\varphi}_m + R^{-1/2}\tilde{\varphi}_{m1}\right) + \dots\right\} = 0,$$

$$Lim_2\left\{\left(-\hat{n}\times(\hat{n}\times\nabla) - R^{1/2}\frac{\partial}{\partial\zeta}\hat{n}\right)\cdot\left(\tilde{\vec{q}}_m + R^{-1/2}\tilde{\vec{q}}_{m1} + \dots\right)\right\} = 0.$$

It follows that the terms of $O(R^{1/2})$ contribute the primary balance,

$$\frac{\partial \tilde{\varphi}_m}{\partial \zeta} = 0, \qquad \frac{\partial}{\partial \zeta}\, \hat{n}\cdot\vec{\tilde{q}}_m = 0. \tag{2.4}$$

whereas those of O(1) constitute the general formulation of the Ekman layer problem:

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \zeta^2}\right)\vec{\tilde{q}}_m + 2\hat{k}\times\vec{\tilde{q}}_m - \hat{n}\,\frac{\partial \varphi_{m'}}{\partial \zeta} &= 0, \\ \frac{\partial}{\partial \zeta}\,\hat{n}\cdot\vec{\tilde{q}}_{m'} &= -\hat{n}\times(\hat{n}\times\nabla)\cdot\vec{\tilde{q}}_m \,. \end{aligned}\right\} \tag{2.5}$$

From the above equations (2.2) - (2.5) we arrive at the following problem sequence. For the O(1) boundary layer and interior functions,

(i) $\quad \frac{\partial}{\partial \zeta}\tilde{\varphi}_m = 0, \qquad \frac{\partial}{\partial \zeta}\hat{n}\cdot\vec{q}_{m'} = 0$

(ii) $\quad i\lambda_m\vec{Q}_m + 2\hat{k}\times\vec{Q}_m + \nabla\Phi_m = 0$

$\nabla\cdot\vec{Q}_m = 0$

with $\hat{n}\cdot\vec{Q}_m = 0$ on S.

The requirement that the boundary layer functions be transcendentally small just outside the boundary layer means that

$$\tilde{\varphi}_m = 0, \qquad \hat{n}\cdot\vec{\tilde{q}}_m = 0.$$

The vanishing of the normal component $\hat{n}\cdot\left(\vec{Q}_m e^{s_m t} + \vec{\tilde{q}}_m\right)$ at the surface S leads to the expected inviscid boundary condition given in (ii). Hence, (ii) is identically satisfied by definition of the natural modes. Later we shall be concerned with solving (ii) for the eigenfunctions. This will entail solving for the pressure Φ_m and then determining $\vec{Q}_m$ as a vector function of $\nabla\Phi_m$. This vector function is easily found by eliminating $\hat{k}\times\vec{Q}_m$ from the momentum equations:

$$\vec{Q}_m = \frac{-i\lambda_m}{4-\lambda_m^2}\left\{ \nabla\Phi_m - \frac{2}{i\lambda_m}\hat{k}\times\nabla\Phi_m - \frac{4}{\lambda_m^2}\left(\hat{k}\cdot\nabla\Phi_m\right)\hat{k} \right\} \qquad (2.6a)$$

The divergence of this expression yields the fundamental eigenvalue equation

$$\nabla^2\Phi_m - \frac{4}{\lambda_m^2}\left(\hat{k}\cdot\nabla\right)^2\Phi_m = 0\,, \qquad (2.6\,b)$$

whereas the condition $\hat{n}\cdot\vec{Q}_m = 0$ on S is equivalent to

$$\hat{n}\cdot\nabla\Phi_m - \frac{2}{i\lambda_m}\hat{n}\cdot\hat{k}\times\nabla\Phi_m - \frac{4}{\lambda_m^2}(\hat{n}\cdot\hat{k})(\hat{k}\cdot\nabla\Phi_m) = 0 \quad \text{on } S. \qquad (2.6\,c)$$

Equations (2.6a, b, c) constitute the basic eigenvalue problem for the interior motion. We shall return to these later when we derive the eigenfunctions for spheroids and cylinders. It will also be shown that these equations require λ_m real and $\lambda_m^2 < 4$ for containers of arbitrary shape.

The next order boundary layer equations in the asymptotic problem sequence are (cf. (2.5))

$$(iii) \quad \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \zeta^2}\right)\tilde{\vec{q}}_m + 2\hat{k}\times\tilde{\vec{q}}_m - \hat{n}\frac{\partial \varphi_{m1}}{\partial \zeta} = 0 ,$$

$$\frac{\partial}{\partial \zeta}\hat{n}\cdot\tilde{\vec{q}}_{m1} = -\hat{n}\times(\hat{n}\times\nabla)\cdot\tilde{\vec{q}}_m = \hat{n}\cdot\nabla\times(\hat{n}\times\tilde{\vec{q}}_m)$$

with $\quad \tilde{\vec{q}}_m + \vec{Q}_m e^{s_m t} = 0 \quad$ on S (*the no-slip condition*)

and $\quad \tilde{\vec{q}}_m = 0 \quad$ at $\quad t = 0$.

The corresponding interior equations are

$$(iv) \quad \frac{\partial}{\partial t}\vec{q}_{m1} + 2\hat{k}\times\vec{q}_{m1} + \nabla\varphi_{m1} = -s_m^{(1)}\vec{Q}_m e^{s_m t} ,$$

$$\nabla\cdot\vec{q}_{m1} = 0$$

with $\quad \hat{n}\cdot(\vec{q}_{m1} + \tilde{\vec{q}}_{m1}) = 0 \quad$ on S.

and $\quad \vec{q}_{m1} = 0 \quad$ at $\quad t = 0$.

The boundary condition here indicates that a divergent or convergent boundary layer motion will in general induce a secondary interior flow. That is, the inviscid normal velocity $\hat{n}\cdot\vec{q}_{m1}$ at the boundary must match the normal outflow from the boundary layer.

Both problems (iii) and (iv) can be solved using the Laplace transform. On the basis of classical perturbation procedure, it may be expected that the forcing term in problem (iv) would give rise to secular terms possessing unacceptable growth rates in time. The parameter $s_m^{(1)}$ will have to be chosen to eliminate these secular terms so that the condition of uniform validity is met. However, the resolution of this difficulty is best deferred until we can establish the actual form of the O(1) boundary correction $\tilde{\vec{q}}_m$ via problem (iii).

Let the Laplace transforms of the relevant boundary layer functions be defined as

$$\tilde{\vec{v}} = \int_0^\infty e^{-st}\tilde{\vec{q}}_m\,dt\,, \qquad \tilde{\vec{v}}_1 = \int_0^\infty e^{-st}\tilde{\vec{q}}_{m1}\,dt\,, \qquad \tilde{\varphi}_1 = \int_0^\infty e^{-st}\tilde{\varphi}_{m1}\,dt\,.$$

It follows directly from (iii) that

$$\left.\begin{aligned} &\left(s - \frac{\partial^2}{\partial\zeta^2}\right)\tilde{\vec{v}} + 2\hat{k}\times\tilde{\vec{v}} - \hat{n}\frac{\partial\tilde{\varphi}_1}{\partial\zeta} = 0 \\ &\frac{\partial}{\partial\zeta}\,\hat{n}\cdot\tilde{\vec{v}}_1 = -\hat{n}\times(\hat{n}\times\nabla)\cdot\tilde{\vec{v}} \\ \text{with}\qquad &\tilde{\vec{v}} = \frac{-\vec{Q}_m}{s-s_m} \quad \text{on } S. \end{aligned}\right\} \tag{2.7}$$

From the first of these we find, since $\hat{n}\cdot\tilde{\vec{v}} = 0$, that

$$\hat{n}\cdot 2\hat{k}\times\tilde{\vec{v}} = \frac{\partial\tilde{\varphi}_1}{\partial\zeta}\;;$$

Hence

$$\left.\begin{aligned} \left(\frac{\partial^2}{\partial\zeta^2} - s\right)\hat{n}\times\tilde{\vec{v}} &= -(2\hat{k}\cdot\hat{n})\tilde{\vec{v}} \\ \left(\frac{\partial^2}{\partial\zeta^2} - s\right)\tilde{\vec{v}} &= 2\hat{k}\times\tilde{\vec{v}} - (\hat{n}\cdot 2\hat{k}\times\tilde{\vec{v}})\hat{n} \\ &= (2\hat{k}\cdot\hat{n})\hat{n}\times\tilde{\vec{v}} \end{aligned}\right\} \tag{2.8}$$

If $\hat{n} \times \tilde{\vec{v}}$ is eliminated between the two equations in (2.8), then

$$\left(\frac{\partial^2}{\partial\zeta^2} - s\right)^2 \tilde{\vec{v}} = -(2\hat{k}\cdot\hat{n})\,\tilde{\vec{v}}\,.$$

For the solution, let

$$\left.\begin{aligned} (s-s_m)\tilde{\vec{v}} &\equiv \tilde{\vec{V}}_m = \vec{a}_1 \exp(-\sqrt{\gamma_+}\,\zeta) + \vec{a}_2 \exp(-\sqrt{\gamma_-}\,\zeta) \\ \gamma_\pm &= s \pm i(2\hat{k}\cdot\hat{n}) \end{aligned}\right\} \qquad (2.9)$$

Then $\vec{a}_1$ and $\vec{a}_2$ are determined from the requirements

$$(s-s_m)\tilde{\vec{v}} = -\vec{Q}_m \quad at \quad \zeta = 0$$

$$\left(\frac{\partial^2}{\partial\zeta^2} - s\right)\tilde{\vec{v}} = (2\hat{k}\cdot\hat{n})\,\hat{n}\times\tilde{\vec{v}} \quad for\ all\ \zeta\,,$$

and therefore

$$\left.\begin{aligned} \vec{a}_1 &= -\tfrac{1}{2}\left(\vec{Q}_m - i\,\hat{n}\times\vec{Q}_m\right)_{\zeta=0} \\ a_2 &= -\tfrac{1}{2}\left(\vec{Q}_m + i\,\hat{n}\times\vec{Q}_m\right)_{\zeta=0} \end{aligned}\right\} \qquad (2.10)$$

Inversion of (2.9) is now readily accomplished with the aid of tables, and the result is

$$\tilde{\vec{q}}_m = \tfrac{1}{2}\vec{a}_1 \mathcal{F}_m(\zeta, t; p_+) + \tfrac{1}{2}\vec{a}_2 \mathcal{F}_m(\zeta, t; p_-)\,, \qquad (2.11)$$

where $\qquad p_\pm = \lambda_m \pm 2\hat{k}\cdot\hat{n}$

and

$$\mathcal{F}_m(\zeta,t;\beta) \equiv e^{s_m t}\Big[\exp\{(i\beta)^{1/2}\zeta\}\,\mathrm{erfc}\Big\{\frac{\zeta}{2t^{1/2}} + (i\beta)^{1/2}t^{1/2}\Big\} +$$

$$+ \exp\{-(i\beta)^{1/2}\zeta\}\,\mathrm{erfc}\Big\{\frac{\zeta}{2t^{1/2}} - (i\beta)^{1/2}t^{1/2}\Big\}\Big] \quad ,$$

$$(i\beta)^{1/2} = \Big|\frac{\beta}{2}\Big|^{1/2}\Big(1 + \frac{i\beta}{|\beta|}\Big).$$

When the container has vertical side-walls, the above analysis is still valid (the boundary layer thickness at the sides being $O(R^{-1/2})$ in the $O(1)$ time scale) but the results are somewhat simplified since now $\hat{n}\cdot\hat{k} = 0$. This means that $p_+ = p_- = \lambda_m$ and the solution (2.11) may be written

$$\tilde{\vec{q}}_m = -\frac{1}{2}\vec{Q}_m\Big|_{\zeta=0} \cdot \mathcal{F}_m(\zeta,t;\lambda_m) \qquad (2.11)'$$

This completes the solution for the $O(1)$ boundary layer corrections as defined by (2.1), for we have already seen that $\tilde{\varphi}_m = 0$.

We now return to the difficulties inherent in the solution of problem (iv). Once again we utilize the Laplace transform and define

$$\vec{v}_1 = \int_0^\infty e^{-st}\vec{q}_{m1}\,dt, \qquad \varphi_1 = \int_0^\infty e^{-st}\varphi_{m1}\,dt.$$

Then we have directly from (iv) that

$$\left.\begin{aligned} s\vec{v}_1 + 2\hat{k}\times\vec{v}_1 + \nabla\varphi_1 &= -s_m^{(1)}\frac{\vec{Q}_m}{s-s_m} \\ \nabla\cdot\vec{v}_1 &= 0 \\ \text{with}\qquad \hat{n}\cdot\vec{v}_1 &= -(\hat{n}\cdot\tilde{\vec{v}}_1)\quad \text{on } \zeta=0. \end{aligned}\right\}\qquad (2.12)$$

A single equation for φ_1 is obtained by solving the first of equations (2.12) for $\vec{v}_1$ and taking the divergence. Moreover, the value of $-(\hat{n}\cdot\tilde{\vec{v}}_1)$ on S $(\zeta = 0)$ is found by integrating the second of equations (2.7) and using (2.9).

In this way it is first established that

$$\vec{v}_1 = \frac{-s}{4+s^2}\left\{\left(\nabla\varphi_1 - \frac{2}{s}\hat{k}\times\nabla\varphi_1 + \frac{4}{s^2}(\hat{k}\cdot\nabla\varphi_1)\hat{k}\right) + \right.$$
$$\left. + \frac{s_m^{(1)}}{s-s_m}\left(\vec{Q}_m - \frac{2}{s}\hat{k}\times\vec{Q}_m + \frac{4}{s^2}(\hat{k}\cdot\vec{Q}_m)\hat{k}\right)\right\}$$

When use is made of the relations (2.6a, b, c), the function φ_1 is seen to satisfy

$$\nabla^2\varphi_1 + \frac{4}{s^2}(\hat{k}\cdot\nabla)^2\varphi_1 = \frac{-4s_m^{(1)}(s+i\lambda_m)}{\lambda_m^2 s^2(s-s_m)}(\hat{k}\cdot\nabla)^2\Phi_m$$

with

$$\hat{n}\cdot\nabla\varphi_1 - \frac{2}{s}\hat{n}\cdot\hat{k}\times\nabla\varphi_1 + \frac{4}{s^2}(\hat{n}\cdot\hat{k})\hat{k}\cdot\nabla\varphi_1 =$$
$$= \frac{4+s^2}{s(s-s_m)}\int_0^\infty \hat{n}\times(\hat{n}\times\nabla)\cdot\tilde{\vec{V}}_m\,d\zeta - \frac{s_m^{(1)}}{s(s-s_m)}\left(\hat{n}\cdot\nabla\Phi_m + \frac{4i}{\lambda_m s}\hat{n}\cdot\hat{k}\;\hat{k}\cdot\nabla\Phi_m\right)\qquad (2.13)$$

on S, where $\vec{\tilde{V}}_m$ is defined in (2.9), (2.10).

Again, (Φ_m, λ_m) is the mth eigenvalue-eigenfunction pair for the inviscid problem. Indeed, we note that since the completely homogeneous problem corresponding to (2.13) possesses nontrivial solutions whenever $s = i\lambda_m$ for any m, then the general solution must have singularities at these values of s. Further, the inhomogeneous terms all have a simple pole at $s = s_m$; hence, the solution must also have this singularity. In particular this means the general solution must have two simple poles a distance $O(R^{-1/2})$ apart in the complex plane, and upon inversion we would obtain a solution which is $O(R^{1/2})$. That is, the function φ_{m1} would behave like

$$\varphi_{m1} \sim R^{1/2} \exp\left(R^{-1/2} s_m^{(1)} t\right)$$

so that

$$R^{-1/2} \varphi_{m1} \sim O(1) ,$$

thereby violating our requirement that the expansions (2.1) be uniformly valid.

We therefore choose $s_m^{(1)}$ so that the terms arising from the two closely situated poles are eliminated. That is, we require that φ_1 have only a simple pole in the neighborhood of $i\lambda_m$, the location being $s = s_m$. Accordingly, we assume that

$$\varphi_1 = \frac{\psi_1}{s-s_m} + \Psi$$

where Ψ is some function regular in a neighborhood of s_m, including $i\lambda_m$. Upon multiplying (2.13) by $(s-s_m)$ and taking the double limit $s \to s_m$, $R \to \infty$ <u>in that order</u>, we find that ψ_1 satisfies

$$\nabla^2\psi_1 - \frac{4}{\lambda_m^2}(\hat{k}\cdot\nabla)^2\psi_1 = \frac{8i\, s_m^{(1)}}{\lambda_m^3}(\hat{k}\cdot\nabla)^2\Phi_m$$

with

$$\hat{n}\cdot\nabla\psi_1 - \frac{2}{i\lambda_m}\hat{n}\cdot\hat{k}\times\nabla\psi_1 - \frac{4}{\lambda_m^2}(\hat{n}\cdot\hat{k})\,\hat{k}\cdot\nabla\psi_1 =$$

$$= \frac{4-\lambda_m^2}{i\lambda_m}\lim_{\substack{s\to s_m\\ R\to\infty}}\int_0^\infty \hat{n}\times(\hat{n}\times\nabla)\cdot\tilde{\vec{V}}_m\,d\zeta - \frac{s_m^{(1)}}{i\lambda_m}\left(\hat{n}\cdot\nabla\Phi_m + \frac{4}{\lambda_m^2}\hat{n}\cdot\hat{k}\;\hat{k}\cdot\nabla\Phi_m\right) \qquad (2.14)$$

on the boundary S.

Now since Φ_m is in fact a solution of the completely homogeneous form of (2.14), the inhomogeneous problem has a solution only when $s_m^{(1)}$ assumes a value consistent with the homogeneous equations. This value is determined by multiplying (2.14) by Φ_m^* and integrating over the volume, using the divergence theorem and the boundary conditions satisfied by Φ_m^* and ψ_1. By this method we first obtain as an intermediate result

$$\iint_S \Phi_m^*\left\{\hat{n}\cdot\nabla\psi_1 - \frac{2}{i\lambda_m}\hat{n}\cdot\hat{k}\times\nabla\psi_1 - \frac{4}{\lambda_m^2}(\hat{n}\cdot\hat{k})\,\hat{k}\cdot\nabla\psi_1\right\}dS =$$

$$= \frac{8i\, s_m^{(1)}}{\lambda_m^3}\left(\iint_S \Phi_m^*(\hat{n}\cdot\hat{k})\,\hat{k}\cdot\nabla\Phi_m\,dS - \iiint_V \left|\hat{k}\cdot\nabla\Phi_m\right|^2 dV\right).$$

If we now substitute the boundary condition for ψ_1 from (2.14), the result is

$$s_m^{(1)} \iiint \left(|\nabla \Phi_m|^2 + \frac{4}{\lambda_m^2} |\hat{k}\cdot\nabla\Phi_m|^2 \right) dV = (4-\lambda_m^2) \iint_S \left(\lim_{\substack{s\to s_m \\ R\to\infty}} \Phi_m^* \int_0^\infty \hat{n}\times(\hat{n}\times\nabla)\cdot\vec{V}_m \, d\zeta \right) dS \qquad (2.15)$$

This is the desired formula for $s_m^{(1)}$, but we shall derive a more useable form from which it shall be clear that $\mathrm{Re}(s_m^{(1)}) < 0$. The derivation of the alternate formula involves somewhat tedious algebra, and the details are given in Appendix A. There it is shown that

$$\left.\begin{aligned} & s_m^{(1)} \cdot \iiint \left(|\nabla \Phi_m|^2 + \frac{4}{\lambda_m^2} |\hat{k}\cdot\nabla\Phi_m|^2 \right) dV = \frac{-(4-\lambda_m^2)}{2} \iint_S J \, dS , \\ & \text{where} \\ & J \equiv \frac{1}{1-(\hat{n}\cdot\hat{k})^2} \Big\{ |\hat{n}\cdot\hat{k}\times\vec{Q}_m - i\hat{k}\cdot\vec{Q}_m|^2 \left(1+\frac{ip_+}{|p_+|}\right)|p_+|^{1/2} + \\ & \qquad + |\hat{n}\cdot\hat{k}\times\vec{Q}_m + i\hat{k}\cdot\vec{Q}_m|^2 \left(1+\frac{ip_-}{|p_-|}\right)|p_-|^{1/2} \Big\} \\ & p_\pm = \frac{\lambda_m}{2} \pm \hat{n}\cdot\hat{k} \end{aligned}\right\} \qquad (2.16)$$

We see at once that when $\lambda_m^2 < 4$ and λ_m is real, then $\mathrm{Re}(s_m^{(1)}) < 0$, and $|\mathrm{Im}(s_m^{(1)})| \leq |\mathrm{Re}(s_m^{(1)})|$.

Greenspan (1965) has in fact demonstrated that the eigenvalues λ_m are real and $\lambda_m^2 \leq 4$. This was established in the following manner:

All inviscid inertial oscillations are solutions of

$$i\lambda_m \vec{Q}_m + 2\hat{k}\times\vec{Q}_m + \nabla\Phi_m = 0, \qquad \nabla\cdot\vec{Q}_m = 0 \qquad (2.17)$$

with $\hat{n}\cdot\vec{Q}_m = 0$ on S. Multiplying by $\vec{Q}_m^*$ and integrating over the volume of the container,

$$i\lambda_m \int_V \vec{Q}_m\cdot\vec{Q}_m^*\,dV + 2\int_V \vec{Q}_m^*\cdot\hat{k}\times\vec{Q}_m\,dV = -\int_V \vec{Q}_m^*\cdot\nabla\Phi_m\,dV.$$

But

$$\int_V \vec{Q}_m^*\cdot\nabla\Phi_m\,dV = \int_V \nabla\cdot(\Phi_m\vec{Q}_m^*)\,dV - \int_V \Phi_m(\nabla\cdot\vec{Q}_m^*)\,dV = 0,$$

and therefore

$$\lambda_m = \frac{2i\int \vec{Q}_m^*\cdot\hat{k}\times\vec{Q}_m\,dV}{\int |\vec{Q}_m|^2\,dV} = \frac{-2\,Im\left(\int \hat{k}\cdot\vec{Q}_m\times\vec{Q}_m^*\,dV\right)}{\int |\vec{Q}_m|^2\,dV}$$

This shows that λ_m is real. The bound $|\lambda_m| \le 2$ is determined by writing $\vec{Q}_m = \vec{A} + i\vec{B}$, and recognizing that $Im(\hat{k}\cdot\vec{Q}_m\times\vec{Q}_m^*) = 2\vec{B}\times\vec{A}$. Then

$$|\lambda_m| \le \frac{4\int |\hat{k}\cdot\vec{A}\times\vec{B}|\,dV}{\int(|\vec{A}|^2+|\vec{B}|^2)\,dV} \le \frac{4\int |\vec{A}|\cdot|\vec{B}|\,dV}{\int(|\vec{A}|^2+|\vec{B}|^2)\,dV}$$

But $4|\vec{A}|\cdot|\vec{B}| \le 2(|\vec{A}|^2+|\vec{B}|^2)$, and this gives the desired result

$$|\lambda_m| \le 2. \tag{2.18}$$

This means that all the modes decay in the dimensionless time $O(R^{1/2})$ with the possible exception of those modes for which $\lambda_m^2 = 4$. For in this latter case, (2.16) is also valid and we have $s_m^{(1)} = 0$. We shall show in Chapter 3, however, that the values $\lambda_m = \pm 2$ are not proper eigenvalues for any finite, closed container. This permits us to replace (2.18) by the strict inequality

$$|\lambda_m| < 2 . \tag{2.19}$$

Equation (2.16) can be derived in a more direct way without using the arguments based on coalescence of poles and nonuniform behavior. This alternate method assumes that all functions, including those in the boundary layers, have the complete exponential time dependence $\exp(s_m t)$. However, a serious mathematical difficulty arises due to nonintegrable singularities at the surface positions $2\hat{n}\cdot\hat{k} = \pm\lambda_m$, and one cannot justify the necessary interchanges of integration involving $\hat{n}$ and ζ at these points. A somewhat less restrictive but equally unsatisfactory aspect of the classical method is that the actual time-dependent behavior of the boundary layer solution is not just a simple exponential function, as equation (2.11) shows. These difficulties are avoided in the Laplace transform analysis. By judicious application of the double limit, $\underset{R\to\infty}{\text{limit}}\cdot\underset{s\to s_m}{\text{limit}}$, all integrals occurring in the derivation remain finite. In order to point out just where the classical method breaks down, we now solve problems (iii) and (iv) again.

For problem (iii), we put $\tilde{\vec{q}}_m = \tilde{\vec{Q}}_m e^{s_m t}$, $\tilde{\varphi}_{m1} = \tilde{\Phi}_{m1} e^{s_m t}$. It is then readily established that

$$\tilde{\vec{Q}}_m = -\tfrac{1}{2}\left(\vec{Q}_m - i\hat{n}\times\vec{Q}_m\right)\exp\left\{-\sqrt{i(\lambda_m+2\hat{n}\cdot\hat{k})}\,\zeta\right\} - \tfrac{1}{2}\left(\vec{Q}_m + i\hat{n}\times\vec{Q}_m\right)\exp\left\{-\sqrt{i(\lambda_m-2\hat{n}\cdot\hat{k})}\,\zeta\right\}. \quad (2.20)$$

From the mass conservation requirement,

$$\frac{\partial}{\partial\zeta}\,\hat{n}\cdot\tilde{\vec{Q}}_{m1} = \hat{n}\cdot\nabla\times\left(\hat{n}\times\tilde{\vec{Q}}_m\right),$$

it follows that

$$\left(\hat{n}\cdot\tilde{\vec{Q}}_{m1}\right)_{\zeta=0} = -\int_0^\infty \hat{n}\cdot\nabla\times\left(\hat{n}\times\tilde{\vec{Q}}_m\right)d\zeta \quad (2.21)$$

To solve problem (iv), we again assume the separable forms $\vec{q}_{m1} = \vec{Q}_{m1}\,e^{s_m t}$, $\varphi_{m1} = \Phi_{m1}e^{s_m t}$. The governing equations then are

$$\left.\begin{aligned} i\lambda_m\vec{Q}_{m1} + 2\hat{k}\times\vec{Q}_{m1} + \nabla\Phi_{m1} &= -s_m^{(1)}\vec{Q}_m \\ \nabla\cdot\vec{Q}_{m1} &= 0 \\ \text{with}\qquad \hat{n}\cdot\vec{Q}_{m1} &= -\hat{n}\cdot\tilde{\vec{Q}}_{m1} \quad \text{on } S. \end{aligned}\right\} \quad (2.22)$$

However, $\vec{Q}_m$ is the solution to the completely homogeneous form of (2.22). A nontrivial solution $\vec{Q}_{m1}$ therefore exists only if $s_m^{(1)}$ assumes a value consistent with the homogeneous equations. This value is found by multiplying (2.22) by $\vec{Q}_m^*$, integrating over the volume, and using the equations satisfied by $\vec{Q}_{m1}$, and $\vec{Q}_m^*$. In this way we arrive at the result

$$s_m^{(1)} \int_V |\vec{Q}_m|^2 \, dV = - \int_S \Phi_m^* \left(\int_0^\infty \hat{n} \cdot \nabla \times (\hat{n} \times \tilde{\vec{Q}}_m) \, d\zeta \right) dS . \qquad (2.23)$$

Because $\tilde{\vec{Q}}_m$ is given by (2.20), the integrand in (2.23) is nonintegrable whenever $2\hat{n}\cdot\hat{k} = \pm \lambda_m$, behaving like $(2\hat{n}\cdot\hat{k} \pm \lambda_m)^{-3/2}$. If we proceed formally and interchange the integrations over ζ and S we will recover formula (2.16). But there is no real justification for interchanging these integrations, and that is why the Laplace transform method is preferred.

CHAPTER 3

THE VALUES $\lambda_m = \pm 2$ ARE NOT PROPER EIGENVALUES

We prove this by establishing the identity

$$(4-\lambda_j\lambda_m)\iiint \vec{Q}_j\cdot\vec{Q}_m^* \, dV = \iiint\left(\nabla\Phi_j\cdot\nabla\Phi_m^* + \frac{4}{\lambda_j\lambda_m}(\hat{k}\cdot\nabla\Phi_j)(\hat{k}\cdot\nabla\Phi_m^*)\right)dV \tag{3.1}$$

which holds for all λ_j, λ_m different from zero. On putting $\lambda_j = \lambda_m$ so that $\Phi_m^* = \Phi_j^*$, we have

$$(4-\lambda_m^2)\iiint |\vec{Q}_m|^2 dV = \iiint\left(|\nabla\Phi_m|^2 + \frac{4}{\lambda_m^2}|\hat{k}\cdot\nabla\Phi_m|^2\right)dV$$

Therefore when $\lambda_m^2 = 4$ and when the container has finite volume, the right-hand side must vanish. This in turn requires that

$$\Phi_m = \text{constant}, \quad \text{when} \quad \lambda_m^2 = 4. \tag{3.2}$$

To establish (3.1), we use the inviscid equations and boundary conditions,

$$i\lambda_j\vec{Q}_j + 2\hat{k}\times\vec{Q}_j + \nabla\Phi_j = 0, \quad \nabla\cdot\vec{Q}_j = 0,$$

$$-i\lambda_m\vec{Q}_m^* + 2\hat{k}\times\vec{Q}_m^* + \nabla\Phi_m^* = 0, \quad \nabla\cdot\vec{Q}_m^* = 0;$$

$$\hat{n}\cdot\vec{Q}_j = \hat{n}\cdot\vec{Q}_m^* = 0, \text{ on } S.$$

It then follows that

$$\nabla\Phi_j \cdot \nabla\Phi_m^* = \{ i\lambda_j \vec{Q}_j + 2\hat{k}\times\vec{Q}_j \}\cdot\{ -i\lambda_m \vec{Q}_m^* + 2\hat{k}\times\vec{Q}_m^* \}$$

$$= \lambda_j\lambda_m \vec{Q}_j\cdot\vec{Q}_m^* + 4(\hat{k}\times\vec{Q}_j)\cdot(\hat{k}\times\vec{Q}_m^*) + \{ i\lambda_j\vec{Q}_j\cdot 2\hat{k}\times\vec{Q}_m^* - i\lambda_m\vec{Q}_m^*\cdot 2\hat{k}\times\vec{Q}_j \}$$

But we may further deduce that

$$\{ i\lambda_j\vec{Q}_j\cdot 2\hat{k}\times\vec{Q}_m^* - i\lambda_m\vec{Q}_m^*\cdot 2\hat{k}\times\vec{Q}_j \} + 2\lambda_j\lambda_m \vec{Q}_j\cdot\vec{Q}_m^* = i\lambda_m\vec{Q}_m^*\cdot\nabla\Phi_m - i\lambda_j\vec{Q}_j\cdot\nabla\Phi_m^* \ .$$

Combining these results, we find

$$\nabla\Phi_j\cdot\nabla\Phi_m^* + \frac{4}{\lambda_j\lambda_m}(\hat{k}\cdot\nabla\Phi_j)(\hat{k}\cdot\nabla\Phi_m^*) = (4-\lambda_j\lambda_m)\vec{Q}_j\cdot\vec{Q}_m^* + \{ i\lambda_m\vec{Q}_m^*\cdot\nabla\Phi_j - i\lambda_j\vec{Q}_j\cdot\nabla\Phi_m^* \},$$

from which (3.1) follows directly upon integration.

Now, equation (3.2) alone doesn't necessarily mean that $\vec{Q}_m \equiv 0$. It may be that Coriolis force and local time acceleration are in balance, as shown by the equation

$$i\lambda_m\vec{Q}_m = -2\hat{k}\times\vec{Q}_m\,, \quad \lambda_m = \pm 2\ . \tag{3.3}$$

However, using this relation between the components of $\vec{Q}_m$, we shall establish that

(i) on the boundary, $\{1-(\hat{n}\cdot\hat{k})^2\}\vec{Q}_m = 0$;

(ii) in polar coordinates the radial component is $u_m = \frac{1}{r}F(re^{i\frac{\lambda_m}{2}\omega}, z)$, where F is an arbitrary function;

(iii) $\nabla_H^2 F = 0$, where $\nabla_H^2 \equiv \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \omega^2}$.

The direct consequence of (i)-(iii) is that F, and hence $\vec{Q}_m$, vanishes identically.

(i) The boundary condition.

Since $i\lambda_m \vec{Q}_m = -2\hat{k}\times\vec{Q}_m$ and $\hat{n}\cdot\vec{Q}_m = 0$ on S,

then $\hat{n}\times\vec{Q}_m = \frac{2}{i\lambda_m}(\hat{n}\cdot\hat{k})\vec{Q}_m = (\hat{n}\cdot\hat{k})\,\hat{k}\times\vec{Q}_m$ on S.

If we multiply this relation by $\hat{n}\times$, the result is

$\hat{n}\times(\hat{n}\times\vec{Q}_m) - (\hat{n}\cdot\hat{k})\hat{n}\times(\hat{k}\times\vec{Q}_m) = 0$ or equivalently,

$$\vec{Q}_m\left\{1-(\hat{n}\cdot\hat{k})^2\right\} = 0 \quad \text{on } S.$$

(ii) The formal solution.

Equation (3.3) shows that

$$\vec{Q}_m \equiv \left(U_m, V_m, W_m\right) = \left(U_m, \frac{i\lambda_m}{2} U_m, 0\right).$$

Mass conservation then requires

$$r\frac{\partial}{\partial r}(rU_m) + \frac{i\lambda_m}{2}\frac{\partial}{\partial\omega}(rU_m) = 0.$$

This has the general solution

$$rU_m = F\left(re^{i\frac{\lambda_m}{2}\omega}, z\right).$$

Note that the arbitrary function F must vanish at r = 0.

(iii) $$\nabla_H^2 F \equiv \frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\omega^2} = 0.$$

This follows directly from the relations

$$r\frac{\partial F}{\partial r} = re^{i\frac{\lambda_m}{2}\omega}\cdot F', \qquad \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial F}{\partial r} = \left\{e^{i\lambda_m\omega}F'' + \frac{1}{r}e^{i\frac{\lambda_m}{2}\omega}F'\right\}$$

$$\frac{1}{r}\frac{\partial F}{\partial \omega} = \frac{i\lambda_m}{2}e^{i\frac{\lambda_m}{2}\omega}\cdot F', \qquad \frac{1}{r^2}\frac{\partial^2 F}{\partial \omega^2} = \frac{-\lambda_m^2}{4}\left\{e^{i\lambda_m\omega}F'' + \frac{1}{r}e^{i\frac{\lambda_m}{2}\omega}F'\right\},$$

where prime denotes differentiation with respect to the argument $re^{i\frac{\lambda_m}{2}\omega}$, and $\lambda_m^2 = 4$.

By the two-dimensional Green's theorem, the equation $\nabla_H^2 F = 0$ implies that

$$\int F^* \nabla_H^2 F \, dA = \oint F^* \frac{\partial F}{\partial n} ds + \int |\nabla_H F|^2 dA = 0.$$

Since $F^* = 0$ on the boundary, then $\nabla_H F \equiv 0$. F is therefore constant with respect to r and ω, and this must hold for any value of z. Hence, $F \equiv 0$, and $\lambda_m = \pm 2$ are not proper eigenvalues of the system.

The assumption on which this proof rests is that the container encloses a finite volume. Normal modes of frequency $\lambda = \pm 2$ do in fact exist, for example, when the fluid is bounded by two infinite horizontal disks rotating about a common vertical axis (Greenspan and Howard, 1963). In this case, not only is the enclosed volume infinite, but also there are no other permissible values of λ different from zero.

CHAPTER 4

EIGENFUNCTIONS AND VISCOUS CORRECTION FACTORS FOR SPHEROIDS

For spheroidal geometries it is natural to use the orthogonal coordinate system ξ_1, ξ_2 given by

$$r = (\xi_1^2 + \alpha^2)^{1/2}(1 - \xi_2^2)^{1/2}, \qquad z = \xi_1 \xi_2 \tag{4.1}$$

(see for example Stewartson and Roberts, 1963). A very useful modification of this transformation, one which, incidentally, renders the resulting ζ, μ coordinate system nonorthogonal, is due to Greenspan:

$$\left.\begin{aligned} r &= \alpha(1-\zeta^2)^{1/2}(1-\mu^2)^{1/2}, \\ z &= (\alpha\beta)\zeta\mu . \end{aligned}\right\} \tag{4.2}$$

Under this transformation, the basic eigenfunction equation (cf. (2.6)),

$$\nabla^2 \varphi_m - \frac{4}{\lambda_m^2}\frac{\partial^2 \varphi_m}{\partial z^2} = 0, \tag{4.3}$$

yields the separable solutions

$$\varphi_m = P_m^k(\zeta)\, P_m^k(\mu)\, e^{ik\omega} \tag{4.4}$$

provided we take

$$\beta^2 = \frac{4}{\lambda_m^2} - 1 = \frac{1}{\xi_m^2} - 1 \text{, say.} \tag{4.5a}$$

If we write the equation of the surface as

$$r^2 + \frac{z^2}{b^2} = 1 ,$$

then by (4.2) the surface is given in the ζ, μ system as

$$\zeta = \left(\frac{b^2}{\alpha^2 \beta^2} \right)^{1/2} = \left(\frac{\xi_m^2}{1 + \varepsilon(1 - \xi_m^2)} \right)^{1/2} \tag{4.5b}$$

where

$$\alpha^2 = \frac{1 + \varepsilon(1 - \xi_m^2)}{(1+\varepsilon)(1 - \xi_m^2)} , \qquad \varepsilon \equiv \frac{1 - b^2}{b^2} . \tag{4.5c}$$

To put the eigenfunctions (4.4) in a more useable form, Toomre has suggested writing the associated Legendre functions $P_m^k(x)$ as

$$P_m^k(x) = \left[C_{mk} (1 - x^2)^{k/2} F(x^2) \right] \cdot \left\{ 1 , x \right\}$$

according as m-k is even or odd. Here $F(x^2)$ is a polynomial in x^2 with real and distinct zeroes x_j^2, $j = 1, \ldots, N$. Accordingly, we can write

$$\left.\begin{aligned}&P_m^k(x) = C_{mk}\, x^{\nu}\,(1-x^2)^{k/2}\prod_{j=1}^{N}(x^2-x_j^2)\,, \qquad C_{mk} = \frac{(2m)!}{2^m\, m!\,(m-k)!}\\&\text{where}\\&N = (m-k)-\nu\,, \quad \text{and} \quad \nu = \begin{cases} 0 & \text{if } m-k \text{ even} \\ 1 & \text{if } m-k \text{ odd}\end{cases}\end{aligned}\right\} \tag{4.6}$$

Using (4.6) and (4.4) together with (4.5a, c) the following result is then readily established (for details, see Appendix B):

$$\left.\begin{aligned}&\varphi_m = P_m^k(\zeta)P_m^k(\mu)e^{ik\omega} = C_{mk}^{\,2}\, e^{ik\omega}\left(\frac{z}{\alpha\beta}\right)^{\nu}\left(\frac{r}{\alpha}\right)^{k}\prod_{j=1}^{N}\left\{D_j + A_j r^2 + B_j z^2\right\}\\&\text{where}\\&D_j = x_j^2\,(x_j^2-1)\\&A_j = \frac{1+\varepsilon}{1+\varepsilon(1-\xi_m^2)}\cdot x_j^2\,(1-\xi_m^2)\\&B_j = \frac{1+\varepsilon}{1+\varepsilon(1-\xi_m^2)}\cdot \xi_m^2\,(1-x_j^2)\end{aligned}\right\} \tag{4.7}$$

and x_j^2 are the $N = (m-k)-\nu$ real and distinct zeroes of $P_m^k(x)$ exclusive of zero and one. α, β, and ξ_m are defined in (4.5).

The eigenvalues $\lambda_m = 2\xi_m$ are determined from the boundary condition

$$\hat{n}\cdot\nabla\varphi_m - \frac{2}{i\lambda_m}\hat{n}\cdot\hat{k}\times\nabla\varphi_m - \frac{4}{\lambda_m^2}(\hat{n}\cdot\hat{k})\hat{k}\cdot\nabla\varphi_m = 0\,, \tag{4.8}$$

which reduces to

$$(1-x^2)\frac{dP_m^k(x)}{dx} = k\left(\frac{1+\varepsilon x^2}{1+\varepsilon}\right)^{1/2} P_m^k(x)\ , \qquad \text{at} \qquad x = \frac{\xi_m}{\{1+\varepsilon(1-\xi_m^2)\}^{1/2}}\ . \qquad (4.9)$$

An alternate form of the eigenvalue relation (4.9), showing more clearly the nature of this relation, is derivable using (4.7). We find that $\xi_m \equiv \lambda_m/2$ is determined by the polynomial equation

$$\nu + \sum_{j=1}^{N} \frac{2x^2}{x^2 - x_j^2} = \frac{k\,\xi_m}{(1+\varepsilon)(1-\xi_m)}\ , \qquad \text{with} \qquad x^2 = \frac{\xi_m^2}{1+\varepsilon(1-\xi_m^2)}\ . \qquad (4.10)$$

This shows that in general there will be several eigenvalues corresponding to given indices (m,k). Therefore, a more complete notation is

$$\varphi_m = P_m^k(\zeta_{mkl})\, P_m^k(\mu)\, e^{ik\omega} \equiv \Phi_{mkl}\, e^{ik\omega}\ , \qquad \zeta_{mkl} \equiv \zeta(r,z;\lambda_{mkl}) \qquad (4.11)$$

the eigenvalues being λ_{mkl} .

Using the above results it is now possible to reduce equation (2.16)--which determines the viscous correction factors $s_m^{(1)}$ for each modal solution--to a form suitable for calculation.

Replacing the notation

$$p_m = \varphi_m\, e^{s_m t}\ , \qquad s_m = i\lambda_m + R^{-1/2} s_m^{(1)}$$

by that used in (4.11),

$$p_m = \Phi_{mkl}\, e^{ik\omega} e^{s_{mkl} t} \;, \qquad s_{mkl} = i\lambda_{mkl} + R^{-1/2} s^{(1)}_{mkl} \;,$$

we find after some algebraic manipulations that

$$s^{(1)}_{mkl} = -(4 - \lambda^2_{mkl}) \frac{I}{J} \tag{4.12}$$

where

$$J = \int_{r=0}^{1} r\,dr \int_{z=-b\sqrt{1-r^2}}^{b\sqrt{1-r^2}} dz \cdot \left\{ \left|\nabla\Phi_{mkl}\right|^2 + \frac{4}{\lambda^2_{mkl}} \left|\frac{\partial\Phi_{mkl}}{\partial z}\right|^2 \right\} \tag{4.13}$$

and, with $\mu = (1+\varepsilon)^{1/2} z$,

$$I = \frac{1}{8}\int_{-1}^{1} \left(\frac{1+\varepsilon\mu^2}{1+\varepsilon}\right)^{1/2} \left[\left| r\frac{\partial\Phi_{mkl}}{\partial r} + \frac{2\Psi^-}{\lambda_{mkl}}(1+\varepsilon) z \frac{\partial\Phi_{mkl}}{\partial z}\right|^2 \cdot \left(1 + \frac{i\Psi^+}{|\Psi^+|}\right) \left|\Psi^+\right|^{1/2} \right.$$

$$\left. + \left| r\frac{\partial\bar{\Phi}_{mkl}}{\partial r} + \frac{2\Psi^+}{\lambda_{mkl}}(1+\varepsilon) z \frac{\partial\Phi_{mkl}}{\partial z}\right|^2 \cdot \left(1 + \frac{i\Psi^-}{|\Psi^-|}\right) \cdot \left|\Psi^-\right|^{1/2} \right] \frac{d\mu}{1-\mu^2} \,. \tag{4.14}$$

In (4.14),

$$\Psi^+ \equiv \frac{\lambda_{mkl}}{2} + \left\{ \frac{(1+\varepsilon)\mu^2}{1+\varepsilon\mu^2} \right\}^{1/2} ,$$

and

$$\Psi^- \equiv \frac{\lambda_{mkl}}{2} - \left\{ \frac{(1+\varepsilon)\mu^2}{1+\varepsilon\mu^2} \right\}^{1/2} .$$

In the special case of a sphere ($\varepsilon = 0$), the above results are of course greatly simplified. Instead of (4.7), the eigenfunctions for the sphere are

$$\varphi_m = C^2_{mk}\, e^{ik\omega}\left(\xi_{mkl} z\right)^{\nu}\left(\frac{r}{\alpha_0}\right)^{k}\prod_{j=1}^{N}\left\{x_j^2(x_j^2-1) + x_j^2(1-\xi^2_{mkl})r^2 + \xi^2_{mkl}(1-x_j^2)z^2\right\} \qquad (4.7)'$$

where

$$\alpha_0^2 = \frac{1}{1-\xi^2_{mkl}} \,.$$

Of practical importance, the eigenvalues $\lambda_{mkl} = 2\,\xi_{mkl}$ are the roots of the equation (cf. (4.9))

$$(1-x^2)\frac{dP^k_m(x)}{dx} = k\,P^k_m(x)\,, \qquad at \quad x = \xi_{mkl}\,. \qquad (4.9)'$$

The viscous correction factor $S^{(1)}_{mkl}$ is similarly modified.

The accompanying tables and graphs show the results of calculations of $S^{(1)}_{mkl}$ for several spheroidal ellipticities ε , illustrating how oblateness affects the decay rate of some of the natural modes. To perform the calculations, the eigenvalues λ_{mkl} must first be determined from (4.9) or (4.9)'; clearly, the larger the value (m-k) is, the greater is the degree of this polynomial, and the computed roots will be correspondingly less accurate.

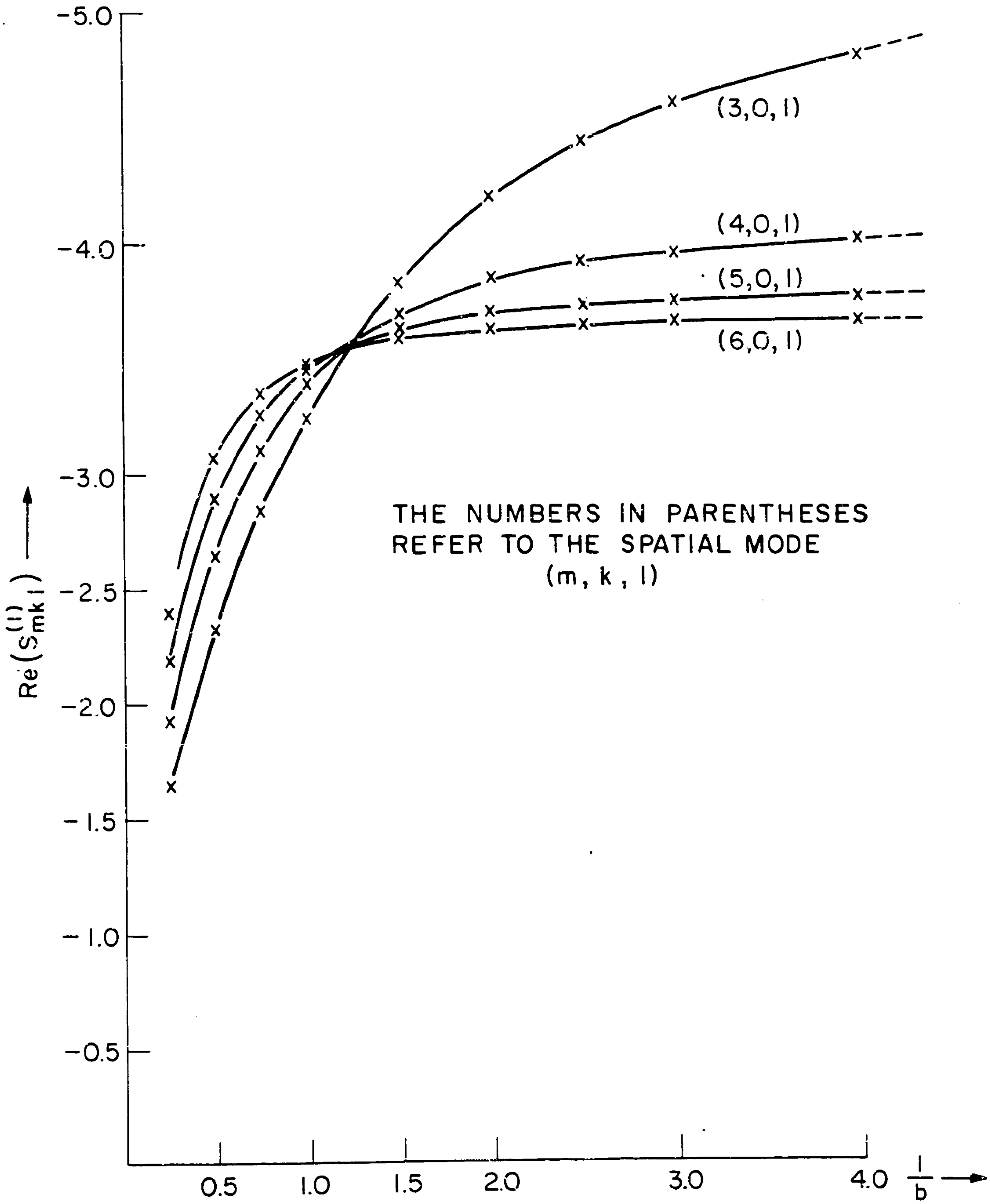
From the few fundamental modes for which data was calculated we see at once the marked effect of varying the radius-to-height ratio $1/b = (1+\varepsilon)^{1/2}$. The eigenfrequencies λ range over the whole interval $-2 < \lambda < 2$ for nearly spherical shapes, but as l/b is increased beyond unity (the configuration becoming more like concentric disks) the eigenvalues asymptotically approach the values ± 2. Similarly, for $l/b \to 0$, the eigenvalues rapidly decrease to zero. Only in the approximate range $\frac{1}{4} < l/b < 4$ is there any appreciable tendency for the eigenfrequency to depend on the spatial structure of the modes.

There is a similar phenomenon in the decay characteristics of these modes. A noticeable change in the decay rates ($\mathrm{Re}(s_{mkl}^{(1)})$) takes place as one passes from prolate to oblate spheroidal shapes. The rates rapidly approach different asymptotic values in the two cases.

The coalescence of the eigenvalues and decay rates means that for forced oscillations of the container at a given frequency, more modes will be in resonance or near-resonance for the extreme shapes than for nearly spherical bodies. However, when the force is removed the theory predicts that these modes will all decay in very nearly the same time.

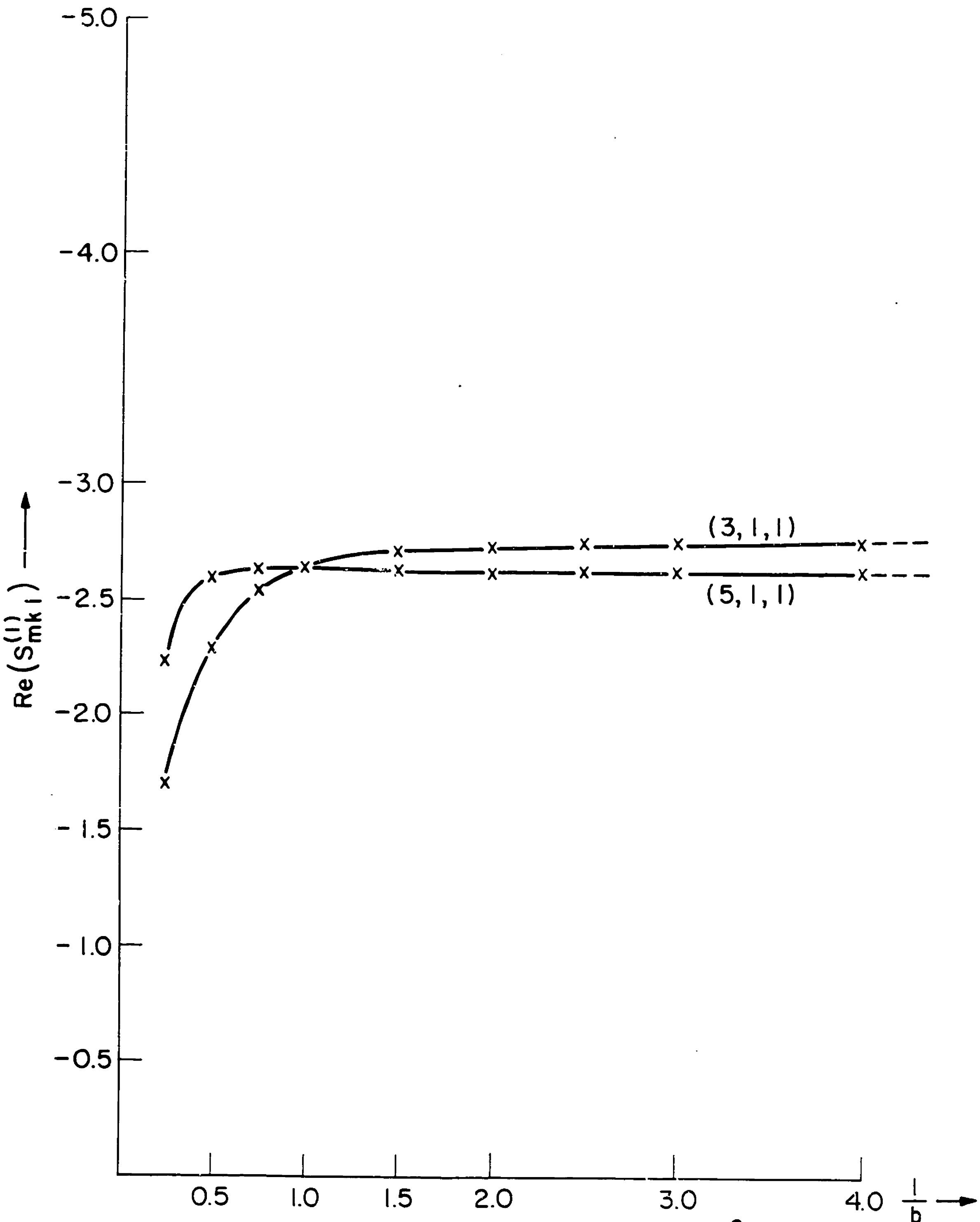
In interpreting and comparing these data with the analogous data given later for a cylinder, it should be noted that a different scaling procedure was applied to the spheroids and cylinders. For spheroids the length has been scaled so that the dimensionless <u>radius</u> is unity, whereas in the cylinders the dimensionless <u>height</u> is unity.

This means that the volume goes to zero as we consider flatter spheroids, but tends toward infinity for flatter cylinders. The antithetical behavior of the decay factor in the two classes of containers is a direct reflection of the different scaling procedures used, and underscores the physical influence of the total fluid volume on the viscous decay process. These remarks do not apply to the eigenfrequencies, which depend only on container shape and not volume.



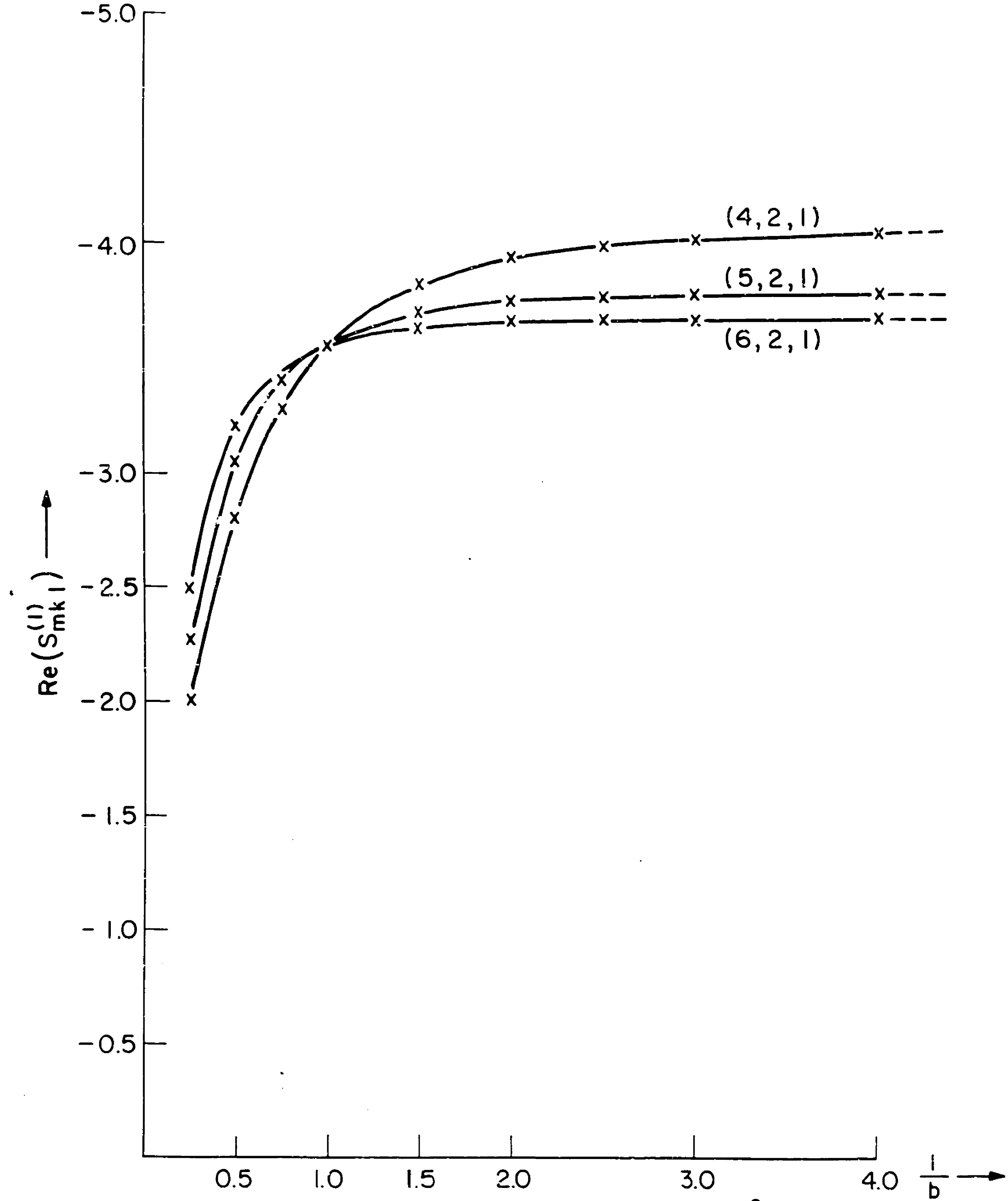
$Re\left(S^{(1)}_{mk1}\right)$ VS. $\frac{1}{b}$ FOR SPHEROID $r^2+\frac{1}{b^2}z^2=1$

FIG. 1 (a)



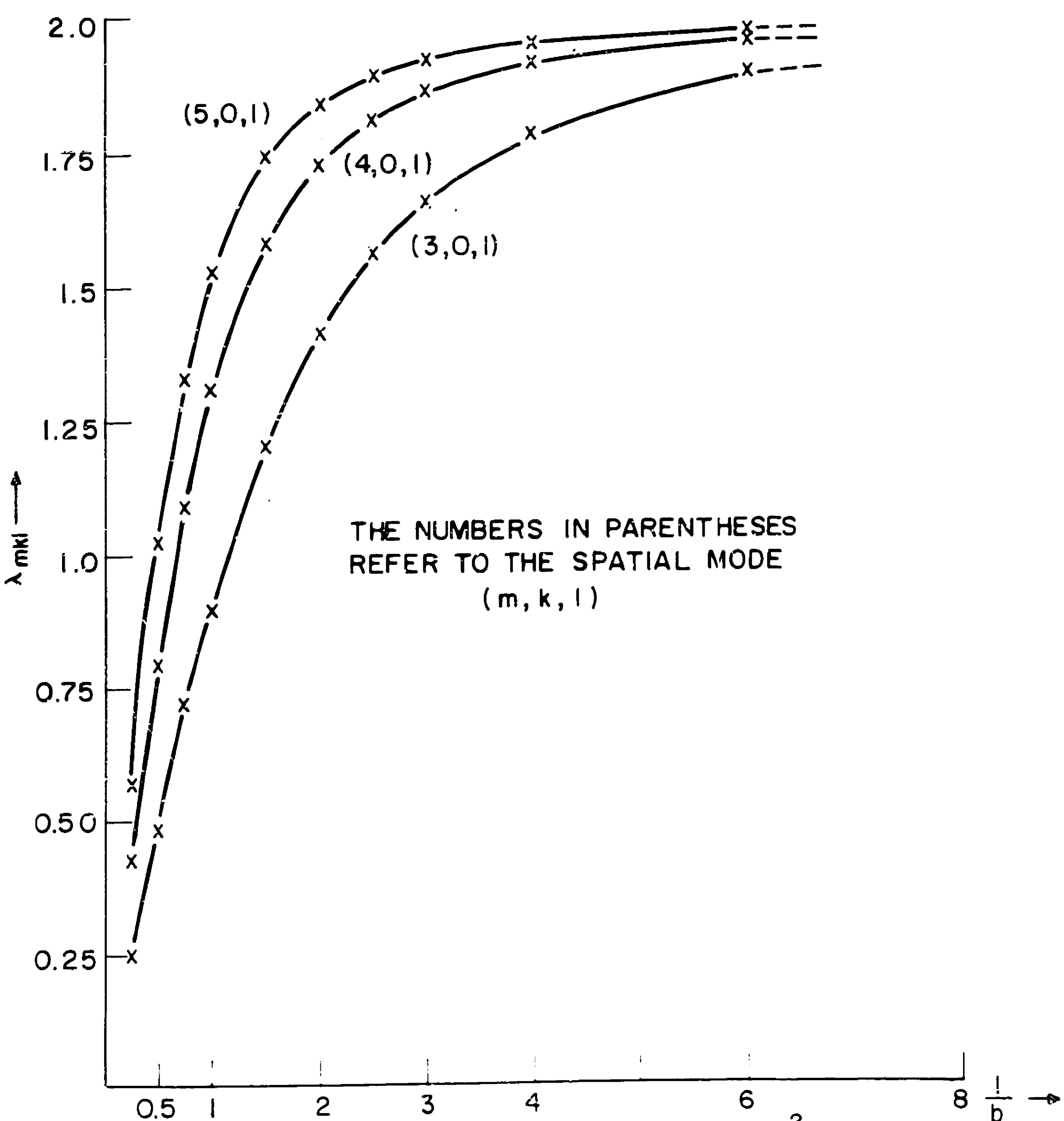
$Re(S^{(1)}_{mk1})$ VS. $\frac{1}{b}$ FOR SPHEROID $r^2 + \frac{z^2}{b^2} z^2 = 1$

FIG. 1 (b)



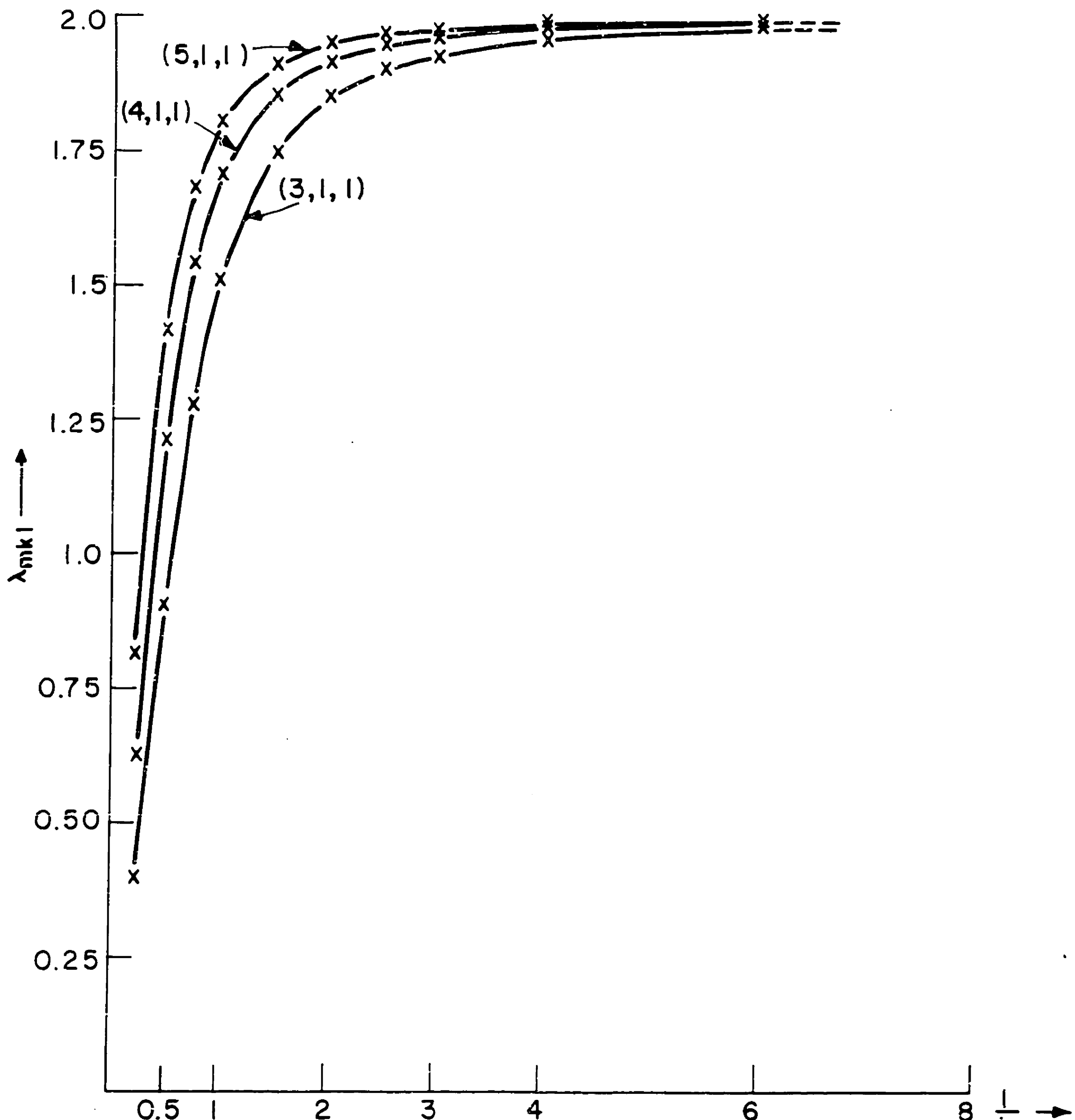
$Re(S^{(1)}_{mk1})$ VS. $\frac{1}{b}$ FOR SPHEROID $r^2 + \frac{z^2}{b^2} = 1$

FIG. 1 (c)



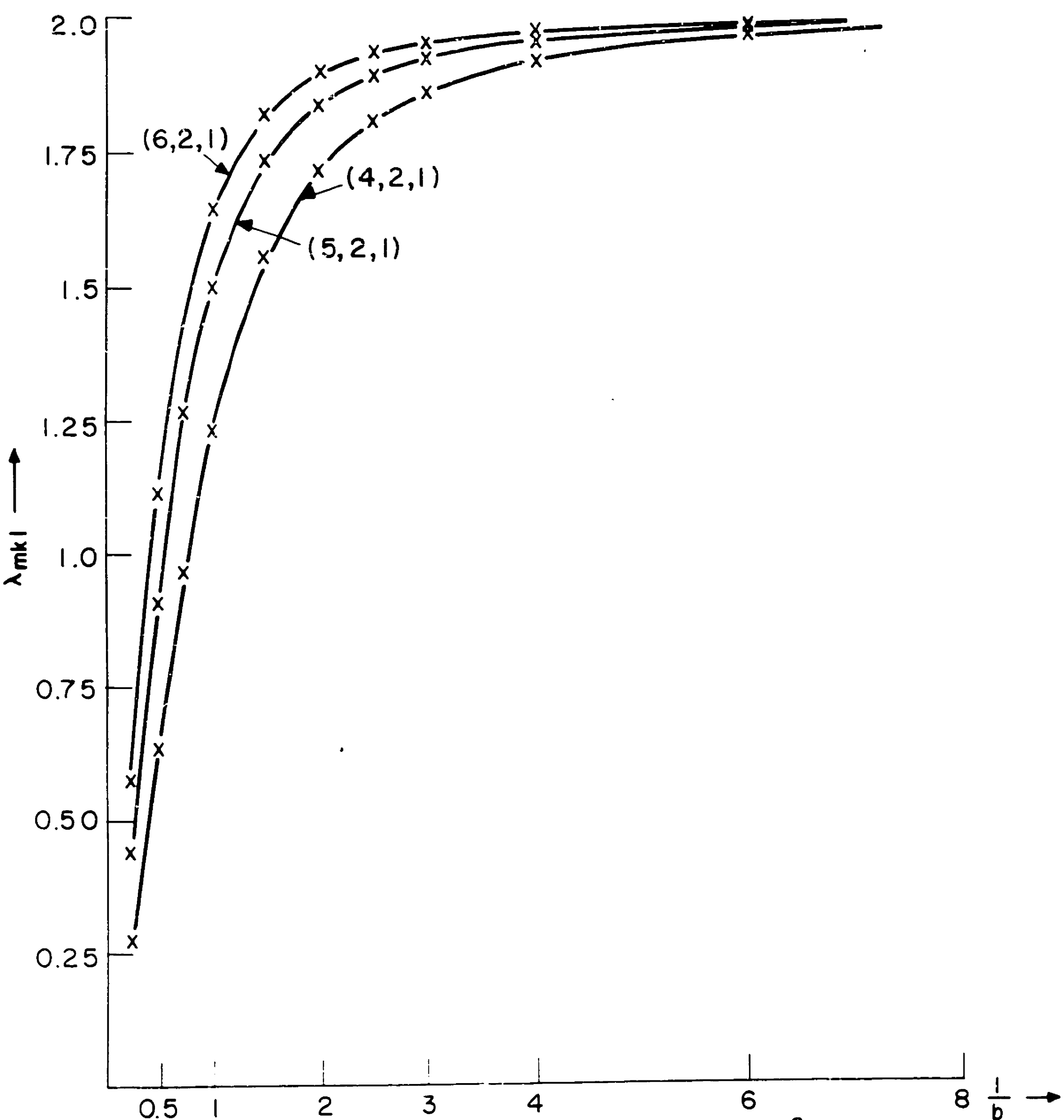
λ_{mkl} VS $\frac{l}{b}$ FOR SPHEROID $r^2 + \frac{z^2}{b^2} = 1$

FIG. 2(a)



λ_{mkl} VS $\frac{1}{b}$ FOR SPHEROID $r^2 + \frac{z^2}{b^2} = 1$

FIG. 2 (b)



λ_{mkl} VS $\frac{1}{b}$ FOR SPHEROID $r^2 + \frac{z^2}{b^2} = 1$

FIG. 2(c)

EIGENVALUES λ_{nkm} AND VISCOUS CORRECTIONS $s^{(1)}_{nkm}$

FOR SPHEROID $r^2 + \frac{z^2}{b^2} = 1$

n,k,m	b^{-1}	λ_{nkm}	$Re(s^{(1)}_{nkm})$	$Im(s^{(1)}_{nkm})$
3,0,1	.25	.2481	-1.639	-0.2089
	.5	.4851	-2.326	-0.1665
	.75	.7022	-2.841	+0.01216
	1.0	.8944	-3.248	+0.2853
	2.0	1.414	-4.205	+0.1584
	4.0	1.789	-4.825	+0.3027
4,0,1	.25	.4232	-1.928	-0.3437
	.5	.7947	-2.651	-0.2214
	.75	1.089	-3.099	+0.07998
	1.0	1.309	-3.385	+0.4258
	2.0	1.732	-3.844	+1.384
	4.0	1.922	-4.025	+1.933
5,0,1	.25	.5694	-2.181	-0.4357
	.5	1.021	-2.892	-0.2119
	.75	1.330	-3.254	+0.1643
	1.0	1.530	-3.447	+0.5054
	2.0	1.843	-3.698	+1.194
	4.0	1.957	-3.779	+1.487
6,0,1	.25	.6979	-2.397	-0.4982
	.5	1.194	-3.065	-0.1704
	.75	1.490	-3.349	+0.2425
	1.0	1.660	-3.479	+0.5536
	2.0	1.896	-3.629	+1.070
	4.0	1.972	-3.673	+1.258

EIGENVALUES λ_{nkm} AND VISCOUS CORRECTIONS $s^{(1)}_{nkm}$

FOR SPHEROID $r^2 + \frac{z^2}{b^2} = 1$

n, k, m	b^{-1}	λ_{nkm}	$\mathrm{Re}(s^{(1)}_{nkm})$	$\mathrm{Im}(s^{(1)}_{nkm})$
3, 1, 1	.25	.4000	-1.699	-0.4777
	.5	.9045	-2.297	-0.3494
	.75	1.277	-2.542	+0.06871
	1.0	1.510	-2.642	+0.4351
	2.0	1.847	-2.734	+1.082
	4.0	1.959	-2.755	+1.321
4, 1, 1	.25	.6221	-2.006	-0.6254
	.5	1.213	-2.492	-0.2549
	.75	1.539	-2.609	+0.2077
	1.0	1.708	-2.639	+0.5037
	2.0	1.917	-2.655	+0.9154
	4.0	1.978	-2.657	+1.043
5, 1, 1	.25	.8126	-2.244	-0.6931
	.5	1.415	-2.590	-0.1326
	.75	1.682	-2.631	+0.3037
	1.0	1.806	-2.633	+0.5382
	2.0	1.947	-2.627	+0.8263
	4.0	1.986	-2.624	+0.9093
6, 1, 1	.25	.9766	-2.423	-0.6960
	.5	1.552	-2.639	-0.01735
	.75	1.768	-2.640	+0.3735
	1.0	1.862	-2.629	+0.5584
	2.0	1.963	-2.613	+0.7723
	4.0	1.991	-2.609	+0.8312

EIGENVALUES λ_{nkm} AND VISCOUS CORRECTIONS $s^{(1)}_{nkm}$

FOR SPHEROID $r^2 + \frac{z^2}{b^2} = 1$

n,k,m	b^{-1}	λ_{nkm}	$\mathrm{Re}(s^{(1)}_{nkm})$	$\mathrm{Im}(s^{(1)}_{nkm})$
4,2,1	.25	.2746	-2.004	-0.3046
	.5	.6345	-2.805	-0.2970
	.75	.9707	-3.285	+0.007221
	1.0	1.232	-3.562	+0.4033
	2.0	1.719	-3.936	+1.441
	4.0	1.920	-4.055	+1.962
5,2,1	.25	.4372	-2.267	-0.4344
	.5	.9127	-3.046	-0.2940
	.75	1.268	-3.404	+0.1206
	1.0	1.496	-3.568	+0.5031
	2.0	1.839	-3.744	+1.219
	4.0	1.957	-3.792	+1.497
6,2,1	.25	.5777	-2.501	-0.5278
	.5	1.118	-3.214	-0.2453
	.75			
	1.0	1.643	-3.565	+0.5567
	2.0	1.894	-3.658	+1.085
	4.0	1.972	-3.681	+1.263

CHAPTER 5

FORCED OSCILLATIONS

We now formulate in general terms the problem of the response of the fluid to forced oscillations of the rotating container. Of particular interest is the question of forced oscillations at a resonant frequency. This will also be dealt with by the present methods.

Let the container be oscillated at frequency α. The Ekman layer flow, being to lowest order confined to the directions parallel to the boundary, causes a small $O(R^{-1/2})$ mass flux from the interior into the boundary layers (mass conservation) thereby stimulating the inviscid modes. The boundary condition for the normal component of interior flow at the boundary S is

$$\vec{q}\cdot\hat{n} = F\,e^{i\alpha t}, \tag{5.1}$$

where F is a known scalar function of the surface velocity.

If we put

$$\vec{q} = \vec{v}(r,\omega,z)\,e^{i\alpha t}, \quad p = \varphi(r,\omega,z)\,e^{i\alpha t}$$

then $\varphi(r,\omega,z)$ satisfies

$$\nabla^2\varphi - \frac{4}{\alpha^2}\frac{\partial^2\varphi}{\partial z^2} = 0 \tag{5.2}$$

subject to the boundary condition

$$\hat{n}\cdot\nabla\varphi - \frac{2}{i\alpha}\hat{n}\cdot\hat{k}\times\nabla\varphi - \frac{4}{\alpha^2}\hat{n}\cdot\hat{k}\ \hat{k}\cdot\nabla\varphi = \frac{4-\alpha^2}{i\alpha}F = G, \text{ say.} \qquad (5.3)$$

Expanding φ in terms of the eigenfunctions $\Phi_{\nu m}$ (where $m = 1,2,\ldots N(\nu)$ indicates that to a given index ν there may correspond more than one eigenvalue), the coefficients $B_{\nu m}$ in the expansion

$$\varphi = \sum_{\nu,m} B_{\nu m}\Phi_{\nu m}$$

must then be determined from equations (5.2) and (5.3). Upon substitution and use of the equations satisfied by the eigenfunctions, these equations become

$$4\sum B_{\nu m}\left(\frac{1}{\lambda_{\nu m}^2} - \frac{1}{\alpha^2}\right)\frac{\partial^2\Phi_{\nu m}}{\partial z^2} = 0 \qquad (5.4)$$

and on S,

$$G = 4\sum B_{\nu m}\left(\frac{1}{\lambda_{\nu m}^2} - \frac{1}{\alpha^2}\right)\hat{n}\cdot\hat{k}\ \hat{k}\cdot\nabla\Phi_{\nu m} - 2i\sum B_{\nu m}\left(\frac{1}{\lambda_{\nu m}} - \frac{1}{\alpha}\right)\hat{n}\cdot\hat{k}\times\nabla\Phi_{\nu m}. \quad (5.5)$$

We multiply (5.4) and (5.5) by $\Phi_{\kappa l}^*$ and integrate over the volume and surface, respectively, Then

$$0 = 4\sum B_{\nu m}\left(\frac{1}{\lambda_{\nu m}^2} - \frac{1}{\alpha^2}\right)\left\{\iiint_V \hat{k}\cdot\nabla\Phi_{\kappa l}^*\ \hat{k}\cdot\nabla\Phi_{\nu m}\,dV - \iint_S \hat{n}\cdot\hat{k}\ \Phi_{\kappa l}^*\ k\cdot\nabla\Phi_{\nu m}\,dS\right\},$$

$$\iint_S \Phi_{\kappa l}^* G\,dS = 4\sum B_{\nu m}\left(\frac{1}{\lambda_{\nu m}} - \frac{1}{\alpha}\right)\left\{\left(\frac{1}{\lambda_{\nu m}} + \frac{1}{\alpha}\right)\iint_S \hat{n}\cdot\hat{k}\ \Phi_{\kappa l}^*\ k\cdot\nabla\Phi_{\nu m}\,dS + \right.$$

$$\left. + \frac{1}{2i}\iint_S \hat{n}\cdot\hat{k}\times\Phi_{\kappa l}^*\nabla\Phi_{\nu m}\,dS\right\}$$

or, as a single equation,

$$4\sum B_{\nu m}\left(\frac{1}{\lambda_{\nu m}}-\frac{1}{\alpha}\right)\left\{\left(\frac{1}{\lambda_{\nu m}}+\frac{1}{\alpha}\right)\iiint_V \hat{k}\cdot\nabla\Phi^*_{\kappa l}\ \hat{k}\cdot\nabla\Phi_{\nu m}\,dV+\frac{1}{2i}\iint_S \hat{n}\cdot\hat{k}\times\Phi^*_{\kappa l}\nabla\Phi_{\nu m}\,dS\right\}=\iint_S \Phi^*_{\kappa l}G\,dS. \tag{5.6}$$

Now for two distinct eigenvalue-eigenfunction pairs $(\lambda_{\kappa l},\Phi^*_{\kappa l})$, $(\lambda_{\nu m},\Phi_{\nu m})$ the orthogonality condition is

$$\frac{4(\lambda_{\kappa l}+\lambda_{\nu m})}{\lambda_{\kappa l}\lambda_{\nu m}}\iiint_V \hat{k}\cdot\nabla\Phi^*_{\kappa l}\ \hat{k}\cdot\nabla\Phi_{\nu m}\,dV-2i\iint_S \hat{n}\cdot\hat{k}\times\Phi^*_{\kappa l}\nabla\Phi_{\nu m}\,dS=0.$$

From this condition it follows that, when $(\kappa,l)\neq(\nu,m)$,

$$\left\{\left(\frac{1}{\lambda_{\nu m}}+\frac{1}{\alpha}\right)\iiint_V \hat{k}\cdot\nabla\Phi^*_{\kappa l}\ \hat{k}\cdot\nabla\Phi_{\nu m}\,dV+\frac{1}{2i}\iint_S \hat{n}\cdot\hat{k}\times\Phi^*_{\kappa l}\nabla\Phi_{\nu m}\,dS\right\}=\frac{\lambda_{\nu m}(\alpha-\lambda_{\nu m})}{2i\alpha(\lambda_{\nu m}+\lambda_{\kappa l})}\iint \hat{n}\cdot\hat{k}\times\Phi^*_{\kappa l}\nabla\Phi_{\nu m}\,dS \tag{5.7a}$$

Let

$$D_{\kappa l}=\frac{\alpha}{\lambda_{\kappa l}}\left\{\left(\frac{1}{\lambda_{\kappa l}}+\frac{1}{\alpha}\right)\iiint_V \left|\hat{k}\cdot\nabla\Phi_{\kappa l}\right|^2 dV+\frac{1}{2i}\iint_S \hat{n}\cdot\hat{k}\times\Phi^*_{\kappa l}\nabla\Phi_{\kappa l}\,dS\right\} \tag{5.7b}$$

Through the use of (5.7a, b) equation (5.6) may be written in the form

$$\frac{4(\alpha-\lambda_{\kappa l})}{\alpha^2}\left[D_{\kappa l}B_{\kappa l}+\frac{1}{2i}\sum_{(\nu,m)\neq(\kappa,l)}B_{\nu m}\left\{\frac{\alpha-\lambda_{\nu m}}{\lambda_{\nu m}+\lambda_{\kappa l}}\iint_S \hat{n}\cdot\hat{k}\times\Phi^*_{\kappa l}\nabla\Phi_{\nu m}\,dS\right\}\right]=\iint_S \Phi^*_{\kappa l}G\,dS. \tag{5.8}$$

Before discussing the consequences of these equations, we show how the effect of a general body force $\vec{\mathcal{F}}e^{i\alpha t}$ may be considered by the same methods. The governing equations are

$$\nabla\cdot\vec{q} = 0, \quad \frac{\partial\vec{q}}{\partial t} + 2\hat{k}\times\vec{q} + \nabla p = \vec{\mathcal{F}} e^{i\alpha t}; \quad \hat{n}\cdot\vec{q} = 0 \text{ on } S.$$

so that, with $p = \varphi e^{i\alpha t}$, $\vec{q} = \vec{Q} e^{i\alpha t}$ the basic forced problem is

$$\nabla^2\varphi - \frac{4}{\alpha^2}\frac{\partial^2\varphi}{\partial z^2} = \mathcal{G},$$

(cf. (5.2)), the boundary condition being

$$\hat{n}\cdot\nabla\varphi - \frac{2}{i\alpha}\hat{n}\cdot\hat{k}\times\nabla\varphi - \frac{4}{\alpha^2}\hat{n}\cdot\hat{k}\;\hat{k}\cdot\nabla\varphi = \mathcal{H}.$$

Here the scalars $\mathcal{G}$ and $\mathcal{H}$ are related to the force $\vec{\mathcal{F}}$ by the equations

$$\mathcal{G} \equiv \nabla\cdot\vec{\mathcal{F}} - \frac{4}{\alpha^2}\frac{\partial}{\partial z}\hat{k}\cdot\vec{\mathcal{F}} + \frac{2}{i\alpha}\hat{k}\cdot(\nabla\times\vec{\mathcal{F}}),$$

$$\mathcal{H} \equiv \hat{n}\cdot\vec{\mathcal{F}} - \frac{2}{i\alpha}\hat{n}\cdot\hat{k}\times\vec{\mathcal{F}} - \frac{4}{\alpha^2}(\hat{n}\cdot\hat{k})\,\hat{k}\cdot\vec{\mathcal{F}} \quad \text{on } S.$$

The expansion

$$\varphi = \sum_{\nu,m} B_{\nu m}\Phi_{\nu m}$$

now leads by the previously-employed methods to a system of equations for $B_{\nu m}$ differing little from (5.8). Only the right-hand side of (5.8) is modified; it now must include the effects of both $\mathcal{G}$ and $\mathcal{H}$. The modified system is

$$\frac{4(\alpha-\lambda_{\kappa l})}{\alpha^2}\left[D_{\kappa l}B_{\kappa l} + \frac{1}{2i}\sum_{\substack{(\nu,m)\\ \neq(\kappa,l)}} B_{\nu m}\left\{\frac{\alpha-\lambda_{\nu m}}{\lambda_{\nu m}+\lambda_{\kappa l}} \iint_S \hat{n}\cdot\hat{k}\times\Phi^*_{\kappa l}\nabla\Phi_{\nu m}\,dS\right\}\right] =$$

$$= \iiint_V \Phi^*_{\kappa l}\mathcal{G}\,dV + \iint_S \Phi^*_{\kappa l}\mathcal{H}\,dS \qquad (5.8)'$$

The Fourier constants $B_{\nu m}$ constitute the solutions of an infinite system of linear equations with constant coefficients. To solve for $B_{\nu m}$ the eigenfunctions must first be found. (This is probably an impossible task if the container is not describeable by a separable coordinate system.) Once the eigenfunctions are found, it is sometimes possible to reduce the infinite sum in (5.8) to a finite sum by re-expanding $\hat{n}\cdot\hat{k}\times\nabla\Phi_{\nu m}$ on the surface as

$$\hat{n}\cdot\hat{k}\times\nabla\Phi_{\nu m} = \sum C_{\nu m p q}\,\Phi_{pq}(S) \qquad (5.9a)$$

For if there exists a surface orthogonality condition such as

$$\iint_S \Phi^*_{\kappa l}\,\Phi_{pq}\,dS = \delta_{\kappa p}\,\Psi(\kappa,p) \qquad (5.9b)$$

then (5.9a) and (5.9b) together may be used to make the sum in (5.8) finite. Relations (5.9a, b) will certainly be applicable when the container is any configuration for which Laplace's equation is separable, for the basic equation

$$\nabla^2\Phi - \frac{4}{\lambda^2}\frac{\partial^2\Phi}{\partial z^2} = 0$$

obviously reduces to Laplace's equation when the z-coordinate is distorted by the factor $(1-4/\lambda^2)^{1/2} = -i\left|4/\lambda^2 - 1\right|^{1/2}$.

When the oscillation frequency α equals an eigenvalue λ_{pq}, it is clear from (5.8) that resonance occurs unless G is such that $\iint \Phi^*_{pq} G \, dS = 0$. It is not difficult to show that the O(1) response at resonance is restricted to the corresponding mode Φ_{pq}, as we would expect from any linear analysis. In this case the equations may be replaced by a single equation for the determination of the resonant amplitude B_{pq} provided we replace λ_{pq} by the eigenvalue corrected for viscosity, $\lambda_{pq} + R^{-1/2} Im\left(s^{(1)}_{pq}\right)$. (The determination of $s^{(1)}_{pq}$ has already been discussed in a previous section.)

We shall demonstrate this for the simple case of spheroids.

In terms of the eigenfunctions

$$\Phi_{\nu m} \equiv \Phi_{\nu k m} e^{ik\omega} = P^k_\nu(\zeta) P^k_\nu(\mu) e^{ik\omega}, \quad k = 0, 1, 2, \ldots \qquad (5.10)$$

where (ζ, μ) are certain transformations of the polar coordinate variables (r,z), see Chapter 4, equation (5.8) reduces to the following:

$$(\alpha - \lambda_{\kappa k l})\left[D_{\kappa k l} B_{\kappa k l} - k \sum_{\substack{m=1 \\ m \neq l}}^{N_{\kappa k}} (\alpha - \lambda_{\kappa k m}) E_{\kappa k m l} B_{\kappa k m} \right] = \frac{\alpha^2}{4} H_{\kappa k l} \qquad (5.11)$$

where $l = 1, 2, \ldots N_{\kappa k}$.

Here $N_{\kappa k}$ is the (finite) number of eigenvalues determined by the boundary condition, and

$$D_{\kappa k l} = \frac{\alpha + \lambda_{\kappa k l}}{\lambda^2_{\kappa k l}} \int_{r=0}^{1} \int_{z=-b\sqrt{1-r^2}}^{b\sqrt{1-r^2}} \frac{1}{b}\left(\frac{\partial \Phi_{\kappa k l}}{\partial z}\right)^2 r\,dr\,dz \;-\; \frac{k\alpha}{2\lambda_{\kappa k l}}\left\{P_\kappa^k(\zeta_l)\right\}^2 \int_{-1}^{1}\left(P_\kappa^k(\mu)\right)^2 d\mu\,,$$

$$E_{\kappa k m l} = \frac{P_\kappa^k(\zeta_l)\,P_\kappa^k(\zeta_m)}{\lambda_{\kappa k l} + \lambda_{\kappa k m}} \cdot \frac{1}{2\kappa + 1} \cdot \frac{(\kappa + k)!}{(\kappa - k)!}\,,$$

$$H_{\kappa k l} = \int_{-1}^{1} \Phi_{\kappa k l}\, e^{-ik\omega}\, G_k \left\{1 + \varepsilon\mu^2\right\}^{1/2} d\mu\,.$$

The spheroid is defined by

$$r^2 + \frac{z^2}{b^2} = 1$$

and $\varepsilon \equiv 1/b^2 - 1$. ζ_m is the value of ζ on the surface, namely

$$\zeta_m = \lambda_{\kappa k m}\left\{4 + \varepsilon\left(4 - \lambda^2_{\kappa k m}\right)\right\}^{-1/2}.$$

G_k indicates that only that Fourier component of G associated with index k is to be taken. The surface orthogonality of the associated Legendre functions $P_\kappa^k(\mu)$,

$$\int_{-1}^{1} P_\kappa^k(\mu)\, P_\nu^k(\mu)\, d\mu = \frac{2}{2\kappa + 1} \cdot \frac{(\kappa + k)!}{(\kappa - k)!}\, \delta_{\kappa k}$$

has been used to obtain $E_{\kappa k m l}$, as indicated formally by (5.9a, b).

To complete the proof that oscillation at a resonant frequency $\lambda_{\kappa k j}$ excites only the corresponding mode $\Phi_{\kappa k j}$, we introduce a few definitions. If the κ, k subscript notation is suppressed $\left(B_{\kappa k l} \equiv B_l, etc.\right)$ and

$$B_j = R^{1/2} B_j^{(0)} + B_j^{(1)}; \qquad \alpha - \lambda_j = \beta_j^{(1)} + R^{-1/2} \beta_j^{(2)}; \quad (\alpha - \lambda_j)(\alpha - \lambda_l) = \gamma_{jl}^{(1)} + R^{-1/2} \gamma_{jl}^{(2)} + \dots,$$

then (5.11) becomes

$$O(R^{1/2}): \qquad \beta_l^{(1)} \left\{ D_l B_l^{(0)} - k \sum_{m \neq l} E_{ml} \beta_m^{(1)} B_m^{(0)} \right\} = 0 \qquad (5.12)$$

$$O(1): \quad \beta_l^{(1)} \left\{ D_l B_l^{(1)} - k \sum_{m \neq l} E_{ml} \beta_m^{(1)} B_m^{(1)} \right\} = \frac{\alpha^2}{4} H_l - \left\{ \beta_l^{(2)} D_l B_l^{(0)} - k \sum_{m \neq l} \gamma_{lm}^{(2)} E_{ml} B_m^{(0)} \right\}. \quad (5.13)$$

At resonance, i.e., $\alpha = \lambda_j$, $\beta_j^{(1)} = 0$, (5.12) becomes a system of $N_\kappa - 1$ homogeneous equations for the $N_\kappa - 1$ coefficients $B_m^{(0)}$, $m \neq j$. These coefficients must therefore vanish:

$$B_m^{(0)} = 0, \quad m \neq j \qquad (5.14)$$

We determine the resonant amplitude $B_j^{(0)}$ by putting $l = j$ in (5.13). Then

$$\frac{\alpha^2}{4} H_j = D_j \beta_j^{(2)} B_j^{(0)} \qquad (5.15)$$

where $\beta_j^{(2)} = -Re\left(s_{\kappa l j}^{(1)}\right)$.

Equations (5.14) and (5.15) show that in a spheroid the O(1) response to forced oscillation at an eigenfrequency

$\lambda_{\kappa k j}$ is restricted to the corresponding mode $\Phi_{\kappa k j}$, and this completes the proof. A similar proof can be given whenever the basic system (5.8) reduces to a finite system as discussed above.

As a particular application of these results we consider a spheroid oscillating at a resonant frequency about its mass center and calculate the amplitude of the excited normal mode.

The oscillation of the surface shall be represented as

$$\vec{Q}(r,z)\,e^{i(k\omega+\alpha t)} = \vec{\psi} \times \vec{r} \tag{5.16}$$

where in polar coordinates (r, ω, z)

$$\vec{\psi} = (0, 0, 1)\cdot e^{ik\omega + i\alpha t} \tag{5.17}$$

and

$$\vec{Q} = (0, r, 0) \tag{5.18}$$

(This oscillation is referred to the <u>rotating</u> reference frame. In addition to the forced oscillation, the spheroid is rotating with dimensionless angular velocity $\hat{z} = (0,0,1)$ measured in <u>inertial</u> space.)

Now, for any surface oscillation of the form

$$\vec{Q}(r,z)\cdot e^{i(k\omega+\alpha t)}$$

the forcing function $H_{\kappa k l}$ in (5.11), which represents the small mass flux from the interior into the boundary layers, can be shown to be

$$H_{\kappa k l} = \iint_S \Phi^*_{\kappa k l}\left(\frac{4-\alpha^2}{i\alpha} \int_0^\infty \hat{n} \times (\hat{n}\times\nabla)\cdot\tilde{\vec{q}}\, d\zeta \right) dS . \qquad (5.19)$$

The Ekman layer velocity $\tilde{\vec{q}}$ is given by

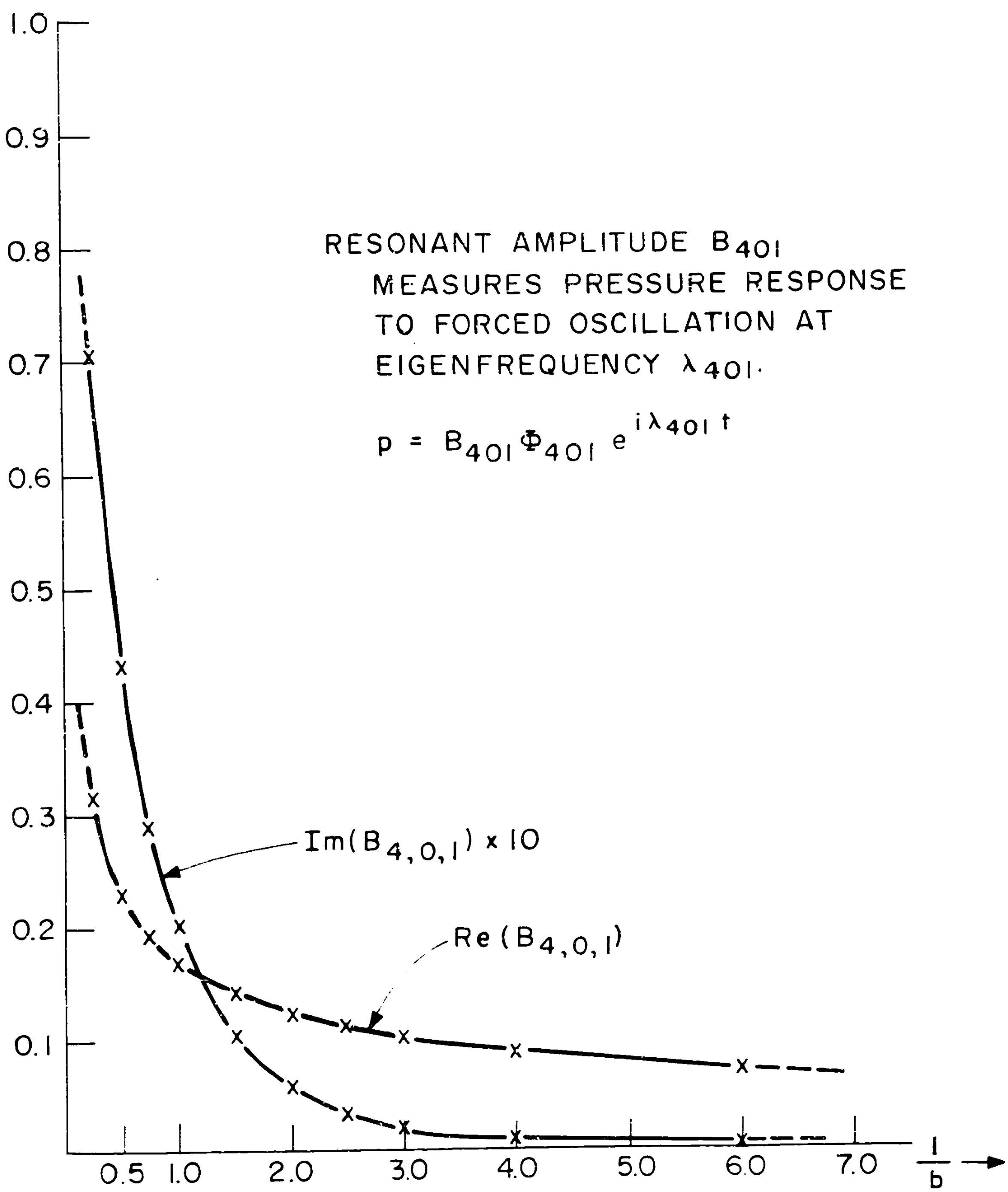
$$\tilde{\vec{q}} = \tfrac{1}{2}\left(\vec{Q} - i\hat{n}\times\vec{Q}\right)\exp -\sqrt{p_+}\,\zeta \;+\; \tfrac{1}{2}\left(\vec{Q} + i\hat{n}\times\vec{Q}\right)\exp -\sqrt{p_-}\,\zeta , \quad p_\pm = i(\alpha \pm 2\hat{n}\cdot\hat{k}).$$

Therefore, once we introduce the explicit formulation of the eigenfunctions from Chapter 4, the system of equations (5.11) will be completely specified. Equation (5.15) can then be used to determine the resonant amplitudes.

Calculations have been made for a few representative modes. These show that the real and imaginary parts (and hence the magnitudes) of the resonant amplitudes are monotonic functions of spheroidal ellipticity. The larger the radius-to-height ratio, the smaller the induced response.

The amplitudes depicted in the graphs are those associated with normalized eigenfunctions. The normalization is such that the kinetic energy of each mode within the container is unity,

$$\int_V \left(\vec{Q}_{\kappa kj}\cdot\vec{Q}^*_{\kappa kj}\right) dV = \frac{1}{4-\lambda^2_{\kappa kj}} \int_V \left\{ \left|\nabla\Phi_{\kappa kj}\right|^2 + \frac{4}{\lambda^2_{\kappa kj}}\left|\frac{\partial\Phi_{\kappa kj}}{\partial z}\right|^2 \right\} dV = 1 .$$



RESONANT AMPLITUDE $B_{\nu k m}$ VS $\frac{1}{b}$ FOR SPHEROID $r^2 + \frac{z^2}{b^2} = 1$

FIG. 3

RESONANT AMPLITUDES B_{nkm} FOR FORCED

OSCILLATION OF THE SPHEROIDS $r^2 + \frac{z^2}{b^2} = 1.$

n,k,m	$\frac{1}{b}$	Inviscid Eigenvalue	$Re(B_{nkm})$	$Im(B_{nkm})$
4,0,1	.25	0.4232	0.3157	0.07064
	.50	0.7947	0.2283	0.04330
	.75	1.089	0.1907	0.02915
	1.0	1.309	0.1682	0.02028
	1.5	1.585	0.1404	0.01041
	2.0	1.732	0.1229	0.005707
	4.0	1.922	0.0880	0.000897
3,1,1	.25	0.4000	2.484	0.9168
	.50	0.9045	0.5922	0.2074
	.75	1.277	0.2845	0.08516
	1.0	1.510	0.1930	0.05111
	1.5	1.745	0.1319	0.03157
	2.0	1.847	0.1086	0.02596
	4.0	1.959	0.0762	0.02050
5,1,1	.25	0.8126	0.2584	0.1017
	.50	1.415	0.06713	0.02307
	.75	1.682	0.05533	0.01802
	1.0	1.806	0.1733	0.05673
	1.5	1.908	0.1667	0.05716
	2.0	1.947	0.1490	0.05309
	4.0	1.986	0.02782	0.01057

CHAPTER 6

ON THE MOTION OF A LIQUID IN A PRECESSING SPHEROID

As a further application of the modal analysis presented in previous sections, we examine the time-dependent motion of a viscous incompressible fluid which wholly fills a precessing, rotating spheroidal container. This problem has also been investigated recently by K. Stewartson and P. H. Roberts (1963, 1965) by a different method.

In the present analysis, the precession problem is considered as a boundary-value problem only. Initial conditions are ignored in deriving a particular solution. The initial-value problem, modified to include the value of the particular solution at the initial instant, can then be solved by Fourier synthesis of the inviscid modes. Our main concern here, however, is to show how the modal analysis picks out the appropriate O(1) interior solution for spheroids, and to note that the special case of a sphere may be dealt with by the same methods. The problem of synthesizing arbitrary initial conditions among the various modes has been resolved by Greenspan (1965) under quite general conditions.

Physically, the precession problem may be stated as follows. The spheroid S and the fluid-filled interior are in uniform rigid rotation, $\vec{\omega} \times \vec{r}$, about the symmetry axis of S. At time $t = 0$ there is imparted to the spheroid a small retrograde angular velocity $\vec{\Omega}$ about a spatially fixed axis which is inclined to the symmetry axis through an arbitrary, constant angle α . We suppose that this motion is started impulsively, so that $\vec{\Omega}$ is an absolute constant, and we want to determine the ensuing time-dependent motion.

We shall consider the equations of motion in a reference frame $\mathcal{E}$ fixed in the precessing, rotating spheroidal shell. $\mathcal{E}$ therefore rotates with angular velocity $\vec{\omega}+\vec{\Omega}$ relative to axes fixed in space and instantaneously coinciding with $\mathcal{E}$. (This procedure differs from that used by Stewartson and Roberts. They chose a reference frame in which both $\vec{\omega}$ and $\vec{\Omega}$ are absolute constants. While there are certain advantages to using such a coordinate system, it is not the natural frame in which to measure departures from solid body rotation $(\vec{\omega}+\vec{\Omega})\times\vec{r}$, and it is not convenient for the formulation of boundary conditions except for axisymmetric containers.)

If $\vec{u}$ is the fluid velocity referred to axes taken in $\mathcal{E}$, then the dimensional equations of motion are

$$\left\{\frac{D}{Dt}+(\vec{\omega}+\vec{\Omega})\times\right\}\left\{\vec{u}+(\vec{\omega}+\vec{\Omega})\times\vec{r}\right\} = -\frac{1}{\rho}\nabla p + \nu\Delta\vec{u} \qquad (6.1)$$

$$\nabla\cdot\vec{u} = 0 , \qquad (6.2)$$

where $D\vec{r}/Dt = \vec{u}$, $D/Dt = \partial/\partial t + \vec{u}\cdot\nabla$.

The polar coordinates (r,θ,z) in $\mathcal{E}$ are chosen with z directed along the symmetry axis so that

$$\vec{\omega} = (0,0,\omega). \tag{6.3}$$

In this system,

$$\vec{\Omega} = |\vec{\Omega}| \left\{ \sin\alpha\cdot\cos(\theta+\omega t),\ -\sin\alpha\cdot\sin(\theta+\omega t),\ -\cos\alpha \right\} \tag{6.4}$$

and

$$\frac{D(\vec{\omega}+\vec{\Omega})}{Dt} = \frac{D\vec{\Omega}}{Dt} = -(\vec{\omega}\times\vec{\Omega}). \tag{6.5}$$

To linearize (6.1) we neglect squares and products of $\vec{u}$ and $\vec{\Omega}$ in accordance with the assumption that $|\vec{\Omega}|$ is small. The result is

$$\frac{\partial\vec{u}}{\partial t} + 2(\vec{\omega}\times\vec{u}) = -\nabla P + \nu\Delta\vec{u} + (\vec{\omega}*\vec{\Omega})\times\vec{r}, \tag{6.6}$$

where $P = p/\rho - \frac{1}{2}\left\{(\vec{\omega}+\vec{\Omega})\times\vec{r}\right\}^2$.

If we introduce the transformations

$$t\rightarrow\omega^{-1}t,\quad \vec{r}\rightarrow L\vec{r},\quad \vec{u}\rightarrow|\vec{\Omega}|L\vec{u},\quad P\rightarrow|\vec{\Omega}|\omega L^2 P,$$

where L is the semi-major axis of the spheroid, equation (6.6) is put into the nondimensional form,

$$\frac{\partial\vec{u}}{\partial t} + 2\hat{k}\times\vec{u} = -\nabla P + R^{-1}\Delta\vec{u} + (\hat{k}\times\vec{\Psi})\times\vec{r}. \tag{6.7}$$

Here $R^{-1} \equiv \nu/(\omega L^2) \ll 1$. Moreover,

$$\vec{\psi} = \{\sin\alpha \cdot \cos(\theta+t), \ -\sin\alpha \cdot \sin(\theta+t), \ -\cos\alpha\} \tag{6.8}$$

$$(\hat{k} \times \vec{\psi}) \times \vec{r} = \mathrm{Re}\left(e^{i(\theta+t)} \sin\alpha \{ z, iz, -r\} \right) \tag{6.9}$$

The O(1) interior solution to (6.7) must also satisfy

$$\nabla \cdot \vec{u} = 0 \tag{6.10}$$

and the boundary condition

$$\vec{u} \cdot \hat{n} = 0 \quad \text{on } S. \tag{6.11}$$

For later usage we record the unit outward normal $\hat{n}$ on S:

$$\hat{n} = \left\{ \frac{(1+\varepsilon)\mu^2}{1+\varepsilon\mu^2} \right\}^{1/2} \left(\frac{r}{(1+\varepsilon)z}, \ 0, \ 1 \right) \tag{6.12}$$

where $\mu^2 \equiv (1+\varepsilon)z^2$ on $r^2 + (1+\varepsilon)z^2 = 1$.

In the interior region, we shall of course neglect the viscous terms in (6.7). (Boundary layer suction induces a secondary interior circulation which is small compared to the O(1) solution.) For the inviscid solution ($R = \infty$) we write

$$\vec{u} = \vec{u}_p + \vec{v} \tag{6.13}$$

Then the two problems to be solved are

$$\left.\begin{aligned} \frac{\partial \vec{u}_p}{\partial t} + 2\hat{k}\times\vec{u}_p &= \mathrm{Re}\left(e^{i(\theta+t)} \sin\alpha \left\{ z, iz, -r \right\} \right) \\ \nabla\cdot\vec{u}_p &= 0 \end{aligned}\right\} \tag{6.15}$$

and

$$\left.\begin{aligned} \frac{\partial \vec{v}}{\partial t} + 2\hat{k}\times\vec{v} &= -\nabla P \\ \nabla\cdot\vec{v} &= 0 \\ \text{with} \quad \hat{n}\cdot\vec{v} &= -\hat{n}\cdot\vec{u}_p \quad \text{on } S. \end{aligned}\right\} \tag{6.16}$$

A solution to (6.15) is

$$\vec{u}_p = -\sin\alpha \left\{ z\sin(\theta+t),\ z\cos(\theta+t),\ r\sin(\theta+t) \right\} \tag{6.17}$$

so that

$$\hat{n}\cdot\vec{u}_p = -\left\{ \frac{(1+\varepsilon)\mu^2}{1+\varepsilon\mu^2} \right\}^{1/2} \left(\frac{2+\varepsilon}{1+\varepsilon} \right) r\sin\alpha\cdot\sin(\theta+t) \tag{6.18}$$

On putting

$$P = \mathrm{Re}\left\{ \varphi(r,z)\, e^{i(\theta+t)} \right\}$$

we can determine the particular solution to (6.16) from the equations

$$\left.\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \varphi}{\partial r} - \frac{1}{r^2}\varphi - 3\frac{\partial^2 \varphi}{\partial z^2} = 0 \\ &\text{with} \\ &r\frac{\partial \varphi}{\partial r} - 3(1+\varepsilon) z \frac{\partial \varphi}{\partial z} + 2\varphi = 3(2+\varepsilon) rz \sin\alpha \\ &\text{on } S. \end{aligned}\right\} \tag{6.19}$$

Clearly one such solution is

$$\left.\begin{aligned} \varphi &= -\frac{2+\varepsilon}{\varepsilon} rz \sin\alpha \\ \vec{v} &= \frac{2+\varepsilon}{\varepsilon}\sin\alpha \cdot \operatorname{Re}\left(e^{i(\theta+t)}\{iz, -z, -ir\}\right) \end{aligned}\right\} \tag{6.20}$$

Now, using the theory of Chapter 5, we are obliged to look for a series solution

$$\varphi = \sum B_{\nu km}\Phi_{\nu km} = \sum B_{\nu km} P_\nu^k(\zeta_{\nu km}) P_\nu^k(\mu), \quad k=1. \tag{6.21}$$

However,

$$P_2^1(\zeta_{21m}) P_2^1(\mu) = A_{21m}\, rz ,$$

where A_{21m} has a known dependence on the eigenvalues, and $P_2^1(\zeta_{21m})$ is constant on S. Therefore all the inhomogeneous terms in the system of equations (5.8) which determines $B_{\nu km}$ vanish unless $\nu = 2$ by virtue of the orthogonality relation among the Legendre functions:

$$\iint_S P_\nu^1(\zeta_{\nu 1m}) P_\nu^1(\mu) \left\{ \frac{(1+\varepsilon)\mu^2}{1+\varepsilon\mu^2} \right\}^{1/2} rz\, dS \quad \propto \int_{-1}^{1} P_\nu^1(\mu) P_2^1(\mu)\, d\mu = 0, \; \nu \neq 2.$$

This means that in (6.20) we have already determined the only nontrivial term of the expansion (6.21).

Combining (6.17) with the real part of (6.20), the particular O(1) solution to the inviscid precession problem is

$$\left. \begin{aligned} \vec{u} &= \frac{-2\sin\alpha}{\varepsilon} \left\{ z(1+\varepsilon)\sin(\theta+t),\; z(1+\varepsilon)\cos(\theta+t),\; -r\sin(\theta+t) \right\} \\ P &= \frac{-(2+\varepsilon)}{\varepsilon}\, rz \sin\alpha \cdot \cos(\theta+t) \end{aligned} \right\} \quad (6.22)$$

The vorticity of this flow has constant magnitude and no vertical component,

$$\operatorname{curl} \vec{u} = \frac{2(2+\varepsilon)}{\varepsilon} \sin\alpha \left\{ \cos(\theta+t),\; -\sin(\theta+t),\; 0 \right\}.$$

Moreover, curl (curl $\vec{u}$) $\equiv 0$. Therefore (6.22) is an exact solution to the viscous equations of motion, though it fails to satisfy the no-slip condition on S.

It is important to note that for the spheroid S, the eigenfunction with $e^{i\theta}$ dependence is

$$\vec{Q} = \left\{ \frac{z}{2-\lambda},\; \frac{iz}{2-\lambda},\; \frac{-r}{\lambda} \right\} e^{i(\theta+\lambda t)} \qquad (6.23)$$

where

$$\lambda \equiv \frac{2(1+\varepsilon)}{2+\varepsilon} . \tag{6.24}$$

For a sphere $(\varepsilon = 0)$ it follows that $\lambda = 1$ and

$$\vec{Q}\Big|_{\varepsilon=0} = \{z, iz, -r\} e^{i(\theta+t)} . \tag{6.25}$$

Therefore in this special case the force $(\hat{k} \times \vec{\Psi}) \times \vec{r}$ (see (6.7), (6.9)) is in resonance with the natural mode (6.25). On the basis of the analysis presented thus far, this is the reason that (6.22) is invalid when $\varepsilon \to 0$.

Now a precessing spherical boundary cannot directly influence the interior fluid motion except through viscous effects. The correct O(1) inviscid solution when $\varepsilon = 0$ must remain solid-body rotation around the initial $(t = 0)$ rotation axis. This means that we cannot let $\varepsilon \to 0$ after putting $R = \infty$ without some further modification to the analysis.

When we chose ω^{-1} for the typical time scale, we made the tacit assumption that $|\vec{\Omega}| / |\vec{\omega}|$ is negligible. The proper time scale, however, is not ω^{-1}, but $(\omega + \Omega)^{-1}$, as Greenspan has shown. For then

$$(\hat{k} \times \vec{\Psi}) \times \vec{r} = \frac{\Omega \sin\alpha}{\omega + \Omega} \operatorname{Re}\left(\{z, iz, -r\} e^{i(\theta + \sigma t)} \right) \tag{6.26}$$

in dimensionless variables, where

$$\sigma \equiv \frac{\omega}{\omega + \Omega} .$$

Even when $\varepsilon = 0$, (6.26) is not in resonance with (6.25), and the solution may be shown to be

$$\vec{u} = \frac{(1-\sigma)\sin\alpha}{\sigma - \lambda} \left\{ \lambda z, \ i\lambda z, \ -(2-\lambda) r \right\} e^{i(\theta + \sigma t)}, \qquad (6.27)$$

where λ is given by (6.24). Therefore,

$$\lim_{\varepsilon \to 0} \left(\vec{u} \right) = -\sin\alpha \left\{ z, \ iz, -r \right\} e^{i(\theta + \sigma t)} . \qquad (6.28)$$

This is in fact the original solid-body rotation referred to the moving $\mathcal{E}$ - axes.

The solution (6.28) shows that the linear theory is capable of producing the correct inviscid solution when $\varepsilon = 0$, provided the three parameters $\varepsilon, R, \Omega/\omega$ are dealt with properly. The order in which one takes the limits $\varepsilon \to 0$, $R \to \infty$, $\Omega/\omega \longrightarrow 0$ forms a critical phase of the analysis and may restrict the range of validity of the solution.

CHAPTER 7

VERTICAL SIDE-WALLS: THE CYLINDER

We determine what modifications are necessary to the general theory for containers with vertical sidewalls by considering a right circular cylinder, $r \leq r_0$, $|z| \leq 1$. ($r_0 = A/H$, where A is the dimensional radius, 2H the dimensional height of the cylinder.)

This section is in two parts: In the first we consider the inertial motions ($\lambda_m \neq 0$), deriving the eigenfunctions and eigenvalue relation. Through numerical computations, the effect of viscosity and of variable r_0 on these solutions is illustrated. In the second part we consider nonaxisymmetric geostrophic motion $(\lambda_m = 0)$. This was not treated in detail for the spheroid because for such a configuration the general theory of geostrophic modes developed by Greenspan is entirely applicable. However, containers of constant height (and therefore with vertical sidewalls) are a special case which must be considered separately from the general theory. We shall see that in a cylinder these modes exhibit an unexpected structure: The boundary layers at $r = r_0$, in reducing the interior tangential velocities to zero, have a net $O(R^{-1/4})$ outflow which drives an interior motion on the same order of magnitude. Jacobs (1964), in a study of the Taylor column problem for viscous steady flow in a rotating cylindrical annulus, found that the same order of magnitude efflux occurs at the walls of the Taylor column.

The relatively large order secondary circulation is not present in the axisymmetric geostrophic modes, and indeed is not predicted by the Ekman layer theory presented in Chapter 2. We shall see that it exists solely to satisfy mass conservation requirements.

<u>The Inertial Modes</u>

The basic eigenvalue problem for the pressure

$$p_{mk} = \varphi_{mk}(r,z)\exp\{i(k\omega + \lambda_{mk}t)\}, \quad \lambda_{mk} \neq 0$$

is (see equation (2.6))

$$\left.\begin{aligned} &\left\{\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{k^2}{r^2} + \left(1 - \frac{4}{\lambda_{mk}^2}\right)\frac{\partial^2}{\partial z^2}\right\}\varphi_{mk} = 0 \\ \text{with} \qquad &\frac{\partial \varphi_{mk}}{\partial z} = 0 \quad \text{on } z = \pm 1 \\ &\left\{\frac{\partial}{\partial r} + \frac{2k}{\lambda_{mk} r}\right\}\varphi_{mk} = 0 \quad \text{on } r = r_0 . \end{aligned}\right\} \tag{7.1}$$

By the method of separation of variables, we find

$$
\left.\begin{aligned}
\varphi_{mk} &= J_k(\alpha_{mk} r)\cos m\pi z \\
u_{mk} &= \frac{-i\cos m\pi z}{4-\lambda_{mk}^2}\left(\alpha_{mk}\lambda_{mk}J_k'(\alpha_{mk}r) + \frac{2k}{r}J_k(\alpha_{mk}r)\right) \\
v_{mk} &= \frac{\cos m\pi z}{4-\lambda_{mk}^2}\left(2\alpha_{mk}J_k'(\alpha_{mk}r) + \frac{\lambda_{mk}k}{r}J_k(\alpha_{mk}r)\right) \\
w_{mk} &= \frac{m\pi\sin m\pi z}{i\lambda_{mk}}J_k(\alpha_{mk}r) \\
\text{where} \quad \alpha_{mk}^2 &= (4-\lambda_{mk}^2)m^2\pi^2/\lambda_{mk}^2 .
\end{aligned}\right\} \quad (7.2)
$$

The eigenvalues λ_{mk} satisfy the transcendental equation determined by the boundary condition at $r = r_o$ (which is the condition that u_{mk} vanish there):

$$(\alpha_{mk}r)J_k'(\alpha_{mk}r) + \frac{2k}{\lambda_{mk}}J_k(\alpha_{mk}r) = 0 \quad at \ r=r_0 . \qquad (7.3a)$$

By the definition of α_{mk} we have that

$$\frac{2}{\lambda_{mk}} = \frac{\alpha_{mk}}{|\alpha_{mk}|}\left|\left(\frac{\alpha_{mk}r_0}{m\pi r_0}\right)^2 + 1\right|^{1/2} . \qquad (7.3b)$$

Substitution for $2/\lambda_{mk}$ in (7.3a) then gives a form suitable for computation,

$$\frac{d}{dx}J_k(x) + k\left\{\left(\frac{1}{m\pi r_0}\right)^2 + \frac{1}{x^2}\right\}^{1/2}J_k(x) = 0, \quad x = \alpha_{mk}r_0 . \qquad (7.3c)$$

By finding the roots of (7.3c) we can then use (7.3b) to determine the eigenvalues. (Comparable calculations have been made by Fultz (1958) for axisymmetric motions.) It is clear from (7.3b, c) that λ_{mk} depends on the product $m \cdot r_o$. Keeping this product fixed while varying m and r_o will result in the same eigenfrequency for given k. Note further that when k = 0 the above analysis still holds except that the eigenvalue relation is simply

$$J_1(\alpha_{mk} r_o) = 0 . \qquad (7.3d)$$

In this case both U_{mk} and V_{mk} vanish at the side wall.

In general, because of (7.3), $\{ U_{mk}, V_{mk}, W_{mk} \}$ are given at $r = r_o$ by

$$U_{mk} = 0; \quad V_{mk} = \frac{-k \cos m\pi z}{\lambda_{mk} r_o} J_k(\alpha_{mk} r_o); \quad W_{mk} = \frac{-i m\pi \sin m\pi z}{\lambda_{mk}} J_k(\alpha_{mk} r_o). \qquad (7.4)$$

The vertical and horizontal sidewalls cause a modification in the general formula for the viscous correction factor $s_{mk}^{(1)}$. This stems from the altered mass outflow condition at $r = r_o$ which in turn is dependent upon the boundary layer solution given by equation (2.11)'. However, on the O(1) time scale, this modification is quantitative, not qualitative, for the boundary layer thickness is still $O(R^{-1/2})$, as it is in the Ekman layers.

A closed analytic expression for $s_{mk}^{(1)}$ is derived in Appendix C on the basis of certain well-known properties of the Bessel functions. This evaluation enables us to compare the relative effects of the two boundary layers on the correction factor

Let I_1 denote the contribution to $s_{mk}^{(1)}$ from the boundary layer at $r = r_o$, and I_2 the corresponding contribution from the Ekman layers at $z = \pm 1$. It is shown in Appendix C that

$$I_1 = \int_{-1}^{1} dz \left\{ \left| \vec{Q}_{mk} \right|^2 \cdot \left(1 + \frac{i\lambda_{mk}}{|\lambda_{mk}|} \right) \cdot \left| \frac{\lambda_{mk}}{2} \right|^{1/2} \right\}_{r = r_0}$$

$$= \left(\frac{J_k(\alpha_{mk} r_0)}{\lambda_{mk}} \right)^2 \cdot \frac{k^2 + m^2\pi^2 r_0^2}{r_0^2} \cdot \left(1 + \frac{i\lambda_{mk}}{|\lambda_{mk}|} \right) \cdot \left| \frac{\lambda_{mk}}{2} \right|^{1/2} \tag{7.5}$$

$$I_2 = \frac{1}{2} \int_0^{r_0} r\,dr \left\{ \left| \frac{\lambda_{mk}}{2} + 1 \right|^{1/2} \cdot \left| \vec{Q}_{mk} - i\hat{k} \times \vec{Q}_{mk} \right|^2 (1 + i) + \right.$$
$$\left. + \left| \frac{\lambda_{mk}}{2} - 1 \right|^{1/2} \cdot \left| \vec{Q}_{mk} + i\hat{k} \times \vec{Q}_{mk} \right|^2 (1 - i) \right\}_{z = 1}$$

$$= \left(\frac{J_k(\alpha_{mk} r_0)}{\lambda_{mk}} \right)^2 \cdot \frac{4 - \lambda_{mk}^2}{2\sqrt{2}} \left[\left(k^2 + m^2\pi^2 r_0^2 - \frac{2k\lambda_{mk}}{2 - \lambda_{mk}} \right) \left(\frac{1 + i}{\{2 + \lambda_{mk}\}^{3/2}} \right) + \right.$$
$$\left. + \left(k^2 + m^2\pi^2 r_0^2 - \frac{2k\lambda_{mk}}{2 + \lambda_{mk}} \right) \left(\frac{1 - i}{\{2 - \lambda_{mk}\}^{3/2}} \right) \right] \tag{7.6}$$

and

$$s_{mk}^{(1)} = \frac{-\left(1 - \frac{\lambda_{mk}^2}{4}\right)\cdot\left\{ I_1 + I_2 \right\}}{\left(\frac{J_k(\alpha_{mk} r_0)}{\lambda_{mk}}\right)^2 \cdot \left(k^2 + m^2\pi^2 r_0^2 - \frac{k\lambda_{mk}}{2}\right)} \tag{7.7}$$

Equations (7.5), (7.6) show that as r_o is increased, the sidewall contribution to the viscous correction factor diminishes with respect to that of the Ekman layers in the ratio $1/r_o^2$:

$$\frac{\mathrm{Re}(I_1)}{\mathrm{Re}(I_2)} \sim \frac{1}{r_0^2}\left\{ \frac{\left(1 - |\lambda_{mk}/2|\right)^{1/2}}{1 - k/(k^2 + m^2\pi^2 r_0^2)} \right\}, \quad r_0 \gg 1 . \tag{7.8}$$

Calculations show that $\mathrm{Re}(s_{mk}^{(1)})$ is sensibly independent of (m,k) except in the neighborhood of $r_o \approx 1$, the values rapidly approaching 0 or $-\infty$ as $r_o \to \infty$ or 0, respectively. In addition, the eigenvalues approach $|\lambda| = 2$ asymptotically for large r_o, and go to zero linearly with r_o.

Thus in the two extreme cases the modes exhibit somewhat unexpected behavior: all the modes are nearly geostrophic $(\lambda \approx 0)$ and are quickly damped for narrow, elongated cylinders; and they tend to persist beyond times of order $R^{1/2}$ for wide, flat cylinders, with frequencies near the critical value $|\lambda| = 2$.

The expectation is that this behavior is essentially independent of (m,k); indeed, in the extreme case $r_o = \infty$

it is known that $\lambda_{mk} = \pm 2$ are the only permissible eigenvalues (corresponding to $s_{mk}^{(1)} = 0$).

In the intermediate range $r_o = O(1)$ the frequency spectrum is probably denumerably infinite in the range $0 < |\lambda| < 2$. Certainly this is true when k = 0, for the eigenvalue equation is then simply $J_1(\alpha_{m1} r_o) = 0$. Now it is known that the equation

$$x J_k'(x) + H(k) J_k(x) = 0 ,$$

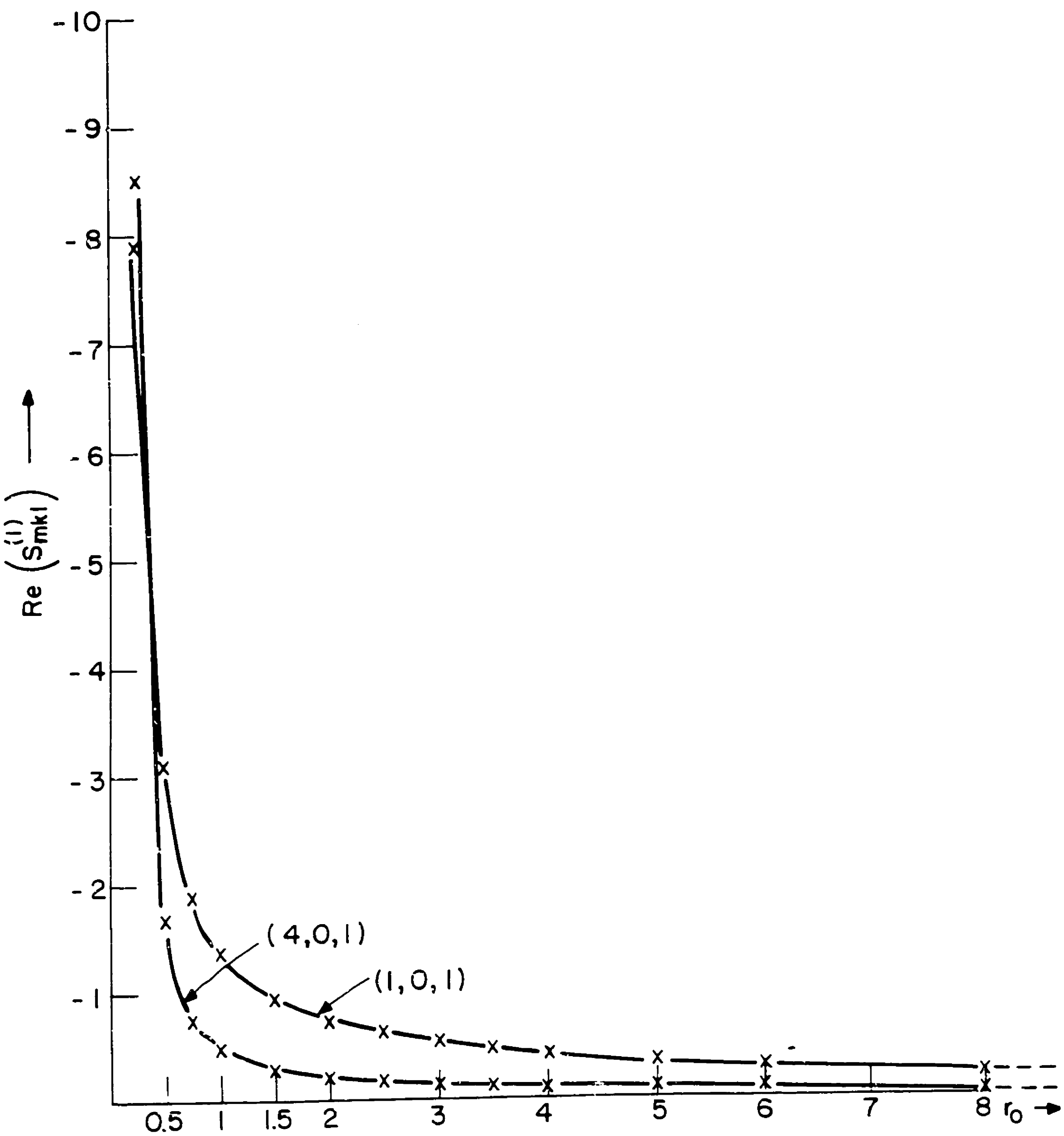
where H(k) is a real constant, has an infinite number of real roots. This is approximately our equation, when $m^2 \pi^2 r_o^2$ is large, and this is the motivation for the above conjecture, for the case $k \neq 0$.

For the important case of axisymmetric motions (k = 0) the explicit formulas (7.7) and (7.5), (7.6) are greatly simplified and this bonus allows us further insight into the nature of $s_{mk}^{(1)}$. We readily see that

$$\operatorname{Re}(s_{mk}^{(1)})\Big|_{k=0} = \frac{-(4-\lambda_{mk}^2)}{4\sqrt{2}} \left\{ \frac{|\lambda_{mk}|^{1/2}}{r_o^2} + \frac{1}{2}\left(\frac{2-\lambda_{mk}}{(2+\lambda_{mk})^{1/2}} + \frac{2+\lambda_{mk}}{(2-\lambda_{mk})^{1/2}} \right) \right\} \qquad (7.9)$$

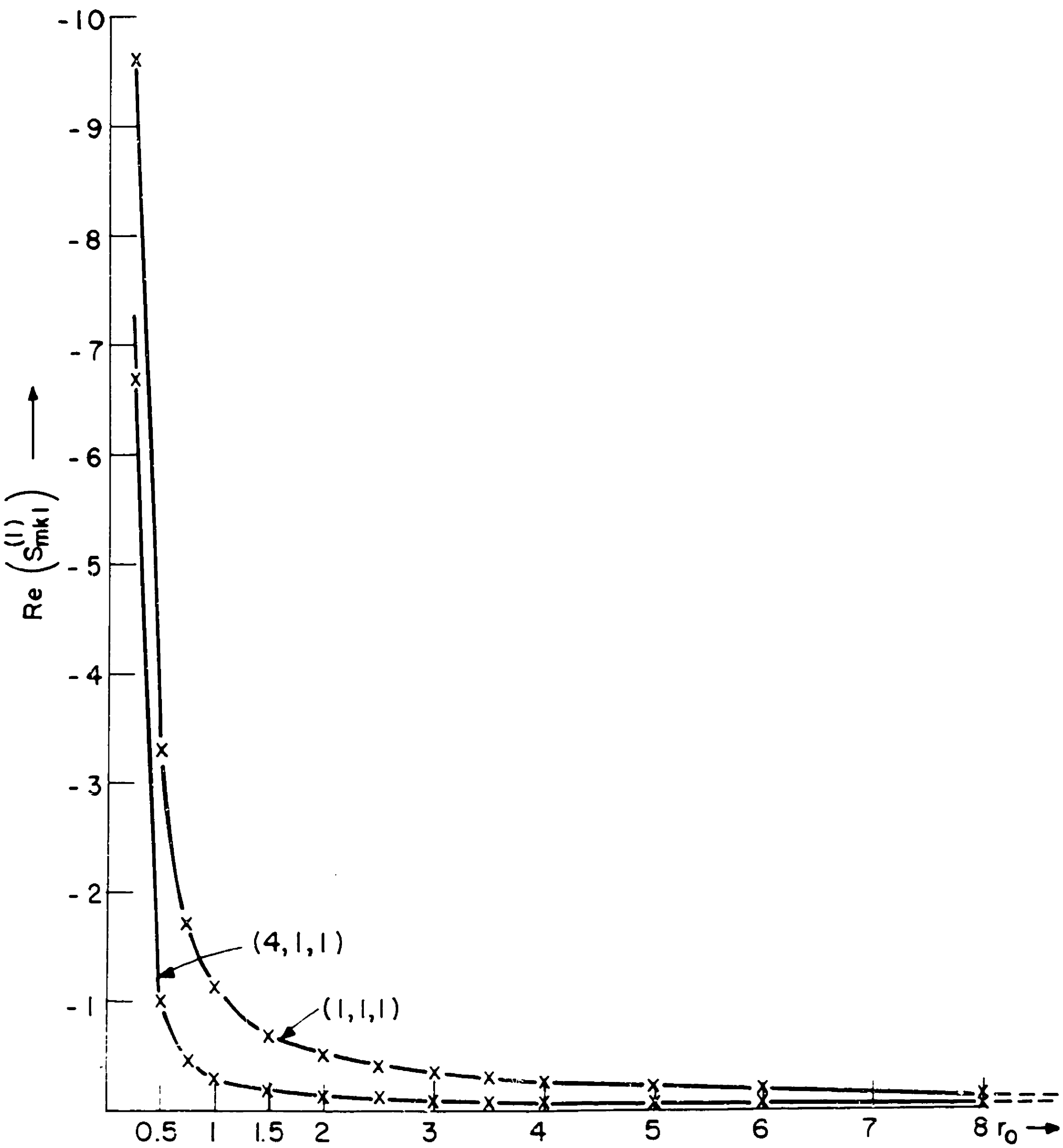
and knowing the λ_{mk} vs. r_o behavior from our numerical results, we have that

$$\left.\begin{aligned} \operatorname{Re}(s_{mk}^{(1)})\Big|_{k=0} &\sim \frac{-1}{\sqrt{2}\, r_o^{3/2}} \quad \text{as} \quad r_o \to 0,\ |\lambda_{mk}| \propto r_o ; \\ \operatorname{Re}(s_{mk}^{(1)})\Big|_{k=0} &\sim -\left\{1 - \left|\frac{\lambda_{mk}}{2}\right|\right\}^{1/2} \quad \text{as} \quad r_o \to \infty,\ |\lambda_{mk}| \to 2. \end{aligned}\right\} \qquad (7.10)$$



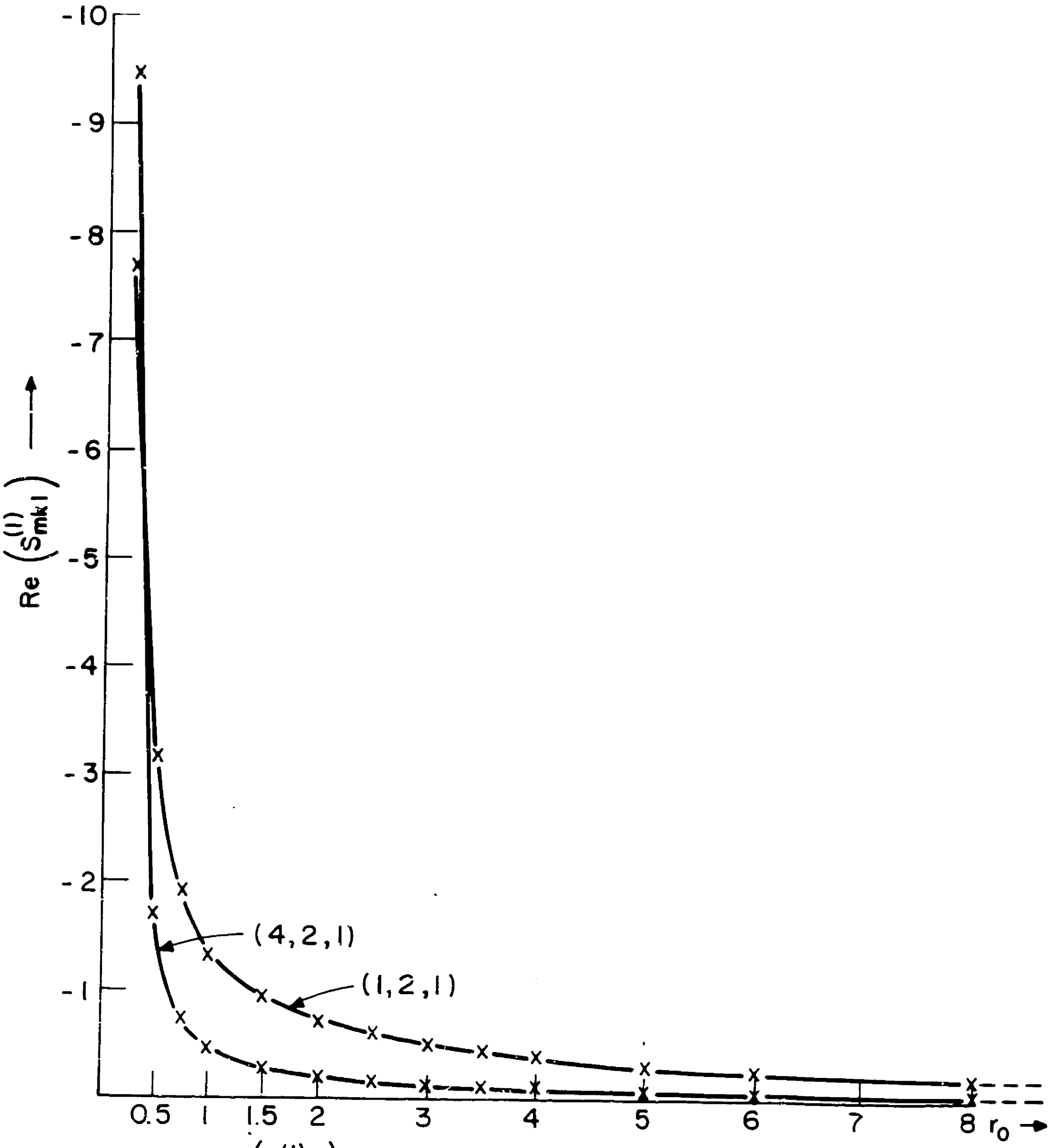
Re $\left(S^{(1)}_{mkl}\right)$ VS r_0 FOR CYLINDER $r \leq r_0$, $|z| \leq 1$

FIG. 4(a)



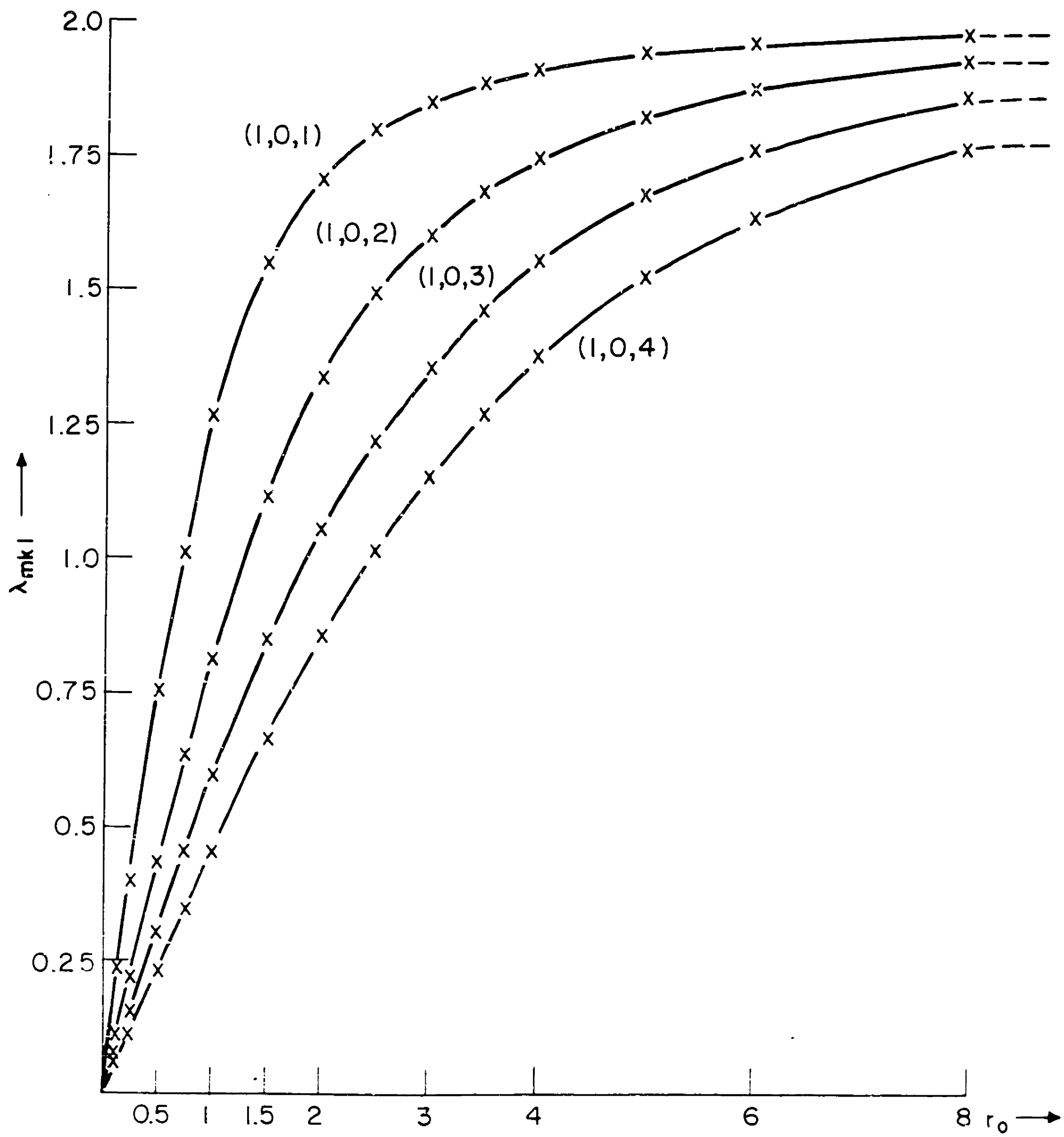
$Re\left(S^{(1)}_{mkl}\right)$ VS r_0 FOR CYLINDER $r \leq r_0$, $|z| \leq 1$

FIG. 4(b)



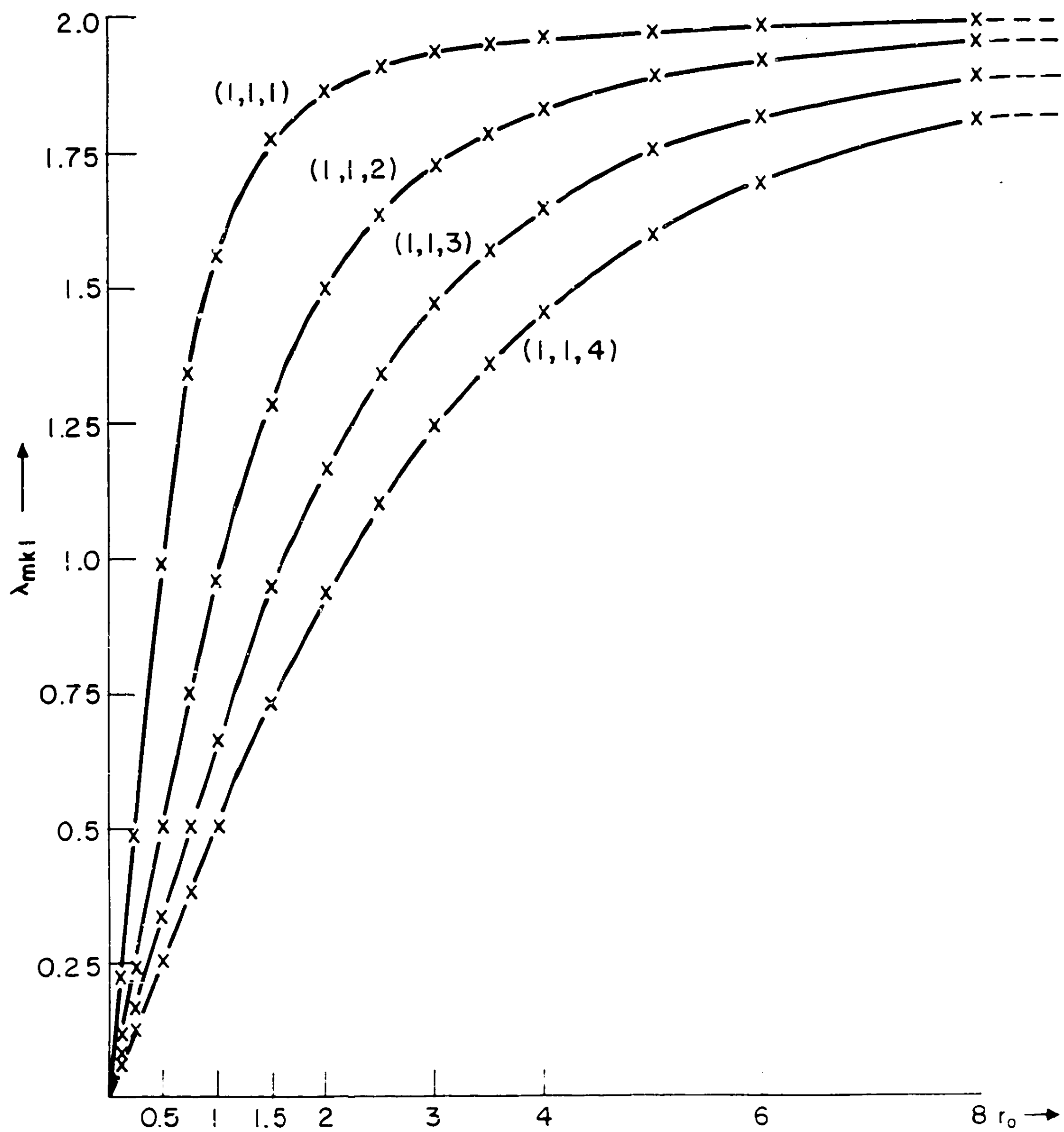
Re $\left(S^{(1)}_{mk1}\right)$ VS r_0 FOR CYLINDER $r \leq r_0$, $|z| \leq 1$

FIG. 4(c)



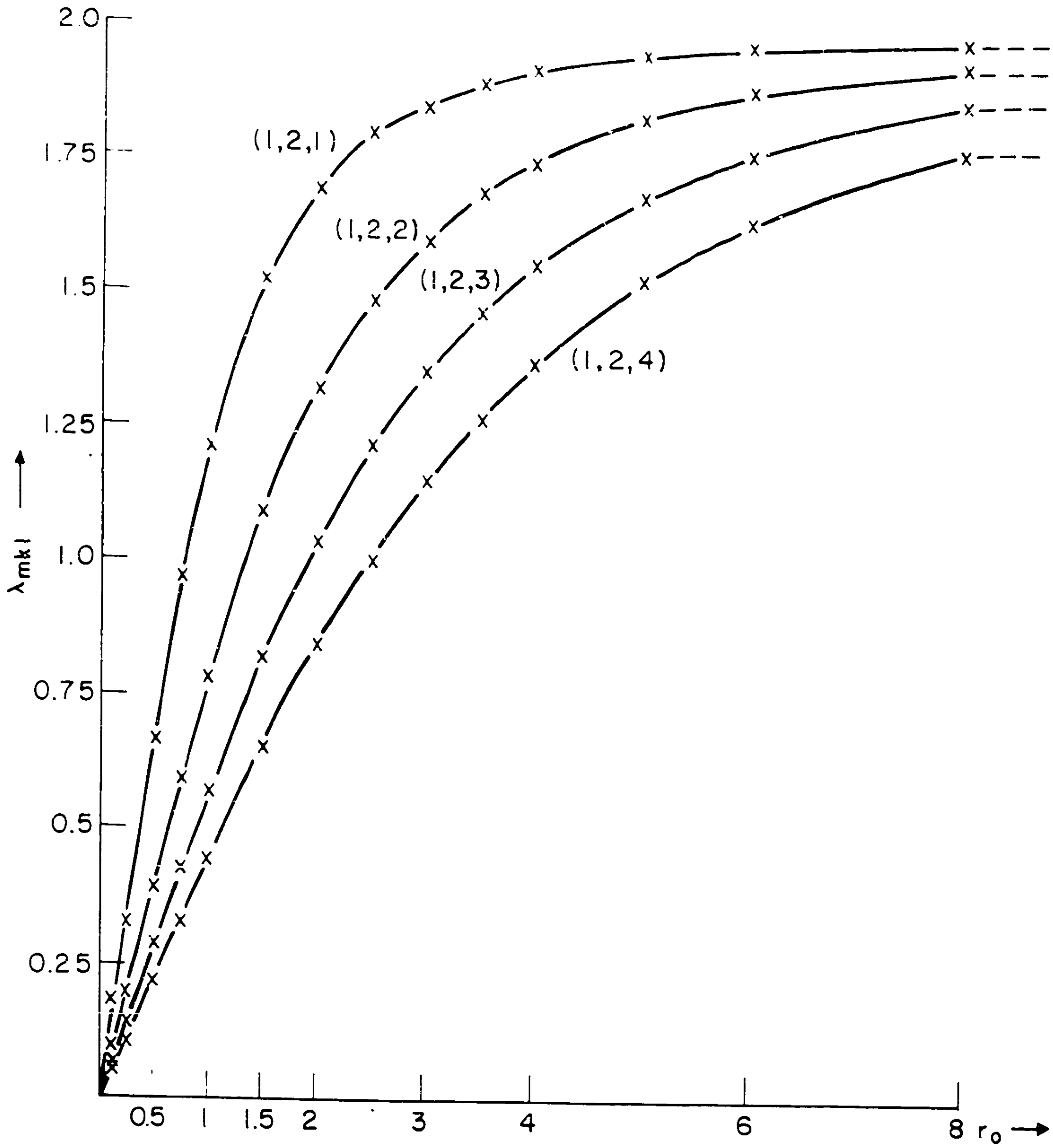
λ_{mkl} VS r_0 FOR CYLINDER $r \leq r_0$, $|z| \leq 1$

FIG. 5(a)



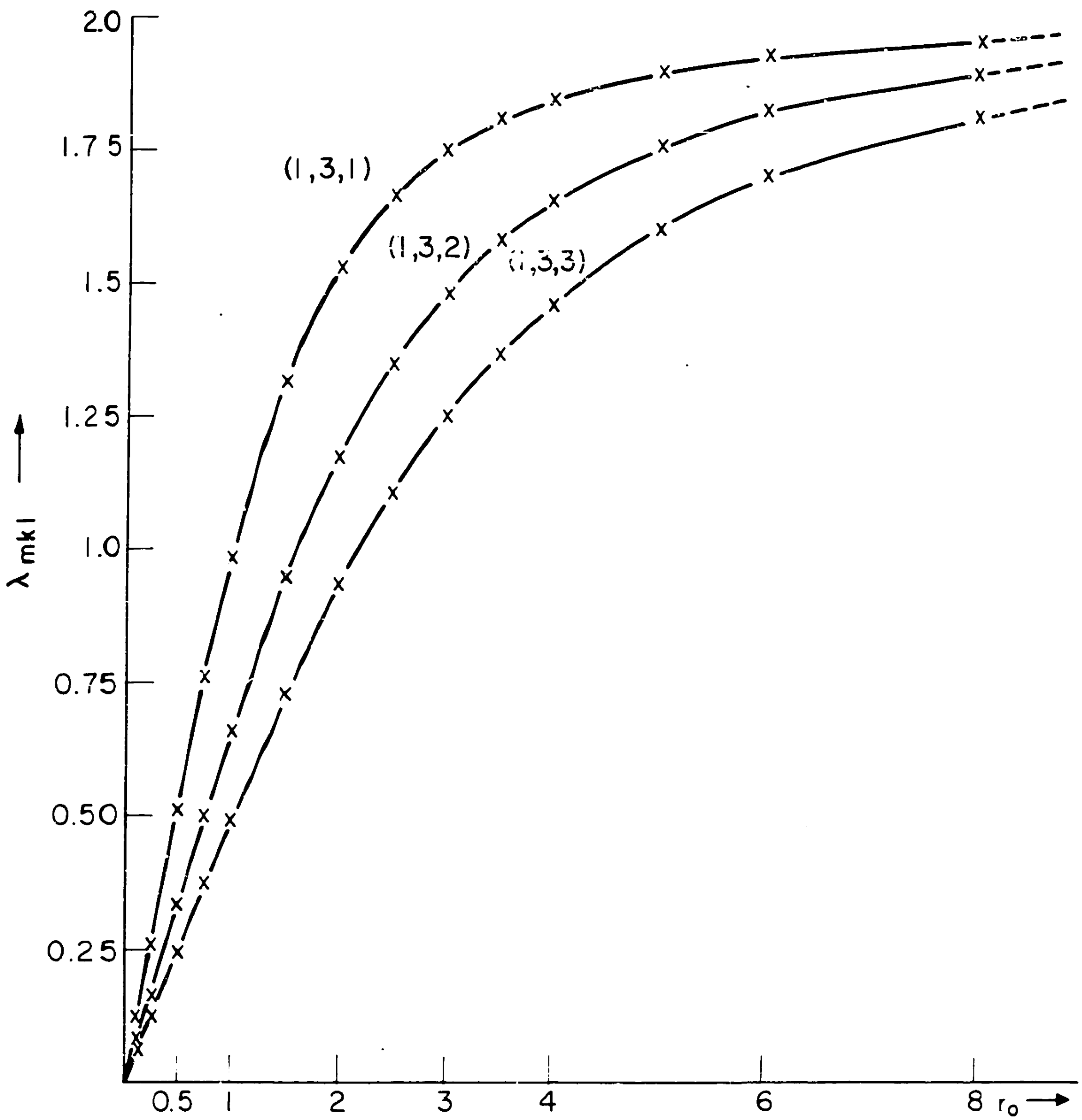
λ_{mkl} VS r_o FOR CYLINDER $r \leq r_o$, $|z| \leq 1$

FIG. 5(b)



λ_{mkl} VS r_0 FOR CYLINDER $r \leq r_0$, $|z| \leq 1$

FIG. 5(c)



λ_{mkl} VS r_o FOR CYLINDER $r \leq r_o$, $|z| \leq 1$

FIG. 5(d)

EIGENVALUES λ_{nkm} AND VISCOUS CORRECTIONS $s^{(1)}_{nkm}$

FOR CYLINDER $\{ r \le r_0 \, , \, |z| \le 1 \}$

n,k,m	r_0	λ_{nkm}	$\mathrm{Re}(s^{(1)}_{nkm})$	$\mathrm{Im}(s^{(1)}_{nkm})$
1,0,1	.25	0.4016	-7.875	-6.586
	.5	0.7586	-3.085	-1.586
	.75	1.048	-1.875	-2.724
	1.0	1.268	-1.369	+0.2459
	2.0	1.708	-0.7342	+0.5799
	4.0	1.913	-0.4056	+0.3926
2,0,1	.25	0.7586	-9.412	-7.913
	.5	1.268	-2.798	-1.183
	.75	1.552	-1.403	+0.08979
	1.0	1.708	-0.9221	+0.3920
	2.0	1.913	-0.4212	+0.3770
	4.0	1.977	-0.2128	+0.2098
3,0,1	.25	1.048	-9.434	-7.741
	.5	1.552	-2.182	-0.6893
	.75	1.758	-1.002	+0.2252
	1.0	1.853	-0.6438	+0.3632
	2.0	1.960	-0.2876	+0.2674
	4.0	1.990	-0.1431	+0.1418
4,0,1	.25	1.268	-8.512	-6.896
	.5	1.708	-1.674	-0.3594
	.75	1.853	-0.7499	+0.2570
	1.0	1.913	-0.4836	+0.3146
	2.0	1.977	-0.2171	+0.2056
	4.0	1.994	-0.1076	+0.1069

EIGENVALUES λ_{mkl} AND VISCOUS CORRECTIONS $s^{(1)}_{mkl}$

FOR THE CYLINDER $\{ r \leq r_0 ,\ |z| \leq 1 \}$

m, k, l	r_o	λ_{mkl}	$Re(s^{(1)}_{mkl})$	$Im(s^{(1)}_{mkl})$
1,1,1	.25	.48982	-9.5968	-8.3959
	.50	.99553	-3.3061	-1.8057
	.75	1.3477	-1.6985	-0.1342
	1.0	1.5616	-1.1188	+0.35179
	1.5	1.7735	-0.69346	0.50158
	2.0	1.8649	-0.51814	0.45158
	4.0	1.9641	-0.26531	0.26074
2,1,1	.25	.99553	-10.742	-9.2418
	.50	1.5616	-2.2335	-0.76297
	.75	1.7735	-0.97235	+0.22268
	1.0	1.8649	-0.61490	0.35482
	1.5	1.9372	-0.37070	0.31564
	2.0	1.9641	-0.27196	0.25410
	4.0	1.9909	-0.13492	0.13378
3,1,1	.25	1.3477	-8.8029	-7.2386
	.50	1.7735	-1.4372	-2.4214
	.75	1.8915	-0.62729	+0.25406
	1.0	1.9372	-0.40484	0.28150
	1.5	1.9715	-0.24777	0.22262
	2.0	1.9839	-0.18216	0.17411
	4.0	1.9959	-0.090178	0.089670
4,1,1	.25	1.5616	-6.6926	-5.2220
	.50	1.8649	-1.0019	-0.032201
	.75	1.9372	-0.45264	-0.23371
	1.0	1.9641	-0.29854	+0.22751
	1.5	1.9839	-1.8528	0.17099
	2.0	1.9909	-0.13662	0.13207
	4.0	1.9977	-0.067680	0.067394

EIGENVALUES λ_{mkl} AND VISCOUS CORRECTIONS $s_{mkl}^{(1)}$

FOR THE CYLINDER $\{ r \leq r_0 \ , \ |z| \leq 1 \}$

m,k,l	r_o	λ_{mkl}	$\mathrm{Re}(s_{mkl}^{(1)})$	$\mathrm{Im}(s_{mkl}^{(1)})$
1,2,1	.25	.32635	-7.6922	-6.5223
	.50	.66724	-3.1710	-1.8026
	.75	.97259	-1.9226	-0.40132
	1.0	1.2151	-1.3887	+0.19229
	1.5	1.5284	-0.94088	0.55797
	2.0	1.6969	-0.73786	0.58784
	4.0	1.9120	-0.40704	0.39513
2,2,1	.25	.66724	-10.039	-8.6709
	.50	1.2151	-3.0055	-1.4246
	.75	1.5284	-1.4558	-0.043052
	1.0	1.6969	-0.93924	+0.38646
	1.5	1.8499	-0.57203	0.44228
	2.0	1.9120	-0.42300	0.37917
	4.0	1.9771	-0.21310	0.21018
3,2,1	.25	.97259	-10.354	-8.8322
	.50	1.5284	-2.3140	-0.81515
	.75	1.7509	-1.0261	+0.21243
	1.0	1.8499	-0.65079	0.36352
	1.5	1.9297	-0.39267	0.33099
	2.0	1.9597	-0.28825	0.26814
	4.0	1.9897	-0.14320	0.14190
4,2,1	.25	1.2151	-9.4730	-7.8920
	.50	1.6969	-1.7448	-0.41909
	.75	1.8499	-0.76105	+0.25326
	1.0	1.9120	-0.48683	0.31533
	1.5	1.9597	-0.29597	0.26042
	2.0	1.9771	-0.21736	0.20591
	4.0	1.9942	-0.10766	0.10694

EIGENVALUES λ_{mkl} AND VISCOUS CORRECTIONS $s^{(1)}_{mkl}$

FOR THE CYLINDER $\{ r \leq r_0 \ , \ |z| \leq 1 \}$

m,k,l	r_o	λ_{mkl}	$\mathrm{Re}(s^{(1)}_{mkl})$	$\mathrm{Im}(s^{(1)}_{mkl})$
1,3,1	.25	.25460	-6.8057	-5.6565
	.50	.51580	-2.9568	-1.6519
	.75	.76436	-1.9223	-0.47811
	1.0	.98417	-1.4654	+0.075388
	1.5	1.3170	-1.0610	0.52400
	2.0	1.5293	-0.86874	0.63195
	4.0	1.8490	-0.51967	0.49741
2,3,1	.25	.51580	-9.0571	-7.7522
	.50	.98417	-3.1956	-1.6548
	.75	1.3170	-1.7153	-0.13023
	1.0	1.5293	-1.1546	+0.34604
	1.5	1.7500	-0.72673	0.51394
	2.0	1.8490	-0.54632	0.47077
	4.0	1.9593	-0.28210	0.27677
3,3,1	.25	.76436	-10.072	-8.6284
	.50	1.3170	-2.8056	-1.2206
	.75	1.6038	-1.3278	+0.11056
	1.0	1.7500	-0.85189	0.38878
	1.5	1.8782	-0.51671	0.41071
	2.0	1.9291	-0.38098	0.34553
	4.0	1.9817	-0.19094	0.18860
4,3,1	.25	.98417	-10.117	-8.5758
	.50	1.5293	-2.2983	-0.79758
	.75	1.7500	-1.0271	+0.21356
	1.0	1.8490	-0.65284	0.36420
	1.5	1.9291	-0.39438	0.33212
	2.0	1.9593	-0.28960	0.26927
	4.0	1.9896	-0.14392	0.14261

EIGENVALUES λ_{mkl} AND VISCOUS CORRECTIONS $s^{(1)}_{mkl}$

FOR THE CYLINDER $\{ r \leq r_0 \, , \, |z| \leq 1 \}$

m,k,l	r_o	λ_{mkl}	$\mathrm{Re}(s^{(1)}_{mkl})$	$\mathrm{Im}(s^{(1)}_{mkl})$
1,4,1	.25	.21130	-6.2496	-5.1179
	.50	.42587	-2.7951	-1.5309
	.75	.63422	-1.8856	-0.49853
	1.0	.82733	-1.4837	+0.003199
	1.5	1.1474	-1.1218	0.46712
	2.0	1.3773	-0.94780	0.62799
	4.0	1.7791	-0.61054	0.57464
2,4,1	.25	.42587	-8.3371	-7.0728
	.50	.82733	-3.1923	-1.7054
	.75	1.1474	-1.8427	-0.25368
	1.0	1.3773	-1.2921	+0.28367
	1.5	1.6462	-0.84513	+0.54845
	2.0	1.7791	-0.64822	0.53696
	4.0	1.9381	-0.34507	0.33670
3,4,1	.25	.63422	-9.5396	-8.1525
	.50	1.1474	-3.0440	-1.4550
	.75	1.4642	-1.5425	-0.0008249
	1.0	1.6462	-1.0130	+0.38061
	1.5	1.8201	-0.62434	0.46902
	2.0	1.8934	-0.46413	0.41064
	4.0	1.9719	-0.23563	0.23199
4,4,1	.25	.82733	-10.027	-8.5399
	.50	1.3773	-2.6694	-1.0936
	.75	1.6462	-1.2479	+0.14564
	1.0	1.7791	-0.79894	0.38623
	1.5	1 8934	-0.48397	0.39081
	2.0	1.9381	-0.35638	0.32540
	4.0	1.9841	-0.17817	0.17613

The Geostrophic Mode

While a general theory for the geostrophic mode has been developed by Greenspan (1965) for containers of arbitrary shape, it is necessary to consider certain special cases separately. One of these is closed containers in which the height h(x,y)--the distance from the bottom surface to the top at any point x,y--is independent of x and y. The cylinder is, of course, the simplest example of such a container.

We recall from Chapter 2 that the Ekman layer thickness is $O(R^{-1/2})$ on the O(1) time scale. This is in fact also true on the geostrophic time scale $t \sim O(R^{1/2})$, as we shall see below. The primary balance is still between Coriolis force and viscous shears which act on tengential velocities of the same order of magnitude.

In the vicinity of the vertical sidewalls $(r \approx r_o)$ however, a different kind of boundary layer is formed. There is an outer boundary layer region of thickness $O(R^{-1/4})$ and an inner one of thickness $O(R^{-1/3})$, each being terminated at the top and bottom by an Ekman layer. The effect of this multiple structure on axisymmetric geostrophic flow in a cylinder has been investigated by Greenspan and Howard (1963). It was shown that viscous diffusion acting through a time $O(R^{1/2})$ affects a region of thickness $O(R^{-1/4})$ near the sidewall, and in so doing creates a fluid vorticity higher than that at the top and bottom surfaces. The result is that fluid is drawn out of the Ekman layers by this high vorticity, creating a relatively large $O(R^{-1/4})$ vertical motion in the $R^{-1/4}$ layer. In this region, however, there do not exist sufficient viscous stresses on the vertical motion to satisfy the no-slip condition at $r = r_0$, and it is only within the thinner $R^{-1/3}$ layer that the vertical velocity is reduced to zero at the wall. Neither layer alone can satisfy both the no-slip condition and the requirement of matching with the interior geostrophic flow.

Much the same physical picture exists in the non-axisymmetric case, with one significant difference: The normal outflow from the side boundary layers is of a higher order than for axisymmetric motions ($O(R^{-1/4})$ vs. $O(R^{-1/2})$). This mass efflux drives an interior circulation of the same order of magnitude. Physically, it is the mass conservation requirement which is at the root of this difference. For within the boundary layer the $O(1)$ azimuthal motion v is a function of angular position ω, and this results in a variable net mass flux across any section perpendicular to the walls. There being no $O(1)$ vertical velocity to absorb this net flow, the only possibility is that it be balanced by the normal gradient of the normal component u :

$$\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \omega} = 0 .$$

Since $\partial/\partial r$ is $O(R^{1/4})$ in the region, then the normal velocity is $O(R^{-1/4})$. It is of course possible that the balance be a perfect one in the sense that at the outer edge of the $R^{-1/4}$ layer there be no net u. However, it is found that this would require v to remain nonzero right up to the wall, and the inner $R^{-1/3}$ layer does not provide sufficient viscous stresses on the azimuthal motion to compensate for this.

It should also be remarked that since there is no induced interior $O(R^{-1/4})$ vertical flow, a qualitatively similar flow pattern exists in the interior at $O(1)$ and $O(R^{-1/4})$. Both flows are strictly two-dimensional, the $O(1)$ flow satisfying the condition of zero normal velocity at $r = r_0$ and the $O(R^{-1/4})$ flow balancing the small mass efflux caused by the azimuthal variation of the boundary layer velocities. The side boundaries also feed the interior at higher orders of magnitude. In particular, the $O(R^{-1/2})$ interior flow induced by suction into the Ekman layers must match the corresponding mass flux at the side boundary.

We now describe the physical balances that take place in each of these different layers.

The equations of motion on the geostrophic time scale $\tau = R^{-1/2}t$, namely,

$$R^{-1/2}\frac{\partial\vec{q}}{\partial\tau} + 2\hat{k}\times\vec{q} = -\nabla p + R^{-1}\Delta\vec{q}$$

$$\nabla\cdot\vec{q} = 0\ ,$$

can be reduced to a single equation for the pressure,

$$R^{-2}(\nabla^2)^3 p - 2R^{-3/2}(\nabla^2)^2 \frac{\partial p}{\partial \tau} + R^{-1}\nabla^2 \frac{\partial^2 p}{\partial \tau^2} + 4\frac{\partial^2 p}{\partial z^2} = 0 . \qquad (7.11)$$

Near $z = \pm 1$ we stretch the normal coordinate by putting $\zeta = R^A(1 \mp z)$ so that, to lowest order,

$$R^{6A-2}\frac{\partial^6 p}{\partial \zeta^6} - 2R^{4A-\frac{3}{2}}\frac{\partial^4}{\partial \zeta^4}\frac{\partial p}{\partial \tau} + R^{2A-1}\frac{\partial^2}{\partial \zeta^2}\frac{\partial^2 p}{\partial \tau^2} + 4R^{2A}\frac{\partial^2 p}{\partial \zeta^2} = 0.$$

The only choice of A which makes the most highly differentiated term at least comparable in order of magnitude to the other terms is clearly determined by requiring $6A - 2 = 2A$, or $A = \frac{1}{2}$. The equations of motion in the Ekman layer, retaining terms of lowest order, are

$$-2v = -\frac{\partial p}{\partial r} + \frac{\partial^2 u}{\partial \zeta^2}$$

$$+2u = -\frac{1}{r}\frac{\partial p}{\partial \omega} + \frac{\partial^2 v}{\partial \zeta^2}$$

$$0 = \pm R^{1/2}\frac{\partial p}{\partial \zeta} + \frac{\partial^2 w}{\partial \zeta^2}$$

$$\frac{1}{r}\left(\frac{\partial}{\partial r} ru + \frac{\partial}{\partial \omega} v\right) \mp R^{1/2}\frac{\partial w}{\partial \zeta} = 0 .$$

These are steady equations, not the same as one obtains on the O(1) time scale. After the first few revolutions in which the Ekman layers are established, there is essentially a steady-state balance between viscous shear and Coriolis force. (The horizontal pressure gradient which appears in these equations reflects the superposition of the interior solution on the boundary layer perturbation quantities; p is essentially constant through the layer.)

Near $r = r_o$ the stretched normal coordinate $\rho = R^A(r-r_o)$ transforms (7.11) into

$$R^{6A-2}\frac{\partial^6 p}{\partial\rho^6} - 2R^{4A-\frac{3}{2}}\frac{\partial^4}{\partial\rho^4}\frac{\partial p}{\partial\tau} + R^{2A-1}\frac{\partial^2}{\partial\rho^2}\frac{\partial^2 p}{\partial\tau^2} + 4\frac{\partial^2 p}{\partial z^2} + \dots = 0,$$

so that now A is determined from the possible balances

$$6A-2 = 4A-3/2\,, \quad 6A-2 = 2A-1, \quad 6A-2 = 0.$$

Clearly, $A = \frac{1}{2}$ is not a solution to these equations. The classical Ekman layer does not exist on the vertical sidewalls for the geostrophic motions. Two other possibilities do exist, however: $A = \frac{1}{3}$ and $A = \frac{1}{4}$.

With the choice $A = \frac{1}{3}$, the pressure equation becomes

$$\frac{\partial^6 p}{\partial\rho^6} + 4\frac{\partial^2 p}{\partial z^2} = 0.$$

The physical meaning of this balance is seen more clearly from the equations of motion. For, with $\rho = R^{1/3}(r-r_o)$ these equations are, to lowest order,

$$\frac{\partial w}{\partial z} = -\frac{1}{2}\frac{\partial^3 v}{\partial \rho^3} , \qquad \frac{\partial v}{\partial z} = \frac{1}{2}\frac{\partial^3 w}{\partial \rho^3} .$$

The large gradients of tangential velocity across the $R^{-1/3}$ layer (viscous shear forces) are in balance with the vertical shears caused by vortex line stretching. The tangential components v and w are both of the same order of magnitude in the $R^{-1/3}$ layer.

With the choice $A = \frac{1}{4}$, a different type of boundary layer becomes possible, one in which the primary balance is essentially geostrophic. The pressure equation,

$$R^{6A-2}\frac{\partial^6 p}{\partial \rho^6} - 2R^{4A-\frac{3}{2}}\frac{\partial^4}{\partial \rho^4}\frac{\partial p}{\partial \tau} + R^{2A-1}\frac{\partial^2}{\partial \rho^2}\frac{\partial^2 p}{\partial \tau^2} + 4\frac{\partial^2 p}{\partial z^2} + \ldots = 0 ,$$

becomes, to lowest order,

$$\frac{\partial^2 p}{\partial z^2} = 0 .$$

The corresponding equations of motion, with $(u,v,w,p) = (R^{-1/4}U, V, R^{-1/4}W, R^{-1/4}P)$, are

$$2V = \frac{\partial P}{\partial \rho}$$

$$2U = -\frac{1}{r}\frac{\partial P}{\partial \omega}$$

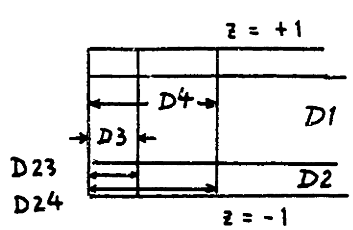
These equations reveal the unusual fact that to lowest order the mechanics of the $R^{-1/4}$ layer is entirely inviscid: The layer is too thick for viscosity to affect the lowest order motion. The region more closely resembles a shear layer of the type considered by Proudman (1956) and Stewartson (1957) than it does a simple boundary layer.

Having now discussed the overall structure of the sidewall layers, we turn to a more explicit formulation of the problem. To describe the different flow regions succinctly, we use the variables introduced by Greenspan and Howard. Let ρ, η, ζ be the stretched normal coordinates given by

$$r = r_0 + R^{-1/4}\rho = r_0 + R^{-1/3}\eta, \qquad \zeta = R^{1/2}(1 \mp z);$$

the relevant domains are

$$\begin{aligned}
D1 &: \ r < r_0, \quad |z| < 1 \\
D2 &: \ r < r_0, \quad \zeta = O(1) \\
D3 &: \ \eta = O(1), \quad |z| < 1 \\
D4 &: \ \rho = O(1), \quad |z| < 1 \\
D23 &: \ \eta = O(1), \quad \zeta = O(1) \\
D24 &: \ \rho = O(1), \quad \zeta = O(1)
\end{aligned}$$



For each domain we determine the solutions as asymptotic expansions and complete the solution by matching the expansions with those of adjacent domains. The results for the six different regions are summarized below; the detailed analysis on which these results depend is contained in Appendix D. (By symmetry the six regions cover the entire cylinder except for an additional corner region where $\zeta = O(1)$, $r - r_0 = O(R^{-1/2})$. No analysis is attempted for this region because of the mathematical difficulty of the equations, but it is not likely that the flow there alters the basic result concerning the $R^{-1/4}$ circulation.)

Of principal interest is the interior circulation. If

$$p_1 = p_1^{(0)} + R^{-1/4} p_1^{(1)} + R^{-1/2} p_1^{(2)} + \dots$$

denotes the interior pressure, we shall show that

$$\nabla^2 p_1^{(0)} = \left(\nabla^2 p_1^{(0)}\right)_{\tau=0} \cdot \exp\{-\tau\}$$

$$\text{with} \qquad \frac{\partial p_1^{(0)}}{\partial \omega} = 0 \quad \text{at} \quad r = r_0 .$$

It is important to note that $\nabla^2 p_1^{(0)} = 2\hat{k} \cdot \operatorname{curl} \vec{q}_1^{(0)}$, twice the vertical component of vorticity. It is this vorticity component which must be prescribed at time $\tau = 0$, not the pressure $p_1^{(0)}$, for otherwise we would not have a properly-posed problem.

A similar boundary-value problem for $p_1^{(1)}$ will also be derived:

$$\nabla^2 p_1^{(1)} = \left(\nabla^2 p_1^{(1)}\right)_{\tau=0} \cdot \exp\{-\tau\}$$

$$\text{with} \qquad \frac{\partial p_1^{(1)}}{\partial \omega} = 2\sqrt{\frac{\tau}{\pi}}\,\frac{\partial^2 p_1^{(0)}}{\partial r \partial \omega} \quad \text{at } r = r_0 .$$

In addition the $O(R^{-1/2})$ solution $\vec{q}_1^{(2)}$ has been shown by Greenspan to be

$$\vec{q}_1^{(2)} = \frac{z}{2}\frac{\partial}{\partial \tau}\nabla\times\vec{q}_1^{(0)} + \frac{1}{2}\frac{\partial}{\partial \tau}\hat{k}\times\vec{q}_1^{(0)} + \nabla\times\vec{B}(r,\omega,\tau) ,$$

with

$$\nabla p_1^{(2)} = -\left\{2\hat{k}\times\vec{q}_1^{(2)} + \frac{\partial}{\partial \tau}\vec{q}_1^{(0)}\right\} .$$

(In this notation the subscript identifies the domain in which the function is defined and the superscript relates the functions to its relative position in the asymptotic expansion $\vec{q} = \sum_k R^{-k/4}\vec{q}^{(k)}$.)

The three-component vector $\nabla\times\vec{B}(r,\omega,\tau)$ must be compatible with the mass flux conditions at the walls, but it is not completely determinate without consideration of higher order terms in the expansions.

At $z = \pm 1$, the mass flux condition has been shown to be

$$w_1^{(2)} = \frac{-z}{|z|}\hat{k}\cdot\nabla\times\vec{q}_i^{(0)} \quad .$$

From this it follows that

$$\hat{k}\cdot\nabla\times\vec{B} = 0$$

or, with $\vec{B} = (B_1, B_2, B_3)$,

$$\frac{\partial}{\partial r} rB_2 = \frac{\partial}{\partial \omega} B_1 \; .$$

For the axisymmetric modes, this means that $B_2 \equiv 0$.

Also, as we shall determine below, the mass flux condition at $r = r_o$ is

$$u_1^{(2)} = \frac{1}{2} v_1^{(0)} + \frac{\tau}{2}\frac{\partial u_1^{(0)}}{\partial r} + 2\sqrt{\frac{\tau}{\pi}}\left\{\frac{\partial u_1^{(1)}}{\partial r} + \frac{u_1^{(1)}}{r}\right\} .$$

This demands that

$$\frac{1}{r_0}\left(\frac{\partial B_3}{\partial \omega}\right)_{r=r_0} = \left\{\frac{\tau}{2}\frac{\partial u_1^{(0)}}{\partial r} + 2\sqrt{\frac{\tau}{\pi}}\left(\frac{\partial u_1^{(1)}}{\partial r} + \frac{u_1^{(1)}}{r}\right)\right\}_{r=r_0} ,$$

a condition which is identically satisfied when $\partial/\partial\omega \equiv 0$.

We now summarize the results in the six domains.

Region D1: The Interior

With $\vec{q} \equiv (u, v, w) = \vec{q}_1^{(0)}(\vec{r}, \tau) + R^{-1/4}\vec{q}_1^{(1)} + R^{-1/2}\vec{q}_1^{(2)} + \ldots$

$$p = p_1^{(0)} + R^{-1/4}p_1^{(1)} + R^{-1/2}p_1^{(2)} + \ldots$$

$$\tau = R^{-1/2}t,$$

the solutions are, with $k = 0, 1$:

$$u_1^{(k)} = -\frac{1}{2r}\frac{\partial p_1^{(k)}}{\partial \omega} \quad ; \quad p_1^{(k)} = p_1^{(k)}(r, \omega, \tau)$$

$$v_1^{(k)} = +\frac{1}{2}\frac{\partial p_1^{(k)}}{\partial r}$$

$$w_1^{(k)} = 0$$

where $p_1^{(0)} = p_1^{(0)}(r, \omega, \tau)$ is the interior $O(1)$ geostrophic pressure. It is clear from this solution that both the $O(1)$ and $O(R^{-1/4})$ circulations are strictly two-dimensional (the Taylor-Proudman theorem).

$O(R^{-1/2})$ solution :

$$u_1^{(2)} = -\frac{1}{2r}\frac{\partial p_1^{(2)}}{\partial \omega} - \frac{1}{2}\frac{\partial v_1^{(0)}}{\partial \tau}$$

$$v_1^{(2)} = +\frac{1}{2}\frac{\partial p_1^{(2)}}{\partial r} + \frac{1}{2}\frac{\partial u_1^{(0)}}{\partial \tau}$$

$$w_1^{(2)} = \frac{-z}{2r}\frac{\partial}{\partial \tau}\left\{\frac{\partial}{\partial r} r v_1^{(0)} - \frac{\partial}{\partial \omega} u_1^{(0)}\right\}.$$

The solution for the vertical component $w_1^{(2)}$ shows the effect of vorticity changes. Interior fluid is drawn vertically into the Ekman layers at $z = \pm 1$ where it is then converted into an $O(1)$ tangential boundary layer velocity. This is one of the primary transient mechanisms to establish the new steady state.

Region D2: The Ekman Layers near z = ± 1

Now write for the total $\vec{q}$ as seen in the boundary layers the expansion

$$\vec{q} = \vec{q}_2^{(0)} + R^{-1/4}\vec{q}_2^{(1)} + R^{-1/2}\vec{q}_2^{(2)} + \ldots$$

Then

$$u_2^{(k)} = u_1^{(k)}\left\{1 - e^{-\zeta}\cos\zeta\right\} - v_1^{(k)}e^{-\zeta}\sin\zeta$$

$$v_2^{(k)} = v_1^{(k)}\left\{1 - e^{-\zeta}\cos\zeta\right\} + u_1^{(k)}e^{-\zeta}\sin\zeta$$

$$w_2^{(k)} = 0$$

$$w_2^{(k+2)} = \pm\frac{1}{r}\int_0^{\zeta}\left(\frac{\partial}{\partial r}ru_2^{(k)} + \frac{\partial}{\partial\omega}v_2^{(k)}\right)d\zeta$$

$$= \mp\frac{1}{2r}\left(1 - e^{-\zeta}\left\{\cos\zeta + \sin\zeta\right\}\right)\left(\frac{\partial}{\partial r}rv_1^{(k)} - \frac{\partial}{\partial\omega}u_1^{(k)}\right)$$

where k = 0,1 and the interior functions (subscript "1") are evaluated at the boundaries z = ± 1. Now $w_2^{(2)}$, the $O(R^{-1/2})$ normal velocity in the Ekman layer, is finite as $\zeta \longrightarrow \infty$. Therefore this net normal flow must be matched by the corresponding $O(R^{-1/2})$ interior w, as we expect from the general Ekman layer analysis of Chapter 2:

$$w_2^{(2)}\Big|_{\zeta=\infty} = w_1^{(2)}\Big|_{z=\pm 1} .$$

From this condition, together with the known form of $w_1^{(2)}$, we determine that the O(1) interior pressure is the solution of

$$\frac{\partial}{\partial \tau}\left(\nabla^2 p_1^{(0)}\right) + \nabla^2 p_1^{(0)} = 0 \tag{7.12}$$

or,

$$\nabla^2 p_1^{(0)} = \left(\nabla^2 p_1^{(0)}\right)_{\tau=0} \cdot \exp\{-\tau\} . \tag{7.13}$$

$\left(\nabla^2 p_1^{(0)}\right)_{\tau=0}$, twice the initial vertical component of vorticity, is given. (7.13) must be solved subject to the boundary condition of vanishing normal velocity at $r = r_0$,

$$u_1^{(0)} = -\frac{1}{2r}\frac{\partial p_1^{(0)}}{\partial \omega} = 0 \quad at \quad r = r_0 . \tag{7.14}$$

The interior solution is then completed by computing $v_1^{(0)} = \frac{1}{2}\cdot\frac{\partial p_1^{(0)}}{\partial r}$. In general, $v_1^{(0)}$ will not vanish at $r = r_0$, whereas both $u_1^{(0)}$ and $w_1^{(0)}$ do. The O(1) sidewall boundary layer solution must therefore reduce $v_1^{(0)}$ to zero and meet the conditions of zero radial and vertical components at the outer edge of the layer.

Now the important point is that the O(1) Ekman layer flow induces an $O(R^{-1/2})$ interior secondary circulation. It is also clear that an $O(R^{-1/4})$ Ekman flow would similarly induce an $O(R^{-3/4})$ interior circulation. Since the interior circulation is still essentially inviscid at this order of magnitude, we may expect that the same type of analysis which was used to derive (7.12) will also be valid to derive an equation for $p_1^{(1)}$. The resulting equation is

$$\frac{\partial}{\partial \tau}\left(\nabla^2 p_1^{(1)}\right) + \nabla^2 p_1^{(1)} = 0 \tag{7.15}$$

or,

$$\nabla^2 p_1^{(1)} = \left(\nabla^2 p_1^{(1)}\right)_{\tau=0} \cdot \exp\{-\tau\} = 0. \tag{7.16}$$

Here we assume that initially there is no $O(R^{-1/4})$ interior vertical vorticity, but this assumption is not crucial to the analysis.

Equation (7.16) must be solved subject to an appropriate boundary condition at $r = r_0$: The normal velocity $u_1^{(1)}$ must match that produced by the sidewall region D4,

$$u_1^{(1)}\Big|_{r=r_0} = u_4^{(1)}\Big|_{\rho=-\infty} .$$

This condition will be seen to be

$$u_1^{(1)} = -\frac{1}{2r}\frac{\partial p_1^{(1)}}{\partial \omega} = 2\sqrt{\frac{\tau}{\pi}}\left(-\frac{1}{2r}\cdot\frac{\partial^2 p_1^{(0)}}{\partial r\,\partial \omega}\right) \quad at \quad r = r_0. \tag{7.17}$$

Region D4: The Outer Boundary Layer

With $\vec{q} \equiv (u,v,w) = \vec{q}_4^{\,(0)} + R^{-1/4}\vec{q}_4^{\,(1)} + R^{-1/2}\vec{q}_4^{\,(2)} + \ldots$,
and $\rho = R^{1/4}(r - r_0)$, we find for the O(1) solution:

$$u_4^{(0)} = 0$$

$$v_4^{(0)} = \left[v_1^{(0)}\right] \operatorname{erf}\left\{-\rho\,(2\tau^{1/2})^{-1}\right\}$$

$$w_4^{(0)} = 0$$

where $v_1^{(0)}$, the interior azimuthal velocity component, includes as we know the factor $\exp(-\tau)$. The bracket notation denotes a function evaluated at $r = r_0$, i.e., $\left[f(r,\omega,z,\tau)\right] \equiv f(r_0,\omega,z,\tau)$.

For the $O(R^{-1/4})$ solution, we find

$$u_4^{(1)} = -\frac{1}{r_0}\int_0^\rho \frac{\partial v_4^{(0)}}{\partial \omega}\,d\rho = \left[\frac{\partial u_1^{(0)}}{\partial r}\right]\left\{2\sqrt{\frac{\tau}{\pi}}\left(1 - \exp\left\{\frac{-\rho^2}{4\tau}\right\}\right) + \rho \operatorname{erf}\left\{\frac{-\rho}{2\tau^{1/2}}\right\}\right\}$$

$$v_4^{(1)} = \rho\left[\frac{\partial v_1^{(0)}}{\partial r}\right] + \left[\frac{v_1^{(0)}}{r}\right]\left(\frac{\rho \operatorname{erfc}\left\{\frac{-\rho}{2\tau^{1/2}}\right\}}{2}\right) + \left[v_1^{(1)}\right]\operatorname{erf}\left\{\frac{-\rho}{2\tau^{1/2}}\right\}$$

$$w_4^{(1)} = -\frac{z}{2}\frac{\partial v_4^{(0)}}{\partial \rho} = \frac{z}{2}\left[v_1^{(0)}\right](\pi\tau)^{-1/2}\exp\left\{\frac{-\rho^2}{4\tau}\right\},$$

and the $O(R^{-1/2})$ solution is given by

$$u_4^{(2)} = -\frac{1}{r_0}\int_0^\rho \frac{\partial v_4^{(1)}}{\partial \omega}\,d\rho - \frac{1}{r_0}\rho u_4^{(1)} + \frac{1}{2}v_4^{(0)}$$

$v_4^{(2)}$ NOT DETERMINED

$$w_4^{(2)} = -\frac{z}{2}\left\{ \frac{\partial v_4^{(1)}}{\partial \rho} + \frac{v_4^{(0)}}{r_0} \right\}.$$

We see at once that the vertical components $w_4^{(1)}$ and $w_4^{(2)}$ do not vanish at the wall $\rho = 0$. This also occurred in the axisymmetric solution of Greenspan and Howard, and it is necessary to solve the inner boundary layer equations in order to completely satisfy the no-slip condition.

From these solutions it may be seen that the mass efflux which drives the interior circulation is given by

$$O(R^{-1/4}): \quad u_4^{(1)} \sim 2\sqrt{\frac{\tau}{\pi}}\left[\frac{\partial u_1^{(0)}}{\partial r}\right] \quad \text{as} \quad \rho \to -\infty,$$

$$O(R^{-1/2}): \quad u_4^{(2)} \sim \frac{1}{2}\left[v_1^{(0)}\right] + \frac{\tau}{2}\left[\frac{\partial u_1^{(0)}}{\partial r}\right] + 2\sqrt{\frac{\tau}{\pi}}\left(\left[\frac{\partial u_1^{(1)}}{\partial r}\right] + \left[\frac{u_1^{(1)}}{r}\right]\right).$$

For axisymmetric modes we know that $u_1^{(0)} = \frac{-1}{2r}\frac{\partial p_1^{(0)}}{\partial \omega} \equiv 0$. Therefore the mass efflux at $r = r_0$ is $O(R^{-1/2})$.

For the nonaxisymmetric modes, $u_1^{(0)} \not\equiv 0$. Hence $\partial u_1^{(0)}/\partial r$ does not necessarily vanish at $r = r_0$, and the mass efflux is $O(R^{-1/4})$. A nontrivial $O(R^{-1/4})$ interior circulation which matches this condition at $r = r_0$ must therefore exist.

Region D24: The Ekman Layer Adjacent to D4

Here $$\vec{q} = \vec{q}_{24}^{(0)}(\rho, \omega, \zeta, \tau) + R^{-1/4}\vec{q}_{24}^{(1)} + \ldots$$

$$\zeta = R^{1/2}(1 \mp z), \quad \rho = R^{1/4}(r - r_0).$$

We find:

$O(1)$:
$$u_{24}^{(0)} = -v_4^{(0)} e^{-\zeta} \sin\zeta$$
$$v_{24}^{(0)} = v_4^{(0)}\left\{1 - e^{-\zeta}\cos\zeta\right\}$$
$$w_{24}^{(0)} = 0$$

$O(R^{-1/4})$:
$$u_{24}^{(1)} = u_4^{(1)}\left\{1 - e^{-\zeta}\cos\zeta\right\} - v_4^{(1)} e^{-\zeta}\sin\zeta$$
$$v_{24}^{(1)} = v_4^{(1)}\left\{1 - e^{-\zeta}\cos\zeta\right\} + u_4^{(1)} e^{-\zeta}\sin\zeta$$
$$w_{24}^{(1)} = -\frac{z}{|z|}\cdot\frac{1}{2}\frac{\partial v_4^{(0)}}{\partial\rho}\left\{1 - e^{-\zeta}(\sin\zeta + \cos\zeta)\right\}$$

$O(R^{-1/2})$:
$u_{24}^{(2)}$, $v_{24}^{(2)}$ NOT DETERMINED
$$w_{24}^{(2)} = -\frac{z}{|z|}\cdot\frac{1}{2}\left(\frac{\partial v_4^{(1)}}{\partial\rho} + \frac{v_4^{(0)}}{r_0}\right)\left(1 - e^{-\zeta}\left\{\sin\zeta + \cos\zeta\right\}\right).$$

These are the typical Ekman layer solutions with the important exception that the vertical outflow which induces further circulation within D4 is $O(R^{-1/4})$. This is, as it must be, consistent with the solution given above for region D4. It bears out an earlier remark that the fluid vorticity in D4 essentially draws fluid out of the Ekman layer, creating a relatively large $O(R^{-1/4})$ vertical motion which does not satisfy the no-slip condition within D4. Also, as $\rho \to -\infty$ there is a net normal $R^{-1/4}$ mass flux corresponding to that found in D4; consequently, a matching $O(R^{-1/4})$ Ekman layer flow is required.

Region D3: The Inner Sidewall Boundary Layer

In this region, $\eta = R^{1/3}(r - r_0)$ and

$$\vec{q} = R^{-1/12}\vec{q}_3^{\,(1)}(\eta,\omega,z,\tau) + R^{-1/4}\vec{q}_3^{\,(3)} + \cdots .$$

The velocity components are determined by Fourier analysis.

$O(R^{-1/12})$:

$$u_3^{(1)} = 0$$

$$v_3^{(1)} = -\left[v_1^{(0)}\right](\pi\tau)^{-1/2}\eta$$

$$w_3^{(1)} = 0$$

$O(R^{-1/4})$:

$$u_3^{(3)} = 0$$

$$v_3^{(3)} = \frac{\left[v_1^{(0)}\right]}{2\tau\,(\pi\tau)^{1/2}}\frac{\eta^3}{3!} - \sum_{n,k}\left\{A_{nk}(r_0,\tau)\,e^{ik\omega}\cdot\cos n\pi z\left(e^{\kappa_n\eta} - f_{nk}(\eta)\right)\right\}$$

$$w_3^{(3)} = \frac{z\left[v_1^{(0)}\right]}{2\,(\pi\tau)^{1/2}} + \sum\left\{A_{nk}(r_0,\tau)\,e^{ik\omega}\cdot\sin n\pi z\left(e^{\kappa_n\eta} + f_{nk}(\eta)\right)\right\}$$

where

$$A_{nk}(r_0,\tau) \equiv \frac{(-1)^n}{2n\pi}\cdot\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\omega}\frac{\left[v_1^{(0)}\right]}{(\pi\tau)^{1/2}}\,d\omega$$

$$f_{nk}(\eta) \equiv \frac{\exp\left(\frac{1}{2}\kappa_n\eta\right)}{n\pi + \frac{ik}{r_0}}\left\{2n\pi\cos\left(\frac{\sqrt{3}}{2}\kappa_n\eta - \frac{\pi}{3}\right) + - \frac{2ik}{\sqrt{3}\,r_0}\sin\left(\frac{\sqrt{3}}{2}\kappa_n\eta - \frac{\pi}{3}\right)\right\}$$

$$\kappa_n \equiv \left|2n\pi\right|^{1/3} .$$

Hence to $O(R^{-1/12})$ the sidewall region D3 has no effect: The $R^{-1/12}$ solution above is just the D4 solution seen in D3. However, the $R^{-1/4}$ components do satisfy the no-slip condition at the wall, $\eta = 0$, and match the corresponding outer boundary layer components at $\eta = -\infty$. The lower order boundary layer solution is thereby completed. It should be noted that a higher order $R^{-7/12}$ mass efflux from D3 induces a comparable circulation in D4.

Region D23: The Corner Ekman Layer Adjacent to D3

The equations in this region (where $\zeta = R^{1/2}(1 \mp z)$, $\eta = R^{1/3}(r - r_0)$ are still of the Ekman layer type through $O(R^{-1/2})$. With

$$\vec{q} = R^{-1/12}\vec{q}_{23}^{\,(1)} + R^{-1/4}\vec{q}_{23}^{\,(3)} + \ldots$$

the solutions are given by

$$O(R^{-1/12}): \quad u_{23}^{(1)} = \frac{[v_1^{(0)}]}{(\pi\tau)^{1/2}}\,\eta\, e^{-\zeta}\sin\zeta$$

$$v_{23}^{(1)} = -\frac{[v_1^{(0)}]}{(\pi\tau)^{1/2}}\,\eta\left(1 - e^{-\zeta}\cos\zeta\right)$$

$$w_{23}^{(1)} = 0$$

$$O(R^{-1/4}): \quad u_{23}^{(3)} = -\left(v_3^{(3)}\right)_{z=1} e^{-\zeta}\sin\zeta$$

$$v_{23}^{(3)} = +\left(v_3^{(3)}\right)_{z=1}\left\{1 - e^{-\zeta}\cos\zeta\right\}$$

$$w_{23}^{(3)} = \frac{z}{|z|}\cdot\frac{[v_1^{(0)}]}{2(\pi\tau)^{1/2}}\left\{1 - e^{-\zeta}(\sin\zeta + \cos\zeta)\right\}.$$

$\left(v_3^{(3)}\right)_{z=1}$ is the azimuthal velocity component in the inner boundary layer region D3, evaluated at the top surface z = 1. By symmetry, it takes the same value at z = +1 and at z = -1.

APPENDIX A

DERIVATION OF FORMULA FOR THE VISCOUS CORRECTION FACTOR $s_m^{(1)}$

It will now be shown that equation (2.15), namely

$$s_m^{(1)} \iiint_V \left(|\nabla\Phi_m|^2 + \frac{4}{\lambda_m^2} |\hat{k}\cdot\nabla\Phi_m|^2 \right) dV = (4-\lambda_m^2) \iint_S \lim_{\substack{s\to s_m \\ R\to\infty}} \left(\Phi_m^* \int_0^\infty \hat{n}\times(\hat{n}\times\nabla)\cdot\tilde{\vec{V}}_m \, d\zeta \right) dS \qquad (A.1)$$

is equivalent to the relation

$$\left.\begin{aligned}
&s_m^{(1)} \iiint_V \left(|\nabla\Phi_m|^2 + \frac{4}{\lambda_m^2} |\hat{k}\cdot\nabla\Phi_m|^2 \right) dV = \frac{-(4-\lambda_m^2)}{2} \iint_S J \, dS \\
&\text{where} \\
&J \equiv \frac{1}{1-(\hat{n}\cdot\hat{k})^2} \left\{ \left| \hat{n}\cdot\hat{k}\times\vec{Q}_m - i\,\hat{k}\cdot\vec{Q}_m \right|^2 \cdot \left(1 + \frac{ip_+}{|p_+|}\right) \cdot |p_+|^{1/2} + \right. \\
&\qquad\qquad \left. + \left| \hat{n}\cdot\hat{k}\times\vec{Q}_m + i\,\hat{k}\cdot\vec{Q}_m \right|^2 \cdot \left(1 + \frac{ip_-}{|p_-|}\right) \cdot |p_-|^{1/2} \right\} \\
&p_\pm = \frac{\lambda_m}{2} \pm \hat{n}\cdot\hat{k}
\end{aligned}\right\} \qquad (A.2)$$

We first show that the surface integral in (A.1) is given by

$$\iint_S \lim \left(\Phi_m^* \int_0^\infty \hat{n}\times(\hat{n}\times\nabla)\cdot\tilde{\vec{V}}_m \, d\zeta \right) dS = \iint_S \left(\nabla\Phi_m^* \cdot \lim \int_0^\infty \tilde{\vec{V}}_m \, d\zeta \right) dS \,. \qquad (A.3)$$

To establish this we need the known form of $\tilde{\vec{V}}_m$ (cf. (2.9)),

$$\tilde{\vec{V}}_m = -\tfrac{1}{2}\left(\vec{Q}_m - i\hat{n}\times\vec{Q}_m\right)\exp\{-\sqrt{\gamma_+}\,\zeta\} - \tfrac{1}{2}\left(\vec{Q}_m + i\hat{n}\times\vec{Q}_m\right)\exp\{-\sqrt{\gamma_-}\,\zeta\} \qquad (A.4)$$

$$\gamma_{\pm} = s \pm i\,(2\hat{n}\cdot\hat{k})$$

Now,

$$\Phi_m^* \,\hat{n}\times(\hat{n}\times\nabla)\cdot\tilde{\vec{V}}_m = -\Phi_m^*\,\hat{n}\cdot\nabla\times\left(\hat{n}\times\tilde{\vec{V}}_m\right)$$

$$= -\hat{n}\cdot\nabla\times\left(\Phi_m^*\,\hat{n}\times\tilde{\vec{V}}_m\right) + \hat{n}\cdot\left\{\nabla\Phi_m^*\times(\hat{n}\times\tilde{\vec{V}}_m)\right\} \qquad (A.5)$$

But by (A.4) and the boundary condition $\hat{n}\cdot\vec{Q}_m = 0$, we must have

$$\hat{n}\cdot\nabla\Phi_m^*\times\left(\hat{n}\times\tilde{\vec{V}}_m\right) = \nabla\Phi_m^*\cdot\tilde{\vec{V}}_m \quad \text{on } S.$$

Using this, we find

$$\Phi_m^*\,\hat{n}\times(\hat{n}\times\nabla)\cdot\tilde{\vec{V}}_m = -\hat{n}\cdot\nabla\times\left(\Phi_m^*\,\hat{n}\times\tilde{\vec{V}}_m\right) + \nabla\Phi_m^*\cdot\tilde{\vec{V}}_m\,, \qquad (A.6)$$

and (A.3) follows at once.

We next derive an alternate expression for the surface integral

$$I_1 \equiv \iint_S \lim_{R\to\infty}\cdot\lim_{s\to s_m}\left\{\nabla\Phi_m^*\cdot\int_0^\infty \tilde{\vec{V}}_m\,d\zeta\right\}dS\,; \qquad s_m = i\lambda_m + R^{-1/2}s_m^{(1)} \qquad (A.7)$$

which will then establish (A.2).

It is clear from the form of $\vec{\tilde{V}}_m$ in (A.4) that

$$\lim_{R\to\infty} \cdot \lim_{s\to s_m} \int_0^\infty \vec{\tilde{V}}_m \, d\zeta = -\left(\frac{\vec{Q}_m - i\hat{n}\times\vec{Q}_m}{2\sqrt{ip_+}} + \frac{\vec{Q}_m + i\hat{n}\times\vec{Q}_m}{2\sqrt{ip_-}} \right), \quad p_\pm = \lambda_m \pm 2\hat{n}\cdot\hat{k} \tag{A.8}$$

Therefore the quantity of primary interest in I_1 is

$$\nabla\Phi_m^* \cdot \left(\vec{Q}_m \mp i\hat{n}\times\vec{Q}_m \right).$$

Now, $\nabla\Phi_m^* = i\lambda_m \vec{Q}_m^* - 2\hat{k}\times\vec{Q}_m^*$.

Thus $\nabla\Phi_m^* \cdot \left(\vec{Q}_m \mp i\hat{n}\times\vec{Q}_m\right) = i\left(\lambda_m \pm 2\hat{n}\cdot\hat{k}\right)\vec{Q}_m\cdot\vec{Q}_m^* + \left(2\hat{k} \pm \lambda_m\hat{n}\right)\cdot\vec{Q}_m\times\vec{Q}_m^*$.

On putting $\hat{k} = -\hat{n}\times(\hat{n}\times\hat{k}) + \hat{n}(\hat{n}\cdot\hat{k})$ and noting that

$$\left\{\hat{n}\times(\hat{n}\times\hat{k})\right\}\cdot\left\{\vec{Q}_m\times\vec{Q}_m^*\right\} = 0 \quad \text{on } S,$$

we find

$$\nabla\Phi_m^* \cdot \left(\vec{Q}_m \mp i\hat{n}\times\vec{Q}_m\right) = \left(\lambda_m \pm 2\hat{n}\cdot\hat{k}\right)\left\{ i\,\vec{Q}_m\cdot\vec{Q}_m^* \pm \hat{n}\cdot\vec{Q}_m\times\vec{Q}_m^* \right\}. \tag{A.9}$$

We shall now show that, on S,

$$\hat{n}\cdot\vec{Q}_m\times\vec{Q}_m^* = \frac{1}{1-(\hat{n}\cdot\hat{k})^2}\left\{ \hat{n}\cdot\hat{k}\times\vec{Q}_m^*\,(\hat{k}\cdot\vec{Q}_m) - \hat{n}\cdot\hat{k}\times\vec{Q}_m\left(\hat{k}\cdot\vec{Q}_m^*\right)\right\}. \tag{A.10}$$

For the proof, we subtract the two vector identities

$$\left(\hat{k}\times\vec{Q}^*\right)\cdot\left(\hat{k}\times(\hat{n}\times\vec{Q})\right) = \vec{Q}^*\cdot\hat{n}\times\vec{Q} - \left(\hat{k}\cdot\vec{Q}^*\right)\hat{k}\cdot\hat{n}\times\vec{Q}$$

$$\left(\hat{k}\times\vec{Q}^*\right)\cdot\left(\hat{k}\times(\hat{n}\times\vec{Q})\right) = \left(\hat{k}\times\vec{Q}^*\right)\cdot\left\{(\hat{k}\cdot\vec{Q})\,\hat{n} - (\hat{k}\cdot\hat{n})\vec{Q}\right\}$$

obtaining

$$\hat{n}\cdot\vec{Q}\times\vec{Q}^* = (\hat{n}\cdot\hat{k})\hat{k}\cdot\vec{Q}\times\vec{Q}^* + (\hat{k}\cdot\vec{Q})\hat{n}\cdot\hat{k}\times\vec{Q}^* - (\hat{k}\cdot\vec{Q}^*)\hat{n}\cdot\hat{k}\times\vec{Q}\,. \qquad (A.11)$$

But

$$\hat{k}\cdot\vec{Q}\times\vec{Q}^* = \left[\hat{n}(\hat{k}\cdot\hat{n}) - \hat{n}\times(\hat{n}\times\hat{k})\right]\cdot\vec{Q}\times\vec{Q}^*$$

$$= (\hat{n}\cdot\hat{k})\,\hat{n}\cdot\vec{Q}\times\vec{Q}^* \quad \text{on } S, \qquad (A.12)$$

since $\hat{n}\cdot\vec{Q} = \hat{n}\cdot\vec{Q}^* = 0$ on S.

Substitution for $\hat{k}\cdot\vec{Q}_m\times\vec{Q}_m^*$ in (A.11) establishes (A.10).

In turn, (A.9) may be rewritten as

$$\left(\vec{Q}_m \mp i\hat{n}\times\vec{Q}_m\right)\cdot\nabla\Phi_m^* = \frac{i(\lambda_m \pm 2\hat{n}\cdot\hat{k})}{1-(\hat{n}\cdot\hat{k})^2}\Big[\left\{1-(\hat{n}\cdot\hat{k})^2\right\}\vec{Q}_m\cdot\vec{Q}_m^* +$$

$$\mp i\left\{\hat{n}\cdot\hat{k}\times\vec{Q}_m^*(\hat{k}\cdot\vec{Q}_m) - \hat{n}\cdot\hat{k}\times\vec{Q}_m(\hat{k}\cdot\vec{Q}_m^*)\right\}\Big] \qquad (A.13)$$

or,

$$\left(\vec{Q}_m \mp i\hat{n}\times\vec{Q}_m\right)\cdot\nabla\Phi_m^* = \frac{i(\lambda_m \pm 2\hat{n}\cdot\hat{k})}{1-(\hat{n}\cdot\hat{k})^2}\left|\hat{n}\cdot\hat{k}\times\vec{Q}_m \mp i\,\hat{k}\cdot\vec{Q}_m\right|^2. \qquad (A.14)$$

Now our basic requirement is that $\sqrt{ip_\pm}$ have positive real part; thus

$$\frac{1}{\sqrt{ip_\pm}} = \frac{1}{2^{1/2}\left|\lambda_m \pm 2\hat{n}\cdot\hat{k}\right|^{1/2}}\left(1 - \frac{i(\lambda_m \pm 2\hat{n}\cdot\hat{k})}{|\lambda_m \mp 2\hat{n}\cdot\hat{k}|}\right).$$

Together with (A.14), this means that

$$-\nabla\Phi_m^* \cdot \left(\frac{\vec{Q}_m \mp i\hat{n}\times\vec{Q}_m}{2\sqrt{ip_\pm}}\right) = \frac{-1}{2\{1-(\hat{n}\cdot\hat{k})^2\}}\left|\hat{n}\cdot\hat{k}\times\vec{Q}_m \mp i\,\hat{k}\cdot\vec{Q}_m\right|^2 \times$$

$$\times\left(1 + \frac{i(\lambda_m \pm 2\hat{n}\cdot\hat{k})}{|\lambda_m \pm 2\hat{n}\cdot\hat{k}|}\right)\times\left|\frac{\lambda_m}{2} \pm \hat{n}\cdot\hat{k}\right|^{1/2}, \qquad (A.15)$$

and this, together with (A.6) and (A.8), shows that

$$I_1 = -\frac{1}{2}\iint_S \frac{dS}{1-(\hat{n}\cdot\hat{k})^2}\left[\left|\hat{n}\cdot\hat{k}\times\vec{Q}_m - i\,\hat{k}\cdot\vec{Q}_m\right|^2\cdot\left(1 + \frac{i(\lambda_m + 2\hat{n}\cdot\hat{k})}{|\lambda_m + 2\hat{n}\cdot\hat{k}|}\right)\cdot\left|\frac{\lambda_m}{2} + \hat{n}\cdot\hat{k}\right|^{1/2} + \right.$$

$$\left. + \left|\hat{n}\cdot\hat{k}\times\vec{Q}_m + i\,\hat{k}\cdot\vec{Q}_m\right|^2\cdot\left(1 + \frac{i(\lambda_m - 2\hat{n}\cdot\hat{k})}{|\lambda_m - 2\hat{n}\cdot\hat{k}|}\right)\cdot\left|\frac{\lambda_m}{2} - \hat{n}\cdot\hat{k}\right|^{1/2}\right] \qquad (A.16)$$

This establishes (A.2).

APPENDIX B

DERIVATION OF SPHEROID EIGENFUNCTIONS AND BOUNDARY CONDITION

Under the transformation

$$r = (\alpha^2 + \eta^2)^{1/2}\,(1-\mu^2)^{1/2}$$

$$\sigma = \eta\mu$$

we have the following operator identity:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\sigma^2} \equiv \frac{1}{\eta^2+\alpha^2\mu^2}\left(\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} + \frac{\partial}{\partial\eta}(\alpha^2+\eta^2)\frac{\partial}{\partial\eta}\right).$$

Now let

$$z = -i\beta\sigma = -i\beta\eta\mu$$

where

$$\beta^2 = 4/\lambda^2 - 1 > 0 .$$

Then the basic eigenfunction equation for (φ, λ), namely

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\varphi}{\partial r} - \frac{k^2}{r^2}\varphi + \left(1 - 4/\lambda^2\right)\frac{\partial^2\varphi}{\partial z^2} = 0, \qquad (B.1)$$

becomes

$$\frac{1}{\eta^2+\alpha^2\mu^2}\left(\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial\varphi}{\partial\mu} + \frac{\partial}{\partial\eta}(\alpha^2+\eta^2)\frac{\partial\varphi}{\partial\eta}\right) - \frac{k^2\varphi}{(\alpha^2+\eta^2)(1-\mu^2)} = 0.$$

Putting $\eta = i\alpha\zeta$ so that

$$\left.\begin{aligned} r &= \alpha(1-\zeta^2)^{1/2}(1-\mu^2)^{1/2} \\ z &= (\alpha\beta)\,\zeta\mu \end{aligned}\right\} \tag{B.2}$$

then

$$\frac{1}{\alpha^2(\mu^2-\zeta^2)}\left(\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial\varphi}{\partial\mu} - \frac{\partial}{\partial\zeta}(1-\zeta^2)\frac{\partial\varphi}{\partial\zeta}\right) - \frac{k^2\varphi}{\alpha^2(1-\zeta^2)(1-\mu^2)} = 0. \tag{B.3}$$

Now

$$\frac{-k^2(\mu^2-\zeta^2)}{(1-\zeta^2)(1-\mu^2)} = -k^2\left(\frac{1}{1-\mu^2} - \frac{1}{1-\zeta^2}\right).$$

Therefore, (B.3) is equivalent to

$$\left(\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial\varphi}{\partial\mu} - \frac{k^2\varphi}{1-\mu^2}\right) - \left(\frac{\partial}{\partial\zeta}(1-\zeta^2)\frac{\partial\varphi}{\partial\zeta} - \frac{k^2\varphi}{1-\zeta^2}\right) = 0, \tag{B.4}$$

whence

$$\varphi = P_m^k(\mu)\,P_m^k(\zeta) \tag{B.5}$$

By the known form of the associated Legendre function $P_m^k(x)$, it follows that

$$P_m^k(x) = C_{mk}\,x^\nu(1-x^2)^{k/2}\prod_{j=1}^{N}(x^2-x_j^2), \tag{B.6}$$

where

$$C_{mk} = \frac{(2m)!}{2^m\, m!\, (m-k)!}$$

$$\nu = \begin{cases} 0 & \text{if } m-k \text{ even} \\ 1 & \text{if } m-k \text{ odd} \end{cases}$$

x_j^2 are the $N = (m-k) - \nu$ distinct squared zeroes of $P_m^k(x)$, exclusive of $0, 1$.

Equation (B.5) is now readily cast into a product of factors involving r and z by substituting directly from (B.6) and (B.2). For, it is clear that

$$\left.\begin{aligned} (\zeta\mu)^\nu \left\{(1-\zeta^2)(1-\mu^2)\right\}^{k/2} &= \left(\frac{z}{\alpha\beta}\right)^\nu \left(\frac{r}{\alpha}\right)^k \\ (\zeta^2 - x_j^2)(\mu^2 - x_j^2) &= x_j^2(x_j^2 - 1) + \frac{x_j^2}{\alpha^2} r^2 + \frac{1-x_j^2}{\alpha^2\beta^2} z^2 . \end{aligned}\right\} \quad (B.7)$$

It may also be recalled from Chapter 4 that

$$\alpha^2 = \frac{1 + \varepsilon(1-\xi_{mk}^2)}{(1+\varepsilon)(1-\xi_{mk}^2)} , \qquad \beta^2 = \frac{1 - \xi_{mk}^2}{\xi_{mk}^2} \quad (B.8)$$

where $\varepsilon \equiv 1/b^2 - 1$ corresponding to the spheroid $r^2 + (z/b)^2 = 1$, and for brevity $\xi_{mk} = \lambda_{mk}/2$. Therefore

$$\varphi = P_m^k(\zeta) P_m^k(\mu) = C_{mk}^2 (\zeta\mu)^\nu (1-\zeta^2)^{k/2} (1-\mu^2)^{k/2} \prod_{j=1}^{N} (\zeta^2 - x_j^2)(\mu^2 - x_j^2)$$

$$= C_{mk}^2 \left(\frac{z}{\alpha\beta}\right)^\nu \left(\frac{r}{\alpha}\right)^k \prod_{j=1}^{N} \left\{ D_j + A_j r^2 + B_j z^2 \right\} \quad (B.9)$$

where
$$D_j \equiv x_j^2 (x_j^2 - 1)$$
$$A_j \equiv \frac{1+\varepsilon}{1+\varepsilon(1-\xi_{mk}^2)} \cdot x_j^2 (1-\xi_{mk}^2)$$
$$B_j \equiv \frac{1+\varepsilon}{1+\varepsilon(1-\xi_{mk}^2)} \cdot \xi_{mk}^2 (1-x_j^2) .$$

This is the result quoted in Chapter 4, Equation (4.7).

We can also derive a simple form of the general boundary condition

$$\hat{n}\cdot\nabla\varphi_m - \frac{2}{i\lambda_m}\hat{n}\cdot\hat{k}\times\nabla\varphi_m - \frac{4}{\lambda_m^2}(\hat{n}\cdot\hat{k})\,\hat{k}\cdot\nabla\varphi_m = 0$$

which on the spheroid is equivalent to

$$r\frac{\partial\varphi_m}{\partial r} + \frac{k}{\xi_m}\varphi_m - \frac{1-\xi_m^2}{\xi_m^2}(1+\varepsilon)z\frac{\partial\varphi_m}{\partial z} = 0, \quad on \quad r^2 + (1+\varepsilon)z^2 = 1.$$

For, using (B.9) we may rewrite this last equation as

$$(C_{mk})^2\left(\frac{z}{\alpha\beta}\right)^\nu\left(\frac{r}{\alpha}\right)^k\Bigg\{\left[k\left(1+\frac{1}{\xi_m}\right) + \nu\left(1-\frac{1}{\xi_m^2}\right)(1+\varepsilon)\right]\prod_{j=1}^{N}\{D_j + A_j r^2 + B_j z^2\} +$$
$$+ 2\sum_{j=1}^{N}\left[\left(A_j r^2 + \left(1-\frac{1}{\xi_m^2}\right)(1+\varepsilon)B_j z^2\right)\left(\prod_{\substack{l=1\\ l\neq j}}^{N}\{D_l + A_l r^2 + B_l z^2\}\right)\right]\Bigg\} = 0. \quad (B.10)$$

Now,

$$k\left(1+\frac{1}{\xi_m}\right)+\nu\left(1-\frac{1}{\xi_m^2}\right)(1+\varepsilon) \equiv \frac{1+\xi_m}{\xi_m}\left\{k-\frac{(1+\varepsilon)(1-\xi_m)}{\xi_m}\nu\right\}. \qquad (B.11)$$

When $r^2+(1+\varepsilon)z^2 = 1$, it follows from (B.2) that

$$r^2 = 1-\mu^2 ,$$

$$(1+\varepsilon)z^2 = \mu^2 .$$

Hence, on the spheroid,

$$A_j r^2 + \left(1-\frac{1}{\xi_m^2}\right)B_j\,(1+\varepsilon)z^2 = \frac{(1+\varepsilon)(1-\xi_m^2)}{1+\varepsilon(1-\xi_m^2)}\,(x_j^2-\mu^2). \qquad (B.12)$$

Moreover,

$$D_j + A_j r^2 + B_j z^2 = (\mu^2-x_j^2)(\zeta^2-x_j^2). \qquad (B.13)$$

The boundary condition (B.10) now becomes greatly simplified upon substitution of (B.11)-(B.13). The result is

$$\left\{\left[\frac{1+\xi_m}{\xi_m}C_{mk}^2\left(\frac{z}{\alpha\beta}\right)^\nu\left(\frac{r}{\alpha}\right)^k\right]\left[\prod_{j=1}^{N}\{D_j+A_jr^2+B_jz^2\}\right]\right\}\times$$

$$\times\left\{\left(k-\frac{(1+\varepsilon)(1-\xi_m)}{\xi_m}\nu\right)-\frac{2\xi_m^2}{1+\varepsilon(1-\xi_m^2)}\cdot\frac{(1+\varepsilon)(1-\xi_m)}{\xi_m}\sum_{j=1}^{N}\frac{1}{\zeta^2-x_j^2}\right\} = 0,$$

or, since the first factor does not vanish identically,

$$\left.\begin{aligned} \nu + 2\zeta^2 \sum_{j=1}^{N}{}' \frac{1}{\zeta^2 - x_j^2} &= \frac{k\,\xi_m}{(1+\varepsilon)(1-\xi_m)} \\ \text{or} \qquad \zeta^2 &= \frac{\xi_m^2}{1+\varepsilon(1-\xi_m^2)} \,. \end{aligned}\right\} \qquad (B.14)$$

APPENDIX C

EVALUATION OF $s_m^{(1)}$ FOR A CYLINDER

As noted in the main text (Chapter 7) it is desirable to derive an explicit evaluation of the viscous correction factor $s_m^{(1)}$ for a cylinder, for this is one of the few geometries which permits such an evaluation and we can determine certain properties of $s_m^{(1)}$ directly without numerical calculations.

First, however, the general formula for $s_m^{(1)}$ (see Chapter 2 or Appendix A) has to be modified. Since $(\hat{n} \cdot \hat{k})^2 = 1$ at $z = \pm 1$, the surface integrand cannot be calculated as given. An alternate formula which circumvents this difficulty is now derived.

We begin the derivation with equation (A.9), Appendix A, namely

$$\left(\vec{Q}_m \mp i\,\hat{n}\times\vec{Q}_m\right)\cdot\nabla\Phi_m^* = \left(\lambda_m \pm 2\hat{n}\cdot\hat{k}\right)\left\{ i\,\vec{Q}_m\cdot\vec{Q}_m^* \pm \hat{n}\cdot\vec{Q}_m\times\vec{Q}_m^* \right\}.$$

In place of the analysis which followed (A.9), we observe that the term on the right may be written

$$\frac{i}{2}\left(\lambda_m \pm 2\hat{n}\cdot\hat{k}\right)\left\{ \vec{Q}_m \mp i\,\hat{n}\times\vec{Q}_m \right\}\cdot\left\{ \vec{Q}_m^* \pm i\,\hat{n}\times\vec{Q}_m^* \right\}$$

as is readily verified by direct calculation. Therefore (cf. (A.14)),

$$\left(\vec{Q}_m \mp i\,\hat{n}\times\vec{Q}_m\right)\cdot\nabla\Phi_m^* = \frac{i}{2}\left(\lambda_m \pm 2\,\hat{n}\cdot\hat{k}\right)\left|\vec{Q}_m \mp i\,\hat{n}\times\vec{Q}_m\right|^2 .$$

From this there follows a revised formula for $s_m^{(1)}$,

$$s_m^{(1)}\iiint_V |\vec{Q}_m|^2\, dV = -\frac{1}{4}\iint_S J\, dS$$

where

$$J = \left|\vec{Q}_m - i\,\hat{n}\times Q_m\right|^2\left(1+\frac{i p_+}{|p_+|}\right)|p_+|^{1/2} + \left|\vec{Q}_m + i\,\hat{n}\times\vec{Q}_m\right|^2\left(1+\frac{i p_-}{|p_-|}\right)|p_-|^{1/2},$$

$$p_\pm = \frac{\lambda_m}{2} \pm \hat{n}\cdot\hat{k} .$$

It is clear that the integral over S must be taken in two parts, one at $z = \pm 1$, the other at $r = r_0$.

Let I_1 denote that part of the surface integral $+\frac{1}{4}\iint J\, dS$ taken over the surface at $r = r_0$, and I_2 the contribution from the surfaces $z = \pm 1$. Then

$$I_1 = 2\pi\int_{-1}^{1} dz\left\{ |\vec{Q}_{mk}|^2\left(1+\frac{i\lambda_{mk}}{|\lambda_{mk}|}\right)\left|\frac{\lambda_{mk}}{2}\right|^{1/2}\right\}_{r=r_0}$$

$$I_2 = 2\pi\cdot\frac{1}{2}\int_0^{r_0} r\,dr\left\{\left|\vec{Q}_{mk} - i\,\hat{k}\times\vec{Q}_{mk}\right|^2(1+i)\left|\frac{\lambda_{mk}}{2}+1\right|^{1/2} + \right.$$

$$\left. + \left|\vec{Q}_{mk} + i\,\hat{k}\times\vec{Q}_{mk}\right|^2(1-i)\left|\frac{\lambda_{mk}}{2}-1\right|^{1/2}\right\}_{z=1} .$$

We recall the following expressions for the inviscid velocity $\vec{Q}_m \equiv \vec{Q}_{mk} = \{U_{mk}, V_{mk}, W_{mk}\}$:

$$\left.\begin{aligned}
\mathcal{U}_{mk} &= \frac{-i\cos m\pi z}{4-\lambda_{mk}^2}\left(\lambda_{mk}\alpha_{mk}J_k'(\alpha_{mk}r) + \frac{2k}{r}J_k(\alpha_{mk}r)\right) \\
V_{mk} &= \frac{\cos m\pi z}{4-\lambda_{mk}^2}\left(2\alpha_{mk}J_k'(\alpha_{mk}r) + \frac{\lambda_{mk}k}{r}J_k(\alpha_{mk}r)\right) \\
W_{mk} &= \frac{m\pi\sin m\pi z}{i\lambda_{mk}}J_k(\alpha_{mk}r) \\
\text{where} \quad \alpha_{mk} &= m\pi\left(\frac{4-\lambda_{mk}^2}{\lambda_{mk}^2}\right)^{1/2}, \quad \lambda_{mk} \neq 0.
\end{aligned}\right\} \qquad (C.1)$$

The vanishing of the normal component $\mathcal{U}_{mk}$ at $r = r_o$ requires

$$J_k'(\alpha_{mk}r) = \frac{-2k}{\lambda_{mk}\alpha_{mk}r}J_k(\alpha_{mk}r) \quad at \quad r = r_o. \qquad (C.2)$$

We proceed first to evaluate I_1 . Substituting from (C.1), we have directly that

$$I_1 = 2\pi\left(1+\frac{i\lambda_{mk}}{|\lambda_{mk}|}\right)\left|\frac{\lambda_{mk}}{2}\right|^{1/2}\left[\left\{\frac{J_k(\alpha_{mk}r_0)}{\lambda_{mk}}\right\}^2 m^2\pi^2\int_{-1}^{1}\sin^2 m\pi z\,dz\; + \right.$$

$$\left. + \left\{\frac{2\alpha_{mk}J_k'(\alpha_{mk}r) + (\lambda_{mk}k/r)J_k(\alpha_{mk}r)}{4-\lambda_{mk}^2}\right\}^2_{r=r_0}\cdot\int_{-1}^{1}\cos^2 m\pi z\,dz\right], \qquad (C.3)$$

and using (C.2) it also follows that

$$\left\{ 2\alpha_{mk} J_k'(\alpha_{mk} r) + \frac{\lambda_{mk}}{r} k J_k(\alpha_{mk} r) \right\}_{r=r_0} = \frac{\lambda_{mk}^2 - 4}{\lambda_{mk}} \cdot \frac{k}{r_0} J_k(\alpha_{mk} r_0)$$

Therefore,

$$I_1 = 2\pi\left(1 + \frac{i\lambda_{mk}}{|\lambda_{mk}|}\right)\left|\frac{\lambda_{mk}}{2}\right|^{1/2} \cdot \left(\frac{J_k(\alpha_{mk} r_0)}{\lambda_{mk}}\right)^2 \left(\frac{k^2 + m^2\pi^2 r_0^2}{r_0^2}\right). \tag{C.4}$$

The calculation of I_2 is somewhat more lengthy, and we shall need the following known properties of Bessel functions:

$$J_k'(x) = \frac{-k}{x} J_k(x) + J_{k-1}(x) = \frac{k}{x} J_k(x) - J_{k+1}(x) \tag{C.5}$$

$$\int x\left(J_k(\alpha x)\right)^2 dx = \frac{x^2}{2}\left\{\left[J_k(\alpha x)\right]^2 - J_{k-1}(\alpha x) J_{k+1}(\alpha x)\right\} \tag{C.6}$$

$$J_{k-1}(x) + J_{k+1}(x) = \frac{2k}{x} J_k(x). \tag{C.7}$$

Using (C.1) we see that, on $z = 1$,

$$\vec{Q}_{mk} = \frac{(-)^m}{4 - \lambda_{mk}^2}\left\{-i\left(\lambda_{mk}\alpha_{mk} J_k' + \frac{2k}{r} J_k\right), \left(2\alpha_{mk} J_k' + \frac{k\lambda_{mk}}{r} J_k\right), 0\right\}$$

$$i\hat{k} \times \vec{Q}_{mk} = \frac{(-)^m}{4 - \lambda_{mk}^2}\left\{-i\left(2\alpha_{mk} J_k' + \frac{k\lambda_{mk}}{r} J_k\right), \left(\lambda_{mk}\alpha_{mk} J_k' + \frac{2k}{r} J_k\right), 0\right\},$$

and from these relations it follows that

$$\left|\vec{Q}_{mk} - i\hat{k}\times\vec{Q}_{mk}\right|^2 = \frac{2\alpha_{mk}^2}{(2+\lambda_{mk})^2}\left\{J_k'(\alpha_{mk}r) - \frac{k}{\alpha_{mk}r}J_k(\alpha_{mk}r)\right\}^2 ,$$

$$\left|\vec{Q}_{mk} + i\hat{k}\times\vec{Q}_{mk}\right|^2 = \frac{2\alpha_{mk}^2}{(2-\lambda_{mk})^2}\left\{J_k'(\alpha_{mk}r) + \frac{k}{\alpha_{mk}r}J_k(\alpha_{mk}r)\right\}^2 .$$

In view of the identities (C.5), these last two equations become

$$\left.\begin{aligned}\left|\vec{Q}_{mk} - i\hat{k}\times\vec{Q}_{mk}\right|^2 &= \frac{2\alpha_{mk}^2}{(2+\lambda_{mk})^2}\left\{J_{k+1}(\alpha_{mk}r)\right\}^2\\ \left|\vec{Q}_{mk} + i\hat{k}\times\vec{Q}_{mk}\right|^2 &= \frac{2\alpha_{mk}^2}{(2-\lambda_{mk})^2}\left\{J_{k-1}(\alpha_{mk}r)\right\}^2\end{aligned}\right\} \qquad (C.8)$$

Now

$$I_2 = 2\pi\cdot\frac{1}{2}\int_0^{r_0} r\,dr\left\{\left|\vec{Q}_{mk} - i\hat{k}\times\vec{Q}_{mk}\right|^2\cdot(1+i)\left|\frac{\lambda_{mk}}{2}+1\right|^{1/2} + \right.$$
$$\left. + \left|\vec{Q}_{mk} + i\hat{k}\times\vec{Q}_{mk}\right|^2\cdot(1-i)\left|\frac{\lambda_{mk}}{2}-1\right|^{1/2}\right\}_{z=1} ;$$

upon substituting from (C.8) we have, equivalently,

$$I_2 = F^{\oplus}\cdot\alpha_{mk}^2\int_0^{r_0} r\left\{J_{k+1}(\alpha_{mk}r)\right\}^2 dr + F^{\ominus}\cdot\alpha_{mk}^2\int_0^{r_0} r\left\{J_{k-1}(\alpha_{mk}r)\right\}^2 dr \qquad (C.9)$$

where

$$F^{\oplus} \equiv 2\pi\,\frac{1+i}{\sqrt{2}}\,\frac{|\lambda_{mk}+2|^{1/2}}{(\lambda_{mk}+2)^2}$$

$$F^{\ominus} \equiv 2\pi\,\frac{1-i}{\sqrt{2}}\,\frac{|\lambda_{mk}-2|^{1/2}}{(\lambda_{mk}-2)^2} .$$

But by the known relation (C.6),

$$\alpha_{mk}^2 \int_0^{r_0} r\left\{J_{k\pm1}(\alpha_{mk}r)\right\}^2 dr = \frac{(\alpha_{mk}r_0)^2}{2}\left\{\left(J_{k\pm1}(\alpha_{mk}r_0)\right)^2 - J_k(\alpha_{mk}r_0)\,J_{k\pm2}(\alpha_{mk}r_0)\right\} \qquad (C.10)$$

We now show that because of the eigenvalue relation (C.2) we can rewrite the integrals (C.10) in terms of just $\left\{J_k(\alpha_{mk}r_0)\right\}^2$. To do this we observe that (C.2) and (C.5) together imply

$$J_{k-1}(\alpha_{mk}r_0) = \frac{\lambda_{mk}-2}{\lambda_{mk}}\cdot\frac{k}{\alpha_{mk}r_0}J_k(\alpha_{mk}r_0),$$

and similarly

$$J_{k+1}(\alpha_{mk}r_0) = \frac{\lambda_{mk}+2}{\lambda_{mk}}\cdot\frac{k}{\alpha_{mk}r_0}J_k(\alpha_{mk}r_0). \qquad (C.11)$$

Then (C.7) and (C.11) show that

$$J_{k-2}(\alpha_{mk}r_0) = J_k(\alpha_{mk}r_0)\left[\frac{2(k-1)}{\alpha_{mk}^2 r_0^2}\cdot\frac{k}{\lambda_{mk}}\cdot(\lambda_{mk}-2) - 1\right]$$

$$J_{k+2}(\alpha_{mk}r_0) = J_k(\alpha_{mk}r_0)\left[\frac{2(k+1)}{\alpha_{mk}^2 r_0^2}\cdot\frac{k}{\lambda_{mk}}\cdot(\lambda_{mk}+2) - 1\right] \qquad (C.12)$$

Collecting the results (C.10)-(C.12) and substituting for $\alpha_{mk}^2 = \left(4-\lambda_{mk}^2\right)\frac{m^2\pi^2}{\lambda_{mk}^2}$, we determine that

$$\alpha_{mk}^2 \int_0^{r_0} r\left\{J_{k\pm 1}(\alpha_{mk} r)\right\}^2 dr = \frac{4-\lambda_{mk}^2}{2\lambda_{mk}^2}\Big(J_k(\alpha_{mk} r_0)\Big)^2\left(k^2 + m^2\pi^2 r_0^2 - \frac{2k\lambda_{mk}}{2\mp\lambda_{mk}}\right). \qquad (C.13)$$

Finally, then, we may substitute (C.13) into (C.9) to obtain the desired result for the integral I_2:

$$I_2 = 2\pi\,\frac{4-\lambda_{mk}^2}{2\sqrt{2}}\left(\frac{J_k(\alpha_{mk} r_0)}{\lambda_{mk}}\right)^2\left[\left\{k^2 + m^2\pi^2 r_0^2 - \frac{2k\lambda_{mk}}{2-\lambda_{mk}}\right\}\cdot\frac{1+i}{(2+\lambda_{mk})^{3/2}} + \right.$$

$$\left. + \left\{k^2 + m^2\pi^2 r_0^2 - \frac{2k\lambda_{mk}}{2+\lambda_{mk}}\right\}\cdot\frac{1-i}{(2-\lambda_{mk})^{3/2}}\right] \qquad (C.14)$$

It is a relatively simple matter now to complete the evaluation of $s_m^{(1)}$, which we recall is given by

$$s_m^{(1)} \iiint_V |\vec{Q}_{mk}|^2 dV = -\left(I_1 + I_2\right). \qquad (C.15)$$

To compute the volume integral $N \equiv \iiint |\vec{Q}_{mk}|^2 dV$, we once again appeal to the identities (C.5)-(C.7) and the consequent result (C.13). These enable us to establish that

$$N = 2\pi \int_0^{r_0} r\,dr \int_{-1}^{1} dz \left|\vec{Q}_{mk}\right|^2 = 2\pi \frac{m^2\pi^2}{\lambda_{mk}^2} \int_{-1}^{1} \sin^2 m\pi z\, dz \times \left[\int_0^{r_0} r \left\{J_k(\alpha_{mk} r)\right\}^2 dr\right] +$$

$$+ \frac{2\pi}{(4-\lambda_{mk}^2)^2} \int_{-1}^{1} \cos^2 m\pi z\, dz \times \left[\frac{(2+\lambda_{mk})^2}{2} \alpha_{mk}^2 \int_0^{r_0} r \left\{J_{k-1}(\alpha_{mk} r)\right\}^2 dr + \right.$$

$$\left. + \frac{(2-\lambda_{mk})^2}{2} \alpha_{mk}^2 \int_0^{r_0} r \left\{J_{k+1}(\alpha_{mk} r)\right\}^2 dr \right]$$

whence

$$N = 2\pi \left(\frac{J_k(\alpha_{mk} r_0)}{\lambda_{mk}}\right)^2 \cdot \frac{4}{4-\lambda_{mk}^2} \cdot \left\{ k^2 + m^2\pi^2 r_0^2 - \frac{k\lambda_{mk}}{2} \right\}. \qquad (C.16)$$

APPENDIX D

THE GEOSTROPHIC MODE IN A CYLINDER

The method of solution for the interior problem (D1) and Ekman layer problem (D2) has already been indicated in the general theory of Chapter 2, and the details need not be repeated here. Indeed, with but slight modification Greenspan's general theory of geostrophic motions readily yields the solutions quoted in Chapter 7, above. Therefore, we turn to discussion of the sidewall boundary layer regions.

The basic method used throughout the following derivations is to assume an asymptotic expansion for the velocity and pressure variables $\vec{q}, p$ in each domain. The form of these expansions is dictated primarily by the knowledge that the Ekman layers induce an $O(R^{-1/2})$ circulation in the interior, and by the requirement that the expansions of one domain join (match) those of adjacent domains in the common overlap regions. (The choice of variables appropriate to each domain has already been discussed in Chapter 7.) These expansions are then substituted into the equations of motion and boundary conditions in order to obtain an asymptotic sequence of problems. The solution of this problem sequence gives the representation of the flow variables in the given domain.

We choose polar coordinates (r, ω, z) and let $\vec{q} = (u, v, w)$, as in Chapter 7. Then the equations of motion on the geostrophic time scale $\tau = R^{-1/2} t$ are

$$\left.\begin{aligned}
-2v + \frac{\partial p}{\partial r} &= -R^{-1/2}\frac{\partial u}{\partial \tau} + R^{-1}\left\{\nabla^2 u - \frac{1}{r^2}\left(u + 2\frac{\partial v}{\partial \omega}\right)\right\} \\
2u + \frac{1}{r}\frac{\partial p}{\partial \omega} &= -R^{-1/2}\frac{\partial v}{\partial \tau} + R^{-1}\left\{\nabla^2 v - \frac{1}{r^2}\left(v - 2\frac{\partial u}{\partial \omega}\right)\right\} \\
\frac{\partial p}{\partial z} &= -R^{-1/2}\frac{\partial w}{\partial \tau} + R^{-1}\nabla^2 w \\
\frac{\partial u}{\partial r} + \frac{1}{r}\left(u + \frac{\partial v}{\partial \omega}\right) + \frac{\partial w}{\partial z} &= 0 \\
\text{where} \qquad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \omega^2} &+ \frac{\partial^2}{\partial z^2}\,.
\end{aligned}\right\} \qquad (D.1)$$

The boundary conditions are that $\vec{q} = 0$ at solid boundaries, and $\vec{q}\,(\vec{r}, \tau) = \vec{q}_*(\vec{r})$ at time $\tau = 0$.

Region D4:

The interior flow has already been determined as

$$\vec{q}_1 = \vec{q}_1^{(0)} + R^{-1/4}\vec{q}_1^{(1)} + R^{-1/2}\vec{q}_1^{(2)} + \cdots$$

with

$$\vec{q}_1^{(0)} \equiv \left\{ u_1^{(0)}, v_1^{(0)}, w_1^{(0)} \right\} = \left\{ -\frac{1}{2r}\frac{\partial \varphi_1^{(0)}}{\partial \omega}, \frac{1}{2}\frac{\partial \varphi_1^{(0)}}{\partial r}, 0 \right\} e^{-\tau} .$$

In D4, where $\rho = R^{1/4}(r - r_0)$, this interior solution is represented as

$$\vec{q}_1 = \left[\vec{q}_1^{(0)}\right] + R^{-1/4}\left\{\rho\left[\frac{\partial \vec{q}_1^{(0)}}{\partial r}\right] + \left[\vec{q}_1^{(1)}\right]\right\} + R^{-1/2}\left\{\frac{\rho^2}{2}\left[\frac{\partial^2 \vec{q}_1^{(0)}}{\partial r^2}\right] + \rho\left[\frac{\partial \vec{q}_1^{(1)}}{\partial r}\right] + \left[\vec{q}_1^{(2)}\right]\right\} + \cdots$$

where $\left[f(r,\omega,z,\tau)\right] \equiv f(r_0,\omega,z,\tau)$. (This bracket notation will be used extensively throughout this section.)

The requirement of matching at the outer edge of D4 suggests an expansion in powers of $R^{-1/4}$,

$$\vec{q} = \vec{q}_4^{(0)} + R^{-1/4}\vec{q}_4^{(1)} + R^{-1/2}\vec{q}_4^{(2)} + \cdots$$

$$p = p_4^{(0)} + R^{-1/4}p_4^{(1)} + R^{-1/2}p_4^{(2)} + \cdots$$

$$\rho = R^{1/4}(r - r_0) .$$

When these expansions are substituted into (D.1), an asymptotically valid problem sequence can be derived. For the radial momentum equation, substitution shows that

$$-2\left\{v_4^{(0)}+R^{-1/4}v_4^{(1)}+\dots\right\} + R^{+1/4}\frac{\partial}{\partial\rho}\left\{p_4^{(0)}+R^{-1/4}p_4^{(1)}+\dots\right\} =$$

$$= -R^{-1/2}\frac{\partial}{\partial\tau}\left\{u_4^{(0)}+R^{-1/4}u_4^{(1)}+\dots\right\} + \left\{R^{-1/2}\frac{\partial^2}{\partial\rho^2}+R^{-3/4}\frac{1}{r_0}\frac{\partial}{\partial\rho}\right\}\left\{u_4^{(0)}+R^{-1/4}u_4^{(1)}+\dots\right\} + O(R^{-1}).$$

Here, we have written

$$R^{-1}\nabla^2 \equiv R^{-1}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\omega^2}+\frac{\partial^2}{\partial z^2}\right) = R^{-1/2}\frac{\partial^2}{\partial\rho^2}+R^{-3/4}\frac{1}{r_0}\frac{\partial}{\partial\rho} + O(R^{-1}).$$

By regrouping terms, it then follows that

$$0 = R^{1/4}\left(\frac{\partial p_4^{(0)}}{\partial\rho}\right) + \left(-2v_4^{(0)}+\frac{\partial p_4^{(1)}}{\partial\rho}\right) + R^{-1/4}\left(-2v_4^{(1)}+\frac{\partial p_4^{(2)}}{\partial\rho}\right) +$$

$$+ R^{-1/2}\left(-2v_4^{(2)}+\frac{\partial p_4^{(3)}}{\partial\rho}+\left\{\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial\rho^2}\right\}u_4^{(0)}\right) +$$

$$+ R^{-3/4}\left(-2v_4^{(3)}+\frac{\partial p_4^{(4)}}{\partial\rho}+\left\{\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial\rho^2}\right\}u_4^{(1)} - \frac{1}{r_0}\frac{\partial u_4^{(0)}}{\partial\rho}\right) +$$

$$+ O(R^{-1}).$$

Entirely similar operations are applied to the remaining equations of motion. In each equation, we equate like powers of $R^{-1/4}$, and thereby derive the following asymptotic problem sequence:

(i) $$\frac{\partial p_4^{(0)}}{\partial \rho} = 0$$

$$2u_4^{(0)} + \frac{1}{r_0}\frac{\partial p_4^{(0)}}{\partial \omega} = 0$$

$$\frac{\partial p_4^{(0)}}{\partial z} = 0$$

$$\frac{\partial u_4^{(0)}}{\partial \rho} = 0, \quad u_4^{(0)} \to 0 \quad \text{as} \quad \rho \to -\infty$$

(ii) $$-2v_4^{(0)} + \frac{\partial p_4^{(1)}}{\partial \rho} = 0$$

$$2u_4^{(1)} + \frac{1}{r_0}\frac{\partial p_4^{(1)}}{\partial \omega} = \frac{\rho}{r_0^2}\frac{\partial p_4^{(0)}}{\partial \omega}$$

$$\frac{\partial p_4^{(1)}}{\partial z} = 0$$

$$\frac{\partial u_4^{(1)}}{\partial \rho} = -\frac{1}{r_0}\left\{\frac{\partial v_4^{(0)}}{\partial \omega} + u_4^{(0)} + r_0\frac{\partial w_4^{(0)}}{\partial z}\right\}$$

(iii) $$-2v_4^{(1)} + \frac{\partial p_4^{(2)}}{\partial \rho} = 0$$

$$2u_4^{(2)} + \frac{1}{r_0}\frac{\partial p_4^{(2)}}{\partial \omega} = \frac{\rho}{r_0^2}\frac{\partial p_4^{(1)}}{\partial \omega} + \left(\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\partial \tau}\right)v_4^{(0)}$$

$$\frac{\partial p_4^{(2)}}{\partial z} = \left(\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\partial \tau}\right)w_4^{(0)}$$

$$\frac{\partial u_4^{(2)}}{\partial \rho} = -\frac{1}{r_0}\left(\frac{\partial v_4^{(1)}}{\partial \omega} + u_4^{(1)} + r_0\frac{\partial w_4^{(1)}}{\partial z}\right) + \frac{\rho}{r_0^2}\cdot\frac{\partial v_4^{(0)}}{\partial \omega}$$

$$(iv) \quad -2v_4^{(2)} + \frac{\partial p_4^{(3)}}{\partial \rho} = \left(\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\partial \tau}\right) u_4^{(0)}$$

$$2u_4^{(3)} + \frac{1}{r_0}\frac{\partial p_4^{(3)}}{\partial \omega} = \left(\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\partial \tau}\right) v_4^{(1)} + \frac{\rho}{r_0^2}\frac{\partial p_4^{(2)}}{\partial \omega} - \frac{\rho^2}{r_0^3}\frac{\partial p_4^{(1)}}{\partial \omega} + \frac{1}{r_0}\frac{\partial v_4^{(0)}}{\partial \rho}$$

$$\frac{\partial p_4^{(3)}}{\partial z} = \left(\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\partial \tau}\right) w_4^{(1)} + \frac{1}{r_0}\frac{\partial w_4^{(0)}}{\partial \rho}$$

$$\frac{\partial u_4^{(3)}}{\partial \rho} = \frac{-1}{r_0}\left(\frac{\partial v_4^{(2)}}{\partial \omega} + u_4^{(2)} + r_0\frac{\partial w_4^{(2)}}{\partial z}\right) + \frac{\rho}{r_0^2}\left(u_4^{(1)} + \frac{\partial v_4^{(1)}}{\partial \omega}\right) - \frac{\rho^2}{r_0^3}\frac{\partial v_4^{(0)}}{\partial \omega}.$$

Problem (i) says simply that the normal velocity and the pressure do not change to first order through the boundary layer. Therefore,

$$u_4^{(0)} \equiv 0\,; \qquad p_4^{(0)} = \left[p_1^{(0)}\right], \quad \textit{the interior pressure at } r = r_0.$$

Problem (ii) then shows that

$$\frac{\partial w_4^{(0)}}{\partial z} = 0\,, \qquad \frac{\partial v_4^{(0)}}{\partial z} = 0. \tag{D.2}$$

Hence, the condition $w_4^{(0)} = 0$ at $z = \pm 1$ requires

$$w_4^{(0)} \equiv 0. \tag{D.3}$$

The mass conservation requirement is satisfied if

$$u_4^{(1)} = \frac{-1}{r_0}\int_0^{\rho} \frac{\partial v_4^{(0)}}{\partial \omega}\, d\rho\,. \tag{D.4}$$

It is necessary that $u_4^{(1)} = 0$ at $\rho = 0$. As subsequent analysis will show, there can be no $O(R^{-1/4})$ mass efflux from the $R^{-1/3}$ layer. However, we still must determine $v_4^{(0)}$. For this we need to impose the condition that the solution match the Ekman layer mass flux at z = $\pm$1. But before doing this we formally complete the solution in D4.

By eliminating $u_4^{(2)}$ in the equations of problem (iii), it is found that

$$\frac{\partial w_4^{(1)}}{\partial z} = -\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right)\frac{\partial v_4^{(0)}}{\partial\rho},$$

whence

$$w_4^{(1)} = \frac{-z}{2}\left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right)\frac{\partial v_4^{(0)}}{\partial\rho} + \hat{w}_4^{(1)}(\rho,\omega,\tau) \tag{D.5}$$

This and the above results are sufficient to establish a relation for $u_4^{(2)}$,

$$\frac{\partial u_4^{(2)}}{\partial\rho} = \frac{-1}{r_0}\frac{\partial v_4^{(1)}}{\partial\omega} - \frac{1}{r_0}\frac{\partial}{\partial\rho}\left(\rho u_4^{(1)}\right) + \frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right)\frac{\partial v_4^{(0)}}{\partial\rho}.$$

$v_4^{(1)}$ must be found by invoking the matching conditions at z = $\pm$1. In addition, problem (iii) entails that

$$\frac{\partial v_4^{(1)}}{\partial z} = 0.$$

We shall need one more component of the solution, the $O(R^{-1/2})$ vertical velocity in D4. This is determined by problem (iv):

$$\frac{\partial w_4^{(2)}}{\partial z} = -\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right)\frac{\partial v_4^{(1)}}{\partial\rho} - \frac{1}{2r_0}\left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right)v_4^{(0)} - \frac{1}{2r_0}\frac{\partial^2 v_4^{(0)}}{\partial\rho^2},$$

$$w_4^{(2)} = -\frac{z}{2}\left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right)\left(\frac{\partial v_4^{(1)}}{\partial\rho} + \frac{v_4^{(0)}}{r_0}\right) - \frac{z}{2r_0}\frac{\partial^2 v_4^{(0)}}{\partial\rho^2} + \hat{w}_4^{(2)}(\rho,\omega,\tau).$$

Region D24:

In what follows the meaning of the double sign $\mp$ or $\pm$ is that the upper sign is used for the Ekman layer near z = +1, the lower sign for the layer near z = -1.

We write

$$\vec{q} = \vec{q}_{24}^{(0)}(\rho,\omega,\zeta,\tau) + R^{-1/4}\vec{q}_{24}^{(1)} + R^{-1/2}\vec{q}_{24}^{(2)} + \ldots$$

with $\rho = R^{1/4}(r-r_0)$, $\zeta = R^{1/2}(1 \mp z)$.

The solutions in this region must (a) match with the adjacent Ekman layer (D2) solutions; (b) match tangential components with the adjacent outer boundary layer (D4) solutions; and (c) satisfy no-slip conditions at $\zeta = 0$.

In addition the small normal outflow $R^{-1/4}w_{24}^{(1)} + R^{-1/2}w_{24}^{(2)} + \ldots$ will give us the necessary conditions to complete the solution in D4.

The governing problem sequence is seen to be:

(i) $$\frac{\partial p_{24}^{(0)}}{\partial \rho} = 0$$

$$\frac{\partial p_{24}^{(0)}}{\partial \zeta} = 0$$

$$\frac{\partial w_{24}^{(0)}}{\partial \zeta} = 0$$

(ii)
$$\frac{\partial^2 u_{24}^{(0)}}{\partial \zeta^2} + 2 v_{24}^{(0)} = \frac{\partial p_{24}^{(1)}}{\partial \rho}$$

$$\frac{\partial^2 v_{24}^{(0)}}{\partial \zeta^2} - 2 u_{24}^{(0)} = \frac{1}{r_0} \frac{\partial p_{24}^{(0)}}{\partial \omega}$$

$$\frac{\partial p_{24}^{(1)}}{\partial \zeta} = 0 , \qquad \frac{\partial w_{24}^{(1)}}{\partial \zeta} = \pm \frac{\partial u_{24}^{(0)}}{\partial \rho}$$

(iii)
$$\frac{\partial^2 u_{24}^{(1)}}{\partial \zeta^2} + 2 v_{24}^{(1)} = \frac{\partial p_{24}^{(2)}}{\partial \rho}$$

$$\frac{\partial^2 v_{24}^{(1)}}{\partial \zeta^2} - 2 u_{24}^{(1)} = \frac{1}{r_0} \frac{\partial p_{24}^{(1)}}{\partial \omega}$$

$$\frac{\partial p_{24}^{(2)}}{\partial \zeta} = 0 , \qquad \frac{\partial w_{24}^{(2)}}{\partial \zeta} = \pm \left\{ \frac{\partial u_{24}^{(1)}}{\partial \rho} + \frac{1}{r_0} \left(u_{24}^{(0)} + \frac{\partial v_{24}^{(0)}}{\partial \omega} \right) \right\} .$$

These equations show that, through $O(R^{-1/2})$, the pressure is constant within the layer:

$$p_{24}^{(k)} = \left(p_4^{(k)} \right)_{z=\pm 1} , \qquad k = 0, 1, 2.$$

Moreover, from the form of the solutions in the adjacent region D2, namely

$$u_2^{(j)} = u_1^{(j)} \left\{ 1 - e^{-\zeta} \cos \zeta \right\} - v_1^{(j)} e^{-\zeta} \sin \zeta$$

$$v_2^{(j)} = v_1^{(j)} \left\{ 1 - e^{-\zeta} \cos \zeta \right\} + u_1^{(j)} e^{-\zeta} \sin \zeta , \qquad j = 0, 1$$

we must require at $\rho = -\infty$ that

$$u_{24}^{(0)} \sim -[v_1^{(0)}]e^{-\zeta}\sin\zeta$$
$$v_{24}^{(0)} \sim +[v_1^{(0)}]\{1-e^{-\zeta}\cos\zeta\}$$

and

$$u_{24}^{(1)} \sim \left(\rho\left[\frac{\partial u_1^{(0)}}{\partial r}\right]+[u_1^{(1)}]\right)\left(1-e^{-\zeta}\cos\zeta\right) - \left(\rho\left[\frac{\partial v_1^{(0)}}{\partial r}\right]+[v_1^{(1)}]\right)e^{-\zeta}\sin\zeta$$
$$v_{24}^{(1)} \sim \left(\rho\left[\frac{\partial v_1^{(0)}}{\partial r}\right]+[v_1^{(1)}]\right)\left(1-e^{-\zeta}\cos\zeta\right) + \left(\rho\left[\frac{\partial u_1^{(0)}}{\partial r}\right]+[u_1^{(1)}]\right)e^{-\zeta}\sin\zeta .$$

We shall write the solutions in the form

$$u_{24}^{(0)} = -v_4^{(0)}e^{-\zeta}\sin\zeta$$
$$v_{24}^{(0)} = v_4^{(0)}\{1-e^{-\zeta}\cos\zeta\}$$

$$u_{24}^{(1)} = u_4^{(1)}\{1-e^{-\zeta}\cos\zeta\} - v_4^{(1)}e^{-\zeta}\sin\zeta$$
$$v_{24}^{(1)} = v_4^{(1)}\{1-e^{-\zeta}\cos\zeta\} + u_4^{(1)}e^{-\zeta}\sin\zeta .$$

To satisfy the matching conditions at $\rho=-\infty$ we take

$$v_4^{(0)} = [v_1^{(0)}] + V_4^{(0)}$$

$$u_4^{(1)} = \rho\left[\frac{\partial u_1^{(0)}}{\partial r}\right] + [u_1^{(1)}] + U_4^{(1)}$$
$$v_4^{(1)} = \rho\left[\frac{\partial v_1^{(0)}}{\partial r}\right] + [v_1^{(1)}] + V_4^{(1)}$$

and require that $U_4^{(1)} \longrightarrow 0$, $V_4^{(0)} \longrightarrow 0$, $V_4^{(1)} \longrightarrow 0$ as $\rho \to -\infty$.

However, in order to determine $U_4^{(1)}$ and $V_4^{(0)}, V_4^{(1)}$ we must first consider the mass outflow conditions at $\zeta = \infty$.

From problem (ii) and the above solution functions, it follows that

$$\frac{\partial w_{24}^{(1)}}{\partial \zeta} = \pm \frac{\partial u_{24}^{(0)}}{\partial \rho} = \mp \frac{\partial v_4^{(0)}}{\partial \rho} e^{-\zeta} \sin \zeta \, .$$

This can be integrated, with the result

$$w_{24}^{(1)} = \mp \frac{1}{2} \frac{\partial v_4^{(0)}}{\partial \rho} \left\{ 1 - e^{-\zeta} (\sin \zeta + \cos \zeta) \right\} .$$

Hence,

$$w_{24}^{(1)} \Big|_{\zeta = \infty} = \mp \frac{1}{2} \frac{\partial v_4^{(0)}}{\partial \rho} = w_4^{(1)} \Big|_{z = \pm 1} . \qquad \text{(D.6)}$$

By similar methods we find from problem (iii) that

$$w_{24}^{(2)} = \mp \frac{1}{2} \left(\frac{\partial v_4^{(1)}}{\partial \rho} + \frac{v_4^{(0)}}{r_0} \right) \left(1 - e^{-\zeta} \{ \sin \zeta + \cos \zeta \} \right)$$

$$w_{24}^{(2)} \Big|_{\zeta = \infty} = \mp \frac{1}{2} \left(\frac{\partial v_4^{(1)}}{\partial \rho} + \frac{v_4^{(0)}}{r_0} \right) = w_4^{(2)} \Big|_{z = \pm 1} . \qquad \text{(D.7)}$$

Now, in the discussion of the solutions in region D4, it was shown that

$$w_4^{(1)} = \frac{-z}{2} \left(\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\partial \tau} \right) \frac{\partial v_4^{(0)}}{\partial \rho} + \hat{w}_4^{(1)}(\rho, \omega, \tau)$$

and

$$w_4^{(2)} = \frac{-z}{2}\left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right)\left(\frac{\partial v_4^{(1)}}{\partial\rho} + \frac{v_4^{(0)}}{r_0}\right) - \frac{z}{2r_0}\frac{\partial^2 v_4^{(0)}}{\partial\rho^2} + \hat{w}_4^{(2)}(\rho,\omega,\tau) \ .$$

By substituting these functions into (D.6) and (D.7), respectively, we derive the equations governing the flow in D4:

$$\left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right)\frac{\partial v_4^{(0)}}{\partial\rho} - \frac{\partial v_4^{(0)}}{\partial\rho} = 0 \ , \qquad (D.8)$$

$$\left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right)\frac{\partial v_4^{(1)}}{\partial\rho} - \frac{\partial v_4^{(1)}}{\partial\rho} = \frac{1}{r_0}\left\{ v_4^{(0)} - \left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right) v_4^{(0)} - \frac{\partial^2 v_4^{(0)}}{\partial\rho^2}\right\}. \qquad (D.9)$$

The solution to (D.8) subject to the boundary conditions

$$v_4^{(0)}\Big|_{\rho=0} = 0 ; \quad v_4^{(0)} \sim \left[v_1^{(0)}\right] \text{ as } \rho \to -\infty ; \quad v_4^{(0)}\Big|_{\tau=0} = 0$$

is readily obtained by Laplace transform calculus:

$$v_4^{(0)} = \left[v_1^{(0)}\right] \operatorname{erf}\left\{\frac{-\rho}{2\tau^{1/2}}\right\} . \qquad (D.10)$$

With the boundary conditions

$$v_4^{(1)}\Big|_{\rho=0} = 0 ; \quad v_4^{(1)} \sim \rho\left[\frac{\partial v_1^{(0)}}{\partial r}\right] + \left[v_1^{(1)}\right] \text{ as } \rho \to -\infty ; \quad v_4^{(1)}\Big|_{\tau=0} = 0$$

the solution to (D.9) is by similar methods seen to be

$$v_4^{(1)} = \rho\left[\frac{\partial v_1^{(0)}}{\partial r}\right] + \left[v_1^{(1)}\right] \operatorname{erf}\left\{\frac{-\rho}{2\tau^{1/2}}\right\} + \left[\frac{v_1^{(0)}}{r}\right] \frac{\rho \operatorname{erfc}\left\{\frac{-\rho}{2\tau^{1/2}}\right\}}{2} . \quad \text{(D.11)}$$

To complete the solution in region D4 we must determine $u_4^{(1)}$ and $u_4^{(2)}$ from the relations

$$\frac{\partial u_4^{(1)}}{\partial \rho} = \frac{-1}{r_0} \frac{\partial v_4^{(0)}}{\partial \omega} ; \quad u_4^{(1)}\Big|_{\rho=0} = 0 \qquad \text{(D.12)}$$

$$\frac{\partial u_4^{(2)}}{\partial \rho} = \frac{-1}{r_0} \frac{\partial v_4^{(1)}}{\partial \omega} - \frac{1}{r_0}\frac{\partial}{\partial \rho}\left(\rho u_4^{(1)}\right) + \frac{1}{2}\left(\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\partial \tau}\right)\frac{\partial v_4^{(0)}}{\partial \rho} ; \quad u_4^{(2)}\Big|_{\rho=0} = 0 . \quad \text{(D.13)}$$

From the first of these we obtain

$$u_4^{(1)} = \left[\frac{\partial u_1^{(0)}}{\partial r}\right]\left\{2\sqrt{\frac{\tau}{\pi}}\left(1 - \exp\left\{\frac{-\rho^2}{4\tau}\right\}\right) + \rho \operatorname{erf}\left\{\frac{-\rho}{2\tau^{1/2}}\right\}\right\} , \qquad \text{(D.14)}$$

since at $r = r_0$ we must have $\partial u_1^{(0)}/\partial r = -(1/r)\, \partial v_1^{(0)}/\partial \omega$.

Clearly, at the outer edge of the boundary layer $(\rho \to -\infty)$ the matching requirement at $O(R^{-1/4})$ is

$$u_4^{(1)} \sim 2\sqrt{\frac{\tau}{\pi}} \left[\frac{\partial u_1^{(0)}}{\partial r}\right] = \left[u_1^{(1)}\right] . \qquad \text{(D.15)}$$

Integration of (D.13) is also a straightforward matter once we make the substitution (cf. (D.8))

$$\left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\tau}\right)\frac{\partial v_4^{(0)}}{\partial\rho} = \frac{\partial v_4^{(0)}}{\partial\rho} .$$

For then,

$$u_4^{(2)} = \frac{-1}{r_0}\int_0^\rho \frac{\partial v_4^{(1)}}{\partial\omega}d\rho - \frac{1}{r_0}\rho u_4^{(1)} + \frac{1}{2}v_4^{(0)} . \qquad (D.16)$$

Now we are primarily interested in the value of $u_4^{(2)}$ at $\rho = -\infty$, for it is this normal mass flux which drives the interior $O(R^{-1/2})$ circulation. We find that as $\rho \to -\infty$,

$$-\frac{1}{r_0}\int_0^\rho \frac{\partial v_4^{(1)}}{\partial\omega}d\rho \sim \frac{\rho^2}{2}\left[\frac{-1}{r_0}\frac{\partial^2 v_1^{(0)}}{\partial r\partial\omega}\right] + \left(\rho + 2\sqrt{\frac{\tau}{\pi}}\right)\left[\frac{-1}{r_0}\frac{\partial v_1^{(0)}}{\partial\omega}\right] + \frac{\tau}{2}\left[\frac{-1}{r_0}\frac{\partial v_1^{(0)}}{\partial\omega}\right]$$

$$-\frac{1}{r_0}\rho u_4^{(1)} \sim \frac{\rho^2}{2}\left[\frac{-2}{r_0}\frac{\partial u_1^{(0)}}{\partial r}\right] + \rho\left[-2\sqrt{\frac{\tau}{\pi}}\cdot\frac{1}{r_0}\frac{\partial u_1^{(0)}}{\partial r}\right]$$

$$\frac{1}{2}v_4^{(0)} \sim \frac{1}{2}\left[v_1^{(0)}\right].$$

At $r = r_0$ we must have

$$\frac{\partial u_1^{(0)}}{\partial r} = -\frac{1}{r_0}\frac{\partial v_1^{(0)}}{\partial\omega}, \qquad \frac{\partial u_1^{(1)}}{\partial r} + \frac{u_1^{(1)}}{r_0} = -\frac{1}{r_0}\frac{\partial v_1^{(1)}}{\partial\omega}$$

with

$$\frac{u_1^{(1)}}{r_0} = 2\sqrt{\frac{\tau}{\pi}}\cdot\frac{1}{r_0}\frac{\partial u_1^{(0)}}{\partial r} .$$

Therefore, as $\rho \to -\infty$,

$$u_4^{(2)} \sim \frac{\rho^2}{2}\left[\frac{\partial^2 u_1^{(0)}}{\partial r^2}\right] + \rho\left[\frac{\partial u_1^{(1)}}{\partial r}\right] + \left\{\frac{1}{2}\left[v_1^{(0)}\right] + \frac{\tau}{2}\left[\frac{\partial u_1^{(0)}}{\partial r}\right] + 2\sqrt{\frac{\tau}{\pi}}\left[\frac{\partial u_1^{(1)}}{\partial r} + \frac{u_1^{(1)}}{r}\right]\right\}.$$

The interior solution will match this mass efflux, provided that

$$\left[u_1^{(2)}\right] = \frac{1}{2}\left[v_1^{(0)}\right] + \frac{\tau}{2}\left[\frac{\partial u_1^{(0)}}{\partial r}\right] + 2\sqrt{\frac{\tau}{\pi}}\left(\left[\frac{\partial u_1^{(1)}}{\partial r}\right] + \left[\frac{u_1^{(1)}}{r}\right]\right). \qquad (D.17)$$

We turn now to consider how the $R^{-1/3}$ layer brings the vertical velocity to zero at the wall.

Region D3:

Expanding the outer (D4) solution in terms ofthe inner variable $\eta = R^{1/3}(r-r_0) = R^{1/12}\rho$,

$$\vec{q}_4 = \left(\vec{q}_4^{(0)}\right)_{\rho=0} + R^{-1/12}\eta\left(\frac{\partial\vec{q}_4^{(0)}}{\partial\rho}\right)_{\rho=0} + R^{-1/6}\frac{\eta^2}{2}\left(\frac{\partial^2\vec{q}_4^{(0)}}{\partial\rho^2}\right)_{\rho=0} + R^{-1/4}\left\{\frac{\eta^3}{3!}\left(\frac{\partial^3\vec{q}_4^{(0)}}{\partial\rho^3}\right)_{\rho=0} + \left(\vec{q}_4^{(1)}\right)_{\rho=0}\right\} + \dots,$$

we have from our already-determined D4 solution that

$$\vec{q}_4 = R^{-1/12}\left\{0, \frac{-\eta[v_1^{(0)}]}{(\pi\tau)^{1/2}}, 0\right\} + R^{-1/4}\left\{0, \frac{\eta^3}{3!}\cdot\frac{[v_1^{(0)}]}{2\tau(\pi\tau)^{1/2}}, \frac{z[v_1^{(0)}]}{2(\pi\tau)^{1/2}}\right\} + \dots .$$

This suggests the expansion to be taken in D3:

$$\vec{q} = R^{-1/12}\vec{q}_3^{(1)} + R^{-1/4}\vec{q}_3^{(3)} + \dots + R^{-k/12}\vec{q}_3^{(k)} + \dots$$

The corresponding asymptotic problem sequence is then seen to be

(i)
$$\frac{\partial^2 w_3^{(1)}}{\partial\eta^2} = \frac{\partial p_3^{(5)}}{\partial z}$$

$$\frac{\partial p_3^{(5)}}{\partial\eta} = 2v_3^{(1)}$$

$$2u_3^{(5)} + \frac{1}{r_0}\frac{\partial p_3^{(5)}}{\partial\omega} = \frac{\partial^2 v_3^{(1)}}{\partial\eta^2}$$

$$\frac{\partial u_3^{(5)}}{\partial\eta} + \frac{1}{r_0}\frac{\partial v_3^{(1)}}{\partial\omega} + \frac{\partial w_3^{(1)}}{\partial z} = 0$$

(iii) $$\frac{\partial^2 w_3^{(3)}}{\partial \eta^2} = \frac{\partial p_3^{(7)}}{\partial z} + \frac{\partial w_3^{(1)}}{\partial \tau}$$

$$\frac{\partial p_3^{(7)}}{\partial \eta} = 2 v_3^{(3)}$$

$$2 u_3^{(7)} + \frac{1}{r_0}\frac{\partial p_3^{(7)}}{\partial \omega} = \frac{\partial^2 v_3^{(3)}}{\partial \eta^2} - \frac{\partial v_3^{(1)}}{\partial \tau}$$

$$\frac{\partial u_3^{(7)}}{\partial \eta} + \frac{1}{r_0}\frac{\partial v_3^{(3)}}{\partial \omega} + \frac{\partial w_3^{(3)}}{\partial z} = 0$$

A solution consistent with (i) and the no-slip conditions at $\eta = 0$ is simply

$$u_3^{(5)} = -\frac{1}{r_0}\int_0^{\eta} \frac{\partial v_3^{(1)}}{\partial \omega}\, d\eta$$

$$v_3^{(1)} = -\eta\, [v_1^{(0)}]\, (\pi\tau)^{-1/2}$$

$$w_3^{(1)} = 0 .$$

Thus the $R^{-1/3}$ layer does not affect the solution to $O(R^{-1/12})$. Nevertheless we record these variables for they are needed in the solution of problem (ii). Indeed we see at once from (ii) that

$$\frac{\partial^3 w_3^{(3)}}{\partial \eta^3} = 2\,\frac{\partial v_3^{(3)}}{\partial z}$$

$$\frac{\partial^3 v_3^{(3)}}{\partial \eta^3} = -2\,\frac{\partial w_3^{(3)}}{\partial z} + \frac{\partial}{\partial \tau}\frac{\partial v_3^{(1)}}{\partial \eta} ,$$

or as a pair of uncoupled equations,

$$\left(\frac{\partial^6}{\partial\eta^6} + 4\frac{\partial^2}{\partial z^2}\right)v_3^{(3)} = 0$$

$$\left(\frac{\partial^6}{\partial\eta^6} + 4\frac{\partial^2}{\partial z^2}\right)w_3^{(3)} = 0 .$$

We determine the solution by separation of variables, subject to the no-slip conditions at $\eta = 0$ and matching conditions at $\eta = -\infty$:

$$\left.\begin{aligned} &\text{At } \eta = 0: \quad && v_3^{(3)} = w_3^{(3)} = \frac{\partial u_3^{(7)}}{\partial z} = 0 \\ &\text{At } \eta = -\infty: && v_3^{(3)} \sim \frac{\eta^3}{3!}\cdot\frac{[v_1^{(0)}]}{2\tau(\pi\tau)^{1/2}} \\ & && w_3^{(3)} \sim \frac{z\,[v_1^{(0)}]}{2(\pi\tau)^{1/2}} . \end{aligned}\right\} \qquad (D.18)$$

Note that the condition $\partial u_3^{(7)}/\partial z = 0$ is in general necessary but not sufficient to ensure $u_3^{(7)} = 0$ at $\eta = 0$; but it shall be evident from the solution that in fact $u_3^{(7)} = 0$ at $\eta = 0$.

Let

$$w_3^{(3)} = \sum_{n,k}\left(B_{nk}(\eta)\,e^{ik\omega}\sin n\pi z\right) + \frac{z\,[v_1^{(0)}]}{2(\pi\tau)^{1/2}} . \qquad (D.19)$$

Then

$$v_3^{(3)} = -\frac{1}{2}\sum\left(B_{nk}'''(\eta)\, e^{ik\omega}\frac{\cos n\pi z}{n\pi}\right) + \frac{\eta^3}{3!}\cdot\frac{[v_1^{(0)}]}{2\tau(\pi\tau)^{1/2}} \qquad (D.20)$$

$$\frac{\partial u_3^{(3)}}{\partial z} = \frac{1}{2}\left\{\frac{\partial^2}{\partial\eta^2}\frac{\partial v_3^{(3)}}{\partial z} - \frac{1}{r_0}\frac{\partial}{\partial\omega}\frac{\partial^2 w_3^{(3)}}{\partial\eta^2}\right\}$$

$$= \frac{1}{2}\sum\left(e^{ik\omega}\sin n\pi z\left\{\frac{1}{2}\frac{d^5 B_{nk}}{d\eta^5} - \frac{ik}{r_0}\frac{d^2 B_{nk}}{d\eta^2}\right\}\right) \qquad (D.21)$$

The coefficients B_{nk} satisfy the sixth-order equation

$$\frac{d^6 B_{nk}}{d\eta^6} + 4n^2\pi^2 B_{nk} = 0.$$

Solutions such that $B_{nk} \to 0$ as $\eta \to -\infty$ are, with $\kappa_n \equiv |2n\pi|^{1/3}$,

$$B_{nk}(\eta) = a_{nk1}\exp\{\kappa_n\eta\} + a_{nk2}\exp\{e^{i\pi/3}\kappa_n\eta\} + a_{nk3}\exp\{e^{-i\pi/3}\kappa_n\eta\}. \quad (D.22)$$

The boundary conditions (D.18) at $r_1 = 0$ together with the expansions (D.19)-(D.21) give us three equations for the determination of the constant coefficients a_{nk1}, a_{nk2}, a_{nk3}:

$$\sum e^{ik\omega}\sin n\pi z\left(a_{nk1} + a_{nk2} + a_{nk3}\right) = \frac{-z\,[v_1^{(0)}]}{2\,(\pi\tau)^{1/2}}$$

$$\sum e^{ik\omega}\cos n\pi z\left(a_{nk1} - a_{nk2} - a_{nk3}\right) = 0$$

$$\sum \kappa_n^2\, e^{ik\omega}\sin n\pi z\left\{\left(n\pi - \frac{ik}{r_0}\right)a_{nk1} - \left(n\pi + \frac{ik}{r_0}\right)\left(e^{i2\pi/3}a_{nk2} + e^{-i2\pi/3}a_{nk3}\right)\right\} = 0.$$

Since

$$-\frac{z}{2} = \sum_{1}^{\infty} (-)^n \frac{\sin n\pi z}{n\pi},$$

the above equations are equivalent to

$$a_{nk1} + a_{nk2} + a_{nk3} = \frac{(-)^n}{n\pi} \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\omega} \frac{[v_1^{(0)}]}{(\pi\tau)^{1/2}} \, d\omega$$

$$a_{nk1} - a_{nk2} - a_{nk3} = 0$$

$$n\pi\left(a_{nk1} - e^{i2\pi/3} a_{nk2} - e^{-i2\pi/3} a_{nk3}\right) = \frac{ik}{r_0}\left(a_{nk1} + e^{i2\pi/3} a_{nk2} + e^{-i2\pi/3} a_{nk3}\right).$$

From this we readily find that

$$a_{nk1} = A_{nk}(r_0,\tau)$$

$$a_{nk2} = A_{nk}(r_0,\tau) \left\{ \frac{e^{-i\pi/3}\left(n\pi - \frac{k}{\sqrt{3}\, r_0}\right)}{n\pi + \frac{ik}{r_0}} \right\}$$

$$a_{nk3} = A_{nk}(r_0,\tau) \left\{ \frac{e^{i\pi/3}\left(n\pi + \frac{k}{\sqrt{3}\, r_0}\right)}{n\pi + \frac{ik}{r_0}} \right\}$$

where

$$A_{nk}(r_0,\tau) \equiv \frac{(-)^n}{2n\pi} \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\omega} \frac{[v_1^{(0)}]}{(\pi\tau)^{1/2}} \, d\omega .$$

Substituting these back into (D.22) and (D.19)-(D.21) yields the $O(R^{-1/4})$ solution in D3 which satisfies the no-slip conditions at the wall:

$$v_3^{(3)} = \frac{\eta^3}{3!}\cdot\frac{[v_1^{(0)}]}{2\tau(\pi\tau)^{1/2}} - \sum_{n,k} A_{nk}(r_0,\tau)e^{ik\omega}\cos n\pi z\left\{\exp(\kappa_n\eta) - f_{nk}(\eta)\right\}$$

$$w_3^{(3)} = \frac{z\,[v_1^{(0)}]}{2(\pi\tau)^{1/2}} + \sum A_{nk}(r_0,\tau)\,e^{ik\omega}\sin n\pi z\left\{\exp(\kappa_n\eta) + f_{nk}(\eta)\right\}$$

where

$$f_{nk}(\eta) \equiv \frac{2\exp(\frac{1}{2}\kappa_n\eta)}{n\pi + \frac{ik}{r_0}}\left\{n\pi\cos\left(\frac{\sqrt{3}}{2}\kappa_n\eta - \frac{\pi}{3}\right) - \frac{ik}{\sqrt{3}\,r_0}\sin\left(\frac{\sqrt{3}}{2}\kappa_n\eta - \frac{\pi}{3}\right)\right\}.$$

This circulation is, as we have seen, balanced by an $O(R^{-7/12})$ normal component:

$$u_3^{(7)} = -\sum\left(n\pi - \frac{ik}{r_0}\right)A_{nk}(r_0,\tau)\,e^{ik\omega}\left(\frac{\cos n\pi z}{\kappa_n}\right)\left(\exp\{\kappa_n\eta\} - g_{nk}\{\eta\}\right),$$

where

$$g_{nk}(\eta) \equiv \frac{2\exp(\frac{1}{2}\kappa_n\eta)}{n\pi - \frac{ik}{r_0}}\left\{n\pi\cos\left(\frac{\sqrt{3}}{2}\kappa_n\eta + \frac{\pi}{3}\right) - \frac{ik}{\sqrt{3}\,r_0}\sin\left(\frac{\sqrt{3}}{2}\kappa_n\eta + \frac{\pi}{3}\right)\right\}.$$

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BIOGRAPHY

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