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FOURIER TRANSFORM AS A TRIANGULAR MATRIX

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ABSTRACT. Let V be a finite dimensional vector space over the field with two elements with a given nondegenerate symplectic form. Let $[V]$ be the vector space of complex valued functions on V, and let $[V]_{\mathbf{Z}}$ be the subgroup of $[V]$ consisting of integer valued functions. We show that there exists a $\mathbf{Z}$-basis of $[V]_{\mathbf{Z}}$ consisting of characteristic functions of certain isotropic subspaces of V and such that the matrix of the Fourier transform from $[V]$ to $[V]$ with respect to this basis is triangular. We show that this is a special case of a result which holds for any two-sided cell in a Weyl group.

INTRODUCTION

0.1. Let V be a vector space of finite even dimension $D = 2d \geq 0$ over the field $\mathbf{F}_2$ with 2 elements with a fixed nondegenerate symplectic form $(,) : V \times V \to \mathbf{F}_2$. Let $[V]$ be the $\mathbf{C}$-vector space of functions $V \to \mathbf{C}$ and let $[V]_{\mathbf{Z}}$ be the subgroup of $[V]$ consisting of the functions $V \to \mathbf{Z}$. For $f \in [V]$ the Fourier transform $\Phi(f) \in [V]$ is defined by $\Phi(f)(x) = 2^{-d} \sum_{y \in V} (-1)^{(x,y)} f(y)$. Now $\Phi : [V] \to [V]$ is a linear involution whose trace is $2^{-d} \sum_{x \in V} 1 = 2^d$. Hence Φ has $2^{D-1} + 2^{d-1}$ eigenvalues equal to 1 and $2^{D-1} - 2^{d-1}$ eigenvalues equal to -1. Here is one of our main results.

Theorem 0.2. *There exists a $\mathbf{Z}$-basis β of $[V]_{\mathbf{Z}}$ consisting of characteristic functions of certain explicit isotropic subspaces of V such that the matrix of $\Phi : [V] \to [V]$ with respect to β is upper triangular (with diagonal entries ± 1) for a suitable order on β.*

Assume for example that $D = 2$. For $x \in V$ let $f_x \in [V]$ be the function whose value at $y \in V$ is 1 if $y = x$ and 0 if $y \neq x$. Let β be the $\mathbf{Z}$-basis of $V_{\mathbf{Z}}$ consisting of $f'_0 = f_0$ and of $f'_x = f_0 + f_x$ for $x \in V - \{0\}$. We have $\Phi(f'_0) = -f'_0 + (1/2) \sum_{x \in V - \{0\}} f'_x$ and $\Phi(f'_x) = f'_x$ for $x \in V - \{0\}$. Thus, the matrix of $\Phi : [V] \to [V]$ with respect to β is upper triangular (with diagonal entries $-1, 1, 1, 1$).

The proof of the theorem is given in §1; we take β to be the new basis $\mathcal{F}(V)$ of $[V]$ defined in [Lus20]. In §2 we compute explicitly the signs ± 1 appearing in the theorem for this β. In §3 we give some tables for $\beta = \mathcal{F}(V)$. In §4 we show that $\mathcal{F}(V)$ has a certain dihedral symmetry which was not apparent in [Lus20]. In §5 we show that the theorem is a special case of a result which applies to any two-sided cell in an irreducible Weyl group.

0.3. *Notation.* For a, b in $\mathbf{Z}$ we set $[a, b] = \{z \in \mathbf{Z}; a \leq z \leq b\}$. For a finite set Y let $|Y|$ be the cardinal of Y.

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1. Proof of Theorem 0.2

1.1. When $D \ge 2$ we fix a subset $\{e_i; i \in [1, D+1]\} \subset V$ such that for $i \ne j$ in $[1, D+1]$ we have $(e_i, e_j) = 1$ if $i - j = \pm 1 \mod D+1$, $(e_i, e_j) = 0$ if $i - j \ne \pm 1 \mod D+1$. (Such a subset exists and is unique up to the action of some isometry of $(,)$.) We say that this subset is a *circular basis* of V. We must have $e_1 + e_2 + \cdots + e_{D+1} = 0$ and any D elements of $\{e_i; i \in [1, D+1]\}$ form a basis of V. For any $I \subset [1, D+1]$ let $e_I = \sum_{i \in I} e_i \in V$. When $D \ge 2$ (resp. $D \ge 4$) we denote by V' (resp. V'') an $\mathbf{F}_2$-vector space with a nondegenerate symplectic form $(,)$. When $D \ge 4$ (resp. $D \ge 6$) we assume that V' (resp. V'') has a given circular basis $\{e'_i; i \in [1, D-1]\}$ (resp. $\{e''_i; i \in [1, D-3]\}$).

When $D \ge 2$, for any $i \in [1, D+1]$ there is a unique linear map $\tau_i : V' \to V$ such that $\tau_i = 0$ for $D = 2$, while for $D \ge 4$, the sequence $\tau_i(e'_1), \tau_i(e'_2), \dots, \tau_i(e'_{D-1})$ is:
$e_1, e_2, \dots, e_{i-2}, e_{i-1} + e_i + e_{i+1}, e_{i+2}, e_{i+3}, \dots, e_D, e_{D+1}$ (if $1 < i \le D$),
$e_3, e_4, \dots, e_D, e_{D+1} + e_1 + e_2$ if $i = 1$,
$e_2, e_3, \dots, e_{D-1}, e_D + e_{D+1} + e_1$ if $i = D+1$.

This map is injective and compatible with $(,)$. Similarly, when $D \ge 4$, for any $i \in [1, D-1]$ there is a unique linear map $\tau'_i : V'' \to V'$ such that $\tau'_i = 0$ for $D = 4$, while for $D \ge 6$, the sequence $\tau'_i(e''_1), \tau'_i(e''_2), \dots, \tau'_i(e''_{D-3})$ is:
$e'_1, e'_2, \dots, e'_{i-2}, e'_{i-1} + e'_i + e'_{i+1}, e'_{i+2}, e'_{i+3}, \dots, e'_{D-2}, e'_{D-1}$ (if $1 < i \le D-2$),
$e'_3, e'_4, \dots, e'_{D-2}, e'_{D-1} + e'_1 + e'_2$ if $i = 1$,
$e'_2, e'_3, \dots, e'_{D-3}, e'_{D-2} + e'_{D-1} + e'_1$ if $i = D-1$.

This map is injective and compatible with $(,)$. Note that

(a) if $D \ge 2$, then $\tau_i(V')$ is a complement of the line $\mathbf{F}_2 e_i$ in $\{x \in V; (x, e_i) = 0\}$.

Assuming that $D \ge 4$ and $i \in [1, D-2]$, we show:

(b) $\tau_{D+1}\tau'_i = \tau_j \tau'_{D-1}$ where $j = i+1$ if $1 < i \le D-2$ and $j = i$ if $i = 1$.

If $D = 4$ the result is trivial. Assume now that $D \ge 6$. Assume first that $1 < i \le D-2$. Both sequences

$$(\tau_{D+1}\tau'_i(e''_1), \tau_{D+1}\tau'_i(e''_2), \dots, \tau_{D+1}\tau'_i(e''_{D-3}))$$
$$(\tau_{i+1}\tau'_{D-1}(e''_1), \tau_{i+1}\tau'_{D-1}(e''_2), \dots, \tau_{i+1}\tau'_{D-1}(e''_{D-3}))$$

are equal to

$$(e_2, e_3, \dots, e_{i-1}, e_i + e_{i+1} + e_{i+2}, e_{i+3}, e_{i+4}, \dots, e_{D-1}, e_D + e_{D+1} + e_1)$$

if $1 < i < D-2$ and to

$$(e_2, e_3, \dots, e_{D-3}, e_{D-2} + e_{D-1} + e_D + e_{D+1} + e_1)$$

if $i = D-2$. Next we assume that $i = 1$. Both sequences

$$(\tau_{D+1}\tau'_i(e''_1), \tau_{D+1}\tau'_i(e''_2), \dots, \tau_{D+1}\tau'_i(e''_{D-3}))$$
$$(\tau_i\tau'_{D-1}(e''_1), \tau_i\tau'_{D-1}(e''_2), \dots, \tau_i\tau'_{D-1}(e''_{D-3}))$$

are equal to

$$(e_4, e_5, \dots, e_{D-2}, e_D + e_{D+1} + e_1 + e_2 + e_3).$$

This proves (b).

In the setup of (b) we show that for a subspace $E'' \subset V''$ we have

(c) $$\tau_{D+1}(\tau'_i(E'') \oplus \mathbf{F}_2 e'_i) \oplus \mathbf{F}_2 e_{D+1} = \tau_j(\tau'_{D-1}(E'') \oplus \mathbf{F}_2 e'_{D-1}) \oplus \mathbf{F}_2 e_j.$$

Using (b) it is enough to show that

$$\mathbf{F}_2 \tau_{D+1}(e'_i) \oplus \mathbf{F}_2 e_{D+1} = \mathbf{F}_2 \tau_j(e'_{D-1}) \oplus \mathbf{F}_2 e_j$$

or that

$$\mathbf{F}_2 e_{i+1} \oplus \mathbf{F}_2 e_{D+1} = \mathbf{F}_2 e_{D+1} \oplus \mathbf{F}_2 e_{i+1}$$

if $i > 1$ and

$$\mathbf{F}_2 e_2 \oplus \mathbf{F}_2 e_{D+1} = \mathbf{F}_2(e_{D+1} + e_1 + e_2) + \mathbf{F}_2 e_1$$

if $i = 1$. This is clear.

1.2. If $D \geq 2$, for any $k \in [0, d]$ let E_k be the subspace of V with basis
$\{e_{[1,D]}, e_{[2,D-1]}, \dots, e_{[k,D+1-k]}\}$.
When $D = 0$ we set $E_0 = 0 \subset V$. If $D \geq 4$ and $k \in [0, d-1]$ let E'_k be the subspace of V' with basis
$\{e'_{[1,D-2]}, e'_{[2,D-3]}, \dots, e'_{[k,D-1-k]}\}$
where for any $I' \subset [1, D-1]$ we set $e'_{I'} = \sum_{i \in I'} e'_i \in V'$. When $D = 2$ we set $E' = 0 \subset V'$.

Following [Lus20], we define a collection $\mathcal{F}(V)$ of subspaces of V by induction on D. If $D = 0$, $\mathcal{F}(V)$ consists of the subspace $\{0\}$. If $D \geq 2$, a subspace E of V is in $\mathcal{F}(V)$ if either

(i) there exists $i \in [1, D]$ and $E' \in \mathcal{F}(V')$ such that $E = \tau_i(E') \oplus \mathbf{F}_2 e_i$, or

(ii) there exists $k \in [0, d]$ such that $E = E_k$.

We now define a collection $\mathcal{F}'(V)$ of subspaces of V by induction on D. If $D = 0$, $\mathcal{F}'(V)$ consists of the subspace $\{0\}$. If $D \geq 2$, a subspace E of V is in $\mathcal{F}'(V)$ if either $E = 0$ or if

(iii) there exists $i \in [1, D+1]$ and $E' \in \mathcal{F}(V')$ such that $E = \tau_i(E') \oplus \mathbf{F}_2 e_i$.

Lemma 1.3. *We have $\mathcal{F}(V) = \mathcal{F}'(V)$.*

We argue by induction on D. If $D = 0$ the result is obvious. Assume that $D \geq 2$. We show that

(a) $\mathcal{F}'(V) \subset \mathcal{F}(V)$.

Let $E \in \mathcal{F}'(V)$. If $E = 0$ then clearly $E \in \mathcal{F}(V)$. Thus we can assume that $E = \tau_i(E') \oplus \mathbf{F}_2 e_i$ for some $i \in [1, D+1]$ and some $E' \in \mathcal{F}'(V)$. By the induction hypothesis we have $E' \in \mathcal{F}(V)$. If $i \in [1, D]$ then by definition we have $E \in \mathcal{F}(V)$. Thus we can assume that $i = D+1$. If $E' = 0$ then $E = \mathbf{F}_2 e_{D+1} = \mathbf{F}_2 e_{[1,D]} = E_1 \in \mathcal{F}(V)$. Thus we can assume that $E' \neq 0$ so that $D \geq 4$. Since $E' \in \mathcal{F}(V')$ we have $E' = \tau'_h(E'') \oplus \mathbf{F}_2 e'_h$ for some $h \in [1, D-2]$ and some $E'' \in \mathcal{F}(V'')$. Thus we have

$$E = \tau_{D+1}(\tau'_h(E'') \oplus \mathbf{F}_2 e'_h) \oplus \mathbf{F}_2 e_{D+1} = \tau_{h'}(E_1) \oplus \mathbf{F}_2 e_{h'}$$

where $E_1 = \tau'_{D-1}(E'') \oplus \mathbf{F}_2 e_{D-1}$ (we have used 1.1(c)); here $h' = h+1$ if $h > 1$ and $h' = h$ if $h = 1$. By the definition of $\mathcal{F}'(V')$ we have $E_1 \in \mathcal{F}'(V')$ hence $E_1 \in \mathcal{F}(V')$, by the induction hypothesis. It follows that $\tau_{h'}(E_1) \oplus \mathbf{F}_2 e_{h'} \in \mathcal{F}(V)$, so that $E \in \mathcal{F}(V)$. This proves (a).

We show that

(b) $\mathcal{F}(V) \subset \mathcal{F}'(V)$.

Let $E \in \mathcal{F}(V)$. Assume first that $E = E_k$ for some $k \in [1, d]$. From the definition we have $E_k = \tau_{D+1}(E'_{k-1}) \oplus \mathbf{F}_2 e_{D+1}$. We have $E'_{k-1} \in \mathcal{F}(V')$ hence by the induction hypothesis we have $E'_{k-1} \in \mathcal{F}'(V')$ and using the definition we have $E_k \in \mathcal{F}'(V)$. If $E = E_0$ then $E = 0$ so that again $E \in \mathcal{F}(V)$. Next we assume that E is not of the form E_k with $k \in [0, d]$. We can find $i \in [1, D]$ and $E' \in \mathcal{F}(V')$ such that $E = \tau_i(E') \oplus \mathbf{F}_2 e_i$. By the induction hypothesis we have $E' \in \mathcal{F}'(V')$. From the definition we have $E \in \mathcal{F}'(V)$. This proves (b).

1.4. For any subset $X \subset V$ let $\psi_X \in [V]$ be the function such that $\psi_X(x) = 1$ if $x \in X$, $\psi_X(x) = 0$ if $x \in V - X$. According to [Lus20],

(a) $\{\psi_E; E \in \mathcal{F}(V)\}$ *is a* $\mathbf{Z}$*-basis of* $[V]_{\mathbf{Z}}$.

Using Lemma 1.3, we deduce:

(b) $\{\psi_E; E \in \mathcal{F}'(V)\}$ *is a* $\mathbf{Z}$*-basis of* $[V]_{\mathbf{Z}}$.

We will no longer distinguish between $\mathcal{F}(V)$ and $\mathcal{F}'(V)$.

1.5. Assume that $D \geq 2$. Let $[V']$, $\Phi' : [V'] \to [V']$ be the analogues of $[V], \Phi : [V] \to [V]$ when V is replaced by V'. For $X' \subset V'$ let $\psi'_{X'} \in [V']$ be the function such that $\psi'_{X'}(y) = 1$ if $y \in X'$, $\psi'_{X'}(x) = 0$ if $y \in V' - X'$.

For $i \in [1, D+1]$ there is a unique linear map $z_i : [V'] \to [V]$ such that $z_i(\psi'_y) = \psi_{\tau_i(y)} + \psi_{\tau_i(y)+e_i}$ for all $y \in V'$. If E' is a subspace of V' we have $z_i(\psi'_{E'}) = \psi_{\tau_i(E') \oplus \mathbf{F}_2 e_i}$. We show:

(a) For $f \in [V']$ we have $\Phi(z_i(f)) = z_i(\Phi'(f))$.

We can assume that $f = \psi'_y$ with $y \in V'$. We have

$$z_i(\Phi'(f)) = 2^{-d+1} \sum_{y_1 \in V'} (-1)^{(y,y_1)} z_i(\psi'_{y_1})$$
$$= 2^{-d+1} \sum_{y_1 \in V'} (-1)^{(y,y_1)} (\psi_{\tau_i(y_1)} + \psi_{\tau_i(y_1)+e_i}),$$

$$\Phi(z_i(f)) = \Phi(\psi_{\tau_i(y)} + \psi_{\tau_i(y)+e_i}) = 2^{-d} \sum_{x \in V} ((-1)^{(\tau_i(y),x)} + (-1)^{(\tau_i(y)+e_i,x)})\psi_x$$
$$= 2^{-d+1} \sum_{x \in V; (e_i,x)=0} (-1)^{(\tau_i(y),x)} \psi_x.$$

In the last sum x can be written uniquely as $x = \tau_i(y_1) + ce_i$ with $y_1 \in V', c \in \mathbf{F}_2$. Thus

$$\Phi(z_i(f)) = 2^{-d+1} \sum_{y_1 \in V', c \in \mathbf{F}_2} (-1)^{(\tau_i(y), \tau_i(y_1)+ce_1)} \psi_{\tau_i(y_1)+ce_i}$$

which is equal to $z_i(\Phi'(f))$. This proves (a).

For $E \in \mathcal{F}(V)$ we write

(b) $\Phi(\psi_E) = \sum_{E_1 \in \mathcal{F}(V)} c_{E,E_1} \psi_{E_1}$

with $c_{E,E_1} \in \mathbf{C}$ are uniquely determined. (We use 1.4(b).)

Lemma 1.6. *Let* $E \in \mathcal{F}(V), E_1 \in \mathcal{F}(V)$ *be such that* $c_{E,E_1} \neq 0$. *Then either* $E_1 = E$ *or* $|E_1| > |E|$.

We argue by induction on D. If $D = 0$ the result is obvious. Assume now that $D \geq 2$. If $E = 0$, the result is obvious since for any $E_1 \in \mathcal{F}(V)$ we have either $E_1 = E$ or $|E_1| > |E|$. Assume now that $E \neq 0$. We can find $i \in [1, D+1]$ and $E' \in \mathcal{F}(V')$ such that $E = \tau_i(E') \oplus \mathbf{F}_2 e_i$. Recall from 1.5 that $z_i(\psi'_{E'}) = \psi_{\tau_i(E') \oplus \mathbf{F}_2 e_i} = \psi_E$. By the induction hypothesis we have

$$\Phi'(\psi'_{E'}) = c'_{E',E'} \psi'_{E'} + \sum_{E'_1 \in \mathcal{F}(V'); |E'_1| > |E'|} c'_{E',E'_1} \psi'_{E'_1}$$

with $c'_{E',E'} \in \mathbf{C}$, $c'_{E',E'_1} \in \mathbf{C}$. Applying z_i and using 1.5(a) we deduce

$$\Phi(z_i(\psi'_{E'})) = c_{E',E'} z_i(\psi'_{E'}) + \sum_{E'_1 \in \mathcal{F}(V');|E'_1|>|E'|} c'_{E',E'_1} z_i(\psi'_{E'_1})$$

$$= c_{E',E'} \psi_E \sum_{E'_1 \in \mathcal{F}(V');|E'_1|>|E'|} c'_{E',E'_1} \psi_{\tau_i(E'_1)\oplus \mathbf{F}_2 e_i}$$

and the result follows in this case since for E'_1 in the last sum we have

$$|\tau_i(E'_1) \oplus \mathbf{F}_2 e_i| = |E'_1| + 1 > |E'| + 1 = |E|.$$

This completes the proof of the lemma.

1.7. We prove Theorem 0.2. By results of [Lus20], the basis 1.4(b) of $[V]$ is a $\mathbf{Z}$-basis of $[V]_{\mathbf{Z}}$. By 1.6, the matrix of Φ with respect to the basis 1.4(b) is upper triangular for a suitable order on the basis. The diagonal entries of this matrix are necessarily ± 1 since $\Phi^2 = 1$. This completes the proof.

2. Sign computation

2.1. Let $E \in \mathcal{F}(V)$. According to [Lus20] there is a unique basis b_E of E which consists of vectors of the form e_I with I of the form $[a,b]$ with $a \le b$ in $[1,D]$. Let n_E be the number of vectors $e_I \in b_E$ such that $|I|$ is even.

For k, k' in $[0,d]$ let $\mathcal{F}_k(V)$ (resp. $\mathcal{F}^{k'}(V)$) be the set of all $E \in \mathcal{F}(V)$ such that $\dim(E) = k$ (resp. $n_E = k'$); let $\mathcal{F}_k^{k'}(V) = \mathcal{F}_k(V) \cap \mathcal{F}^{k'}(V)$.

If $E \in \mathcal{F}(V)$ we denote by $E^!$ the subspace of E spanned by the vectors $e_I \in b_E$ such that $|I|$ is odd; we have $E^! \in \mathcal{F}^0(V)$. We have the following result.

(a) *Let $\mathfrak{E} \in \mathcal{F}^0_{d-k}(V)$ where $k \in [0,d]$ and let $\mathcal{M}(\mathfrak{E}) = \{E \in \mathcal{F}(V); E^! = \mathfrak{E}\}$. Then $\mathcal{M}(\mathfrak{E})$ consists of $k+1$ subspaces $\mathfrak{E} = \mathfrak{E}(0) \subset \mathfrak{E}(1) \subset \ldots \subset \mathfrak{E}(k)$; we have $\mathfrak{E}(t) \in \mathcal{F}^t_{d-k+t}(V)$ for $t \in [0,k]$.* We argue by induction on D. If $D = 0$ the result is obvious. Assume now that $D \ge 2$. If $\mathfrak{E} = 0$ then $k = d$ and $\mathcal{M}(\mathfrak{E}) = \{E_0, E_1, \ldots, E_d\}$ (see 1.2) and the result is obvious. Assume now that $\mathfrak{E} \ne 0$. We can find $i \in [1,D]$ and $\mathfrak{E}' \in \mathcal{F}(V')$ such that $\mathfrak{E} = \tau_i(\mathfrak{E}') \oplus \mathbf{F}_2 e_i$. We have $\mathfrak{E}' \in \mathcal{F}^0_{d-1-k}$ so that by the induction hypothesis $\mathcal{M}(\mathfrak{E}')$ consists of $k+1$ subspaces $\mathfrak{E}' = \mathfrak{E}'(0) \subset \mathfrak{E}'(1) \subset \ldots \subset \mathfrak{E}'(k)$ and we have $\mathfrak{E}'(t) \in \mathcal{F}^t_{d-1-k+t}(V')$ for $t \in [0,k]$. For $t \in [0,k]$ we set $\mathfrak{E}(t) = \tau_i(\mathfrak{E}'(t)) \oplus \mathbf{F}_2 e_i$; we have $\mathfrak{E} = \mathfrak{E}(0) \subset \mathfrak{E}(1) \subset \ldots \subset \mathfrak{E}(k)$ and $\mathfrak{E}(t) \in \mathcal{F}^t_{d-k+t}(V')$, $\mathfrak{E}(t)^! = \mathfrak{E}$. Thus $\{\mathfrak{E}(0), \mathfrak{E}(1), \ldots, \mathfrak{E}(k)\} \subset \mathcal{M}(\mathfrak{E})$. Now let $E \in \mathcal{M}(\mathfrak{E})$. Since $e_i \in \mathfrak{E}$ we have $e_i \in E$ and, using [Lus20, 1.3(f)], we see that there exists $E' \in \mathcal{F}(V')$ such that $E = \tau_i(E') \oplus \mathbf{F}_2 e_i$. From the definitions we have $E' \in \mathcal{M}(\mathfrak{E}')$ so that $E' = \mathfrak{E}'(t)$ for some $t \in [0,k]$ and $E = \mathfrak{E}(t)$ for some $t \in [0,k]$. This proves (a).

From (a) we see that there is a unique involution $\kappa : \mathcal{F}(V) \to \mathcal{F}(V)$ such that for any $\mathfrak{E} \in \mathcal{F}^0_{d-k}(V)$ we have $\kappa(\mathfrak{E}(t)) = \mathfrak{E}(k-t)$ for $t \in [0,k]$. This involution restricts to a bijection

(b) $$\mathcal{F}^t(V) \xrightarrow{\sim} \mathcal{F}_{d-t}(V)$$

for any $t \in [0,d]$.

The following equality follows from [Lus20, 1.27(a)]:

(c) $|\mathcal{F}^k(V)| = \binom{D+1}{d-k}$ *for* $k \in [0,d]$.

Using (b),(c) we deduce

(d) $|\mathcal{F}_k(V)| = \binom{D+1}{k}$ *for* $k \in [0,d]$.

2.2. For any integer N we set $\delta(N)=(-1)^{N(N+1)/2}$. We have the following identity:

(a) $\sum_{k\in[0,d]}\delta(d-k)\binom{D+1}{k}=2^d$.

We prove (a) by induction on D. If $D=0$ the result is obvious. Assume now that $D\ge 2$. We must show that

$$\binom{2d+1}{d}-\binom{2d+1}{d-1}-\binom{2d+1}{d-2}+\binom{2d+1}{d-3}+\binom{2d+1}{d-4}-\binom{2d+1}{d-5}-\dots=2^d$$

or that

$$\begin{aligned}&(\binom{2d}{d}+\binom{2d}{d-1})-(\binom{2d}{d-1}+\binom{2d}{d-2})-(\binom{2d}{d-2}+\binom{2d}{d-3})\\&+(\binom{2d}{d-3}+\binom{2d}{d-4})+(\binom{2d}{d-4}+\binom{2d}{d-5})-(\binom{2d}{d-5}+\binom{2d}{d-6})-\dots\\&=2^d\end{aligned}$$

or that

$$\binom{2d}{d}-2\binom{2d}{d-2}+2\binom{2d}{d-4}-2\binom{2d}{d-6}+\dots=2^d$$

or that

$$\begin{aligned}&(\binom{2d-1}{d}+\binom{2d-1}{d-1})-2(\binom{2d-1}{d-2}+\binom{2d-1}{d-3})\\&+2(\binom{2d-1}{d-4}+\binom{2d-1}{d-5})-\dots=2^d\end{aligned}$$

or that

$$2\binom{2d-1}{d-1}-2\binom{2d-1}{d-2}-2\binom{2d-1}{d-3}+2\binom{2d-1}{d-4}+2\binom{2d-1}{d-5}-\dots=2^d.$$

But this is known from the induction hypothesis. This proves (a).

2.3. The following result describes the diagonal entries of the upper triangular matrix in 1.7.

Proposition 2.4. *Let $E\in\mathcal{F}(V)$ and let $c_{E,E}$ be as in 1.5(b). We have $c_{E,E}=\delta(d-\dim E)$.*

We argue by induction on D. If $D=0$ the result is obvious. Assume now that $D\ge 2$. Assume first that $E\ne 0$. We can find $i\in[1,D+1]$ and $E'\in\mathcal{F}(V')$ such that $E=\tau_i(E')\oplus\mathbf{F}_2e_i$. By the proof of 1.6 we have $c_{E,E}=c'_{E',E'}$ (notation of 1.6). The proposition applies to $c'_{E',E'}$ by the induction hypothesis. The desired result for E follows since $d-\dim E=d-1-\dim E'$. We now assume that $E=0$. The trace of Φ is equal to $\sum_{E_1\in\mathcal{F}(V)}c_{E_1,E_1}$ and on the other hand is equal to 2^d (see 0.1). Thus we have $\sum_{E_1\in\mathcal{F}(V)}c_{E_1,E_1}=2^d$. In the last sum all terms with $E_1\ne 0$ are already known. Hence the term with $E_1=0$ is determined by the last equality. Thus to prove the proposition it is enough to verify the identity

$$\sum_{E_1\in\mathcal{F}(V)}\delta(d-\dim E_1)=2^d$$

or equivalently

$$\sum_{k\in[0,d]}|\mathcal{F}_k(V)|\delta(d-k)=2^d.$$

This follows from 2.1(d), 2.2(a). This completes the proof.

3. Tables

3.1. In this section we assume that $D \geq 2$. Let $E \in \mathcal{F}(V)$. Recall that the basis b_E consists of certain vectors e_I where I is of the form $[a, b]$ with $a \leq b$ in $[1, D]$. We have $e_I = e_{I'}$ where $I' \subset [1, D+1]$ is defined by $I' = I$ if $|I|$ is odd and $I' = [1, D+1] - I$ if I is even. Note that $|I'|$ is always odd. Now E is completely described by the list of all subsets I' defined as above. In the following three sections we describe each $E \in \mathcal{F}(V)$ as a list of such I' assuming that D is $2, 4$ or 6. (This list is more symmetric than the corresponding list of the I which is given in [Lus20].) In each of these tables each horizontal line represents the various $\mathfrak{E}(0), \mathfrak{E}(1), \dots, \mathfrak{E}(k)$ with a fixed $\mathfrak{E} \in \mathcal{F}^0(V)$ as in 2.1. For example the second line $< 1 >, < 1, 512 >$ in 3.3 represents two subspaces in $\mathcal{F}(V)$; one spanned by e_1 and the other spanned by e_1 and $e_5 + e_1 + e_2$.

3.2. **The table for $D = 2$.**

$\emptyset, < 3 >$

$< 1 >$

$< 2 >$.

3.3. **The table for $D = 4$.**

$\emptyset, < 5 >, < 5, 451 >$

$< 1 >, < 1, 512 >$

$< 2 >, < 2, 5 >$

$< 3 >, < 3, 5 >$

$< 4 >, < 4, 345 >$

$< 1, 3 >$

$< 1, 4 >$

$< 2, 4 >$

$< 2, 123 >$

$< 3, 234 >$.

3.4. **The table for $D = 6$.**

$\emptyset, < 7 >, < 7, 671 >, < 7, 671, 56712 >$

$< 1 >, < 1, 712 >, < 1, 712, 67123 >$

$< 2 >, < 2, 7 >, < 2, 7, 67123 >$

$< 3 >, < 3, 7 >, < 3, 7, 671 >$

$< 4 >, < 4, 7 >, < 4, 7, 671 >$

$< 5 >, < 5, 7 >, < 5, 7, 45671 >$

$< 6 >, < 6, 567 >, < 6, 567, 45671 >$

$< 1, 3 >, < 1, 3, 71234 >$

$< 1, 4 >, < 1, 4, 712 >$

$< 1, 5 >, < 1, 5, 712 >$

$< 1, 6 >, < 1, 6, 56712 >$

$< 2, 4 >, < 2, 4, 7 >$

$< 2, 5 >, < 2, 5, 7 >$

$< 2, 6 >, < 2, 6, 567 >$

$< 3, 5 >, < 3, 5, 7 >$

$< 3, 6 >, < 3, 6, 567 >$

$< 4, 6 >, < 4, 6, 34567 >$

$< 2, 123 >, < 2, 123, 71234 >$

$<3,234>, <3,7,234>$
$<4,345>, <4,7,345>$
$<5,456>, <5,456,34567>$
$<1,3,5>$
$<1,3,6>$
$<1,4,6>$
$<2,4,6>$
$<1,4,345>$
$<1,5,456>$
$<2,5,123>$
$<2,5,456>$
$<2,6,123>$
$<3,6,234>$
$<2,4,12345>$
$<3,5,23456>$
$<3,234,12345>$
$<4,345,23456>$.

4. Dihedral symmetry

4.1. There is a unique linear map $R : V \to V$ such that if $D = 0$ we have $R = 0$ while if $D \geq 2$, $R(e_1), R(e_2), \dots, R(e_{D+1})$ is $e_2, e_3, \dots, e_{D+1}, e_1$. If $D \geq 2$, there is a unique linear map $R' : V' \to V'$ such that if $D = 2$ we have $R' = 0$ while if $D \geq 4$, $R'(e'_1), R'(e'_2), \dots, R'(e'_{D-1})$ is $e'_2, e'_3, \dots, e'_{D-1}, e'_1$. From the definitions we see that if $D \geq 2$, $i \in [1, D+1]$ we have

(a) $R\tau_i = \tau_{i+1}R' : V' \to V$ if $i \in [1, D]$, $R\tau_i = \tau_1 : V' \to V$ if $i = D+1$.

4.2. Let $E \in \mathcal{F}(V)$. We show:

(a) $R(E) \in \mathcal{F}(V)$.

We argue by induction on D. If $D = 0$ the result is obvious. Assume that $D \geq 2$. If $E = 0$ we have $R(E) = 0$ and the result is clear. Assume now that $E \neq 0$. We can find $i \in [1, D+1]$ and $E' \in \mathcal{F}(V')$ such that $E = \tau_i(E') \oplus \mathbf{F}_2 e_i$. Applying R we deduce $R(E) = R\tau_i(E') \oplus \mathbf{F}_2 e_{i+1}$ if $i \in [1, D]$, $R(E) = R\tau_i(E') \oplus \mathbf{F}_2 e_1$ if $i = D+1$. Using 4.1(a) we deduce $R(E) = \tau_{i+1}R'(E') \oplus \mathbf{F}_2 e_{i+1}$ if $i \in [1, D]$, $R(E) = \tau_1(E') \oplus \mathbf{F}_2 e_1$ if $i = D+1$. By the induction hypothesis we have $R'(E') \in \mathcal{F}(V')$. It follows that $R(E) \in \mathcal{F}(V)$, as required.

4.3. There is a unique linear map $S : V \to V$ such that if $D = 0$ we have $S = 0$, while if $D \geq 2$ we have $S(e_i) = e_{D+1-i}$ if $i \in [1, D]$, $S(e_{D+1}) = e_{D+1}$. If $D \geq 2$, there is a unique linear map $S' : V' \to V'$ such that if $D = 2$ we have $S' = 0$ while if $D \geq 4$ we have $S'(e_i) = e_{D-1-i}$ if $i \in [1, D-2]$, $S'(e_{D-1}) = e_{D-1}$. From the definitions we see that if $D \geq 2$, $i \in [1, D+1]$ we have

(a) $S\tau_i = \tau_{D+1-i}S' : V' \to V$ if $i \in [1, D]$, $S\tau_i = \tau_i S' : V' \to V$ if $i = D+1$.

4.4. Let $E \in \mathcal{F}(V)$. We show:

(a) $S(E) \in \mathcal{F}(V)$.

We argue by induction on D. If $D = 0$ the result is obvious. Assume that $D \geq 2$. If $E = 0$ we have $S(E) = 0$ and the result is clear. Assume now that $E \neq 0$. We can find $i \in [1, D+1]$ and $E' \in \mathcal{F}(V')$ such that $E = \tau_i(E') \oplus \mathbf{F}_2 e_i$. Applying S we deduce $S(E) = S\tau_i(E') \oplus \mathbf{F}_2 e_{D+1-i}$ if $i \in [1, D]$, $S(E) = S\tau_i(E') \oplus \mathbf{F}_2 e_i$ if $i = D+1$. Using 4.3(a) we deduce $S(E) = \tau_{D+1-i}S'(E') \oplus \mathbf{F}_2 e_{D+1-i}$ if $i \in [1, D]$,

$S(E) = \tau_i S'(E') \oplus \mathbf{F}_2 e_i$ if $i = D+1$. By the induction hypothesis we have $S'(E') \in \mathcal{F}(V')$. It follows that $S(E) \in \mathcal{F}(V)$, as required.

4.5. Assume that $D \geq 2$. Let $Sp(V)$ be the group of automorphisms of $V, (,)$. Let Δ be the subgroup of $Sp(V)$ generated by R, S (a dihedral group of order $2(D+1)$). From 4.2(a), 4.4(a) we see that the Δ-action on V induces a Δ-action on $[V]$ which keeps stable the basis $\mathcal{F}(V)$.

4.6. We now restate the definition of $\mathcal{F}(V)$ in 3.2 in more invariant terms. (In this definition the dihedral symmetry in 4.5 is obvious.)

When $D \geq 2$, we consider a connected graph with $D+1$ vertices and $D+1$ edges such that any vertex touches exactly two edges (this is a graph of affine type A_D). Let Γ be the set of vertices and let Λ be the set of edges. We assume that we are given an imbedding $\Gamma \subset V$ such that for $\gamma_1 \neq \gamma_2$ in Γ we have $(\gamma_1, \gamma_2) = 1$ if γ_1, γ_2 are joined by an edge and $(\gamma_1, \gamma_2) = 0$ if γ_1, γ_2 are not joined by an edge. We then say that (Γ, Λ) is an un-numbered circular basis (or u.c.b.) of V. Note that a u.c.b. exists; in particular the circular basis $\{e_i; i \in [1, D+1]\}$ in 1.1 can be viewed as a u.c.b. in which $\Gamma = \{e_i; i \in [1, D+1]\}$ and e_i, e_j are joined whenever $i - j = \pm 1$ mod $D+1$.

When $D \geq 4$ we assume that V' in 1.1 has a given u.c.b. with set of vertices Γ' and set of edges Λ'. When $D \geq 4$ for any $\gamma' \in \Gamma', \gamma \in \Gamma$ there is a unique linear map $\tilde{\tau} = \tilde{\tau}_{\gamma', \gamma} : V' \to V$ compatible with the symplectic forms and such that, setting $[\gamma] = \{\gamma\} \sqcup \{\gamma_1 \in \Gamma; (\gamma_1, \gamma) = 1\} \subset \Gamma$, we have $\tilde{\tau}(\gamma') = \sum_{\tilde{\gamma} \in [\gamma]} \tilde{\gamma}$ and $\tilde{\tau}$ restricts to a bijection $\Gamma' - \{\gamma'\} \xrightarrow{\sim} \Gamma - [\gamma]$. This map is injective.

We now define a collection $\mathcal{F}''(V)$ of subspaces of V by induction on D. If $D = 0$, $\mathcal{F}''(V)$ consists of the subspace $\{0\}$. If $D = 2$, $\mathcal{F}''(V)$ consists of the subspaces of V of dimension 0 or 1. If $D \geq 4$, a subspace E of V is in $\mathcal{F}''(V)$ if either $E = 0$ or if there exists $\gamma' \in \Gamma', \gamma \in \Gamma$ and $E' \in \mathcal{F}''(V')$ such that $E = \tilde{\tau}_{\gamma', \gamma}(E') \oplus \mathbf{F}_2 \gamma$. We show:

(a) *If $D \geq 2$ and the u.c.b. of V is numbered as in 1.1 so that $\mathcal{F}(V)$ is defined, we have $\mathcal{F}''(V) = \mathcal{F}(V)$.*

We argue by induction on D. If $D = 2$ the result is obvious. Assume now that $D \geq 4$. We can assume that the u.c.b. of V' is numbered as in 1.1. For $i \in [1, D+1]$ we have

$\tau_i = \tilde{\tau}_{i-1,i}$ if $2 \leq i \leq D$,
$\tau_i = \tilde{\tau}_{D-1,1}$ if $i = 1$,
$\tau_i = \tilde{\tau}_{D-1,D+1}$ if $i = D+1$.

Using this and the induction hypothesis we see that $\mathcal{F}(V) \subset \mathcal{F}''(V)$. If $i \in [1, D-1]$ and $j \in [1, D+1]$ then for some $s \geq 0$, $\tilde{\tau}_{i,j}$ is of the form $R^s \tilde{\tau}_{i,j'}$ where $\tilde{\tau}_{i,j'}$ is as in one of the three equalities above and R is as in 4.1. Using this, together with 4.2(a) and the induction hypothesis we see that $\mathcal{F}''(V) \subset \mathcal{F}(V)$. This proves (a).

5. Cells in Weyl groups

5.1. For any finite group Γ, let $M(\Gamma)$ be the set consisting of pairs (x, ρ) where $x \in \Gamma$ and ρ is an irreducible representation over $\mathbf{C}$ of the centralizer of x; these pairs are taken up to Γ-conjugacy; let $\mathbf{C}[M(\Gamma)]$ be the $\mathbf{C}$-vector space with basis $M(\Gamma)$ and let $A_\Gamma : \mathbf{C}[M(\Gamma)] \to \mathbf{C}[M(\Gamma)]$ be the "non-abelian Fourier transform" (as in [Lus79]). Let $\mathbf{Z}[M(\Gamma)]$ be the free abelian subgroup of $\mathbf{C}[M(\Gamma)]$ with basis $M(\Gamma)$.

5.2. In this section we fix an irreducible Weyl group W and a family c of irreducible representations of W (in the sense of [Lus79]). This is the same as fixing a two-sided cell of W. To c we associate a finite group $\mathcal{G}_c$ as in [Lus79], [Lus84]. Let $\widetilde{\mathbf{B}}_c$ be the "new basis" of $\mathbf{C}[M(\mathcal{G}_c)]$ defined in [Lus20]. (It is actually a $\mathbf{Z}$-basis of $\mathbf{Z}[M(\mathcal{G}_c)]$.) This basis is in canonical bijection with $M(\mathcal{G}_c)$, see [Lus20]. Let $\widehat{(x,\rho)}$ be the element of $\widetilde{\mathbf{B}}_c$ corresponding to $(x,\rho) \in M(\mathcal{G}_c)$. We write F for the non-abelian Fourier transform $A_{\mathcal{G}_c}$. We have the following result.

Theorem 5.3. *The matrix of the non-abelian Fourier transform $F : \mathbf{C}[M(\mathcal{G}_c)] \to \mathbf{C}[M(\mathcal{G}_c)]$ with respect to the new basis $\widetilde{\mathbf{B}}_c$ is upper triangular for a suitable order on $\widetilde{\mathbf{B}}_c$.*

From the theorem we see that there is a well defined function $\widetilde{\mathbf{B}}_c \to \{1,-1\}$ (called the *sign function*) whose value at $\widehat{(x,\rho)} \in \widetilde{\mathbf{B}}_c$ is the diagonal entry of the matrix of F at the place indexed by $\widehat{(x,\rho)}$. (We use that $F^2 = 1$.)

In the case where W is of classical type, the theorem follows from Theorem 0.2 and its proof. In the remainder of this section we assume that W is of exceptional type. In this case, $\mathcal{G}_c$ is a symmetric group S_n in n letters where $n \in [1,5]$. If n is 1 or 2 the result is immediate. The case where $n \in [3,5]$ is considered in 5.4-5.6. We shall use the notation of [Lus84] for the elements of $M(\mathcal{G}_c)$. Let θ, i, ζ be a fixed primitive root of 1 (in $\mathbf{C}$) of order $3,4,5$ respectively.

5.4. In this subsection we assume that $\mathcal{G}_c = S_3$. We partition the new basis $\widetilde{\mathbf{B}}_c$ in three pieces (1)-(3) as follows:

(1) $$\widehat{(1,1)}$$

(2) $$\widehat{(1,r)}$$

(3) $$\widehat{(1,\epsilon)}, \widehat{(g_2,1)}, \widehat{(g_2,\epsilon)}, \widehat{(g_3,1)}, \widehat{(g_3,\theta)}, \widehat{(g_3,\theta^2)}.$$

Then

(a) *F applied to an element in the n-th piece is $\pm$ that element plus a $\mathbf{Q}$-linear combination of elements in m-th pieces with $m > n$.*

We have

$$\begin{aligned}
&F(\widehat{(1,r)})\\
&= F((1,1)+(1,r)) = (1,1)/2 + (1,r) + (1,\epsilon)/2 + (g_2,1)/2 + (g_2,\epsilon)/2\\
&= ((g_2,\epsilon)/2 + (1,r)/2 + (1,1)/2) + ((1,\epsilon)/2 + (1,r) + (1,1)/2) + ((g_2,1)/2\\
&+ (1,r)/2 + (1,1)/2) - ((1,r) + (1,1))\\
&= \widehat{(g_2,\epsilon)}/2 + \widehat{(1,\epsilon)}/2 + \widehat{(g_2,1)}/2 - \widehat{(1,r)}.
\end{aligned}$$

The formula for $F(\widehat{(1,1)})$ is as follows. If W is of type G_2 then

$$
\begin{aligned}
&F(\widehat{(1,1)}) = F(1,1)\\
&= (1,1)/6 + (1,r)/3 + (1,\epsilon)/6 + (g_2,1)/2 + (g_2,\epsilon)/2 + (g_3,1)/3\\
&+ (g_3,\theta)/3 + (g_3,\theta^2)/3\\
&= ((g_3,\theta)/3 + (g_2,1)/3 + (1,1)/3) + ((g_3,\theta^2)/3 + (g_2,1)/3 + (1,1)/3)\\
&+ ((g_3,1)/3 + (g_2,1)/3 + (1,1)/3) + ((g_2,\epsilon)/2 + (1,r)/2 + (1,1)/2) + ((1,\epsilon)/6\\
&+ (1,r)/3 + (1,1)/6) - ((g_2,1)/2 + (1,r)/2 + (1,1)/2) - (1,1)\\
&= \widehat{(g_3,\theta)}/3 + \widehat{(g_3,\theta^2)}/3 + \widehat{(g_3,1)}/3 + \widehat{(g_2,\epsilon)}/2 + \widehat{(1,\epsilon)}/6 - \widehat{(g_2,1)}/2 - \widehat{(1,1)}.
\end{aligned}
$$

If W is of type E_6, E_7 or E_8 then

$$
\begin{aligned}
&F(\widehat{(1,1)}) = F(1,1)\\
&= (1,1)/6 + (1,r)/3 + (1,\epsilon)/6 + (g_2,1)/2 + (g_2,\epsilon)/2 + (g_3,1)/3\\
&+ (g_3,\theta)/3 + (g_3,\theta^2)/3\\
&= ((g_3,\theta)/3 + (g_2,\epsilon)/3 + (1,1)/3) + ((g_3,\theta^2)/3 + (g_2,\epsilon)/3 + (1,1)/3)\\
&+ ((g_3,1)/3 + (g_2,1)/3 + (1,1)/3) - ((g_2,\epsilon)/6 + (1,r)/6 + (1,1)/6) + ((1,\epsilon)/6\\
&+ (1,r)/3 + (1,1)/6) + ((g_2,1)/6 + (1,r)/6 + (1,1)/6) - (1,1)\\
&= \widehat{(g_3,\theta)}/3 + \widehat{(g_3,\theta^2)}/3 + \widehat{(g_3,1)}/3 - \widehat{(g_2,\epsilon)}/6 + \widehat{(1,\epsilon)}/6 + \widehat{(g_2,1)}/6 - \widehat{(1,1)}.
\end{aligned}
$$

We see that the matrix of F in the new basis is upper triangular. This proves 5.3 in our case.

(b) *The sign function on* $\widetilde{\mathbf{B}}_c$ *is constant on each piece; its value on the piece (1),(2),(3) is* $-1,-1,1$ *respectively.*

5.5. In this subsection we assume that $\mathcal{G}_c = S_4$ so that W is of type F_4. We partition the new basis $\widetilde{\mathbf{B}}_c$ in five pieces (1)-(5) as follows:

(1) $\widehat{(1,1)}$

(2) $\widehat{(1,\lambda^1)}$

(3) $\widehat{(1,\sigma)}$

(4) $\widehat{(1,\lambda^2)}, \widehat{(g_2,1)}, \widehat{(g_2',1)}, \widehat{(g_2,\epsilon'')}, \widehat{(g_2,\epsilon')}$

(5) $\widehat{(g_3,1)}, \widehat{(g_4,1)}\widehat{(g_2',\epsilon'')}, \widehat{(g_2',\epsilon')}, \widehat{(g_2',r)}, \widehat{(g_4,-1)}, \widehat{(1,\lambda^3)}, \widehat{(g_2,\epsilon)}, \widehat{(g_2',\epsilon)},$
$\widehat{(g_3,\theta)}, \widehat{(g_3,\theta^2)}, \widehat{(g_4,i)}, \widehat{(g_4,-i)}.$

Then

(a) *F applied to an element in the n-th piece is $\pm$ that element plus a $\mathbf{Q}$-linear combination of elements in m-th pieces with $m > n$.*

We see that the matrix of F in the new basis is upper triangular. This proves 5.3 in our case.

(b) *The sign function on* $\widetilde{\mathbf{B}}_c$ *is constant on each piece; its value on the piece (1),(2),(3),(4),(5) is* $1,-1,1,-1,1$ *respectively.*

5.6. In this subsection we assume that $\mathcal{G}_c = S_5$ so that W is of type E_8. We partition the new basis $\widetilde{\mathbf{B}}_c$ in eight pieces (1)-(8) as follows:

$$\widehat{(g_5,\zeta)} \tag{1}$$

$$\widehat{(1,1)} \tag{2}$$

$$\widehat{(1,\lambda^1)} \tag{3}$$

$$\widehat{(1,\nu)} \tag{4}$$

$$\widehat{(1,\nu')} \tag{5}$$

$$\widehat{(1,\lambda^2)}, \widehat{(g_2,1)}, \widehat{(g_2,-1)} \tag{6}$$

$$\widehat{(1,\lambda^3)}, \widehat{(g_2,r)}, \widehat{(g_3,1)}, \widehat{(g_2',1)}, \widehat{(g_2,-r)}, \widehat{(g_2',r)}, \widehat{(g_3,\theta)}, \widehat{(g_3,\theta^2)} \tag{7}$$

$$\begin{gathered}\widehat{(g_2',\epsilon'')}, \widehat{(g_6,1)}, \widehat{(g_2,\epsilon)}, \widehat{(g_3,\epsilon)}, \widehat{(g_4,1)}, \widehat{(g_5,1)}, \widehat{(g_2',\epsilon')}, \widehat{(g_4,-1)},\\ \widehat{(g_6,-1)}, \widehat{(g_6,\theta)}, \widehat{(g_6,\theta^2)}, \widehat{(1,\lambda^4)}, \widehat{(g_2,-\epsilon)}, \widehat{(g_3,\epsilon\theta)}, \widehat{(g_3,\epsilon\theta^2)}, \widehat{(g_2',\epsilon)},\\ \widehat{(g_6,-\theta)}, \widehat{(g_6,-\theta^2)}, \widehat{(g_4,i)}, \widehat{(g_4,-i)}, \widehat{(g_5,\zeta^2)}, \widehat{(g_5,\zeta^3)}, \widehat{(g_5,\zeta^4)}.\end{gathered} \tag{8}$$

Then

(a) *F applied to an element in the n-th piece is $\pm$ that element plus a $\mathbf{R}$-linear combination of elements in m-th pieces with $m > n$.*

(If $n \geq 2$ we can replace $\mathbf{R}$ by $\mathbf{Q}$ in (a). If $n = 1$ the coefficients in the linear combination can involve the golden ratio.) We see that the matrix of F in the new basis is upper triangular. This proves 5.3 in our case.

(b) *The sign function on $\widetilde{\mathbf{B}}_c$ is constant on each piece; its value on the piece (1),(2),(3),(4),(5),(6),(7),(8) is $-1, -1, 1, 1, 1, -1, -1, 1$ respectively.*

We now give some indication of how (a) can be verified. Let $\mathcal{H}$ be the hyperplane in $\mathbf{C}[M(S_5)]$ consisting of all sums $\sum_{(x,\rho)\in M(S_5)} a_{x,\rho}(x,\rho)$ where $a_{x,\rho} \in \mathbf{C}$ satisfy the equation

$$a_{g_5,\zeta} + a_{g_5,\zeta^4} = a_{g_5,\zeta^2} + a_{g_5,\zeta^3}.$$

One can check that $F(\mathcal{H}) = \mathcal{H}$. Moreover one can check that $\widehat{(x,\rho)} \in \mathcal{H}$ for any (x,ρ) in $M(S_5)$ other than $\widehat{(g_5,\zeta)}$. It follows that to verify (a) we can assume that $n \geq 2$. In that case the proof of (a) is similar to that of 1.6; the role of z_i in 1.6 is now played by the maps $\mathbf{s}_{H,H'}$ in [Lus20, 3.1]; the commutation of z_i with Fourier transform (see 1.5(a)) is replaced by the commutation of $\mathbf{s}_{H,H'}$ with the non-abelian Fourier transform (see [Lus20, 3.1(b),(e)]). A similar argument (except for the reduction to the case $n \geq 2$ which is not needed in this case) applies to the proof of 5.5(a). The proof of (b) is similar to that of 2.4; we use an induction hypothesis where S_5 is replaced by $S_4, S_3 \times S_2, S_3, S_2 \times S_2$ or S_2. Using the known equality $\text{tr}(F, \mathbf{C}[M_5]) = 13$, we see that the values of the sign function on the elements not covered by the induction hypothesis (that is those in the pieces (1),(2)) have sum equal to -2. It follows that both these values are -1. A similar argument applies to the proof of 5.5(b) (in this case the only element not covered by the induction hypothesis is that in piece (1)).

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