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THE PERVERSE FILTRATION FOR THE HITCHIN FIBRATION IS LOCALLY CONSTANT

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ABSTRACT. We prove that the perverse Leray filtration for the Hitchin morphism is locally constant in families, thus providing some evidence towards the validity of the $P = W$ conjecture due to de Cataldo, Hausel and Migliorini in non Abelian Hodge theory.

CONTENTS

1. INTRODUCTION

Let S be an algebraic variety, and let $f : X \to Y$ be an S-morphism of algebraic varieties. Let F be a constructible complex of rational vector spaces on X. For each $s \in S$, the vector spaces $H^\bullet(X_s, F_s)$ carry the perverse Leray filtrations P^{f_s} associated with the morphism $f_s : X_s \to Y_s$. It is natural to ask how these filtered vector spaces $(H^\bullet(X_s, F_s), P^{f_s})$ behave as s varies in S. The main goal of this paper is to provide criteria (Theorem 3.2.1) for checking local constancy of these filtrations.

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1.1. **Perverse filtration for the Hitchin morphism.**

Our motivating example arises from perverse filtrations associated to the Hitchin morphism for the moduli space of Higgs bundles for a smooth projective variety. We refer to §4.1 for more details on what follows.

Let $\mathscr{X} \to S$ be a smooth projective morphism with connected fibers over a connected quasi projective variety, and let G be a reductive group. Associated to this family is the so-called Dolbeault moduli space $\pi_D = \pi_D(\mathscr{X}/S, G) : M_D(\mathscr{X}/S, G) \to S$ and the Hitchin projective S-morphism $h = h(\mathscr{X}/S, G) : M_D(\mathscr{X}/S, G) \to \mathcal{V}(\mathscr{X}/S, G)$, which induces, for every point $s \in S$, the Hitchin morphism $h_s = h(\mathscr{X}_s, G) : M_D(\mathscr{X}_s, G) \to \mathcal{V}(\mathscr{X}_s, G)$.

It follows from non-abelian Hodge theory that the the structural morphism π_D is topologically locally trivial over S. It follows that the intersection cohomology groups $IH^\bullet(M_D(\mathscr{X}_s, G), \mathbb{Q})$ on the Dolbeault moduli spaces $M_D(\mathscr{X}_s, G)$, give rise to local systems $IH_D^\bullet(\mathscr{X}/S, G)$ on S.

For every $s \in S$, the Hitchin morphism h_s induces the perverse Leray filtration $P_{D,\star}^{h_s}(\mathscr{X}_s, G)$ on the intersection cohomology groups $IH^\bullet(M_D(\mathscr{X}_s, G), \mathbb{Q})$.

It does not seem immediately clear that these perverse Leray filtrations should match-up and give rise to locally constant subsheaves of the local systems $IH_D^\bullet(\mathscr{X}/S, G)$ on S. A priori, the perverse filtrations could jump at special point in S. For example, a priori, one could have un-expected direct summands in $D(\mathcal{V}(\mathscr{X}_s, G))$ in the decomposition theorem for the Hitchin morphism $h(\mathscr{X}_s, G)$ at special points $s \in S$. For a brief discussion, see the beginning of §3.

In this paper, we prove the following result, which can be viewed as some evidence towards the validity of the $P = W$ Question and Conjecture, see Question 4.1.7 and Remark 4.1.8: the weight filtration on the intersection cohomology of the Betti moduli spaces of a family of projective manifolds gives rise to locally constant sheaves on the base of the family; if $P = W$, then the same would be true for the corresponding perverse Leray filtrations associated with the family of Hitchin morphisms, and this is what the following theorem asserts.

Theorem 1.1.1. (**The perverse Leray filtration is locally constant**) *The perverse leray filtration gives rise to locally constant subsheaves:*

$$P_{D,\star}^h(\mathscr{X}/S, G) \subseteq IH_D^\bullet(\mathscr{X}/S, G). \tag{1}$$

Theorem 1.1.1 follows at once from Theorem 1.1.2 below, to the effect that specialization is a filtered isomorphism for the perverse Leray filtrations.

Theorem 1.1.2. *Let S be a nonsingular connected curve, let $s \in S$ be any point, let Δ be a suitably general disk centered at s and let $t \in \Delta \setminus \{s\}$ be a suitably general point. The specialization morphism $IH^\bullet(M_D(\mathscr{X}_s, G), \mathbb{Q})) \to IH^\bullet(M_D(\mathscr{X}_t, G), \mathbb{Q}))$ is well-defined and gives a filtered isomorphism for the respective perverse Leray filtrations $P_D^{h_s}(\mathscr{X}_s, G)$ and $P_D^{h_t}(\mathscr{X}_t, G)$. In particular, the dimensions of the graded spaces $\mathrm{Gr}_\star^{P_D^{h_s}} IH^\bullet(M_D(\mathscr{X}_s, G), \mathbb{Q}))$ are independent of $s \in S$.*

Remark 1.1.3. *Theorems 1.1.1 and 1.1.2 hold, with the same proof, if we replace the Dolbeault moduli spaces with their twisted counterparts (cf. Remark 4.1.8) in the cases of*

relative dimension $\dim \mathscr{X}/S = 1$ *and* $G = GL_n, SL_n, PGL_n$. *In these cases the Dolbeault moduli spaces are integral orbifolds, even nonsingular in the cases* $G = GL_n, SL_n$.

We may say, informally, that the perverse Leray filtration for the Hitchin morphism associated with a family of projective manifolds is independent of the members of the family. In particular, by applying this to curves and their moduli space, we may say that the perverse Leray filtration on the intersection cohomology of the Dolbeault moduli space associated with the curve and the reductive group G is independent of the curve.

Theorem 1.1.2 concerning the Hitchin morphism is proved in §4.2 as a corollary to the more general Theorem 3.2.1, which provides a sufficient condition for when the specialization morphism is well-defined and is a filtered isomorphism.

1.2. **Motivation from Gopakumar-Vafa invariants.**

One motivation for this work is to help understand a conjecture on Gopakumar-Vafa invariants proposed in [MT], following earlier work of Kiem-Li [KL]. In this section, we briefly sketch this connection.

Let X denote a Calabi-Yau threefold and let $\beta \in H_2(X, \mathbb{Z})$ denote a curve class on X. There are various approaches to defining virtual counts of curves on X in class β – Gromov-Witten or Pandharipande-Thomas invariants, for example; While these typically produce an infinite series of invariants, a physics proposal of Gopakumar-Vafa suggests that they should be governed by finitely many integers determined from the cohomology of a space of one-dimensional sheaves.

In [MT], the authors suggest a rigorous approach to their proposal using the perverse filtration as follows. Consider the moduli space $M_\beta(X)$ of stable one-dimensional sheaves on X with support cycle class β and Euler characteristic 1, and let

$$\pi : M_\beta(X) \to \mathrm{Chow}_\beta(X)$$

denote the Hilbert-Chow morphism to the Chow variety of X which remembers the support cycle. [1] Under certain assumptions, $M_\beta(X)$ carries a natural perverse sheaf ϕ_M associated to its shifted symplectic structure, and one can study the perverse cohomology sheaves of its pushforward. More precisely, in [MT], the authors define Gopakumar-Vafa invariants $n_{g,\beta}$ of X in class β via the identity

$$\sum_{i \in \mathbb{Z}} \chi(^p\mathcal{H}^i(R\pi_*\phi_M))y^i = \sum_{g \geq 0} n_{g,\beta}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}. \tag{2}$$

Conjecturally, these GV invariants should, after a universal change of variables, agree with the Gromov-Witten invariants of X in class β. Since the precise relationship is somewhat intricate, we refer the reader there for details.

In order for this picture to be plausible, it is necessary for GV invariants to be invariant under deformations of the Calabi-Yau threefold X – indeed such deformation invariance is built into the intersection-theoretic constructions of GW/PT invariants. One can view the main theorem of this paper as evidence for this invariance; for example, one consequence of our main result is the deformation invariance of $n_{g,\beta}$ for local del Pezzo surfaces. While

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[1]One needs to pass to seminormalizations to define this.

there is no discussion of the sheaf ϕ_M in this paper, one might hope to extend the technique here to this more general setting.

1.3. **Outline.**

§2 sets up the notation. More precisely: §2.1 sets up the notation in the context of the constructible derived category; §2.2 is concerned with the formalism of the vanishing/nearby cycles functors and of the specialization morphism.

§3 deals with the perverse Leray filtration and specialization: §3.1 contains Proposition 3.1.4, which provides a sufficient condition for when perverse truncation and (shifted) restriction to a Cartier divisor commute; §3.2 contains our main technical result, the aforementioned Theorem 3.2.1

§4 contains our main application to the Hitchin morphism: §4.1 is a refresher on Dolbeault and Betti moduli spaces and also proves some preliminary facts needed in the proof, given in §4.2, of Theorem 1.1.2, which in turn implies Theorem 1.1.1 on the local constancy of the perverse Leray filtration for the Hitchin morphism over a base.

2. Notation

2.1. **General notation.**

By variety, we mean a separated scheme of finite type over the field of complex numbers $\mathbb{C}$. By point, we mean a closed point. See [d-M-2009] for a quick introduction and for standard references for many of the concepts and objects in this subsection.

Given a variety Y, we denote by $D(Y)$ the constructible bounded derived category of sheaves of rational vector spaces on Y, endowed with the middle perversity t-structure. We denote derived functors using the un-derived notation, e.g. if $f : X \to Y$ is a morphism of varieties, then the derived direct image (push-forward) functor Rf_* is denoted by f_*, etc. Distinguished triangles in $D(Y)$ are denoted $G' \to G \to G'' \rightsquigarrow$. The full subcategory of perverse sheaves is denoted by $P(Y)$. We employ the following standard notation for the objects associated with this t-structure: the full subcategories ${}^p D^{\leq j}(Y)$ and ${}^p D^{\geq j}(Y)$, $\forall j \in \mathbb{Z}$, of $D(Y)$, and ${}^p D^{[j,k]}(Y) := {}^p D^{\geq j}(Y) \cap {}^p D^{\leq k}(Y)$, $\forall j \leq k \in \mathbb{Z}$; the truncation functors ${}^p \tau_{\leq j} : D(Y) \to {}^p D^{\leq j}(Y)$ and ${}^p \tau_{\geq j} : D(Y) \to {}^p D^{\geq j}(Y)$; the perverse cohomology functors ${}^p H^j : D(Y) \to P(Y)$. At times, we drop the space variable Y from the notation.

The following operations preserve constructibility of complexes: ordinary and extraordinary push-forward and pull-backs, hom and tensor product, Verdier duality. The nearby and vanishing cycle functors also preserves constructibility, with the provision that, when dealing with these functors, one is working over a disk.

The k-th (hyper)cohomology groups of Y with coefficients in $G \in D(Y)$ are denoted by $H^k(Y, G)$. The complex computing this cohomology is denoted by $R\Gamma(Y, G)$ and it lives in the bounded derived category $D(\text{point})$ whose objects are complexes of vector spaces with cohomology given by finite dimensional rational vector spaces.

The filtrations we consider are finite and increasing. For every $G \in D(Y)$, the t-structure defines a natural filtered object $(R\Gamma(Y, G), P)$, and P is called the perverse filtration.

If a statement is valid for every value of an index, e.g. the degree of a cohomology group, or the step of a filtration, then we denote such an index by a bullet-point symbol, or by a star symbol: $H^\bullet(X, F)$, $P_\bullet$, ${}^p\tau_{\leq \bullet}$, $P_\star H^\bullet(X, F)$, $\text{Gr}_\star^{P_f} H^\bullet(X, F)$.

Given $G \in D(Y)$ we have the system of truncation morphisms:

$$\ldots \longrightarrow {}^p\tau_{\leq k-1}G \longrightarrow {}^p\tau_{\leq k}G \longrightarrow \ldots \longrightarrow G \tag{3}$$

A morphism $G \to G'$ in $D(Y)$ gives rise to a system of morphisms:

$$ {}^p\tau_{\leq \bullet}G \longrightarrow {}^p\tau_{\leq \bullet}G' \tag{4}$$

which are compatible with (3) is the evident manner. We say that the system (4) is a system of compatible morphisms. It gives rise to a morphisms of filtered objects:

$$(R\Gamma(Y, G), P)) \longrightarrow (R\Gamma(Y, G'), P)) \qquad \text{in } DF(\text{pt}), \tag{5}$$

where $DF(-)$ denotes the filtered derived category [Il-1971].

Recall that $P(Y)$ is Artinian, so that the Jordan-Holder theorem holds in it. The constituents of a non-zero perverse sheaf $G \in P(Y)$ are the isomorphisms classes of the perverse sheaves appearing in the unique and finite collection of non-zero simple perverse sheaves appearing as the quotients in a Jordan-Holder filtration of G. The constituents of a non-zero complex $G \in D(Y)$ are defined to be the constituents of all of its non-zero perverse cohomology sheaves.

In general, we drop decorations (indices, parentheses, space variables, etc.) if it seems harmless in the context.

In the context of a Cartesian diagram of varieties, we denote parallel arrows by the same symbol. This should not lead to confusion in expressions like the base change morphism $g^* f_* \to f_* g^*$.

We are going to make heavy use of the nearby/vanishing cycle functors. See §2.2 for the basic facts.

We are going to use the decomposition theorem for semisimple complexes, i.e. isomorphic to direct sums of shifted simple perverse sheaves, due to Mochizuki, Sabbah and others; see the references in [de-2017].

Theorem 2.1.1. *Let $f : X \to Y$ be a proper morphism of varieties and let $F \in D(X)$ be semisimple. Then $f_* F$ is semisimple.*

2.2. **The vanishing and nearby cycles formalism.**
Standard references for this section are [De-1972, XIII, XIV] and [Ka-Sh-1990, Ch. 8,10].

Let S be a nonsingular and connected curve and consider a morphism of varieties:

$$v = v_Y : Y \to S.$$

Let $s \in S$ and let $i : s \to S$ be the closed embedding; this is what we call the special point. Objects restricted to s maybe be denoted with a subscript s; e.g. $Y_s := v^{-1}(s)$.

In what follows, in Choice 2.2.9, we choose a disk $\Delta \subseteq S$, centered at s. For the relevance of this choice to this paper, see Remark 2.2.14.

Let $Y_\Delta := v_Y^{-1}(\Delta)$ and let $v_{Y_\Delta} := v_{Y|Y_\Delta} : Y_\Delta \to \Delta$.

We have the functors $i^*, i^!, \psi := \psi_{Y_\Delta}$ and $\phi := \phi_{Y_\Delta}$:

$$i^*, i^!, \psi, \phi : D(Y) \to D(Y_s), \tag{6}$$

where ψ is the nearby cycle functor and ϕ is the vanishing cycle functor.

We have the two, Verdier-dual, canonical distinguished triangle of functors: (we denote by the same symbol the dual arrows σ)

$$i^*[-1] \xrightarrow{\sigma} \psi[-1] \longrightarrow \phi \rightsquigarrow , \qquad \phi \longrightarrow \psi[-1] \xrightarrow{\sigma} i^![1] \rightsquigarrow . \tag{7}$$

Recall that:

$$[\star] \circ {}^p\tau_{\leq \bullet} = {}^p\tau_{\leq \bullet - \star} \circ [\star], \qquad \text{ditto for } {}^p\tau_{\geq \bullet} \text{ and } {}^pH^\bullet. \tag{8}$$

Fact 2.2.1. (**t-exactness for $\psi[-1]$ and ϕ**) *The functors $\psi[-1]$ and ϕ are exact functors of triangulated categories, are t-exact, and they commute with Verdier duality. In particular, they commute with the formation of the perverse cohomology sheaves functors ${}^pH^\bullet$ and with the perverse truncation functors ${}^p\tau_{\leq \bullet}$ and ${}^p\tau_{\geq \bullet}$. We thus have the following canonical identifications:*

$$ {}^p\tau_{\leq \bullet}\phi = \phi\, {}^p\tau_{\leq \bullet}; \qquad {}^p\tau_{\leq \bullet}\psi[-1] = \psi[-1]\, {}^p\tau_{\leq \bullet}; \qquad \text{ditto for } {}^p\tau_{\geq \bullet} \text{ and } {}^pH^\bullet. \tag{9}$$

The following is a key property of the vanishing cycle functor.

Fact 2.2.2. (**Vanishing of ϕ for smooth morphism and lisse sheaves**) *If $v_{Y_\Delta} : Y_\Delta \to \Delta$ is smooth and $G \in D(Y)$ has locally constant cohomology sheaves on Y_Δ, then $\phi\, G = 0 \in D(Y_s)$. See [De-1972, XIII, 2.1.5]. Note the special case where $Y = S$ and v_Y is the identity.*

Fact 2.2.3. *The composition $i^*[-1] \to \psi[-1] \to i^![1]$ yields a morphism of functors $D(Y) \to D(Y_s)$:*

$$i^*[-1] \longrightarrow i^![1] \tag{10}$$

The morphism (10) is an isomorphism when evaluated on a complex $G \in D(Y)$ such that $\phi\, G = 0$; in this case, we have isomorphisms:

$$i^*[-1]G \xrightarrow{\;\simeq\;} \psi[-1]G \xrightarrow{\;\simeq\;} i^![1]G \tag{11}$$

The morphism (10) coincides with the morphism obtained via Verdier's specialization functor, (cf. [Sc-2003], for example), so that it depends only on the closed embedding $Y_s \to Y$, i.e. it is independent of v_Y and of the choice of the disk Δ.

Fact 2.2.4. *(Base change diagrams for ψ and ϕ) Let $f : X \to Y$ be an S-morphism and let $v_X = v_Y \circ f : X \to S$. The base change morphisms associated with i and f give rise to morphisms of distinguished triangles of morphisms as follows (cf.* [De-1972, *XIII, 2.1.7]):*

(1)

$$i^*[-1]f_* \xrightarrow{\sigma \circ f_*} \psi_Y[-1]f_* \longrightarrow \phi_Y f_* \rightsquigarrow \qquad (12)$$

$$f_* i^*[-1] \xrightarrow{f_* \circ \sigma} f_* \psi_X[-1] \longrightarrow f_* \phi_X \rightsquigarrow .$$

When f is proper, the morphism (12) *is an isomorphism.*

(2)

$$\phi_Y f_* \longrightarrow \psi_Y[-1]f_* \xrightarrow{\sigma \circ f_*} i^![1]f_* \rightsquigarrow \qquad (13)$$

$$f_* \phi_X \longrightarrow f_* \psi_Y[-1] \xrightarrow{f_* \circ \sigma} f_* i^![1] \rightsquigarrow ,$$

When f is proper, the morphism (13) *is an isomorphism.*

(3) *by combining the* (12) *with* (13)*, we obtain the following commutative diagram:*

$$i^*[-1]f_* \longrightarrow \psi[-1]f_* \longrightarrow i^![1]f_* \qquad (14)$$

$$f_* i^*[-1] \longrightarrow f_* \psi[-1] \longrightarrow f_* i^![1].$$

Let $t \in S$ be another point and, by abuse of notation, denote the closed embedding $t \to S$ also by t. There is the natural morphism (cf. (10)):

$$t^*[-1] \longrightarrow t^![1]. \qquad (15)$$

Fact 2.2.5. *Let $G \in D(Y)$ and let $t \in S$ be a general point. Then the natural morphism $t^*[-1]G \to t^![1]G$ is an isomorphism in $D(Y_t)$. To see this, recall* (10)*, and use the vanishing of vanishing cycle functor "translated to t"* [Sc-2003, *Rmk. 4.2.4]. As usual, here "general" means that it can be chosen to be any point of a suitable Zariski-dense open subset $S^o \subseteq S$ that depends on $G \in D(Y)$.*

The following follows from the fact that for t general, $t^*[-1], t^![1]$ commute with all the functors involved in the constructions of perverse truncation.

Fact 2.2.6. *(t-exactness and $t^*, t^!$) Let $G \in D(Y)$ and let $t \in S$ be general. Then we have:*

$$t^*[-1] \,{}^p\tau_{\leq\bullet}G = {}^p\tau_{\leq\bullet}t^*[-1]G, \quad t^![1] \,{}^p\tau_{\leq\bullet}G = {}^p\tau_{\leq\bullet}t^![1]G, \qquad \text{ditto for } {}^p\tau_{\geq\bullet} \text{ and } {}^pH^\bullet. \quad (16)$$

Fact 2.2.7. *What follows is a consequence of Deligne's generic base change theorem* [De-1976, *Thm. 9.1] and of stratification theory; see also the discussion in* [d-M-2009b]. *Given a finite collection of morphisms $v_i : Y_i \to S$ and complexes $G_i \in D(Y_i)$, there is a Zariski-dense open subset $S^o \subseteq S$ such that the direct images $v_{i*}G_i$ are lisse, and such that*

their formation commutes with arbitrary base change. By shrinking S^o if necessary, we can further assume that the G_i have no constituent supported on fibers of the morphisms v_i over S^o, and that the strata on the Y_i, of stratifications with respect to which the G_i are constructible, map smoothly to S over S^o (cf. [Sc-2003, Rmk. 4.2.4]). The points $t \in S^o$ are said to be general for the finite collection $\{G_i\}$.

Definition 2.2.8. *We say that $t \in S$ is general (for the collection of G_i's) if $t \in S^o = S^o(G)$ (cf. Fact 2.2.7).*

Choice 2.2.9. *Let $s \in S$. Let $S^o \subseteq S$ be the Zariski-dense open subset of points of the connected curve S which are general with respect to some finite collection of G_i's as above. Let $t \in S^o$. Choose a disk $s, t \in \Delta \subseteq S$ such that $\Delta^* := \Delta \setminus \{s\} \subseteq S^o$. Chose a pointed universal covering $(\widetilde{\Delta^*}, \widetilde{t}) \to (\Delta^*, t)$.*

Remark 2.2.10. *In the special case where $v_Y : Y \to S$ is the identity on S, we have that $i^*, i^!, \psi, \phi : D(S) \to D(s)$ and that $t^*, t^! : D(S) \to D(t)$. We have canonical identifications $D(s) = D(\mathrm{pt}) = D(t)$, where pt is just a point, so that all three categories are naturally equivalent to the bounded derived category of finite dimensional rational vector spaces. Similarly, in the filtered case: $DF(s) = DF(\mathrm{pt}) = DF(t)$.*

When $v_Y : Y \to S$ is the identity, we have the following:

Fact 2.2.11. (Fundamental isomorphism) *Let things be as in Choice 2.2.9. There are the natural isomorphisms:*

$$t^*[-1]G \xrightarrow{\simeq} \psi_\Delta[-1]G \xrightarrow{\simeq} t^![1]G, \qquad \text{in } D(t) = D(\mathrm{pt}) = D(s), \qquad (17)$$

where all three terms are well-defined, up to canonical isomorphism, independently of the choices, but where the isomorphisms, which depend on the choice of $\widetilde{t}$, are uniquely defined modulo the monodromy action of the fundamental group $\pi_1(\Delta^*, t)$. See the fundamental identity [De-1972, XIV, 1.3.3.1]. Note that (17) is Verdier self-dual.

Fact 2.2.12. (**Specialization morphism**) *Consider, in the special case where f is v_Y, the morphism of distinguished triangles (12). We place ourselves in the set-up of Choice 2.2.9. One would like to specialize cohomology from the special point $s \in \Delta$, to the general point $t \in \Delta^*$. This is not always possible, as we now discuss.*

Diagram (12) yields the functorial morphism of distinguished triangles in $D(\mathrm{pt})$:

$$
\begin{array}{ccccccc}
R\Gamma(s, i^* v_* G) & \xrightarrow{\sigma_{Y_\Delta}} & R\Gamma(s, \psi_\Delta v_* G) = R\Gamma(Y_t, t^* G) & \longrightarrow & R\Gamma(s, \phi_\Delta[1] v_* G) & \rightsquigarrow & (18) \\
\downarrow & \overset{\mathrm{sp?}}{\dashrightarrow} & \downarrow & & \downarrow & & \\
R\Gamma(Y_s, i^* G) & \xrightarrow{\sigma_\Delta} & R\Gamma(Y_s, \psi_{Y_\Delta} G) & \longrightarrow & R\Gamma(Y_s, \phi_{Y_\Delta}[1] G) & \rightsquigarrow,
\end{array}
$$

where the canonical identification in the middle of the first row stems from Fact 2.2.11 and the fact that base change is an isomorphism for general t (cf. Fact 2.2.7).

Note that while this identification depends on the choice of the pre-image $\widetilde{t} \in \widetilde{\Delta^*}$ of $t \in \Delta^*$, the resulting restriction morphism $\sigma_{Y_\Delta} : R\Gamma(Y_\Delta, G) \to R\Gamma(Y_t, G)$ depends only on t, and not on the choice $\widetilde{t}$: in fact, monodromy acts on the target, but the domain maps into the invariants.

Let us emphasize an important piece of diagram (18): *(note that the middle term below does not change when we shrink the disk Δ centered about s)*

$$R\Gamma(Y_s, i^*G) \xleftarrow{\text{restriction to } s} R\Gamma(Y_\Delta, G) \xrightarrow{\text{restriction to } t} R\Gamma(Y_t, t^*G). \tag{19}$$

In general, i.e. without any further hypothesis ensuring that the restriction to s is an isomorphism, there is no resulting natural morphism $R\Gamma(Y_s, G) \to R\Gamma(Y_t, G)$. When restriction to s is an isomorphism, e.g. when v_Y is proper, then we call the resulting morphism the specialization morphism:

$$R\Gamma(Y_s, G) \xrightarrow{\text{sp}} R\Gamma(Y_t, G). \tag{20}$$

Of course, even if v_Y is not proper, it may happen that there is a well-defined specialization morphism for some $G \in D(Y)$.

When the specialization morphism is well-defined, we have the distinguished triangle:

$$v_* i^* G \simeq i^* v_* G \xrightarrow{\text{sp}} \psi v_* G \longrightarrow \phi v_* G \rightsquigarrow . \tag{21}$$

Fact 2.2.13. *If $G \in D(S)$ has locally constant cohomology sheaves, then a specialization morphism is an isomorphism. This follows at once from Fact 2.2.2.*

Remark 2.2.14. *The Choice 2.2.9 is harmless for our purposes: in fact, when defined, the specialization morphism depends only on s and on $t \in S^o \setminus \{s\}$. Note also that the morphisms* (15) *are independent of the choice of the disk Δ.*

3. Perverse filtration and specialization

If one analyzes the behavior of the perverse filtration in families via the specialization morphism, there are two issues. First, the specialization morphism is not defined in general if X is not proper over S, which is the case for families of Dolbeault moduli spaces. The second issue is that, even when the specialization morphism is defined, as it is in the case of the Dolbeault moduli spaces, it gives rise to a filtered morphism for the perverse Leray filtrations. Its failure to be a filtered isomorphism is detected by the filtered cone; however, in general, this is not the filtered cone associated with the natural morphism of functors $i^* \to \psi$ (i^* the pull-back to the special fiber, ψ the nearby fiber functor). This is due to the fact that perverse truncation does not commute with restriction/pull-back: e.g. when one has a direct summand supported on the special fiber.

The goal of this section is to prove Theorem 3.2.1 in §3.2, which is a criterion to have a well-defined specialization morphism which is a filtered isomorphism for the corresponding perverse filtrations. To this end, in §3.1, we study a bit the relationship between perverse truncation and restriction to a Cartier divisor.

3.1. **Cartier divisors and partial t-exactness.**

The purpose of this section is to prove Proposition 3.1.4, especially equation (30). We recommend that readers skip this section at a first reading.

The next lemma records some general t-exactness properties related to embeddings of effective Cartier divisors on varieties.

Lemma 3.1.1. *Let* $\iota : T' \to T$ *be a closed embedding of varieties such that the open embedding* $T \backslash T' \to T$ *is an affine morphism (e.g.* T' *is an effective Weil divisor supporting an effective Cartier divisor). Then: (we omit the space variables)*

(1) *The functor* ι^* *is right* t*-exact and the functor* $\iota^!$ *is left* t*-exact, i.e.:*

$$\iota^* : {}^pD^{\leq\bullet} \to {}^pD^{\leq\bullet}, \qquad \iota^! : {}^pD^{\geq\bullet} \to {}^pD^{\geq\bullet}. \tag{22}$$

(2) *The functor* $\iota^*[-1]$ *is left* t*-exact and the functor* $\iota^![1]$ *is right* t*-exact:*

$$\iota^* : {}^pD^{\geq\bullet} \to {}^pD^{\geq\bullet-1}, \qquad \iota^! : {}^pD^{\leq\bullet} \to {}^pD^{\leq\bullet+1}. \tag{23}$$

Proof. The inequalities (22) are [Be-Be-De-1982, 4.2.4]), in fact, they are valid for any immersion. What follows is specific to the situation of the Lemma.

We prove the inequalities (23). Recall that, by [Be-Be-De-1982, 4.1.10.ii], we have: if $G \in P(T)$, than $\iota^*G \in {}^pD^{[-1,0]}$. The desired inequality for ι^* follows from this fact and a simple descending induction on $\bullet$, by using ι^* of the truncation distinguished triangles ${}^pH^\bullet[-\bullet] \to {}^p\tau_{\geq\bullet} \to {}^p\tau_{\geq\bullet+1} \rightsquigarrow$, and $\iota^* {}^pH^{\bullet+1} \in {}^pD^{[-1,0]}$.

Since, for $G \in P(T)$, we have that $\iota^!G \in {}^pD^{[0,1]}$, the proof for $\iota^!$ is analogous. Alternatively, it can also be deduced from the one for ι^* by Verdier duality. □

The following lemma is a technical preliminary to Proposition 3.1.4.

Lemma 3.1.2. (Canonical factorization of ${}^p\tau_{\leq\bullet-1}i^* \to {}^p\tau_{\leq\bullet}i^*$**)** *Let* $\iota : T' \to T$ *be as in Lemma 3.1.1. The natural morphism* $\gamma : {}^p\tau_{\leq\bullet-1}\iota^* \to {}^p\tau_{\leq\bullet}\iota^*$ *admits a canonical factorization:*

$$\gamma : {}^p\tau_{\leq\bullet-1}\iota^* \xrightarrow{\ \delta\ } \iota^* {}^p\tau_{\leq\bullet} \xrightarrow{\ \epsilon\ } {}^p\tau_{\leq\bullet}\iota^*. \tag{24}$$

Similarly, for the dual natural morphism ${}^p\tau_{\geq\bullet+1}\iota^! \longleftarrow {}^p\tau_{\geq\bullet}\iota^! : \gamma'$:

$$ {}^p\tau_{\geq\bullet+1}\iota^! \xleftarrow{\ \delta'\ } \iota^! {}^p\tau_{\geq\bullet} \xleftarrow{\ \epsilon'\ } {}^p\tau_{\geq\bullet}\iota^! : \gamma' \tag{25}$$

Proof. We start with the following diagram of distinguished triangles in $D(T')$: (the arrows γ and δ are not part of either distinguished triangle; they are there to help visualize the situation)

$$
\begin{array}{ccccc}
 & \iota^* {}^p\tau_{\leq\bullet} & \longrightarrow & \iota^* & \longrightarrow \iota^* {}^p\tau_{\geq\bullet+1} \rightsquigarrow \\
\delta \nearrow & \downarrow \epsilon & & \Vert & \downarrow \\
({}^p\tau_{\leq\bullet-1}\iota^* \xrightarrow{\gamma}) & {}^p\tau_{\leq\bullet}\iota^* & \longrightarrow & \iota^* & \longrightarrow {}^p\tau_{\geq\bullet+1}\iota^* \rightsquigarrow .
\end{array}
\tag{26}
$$

By the l.h.s. of (23), we have that:

$$\iota^* {}^p\tau_{\geq\bullet+1} : D(T) \to {}^pD^{\geq\bullet}(T'). \tag{27}$$

By [Be-Be-De-1982, Prop. 1.1.9, p.23], the diagram (26) can then be completed uniquely to a morphism of distinguished triangles; this is visualized by means of the dotted arrows.

The conclusion follows from the inequality (27), by taking the long exact sequence associated with $\mathrm{Hom}_{D(T')}({}^p\tau_{\leq\bullet-1}\iota^*, -)$ applied to the distinguished triangle on the top of (26): in fact, for every $G \in D(Y)$, we have $\mathrm{Hom}_{D(T')}({}^p\tau_{\leq\bullet-1}\iota^*G, -) = 0$, when evaluated at something in ${}^pD^{\geq\bullet}(T')$.

The proof for $\iota^!$ is analogous. Alternatively, it can also be deduced from the one for ι^* by Verdier duality. $\qquad\square$

Remark 3.1.3. *In view of the l.h.s. inequality in (23), the r.h.s. vertex in (26) satisfies the inequality $\iota^* \, {}^p\tau_{\geq\bullet+1} : D \to {}^pD^{\geq\bullet}$. By taking the long exact sequence of perverse cohomology of the top distinguished triangle in (26), the aforementioned inequality yields the natural isomorphism of functors:*

$$ {}^p\tau_{\leq\bullet-1}\iota^* \, {}^p\tau_{\leq\bullet} \xrightarrow{\;\simeq\;} {}^p\tau_{\leq\bullet-1}\iota^*. \tag{28} $$

Proposition 3.1.4. (No constituents on divisors and t-exactness) *Let $\iota : T' \to T$ be as in Lemma 3.1.1. If $G \in D(T)$ has no constituent supported on T', then the morphisms δ (cfr. (24)) are isomorphisms, and we get natural isomorphisms:*

$$ {}^p\tau_{\leq\bullet-1}\iota^*G \xrightarrow[\simeq]{\delta} \iota^* \, {}^p\tau_{\leq\bullet}G, \qquad {}^p\tau_{\geq\bullet-1}\iota^*G \xrightarrow{\;\simeq\;} \iota^* \, {}^p\tau_{\geq\bullet}G, \qquad {}^pH^{\bullet-1}\iota^*G \xrightarrow{\;\simeq\;} \iota^*[-1]{}^pH^\bullet G. \tag{29} $$

The same holds if we replace ι^* with $\iota^!$ and $(\bullet - 1)$ with $(\bullet + 1)$.

Equivalently, we have:

$$ {}^p\tau_{\leq\bullet}\iota^*[-1]G \xrightarrow[\simeq]{\delta} \iota^*[-1] \, {}^p\tau_{\leq\bullet}G, \qquad {}^p\tau_{\leq\bullet}\iota^![1]G \xleftarrow[\simeq]{\delta} \iota^![1] \, {}^p\tau_{\leq\bullet}G, \qquad {}^pH^{\bullet+1}\iota^!G \xleftarrow{\;\simeq\;} \iota^![1]{}^pH^\bullet G. \tag{30} $$

Proof. It is enough to prove (29): the statement for $\iota^!$ follows by Verdier duality; The equivalent statements are mere reformulations by means of (9).

It is enough to prove the first statement on the l.h.s. of (29), for the remaining ones follow formally by consideration of the truncation distinguished triangles.

We have that ${}^p\tau_{\leq\bullet}G \in {}^pD^{\leq\bullet}$, so that, by (22), we have that $\iota^* \, {}^p\tau_{\leq\bullet}G \in {}^pD^{\leq\bullet}$, and then, clearly, we have that

$$ \iota^* \, {}^p\tau_{\leq\bullet}G = {}^p\tau_{\leq\bullet}\iota^* \, {}^p\tau_{\leq\bullet}G. \tag{31} $$

In view of (31) and of (28), and by considering the truncation triangle ${}^p\tau_{\leq\bullet-1} \to {}^p\tau_{\leq\bullet} \to {}^pH^\bullet[-\bullet] \rightsquigarrow$ applied to $\iota^* \, {}^p\tau_{\leq\bullet}G$, in order to prove the first equality on the lhs in (29), it is necessary and sufficient to show that ${}^pH^\bullet(\iota^* \, {}^p\tau_{\leq\bullet}G) = 0$.

This can be argued as follows. By taking the long exact sequence of perverse cohomology of the distinguished triangle $\iota^* \, {}^p\tau_{\leq\bullet-1}G \to \iota^* \, {}^p\tau_{\leq\bullet}G \to \iota^* {}^pH^\bullet G[-\bullet] \rightsquigarrow$, we see that it is necessary and sufficient to show that $\iota^* {}^pH^\bullet G[-\bullet] \in {}^pD^{\leq\bullet-1}$, or, equivalently, that $\iota^* {}^pH^\bullet G \in {}^pD^{\leq-1}$.

By [Be-Be-De-1982, 4.1.10.ii], we have the distinguished triangle ${}^pH^{-1}(\iota^* {}^pH^\bullet G)[1] \to \iota^* {}^pH^\bullet G \to {}^pH^0(\iota^* {}^pH^\bullet G) \rightsquigarrow$ and an epimorphism ${}^pH^\bullet G \to {}^pH^0(\iota^* {}^pH^\bullet G)$. Since G is assumed to not have constituents supported at Y_s, we must have ${}^pH^0(\iota^* {}^pH^\bullet G) = 0$, so

that $\iota^{*p}H^\bullet G[-\bullet] \in {}^pD^{\leq\bullet-1}$, as requested. The l.h.s. equality in (29) follows, and we are done. $\qquad\square$

We shall use the following simple lemma, where we employ the notation in §2.2.

Lemma 3.1.5. *Let $G \in D(Y)$ be such that $\phi_{Y_\Delta}G = 0$. Then no constituent of G is supported on Y_s.*

Proof. By chasing the definitions, we see that we may assume that G is perverse. By using the fact that ϕ is t-exact, and by an easy induction on the length of a Jordan-Hölder filtration of G, we see that $\phi G' = 0$ for every constituent G' of G. The conclusion follows from the fact that if $G' \in D(Y)$ is a non-zero complex supported on Y_s, then $i_*\phi G' = G'$. $\qquad\square$

By combining Lemma 3.1.5 with Proposition 3.1.4, we obtain the following

Corollary 3.1.6. *Let $G \in D(Y)$ be such that $\phi_{Y_\Delta}G = 0$. Then we have a natural isomorphism:*

$$\delta : {}^p\tau_{\leq\bullet}i^*[-1]G \xrightarrow{\simeq} i^*[-1]\,{}^p\tau_{\leq\bullet}G; \quad \text{similarly, for } {}^p\tau_{\geq\bullet}, \text{ and for } {}^pH^\bullet; \text{ ditto, for } i^![1]. \quad (32)$$

3.2. Compatibility of the specialization morphism with the perverse filtration.

Theorem 3.2.1. *Let $f : X \to Y$ be proper morphism, let $v_Y : Y \to S$ be a morphisms onto a nonsingular connected curve and let $F \in D(X)$. Choose $s, t \in S$ and a disk Δ as in Choice 2.2.9. Assume one of the following conditions:*

(i) *$\phi F = 0$ and v_Y is proper, or*

(ii) *$\phi F = 0$ and the $v_*\,{}^p\tau_{\leq\bullet}f_*F$ have locally constant cohomology sheaves on S, or*

(iii) *F is semisimple, $\phi F = 0$ and v_*f_*F has locally constant cohomology sheaves on S.*

Then the specialization morphism is well-defined and it is a filtered isomorphism for the respective perverse Leray filtrations:

$$\mathrm{sp} : \left(R\Gamma(X_s, i^*F), P^{f_s}\right) \xrightarrow{\simeq} \left(R\Gamma(X_t, t^*F), P^{f_t}\right). \quad (33)$$

Proof. By applying (14) to the ${}^p\tau_{\leq\bullet}f_*$, we obtain the following commutative diagrams:

$$
\begin{array}{ccccc}
i^*[-1]v_*\,{}^p\tau_{\leq\bullet}f_* & \xrightarrow{\;1\;} & \psi[-1]v_*\,{}^p\tau_{\leq\bullet}f_* & \xrightarrow{\;2\;} & i^![1]v_*\,{}^p\tau_{\leq\bullet}f_* \\
\Big\downarrow{\scriptstyle 3} & {\scriptstyle\mathrm{sp?}}\nearrow & \Big\downarrow{\scriptstyle 4} & & \Big\downarrow{\scriptstyle 5}{\scriptstyle =} \\
v_*i^*[-1]\,{}^p\tau_{\leq\bullet}f_* & \xrightarrow{\;1'\;} & v_*\psi[-1]\,{}^p\tau_{\leq\bullet}f_* & \xrightarrow{\;2'\;} & v_*i^![1]\,{}^p\tau_{\leq\bullet}f_*
\end{array} \quad (34)
$$

Up to shift: the cones of 1 and 2 coincide with $\phi v_*\,{}^p\tau_{\leq\bullet}f_*$; the cones of $1'$ and $2'$ coincide with $v_*\phi\,{}^p\tau_{\leq\bullet}f_*$.

Let us prove (i).

In this case, we only need the commutative square on the l.h.s. of (34).

Since v is assumed to be proper, the base change morphisms 3 and 4 are isomorphisms. In particular, the specialization morphism sp is well-defined and it gives rise to a system of compatible morphisms.

The assumption $\phi F = 0$, which is common to (i) and (ii), implies, by the t-exactness of ϕ and the properness of f, that $v_* \phi \, {}^p\tau_{\leq \bullet} f_* F = v_* \, {}^p\tau_{\leq \bullet} f_* \phi F = 0$, i.e. that the cone of $1'$ is zero, so that so is the cone of 1.

It follows that (i) is a system of compatible isomorphisms and, as such, it gives rise to an isomorphism in the filtered derived category.

The filtered complex $R\Gamma(\mathscr{X}_s, i^*[-1]F), P^{f_s})$ arises in connection with the cohomology of the compatible system of complexes $v_* \, {}^p\tau_{\leq \bullet} f_* i^*[-1]F)$. Similarly, the filtered complex $R\Gamma(\mathscr{X}_t, t^*[-1]F), P^{f_t})$ arises in connection with the cohomology of the compatible system of complexes $v_* \, {}^p\tau_{\leq \bullet} f_* t^*[-1]F)$.

It remains to identify:

(a) $v_* i^*[-1] \, {}^p\tau_{\leq \bullet} f_* F$ with $v_* \, {}^p\tau_{\leq \bullet} f_*[-1]i^*F$, and
(b) $\psi[-1] v_* \, {}^p\tau_{\leq \bullet} f_* F$ with $v_* \, {}^p\tau_{\leq \bullet} f_* t^*[-1]F$.

To achieve (b), we argue as follows. The choice of t general for F, made in Definition 2.2.8, allows us to: replace ψ with t^* (cf. (17)); use the identification $t^* v_* = v_* t^*$; use the identification $t^*[-1] \, {}^p\tau_{\leq \bullet} = \, {}^p\tau_{\leq \bullet} t^*[-1]$ (cf. 16)); use the identification $t^* v_* = v_* t^*$. Then (b) follows.

To achieve (a), we argue as follows. We first apply Corollary 3.1.6 to $G = f_* F$; the condition $\phi f_* F = 0$ is met in view of the properness of f and of $\phi F = 0$. We then apply proper base change $i^* f_* \stackrel{\sim}{\to} f_* i^*$. Then (a) follows, and (i) is proved.

Let us prove (ii).

As it has been seen above, the assumption $\phi F = 0$ implies that the cones of $1'$ and $2'$ are zero, so that $1'$ and $2'$ are isomorphisms.

Since we are assuming that the $v_* \, {}^p\tau_{\leq \bullet} f_* F$ have locally constant cohomology sheaves on Δ, we have that the cones of 1 and 2 are zero as well (cf. Fact 2.2.2), so that 1 and 2 are isomorphisms.

Since the morphism 5 is an isomorphism, all the morphisms in (34) are isomorphisms.

We conclude as in case (i).

Case (iii) is weaker than case (ii); we can also prove it without resorting to the use of Corollary 3.1.6. The proof is very similar, except that in order to achieve the critical commutation property (a) via Corollary 3.1.6, we use that: F semisimple implies $f_* F$ semisimple (cf. the decomposition Theorem 2.1.1); the assumption $\phi F = 0$ implies that no simple summand of $f_* F$ is supported on Y_s; the commutation property for a simple, un-shifted, simple perverse summand P, which we know not to be supported on Y_s follows from [Be-Be-De-1982, 4.10.1], to the effect that $i^*[-1]P$ is perverse.

□

4. The Hitchin morphism and specialization

4.1. The Betti and Dolbeault moduli spaces: the $P = W$ conjecture.

Let $X \to S$ be a smooth projective morphism over a variety S and let G be a reductive group.

The Betti moduli space $M_B(\mathscr{X}/S, G)$ (cf. [Si-1994, pp.12-15]) is a complex analytic space over S. The fiber over a point $s \in S$ is the moduli space (a.k.a. the character variety) $M_B(\mathscr{X}_s, G)$ of representations of the fundamental group of $\mathscr{X}_s$ into G.

The Dolbeault moduli space $M_D(\mathscr{X}/S, G)$ (cf. [Si-1994, §9, esp. Prop. 9.7]) is quasi-projective over S; in general, it is not proper over S. The fiber over a point $s \in S$ is the moduli space $M_D(\mathscr{X}_s, G)$ of principal Higgs bundles of semiharmonic type on $\mathscr{X}_s$ for the group G.

Fact 4.1.1. (Non-Abelian Hodge Theorem) *There is a natural S-homeomorphism of the underlying topological spaces: (cf.* [Si-1994, Thm. 9.11, and Lm. 9.14]*)*

$$\Psi(\mathscr{X}/S, G) : M_B(\mathscr{X}/S, G) \xrightarrow{\simeq} M_D(\mathscr{X}/S, G). \tag{35}$$

To fix ideas, in what follows, we tacitly assume that S and the fibers of the family $\mathscr{X}/S$ are connected; such an assumption is for ease of exposition only; see [Si-1994, pp. 14-15].

Choose any point $s_o \in S$. The structural morphism $\pi_B(\mathscr{X}/S, G) : M_B(\mathscr{X}/S, G) \to S$ is analytically locally trivial over S, with transition functions with values in the group of $\mathbb{C}$-scheme automorphisms of the fiber $M_B(\mathscr{X}_{s_o}, G)$; see [Si-1994, Lm 6.2, p.13]. More precisely: let $(\widetilde{S}, \widetilde{s}_o) \to (S, s_o)$ be a pointed universal covering space with associated identification of the deck transformation group with the fundamental group $\pi_1(S, s_o)$; the fundamental group acts on M_{s_o} via $\mathbb{C}$-scheme automorphisms; $M_B(\mathscr{X}/S, G)$ is constructed as the quotient of $M_B(\mathscr{X}_{s_o}, G) \times \widetilde{S}$ under the usual action of the fundamental group $\pi_1(S, s_o)$.

Fact 4.1.2. (Local triviality of the Dolbeault moduli space over the base) *The local triviality of the Betti moduli space over the base, coupled with the Non-Abelian Hodge Theorem S-homeomorphism Ψ (35), implies that the structural morphism $\pi_D(\mathscr{X}/S, G) : M_D(\mathscr{X}/S, G) \to S$ is topologically locally trivial over the base S.*

Recall that, for irreducible varieties, the intersection cohomology complexes/groups are topological invariants, independent of the stratification (cf. [Go-Ma-1983, §4.1]). Note that [Go-Ma-1983] deals with irreducible analytic varieties; on the other hand, as the forthcoming Lemma 4.1.3 shows, if we define the intersection complex of a variety as the direct sum of the intersection complex of its irreducible components, then the topological invariance of the intersection cohomology complexes/groups is still valid.

In particular, given a topologically locally trivial fibration with fibers varieties, the intersection cohomology groups of the fibers give rise to locally constant sheaves on the base.

We thank G. Williamson for suggesting the definition of the set X' in the proof of the following lemma. Our original set X' was defined using the notion of local irreducibility and lead to a more cumbersome proof.

Lemma 4.1.3. (Topological invariance of intersection cohomology for reducible varieties) *Let X and Y be complex analytic set and let $g : X \simeq Y$ be a homeomorphism of the underlying topological spaces endowed with the classical topology. Then:*

(1) *The homeomorphism g induces a natural bijection $\gamma : I \simeq B$ on the sets of irre-ducible components of X and Y such that g induces homeomorphisms $X_i \simeq Y_{\gamma(i)}$, for every $i \in I$.*

(2) *Define the intersection complex IC_X of a complex analytic set X to be the direct product of the intersection cohomology complexes of the irreducible components of X. Then the homeomorphism u induces a natural isomorphism $IC_X = u^* IC_Y$.*

Proof. The case when X and Y are irreducible is proved by M. Goresky and R. MacPherson in [Go-Ma-1983]. It is thus clear that it is enough to prove the first statement of the lemma, which is what we do next, leaving some elementary details to the reader.

Let X' (Y', resp.) the be open subset of those points of X (Y, resp.) admitting a classical open neighborhood all of whose points admit a classical open neighborhood homeomorphic to a Euclidean space of some dimension.

Since X' is defined topologically, it is clear that $g(X') = Y'$.

We have the following inclusions of classical open subsets:

$$X^{\mathrm{sm}} \subseteq X' \subseteq X^o \subseteq X, \quad X_i^{\mathrm{sm}} \subseteq X_i' \subseteq X_i^o \subseteq X_i,$$

where X^{sm} is the set of smooth points of the complex analytic set X, and X^o is the complement of the union of all intersections $X_i \cap X_j$, $i, j \in I$, $i \neq j$, and where $-_i$ denotes intersection with the irreducible component X_i.

The classical open subsets X_i^{sm} and X_i^o are also Zariski open, irreducible and connected. The classical open subset X_i' is connected. The X_i' are the connected components of X'. The homeomorphism g must respect the decomposition of X' and Y' into their connected components, so that we obtain the desired bijection $\gamma : I \simeq B$. The classical closure of X_i' is X_i and since g is a homeomorphism, we must have that g induces a homeomorphism $X_i \cong Y_{\gamma(i)}$. $\qquad \square$

Lemma 4.1.3 implies at once the following

Corollary 4.1.4. *Let F, S be a varieties, let X be an S-variety, and let $\tau : X \to F \times S$ be an S-homeomorphism. Then, for every $s \in S$, we have canonical isomorphisms:*

$$(\tau^* pr_F^* IC_F)_{|X_s} = IC_{X_s}$$

Remark 4.1.5. *The definition of intersection cohomology complex for reducible varieties stemming from Lemma 4.1.3 is reasonable in view of the fact that it satisfies virtually all the usually properties of the usual intersection cohomology complex for irreducible varieties, e.g. purity, mixed and pure Hodge structures, Artin vanishing, Lefschetz theorems, relative hard Lefschetz, Hodge Riemann relations, decomposition theorem. See [d-2012, §4.6], for example.*

The intersection cohomology groups of the fibers of $\pi_B(\mathscr{X}/S, G)$ give rise to locally constant sheaves $IH_B^\bullet(\mathscr{X}/S, G)$ on S. In view of the homeomorphism $\Psi(\mathscr{X}/S, G)$, and of the topological invariance of intersection cohomology, the same applies to the Dolbeault picture, and we get locally constant sheaves $IH_D^\bullet(\mathscr{X}/S, G)$ on S.

Fact 4.1.6. (The Betti weight filtration is locally constant) *By the local triviality of $\pi_B(\mathscr{X}/S, G)$ over S, and since the transition automorphisms are compatible with mixed*

Hodge structures in (intersection) cohomology, we see that the weight filtration W for the mixed Hodge structure for the intersection cohomology of the fibers of $\pi_B(\mathscr{X}/S, G)$ gives rise to locally constant subsheaves $W_{\star,B}(\mathscr{X}/S, G) \subseteq IH^\bullet_B(\mathscr{X}/S, G)$ on S.

The Dolbeault moduli space is endowed with a natural $\mathbb{G}_m$-action (cf. [Si-1994, p.62, and p.17]), given by scalar multiplication on the Higgs field.

The reference in this paragraph is [Si-1994, p.20-23], which deals with the case of $G = GL_n$, suitably adapted to an arbitrary reductive G. The Dolbeault moduli space admits the Hitchin S-morphism

$$h(\mathscr{X}/S, G) : M_D(\mathscr{X}/S, G) \longrightarrow \mathcal{V}(\mathscr{X}/S, G). \tag{36}$$

Here, $\mathcal{V}(\mathscr{X}/S, G)$ is the quasi projective S-variety representing the functor $(S' \to S) \mapsto \oplus_{i=1}^{\mathrm{rk}G} H^0(X'/S', \mathrm{Sym}^{e_i(G)}\Omega^1_{X'/S'})$, where the positive integers e_i are the degrees of the generators of conjugation-invariant polynomials on the Lie algebra of G. The Hitchin morphism assigns to a G-principal Higgs bundle the symmetric polynomials appearing as the coefficients of the "characteristic polynomial" of the Higgs field, viewed as sections of the appropriate sheaf. The Hitchin morphism is proper over S, hence projective over S (the Dolbeault moduli space is quasi projective over S). Domain and target of the Hitchin morphism are endowed with a natural $\mathbb{G}_m$-action (cf. [Si-1994, p.62]), which covers the trivial action on S. The $\mathbb{G}_m$-action on the target is contracting. The Hitchin morphism is $\mathbb{G}_m$-equivariant.

We observe that the mixed Hodge structure on the intersection cohomology groups of each Dolbeault moduli space is pure (the starting point is the $\mathbb{G}_m$-equivariance and the contracting action; then one can imitate the proof of [de-Hai-Li-2017, Lemma 6.1.1 and references therein, and the proof of Thm 2.4.1]). We do not need this fact, except to point out that it is in sharp contrast with the expected (known in some cases) non purity of the corresponding intersection cohomology groups of each Betti moduli space.

Question 4.1.7. (**Is $P = W$ in the non-Abelian Hodge theory in arbitrary dimensions?**) *Let X be a connected smooth projective variety. Then for the weight filtration $W_B(X, G)$, do we have $W_{B,2\star+1}(X, G) = W_{2\star}(X.G) \subseteq IH^\bullet(M_B(X, G))$? Via the Non Abelian Hodge Theorem isomorphism Ψ_*, do we have that $\Psi_* W_{B,2\star}(X, G) = P^h_{D,\star}(X, G)$, where $P^h_D(X, G)$ is the perverse Leray filtration for the Hitchin morphism (suitably normalized, so that it "starts at zero")?*

Remark 4.1.8. *Actually, the $P = W$ conjecture, which is due to M.A. de Cataldo, T. Hausel and L. Migliorini, is concerned with a twisted version of the Betti/Dolbeault moduli spaces for curves of genus $g \geq 2$. The paper [de-Hau-Mi-2012] proves the validity of the conjecture in this twisted case when X is a curve and $G = GL_2, SL_2$ and PGL_2. In this twisted case over a curve of genus $g \geq 2$, the moduli spaces for $G = GL_n, SL_n$ are nonsingular, and for $G = PGL_n$ they are orbifolds; Theorem 1.1.2 applies to this situation.*

In the next section, we prove Theorem 1.1.2, to the effect that the perverse filtration gives rise to locally constant subsheaves subsheaves $P^h_{D,\star}(\mathscr{X}/S, G) \subseteq IH^\bullet(\mathscr{X}/S, G)$ on S.

We need the following

Lemma 4.1.9. *Let things be as in Theorem 1.1.2. Let F be the intersection complex of the Dolbeault moduli space $M_D(\mathscr{X}/S, G)$. Then*

$$\phi F = 0, \tag{37}$$

the specialization morphism is defined and it is an isomorphism:

$$\mathrm{sp} : R\Gamma(\mathscr{X}_s, i^*F) \xrightarrow{\simeq} R\Gamma(\mathscr{X}_t, t^*F). \tag{38}$$

Proof. The Betti moduli space is topologically locally trivial over any disk contained in S. By the Non Abelian Hodge Theorem, the Dolbeault moduli space is also topologically locally trivial over our disk Δ. Let $M_D(\mathscr{X}_s, G) \times \Delta \cong M_D(X_\Delta/\Delta, G)$ be any topological trivialization. Then the intersection complex on $M_D(X_\Delta/\Delta, G)$ is the pull-back of the intersection complex of $M_D(\mathscr{X}_s, G)$ via the first projection associated with the chosen trivialization. Both the conclusions of the lemma follow.

$\square$

4.2. **Proof of Theorem 1.1.2 on the Hitchin morphism and specialization.**

Proof. We denote the Hitchin morphism (36) simply by $h : M \to \mathcal{V}$, we denote the structural S-morphisms by $\pi : M \to S$ and $\rho : \mathcal{V} \to S$. Let $\mathcal{IC}_M$ be the intersection complex of the Dolbeault moduli space M.

Our first goal is to verify that we are now in the situation of Theorem 3.2.1.(ii).

We set (X, Y, S, f, v, F) to be $(M, \mathcal{V}, S, h, \rho, \mathcal{IC}_M)$.

The requirement $\phi F = 0$ is met by Lemma 4.1.9.

As in the proof of Lemma 4.1.9, the intersection complex $\mathcal{IC}_M$ is, locally over the nonsingular S, the pull-back from of the intersection complex of a fiber. It follows that the direct image sheaves $R^\bullet \pi_* \mathcal{IC}_M = IH_D^\bullet(\mathscr{X}/S, G)$ are locally constant on S, with stalks the intersection cohomology groups of the fibers of $\pi : M \to S$.

By the decomposition theorem [Be-Be-De-1982], applied to the projective h and the simple perverse sheaf $\mathcal{IC}_M$, we have that the truncated ${}^p\tau_{\leq \bullet} h_* \mathcal{IC}_M$ are direct summands of the direct image $h_* F$. It follows that $\rho_* {}^p\tau_{\leq \bullet} h_* \mathcal{IC}_M$ are direct summands of $\pi_* \mathcal{IC}_M = \rho_* h_* \mathcal{IC}_M$ in $D(S)$.

By combining the two last paragraphs, we have that the $\rho_* {}^p\tau_{\leq \bullet} h_* \mathcal{IC}_M$ have locally constant cohomology sheaves.

We can thus apply Theorem 3.2.1.(ii) (or its weaker variant (iii)), the conclusion of which is that the specialization morphism $R\Gamma(M_s, i^* \mathcal{IC}_M) \to R\Gamma(M_t, t^* \mathcal{IC}_M)$ is defined and it is a filtered isomorphism for the respective perverse Leray filtrations.

Since, as it has been observed above, the intersection complex of M restricts to the intersection complexes of the fibers M_s and M_t, the first assertion of Theorem 1.1.2 follows.

The second assertion, i.e. the independence of $s \in S$ on a connected S, follows easily: pick any two points s, s' and a suitably general point $t \in S$. Then apply what we have proved to the pairs (s, t) and (s', t).

$\square$

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