Applications and Extensions of Fomin's Generalization of the Robinson-Schensted Correspondence to Differential Posets

by

Thomas Walton Roby V

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 1991

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Abstract

The Robinson-Schensted (R-S) correspondence and its many variations lie at the combinatorial heart of many facts from representation theory and symmetric function theory. They provide concrete bijective proofs of results that were often originally obtained in much more algebraic or abstract ways. Most of these results can be viewed as counting Hasse walks in certain partially ordered sets. Stanley was able to derive many enumerative results on the class of *differential posets* (of which Young's lattice is a member) using a highly algebraic approach which converted certain enumerative problems to (solvable) partial differential equations. Fomin (independently) defined essentially the same class of graphs and constructed a generalization of the R-S correspondence to differential posets. We show how Fomin's construction can be used to unify many of the R-S variants, including Knuth's generalization to semi-standard tableaux, the skew algorithms of Sagan and Stanley, the oscillating algorithms of Sundaram, and the oscillating Knuth algorithm of Gessel. It allows one to view all these variants as natural constructions.

Besides Young's lattice, the other interesting example of a differential poset is the Fibonacci lattice. We use Fomin's methods to construct a R-S type bijection and prove some of its properties. In particular, we are able to give an equivalent insertion algorithm and an analogue of the Greene-Kleitman-Fomin correspondence for turning a permutation into a poset.

Sequentially differential posets are a more general class of posets which include the differential posets as a special case. There are many more interesting examples of sequentially differential posets, but the enumeration of their Hasse walks is more complicated. By generalizing Fomin's construction, we are able to give bijective proofs of Stanley's results and derive some new results as well.

Thesis Supervisor: Richard P. Stanley
Title: Professor of Applied Mathematics

Acknowledgments

Mathematically, I would like to thank my advisor, Richard Stanley, who was always generous with his time and ideas. I learned a great deal from him and the other faculty. I am also indebted to my fellow graduate students, especially Art Duval, Vic Reiner, William Jockusch, and Jan Pedersen. Ira Gessel deserves special mention for being willing to explain things after hours at folkdancing. Steve Maurer originally got me interested in combinatorics at Swarthmore. Anders Björner inspired me to choose combinatorics over algebraic topology by demonstrating during my first year at MIT that I could have my cake and eat it too (not that I did). Steve Viktora and the rest of Kenwood's math department made sure that I was challanged during my high school years. Arnold Ross and the Ohio State Program provided mathematical and teaching challenges over five summers.

Financial support was provided by MIT, by teaching jobs at Hampshire College's Summer Studies in Mathematics, Boston University's ProMYS program, Tufts University, and Harvard's Quantitative Reasoning Requirement, and by my parents.

For typesetting, I would like to thank Donald Knuth for TEX, Leslie Lamport for LATEX, and Mark Haiman for UltraTEX mode in emacs. Ira Gessel and Mark Haiman also provided macros and advice. As a Project Athena user consultant, I cut my teeth on many other students' LATEX thesis questions long before I started to have my own. Typesetting thus became a far more pleasant task than it would have been otherwise.

Richard Stanley, Art Duval, Ira Gessel, and Alex Doohovskoy read preliminary drafts and suggested many corrections and improvements.

I would like to thank the many who make ethnic dancing in the Boston area such a treat, especially those, too numerous to mention, who have kept it going at MIT so successfully for so long. My recreational hours have been greatly enriched by the quality of the folkdance community.

I would like to thank Sat Darshan Khalsa for helping me learn to relax in a high-pressure environment.

I would like to thank my advisor and the MIT Japan Program for making it possible for me to spend the next year in Tokyo, thus bypassing what was (according to Professor Rota) the worst U. S. job market for new Ph.D.'s since 1932.

Finally I would like to thank my parents, Anne, Tom, and Mary, for their continued support over the course of my academic career.

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Chapter 1

The Original R-S Correspondence

This chapter contains background material for those unfamiliar with the Robinson-Schensted correspondences. Many readers may prefer to skip this section and just consult it as necessary to understand my taste in notation. My choices of notation and presentation have been lifted (sometimes almost verbatim) from several of my predecessors, especially B. Sagan [SS], R. P. Stanley [Sta1], J. R. Stembridge [Ste], and S. Sundaram [Sun].

1.1 Partitions and shapes

We begin by defining objects which are ubiquitous in the subject.

Definition 1.1.1 A **partition** λ is a sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

such that:

1. The terms are weakly decreasing, i.e., $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$
2. Only a finite number of the terms are nonzero.

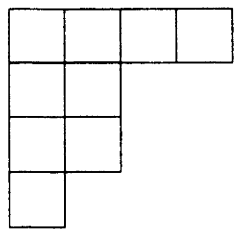


Figure 1-1: The Young diagram for 4221

The nonzero terms are called the **parts** of λ. We keep track of the multiplicity of each part via

$$m_i(\lambda) := \#\{j \geq 1 : \lambda_j = i\}$$

The number of parts is called the **length** of λ and is written $\ell(\lambda)$. The sum of the parts is called the **weight**, and we write

$$|\lambda| = \sum_{i \geq 1} \lambda_i$$

The unique partition of weight 0 is denoted by $\emptyset$. If the weight of λ is n, then we say λ is a **partition of** n and write $|\lambda| = n$ or $\lambda \vdash n$.

Example 1.1.2 In practice, one suppresses the commas and trailing zeroes in a partition, and even removes the parentheses. With such a convention the partitions of 5 are

$$5, \quad 41, \quad 32, \quad 311, \quad 221, \quad 2111, \quad 11111$$

Geometrically one can view a partition as a left-justified array of dots or empty boxes. The former are called **Ferrers diagrams** and the latter **Young diagrams**. See Figure 1-1. More technically, we have the following definition.

Definition 1.1.3 The **diagram** or **shape** of a partition λ is the set

$$D_\lambda = \{(i, j) \in \mathbf{Z}^2 : 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$$

We must think of $\mathbb{Z}^2$ as being ordered so that the positive axes point downward and to the right (as with matrices) in order for this to agree with our current pictures; the usual coordinate geometry convention is to let the positive axes point upwards and to the right.

We will often blur the distinction between a partition and its shape; the notation $x \in \lambda$ should be taken to mean $x \in D_\lambda$. We call each empty box in this representation of a shape a **cell**.

Definition 1.1.4 The **conjugate** of a partition λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ whose diagram is given by

$$D_{\lambda'} = \{(i,j) \in \mathbb{Z}^2 : (j,i) \in D_\lambda\}$$

In other words, the diagram of λ' is obtained from that of λ by exchanging the rows and columns. Algebraically it may be characterized by

$$n_i(\lambda') = \lambda_i - \lambda_{i+1}$$

The conjugate of the partition $\lambda = 4221$ in Figure 1-1 is $\lambda' = 4311$.

Definition 1.1.5 We may define a partial order $\subseteq$ on partitions by $\mu \subseteq \lambda$ if and only if $D_\mu \subseteq D_\lambda$. Equivalently,

$$\mu \subset \lambda \iff \mu_i \leq \lambda_i \qquad \forall \quad i \geq 1.$$

This partial order is easily seen to be a distributive lattice, Y, which we will call **Young's Lattice** and which will play a significant role in what follows.

Definition 1.1.6 If $\mu \subset \lambda$, we let λ/μ denote the set-theoretic difference

$$D_\lambda \backslash D_\mu = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i\}$$

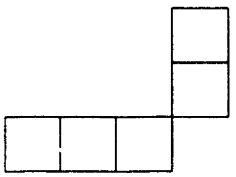


Figure 1-2: The skew shape 443/32

and call λ/μ a **skew diagram** or a **skew shape**. Equivalently we may define a skew shape as a convex set in the in the poset $\mathbf{Z}^2$ (where we view the ordering of $\mathbf{Z}^2$ as in Definition 1.1.3. The notion of weight and conjugate are extended to skew shapes in the natural way:

$$|\lambda/\mu| = |\lambda| - |\mu|$$

$$(\lambda/\mu)' = \lambda'/\mu'$$

Definition 1.1.7 The skew shape λ/μ is called a **horizontal strip** if no two cells share the same column, and a **vertical strip** if no two cells share the same row.

Definition 1.1.8 Let λ/μ be a skew shape. A **tableau** T **of shape** λ/μ is an order preserving map

$$T : D_{\lambda/\mu} \mapsto \mathbf{Z}^+$$

i.e., an assignment of positive integers to the cells of λ which is weakly increasing along the columns (from left to right) and down the rows. We define the **weight** or **type** of a tableau to be the sequence

$$w(T) = (n_1, n_2, n_3, \ldots)$$

where n_i is the number of cells of λ/μ assigned to the integer i by T. Equivalently, we may consider T to be a multichain in Young's lattice, i.e., a weakly increasing sequence of shapes

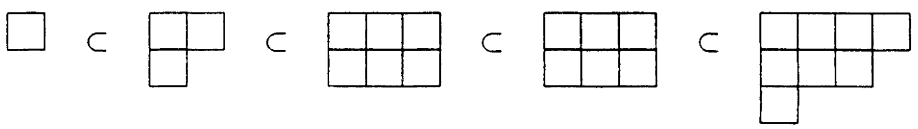
$$\mu = \lambda^0 \subseteq \lambda^1 \subseteq \ldots \subseteq \lambda^k = \lambda$$

by simply filling each skew shape λ^i/λ^{i-1} with the integer i. We write $sh(T) = \lambda/\mu$.

Example 1.1.9 The tableaux

$$T = \begin{array}{|c|c|c|c|}\hline & 1 & 2 & 4 \\ \hline 1 & 2 & 2 \\ \cline{1-3} 4 & 4 \\ \cline{1-2}\end{array}$$

corresponds to the sequence of shapes



Sometimes tableaux are written by just aligning the numbers and omitting the surrounding boxes. We do this when the context suggests it might improve the readability. With this convention, the tableau in Example 1.1.9 is written

$$T = \begin{array}{cccc} & 1 & 2 & 4 \\ 1 & 2 & 2 & \\ 4 & 4 & & \end{array}$$

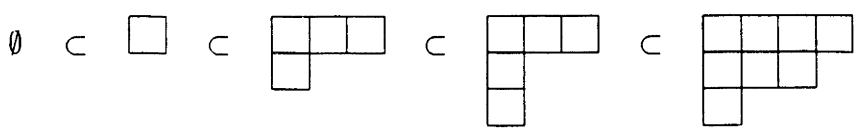
We shall be particularly interested in some special kinds of tableaux.

Definition 1.1.10 A tableaux T is said to be **column-strict** or **semi-standard** or **generalized** if it is strictly increasing along the columns. When the tableaux is viewed as a multichain, this says exactly that each λ^i/λ^{i-1} is a horizontal strip. T is called **standard** if it has type $(1, 1, \ldots, 1)$, i.e., if all the entries are distinct (so we may take them to be the numbers $1, 2, \ldots, |T|$). When viewed as a multichain, this says exactly that each λ^i/λ^{i-1} is a single cell; hence, it represents a saturated chain in $\mathbf{Y}$. The sets of generalized and standard tableaux will be denoted $ST(\lambda/\mu)$ and $GT(\lambda/\mu)$, respectively.

Example 1.1.11 The column-strict tableau

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 4 \\ \hline 2 & 4 & 4 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}$$

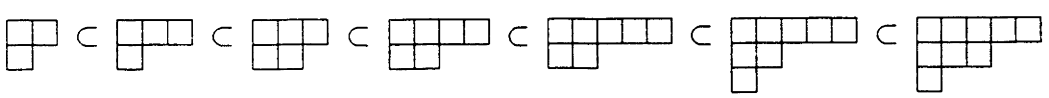
corresponds to the sequence of shapes



While the standard tableau

$$T = \begin{array}{|c|c|c|c|}\hline & & 1 & 3 & 4 \\ \hline & 2 & 6 \\ \cline{2-3} 5 \\ \cline{1-1}\end{array}$$

corresponds to the saturated chain



We will be interested in counting the number of chains in Young's lattice which satisfy certain conditions. The next two definitions count the special tableaux we just defined.

Definition 1.1.12 $f^{\lambda/\mu} := \#\{\text{standard tableaux } T \text{ of shape } \lambda/\mu\}$

Definition 1.1.13 The generating function for column-strict tableaux of a given shape is called the **Schur function**. For convenience we take an infinite set of variables $\{x_1, x_2, \ldots\}$. We have

$$s_{\lambda/\mu}(x_1, x_2, \ldots) := \sum_{T \in GT(\lambda/\mu)} (x_1^{n_1} x_2^{n_2} \ldots)$$

where n_k is the number of elements of T equal to k (as in the definition of **type** above).

It turns out that these Schur functions are symmetric and that $\{s_\lambda : \lambda \vdash n\}$ forms an orthogonal basis for the space Λ of symmetric functions, with respect to a natural scalar product. Since symmetric functions are not the main focus of our discussion, we refer the reader to [Mac] for more information about symmetric functions and their connections with representation theory.

Next we define permutation-like objects.

Definition 1.1.14 A **biword**, π, is a sequence of vertical pairs of positive integers

$$\pi = \begin{matrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{matrix}$$

with $i_1 \leq i_2 \leq \ldots \leq i_k$. We denote the top and bottom lines of π by $\hat{\pi} = i_1 i_2 \ldots i_k$ and $\check{\pi} = j_1 j_2 \ldots j_k$. We will consider three types of biwords. **Partial permutations of n** have no entries greater than n, and within each line the entries are distinct. **Permutations of n** are partial permutations with top line $1, 2, \ldots, n$. Finally, $n \times n$ **matrix words** have no entries greater than n with the pairs arranged lexicographically: $i_r = i_{r+1}$ implies $j_r \leq j_{r+1}$. The sets of partial permutations, permutations, and $n \times n$ matrix words will be denoted PS_n, S_n, and $\mathrm{MAT}(n)$, respectively.

Example 1.1.15 An example of each of these when $n = 5$:

$$\pi = \begin{matrix} 1 & 2 & 4 \\ 4 & 2 & 3 \end{matrix}, \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{matrix}, \quad \begin{matrix} 1 & 1 & 1 & 2 & 2 & 4 & 4 & 4 \\ 1 & 3 & 3 & 2 & 5 & 1 & 5 & 5 \end{matrix}$$

Sometimes the top row $\hat{\pi}$ of a permutation is suppressed, and we just write $\pi = 42531$ to mean $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{matrix}$. Note that there is a bijective correspondence between matrix words and $n \times n$ matrices $M = (m_{ij})$ with nonnegative integral entries given by m_{ij} is the number of times the pair $\binom{i}{j}$ appears in π.

Example 1.1.16 The matrix corresponding to the matrix word above is

$$M = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1.2 The basic algorithm via insertion

The basic Robinson-Schensted algorithm is a bijection between permutations and pairs of standard tableaux of the same shape. More precisely, we have the following

Theorem 1.2.1 Fix an integer n. There is a bijection between permutations in S_n and pairs of standard Young tableaux (P, Q) of the same shape λ, where λ runs over all partitions of n:

$$\pi \quad \overset{\text{R-S}}{\longleftrightarrow} \quad (P, Q)$$

The usual approach to the R-S algorithm is through repeated use of a process called *insertion*. To insert a number a into a given tableaux T, we look in the top row of T for the least number which is greater than a, call it b. We replace ("bump") b with a in the top row, and view b as living temporarily in the crack between the two rows. Now proceed with b and the second row in exactly the same way; i.e., if c is the least number in the second row bigger than, then b bumps c, and we view c as temporarily homeless between rows two and three. Eventually, and this can happen even in the first step, the homeless number will be greater than all the others in the row below it, and we place it at the end of this row, terminating the insertion procedure. An example follows forthwith.

Example 1.2.2 Suppose we insert 4 into the tableau

$$\begin{array}{lll} 1 & 3 & 5 \\ 2 & 7 & \\ 6 & 9 & \\ 8 & & \end{array}$$

We get (showing the intermediate steps)

$$\begin{array}{lll} 1 & 3 & {\scriptstyle 4}5 \\ 2 & 7 & \\ 6 & 9 & \\ 8 & & \end{array} \qquad \begin{array}{lll} 1 & 3 & 4 \\ 2 & {\scriptstyle 5}7 & \\ 6 & 9 & \\ 8 & & \end{array} \qquad \begin{array}{lll} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & {\scriptstyle 7}9 & \\ 8 & & \end{array} \qquad \begin{array}{lll} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & 7 & \\ 8 & 9 & \end{array}$$

The last tableau on the right is the result of this insertion.

If we start with a permutation π viewed as a biword, we proceed as follows. Initialize the tableaux P and Q to be empty. Successively insert the elements of $\check{\pi}$ from left to right into the tableaux P. At the ith step, when the bumping process ceases, a new cell will have been added to P; add a cell to Q in the corresponding position, and put the value i inside it. Hence, at each step, P and Q have the same shape, and Q records the order in which cells were added to P. An example follows forthwith.

Example 1.2.3 If we apply the above to the permutation

$$\begin{array}{lllll} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{array}$$

We get the following sequence of tableaux:

$$P: \quad \emptyset \qquad \begin{array}{l} 3 \end{array} \qquad \begin{array}{l} 1 \\ 3 \end{array} \qquad \begin{array}{ll} 1 & 4 \\ 3 & \end{array} \qquad \begin{array}{ll} 1 & 2 \\ 3 & 4 \end{array} \qquad \begin{array}{lll} 1 & 2 & 5 \\ 3 & 4 & \end{array}$$

$$Q: \quad \emptyset \quad \begin{array}{c} 1 \end{array} \quad \begin{array}{cc} 1 & \\ 2 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array} \quad \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \end{array}$$

It is easy to see from this definition of R-S that we have a bijection. At each stage, we know by looking at the highest entry of Q which cell was added to P. We can then reverse the bumping procedure from the element, say c in that cell. Now c must have been bumped by the greatest element smaller than c in the row above it, say b. So we replace b with c, and view b as temporarily homeless between rows. Then we repeat this, allowing b to unbump the greatest element smaller than b in the row above. Eventually, we force an element out the top of the tableau, say a, which must have been the element we originally inserted into P. Meanwhile, by undoing each stage of the bumping process, we have restored P to its condition before a was inserted.

By counting the items on each side of this bijection we obtain

Corollary 1.2.4

$$\sum_{\lambda \vdash n} f_\lambda^2 = n! \tag{1.1}$$

1.3 Properties of the correspondence

The Robinson-Schensted correspondence has many important properties which are not immediately obvious from the above description. We defer proofs until later (2.7 when we have built up the machinery of Fomin's approach. The reader who wishes to see how these results are derived from the insertion algorithm is urged to consult Sagan's excellent, readable account [Sag].

Schensted was originally interested in studying increasing and decreasing subsequences in a permutation. After a definition, we give Schensted's original main result.

Definition 1.3.1 Suppose the permutation π has bottom row $\tilde{\pi} = w_1 w_2 \cdots w_n$. An

increasing subsequence of π is

$$w_{i_1} < w_{i_2} < \cdots < w_{i_j} \tag{1.2}$$

where

$$i_1 < i_2 < \cdots i_j$$

j is called the *length* of the subsequence. One defines *decreasing subsequences* analogously, by replacing "<" with ">" in (1.2) above.

Theorem 1.3.2 (Schensted) Let $\pi \overset{\text{R-S}}{\longleftrightarrow} (P,Q)$. The length of the longest increasing subsequence of π equals the length of the first row of P, and the length of the longest decreasing subsequence of π equals the length of the first column of P.

Checking this against our Example 1.2.3, we find that the longest increasing subsequence is $1,2,5$ of length 3, and the longest decreasing subsequences are $3,1$ and $4,2$ of length 2. Correspondingly, P's first row is length 3, and its first column is length 2.

The algorithm behaves surprisingly well with respect to inverses.

Theorem 1.3.3 ([Scü1]) If

$$\pi \overset{\text{R-S}}{\longleftrightarrow} (P,Q)$$

then

$$\pi^{-1} \overset{\text{R-S}}{\longleftrightarrow} (Q,P)$$

The reader is invited to check that this works for Example 1.2.3, i.e that

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{matrix} \overset{\text{R-S}}{\longleftrightarrow} \left(\begin{matrix} 1 & 3 & 5 \\ 2 & 4 \end{matrix} , \begin{matrix} 1 & 2 & 5 \\ 3 & 4 \end{matrix} \right)$$

By Theorem 1.3.3 when we restrict the correspondence to permutations which are involutions, we insist that $P = Q$, yielding

Corollary 1.3.4 Let Inv_n denote the set of involutions of S_n (i.e., all permutations σ such that $\sigma{=}\sigma^{-1}$). Then we have a bijection between permutations in Inv_n and standard tableau of shape λ, where λ runs over all partitions of n:

$$\sigma \overset{\text{R-S}}{\longleftrightarrow} P$$

Example 1.3.5 The reader is invited to check that

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 3 & 6 & 2 & 4 \end{array} \overset{\text{R-S}}{\longleftrightarrow} \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

Counting these gives us an analogue of (1.1).

Corollary 1.3.6

$$\sum_{\lambda \vdash n} f_\lambda = |Inv_n| \tag{1.3}$$

Theorem 1.3.7 (Schützenberger) Let π be an involution in S_n and assume that

$$\pi \overset{\text{R-S}}{\longleftrightarrow} P$$

Then the number of fixed points of π is equal to the number of columns of P of odd length.

In Example 1.3.5 our involution had two fixed points (1 and 3), and column lengths 3, 2, 1.

Chapter 2

Fomin's approach to Schensted

In this chapter we outline S. V. Fomin's approach to the Robinson-Schensted correspondence, which is in many ways more natural and general than the usual bumping procedure. It provides a pictorial way of looking at things that is often helpful in understanding certain properties, some of which are difficult to show from the bumping description. Fomin's results apply not only to Young's lattice but to an entire class of posets called *Differential Posets* or *Y-graphs*. These were discovered by Fomin in his development of the theory we present here, and independently by Stanley in a quite different context. In [Sta1] Stanley was able to derive many enumerative results involving walks or chains in a differential poset by constructing an algebra of operators on the poset. The (formal) solution of certain partial differential equations involving these operators yielded generating functions counting such walks. Stanley's results are powerful but entirely algebraic. Fomin's approach gives bijective proofs of some of Stanley's results.

Our exposition largely follows that of Fomin [Fom2], but we have followed Stanley and others in certain places. In particular, Fomin's "graded graph" is our "graded poset", and his "Y-graph" is (essentially) our "differential poset".

2.1 Differential Posets

Much of this section follows Stanley's original paper [Sta1] almost verbatim.

In general if P is any graded poset, then we let ρ denote its rank function, i.e., if $x \in P$ then $\rho(x)$ is the length l of the longest chain $x_0 < x_1 < \cdots < x_l = x$ in P with top element x. Write

$$P_i = \{x \in P : \rho(x) = i\}$$

So $P = P_0 \uplus P_1 \uplus P_2 \uplus \cdots$ (disjoint union).

Definition 2.1.1 Let r be a positive integer. A poset P is called **r-differential** if it satisfies the following three conditions:

(D1) P is locally finite, graded, and has a $\hat{0}$ element.

(D2) If $x \neq y$ in P and there are exactly k elements of P which are covered by both x and y, then there are exactly k elements of P which cover both x and y.

(D3) If $x \in P$ and x covers exactly k elements of P, then x is covered by exactly $k + r$ elements of P.

When $r = 1$, we will sometimes omit the r in r-differential.

Property (D2) is essentially a modularity condition.

Proposition 2.1.2 ([Sta1]) If P is a poset satisfying (D1) and (D2), then for $x \neq y$ in P the integer k of (D2) is equal to zero or one.

Proof: Suppose the contrary. Let x and y be elements of minimal rank for which $k > 1$. Then x and y both cover elements $x_1 \neq y_1$ of P. But x_1 and y_1 are elements of smaller rank with $k > 1$, a contradiction. □

For a lattice L satisfying (D1), condition (D2) is equivalent to L being modular (See, e.g., [Bir, Theorem 16, p. 41]).

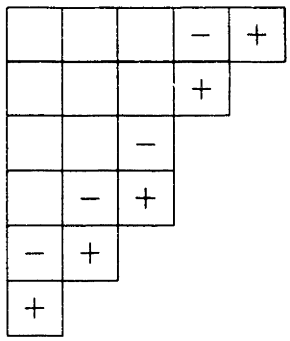


Figure 2-1: Y has the property (D3)

The prototypical example of a differential poset is Young's lattice Y. Property (D1) is clear. Since Y is distributive, being the lattice of finite order ideals of the poset $\mathbb{N}^2$, it is modular. Property (D3) may be seen geometricaliy. Suppose the partition λ has d rows of distinct length. One obtains all the d shapes which cover that of λ by deleting one square from each row which has a strictly smaller row below it. To get the $d+1$ shapes which cover λ, one adds a square to any row which has a strictly larger row above it, including the first length 0 row. See Figure 2-1, where $\lambda = 43321$, and the squares which can be deleted or added are indicated with $-$ and $+$, respectively. In fact, it is not much harder to show that $\mathbf{Y}^r$ is r-differential [Sta1, Cor. 1.4].

2.2 Fibonacci Differential Posets

First we give Stanley's definition [Sta1, Def. 5.2] almost verbatim.

Definition 2.2.1 Let r be a positive integer. Let A_r be an alphabet of $r+1$ letters. We will regard the letters as consisting of the number 1 with r different "colors," denoted $1_1, 1_2, \ldots, 1_r$, together with the number 2. Thus $A_r = \{1_1, 1_2, \ldots, 1_r, 2\}$. Let A_r^* denote the free monoid generated by A_r, i.e., A_r^* consists of all finite words $a_1, a_2, \ldots, a_k$ (including the empty word $\emptyset$) of elements of A_r.

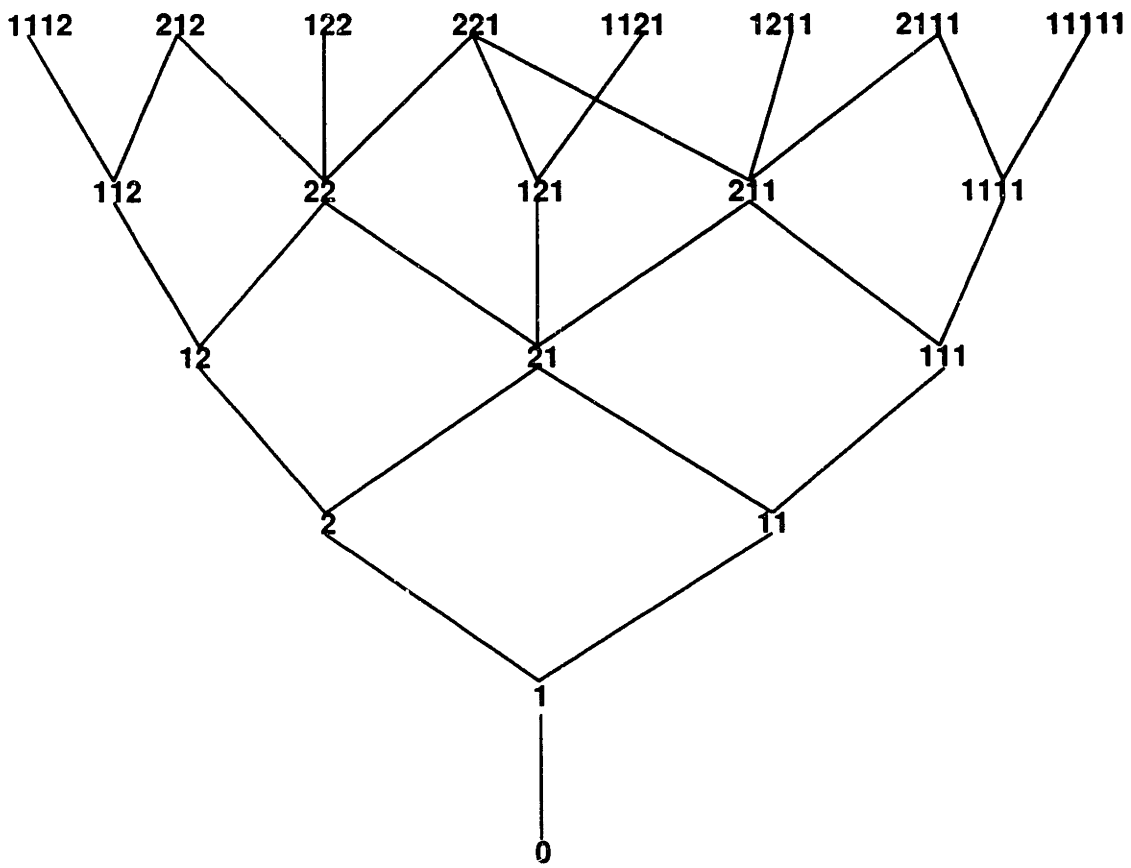


Figure 2-2: The Fibonacci Lattice $Z(1)$

Define a poset $Z(r)$ as follows. As a set, $Z(r)$ coincides with A_r^*. If $w \in Z(r)$, then define w' to be covered by w in $Z(r)$ if either:

(a) w' is obtained from w by changing a 2 to some 1_i, provided that the only letters to the left of this 2 are also 2's, or

(b) w' is obtained from w by deleting the first letter of the form 1_i.

This defines the cover relations in $Z(r)$, and hence by transitivity a partial ordering of $Z(r)$. We call $Z(r)$ the *Fibonacci r-differential poset.*

Geometrically, one can think of $Z(r)$ as being built up inductively in the following manner. Start with a single element which will be the $\hat{0}$ of the poset P. Add the elements $1_1, 1_2, \ldots, 1_r$ above it, to yield the rank 1 elements. Now reflect P about the rank one element, which in this case places exactly one element (the reflection of $\hat{0}$) at level two, and label that element by prepending the letter 2. Then add elements $1_i 1_j$ by prepending each of $1_1, 1_2, \ldots, 1_r$ to each the rank 1 elements. In general, once $P_{[0,i]}$ has been constructed, we construct $P_{[i,i+1]}$ as follows. First reflect $P_{[i-1,i]}$, and prepend the letter 2 to each element at level $i+1$; this insures that condition (D2) will continue to be satisfied. Then add elements above each of the elements of P_i by appending in turn $1_1, 1_2, \ldots, 1_r$; this insures that (D3) will continue to be satisfied.

This process of *reflection extension*, due to D. Wagner, actually applies more generally to any *partial r-differential poset*, i.e. a poset which is r-differential up to a certain rank n. Any such poset may be truncated at the nth level, and then the above process applied to yield a full-fledged r-differential poset. In particular, one can truncate Υ^r at any level and the rebuild it by reflection extension, yielding a large collection of non-isomorphic posets. This is one reason to suspect that the classification of all r-differential posets may not have a reasonable answer. See [Sta1, p. 957ff] for more information.

2.3 Growths

In any graded poset, if y covers x (i.e., $x \leq y$ and $\rho(y) = \rho(x) + 1$) we will sometimes write "$x \lessdot y$".

Definition 2.3.1 Let P and Q be graded posets. A map $g : P \mapsto Q$ is called a *growth* if it preserves the relation $\leq$:

$$x \leq y \longrightarrow g(x) \leq g(y)$$

Not every order preserving map is a growth, and not every growth is one-to-one.

Example 2.3.2

(1) The rank function $\rho : P \mapsto \mathbf{Z}$ is a growth.

(2) A multichain in P is a growth from $\mathbf{Z}$ to P.

(3) The composition of two growths is a growth.

Definition 2.3.3 Let $g : P \mapsto Q$ be any growth, and $\rho : Q \mapsto \mathbf{Z}$ be the rank function of Q. By composing these we get a new growth called the *modulus* of g and written

$$|g| : P \mapsto \mathbf{Z}$$

We will be concerned primarily with growths on skew shapes.

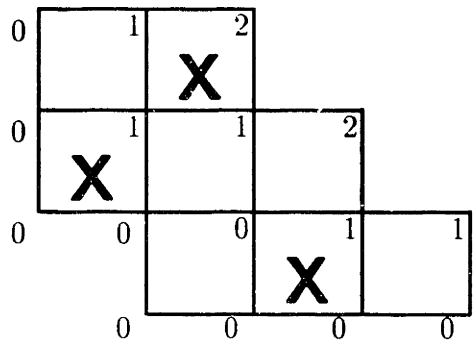
Definition 2.3.4 If S is a skew diagram (i.e., a finite convex subset of $\mathbf{Z}^2$), then a growth on S is called *two-dimensional.* For a skew diagram S we define the *upper and lower boundaries* of S by

$$\partial^+(S) = \{(x,y) \in S : (x+1, y+1) \notin S\}$$

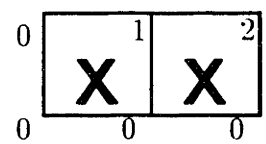
$$\partial^-(S) = \{(x,y) \in S : (x-1, y-1) \notin S\}$$

There are two important caveats associated with this seemingly innocuous definition. First, although a skew diagram is the same thing as a skew shape, we will view our skew diagrams with the usual coordinate geometry orientation rather than the matrix one, i.e., upside down (cf. Def 1.1.3). Second, we view the preimage of the growth as the *vertices* of the diagram, rather than the cells (cp. Definition 1.1.8 of "tableau"). We will reserve the cells for other (related) uses. Since we will be using skew shapes and skew diagrams in completely different ways, no confusion should result.

Example 2.3.5 In the skew diagram below, the growth $g : S \mapsto \mathbb{Z}$ is given by $g(x) = \#$cells marked with an X which are below and to the left of v. The image of g on a vertex v of S is given just slightly below and to the left of v.



The above example reflects a general class of two-dimensional growths; the cells marked with an X cannot share a row or column of S since then we could find a pair of adjacent vertices whose g values differed by at least two. A prototypical example is



where the right edge of the diagram is disallowed in the definition of "growth".

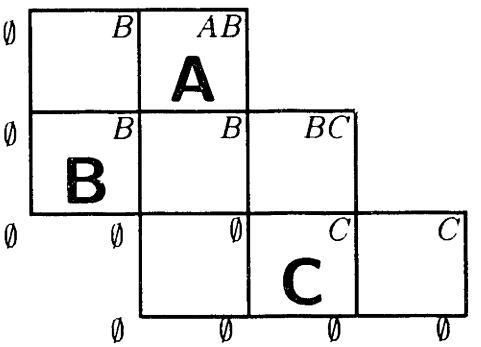
Definition 2.3.6 A *generalized permutation* is a finite set of cells in the skew diagram S which do not share any row or column. Other commonly used terms include *nontaking rook placement* or *permutation with restricted positions.*

Our previous definitions of "permutation" and "partial permutation" are examples of generalized permutations for the skew shape S being a $n \times n$ square of cells. For an elementary enumerative approach to nontaking rooks we refer the interested reader to [Sta3].

The two-dimensional growth above is really the modulus of the following (more interesting) growth. For a fixed skew diagram S, let $\mathcal{C}(S)$ denote the set of cells of S. The collection of all subsets of $\mathcal{C}(S)$ ordered by inclusion forms a poset $\mathcal{B}(S)$ (isomorphic to a boolean algebra on the number of cells of S). Let $\mathcal{G}(S)$ denote the collection of all generalized permutations on the cells of S considered as a subposet of $\mathcal{B}(S)$. $\mathcal{G}(S)$ has rank function $\rho(\sigma)$ = the number of cells in σ. Hence, we can consider the growth given by

$$g(v) = \{\text{cells in the generalized permutation which are below and to the left of } v\}$$

Example 2.3.7 If we give the marked cells in the example above distinguished labels, then we have:



2.4 Permutations to posets to partitions

We can turn any generalized permutation σ on the cells of a skew diagram S into a poset in a natural way as follows. If the coordinates of the cells are given by (i_1, j_1) and (i_2, j_2), then

$$(i_1, j_1) \leq (i_2, j_2) \iff i_1 \leq i_2 \quad \text{and} \quad j_1 \leq j_2.$$

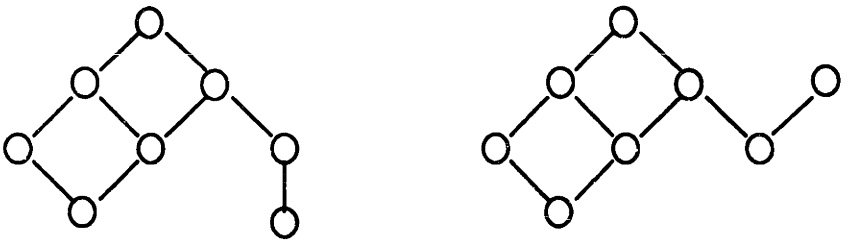


Figure 2-3: Two posets and their partitions

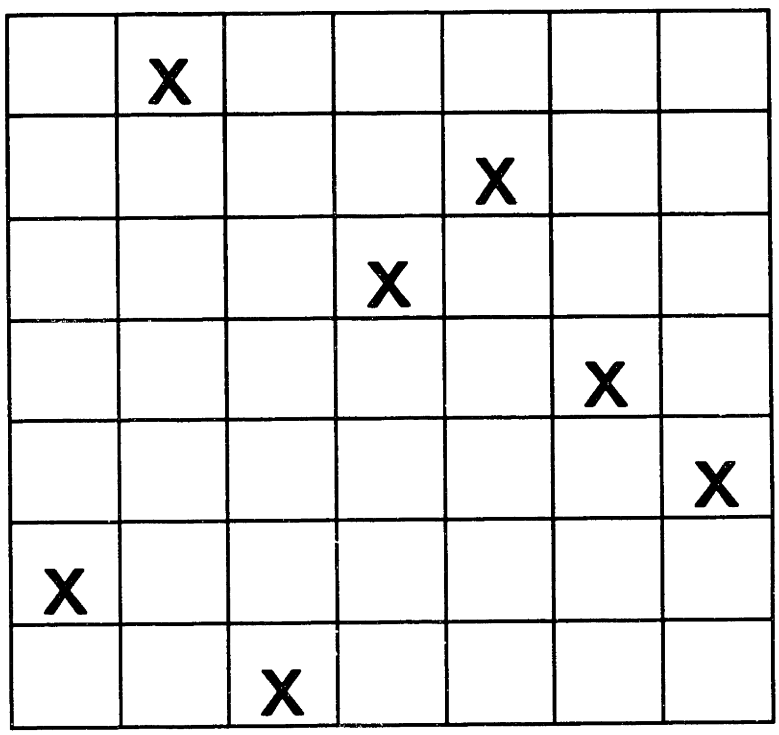
In other words, a cell is bigger if it is above and to the right of another in S. In Example 2.3.7, for instance, the partial order is simply "A covers B". Next, we turn a poset into a partition via the following theorem, due independently to C. Greene [Gre1] and Fomin [Fom1].

Theorem 2.4.1 Let P be any finite poset. For k a positive integer, set $c_k(P)$ (resp. $a_k(P)$) to be the size of the largest number of elements which is the union of k chains (resp. antichains) of P. Now, let $\lambda_k(P) = c_k(P) - c_{k-1}(P)$ and $\mu_k(P) = a_k(P) - a_{k-1}(P)$. Then $\lambda(P) = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ are *partitions*, and μ is the conjugate of λ.

The interested reader may find a proof of this in either of the references cited above. Two examples of posets with eight elements and their corresponding partitions are given in Figure 2-3.

Definition 2.4.2 If σ is a generalized permutation of the cells of a skew diagram S, then we let $\lambda(\sigma)$ be the partition given by the above theorem applied to σ (regarded as a poset in the way described above).

Example 2.4.3 Let $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 1 & 5 & 6 & 4 & 3 \end{matrix}$. We represent π as a generalized permutation of the n-cell by n-cell skew shape Sq_n as follows:



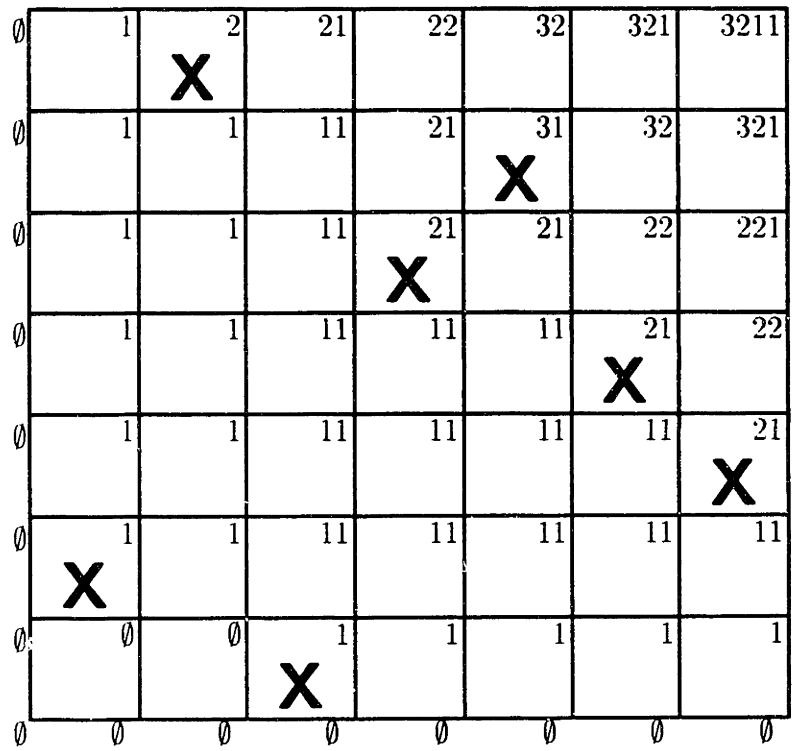
The element $\begin{matrix} i \\ j \end{matrix}$ appears in the ith column and the jth row as read from the lower left hand corner of the diagram, as per our convention. We compute $\lambda(\sigma)$ by inspection to be $(3,2,1,1)$.

We can extend the above example in a natural way to obtain a two-dimensional growth on Sq_n.

Example 2.4.4 In the above example we take define a growth g as follows. For each vertex of v of S, let $\mathcal{C}(v)$ denote the set of cells of S below and to the left of v (as we did in Example 2.3.7). If we restrict out attention to those cells which are in π and take the poset corresponding to this generalized permutation, then we get a map

$$v \mapsto \lambda(\mathcal{C}(v) \cap \pi)$$

which is a two dimensional growth $y : S \to \mathbf{Y}$:



What is particularly interesting about this growth is to consider its restriction to the upper boundary $\partial^+(Sq_n)$. The top edge T and the right edge R can each be interpreted as a saturated chain in $\mathbf{Y}$ from $\emptyset$ to $(3,2,1,1)$, i.e., as a standard tableau. We have

$$R = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array} \qquad T = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

This is the same pair of tableaux we would have obtained by applying the bumping algorithm of Section 1.2 to the permutation π.

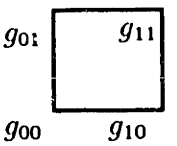


Figure 2-4: Close up of the growth around one cell

2.5 Semimodular growth

We would like to characterize those two-dimensional growths which come from a generalized permutation.

Definition 2.5.1 A two-dimensional growth $g : S \mapsto \mathbf{Z}$ is called *semimodular* if the values of g on the vertices of any cell, say g_{00}, g_{01}, g_{10}, g_{11} (see Figure 2-4) satisfy the following inequality:

$$g_{00} + g_{11} \geq g_{01} + g_{10} \tag{2.1}$$

When the inequality is strict, we call the cell in question an *atom* of g.

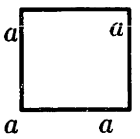
Example 2.3.5 shows a semimodular growth with the atoms marked with X's. This will be our typographic convention in many of the figures to follow.

Lemma 2.5.2 Any semimodular growth $g : S \mapsto \mathbf{Z}$ is uniquely determined by its restriction to the lower boundary $g|_{\partial^-(S)}$ and by a generalized permutation of the cells of S, which is the set of atoms for g.

Proof: The reader is encouraged to experiment with a couple of examples similar to the ones given above prior to (or instead of) trying to follow the technicalities we now present.

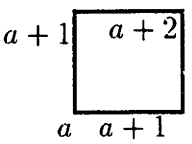
Given the semimodular growth g, mark the cells which are atoms with the symbol X. What can a semimodular growth look like locally? Consider the values of g at the

vertices of a cell as in Figure 2-4. If $g_{11} = g_{00} =: a$, then all four values are the same:



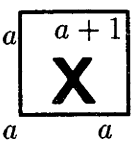
(2.2)

If $g_{11} = g_{00} + 2 := a + 2$, then the other two vertices must be $a + 1$ by definition of growth:

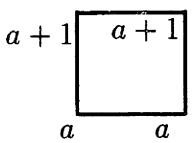


(2.3)

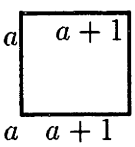
The only remaining case is $g_{11} = g_{00} + 1 := a + 1$. Here the definition of growth allows four possibilites:



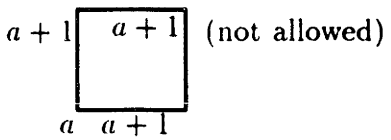
(2.4)



(2.5)

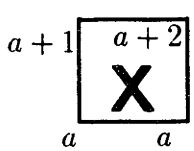


(2.6)



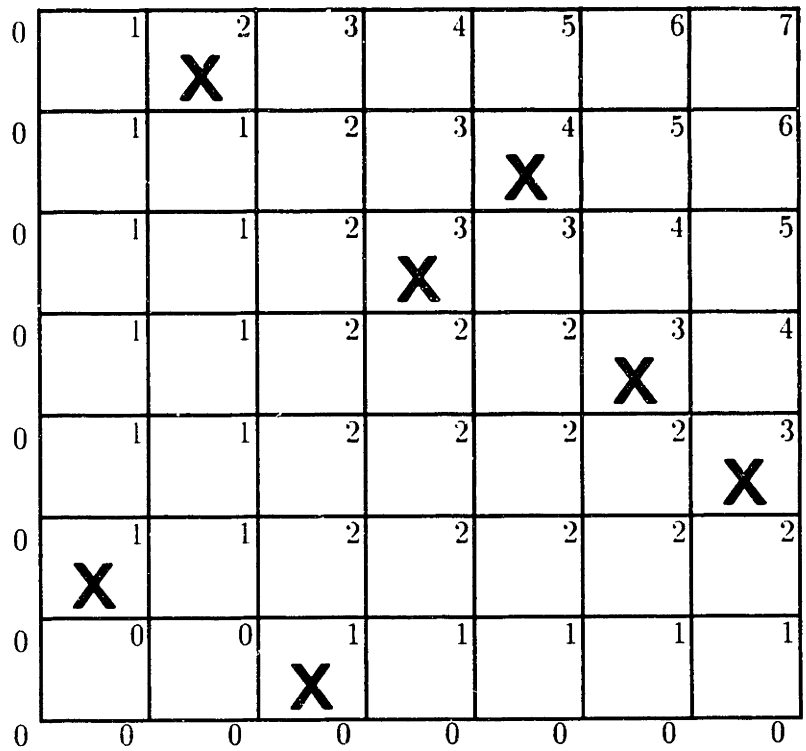
(not allowed)

The last possibility is eliminated by the definiton of semimodular, but the first three are all possible. Now notice that in the five total allowed possible cell configurations that the values g_{00}, g_{01}, g_{10} determine g_{11}, except for the ambiguous case when all three values are the same and g_{11} can be either a or $a+1$. But in this case we only need to know whether the cell is an atom and the value of g_{11} is completely determined. Hence, given the restriction of g to $\partial^-(S)$ and a generalized permutation, we can extend the growth inductively cell by cell. On the other hand, any semimodular growth g naturally determines a set of atoms and its restriction to $\partial^-(S)$. We claim that this set of atoms must always be a generalized permutation. For if two atoms shared a row of S, then the first must occur at a cell of type (2.4). Now as we proceed along the row, the value of g_{00} will continue to be strictly less than g_{01} since we eliminated the only case above which allows g_{00} to increase while g_{01} remains the same. When we reach the next atom, we find that we still have this strict inequality along the left edge of the cell, which means our cell would have to look like



contradicting the growth condition on g. □

Example 2.5.3 Take the modulus of the growth in Example 2.4.4, we get the semimodular growth:



In this case the lower boundary is trivial.

As consequences of the lemma we have the following.

Corollary 2.5.4 A semimodular growth $g : S \mapsto \mathbf{Z}$ whose restriction to $\partial^-(S)$ is zero is given simply by a generalized permutation σ of the cells of S. In this case we write $g = g_\sigma$.

By re-examining the figures in the proof above, we see that we can extend down as well as up.

Porism 2.5.5 Any semimodular growth $g : S \mapsto \mathbf{Z}$ is uniquely determined by its restriction to the upper boundary $g|_{\partial^+(S)}$ and by a generalized permutation of the

cells of S, which is the set of atoms for g.

2.6 Growths in differential posets

It turns out that differential posets have exactly the right structure to extend growths nicely from either boundary of a skew diagram. In order to do this we first look at what can happen locally as we attempt to extend a growth up or down.

Let $g : S \to P$ be a growth in any differential poset P. Consider the value of g at the four vertices of a cell as in Figure 2-4. If we attempt to compute the value of g_{00} given the other three vertices, we have several cases:

Case 1: $|g_{01}| < |g_{10}|$. This forces $g_{00} = g_{01}$ and $g_{11} = g_{10}$ by the definition of growth.

Case 2: $|g_{10}| < |g_{01}|$. This forces $g_{00} = g_{10}$ and $g_{11} = g_{01}$ as in case 1.

Case 3: $|g_{01}| = |g_{10}|$ but $g_{01} \neq g_{10}$. By modularity (D2) of P, g_{11} determines g_{00} and vice-versa.

Case 4: $g_{01} = g_{10} := x \in P$. This is the interesting case. Set $C^+(x) = \{y \in P : x \lessdot y\}$ and $C^-(x) = \{y \in P : y \lessdot x\}$. By axiom (D3) $C^+(x)$ has one more element than $C^-(x)$, so there is not a bijection between these sets. But there almost is. In fact, we can construct a bijection Φ_x between $C^+(x)$ and $C^-(x) \cup \{x\}$. Now set $\Phi_x(x) = x$. If we are extending down from g_{11} we set $g_{00} = \Phi_x(g_{11})$. This is well-defined since g_{11} must be something which covers or equals x. In fact, the growth will be semi-modular at this cell; for in the generic case, $|g_{00}| + 1 = |g_{01}| = |g_{11}| - 1$. If $g_{11} = x$, then all four vertices will have the same value. Finally, if $x \lessdot g_{11}$ and $\Phi_x(g_{11}) = x$, then we have

$$|g_{00}| + |g_{11}| > |g_{01}| + |g_{10}|$$

and the cell is an atom of the semimodular growth $|g|$.

To extend a growth upwards we will examine the same four cases. The first three are the same. In case 4 we find that our correspondence Φ gives a well-defined answer as long as $g_{00} \neq x$. If $g_{00} = x$, then $\Phi_x^{-1}(g_{00})$ could be either x or a certain element of $C^+(x)$. To decide between these we need an extra piece of information, which is whether or not the cell in question contains an X, i.e., whether it should be an atom of the semimodular growth $|g|$.

Definition 2.6.1 Let P be a differential poset. At each $x \in P$ define a correspondence Φ as in case 4 above. We call the collection $\Phi := \{\Phi_x : x \in P\}$ an *R-correspondence*. A growth $g : S \to P$ is said to be *compatible* with the R-correspondence if at each cell such that $g_{01} = g_{10}$, we have $g_{00} = \Phi_{g_{01}}(g_{11})$. We call such a g a Φ-*growth*.

An R-correspondence is defined in a purely local way, and a differential poset will have many different ones.

Example 2.6.2 Consider the element $x = (2,1)$ of Young's lattice $\mathbf{Y}$. One possibility for the R-correspondence Φ around this element is given by:

$$\Phi_x : (2,1,1) \mapsto (1,1) \quad (2,2) \mapsto (2,1) \quad (3,1) \mapsto (2)$$

Another is

$$\Phi_x : (2,1,1) \mapsto (2) \quad (2,2) \mapsto (1,1) \quad (3,1) \mapsto (2,1)$$

Example 2.6.3 While not essential for our extensions to work, it is convenient to have a coherent way to define an R-correspondence over all of $\mathbf{Y}$. One way to do this is as follows; see Figure 2-1 on page 23. If $\mu \lessdot \lambda$ and $\lambda_i = \mu_i + 1$, then set

$$\Phi_\mu : \lambda \mapsto \nu$$

where ν agrees with μ except $\nu_{i-1} = \mu_{i-1} - 1$ if $i > 1$. In terms of Young diagrams we are saying that if in going from λ to μ we delete a square in some row, then we

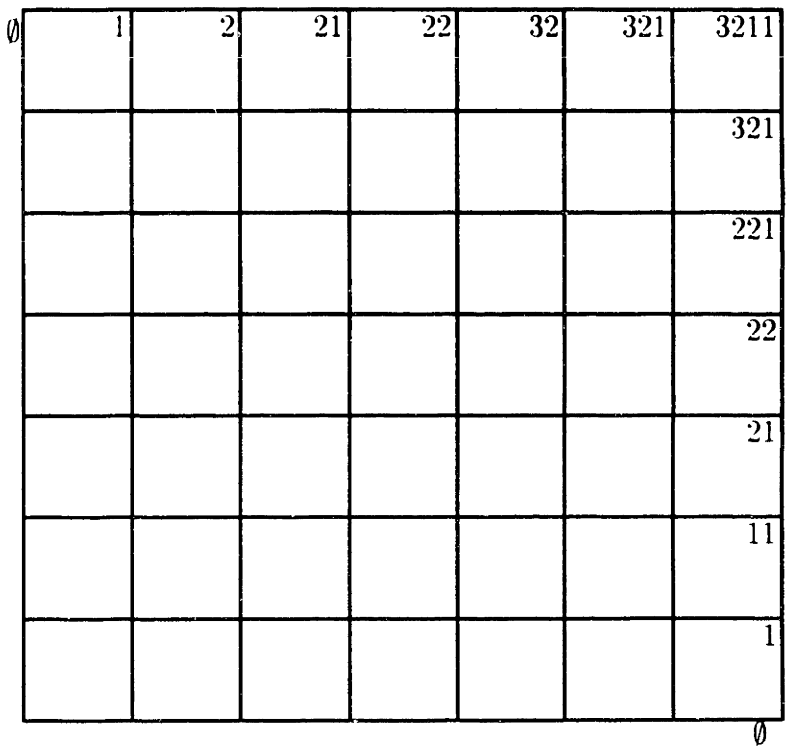
next delete in the row above (when possible). So if λ and μ differ in the first row (i.e., $i = 1$), then $\Phi_\mu(\lambda) = \mu$. It is easy to see that the sequence ν thus defined is a partition, so this map is well-defined. It coincides with the second correspondence given in Example 2.6.2 above.

Lemma 2.6.4 Let P be a differential poset and S a skew shape. Any growth $g^+ : \partial^+(S) \to P$ on the upper boundary of S can be extended uniquely to a Φ-growth on all of S. Furthermore, $|g|$ will be semimodular.

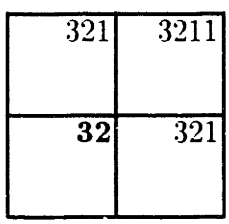
Proof: Using the cases listed above, we extend our growth cell by cell from the top down and from right to left. Semimodularity is clear in cases 1–3 (in fact, we have $|g_{00}| + |g_{11}| = |g_{01}| + |g_{10}|$), and in case 4 we have shown semimodularity above. □

Example 2.6.5 Suppose we start with the following growth on the upper boundary

of Sq_7.



The reader should attempt to extend the growth using the four cases listed above, with the R-correspondence given in Example 2.6.3. With a bit of practice one can do this quite quickly. One hint is to notice that whenever $g_{01} \neq g_{10}$, then one just fills in g_{00} with the meet (or greatest lower bound) of g_{01} and g_{10}; this takes care of cases 1–3. Case 4 requires a bit more attention, but it also easy. For example, the first vertex one fills in above is



since the decrease in row 4 implies the next decrease in row 3. The solution is given in Example 2.4.4. Note that the atoms of the modulus of this growth occur when the initial decrease is in the top row and (consequently) the three lower values coincide.

What about extensions from the lower boundary? We have:

Lemma 2.6.6 Let P be a differential poset, S a skew shape, and σ a generalized permutation of the cells of S. Any growth $g^- : \partial^-(S) \to P$ on the lower boundary of S can be extended uniquely to a Φ-growth on all of S, in such a way that $|g|$ is semimodular with the atoms of g being the cells in σ.

Proof: Again we use cases 1–4 above, this time to extend cell by cell from the bottom up and from left to right. So given g_{00}, g_{01}, g_{10} surrounding a cell C, we want a canonical way to decide the value of g_{11}. Cases 1–3 are clear. In case 4 (where $g_{01} = g_{10} := x \in P$) we use our R-correspondence as follows. If $g_{00} \neq x$, then we set $g_{11} := \Phi_x^{-1}(g_{00})$. If $g_{00} = x$, then we have two subcases.

Case 4.1: $C \notin \sigma$. Then set $g_{11} = g_{00}$, so all four vertices around C are equal to x.

Case 4.2: $C \in \sigma$. By definition, $\Phi_x^{-1}(g_{00})$ consists of a two element set $\{x, y\}$ where $x \lessdot y$. So we set $g_{11} = y$.

It is clear that this extension is well defined and uniquely determined. $|g|$ is semimodular because in cases 1–3 and 4.1 we have $|g_{00}| + |g_{11}| = |g_{01}| + |g_{10}|$; in case 4.2, we get $|g_{00}| + |g_{11}| = |g_{01}| + |g_{10}| + 1$. Hence, the atoms of g are the cells of σ. □

We are now in a position to state Fomin's main result.

Theorem 2.6.7 (Fomin) Let P be a differential poset and S any skew diagram. Fix an R-correspondence Φ on S. We have a bijection between growths $g^+ : \partial^+(S) \mapsto P$ and pairs (g^-, σ) where $g^- : \partial^-(S) \mapsto P$, and σ is a generalized permutation on S. Each of these is also associated with a uniquely defined two-dimensional Φ-growth

$g : S \mapsto P$, whose restriction to $\partial^+(S)$ and $\partial^-(S)$ is g^+ and g^-, respectively, and whose modulus $|g|$ is the semimodular growth whose atoms are σ.

Proof: The two preceeding lemmas yield an algorithmic way to construct g from either g^+ or (g^-, σ). □

The power of this general result will become clear as we exploit it and extend it in the later chapters. First we present a few simple applications.

Example 2.6.8 Let P be any differential poset and suppose that the skew diagram S has the same lower border as the square shape Sq_n; see Figure 2-5. Let the lengths of the segments which make up $\partial^+(S)$ be given by the numbers $a_1, a_2, \ldots, a_{2k}$ as shown. Then a growth g^+ on the upper boundary of S is exactly a walk in (the Hasse diagram of) P starting at $\hat{0}$, going up by a_1 steps, then down by a_2 steps, ... down by a_{2k} steps, ending at $\hat{0}$. Using Theorem 2.6.7 with $g^- \equiv 0$ we obtain a bijection between such walks and generalized permutations σ of the cells of S. The latter may be counted via elementary sieve methods; see [Sta3, Section 2.4] for more information. In particular, when $S = Sq_n$, we get a bijection between pairs of saturated chains in P which go from $\hat{0}$ to the same $x \in P_n$ and permutations in S_n. We have already seen a special case of this in Examples 2.4.4 and 2.6.5 above.

Actually, when we originally introduced the growth g given in Example 2.4.4, we determined $g(v)$ by looking below and to the left of v rather than by building it up locally according to Lemma 2.6.6. The reader may want to check that following the procedure given by the lemma does agree with the result we obtained in the first place. That these two procedures yield the same result for **Y** is not at all obvious; we have relegated the proof to Appendix A so as not to interrupt the flow of the main discussion. Further applications and consequences of this theory will form the basis for most of the rest of this thesis.

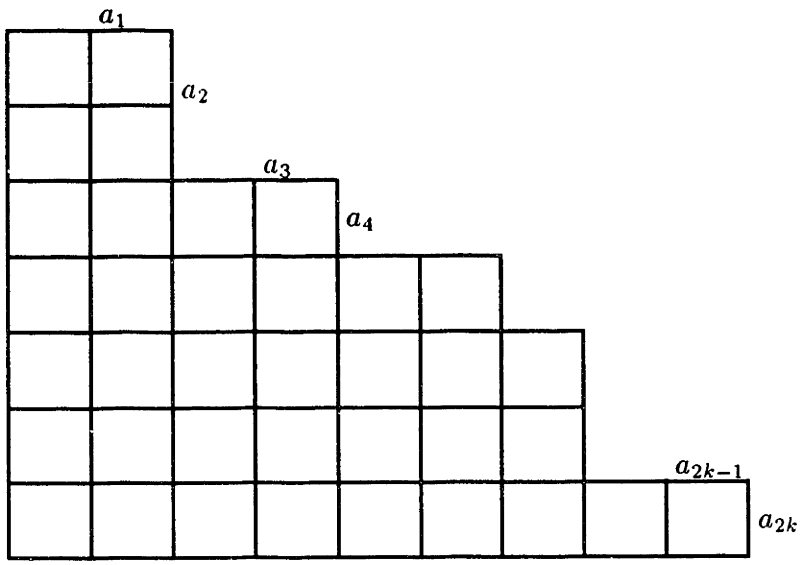


Figure 2-5: A partition-like skew diagram

2.7 Elementary proofs of basic properties

In this section we derive most of the properties which we stated without proof in Section 1.3.

One important property of these algorithms which is much easier to see via Fomin's methods is Theorem 1.3.3: If

$$\pi \quad \overset{\text{R-S}}{\longleftrightarrow} \quad (P, Q)$$

then

$$\pi^{-1} \quad \overset{\text{R-S}}{\longleftrightarrow} \quad (Q, P)$$

Proof: Replacing π with π^{-1} corresponds to interchanging the rows and columns of our skew diagram (Sq_n in this case), i.e., transposing the diagram. In this case it is

clear that the entire correspondence is transposed, switching P and Q. □

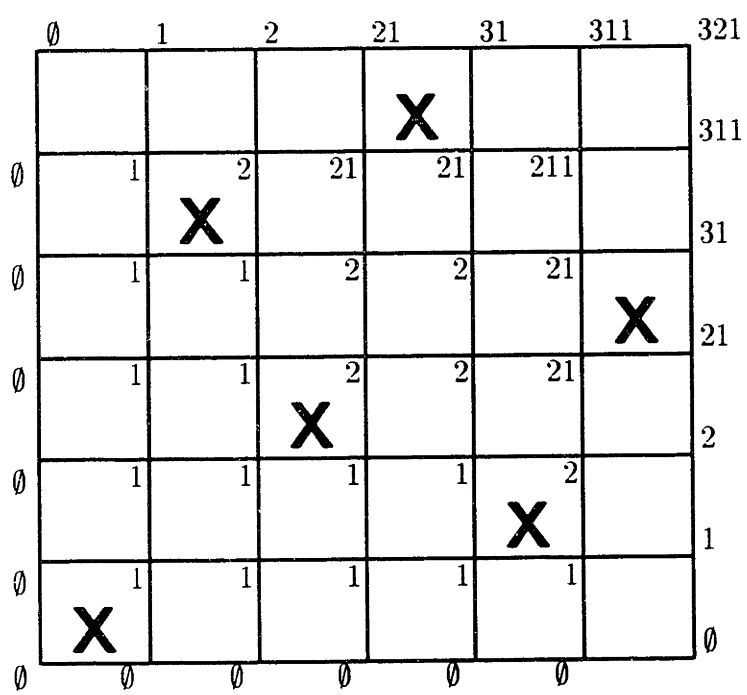
The corollaries then follow as before.

Schützenberger's Theorem 1.3.7, that for an involution the number of fixed points equals the number of odd length columns, is also quite easy to show.

Example 2.7.1 The correspondence

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 3 & 6 & 2 & 4 \end{array} \quad \overset{\text{R-S}}{\longleftrightarrow} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

is represented by the picture below.



Proof: First of all, since π is an involution, the entire growth is symmetric about the main diagonal of Sq_n. We claim in fact that at each shape along the main diagonal the number of fixed points so far is equal to the number of columns of odd length in the shape. Call these shapes $\mu_1, \mu_2, \ldots, \mu_n$. In the example above, these shapes are $\emptyset$, 1, 1, 2, 2, 211, 321; and fixed points occur in the first and third columns. As we proceed along the diagonal, one of three things happens. If the cell C between μ_i and μ_{i+1} contains an X, then μ_{i+1} and μ_i differ by a cell in the first row. This means that μ_{i+1} has exactly one more column than μ_i, and it is of length 1. In this case, the cell is a fixed point of the involution. Otherwise, there is one X in C's row, and another in C's column, which are the only ones we have to worry about as we proceed from μ_i to μ_{i+1}. If the X's are placed (symmetrically) above and to the right of C, then all the vertices of C are identical and $\mu_i = \mu_{i+1}$. Finally, if the X's are placed (symmetrically) below and to the left of C, then let λ be the shape on the other two vertices of the cell. If λ and μ_i differ in row k, then (using the standard R-correspondence) μ_{i+1} and λ differ in row $k+1$. This leaves the number of odd length columns unchanged, since one column will increase by one to length k, and another by one to length $k+1$. Hence, by induction, we obtain the theorem. □

Chapter 3

Skew tableaux

In this chapter we discuss how the methods of Fomin may be applied to explain certain algorithms of Sagan and Stanley for skew tableaux [SS]. The search for these algorithms was originally inspired by some of the enumerative results of [Sta1], which Stanley had obtained algebraically rather than bijectively. Sagan and Stanley's bijections are based on a generalized bumping procedure. Basically, they define two notions of insertion, the usual one, which they call "external", and a new one, which they call "internal". We bypass this description, however, and use the machinery we have built up in Chapter 2 to obtain their results.

3.1 The fundamental algorithm

Recall that PS_n denotes the set of partial permutations on n. We have a similar notion for tableaux.

Definition 3.1.1 A tableaux of shape λ/μ is called *partial* if its elements are distinct (but not necessarily the numbers $1, 2, \ldots, n$). Let $PT(\lambda/\mu)$ denote the set of all partial tableaux of shape λ/μ.

The basic result of [SS] follows. In the following, "$\uplus$" denotes "disjoint union". Also, recall the notation $\hat{\pi}$ and $\check{\pi}$ for the top and bottom lines of a biword (see

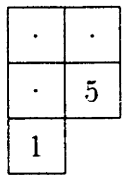
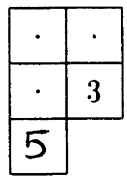
Definition 1.1.14).

Theorem 3.1.2 Let n be a fixed positive integer and α a fixed partition (not necessarily of n). Then there is a map

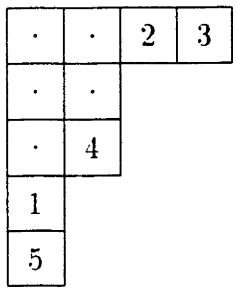
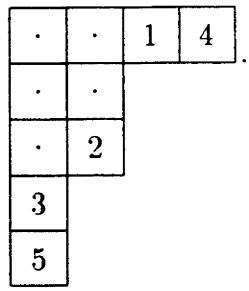
$$(\pi, T, U) \longleftrightarrow (P, Q)$$

defined below which is a bijection between $\pi \in PS_n$ with $T, U \in PT(\alpha/\mu)$ such that $\check{\pi} \uplus T = \hat{\pi} \uplus U = \{1.2.\ldots, n\}$, on the one hand, and $P, Q \in ST(\lambda/\alpha)$ such that $\lambda/\alpha \vdash n$, on the other.

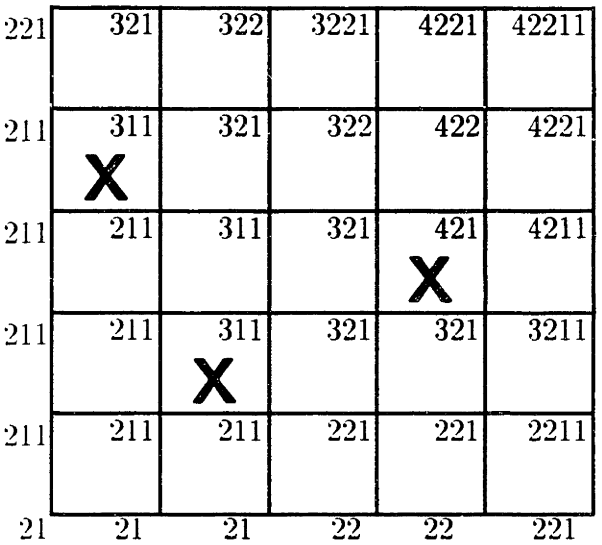
Example 3.1.3 This is the same example Sagan and Stanley give in their paper [SS, p. 165], but we reinterpret it using Fomin's approach. Let $n = 5$, $\alpha = (2,2,1)$, $\pi = \begin{matrix} 1 & 2 & 4 \\ 4 & 2 & 3 \end{matrix}$,

$T =$  and $U =$  .

Then we get

$P =$  and $Q =$  .

In the picture below, P is the right edge, Q the upper edge, T the left edge, and U the lower edge. The partial permutation is represented by X's, as usual.



Proof: All the hard work has already been done, and this is just a simple application of Theorem 2.6.7. The skew tableaux P and Q represent the growth on the upper boundary, while T and U represent the lower boundary and π represents the atoms of the semimodular growth $|g|$. The condition that $\check{\pi} \uplus T = \hat{\pi} \uplus U = \{1, 2, \ldots, n\}$ insures that the tableaux on the upper boundary are standard, and vice-versa. □

The enumerative corollary of the above is

Corollary 3.1.4 Let n be a fixed positive integer and α be a fixed partition. Then

$$\sum_{\lambda/\alpha \vdash n} f^2_{\lambda/\alpha} = \sum_{k=0}^{n} \binom{n}{k}^2 k! \sum_{\alpha/\mu \vdash n-k} f^2_{\alpha/\mu}.$$

These results reduce to the original Robinson-Schensted results when $\alpha = \emptyset$.

3.2 Inverting permutations and tableaux

Sagan and Stanley originally gave a somewhat lengthy proof that their algorithm enjoys an analogue of Theorem 1.3.3 by showing how to mimic their insertion procedure with the original Robinson-Schensted correspondence, then applying the original inverting theorem. They obtained:

Theorem 3.2.1 If $(\pi, T, U) \longleftrightarrow (P, Q)$ by skew Robinson-Schensted then $(\pi^{-1}, U, T) \longleftrightarrow (Q, P)$.

Proof: From our standpoint, the theorem is almost obvious. Just transpose the skew diagram Sq_n. □

In particular, if we restrict to the case where $\pi = \pi^{-1}$ and $T = U$ we get

Corollary 3.2.2 If π is an involution then we have a bijection

$$(\pi, T) \longleftrightarrow P$$

between $\pi \in PS_n$ with $T \in PT(\alpha/\mu)$ such that $\pi \uplus T = \{1, 2, \ldots, n\}$, on the one hand, and $P \in ST(\lambda/\alpha)$ such that $\lambda/\alpha \vdash n$, on the other.

Corollary 3.2.3 Let n be a fixed positive integer and α be a fixed partition. Then

$$\sum_{\lambda/\alpha \vdash n} f_{\lambda/\alpha} = \sum_{k=0}^{n} \binom{n}{k} \mathrm{Inv}(k) \sum_{\alpha/\mu \vdash n-k} f_{\alpha/\mu}$$

where $\mathrm{Inv}(k)$ denotes the number of involutions in S_k.

We also have an analogue of Schützenberger's Theorem 1.3.7, as Sagan and Stanley noted. It follows immediately by reconsidering the proof of 1.3.7 given on page 44.

Porism 3.2.4 In the bijection of Corollary 3.2.2 we always have the number of odd-length columns of λ equals the number of odd-length columns of μ plus the number of fixed points of π. We write

$$\mathrm{fix}(\pi) + \mathrm{odd}(\mu) = \mathrm{odd}(\lambda)\,.$$

In particular, one can restrict the bijection to the case where π is fixed point free and λ and μ have no odd-length columns to obtain a few more identities, given in [SS].

3.3 Iterated skew maps

Sagan and Stanley have also an iterated version of their bijection, which they used to give a bijective proof of an identity from [Stal]. It turns out that this version is also easier to understand via Fomin's approach, which will allow us to generalize it later to the case of sequentially differential posets.

Definition 3.3.1 A *weighted permutation* π is a permutation in which a nonnegative integer k has been associated to each $\begin{pmatrix} i \\ j \end{pmatrix} \in \pi$. We write $\mathrm{wt}\begin{pmatrix} i \\ j \end{pmatrix} = q^k$. The *weight* of the permutation is given by $\mathrm{wt}(\pi) = \prod \mathrm{wt}\begin{pmatrix} i \\ j \end{pmatrix}$ where the product is over all pairs in π. If $\mathrm{wt}\begin{pmatrix} i \\ j \end{pmatrix} = q^k$, we write the term as $\begin{pmatrix} i \\ j^{(k)} \end{pmatrix}$, omitting the symbol $^{(k)}$ if $k = 0$.

Example 3.3.2 Let

$$\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 5^{(4)} & 2^{(2)} & 4 & 1^{(7)} & 3^{(2)} \end{matrix}\,.$$

Then $\mathrm{wt}(\pi) = q^4 q^2 q^0 q^7 q^2 = q^{15}$.

We also give a partition ν the weight $q^{|\nu|}$, and to a pair (π, ν) the product $\mathrm{wt}(\pi)\,\mathrm{wt}(\nu)$. To a pair (P, Q) of standard tableaux of shape λ/μ we assign the weight $\mathrm{wt}(P, Q) = q^{|\mu|}$. Sagan and Stanley's theorem is as follows:

Theorem 3.3.3 There is a weight preserving bijection

$$(\pi, \nu) \longleftrightarrow (P, Q)$$

between pairs $\pi \in S_n$ with ν a partition and pairs $P, Q \in ST(\lambda/\mu)$, where $\lambda/\mu \vdash n$.

Proof: Suppose we start with the pair $(P, Q) := (P^0, Q^0)$. By the skew map (Theorem 3.1.2), we can associate a triple (π, T, U) which we relabel as $(\pi^{(0)}, P^1, Q^1)$. At the kth step apply the skew map to the pair (P^{k-1}, Q^{k-1}) to obtain $(\pi^{(k-1)}, P^k, Q^k)$. At each stage we obtain a partial permutation (which may be empty). Set λ^k/λ^{k+1} to be the shape of P^k. The process terminates when $\lambda^k = \lambda^{k+1} = \nu$, which it must eventually since the λ^i's are decreasing. The partial permutations will fit together to give a weighted permutation, since at each stage we have $\check{\pi}^{(k)} \uplus P^k = \hat{\pi}^{(k)} \uplus Q^k = \{1, 2, \ldots, n\}$. We can also view the bijection as starting from the pair (π, ν), starting with the growth on $\partial^-(Sq_n)$ being identically ν and the partial permutation of highest weight in π. We then use the resulting upper border as the lower border in the next iteration, along with the terms of π of next highest weight. When we finally use up the terms of weight 0 in π, we are done.

Why do the weights work properly? The example below should make it clear. Consider the bijection going forward from (π, ν) to (P, Q). If the term $\binom{i}{j^{(k)}} \in \pi$, then it contributes one cell to λ^k/λ^{k+1} (as part of $\pi^{(k)}$); then at each stage afterwards, it also contributes a cell to λ^t/λ^{t+1} $(1 \leq t < k)$, because of its affect on the lower border. Since the weight of (P, Q) depends only on $\mu = \lambda^1$, the terms of $\pi^{(0)}$ (correctly) contribute nothing to the weight of (P, Q). Hence, $|\lambda^1| = |\nu| + \mathrm{wt}(\pi)$ which says exactly that the bijection is weight preserving. □

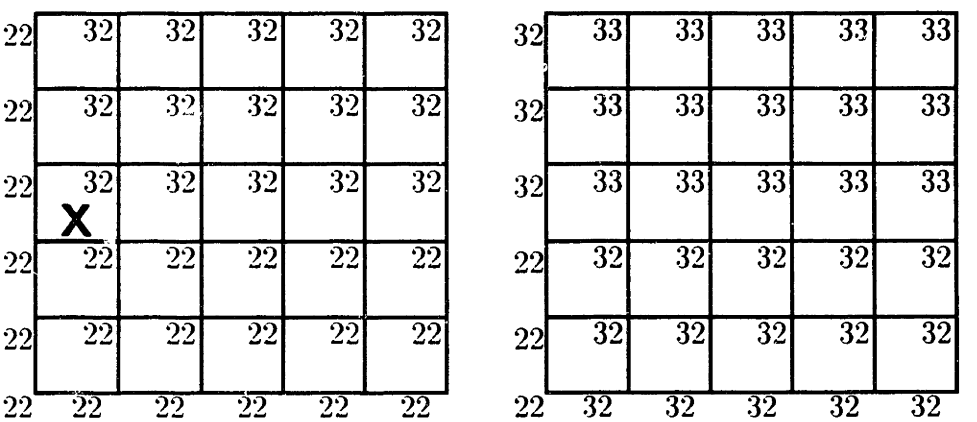
Example 3.3.4 Let

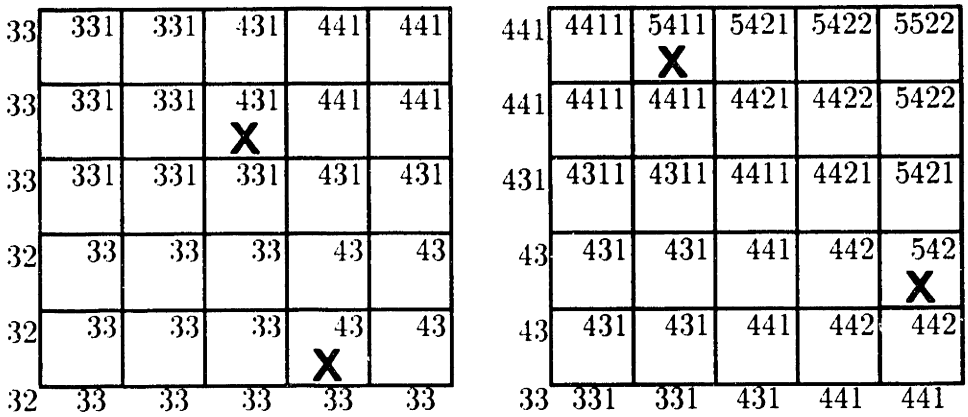
$$\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3^{(3)} & 5 & 4^{(1)} & 1^{(1)} & 2 \end{matrix} \quad \text{and} \quad \nu = \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \end{array}.$$

We obtain

$$P = \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & \cdot & 2 \\ \hline \cdot & \cdot & \cdot & \cdot & 5 \\ \hline \cdot & 1 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} \end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & \cdot & 2 \\ \hline \cdot & \cdot & \cdot & \cdot & 5 \\ \hline \cdot & 3 \\ \cline{1-2} 1 & 4 \\ \cline{1-2} \end{array},$$

via





The enumerative consequence of this theorem is

Corollary 3.3.5

$$\sum_{k,n}\left(\sum_{\substack{\lambda/\mu\vdash n\\ \mu\vdash k}} f^2_{\lambda/\mu}\right) q^k t^n/n! = \frac{1}{1-t/(1-q)}\prod_i (1-q^i)^{-1}.$$

Proof: Consider the coefficient of $t^n/n!$ on each side. On the left we have pairs (P,Q) with $P,Q \in SYT(\lambda/\mu)$, with each pair weighted by the factor $q^{|\mu|}$. On the right, the product counts partitions ν with weight $q^{|\nu|}$, and the other factor counts permutations $\pi \in S_n$ where each term is allowed a nonnegative weight q^k. By the theorem above, these are in bijective correspondence with one another. □

It is clear that this iterated algorithm also enjoys the property that inverting the permutation interchanges the two tableaux. If π is weighted then π^{-1} is the weighted permutation defined by

$$\binom{i}{j^{(k)}} \in \pi \iff \binom{j}{i^{(k)}} \in \pi^{-1}.$$

We call π an *involution* when $\pi^{-1} = \pi$. Sagan and Stanley then obtain the following corollaries.

Corollary 3.3.6 If π is a weighted involution, then the mapping of Theorem 3.3.3 restricts to a weight preserving bijection

$$(\pi, \nu) \longleftrightarrow P$$

between $\pi \in S_n$ with ν a partition, and $P \in ST(\lambda/\mu)$ with $\lambda/\mu \vdash n$.

Corollary 3.3.7

$$\sum_{k,n} \left(\sum_{\substack{\lambda/\mu \vdash n \\ \mu \vdash k}} f_{\lambda/\mu} \right) q^k t^n/n! = \exp\left(\frac{t}{1-q} + \frac{t^2}{2(1-q^2)} \right) \prod_i (1-q^i)^{-1} .$$

Corollary 3.3.8 In the bijection of Corollary 3.3.6 we always have

$$\mathrm{fix}(\pi) + 2\,\mathrm{odd}(\nu) = \mathrm{odd}(\mu) + \mathrm{odd}(\lambda) .$$

Proof: Using the notation of Theorem 3.3.3, apply Porism 3.2.4 to each iteration of the skew algorithm, to get

$$\mathrm{odd}(\lambda^k) = \mathrm{fix}(\pi^{(k)}) + \mathrm{odd}(\lambda^{k-2}) .$$

Summing over all k and cancelling terms which occur on each side of the equation, we obtain:

$$\mathrm{odd}(\lambda^0) + \mathrm{odd}(\lambda^1) = \mathrm{fix}(\pi) + 2\,\mathrm{odd}(\nu) ,$$

which is equivalent to the corollary. □

3.4 A rectangular skew algorithm

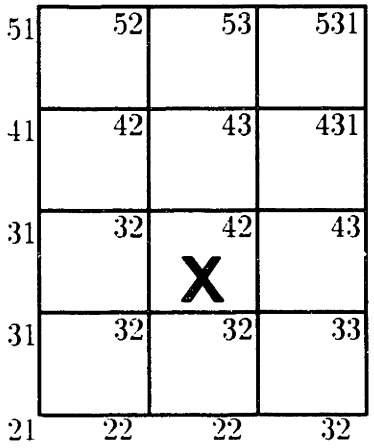
In order for the iterated algorithm to work properly, the partitions on the upper-left and bottom-right corners of Sq_n must be the same. (This is, after all, the intersection of the upper and lower boundaries.) But in general there is no reason this must be the case; i.e., the skew tableaux P and Q may have different (inner) shapes. The following are from [SS, Section 5].

Theorem 3.4.1 Let n and m be fixed integers and let α and β be fixed partitions (where $|\alpha|$ need not equal $|\beta|$). Then there is a bijection

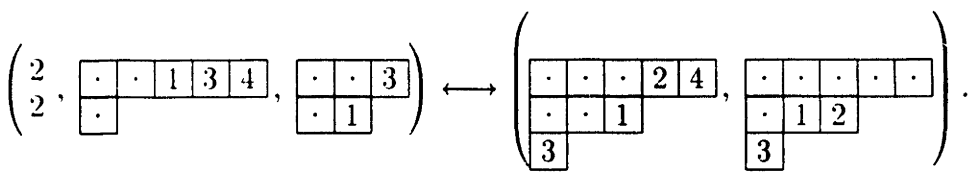
$$(\pi, T, U) \longleftrightarrow (P, Q)$$

between partial permutations π with $T \in PT(\alpha/\mu)$, $U \in PT(\beta/\mu)$, such that $\hat{\pi} \uplus T = \{1, 2, \ldots, n\}$, $\hat{\pi} \uplus U = \{1, 2, \ldots, m\}$, on the one hand, and $P \in ST(\lambda/\beta), Q \in ST(\lambda/\alpha)$, such that $\lambda/\beta \vdash n, \lambda/\alpha \vdash m$, on the other.

Example 3.4.2 Let $n = 4, m = 3, \mu = (2,1), \alpha = (5,1), \beta = (3,1)$. Our picture is



which says that



Corollary 3.4.3 Let n, m be fixed integers and α, β fixed partitions. Then

$$\sum_{\substack{\lambda/\beta \vdash n \\ \lambda/\alpha \vdash m}} f_{\lambda/\beta} f_{\lambda/\alpha} = \sum_{k \geq 0} \binom{n}{k} \binom{m}{k} k! \sum_{\substack{\alpha/\mu \vdash n-k \\ \beta/\mu \vdash m-k}} f_{\alpha/\mu} f_{\beta/\mu}.$$

The following corollary interpolates between Corollary 1.2.4 and 1.3.6. It is also a special case of [Sta1, Thm. 3.7].

Corollary 3.4.4 Let n and m be fixed integers. Then

$$\sum_{\substack{\lambda \vdash n \\ \alpha \vdash n-m}} f_\lambda f_{\lambda/\alpha} = \binom{n}{m} m! \operatorname{Inv}(n-m).$$

A interpretation of the above formula in terms of symmetric group characters is given in [SS].

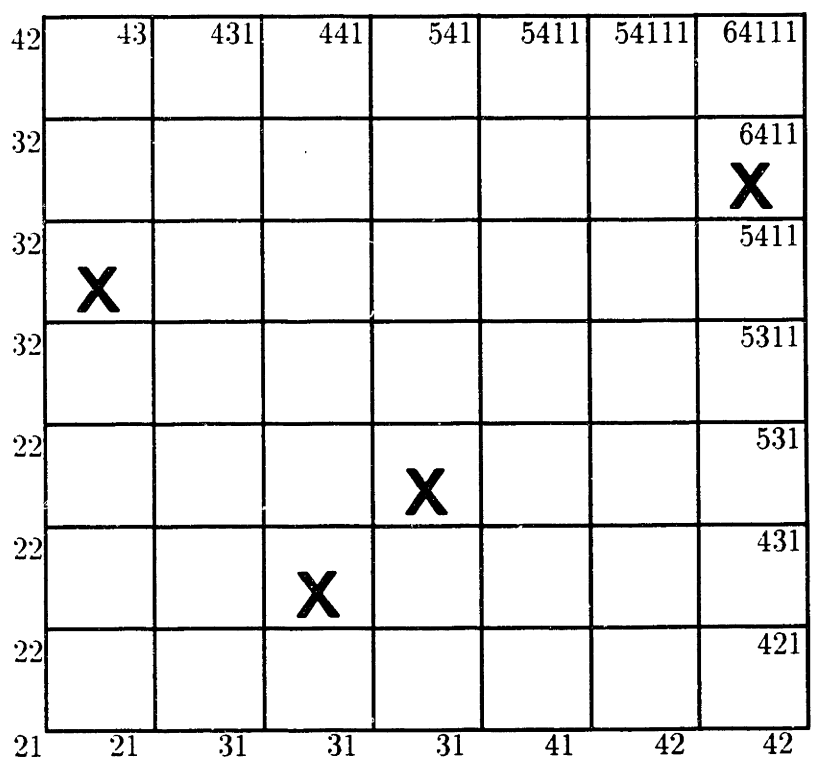
One fact which Sagan and Stanley were unable to generalize to the skew case was Schensted's Theorem 1.3.2 that the length of the longest increasing subsequence of the permutation is the length of the first row of the output tableaux. In general, the length of the first row of P depends nontrivialy on π, T, and U. The following result seems to be new.

Theorem 3.4.5 Assume that $(\pi, T, U) \longleftrightarrow (P, Q)$ by skew Robinson-Schensted. Let the first row of T be given by cells labeled $t_1, t_2, \ldots, t_i$, and the first row of U by $u_1, u_2, \ldots, u_j$. We can conveniently view these in our diagram by marking X's up the left side of Sq_n at heights t_k $(1 \leq k \leq i)$ and along the bottom of Sq_n at places u_k $(1 \leq k \leq j)$. View this extended diagram as a poset as usual (things higher and

to the right are greater), except also consider $t_k < t_l$ and $u_k < u_l$, for $k < l$. For a shape ν write $R_1(\nu)$ for the length of the first row of ν, and let M be the maximum length of a chain in the above poset. Then we have

$$R_1(\lambda) = R_1(\mu) + M\,.$$

Example 3.4.6 In the example below, $\mu = 21$ and $\lambda = 64111$. One should mark X's on the left side at heights four and seven, and along the bottom side in the second and fifth spots. The maximum length of a chain in this extended poset has four elements.



Proof: In the Schensted-Fomin algorithm (with the usual R-correspondence), a shape increases in the first row when there is an X in the cell; see Equation 2.4. Suppose that the vertex v is the smallest one above and to the right of the two cells A and B of

the generalized permutation. If $A < B$, then each cell increases the first row by one, for a total increase of two. If, on the other hand, A and B are incomparable, then each one will increase the first row by one, but the total effect will not be cumulative. Instead v will grow somewhere other than the first row.

So to find the final shape of λ, one starts with μ and finds the longest chain of first row increases, using increases both along the side of Sq_n and inside. □

Chapter 4

Knuth Analogues and Oscillating Tableaux

4.1 A Knuth analogue

Knuth was able to extend the original Robinson-Schensted algorithm to tableaux with repetitions [Knu]. More precisely, he exhibited a bijection between $n \times n$ matrix words and pairs of semi-standard tableaux of the same shape. A natural question to ask is how this fits with Fomin's approach. It turns out that we can fit this generalization nicely into Fomin's picture for Young's lattice. Unfortunately, our method does not seem to work in an arbitrary differential poset. It is not clear what the analogue of a "horizontal strip" is even for the Young-Fibonacci lattice $Z(1)$. Nonetheless, we are able to simplify and render more pictorial the description of many algorithms for Young's lattice.

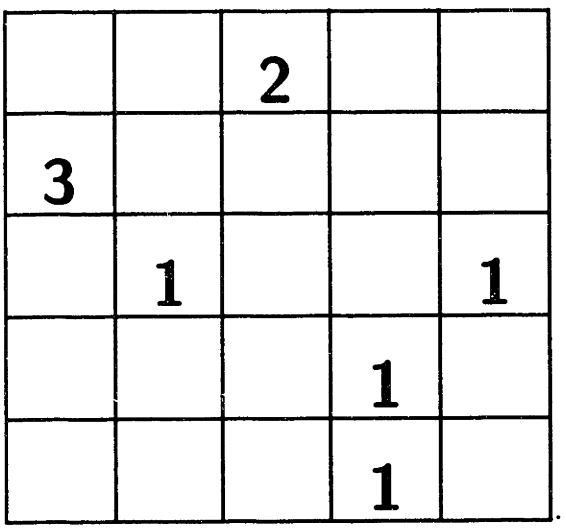
Definition 4.1.1 Recall the notations $\mathrm{MAT}(n)$ for the set of $n \times n$ matrix words and $GT(\lambda)$ for the set of generalized or semi-standard tableaux of shape λ. If $T \in GT(\lambda)$, let the **content** of T, denoted $\mathrm{cont}\, T$, be the multiset of integers which is mapped to the cells of λ. So if $w(T) = (n_1, n_2, n_3, \ldots)$ is the weight of T, then $\mathrm{cont}\, T =$

$\{1^{n_1}, 2^{n_2}, 3^{n_3}, \ldots\}$. Similarly, we can call the multiset of elements which appear in the top (resp. bottom) line of a matrix word the **contents** of $\hat{\pi}$ (resp. $\check{\pi}$).

Theorem 4.1.2 (Knuth) There is a bijection between $\pi \in \mathrm{MAT}(n)$ and pairs of semi-standard tableaux $P, Q \in GT(\lambda)$ such that $\mathrm{cont}\,\check{\pi} = \mathrm{cont}\,P$ and $\mathrm{cont}\,\hat{\pi} = \mathrm{cont}\,Q$.

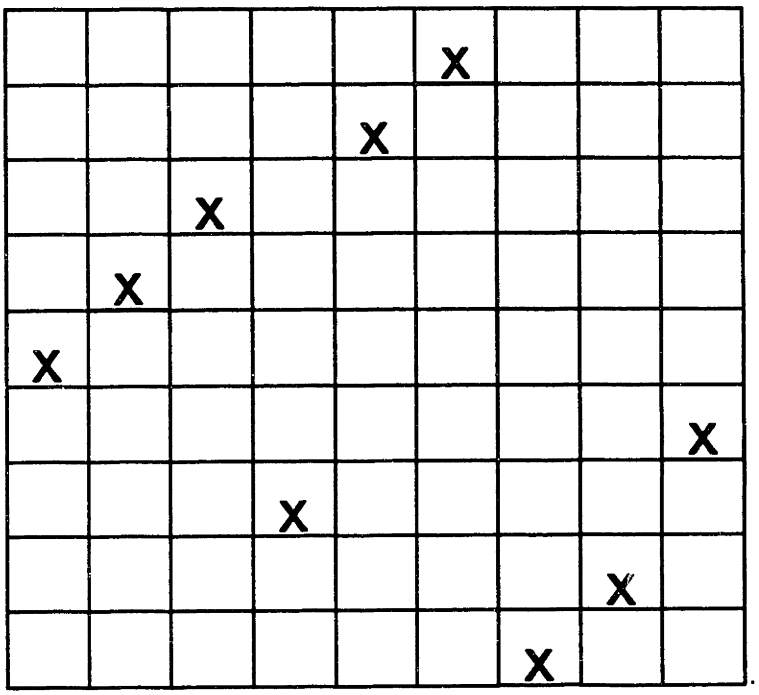
The usual proof of this is the same as the proof for the standard case. The insertion algorithm goes through essentially unchanged, with elements bumping only those elements which are strictly greater.

Example 4.1.3 Let $n = 5$ and $\pi = \begin{matrix} 1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 5 \\ 4 & 4 & 4 & 3 & 5 & 5 & 1 & 2 & 3 \end{matrix}$. We can represent π in the cells of a square skew diagram as



But this is no longer a generalized permutation of Sq_5. To fix this, we introduce the minimum number of horizontal and vertical dividers necessary to view each element of π as being in its own row and column. Summing the integers in a given row tells

us how many dividers we need in that row, and similarly for columns. We obtain:



Now we can perform the usual algorithm to get a pair of standard Young tableaux as the top boundary of the our refined square. It turns out that these tableaux always increase by a horizontal strip where we put the dividers. In the diagram below, if the

value of the growth at a vertex is the empty tableaux, we have just left it blank.

1	2	3	31	41	51 X	511	521	531
1	2	3	31	41 X	41	411	421	431
1	2	3 X	31	31	31	311	321	331
1	2 X	2	21	21	21	211	221	321
1 X	1	1	11	11	11	111	211	311
			1	1	1	11	21	31 X
			1 X	1	1	11	21	21
						1	2 X	2
						1 X	1	1

In terms of our original diagram we have simply

3	31	51 2	521	531
3 3	31	31	321	331
	1 1	1	21	31 1
			2 1	2
			1 1	1

So we get that applying the algorithm to π yields the pair of semi-standard tableaux

$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 5 \\ \hline 3 & 4 & 4 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}, \quad Q = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 4 & 5 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}.$$

Conversely, suppose we start with P and Q. Then each integer i appears in each tableaux in a horizontal strip. Suppose that P has weight $w(T) = (n_1, n_2, n_3, \ldots)$. Then we split the ith row of cells into n_i rows of cells, by placing $n_i - 1$ dividers. We can assign a shape to each vertex on this refined boundary by the rule that cells in a horizontal strip get built up from left to right. Similarly, split the columns up according to Q. This gives us our refined square skew diagram. We can then use the normal reverse algorithm to get a permutation, which we view as a matrix word by taking out the dividers. Readers familiar with the algorithm will notice that this is essentially the relabeling scheme one can use to mimic the Knuth analogue with the original version.

To show that this works in general, we need only the following lemma.

Lemma 4.1.4 In the usual Schensted-Fomin algorithm, let $\mu^i, \mu^{i+1}, \ldots, \mu^j$ be consecutive shapes growing along the top side of Sq_n, and let the partial permutation below these be given by $\begin{matrix} i & i+1 & \cdots & j \\ w_i & w_{i+1} & \cdots & w_j \end{matrix}$. See Figure 4-1. If

$$w_i < w_{i+1} < \ldots < w_j ,$$

then the shapes μ^j and μ^i differ by a horizontal strip and the cells are added sequentially in $\mu^i, \mu^{i+1}, \ldots, \mu^j$ from left to right.

Proof: The proof hinges on the fact we have relegated to Appendix A, that the global description of a shape in terms of looking at the cells below and to the left of the vertex in question, as given in Section 2.4, coincides with the local description.

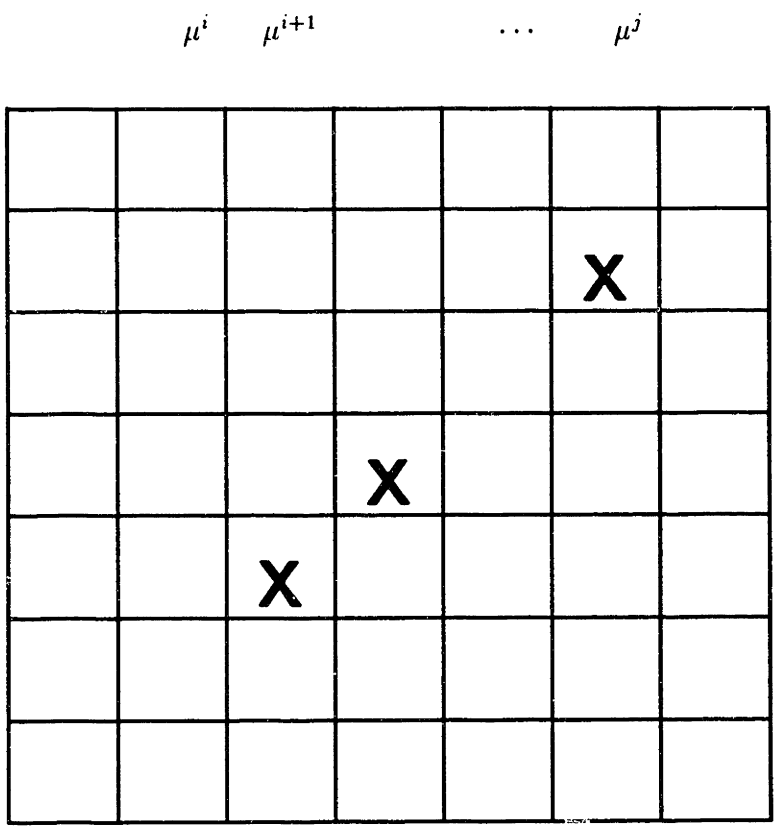


Figure 4-1: Increasing subsequences and horizontal strips

Suppose that μ^k and μ^{k+1} differ in the lth part, i.e.,

$$\mu^{k+1} = (\mu_1^k, \mu_2^k, \ldots, \mu_l^k + 1, \ldots, \mu_t^k).$$

This means exactly that by adding the term $\overset{l}{w_k}$ to the permutation (viewed as a poset), that we did not increase the maximum cardinality of a union of s chains for $s < l$, but that we did increase that for a union of l chains. Now, if $w_k < w_{k+1}$, then clearly the maximum cardinality of a union of l chains must be one greater than before, since we can just add the new element to a chain system which attained the maximum for μ_{k+1}. It is also possible that we have allowed the maximum to increase for some $s < l$. But in either case, we must have that μ_{k+2} and μ_{k+1} differ in the sth part for some $s \leq l$. Now by induction, we find that as long as the permutation is increasing, the shapes we get grow by the cells of a horizontal strip from left to right. □

We can now begin to combine options in useful ways. If we combine the skew algorithm with the Knuth analogue, we get a Knuth version of the skew analogue which coincides with Sagan and Stanley's. We can also iterate this skew Knuth version. See [SS, Section 6] for the statements of these theorems, whose proofs are essentially immediate by drawing the appropriate Fomin picture.

4.2 Oscillating tableaux

Oscillating tableaux also fit into Fomin's picture in a nice way. Our source for much of the following material is S. Sundaram's thesis [Sun].

Definition 4.2.1 An **oscillating** or **up-down tableau** of length k is a sequence of shapes

$$_\lambda S_\mu^k := (\lambda = \mu^0, \mu^1, \ldots, \mu^k = \mu)$$

such that any two shapes differ by exactly one cell. We define ${}_\lambda\tilde{F}^k_\mu = \{$all oscillating tableaux of length k from shape λ to shape $\mu\}$, and ${}_\lambda\tilde{f}^k_\mu := |{}_\lambda\tilde{F}^k_\mu|$. If $\lambda = \emptyset$, then we omit the first subscript.

Example 4.2.2

$$S^9_{21} = (\emptyset, 1, 11, 21, 2, 21, 31, 3, 2, 21)$$

is one possible oscillating tableau of shape (2,1) and length 9.

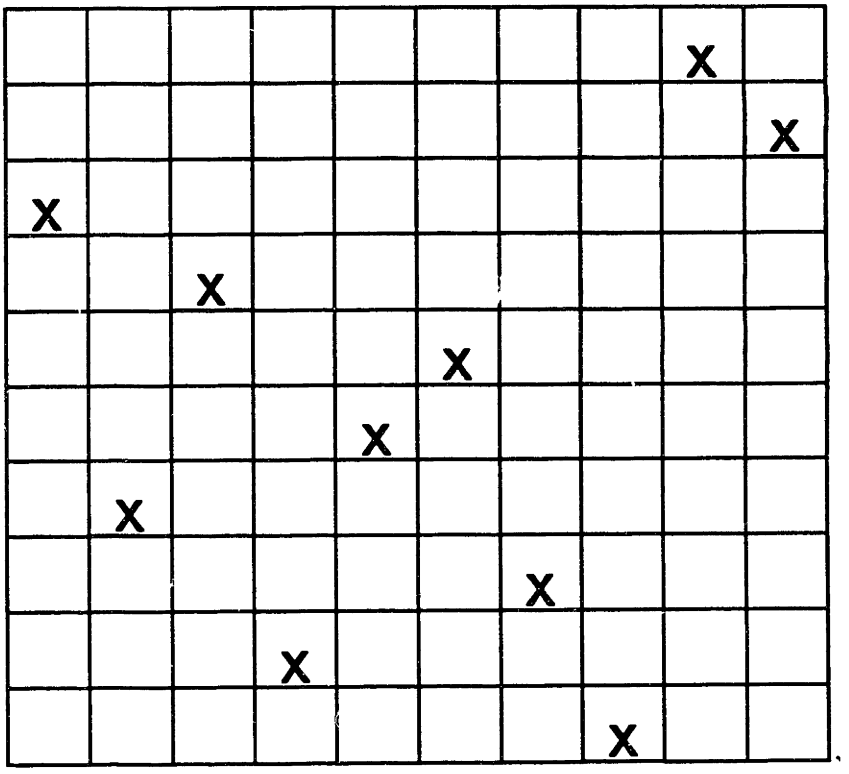
The following bijection seems to be originally due to R. P. Stanley.

Proposition 4.2.3 There is a bijection between fixed point free involutions of S_{2j} and $\tilde{F}^{2j}_\emptyset$.

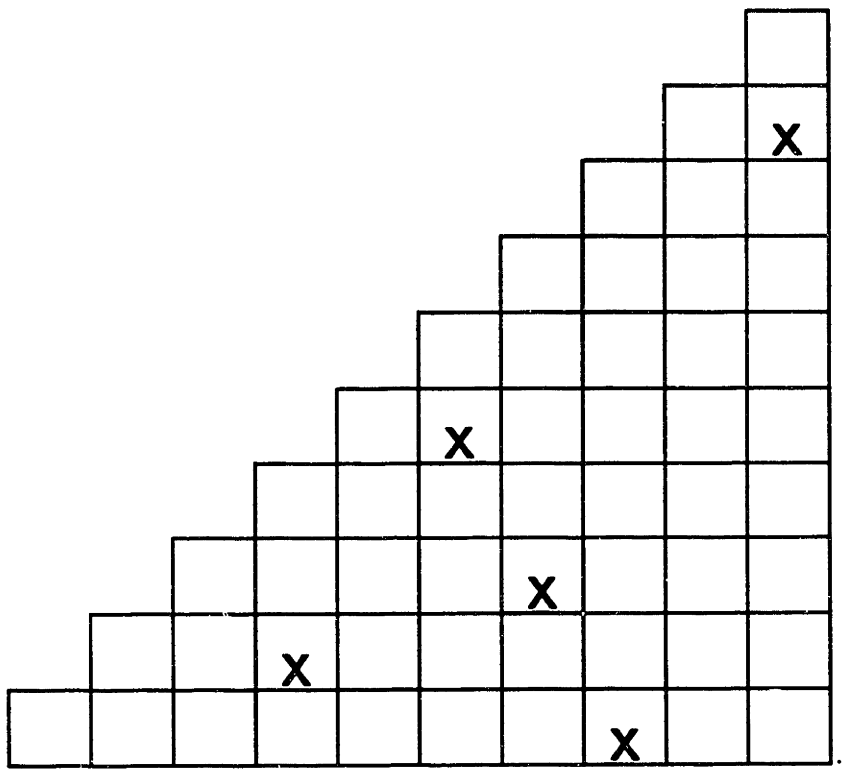
Proof: The right hand side counts the number of fixed point free involutions of S_{2j}. Stanley's original bijection involved viewing such an involution as the number of ways of pairing $2j$ points on a line segment with arcs, i.e. a matching of $2j$ vertices. Then by labeling the vertices in a certain way and using Schensted insertion and jeu de taquin, a bijection may be constructed. Our approach involves looking at the main diagonal of our skew diagram Sq_n.

Example 4.2.4 This is the same example given in [Sun, Example 8.4]. A fixed point free involution is represented by marking X's in the cells of Sq_n symmetrically about the main diagonal, so it is determined by the cells below the main diagonal. Suppose

our involution is $\sigma = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 4 & 7 & 2 & 6 & 5 & 3 & 1 & 10 & 9 \end{matrix}$. Then we have the picture

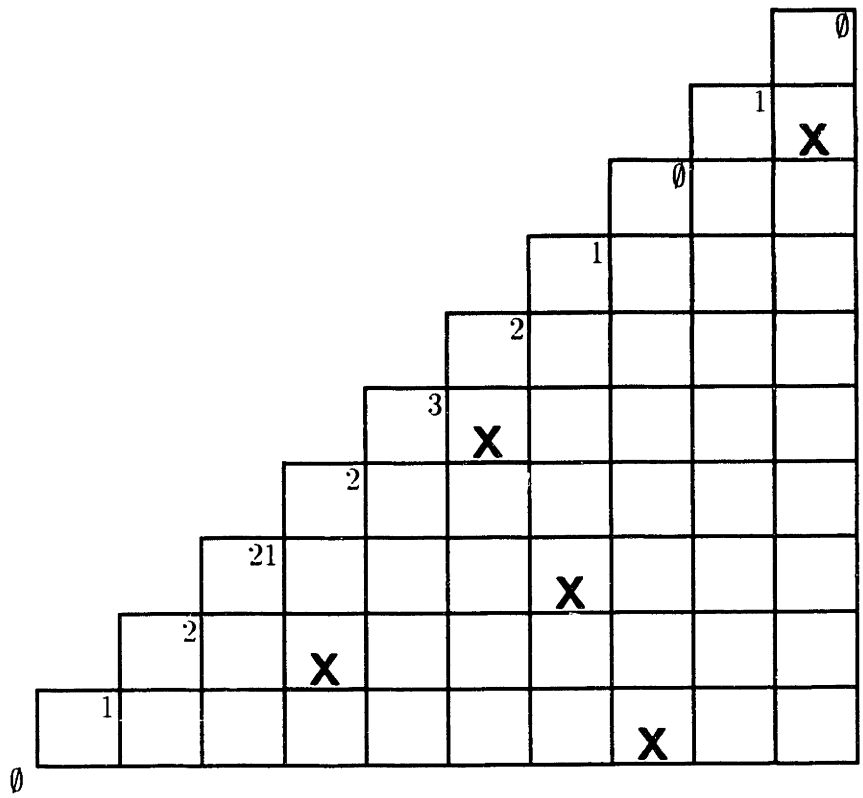


which becomes



Now we define a growth on this triangular shape (call it Tr_n) as follows. Change orientations so that going up and *left* is considered positive; note that in this orientation Tr_n is a skew diagram. Set each vertex on the lower boundary to have value $\emptyset$. Now use the standard method to compute the value of the growth on the cells along the

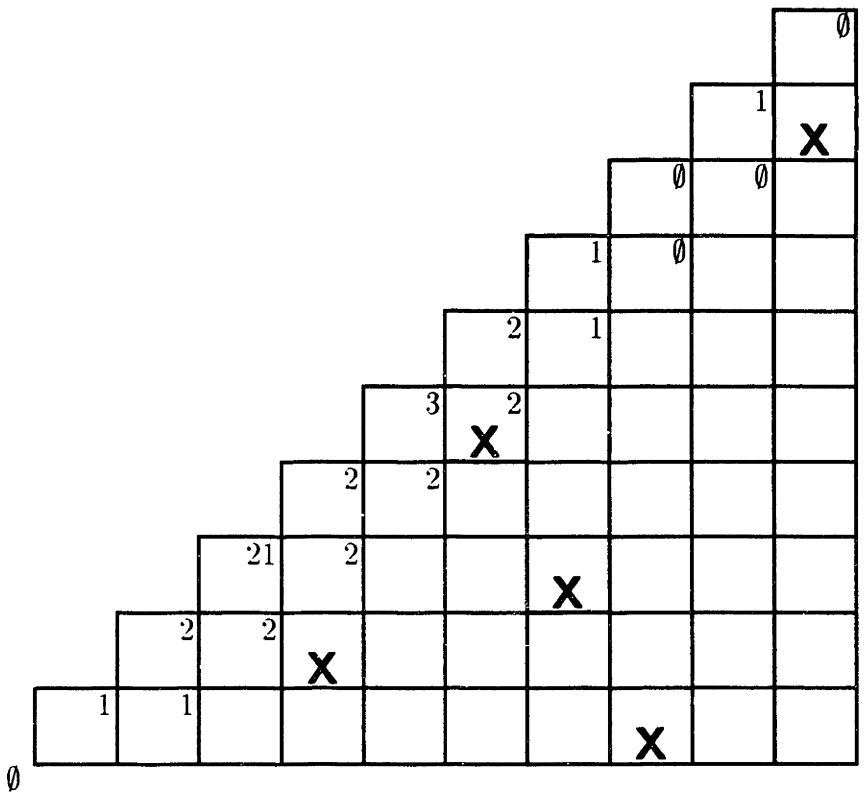
diagonal. We get:



The shapes we obtain are the conjugates of those obtained by Sundaram, but otherwise the algorithm is equivalent.

Why do we always get an oscillating tableau? Since the permutation is symmetric, as we proceed along the diagonal we either gain an X on our left or we lose one below. Since it is fixed point free, these two cases cannot coincide. Hence, the sequence along the diagonal is always an oscillating tableaux.

To reverse the bijection we proceed as follows. We fill in each vertex of the subdiagonal with the smaller of the two tableau above it. In our example this gives:



Now we just use the normal backwards algorithm to fill in the other vertices. Since the intersection of upper and lower boundary of this shape has the value $\emptyset$, the lower boundary must end up identically $\emptyset$. The atoms of this growth will give us the fixed point free involution. □

The enumerative consequence is

Corollary 4.2.5

$$\tilde{f}_{\emptyset}^{2j} = (2j-1)!!$$

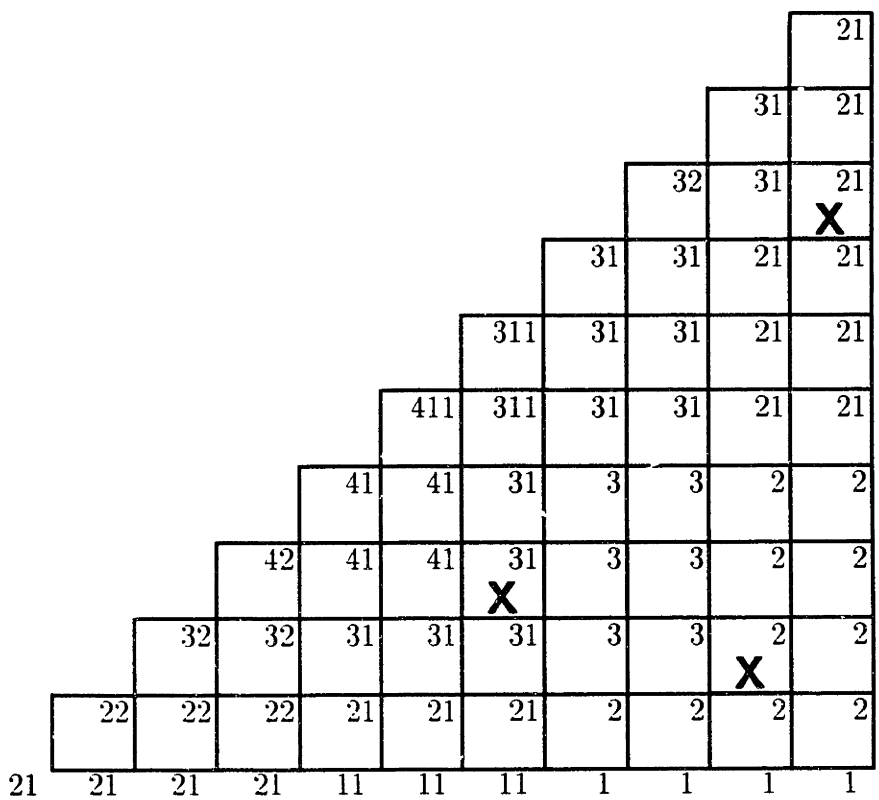
where $(2j-1)!!$ denotes the product of all odd positive integers $\leq (2j-1)$.

To construct a skew oscillating bijection, we only need allow our lower boundary to be nontrivial.

Example 4.2.6 Suppose we start with the oscillating tableaux

$$ {}_{21}P^{10}_{21} = (21, 22. 32, 42, 41, 411, 311, 31, 32, 31, 21) . $$

Putting these shapes along the diagonal and using Fomin's algorithm we obtain



Hence, ${}_{21}P^{10}_{21}$ corresponds with the following partial fixed point free involution σ and pair of tableaux of shape $(2,1)/(1)$

$$ \sigma = \begin{matrix} 2 & 3 & 6 & 8 & 9 & 10 \\ 9 & 6 & 3 & 10 & 2 & 8 \end{matrix} , \quad \begin{array}{|c|c|} \hline \cdot & 1 \\ \hline 5 \\ \cline{1-1} \end{array} , \quad \begin{array}{|c|c|} \hline \cdot & \bar{4} \\ \hline \bar{7} \\ \cline{1-1} \end{array} $$

where the barred numbers $\bar{7}, \bar{4}$ indicate the dual ordering on the set $\{1, 2, \ldots, 10\}$, so $\bar{10} < \bar{9} < \cdots < \bar{1}$. The reason we need this dual ordering is that we have reoriented the horizontal axis to point to the left for the purposes of our growth, but for our permutation it still points to the right.

Proposition 4.2.7 Fix a shape α. There is a bijection between oscillating tableaux $P \in {}_{\alpha}\tilde{F}^{2j}_{\alpha}$ and triples (σ, T, U) where σ is a partial fixed point free involution, and $T, U \in PT(\alpha/\mu)$ such that $T \uplus \hat{\pi} \uplus U = \{1, 2, \ldots, 2j\}$, where μ ranges over all shapes $\supseteq \alpha$. The elements of U (but not those of T) are taken with the dual order on $\{1, 2, \ldots, 2j\}$.

Proof: We proceed as in the example above. Because the tableaux is oscillating, we get a symmetric partial permutation and we get that the entries of T and U are disjoint sets, the former corresponding to increases and the latter to decreases. The partition μ which we get in the bottom right corner is determined by P, but in general different P will give different μ. Conversely, we may start with a triple satisfying the above conditions and obtain an oscillating tableau P. □

The enumerative corollary is

Corollary 4.2.8 Fix a shape α. Then

$$ {}_{\alpha}\tilde{f}^{2j}_{\alpha} = \sum_{i=1}^{j} \binom{2j}{2i} (2i-1)!! \sum_{\substack{\mu \\ \alpha/\mu \vdash (j-i)}} \tilde{f}^{2}_{\alpha/\mu}. $$

Example 4.2.9 The above bijection is really more general: there is no need for the oscillating tableaux to begin and end at the same shape α, and then the number of steps need not be even. If we erase the bottom row in the example above, we get

$P \in {}_{22}F^9_{21}$ and

$$\sigma = \begin{matrix} 1 & 2 & 5 & 7 & 8 & 9 \\ 8 & 5 & 2 & 9 & 1 & 7 \end{matrix}, \quad T = \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline 4 \\ \cline{1-1} \end{array}, \quad U = \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \bar{6} & \bar{3} \\ \hline \end{array}.$$

Thus we get the more general

Theorem 4.2.10 Fix shapes α and β. There is a bijection between oscillating tableaux $P \in {}_\beta\tilde{F}^k_\alpha$ and triples (σ, T, U) where σ is a partial fixed point free involution, and $T \in PT(\alpha/\mu), U \in PT(\beta/\mu)$ such that $T \uplus \hat{\pi} \uplus U = \{1, 2, \ldots, 2j\}$, where μ ranges over all shapes $\supseteq \alpha$. The elements of U (but not those of T) are taken with the dual order on $\{1, 2, \ldots, 2j\}$.

Corollary 4.2.11 Fix shapes α and β. Then

$${}_\beta\tilde{f}^k_\alpha = \sum_{i=1}^{\lfloor k \rfloor} \binom{k}{2i} (2i-1)!! \sum_\mu \tilde{f}_{\alpha/\mu} \tilde{f}_{\beta/\mu}$$

where the sum is over all μ such that $\alpha/\mu \vdash a, \beta/\mu \vdash b$ with $a + b = k - 2i$.

Note that to make these bijections work, one has to use different orderings for the involutions and the growth. If we use our usual convention for permutations, then we have to grow up and to the left. We could also have stood all the diagrams in this section on their sides, used our usual convention for growth, but reversed one of the orders when we marked the cells of the permutation. We will adopt this latter convention sometimes in what follows.

4.3 More oscillating tableaux

In her investigation of properies of Berele's algorithm for symplectic tableaux, Sundaram derives a number of bijections involving oscillating tableaux. It turns out that several of these can be combined into one nice pictorial interpretation which we now present.

Proposition 4.3.1 There is a bijection between oscillating tableaux in $\tilde{F}^k_\mu$ and pairs (σ, U) where σ is a partial fixed point free involution, and $U \in ST(\mu)$, with $\hat{\pi} \uplus U = \{1, 2, \ldots, k\}$. Hence,

$$\tilde{f}^k_\mu = \binom{k}{|\mu|}(2r-1)!!f^\mu, \quad \mu \vdash (k-2r).$$

Proposition 4.3.2 There is a bijection between oscillating tableaux in $\tilde{F}^k_\mu$ and pairs (L, U) where $U \in ST(\mu)$, $L \in PT(\beta)$, β' is even, and $L \uplus U = \{1, 2, \ldots, k\}$. Hence,

$$\tilde{f}^k_\mu = \sum_{\substack{\beta \vdash (k-|\mu|) \\ \beta' even}} \binom{k}{|\mu|} f^\beta f^\mu.$$

We need one more definition to state the third result.

Definition 4.3.3 A **lattice permutation** w is finite sequence of positive integers $w_1 w_2 \cdots w_n$ such that, as one reads the word from left to right, the integer i always occurs at least as many times as the integer $i+1$. The **weight** of w is the the vector (in fact a partition) $\alpha = (\alpha_1, \alpha_2, \ldots,)$ where $\alpha_i :=$ the number of times i occurs in w.

Example 4.3.4 The lattice permutation $w = 1211324213142$ has weight $(5, 4, 2, 2)$.

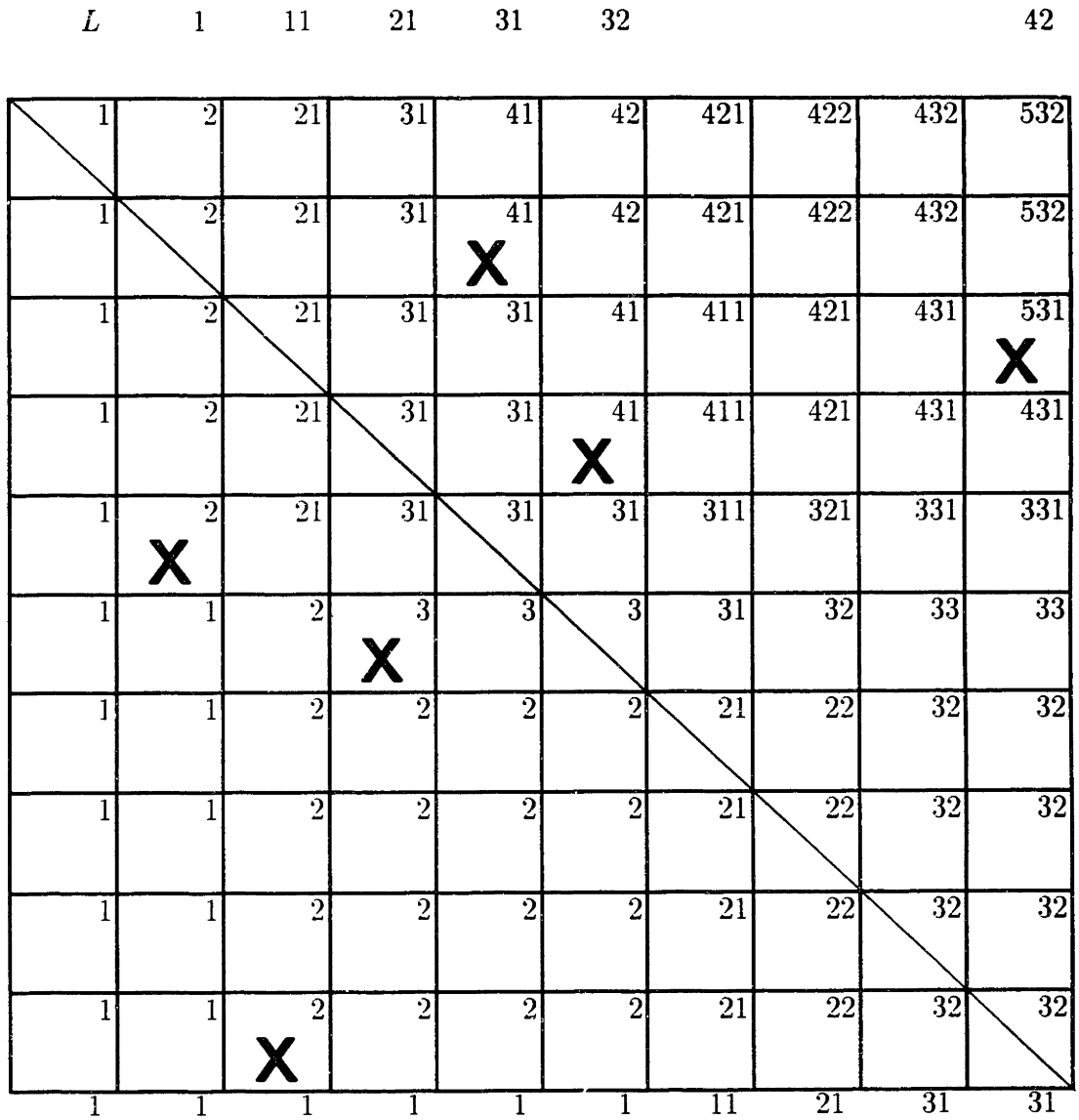
Proposition 4.3.5 There is a bijection between oscillating tableaux in $\tilde{F}^k_\mu$ and pairs $(Q, T_{\lambda/\mu}(\beta))$ where $Q \in ST(\lambda), \lambda \vdash k$, β is a shape with even columns, and $T_{\lambda/\mu}(\beta)$ is a lattice permutation of weight β which fits the skew shape λ/μ. Hence,

$$\tilde{f}^k_\mu = \sum_{\lambda \vdash k} f^\lambda \sum_{\substack{\beta \vdash (k-|\mu|) \\ \beta' even}} c^\lambda_{\mu,\beta}$$

where $c^\lambda_{\mu,\beta} = |\{\text{lattice permuations of shape } \lambda/\mu \text{ and weight } \beta\}|$.

Example 4.3.6 This is the same example given by Sundaram to illustrate this bijections, re-interpreted pictorially. Actually, we always get shapes conjugate to those

in her original example. This time our growth proceeds up and to the right, and we view our permutation in the dual order, greater elements being below and to the right (the usual matrix convention).



We start with the oscillating tableaux

$$S^{10}_{(2,1,1)} = (\emptyset, 1, 2, 21, 31, 3, 2, 21, 22, 32, 31).$$

In our picture this is represented by the diagonal from the upper left corner to the lower right one. By the first proposition $S^{10}_{(2,1,1)}$ maps to the pair (σ, U) given by

$$\sigma = \begin{matrix} 2 & 3 & 4 & 5 & 6 & 10 \\ 5 & 10 & 6 & 2 & 4 & 3 \end{matrix}, \quad U = \begin{array}{|c|c|c|} \hline 1 & 8 & 9 \\ \hline 7 \\ \cline{1-1} \end{array}.$$

In our picture σ is marked with X's and U is the partial tableau we get along the bottom edge of the square. That this correspondence is bijective in general is clear by Fomin: we can either work from the diagonal downwards to get σ (which is symmetric about this diagonal) and U or the other way around. In fact, Proposition 4.3.1 is just a special case of Theorem 4.2.10.

By the second proposition this maps to (L, U) where

$$L = \begin{array}{|c|c|c|c|} \hline 2 & 4 & 5 & 10 \\ \hline 3 & 6 \\ \cline{1-2} \end{array} \quad \text{and} \quad U = \begin{array}{|c|c|c|} \hline 1 & 8 & 9 \\ \hline 7 \\ \cline{1-1} \end{array}.$$

Here, all we are doing is replacing σ with the the even-columned (rowed, in our picture) tableaux to which it corresponds bijectively. We can place it nicely in our picture above the top row of Sq_{10} as shown. It may be computed most easily by the procedure of Section 2.4, which turns the chains in σ below and to the left of a vertex into a shape. Note that L and U fit together so that when one increases, the other is stable.

By the third proposition $S^{10}_{(2,1,1)}$ maps to the pair $(Q, T_{\lambda/\mu}(\beta))$ where

$$Q = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 10 \\ \hline 3 & 6 & 9 \\ \cline{1-3} 7 & 8 \\ \cline{1-2}\end{array}, \text{ and } T_{\lambda/\mu}(2211) = \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & 3 & 4 \\ \hline \cdot & 1 & 2 \\ \cline{1-3} 1 & 2 \\ \cline{1-2}\end{array}.$$

In our picture Q is the standard tableau we get across the top edge, and $T_{\lambda/\mu}(\beta)$ is obtained by relabeling the partial tableau P we get on the right side in the following way.

$$P = \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & 7 & 8 \\ \hline \cdot & 1 & 5 \\ \cline{1-3} 6 & 9 \\ \cline{1-2}\end{array}.$$

Since $\beta' = (4, 2)$, σ may be thought of as a union of two chains, "A" of length 4, and "B" of length 2. We label the elements of each chain in order, and use these labels for the corresponding growth of P. Our diagram becomes:

L	1	11	21	31	32				42
1	2	21	31	41	42	421	422	432	532
1	2	21	31	41 B_2	42	421	422	432	532
1	2	21	31	31	41	411	421	431	531 A_4
1	2	21	31	31	41 A_3	411	421	431	431
1	2 B_1	21	31	31	31	311	321	331	331
1	1	2	3 A_2	3	3	31	32	33	33
1	1	2	2	2	2	21	22	32	32
1	1	2	2	2	2	21	22	32	32
1	1	2	2	2	2	21	22	32	32
1	1	2 A_1	2	2	2	21	22	32	32
1	1	1	1	1	1	11	21	31	31

Now at each increase along the right side we use the label associated with the element of σ in that row. In other words we relabel P thusly:

$$P = \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & 7_3 & 8_4 \\ \hline \cdot & 1_1 & 5_2 \\ \cline{1-3} 6_1 & 9_2 \\ \cline{1-2}\end{array}.$$

Although our picture is essentially a proof of the bijection given in the first two propositions, it is not immediately clear that Proposition 4.3.5 is a bijection. But the correspondence in the forward direction is easy to see from the picture. In fact, a pictorial proof along these lines would be very interesting, since it would probably give a nice proof of the Littlewood-Richardson Rule.

4.4 A Knuth oscillating correspondence

An interesting and, until recently, unsolved problem is to come up with a Knuth analogue of the Robinson-Schensted correspondence for oscillating tableaux. The enumerative results of this section were first obtained by I. Gessel [Ges]. Gessel generalized Stanley's original approach by using operators, corresponding to symmetric functions, which move up or down by horizontal (or vertical) strips. He then used certain generating function techniques and symmetric function identities to obtain results somewhat more general than those we give here. Almost as an afterthought, he supplied a Robinson-Schensted-Knuth bijection based on insertion. We arrived at the (slightly more general) algorithm presented here independently of Gessel's bijection, using instead the approach of Fomin.

Definition 4.4.1 An **oscillating semistandard** tableau of length $2k$ is a sequence of shapes

$$_\lambda S_\mu^{2k} := (\lambda = \mu^0 \subseteq \mu^1 \supseteq \mu^2 \subseteq \ldots \subseteq \mu^{2k-1} \supseteq \mu^k = \mu)$$

where the shapes alternately increase and decrease (weakly) and any two shapes differ by a horizontal strip. We define

$$_{\lambda}\tilde{G}_{\mu}(u_1,d_1,u_2,d_2,\ldots,u_k,d_k) = \{\text{all oscillating semistandard tableaux of length } 2k \text{ from shape } \lambda \text{ to shape } \mu \text{ which alternately go up by horizontal strips of size } u_i \text{ and down by horizontal strips of size } d_i\}$$

and $_{\lambda}\tilde{g}_{\mu}(u_1,d_1,u_2,d_2,\ldots,u_k,d_k) := |_{\lambda}\tilde{G}_{\mu}(u_1,d_1,u_2,d_2,\ldots,u_k,d_k)|$. If $\lambda = \emptyset$, then we omit the first subscript.

Definition 4.4.2 Let M be a symmetric $k \times k$ matrix. We define modified row- and column-sum vectors, which stop summing at the main diagonal, as follows. Set

$$\rho_i(M) = \sum_{j=1}^{i} a_{ij}, \quad \gamma_i(M) = \sum_{j=i}^{k} a_{ji}\,.$$

Now set $\rho(M) = (\rho_1(M),\ldots,\rho_k(M))$ and $\gamma(M) = (\gamma_1(M),\ldots,\gamma_k(M))$.

Theorem 4.4.3 Fix a positive integer k. There is a bijection between tableaux $R \in \tilde{G}_{\emptyset}(u_1,d_1,u_2,d_2,\ldots,u_k,d_k)$ and symmetric $k \times k$ matrices M with nonnegative integer entries where $\rho(M) = (d_1,\ldots,d_k)$ and $\gamma(M) = (u_1,\ldots,u_k)$.

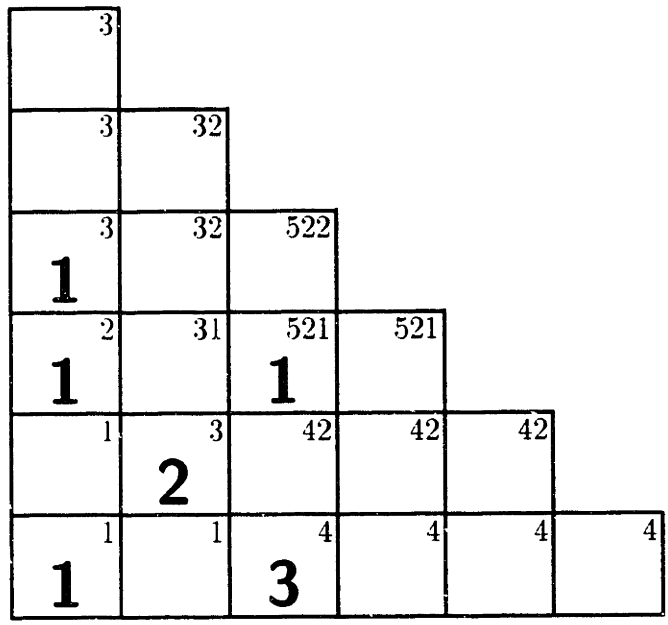
Example 4.4.4 Suppose we start with the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 \end{pmatrix},$$

which can also be thought of as a matrix word

$$\sigma = \begin{matrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 6 & 6 \\ 3 & 4 & 6 & 5 & 5 & 1 & 4 & 6 & 6 & 6 & 1 & 3 & 2 & 2 & 1 & 3 & 3 & 3 \end{matrix},$$

then $\rho(M) = (0,0,1,2,2,4)$ and $\gamma(M) = (3,2,4,0,0,0)$. We have the following diagram:



and we read off the oscillating semistandard tableau along the diagonal:

$$R = (\emptyset, 3, 3, 32, 32, 522, 521, 521, 42, 42, 4, 4, \emptyset)$$

which lies in $\tilde{G}_{\emptyset}(3,0,2,0,4,1,0,2,0,2,0,4)$. If we start with M or σ we can construct R also by simply looking at the chains of σ below and to the left of a vertex (as in Section 2.4). To go the other way, we need to refine this diagram as we did in Example 4.1.3. We omit the details.

Note that the original oscillating bijection used only the vertices on the diagonal, whereas here we use both the diagonal and the subdiagonal of our shape. Fixed points were not allowed since they would have been invisible to the original one, but

here we can allow them without difficulty. We can also allow cells in σ to occur both below and to the left of a diagonal cell at the same time, whereas before this was also forbidden. This may help explain why other attempts to construct a Knuth analogue ran into trouble.

Proof: By the above example. □

Corollary 4.4.5

$$\sum \tilde{g}_{\emptyset}(u_1, d_1, \ldots, u_k, d_k) x_1^{u_1} y_1^{d_1} \cdots x_k^{u_k} y_k^{d_k} = \prod_{1 \le i \le j \le k} (1 - x_i y_j)^{-1}.$$

We can easily combine this with our method for constructing skew algorithms to obtain

Theorem 4.4.6 Fix a positive integer k. There is a bijection between tableaux $R \in {}_{\lambda}\check{G}_{\emptyset}(u_1, d_1, u_2, d_2, \ldots, u_k, d_k)$ and triples (σ, T, U) where σ represents a symmetric $k \times k$ matrix with nonnegative integer entries, $T \in GT(\lambda/\nu), U \in GT(\mu/\nu)$, and ν is allowed to vary.

Example 4.4.7 We can use Example 4.4.4 above to construct one example. Simply truncating the left column and bottom row off our picture we obtain a bijection between

$$R = (3, 32, 32, 522, 521, 521, 42, 42, 4)$$

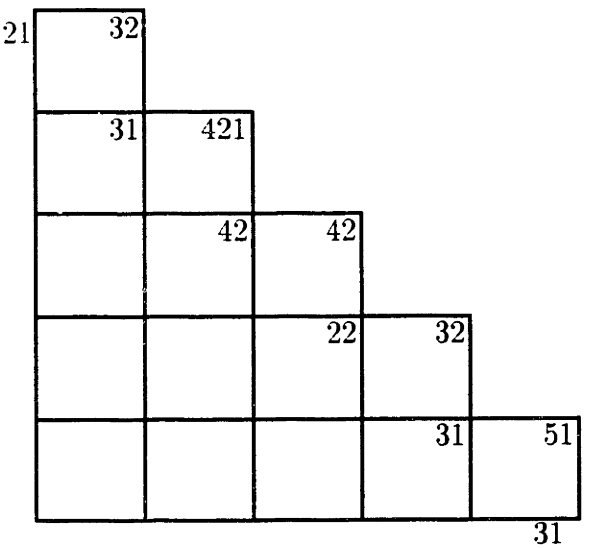
and the triple (σ, T, U), where

$$\sigma = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{array}{|c|c|c|}\hline \cdot & 2 & 3 \\ \hline \end{array}, \quad U = \begin{array}{|c|c|c|c|}\hline \cdot & 2 & 2 & 2 \\ \hline \end{array}.$$

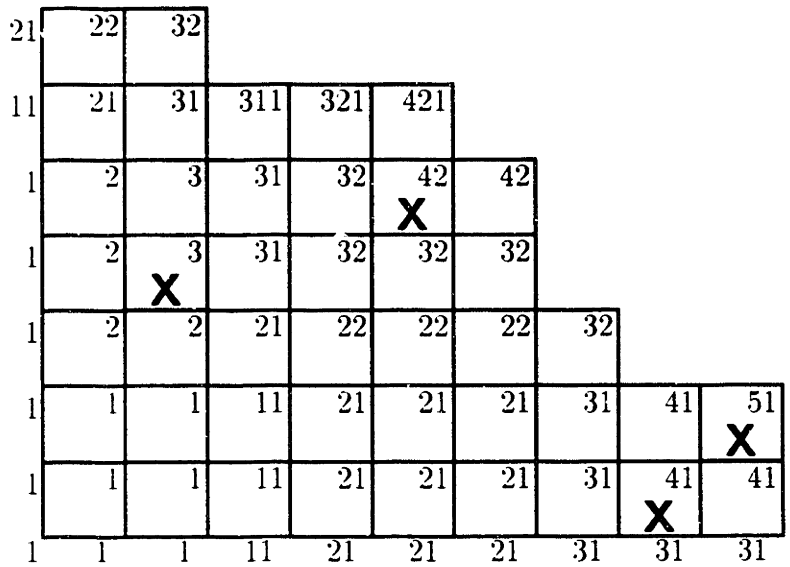
But not every example will have such a form, since in general T and U may have more than one part. For a more interesting example we start with

$$R = (21, 32, 31, 421, 42, 42, 22.32.31.51, 31),$$

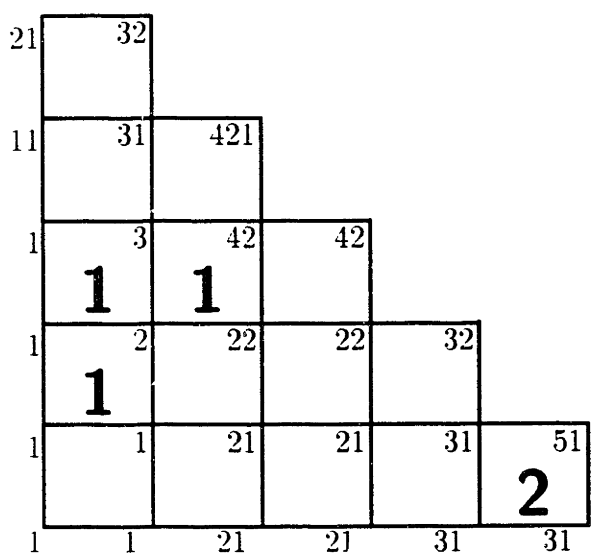
which we represent initially as



We refine it as usual and compute the interior to get



which we rewrite more compactly as



Hence, $R \longleftrightarrow (\sigma, T, U)$ where

$$\sigma = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad T = \begin{array}{|c|c|}\hline \cdot & 5 \\ \hline 4 \\ \cline{1-1} \end{array}, \quad U = \begin{array}{|c|c|c|}\hline \cdot & 2 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}.$$

Corollary 4.4.8

$$\sum {}_{\lambda}\tilde{g}_{\mu}(u_1, d_1, \ldots, u_k, d_k) x_1^{u_1} y_1^{d_1} \cdots x_k^{u_k} y_k^{d_k} = \sum_{\nu} s_{\lambda/\nu}(x) s_{\mu/\nu}(y) \prod_{1 \le i \le j \le k} (1 - x_i y_j)^{-1}$$

where $s_{\lambda/\nu}(x) := s_{\lambda/\nu}(x_1, x_2, \ldots, x_k)$ is the Schur function in k variables.

Proof: The only difference from the corollary above is that we need generating functions for the skew semistandard tableaux T, U. By definition, these are the Schur functions (see Section 1.1). □

Chapter 5

Fibonacci Differential Posets

5.1 The Fibonacci Differential Poset

Much of the discussion in the preceeding chapters is valid for any differential poset. It is interesting to consider how the theory works for the other standard example $Z(1)$, which we defined in Section 2.2.

Example 5.1.1 Let $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 9 & 1 & 8 & 6 & 7 & 3 \end{matrix}$. We have the following diagram:

1	11	21	121 X	221	2211	21211	22211	221211
1	11	21	21	211	1211 X	2211	21211	211211
1	11	21	21	211	211	1211	11211 X	21211
1	11	21	21	211	211	1211 X	1211	2211
1	11 X	21	21	211	211	211	211	221
1	1	11 X	11	21	21	21	21	22
1	1	1	1	2	2	2	2	12 X
1 X	1	1	1	2	2	2	2	2
∅	∅	∅	∅	1 X	1	1	1	1

We use the natural R-correspondence Φ (see Section 2.6) on $Z(1)$ defined as follows. Each $x \in Z(1)$ is covered by a exactly one element which begins with 1, namely $1x$. We set $\Phi_x(1x) = x$. For every other element $2z$ covering x we set $\Phi_x(2z) = z$. If we view $Z(1)$ geometrically as the "reflection-extension" of a point (see Section 2.2), then we are mapping the reflected part back to its reflection around x and the extended part to x. In practice, this R-correspondence is incredibly easy to compute with for the following reason. Suppose as usual that the values of our growth around a cell C are given by $g_{00}, g_{01}, g_{10}, g_{11}$, and that $|g_{00}| < |g_{01}| = |g_{10}| < |g_{11}|$. Then whether

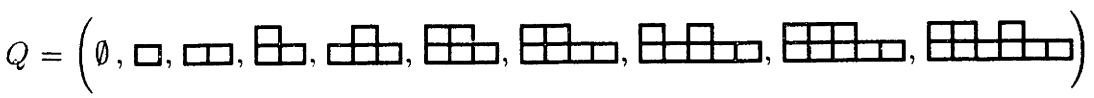
$g_{01} = g_{10}$ or not, we obtain g_{11} from g_{00} by prepending a 2. The only time we prepend a 1 is when the C is marked with an X. So π corresponds with the following pair (P, Q) of paths in Z(1):

$$P = (\emptyset, 1, 2, 12, 22, 221, 2211, 21211, 211211, 221211)$$

$$Q = (\emptyset, 1, 11, 21, 121, 221, 2211, 21211, 22211, 221211)$$

One question which immediately presents itself is if there is some way to represent a path which is more convenient than writing it all out. Young's lattice had the advantage that such a path could be represented canonically as a standard tableau, and any standard tableau represents exactly one path. Here we are less fortunate.

Example 5.1.2 We can represent an element of $Z(1)$ typographically as a sort of Young diagram as follows. Each "2" corresponds to two boxes stacked on top of one another, each "1" to a single box, and we proceed from left to right. With this convention, the shapes in Q above are:

$Q =$ 

So it would be natural to to try represent the above pair of paths as

$P =$

4	9		2		
3	8	7	1	6	5

, $Q =$

5	8		3		
4	7	9	2	6	1

One can do this, but there are several disadvantages compared with the tableaux we get for Young's lattice. While each element in the top row must be greater than the one below it, there are no other obvious rules governing what numberings are allowed for a given shape. But in general, a numbering which follows the obvious rule will

not correspond with a path in $Z(1)$. For example, the "tableau"

$$T = \begin{array}{|c|c|c|c|} \hline 6 & 2 & & 7 \\ \hline 5 & 1 & 3 & 4 \\ \hline \end{array}$$

does not represent a path because 2212 does not cover 2211 in $Z(1)$.

Another way to represent paths in $Z(1)$ is discussed in the next section.

5.2 An insertion algorithm

The following insertion algorithm gives the same results as Fomin's method.

Suppose we wish to insert the number a into a given "partial tableau" T of shape x. We first compare a to the value t_1 of the leftmost square in the bottom row of T. If $a > t_1$, then add a cell to the left of t_1 and put the value a inside, terminating the algorithm (our new tableaux has shape $1x$). If $a < t_1$, then place a in the cell directly above t_1, bumping any element b which may be in that cell. If the cell is empty, we are done. Otherwise, we continue (inductively) by comparing b to the element t_2 to the right of t_1 in the bottom row.

Example 5.2.1 If we insert 8 into the partial tableau

$$\begin{array}{|c|c|c|} \hline 1 & 4 & \\ \hline 9 & 5 & 2 \\ \hline \end{array}$$

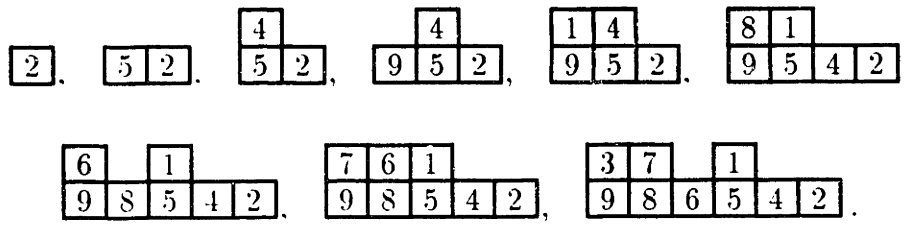
we get

$$\begin{array}{|c|c|c|c|} \hline 8 & 1 & & \\ \hline 9 & 5 & 4 & 2 \\ \hline \end{array}$$

Note that unlike in the original R-S algorithm, the number a may bump elements b in the second row which are larger or smaller than a. The algorithm can terminate in two ways. Suppose the shape x of T has i initial 2's, i.e., $x = 2^i w$ for some word w; let $t_1, \ldots, t_{i+1}$ (resp. $u_1, \ldots, u_{i+1}$) denote the first $i+1$ elements in the bottom (resp. top) row, and set $u_0 := a$. Then if $u_j < t_{j+1}$ for all $0 \leq j \leq i$, each u_j will bump u_{j+1} to the right one place until finally u_i comes to rest on top of t_{i+1}. In this case,

the shape x' of our new tableau T' is $2^{i+1}w$. Otherwise, let q be the least j such that $u_j > t_{j+1}$. Then the algorithm terminates with u_j forming a new column in between t_j and t_{j+1}, and the shape of T' becomes $x' = 2^j 12^{i-j}w$. In either case, x' covers x in $Z(1)$.

Example 5.2.2 Suppose we again start with $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 9 & 1 & 8 & 6 & 7 & 3 \end{matrix}$ as in Example 5.1.1. We have the following sequence of insertions:



Notice that the shapes we get correspond exactly to the path Q from before. As in the Schensted case, we can think of the path Q as a "recording tableau" for the sequence of shapes we get by successive insertions. Our final tableau

$$P = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 7 & & 1 & & \\ \hline 9 & 8 & 6 & 5 & 4 & 2 \\ \hline \end{array}$$

represents the path P from before, but there seems (so far) to be no easy way to translate P into its equivalent path.

Definition 5.2.3 From now on when we need to distinguish between the two ways of representing a path as a "tableau", we shall call the ones from Example 5.1.2, where we just number the elements of a shape x in the same order as they occur in the path, "path-tableaux". The ones from Example 5.2.2, which simply have to satisfy the requirement that elements decreases along the bottom row and up each column, we will call "tableaux". There are no requirements on the top row elements with respect to one another. In fact, these are tableaux in the proper sense of the word, being order preserving maps from a shape x to $\mathbf{Z}^+$. We just consider the order

on $x \in Z(1)$ to be given by $(j,1) < (k,1)$ if $k < j$ and $(j,1) < (j,2)$ for all j. Here (i,j) denotes the cell of x in the ith row (from the left) and the jth column (from the bottom). The advantage tableaux have over path-tableaux is that we can easily write down all the legitimate tableaux of a given shape. Let $ZT(x)$ (resp. $PZT(x)$) denote the set of all tableaux (resp. path-tableaux) of shape x. Set $z_x := |ZT(x)|$.

The most naive form of our insertion algorithm is the following

Proposition 5.2.4 There is a bijection between $\pi \in S_n$ and pairs (P,Q) where $P \in ZT(x)$, $Q \in PZT(x)$, and $|x| = n$. We write $\pi \overset{\text{F-R-S}}{\longleftrightarrow} (P,Q)$.

Proof: First note that the insertion procedure defined above always gives us a legitimate tableau. As elements are inserted, they are bumped across the top of P as long as they are smaller than the elements beneath them. As soon as they become bigger, they fall between two elements in the bottom row and form their own columns. Hence, our output is always a tableau in the sense of the above definition.

We claim that the algorithm given above is invertible. At the kth stage, the path-tableau Q tells us which was the cell most recently created in our tableaux P_k of shape x. If the cell was added in the top row (and, say, the ith column), then x must have the form $2^i w$ for some word w in 1's and 2's. (See the remarks after Example 5.2.1.) Then each u_j in the top row will bump u_{j-1} backwards, until the leftmost u is bumped out of the tableau. This must have been the element which was originally inserted, and we now have our P_{k-1}. If the cell was added in the bottom row (and, say, the ith column), then the element t_i inside it must be smaller than t_{i-1} and P_k has shape $2^{i-1}1w$ for some w. Hence, we bump u_{i-1} with t_i, and continue bumping as before along the top row until the leftmost u is bumped out of the tableau, yielding the originally inserted element and P_{k-1}. □

Corollary 5.2.5

$$\sum_{|x|=n} z_x^2 = n!$$

5.3 Involutions

One property the above algorithm shares with the usual Robinson-Schensted algorithm is the following

Theorem 5.3.1 If $\pi \overset{\text{F-R-S}}{\longleftrightarrow} (P,Q)$, then $\pi^{-1} \overset{\text{F-R-S}}{\longleftrightarrow} (Q,P)$.

Proof: This will be an immediate consequence of Lemma 5.4.4, in which we will prove the bumping algorithm is equivalent to the one derived from Fomin's method. □

We pause a minute to unravel the meaning of the above theorem. If we treat P and Q as representing paths, then the theorem is clear. In terms of our insertion algorithm, $P \in ZT(x)$ and $Q \in PZT(x)$, but their roles are reversed when we replace π with π^{-1}. What this says is that any path from $\emptyset$ to x in $Z(1)$ may be written in either form. If we obtained (P,Q) via the algorithm applied to π, then we can get each represented in the other form by applying the algorithm to π^{-1}. If we start with just say $P \in ZT(x)$ and want to represent it as an element of $PZT(x)$, we may choose any $Q \in PZT(x)$, work backwards to get a permutation σ, then apply the algorithm to σ^{-1} to get $Q_{\sigma^{-1}} \in PZT(x)$ which represents the same path as our original P.

We have the usual

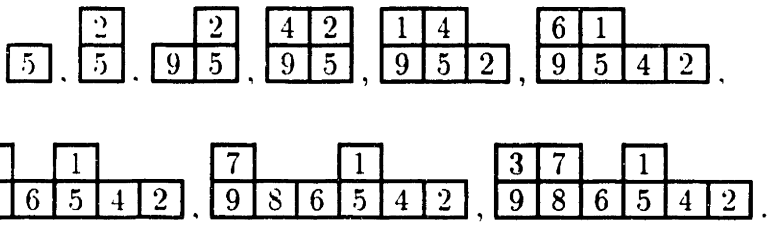
Corollary 5.3.2 Restriction of the above correspondence to involutions of S_n gives a bijection with tableaux in $ZT(x)$ where $|x| = n$. Hence,

$$\sum_{|x|=n} z_x = \mathrm{Inv}(n).$$

Example 5.3.3 If we apply our algorithm to the involution

$$\sigma = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 9 & 4 & 1 & 6 & 8 & 7 & 3 \end{matrix},$$

we get the following sequence of tableau



The cycle decomposition for σ is (93)(87)(6)(51)(4)(2), which corresponds exactly to our P! This serendipitous occurrance is not an accident.

Theorem 5.3.4 Let $\sigma \in S_n$ be an involution, and assume $\sigma \overset{\text{F-R-S}}{\longleftrightarrow} P \in ZT(x)$. Then P represents the cycle decomposition of σ in the obvious manner; to wit, the elements in each column of P form a cycle in σ.

Proof: This follows from the key Lemma 5.4.3, which we prove in the next section. □

5.4 A global description

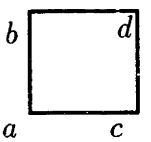
Definition 5.4.1 Let π be a generalized permutation of the cells of a Sq_n, and let v be a any vertex. Define an element $G(v) \in Z(1)$ recursively as follows. Start with the empty word $\emptyset$. If w is the word constructed so far, let R (resp. C) be the cell of π in the highest-occuring row (resp. column). If $R = C$, append a 1 to the end of w; if $R \neq C$, append a 2. Now delete the cells R, C from π, and repeat the whole process. When all cells of π have been deleted, the process terminates (this can take at most n steps). We set $G(v)$ to be the last word we obtain.

Example 5.4.2 The growth which we defined locally in Example 5.1.1, is also an example of a growth of this type, as the interested reader can easily check.

In fact, this global description of growth always coincides with the local one. This is analogous to the situation for Young's lattice, where we could describe a growth either way (see Sections 2.4 and 2.6).

Lemma 5.4.3 Let π be a generalized permutation of the cells of a Sq_n, and let $g : Sq_n \mapsto Z(1)$ be the growth corresponding to π (with the usual R-correspondence on $Z(1)$). Then $g(v) = G(v)$ for every $v \in Sq_n$.

Proof: The reader is encouraged to experiment with a couple of examples similar to the ones given above prior to (or instead of) trying to follow the technicalities we now present. We proceed inductively. Let C be any cell of Sq_n, and let the the vertices of C be given labeled as below:



Now supposing the result is true for all vertices to the left of and below d, we will show it holds also for d. We have several cases.

Case 1: Suppose cells in π occur to the left and below the cell C. Then in the definition of $G(d)$ these cells will be the cells R and C and they will not be equal. Hence, $G(d) = 2G(a)$. On the other hand, the situation forces $|g_a| < |g_b| = |g_c| < |g_d|$, so by the remark after Example 5.1.1 we have $g(d) = 2g(a)$. Hence, $G(a) = g(a)$ implies $G(d) = g(d)$.

Case 2: Suppose there is a cell in π below C, but not one to the left of C. Then clearly $G(c) = G(d)$ since they both see the same generalized permutation below and to the left; $G(a) = G(b)$ for the same reason. Hence, $g(a) = g(b)$ by the induction hypothesis, so $g(c) = g(d)$ by the workings of the local algorithm.

The symmetrical case where there is a cell in π to the left of C but not below is treated the same way.

Case 3: $C \in \pi$. Since π is a generalized permutation, this means that there are no cells of π below or to the left of C. Hence, $g(a) = g(b) = g(c)$ and $G(a) = G(b) = G(c)$. Now in the definition of $G(d)$ the cells R and C will coincide, and then afterwards the situation will be the same as that for $G(a)$; hence $G(d) = 1G(a)$. But our R-correspondence for $Z(1)$ says exactly that when $C \in \pi$, $g(d) = 1g(a)$. Hence, $G(a) = g(a)$ implies $G(d) = g(d)$.

Case 4: Cells in π occur only above or to the right of C. Then clearly $g(a) = g(b) = g(c) = g(d)$ and $G(a) = G(b) = G(c) = G(d)$. Hence, $G(d) = g(d)$.

□

We are now in a position to prove that our insertion algorithm agrees with Fomin's method.

Lemma 5.4.4 Suppose that $\pi \overset{\text{F-R-S}}{\longleftrightarrow} (P, Q)$, where $P \in ZT(x)$, $Q \in PZT(x)$ via the insertion algorithm. Then (P, Q) represents the same pair of paths $(\hat{P}, \hat{Q})$ that we get via Fomin's method.

Proof: This is best seen by referring to an example, which the reader is encouraged to do. The gory details follow. We first show that $Q = \hat{Q}$. Suppose the paths agree up to the $k-1$st stage. Let $\pi(k) = a$. Suppose the shape x of Q_{k-1} has i initial 2's, i.e., $x = 2^i w$ for some word w: let $t_1, \ldots, t_{i+1}$ (resp. $u_1, \ldots, u_{i+1}$) denote the first $i+1$ elements in the bottom (resp. top) row, and set $u_0 := a$. Then if $u_j < t_{j+1}$ for all $0 \leq j \leq i$, each u_j will bump u_{j+1} to the right one place until finally u_i comes to rest on top of t_{i+1}. In this case, the shape x' of our new tableau T' is $2^{i+1}w$. But the global description of Fomin's algorithm says exactly the same thing; for as a replaces u_1 as the rightmost cell, it forces each u_i to pair up with t_{i+1} instead of t_i. In the

other case, let q be the least j such that $u_j > t_{j+1}$. Then the algorithm terminates with u_j forming a new column in between t_j and t_{j+1}, and the shape of T' becomes $x' = 2^j 12^{i-j} w$. The global description again agrees, since u_i will pair up with t_{i+1} only while $u_i < t_{i+1}$. u_j then becomes an unpaired element, and causes a 1 to be added between the jth and $j+1$st columns of x. □

Theorem 5.3.1 now follows by using Fomin's approach and merely transposing the diagram. Theorem 5.3.4 has the following proof:

Proof: (Of Theorem 5.3.4). We use the global description of $G(v)$ at each v along the upwards diagonal of Sq_n. Because σ is symmetric about this diagonal, in the definition of $G(n)$ each cell in σ will always pair up with its image. Hence, fixed points of σ will contribute columns of height 1 in P and two-cycles will contribute columns of height 2. The theorem follows. □

Skew and oscillating versions of this algorithm can also be constructed, and in terms of Fomin's approach one sees immediately how to do so. Skewing means allowing the lower boundary of Sq_n to be nontrivial, and oscillating means looking at tableau along a diagonal. However, $Z(1)$ seems to lack a Knuth version of its algorithm. In the first place, it is unclear what the analogue of a horizontal strip should be. If one attempts instead to generalize the insertion algorithm to matrix words by allowing elements to bump themselves, one finds that correspondence is no longer a bijection [Ker].

Chapter 6

Sequentially Differential Posets

In this section we extend certain results to the class of sequentially differential posets, first defined by Stanley in [Sta2]. Unlike differential posets, which seem to include only two interesting classes of examples ($\mathbf{Y}^r$ and $Z(r)$), sequentially differential posets include many interesting examples. But the enumeration of paths becomes significantly harder and the results do not have the same nice formulation we get in the differential case.

6.1 Definitions and example

Most of the following is straight out of [Sta2].

Definition 6.1.1 Let $\mathbf{r} = (r_0, r_1, \ldots,)$ be an infinite sequence of integers. A poset P is called **r**-*differential* if it satisfies conditions (D1) and (D2) of Definition 2.1.1 as well as

(D3′) If $x \in P_i$ and x covers exactly k elements of P, then x is covered by exactly $k + r_i$ elements of P.

For differential posets we required that the difference between the number of elements covering x and covered by x was a constant function over all of P. Now we let it vary

from rank to rank.

Example 6.1.2 The following are some examples of sequentially differential posets.

(a) An n-element chain ($r_0 = 1, r_i = 0$ for $1 \leq i \leq n-2, r_{n-1} = -1$).

(b) The boolean algebra B_n of rank n ($r_i = n - 2i$, $0 \leq i \leq n$).

(c) A product C_3^n of n 3-element chains C_3 ($r_i = n - i$, $0 \leq i \leq 2n$).

(d) The lattice $L_n(q)$ of subspaces of an n-dimensional vector space over the finite field $\mathbf{F}_q$ ($r_i = 1 + q + \cdots + q^{n-i-1} - (1 + q + \cdots + q^{i-1})$, $0 \leq i \leq n$).

(e) If P is $\mathbf{r}$-differential and finite of rank n, then the dual P^* is $(-r_n, -r_{n-1}, \ldots, -r_0)$-differential.

There are many other examples.

6.2 Enumerative results

Definition 6.2.1 For a fixed poset P let $\alpha(m \to n)$ denote the number of saturated chains from P_m to P_n.

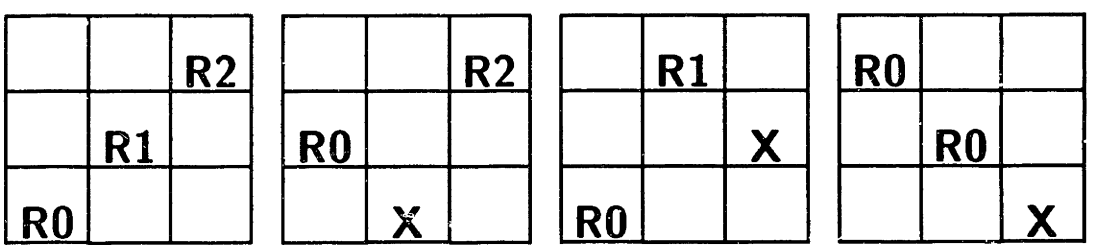
The following theorem was discovered and proved by Stanley using the algebraic machinery of U and D operators. We are able to give a much simpler proof based on Fomin's ideas.

Theorem 6.2.2 Let P be an $\mathbf{r}$-differential poset. Then

$$\alpha(0 \to n) = \sum_w \prod_m r_{\eta(w,m)}$$

where w ranges over all involutions $w_1 w_2 \cdots w_n$ in S_n, m ranges over all *weak excedances* of w (i.e., $w_m \geq m$), and $\eta(m, w)$ is the number of integers j satisfying $j < m$ and $w_j < w_m$.

Example 6.2.3 When $n = 3$ we have four involutions, which may be represented in the following diagrams. In each diagram, the weak excedances, which are simply the cells of the involution above or on the diagonal, are marked by the letter R. The following number is the corresponding value of $\eta(w, m)$, which is simply the number of cells in the involution which are below and to the left of our cell. The other cells of the involution are marked simply with the letter X.



Hence, we conclude that $\alpha(0 \rightarrow n) = r_0 r_1 r_2 + r_0 r_2 + r_0 r_1 + r_0^2$.

We need to generalize somewhat the machinery of Section 2.6. Recall that an R-correspondence Φ was a collection of maps $\{\Phi_x : x \in P\}$, with $\Phi_x : C^+(x) \bigcup \{x\} \mapsto C^-(x) \bigcup \{x\}$. These maps were meant to be "almost-bijections", in that we could define a growth uniquely up to one choice, which we indicated by marking an X in a cell of our skew diagram. Had we been considering r-differential posets at the time, each of these maps would have been bijections except for r elements covering x and our growths would have been uniquely defined up to marking one of the numbers $\{1, 2, \ldots, r\}$ in appropriate cells. More precisely, write $C^+(x) = R(x) \uplus E(x)$, where $\#R(x) = \#C^-(x)$ and $\#E(x) = r$. We define Φ_x to be a bijection between $R(x)$ and $C^-(x)$, and set

$$\Phi_x(y) = x, \quad \forall\, y \in E(x) \cup \{x\}\,.$$

The same argument as before shows that these cells will still form a generalized permutation. Next, because we carry out our construction locally, there is no reason that we have to have the same r for each $x \in P$.

Definition 6.2.4 Let P be an **r**-differential poset, and assume that each $r_i \geq 0$. An **R-correspondence** Φ is a collection of maps $\{\Phi_x : x \in P\}$, with $\Phi_x : C^+(x) \bigcup \{x\} \mapsto C^-(x) \bigcup \{x\}$ which satisfy the following conditions:

(1) $\Phi_x(x) = x$.

(2) If $x \in P_i$, then there are exactly r_i elements $y_1, y_2, \ldots, y_{r_i} \in C^+(x)$ satisfying $\Phi_x(y_i) = x$.

(3) $\Phi_x : C^+(x) \backslash \{y_1, y_2, \ldots, y_{r_i}\} \mapsto C^-(x)$ is a bijection.

It is not hard to think of ways to generalize this definition for the case where some of the r_i's can be negative, but it does not seem to work well with the simple proofs which follow.

Definition 6.2.5 A **weighted generalized permutation** $\tilde{\sigma}$ on a skew diagram S is a generalized permutation of S, each of whose cells is assigned an integer (called the **weight**).

The following theorem is proven analogously to Theorem 2.6.7.

Theorem 6.2.6 Let P be an **r**-differential poset with each $r_i \geq 0$ and S any skew diagram. Fix an R-correspondence Φ on P. Then we have a bijection between growths $g^+ : \partial^+(S) \mapsto P$ and pairs $(g^-, \tilde{\sigma})$ where $g^- : \partial^-(S) \mapsto P$, and $\tilde{\sigma}$ is a weighted generalized permutation on S, with weight function $\eta(C) =$ the number of cells in σ below and to the left of σ.

Proof: The only interesting detail is the description of the weight function. By definition of the R-correspondence, our growth is determined except at cells $C \in \sigma$.

The number of choices at such a cell C is determined by the rank of the element g_{00}, which is $\eta(C)$. □

Now we can make quick work of Theorem 6.2.2.

Proof: (of Theorem 6.2.2). Since we are interested in saturated chains from $\emptyset$, we get in the above theorem that $g^- \equiv \emptyset$, and σ must be a permutation. Since we are interested in chains but not pairs of chains, we restrict to the case where g^+ consists of the same chain along the top and right edges, which forces σ to be an involution. Since the entire diagram must be symmetric about the upwards diagonal, we only are allowed the choices given by our weights on (say) those cells above or on the diagonal. The values at the other cells are then determined. Each saturated path corresponds with one weighting of one involution σ. Hence, we obtain

$$\alpha(0 \rightarrow n) = \sum_w \prod_m r_{\eta(w,m)} \cdot$$

This give a simple bijective proof in the case where all the r_i's are nonnegative. It is easy to see from the machinery of differential posets [Sta2] that the answer in general must be a polynomial in the r_i's. Hence, the polynomial we determined in the special case of all r_i's being nonnegative must hold for general r_i's. □

Next we consider a kind of oscillating analogue, and give a simple bijective proof of another of Stanley's results. Consider a word $w = w(U, D) = w_1 w_2 \cdots w_l$ in the letters U and D. We wish to count the number of Hasse walks $\hat{0} = x_0, x_1, \ldots, x_l = x$ with the cover relations $x_{i-1} \lessdot x_i$ or $x_i \lessdot x_{i-1}$, specified by w.

Definition 6.2.7 Fix a poset P and $x, y \in P$. Consider a word $w = w(U, D) = w_1 w_2 \cdots w_l$ in the letters U and D. Let $\varepsilon(x \xrightarrow{w} y)$ denote the number of Hasse walks from x to y, with the cover relations $x_{i-1} \lessdot x_i$ or $x_i \lessdot x_{i-1}$, specified by w; i.e., $x_{i-1} \lessdot x_i$ if $w_i = U$ and $x_i \lessdot x_{i-1}$ if $w_i = D$. When we omit w and just write

$\varepsilon(x \to y)$, we mean the number of saturated chains from x to y (i.e., $w = U^n$ or D^n). In particular,

$$\alpha(m \to n) = \sum_{\substack{x \in P_m \\ y \in P_n}} \varepsilon(x \to y).$$

Note that $\varepsilon(\hat{0} \to x)$ is often denoted (particularly by Stanley) $e(x)$ (the number of linear "extensions" to x).

Example 6.2.8 If $w = UUDUDDU$ and $x = \hat{0}$, then $\varepsilon(\hat{0} \xrightarrow{w} y)$ is the number of Hasse walks $\hat{0} = x_0 < x_1 < x_2 > x_3 < x_4 > x_5 > x_6 < x_7 = y$.

Clearly $\varepsilon(\hat{0} \xrightarrow{w} x) = 0$ except under the following conditions.

Definition 6.2.9 Fix $x \in P$. We call the word $w = w(U, D)$ a *valid x-word* if the following conditions hold:

(a) For all $1 \leq i \leq l$, the number of D's among $w_1, \ldots, w_i$ does not exceed the number of U's.

(b) The difference between the number of U's and the number of D's in w is the rank $\rho(x)$ of x.

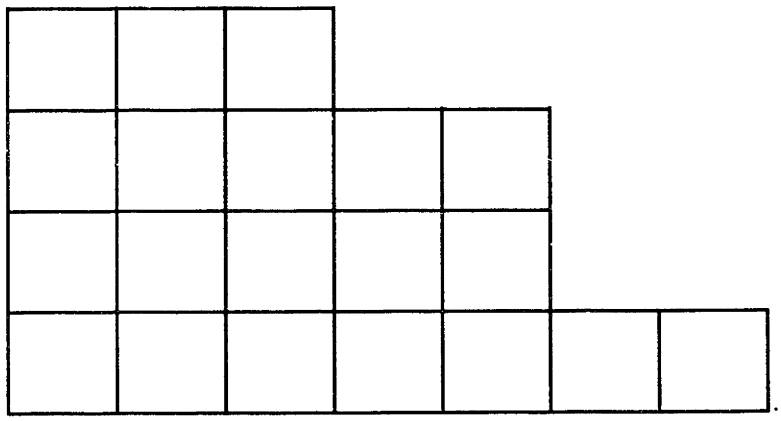
Theorem 6.2.10 Let P be an **r**-differential poset, and let $x \in P$. Let $w = w(U, D) = w_1 w_2 \cdots w_l$ be a valid x-word. Let $S = \{i : w_i = D\}$. For each $i \in S$, let d_i be the number of D's in w to the left of or including w_i, and let u_i be the number of U's in w to the left of or including w_i. Set $f_i = u_i - d_i$. Then

$$\varepsilon(\hat{0} \xrightarrow{w} x) = \varepsilon(\hat{0} \to x) \prod_{i \in S} (r_0 + r_1 + \cdots + r_{f_i}). \tag{6.1}$$

Example 6.2.11 Let $w = UUUDUUDDUUD$. Then $S = \{4, 7, 8, 11\}$, and we easily compute $f_4 = 2$, $f_7 = 3$, $f_8 = 2$, $f_{11} = 3$. If $\rho(x) = 3$, then

$$\varepsilon(\hat{0} \xrightarrow{w} x) = \varepsilon(\hat{0} \to x)(r_0 + r_1 + r_2)^2 (r_0 + r_1 + r_2 + r_3)^2.$$

Proof: First assume that each $r_i \geq 0$. Let S be given as $\{s_1, \ldots, s_d\}$. Construct a skew diagram F, by letting the ith row of F (counting from the top) consist of $f_{s_i} + i$ cells. Then F will be a sort of upside-down Young diagram. Any Hasse walk counted by the left hand side of (6.1) may be represented by a growth g^+ on $\partial^+(F)$, where the upper left corner takes the value $\hat{0}$, and the lower right corner takes the value x. In the preceeding example, we would have the following diagram for F:



By the usual process, g^+ is in bijective correspondence with a pair $(g^-, \tilde{\sigma})$, where g^- is a growth on the lower boundary of F, and $\tilde{\sigma}$ is a weighted generalized permutation on S, with usual weight function $\eta(C)$. Since, g^+ is never constant, each column of F must contain a cell of $\tilde{\sigma}$ or else there is nontrivial growth across the bottom edge of F in that column. Now since the left edge of F must be identically $\hat{0}$, g^- is determined simply the nontrivial growth along the bottom edge of F, which can be thought of as a saturated chain from $\hat{0}$ to x. This accounts for the $\varepsilon(\hat{0} \rightarrow x)$ part of the right hand side.

Next, we need to count the number of weighted generalized permutation on $\tilde{\sigma}$ on S, with usual weight function $\eta(C)$. In the first row, we have a choice of $f_{s_1} + 1$ cells, and picking the jth column will give weight r_{j-1}. Hence, we have $(r_0 + \cdots + r_{f_{s_1}})$ total choices. In the ith row, our previous choices have eliminated $i - 1$ columns, so we have $f_{s_i} + i - (i - 1) = f_{s_i} + 1$ choices left. The weights work exactly as they did

in the first column: picking the jth allowed column will give weight r_{j-1} (since the disallowed column all have cell in $\check{\sigma}$ above the row under consideration. Hence, the total number of weighted generalized permutation is given by $\prod_{i \in S}(r_0 + r_1 + \cdots + r_{f_i})$. (6.1) follows. The extension to the case where some r_i's may be negative proceeds as before. □

Corollary 6.2.12 Let P be an **r**-differential poset. Then

$$\alpha(0 \to n \to 0) = \sum_{x \in P_n} \varepsilon(\hat{0} \xrightarrow{w} x)^2$$

$$= \sum_{w \in S_n} \prod_{m \in w} r_{\eta(w,m)} = \prod_{i=0}^{n-1} (r_0 + r_1 + \cdots + r_i).$$

Proof: The first equality is just the definition. The latter two are the case $w = U^n D^n$ (i.e., the skew diagram F is square) in the above theorem. Compare also Theorem 6.2.2. □

6.3 Iterated skew maps

An open question in [Sta2] was to come up with a more explicit formula for $\alpha(n \to n + k)$ along the lines of Theorem 6.2.2. Using an analogue of Sagan and Stanley's skew iterated algorithm (Section 3.3), we give such an explicit formula, which will be the restriction to the case of involutions of a more general iterated bijection.

Definition 6.3.1 Let $\kappa(k \to n + k \to k)$ denote the number of closed Hasse walks of length $2n$ on a poset P which start and end at the same element of level k, going up to level $n + k$ by a saturated chain and returning. In terms of our earlier notation

we have:

$$\kappa(k \to n+k \to k) = \sum_{x \in P_k} \varepsilon(x \xrightarrow{w} x)$$

where $w = U^n D^n$.

Definition 6.3.2 A **doubly-weighted generalized permutation** $\tilde{\omega}$ on a skew diagram S is a generalized permutation of S, each of whose cells is assigned two integer. The first integer we will call the "q-weight"; it keeps track of which iteration of the skew algorithm we are on, and is defined exactly as in Definition 3.3.1. The second weight (the "r-weight") keeps track of what level we are currently at in the poset. This is the notion used earlier in this chapter. Although we have used the same terminology for both of these ideas, they will coexist peacefully in the results to follow.

The following r-weight function will be the analogue of the weight function η we used before.

Definition 6.3.3 Fix a cell $C^{(t)}$ of q-weight t in the doubly-weighted generalized permutation $\tilde{\omega}$, and fix $\nu \in P$. Let $\tilde{\omega}^{(\geq u)}$ denote the set of cells of $\tilde{\omega}$ which have weight at least u. Set

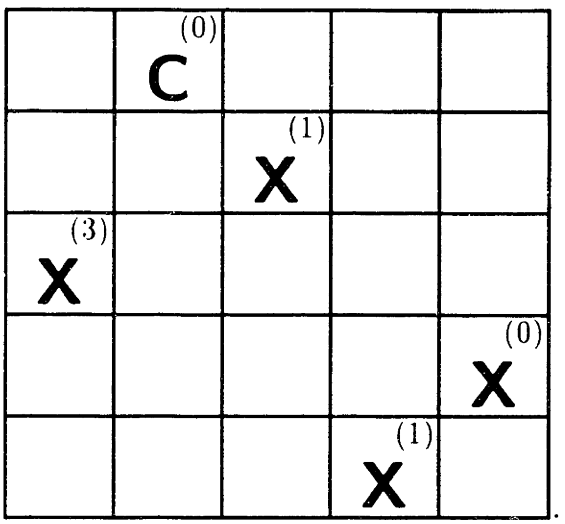
$$\begin{aligned}\theta(\tilde{\omega}, C^{(t)}, \nu) \;=\; & \#Y \in \tilde{\omega}^{(t)} \text{ below and to the left of } C \\ & + \#Y \in \tilde{\omega}^{(\geq t+1)} \text{ below or to the left of } C, \text{ counted with multiplicity} \\ & + \sum_{Y \in \tilde{\omega}^{(\geq t+2)}} wt(Y) - t - 1 \\ & + |\nu|\end{aligned}$$

By "counted with multiplicity" we mean that if the cell Y is both below *and* to the left of C that we count it twice. The third summand is equivalent to $\#Y \in \tilde{\omega}^{(\geq t+2)} + \#Y \in \tilde{\omega}^{(\geq t+2)} + \cdots$.

Example 6.3.4 Let $|\nu| = 4$ and let

$$\tilde{\omega} = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3^{(3)} & 5 & 4^{(1)} & 1^{(1)} & 2 \end{matrix},$$

as in Example 3.3.4. We may represent $\tilde{\omega}$ by the following diagram:



Then $\theta(\tilde{\omega}, C, \nu) = 0 + 4 + 2 + 4 = 10$. This is the same weighted permutation that appeared in Example 3.3.4. The reader may want to refer back to this example and note that the value of the growth on the lower corners of the corresponding cell had rank 10. One can also check that this works for the other cells in $\tilde{\omega}$, i.e., that θ is exactly the right function to determine the rank that occurs just below a cell in an iterated growth.

Theorem 6.3.5 Let P be an **r**-differential poset. Then

$$\sum_{k \geq 0} \kappa(k \rightarrow n+k \rightarrow k) q^k = \sum_{\nu \in P} \left(\sum_{\omega} \prod_{C \in \omega} r_{\theta(\omega, C, \nu)} q^{wt(\omega)} \right) q^{|\nu|} \tag{6.2}$$

where ω ranges over all q-weighted permutations.

Proof: One just mimics the proof of Theorem 3.3.3, but now that our poset is sequentially differential, we have to count the r-weight of each term of our q-weighted permutation separately using the weight function θ. □

Restricting to the case of symmetric growths we obtain:

Corollary 6.3.6 Let P be an **r**-differential poset. Then

$$\sum_{k\geq 0} \alpha(k \to n+k)q^k = \sum_{\nu\in P} \left(\sum_{\omega} \prod_{C\in\omega} r_{\theta(\omega,C,\nu)} q^{wt(w)} \right) q^{|\nu|} \tag{6.3}$$

where ω ranges over all q-weighted involutions.

Appendix A

The equivalence of the approaches for Young's lattice

A.1 Introduction

In this chapter we show that for Young's lattice **Y** the two methods we gave of defining a growth, the "global" approach of Section 2.4 and the local approach of Section 2.6, are equivalent. Our proof closely follows that of Fomin [Fom2, Section 6], and we have preserved most of his notation. In light of the following extension of Schensted's original theorem on the length of the longest increasing (decreasing) subsequence of a permutation by Curtis Greene, this will imply that Fomin's approach, in the particular case of **Y** with the usual R-correspondence, is identical with the usual R-S algorithm (via insertion).

Theorem A.1.1 ([Gre1]) Let $\pi \overset{\text{R-S}}{\longleftrightarrow} (S, T)$ where S and T are of shape λ. For each $k \leq n$, let $a_k(\pi) =$ denote the length of the longest subsequence of π which has no increasing subsequences of length $k+1$. (It can be shown easily that any such sequence is obtained by taking the union of k decreasing subsequences.) Similarly,

define $c_k(\pi)$ to be the length of the longest subsequence consisting of k ascending subsequences. Then

$$\begin{aligned} c_k(\pi) &= \lambda_1 + \lambda_2 + \cdots + \lambda_k \\ a_k(\pi) &= \lambda'_1 + \lambda'_2 + \cdots + \lambda'_k \end{aligned}$$

where λ' denotes the conjugate partition to λ as usual.

A.2 Posets to Partitions

First we recall the basic theorem of Section 2.4.

Theorem A.2.1 Let P be any finite poset. For k a positive integer, set $c_k(P)$ (resp. $a_k(P)$) to be the size of the largest number of elements which is the union of k chains (resp. antichains) of P. Now, let $\lambda_k(P) = c_k(P) - c_{k-1}(P)$ and $\mu_k(P) = a_k(P) - a_{k-1}(P)$. Then $\lambda(P) = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ are *partitions*, and μ is the conjugate of λ.

We will also need the following result, which is proven in Fomin's earlier paper [Fom1].

Lemma A.2.2 Let e be an extremal (maximal or minimal) element of a finite poset P. Then $\lambda(P \backslash \{e\}) \subset \lambda(P)$.

Hence, it is clear that the global definition of growth actually is a growth.

Definition A.2.3 Let $w = (w_1, w_2, \ldots, w_k)$ be a system of disjoint chains in the finite poset P. For each nonnegative integer s define a functional $H_s(w)$ by

$$H_s(w) = \sum_{1 \le i \le k} (\#w_i - s).$$

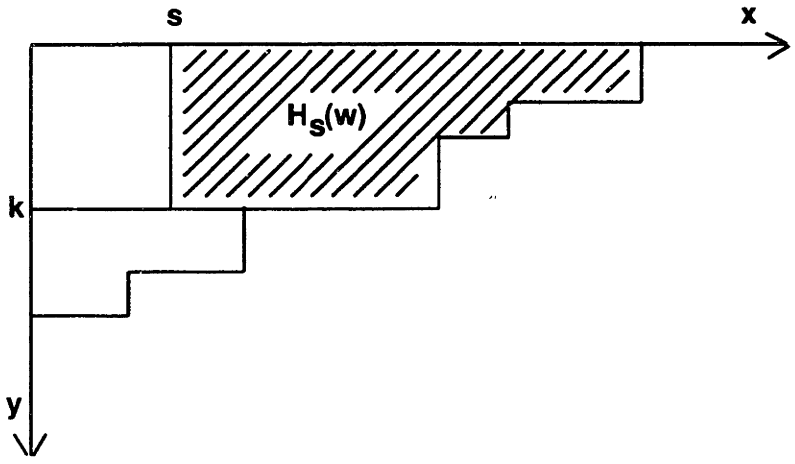


Figure A-1: The functional $H_s(w)$ viewed inside $\lambda(P)$

For fixed k, if w is a maximal system of chains, then $H_s(w) = c_k - sk$ (see Figure A-1). For fixed s, the functional H_s is maximized when k is chosen so that cell (k, s) lies at the boundary of the diagram $\lambda(P)$. A system of chains w in P may be regarded as a collection of edges in the Hasse diagram H of P. If we temporarily disregard the order, we may treat H as an undirected graph, and w as simply a collection of edges in H. Now, if p is any path in H, it makes sense to take the symmetric difference $w' = w \triangle p$. It may or may not be the case that $w \triangle p$ can be regarded as a system of chains of P. Some schematic examples of this appear in Figure A-2, where w is always the three vertical solid line segments, and p is the path indicated by the bold and dashed segments. Note that while we always start with w having 3 chains, that w' may consist of more or fewer chains. In the figure, example a has a 4 chain w', and example d has a 2 chain w'. We have the following "Ford-Fulkerson" type theorem which guarantees the existence of maximal chain systems.

Theorem A.2.4 ([Fom1]) Fix $s \in \mathbb{N}$. If $H_s(w) < \max H_s$ then there exists a path p and a chain system w' such that $w' = w \triangle p$ and $H_s(w) < H_s(w')$.

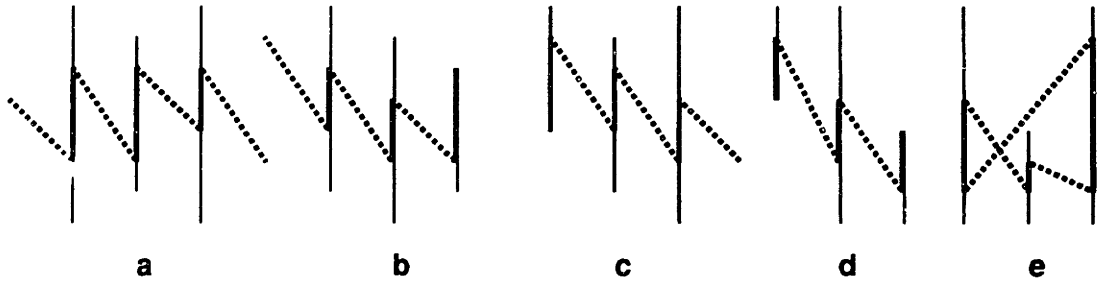


Figure A-2: Replacing a chain system with another via a path

The above theorem will be the main tool used in proving the equivalence of the local and global approaches. Note that in general the chain system w' need not have the same number of chains as w. If they do, we have the following Lemma, which is illustrated by Figure A-2.

Lemma A.2.5 In the situation of Theorem A.2.4, if the chain systems w and w' have the same number of chains, then either w and w' have the same set of maximal elements or they have the same set of minimal elements.

A.3 Adding successive extremal elements

Let e_1 and e_2 be extremal elements of P. Define $\lambda_{00} = \lambda(P\backslash\{e_1, e_2\})$, $\lambda_{01} = \lambda(P\backslash\{e_1\})$, $\lambda_{10} = \lambda(P\backslash\{e_2\})$, $\lambda_{11} = \lambda(P)$. By Lemma A.2.2, we have that $\lambda_{00} \subset \lambda_{01} \subset \lambda_{11}$ and $\lambda_{00} \subset \lambda_{10} \subset \lambda_{11}$. We think of these partitions as representing the growth at the four corners of a cell of some skew diagram, as in Figure A-3. If $\lambda_{01} \neq \lambda_{10}$, then λ_{00} and λ_{11} must be the meet and join (respectively) of λ_{01} and λ_{10}, simply because they form a growth on the corners of the cell. This agrees with our local determination of the growth. The harder case is $\lambda_{01} = \lambda_{10}$, for which we need the following theorem. This will tell us how adding two successive extremal elements to a poset affects the associated partition.

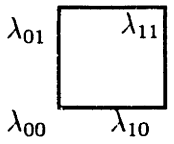


Figure A-3: Close up of the growth around one cell

Theorem A.3.1 Assume $\lambda_{01} = \lambda_{10}$ in the situation above. Let $A = (X_A, Y_A)$ denote the cell $\lambda_{01}/\lambda_{00}$ and $B = (X_B, Y_B)$ denote the cell $\lambda_{11}/\lambda_{01}$.

Case 1: If e_1 and e_2 are extremal elements of different types (i.e., one is maximal and the other minimal) then $x_B = x_A$ or $x_B = x_A + 1$.

Case 2: If e_1 and e_2 are extremal elements of the same type (i.e., both maximal or both minimal), then $x_B \leq x_A$.

See Figure A-4, where the shaded regions indicate the allowed position of the cell B relative to A.

Example A.3.2 Consider the two posets in Figure A-5. Both are obtained by adding two extremal elements to the same five-element poset, but in the first one these elements are of different types. in the second, both are maximal. The corresponding partitions are $\lambda_{00} = 32$, $\lambda_{01} = \lambda_{10} = 33$ in both cases; but $\lambda_{11} = 43$ in the first case, $\lambda_{11} = 331$ in the second.

Proof: Let $c_k = c_k(P\backslash\{e_1, e_2\})$ and $a_k = a_k(P\backslash\{e_1, e_2\})$ (as in the theorem at the beginning of this appendix). So c_k counts the sum of the first k parts of λ_{00}. We consider Case 1 and proceed by contradiction. If the conclusion is false, then we have one of the following two equations:

$$y_B < y_A, \quad x_B > x_A + 1 \tag{A.1}$$

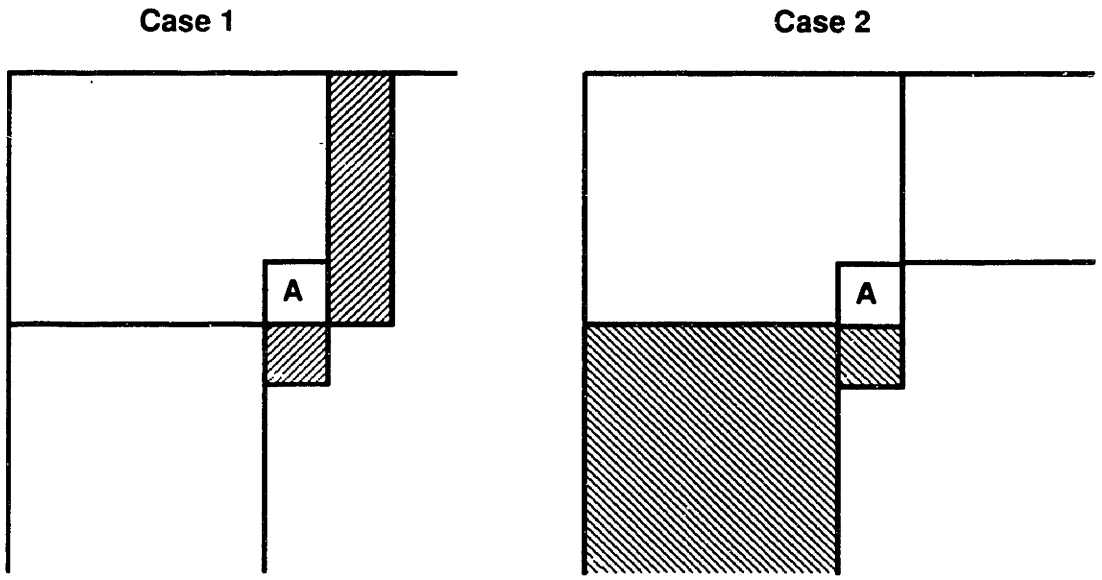


Figure A-4: Allowable positions for the cell B

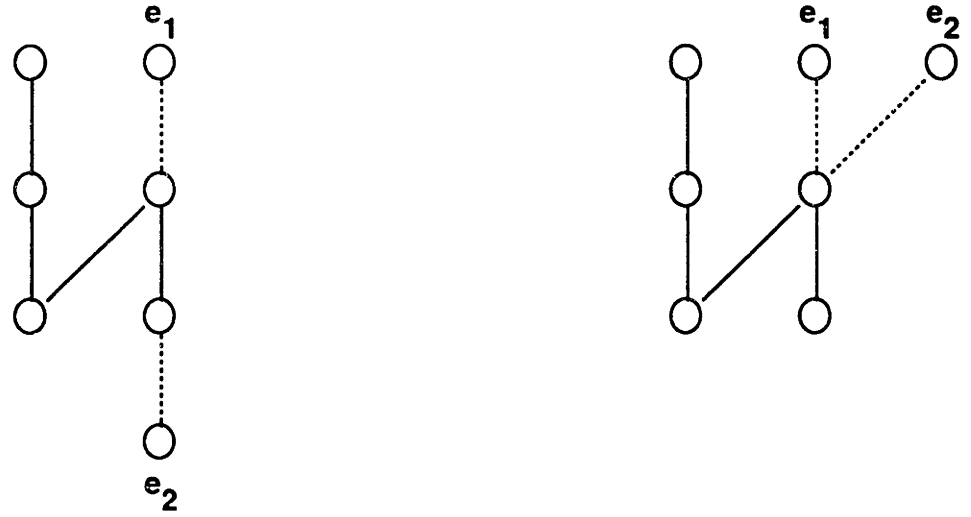


Figure A-5: Examples of extremal elements added to a five-element poset

$$y_B > y_A, \quad x_B < x_A \tag{A.2}$$

We examine each subcase in turn. In subcase A.1, we have that c_{y_B} does not increase upon adding either e_1 or e_2, but does increase when we add both of them. Hence, there exists a system w of y_B chains in $P\backslash\{e_1, e_2\}$ which covers $c_{y_B} - 1$ elements, and to which we may add the elements e_1 and e_2 to get a system of y_B chains covering $c_{y_B} + 1$ elements. Now, set $s = x_B - 2$, so s is 1 less than the length of the y_Bth row of λ_{00}. By Theorem A.2.4, we can construct a system of k chains w' satisfying $H_s(w') > H_s(w)$. Now, k cannot be $y_B - 1$, because even for a maximal system with $y_B - 1$ chains the value of H_s is at most $H_s(w)$. If $k = y_B$, then by Lemma A.2.5, w and w' have either the same set of maximal elements or the same set of minimal elements; but either way we can then enlarge the system w' by adding either e_1 or e_2, contradicting our assumption that c_{y_B} does not increase upon adding either e_1 or e_2. Similarly, we eliminate the case $k = y_B + 1$; even if $y_B + 1 = y_A$, the value of H_s on a maximal system of k chains does not exceed $H_s(w)$.

In subcase A.2, we have that a_{x_B} does not change by adding either e_1 or e_2 singly, but does while adding both together. Hence, there exists an antichain system $s = (s_1, \ldots, s_{x_B})$ which covers $a_{x_B} - 1$, and to which we may add the elements e_1 and e_2 to get a system of antichains which covers $a_{x_B} + 1$ elements. Let $A = (A_1, \ldots, A_{x_B})$ be a maximal system of x_B antichains for $P\backslash\{e_1, e_2\}$. Without loss of generality, we may assume that the antichains in s are ordered so that

$$p_i \in s_i, \quad p_j \in s_j, \quad p_i \leq p_j \Rightarrow i \leq j$$

and that the same is true for the system A. Now set

$$s_i \vee A_i = \{v \in s_i : \exists w \in A_i : v \geq w\} \bigcup \{w \in A_i : \exists v \in s_i : w \geq v\},$$

$$s_i \wedge A_i = \{v \in s_i : \exists w \in A_i : v \leq w\} \bigcup \{w \in A_i : \exists v \in s_i : w \leq v\}.$$

Note that any vertex in $P\backslash\{e_1, e_2\}$ is comparable with at least one element of s_i and at least one element of A_i. It is not difficult to show that both $(s_1 \vee A_1, \ldots, s_{x_B} \vee A_{x_B})$, and $(s_1 \wedge A_1, \ldots, s_{x_B} \wedge A_{x_B})$ are antichain systems, which we denote by J and M respectively. Further, J and M are disjoint, and their union contains all the elements in $s \bigcup A$. Hence, one of the constructed systems is maximal. But if J is maximal, then we can add whichever of e_1 or e_2 is maximal to J to get a larger system; if M is maximal, then adding the minimal e_i will enlarge it. This contradicts our initial assumption that a system of x_B antichains does not change when we add just one of the e_i's.

Case 2 is handled similarly to Case 1, subcase 1, where we set $s = x_A$. □

At last, we are in a position to show the main result.

Theorem A.3.3 Let σ be a generalized permutation of the cells of a skew diagram S. For each vertex of v of S, let $\mathcal{C}(v)$ denote the set of cells of S below and to the left of v (as we did in Example 2.3.7 and Example 2.4.4.). If we restrict out attention to those cells which are in π and take the poset corresponding to this generalized permutation, then we get a map

$$v \mapsto \lambda(\mathcal{C}(v) \cap \pi)$$

which is a two dimensional growth $G : S \to \mathbf{Y}$. We may also define a growth $g : S \to \mathbf{Y}$ by extending locally from the (trivial) lower boundary of S using the usual R-correspondence of Example 2.6.3. Then these two growths are identical on S.

Proof: Let the values of the growth G around a fixed cell C be given as in Figure A-3. We need the show that these elements satisfy the local conditions for growth, i.e., that the value of λ_{11} depends only on the values of the other three corners, and gives the same answer as our local construction. As we remarked above, the case $\lambda_{01} \neq \lambda_{10}$ is

trivial, since then the others corners are determined as the meet and join just because both g and G are growths. First we examine the case where $\lambda_{01} = \lambda_{10} \neq \lambda_{00}$. In this case, there must be a cell in σ both directly below C, which we call e_1, and one directly to the left of C, which we call e_2. e_1 and e_2 are both maximal elements of σ regarded as a poset (ordering up and to the right). Now we apply Theorem A.3.1, Case 2, to get that $x_B \leq x_A$, where A and B are, as before, the cells corresponding to adding one e_i, then the other.

Next, we take the dual order on the poset σ, i.e., we consider one cell to be bigger than another if it is below and to the right of that cell. By Theorem A.2.1, this corresponds to taking the transpose of all our diagrams. But now e_1 and e_2 are extremal elements of different types. Let $A' = (X'_A, Y'_A)$ denote the cell $\lambda'_{01}/\lambda'_{00}$ and $B' = (X'_B, Y'_B)$ denote the cell $\lambda'_{11}/\lambda'_{01}$. By Theorem A.3.1, Case 1, we get that $X'_B = X'_A$ or $X'_B = X'_A + 1$. But conjugating a partition simply interchanges the two coordinates, so this condition translates to $Y_B = Y_A$ or $Y_B = Y_A + 1$. Combining this with the condition $x_B \leq x_A$, we get that the growth G is determined locally according the the standard R-correspondence on $\mathbf{Y}$.

The only other case to consider is when $\lambda_{01} = \lambda_{10} = \lambda_{00}$. In this case, there can be no cells in σ either below C or to the left of C. If $C \in \sigma$, then clearly C may be taken as extending the longest chain of a maximal chain system for λ_{00}; hence, λ_{11} and λ_{00} differ by a cell in the top row. This agrees with the standard R-correspondence. If $C \notin \sigma$, then all the λ_{ij}'s are images of the same poset, so they are all the same, which agrees with the standard R-correspondence. This finishes the proof. □

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