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ON THE HIGH-LOW METHOD FOR NLS ON THE HYPERBOLIC SPACE

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Dedicated to the memory of Jean Bourgain

Abstract. In this paper, we first prove that the cubic, defocusing nonlinear Schrödinger equation on the two dimensional hyperbolic space with radial initial data in $H^s(\mathbb{H}^2)$ is globally well-posed and scatters when $s > \frac{3}{4}$. Then we extend the result to nonlineraities of order $p > 3$. The result is proved by extending the high-low method of Bourgain in the hyperbolic setting and by using a Morawetz type estimate proved by the first author and Ionescu.

1. Introduction

In this paper we consider the cubic nonlinear Schrödinger (NLS) initial value problem on the hyperbolic plane $\mathbb{H}^2$:

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^2} u = |u|^2 u, & t \in \mathbb{R}, \quad x \in \mathbb{H}^2, \\ u(0,x) = \phi(x), \end{cases} \tag{1.1}$$

where $u = u(t,x)$ is a complex-value function in spacetime $\mathbb{R} \times \mathbb{H}^2$ and ϕ is a radial initial datum.

The solution of (1.1) conserves both the mass:

$$M(u(t)) := \int_{\mathbb{H}^2} |u(t,x)|^2 \, dx = M(u_0), \tag{1.2}$$

and the energy:

$$E(u(t)) := \int_{\mathbb{H}^2} \frac{1}{2} |\nabla_{\mathbb{H}^2} u(t,x)|^2 + \frac{1}{4} |u(t,x)|^4 \, dx = E(u_0). \tag{1.3}$$

Conservation laws of mass and energy give the control of the L^2 and $\dot{H}^1$ norms of the solutions, respectively.

Our goal in this paper is to prove the global well-posedness and scattering of (1.1) with the regularity of the initial data below H^1.

In order to best frame the problem and to emphasize its challenges we start by recalling the results in $\mathbb{R}^d$, a setting that has been extensively considered in recent years. Consider the evolution equation in (1.4) with general non-linearities

$$i\partial_t u + \Delta u = |u|^{p-1} u, \quad p > 1 \tag{1.4}$$

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in $\mathbb{R}^d$. Let us first recall that the critical scaling exponent in $\mathbb{R}^d$ is

$$s_c := \frac{d}{2} - \frac{2}{p-1}. \tag{1.5}$$

It is well-known that in the sub-critical and critical regimes ($s > s_c$ and $s = s_c$, respectively), the initial value problem (1.4) is locally well-posed[1], [8, 9, 10, 7]. Thanks to the conservation laws of energy and mass, the H^1-subcritical initial value problem and the L^2-subcritical initial value problem are globally well-posed in the energy space H^1 and mass space L^2, respectively. The questions about scattering[2] are much more delicate.

Before we talk about global results in the more general subcritical case with data with regularity between L^2 (mass) and H^1 (energy), that is in H^s, $0 < s < 1$, let us denote with $g^p_{\mathcal{M}}$ the regularity index above which one obtains global well-posedness for the NLS problem on the manifold $\mathcal{M}$ with power nonlinearity p, and by $s^p_{\mathcal{M}}$ the index above which one obtains scattering (with the global well-posedness) again on the manifold $\mathcal{M}$ with power nonlinearity p.

The very first global well-posedness result in the subcritical case between the two (mass and energy) conservation laws ($0 < s < 1$) was given by Bourgain in [5], where he developed the high-low method to prove global well-posedness for the cubic ($p = 3$) NLS in two dimensions for initial data in H^s, $s > \frac{3}{5}$. According to the above notation the regularity index in [5] is $g^3_{\mathbb{R}^2} = \frac{3}{5}$.

We now describe the high-low method of Bourgain because it is the inspiration for part of our current work. To start the initial datum is decomposed into a (smoother) low frequency part and a (rougher) high frequency part. The first step is to solve the NLS globally for the smoother part, for which the energy is finite, and then solve a difference equation for the rougher part. The *miracle* in this argument, that allows one to continue with an iteration, is that in fact the Duhamel term in the solution to the difference equation is small and smoother in an interval of time that is inverse proportional to the size of the low frequency part of the initial datum. At the next iteration one merges this smoother part with the evolution of the low frequency part of the datum and repeats. It is worth mentioning that in order to obtain the *miracle* step, Bourgain used a Fourier transform based space $X^{s,b}$ [5] that captures particularly well the behavior of solutions with low regularity initial datum. Let us remark that there is no scattering result in the high-low method proposed by Bourgain.

In [12], Colliander-Keel-Staffilani-Takaoka-Tao improved the global well-posedness index $g^3_{\mathbb{R}^2}$ of the initial data to $\frac{4}{7}$ by introducing a different method, now known as I-method. This is also based on an iterative argument. One first defines a Fourier multiplier that smooths out the initial data into the energy space and proves that the energy of the smoothed solution is almost conserved, that is, at each iteration the growth of such modified energy is uniformly small. The index $g^3_{\mathbb{R}^2}$ is derived by keeping the accumulation of energy controlled. As a result, in [12] the authors obtained a polynomial growth of the sub-energy Sobolev norm of the global solution. The cubic NLS in $\mathbb{R}^3$ was also considered in [12] and the index $g^3_{\mathbb{R}^3} = \frac{5}{6}$.

[1]With *local well-posedness* we mean local in time existence, uniqueness and continuous dependence of the data to solution map.

[2]This will be made more precise later, see for example Theorem 1.1, but in general terms, with *scattering* we intend that the nonlinear solution as time goes to infinity approachs a linear one.

Later, in [13] by combining the Morawetz estimate with the I-method and a bootstrapping argument, the same authors were able to lower the index $g^3_{\mathbb{R}^3}$ to $\frac{4}{5}$ and proved, for the first time[3], that the global solution also scatters, hence $g^3_{\mathbb{R}^3} = s^3_{\mathbb{R}^3} = \frac{4}{5}$. To prove scattering, one needs to show that a spacetime norm of the solution is uniformly bounded. To this end, an iteration of local well-posedness would not suffice. Instead, one uses a Morawetz estimate that gives a uniform bound of the L^4 spacetime norm of the solution, combined with the I-method. More in details one splits the time line into a finite number of intervals I_j, of possible infinite length, on which the $L^4_{I_j}$ of the solution is small. The smallness allows for a better spacetime bound of the global solution on each interval I_j, and then one uses an iteration on the finite number of these intervals, which finally gives the desired spacetime uniform bound for the solution and hence scattering.

More results on the high-low method and the I-method both in $\mathbb{R}^d$ or compact manifolds can be found in [13, 16, 18, 19, 20, 21, 23, 24, 27, 28, 30, 31, 40, 43, 46].

We now consider the initial value problem

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^d} u = |u|^{p-1} u, & t \in \mathbb{R}, \quad x \in \mathbb{H}^d, \\ u(0,x) = \phi(x), \end{cases} \tag{1.6}$$

with $p > 1$. Compared to what we recalled above, we expect even better results in $\mathbb{H}^d$. In fact the negative curvature of the ambient manifold allows for more dispersion in $\mathbb{H}^d$ than in the Euclidean spaces. Mathematically we can see this in the Strichartz estimates on $\mathbb{H}^d$, a family of estimates that is broader than the one obtained for the Euclidean space, see [1, 35]. The fact that the family of Strichartz estimates is larger in $\mathbb{H}^d$ reminds us of another case in which this is true. In fact also for the wave equation the Strichartz estimates form a larger family. In this case though it is not the curvature of the ambient manifold that generates a larger number of estimates, but instead it is the fact that the wave operator has a strong smoothing effect pointwise in time, a property that is not enjoyed by the Schrödinger operator. As a consequence when one considers a nonlinear wave equation, the smoother and more plentiful estimates provide more suitable control of the nonlinear terms, and this is the reason why in the nonlinear wave setting, the *miracle* step in the high-low method in [38] does not need the Fourier type spaces $X^{s,b}$ mentioned above. However, in contrast, the larger range of the Strichartz estimates for the Schrödinger operator in the hyperbolic space still is not readily enough to handle the *miracle* step since although one obtains better spacetime estimates, there is no pointwise smoothing effect, hence the context we work in is more challenging than the one in [38]. At this point one may guess that using some hyperbolic version of the space $X^{s,b}$ may do the trick. While this is indeed the case when the problem is posed in $\mathbb{T}^d$, see for example [19], in $\mathbb{H}^d$ the space it is not clear how to define the Fourier transform based $X^{s,b}$ type spaces in a way that is useful to handle nonlinearities. A naive definition using the Helgason-Fourier transform in [32] is deficient because of the following two reasons: first, the eigenfunctions of the Laplace-Beltrami operator on $\mathbb{H}^d$ lead to a very different Fourier inversion formula and Plancherel theorem. In particular we cannot claim that the Fourier transform of a product is the convolution of Fourier transforms, which is a fundamental fact used in the estimates of nonlinear terms via the space $X^{s,b}$. Second, the frequency localization based on the Helgason Fourier transform does not behave well in $L^p(\mathbb{H}^d)$, which causes difficulties in defining an effective Littlewood-Paley decomposition. We

[3]Actually in [6] Bourgain proved the global well-posedness for general data with index $g^3_{\mathbb{R}^3} = \frac{11}{13}$ and scattering for radial data with index $s^3_{\mathbb{R}^3} = \frac{5}{7}$.

anticipate here that our approach to recover the *miracle* step, where one has to prove a gain of smoothness for the solution to the nonlinear difference equation, takes advantage of a Kato type smoothing effect. This smoothing is not pointwise in time, like for the wave operator, but in average in time, hence much weaker. In oder to make up for this weakness we need to use a maximal function estimate combined with a better Sobolev embedding, which in turn forces us to assume radial symmetry for our initial data. We expect though that our global well-posedness and scattering results are true in general and we believe that the more sophisticated smoothing effect in [39] may play an important role.

We now move to a summary of results that have been proved in the context of well-posedness and scattering for NLS in $\mathbb{H}^d$. Although the initial value problem (1.6) cannot be properly scaled, we still use the same index s_c defined in (1.5) to guide us in gauging the difficulty of proving global well-posedness and scattering for (1.6). The subcritical initial value problem in the hyperbolic setting was first considered in [3], where the authors proved scattering for a family of power-type nonlinearity NLS with radial H^1 data. In [4] the authors showed global well-posedness, scattering and blow-up results for energy-subcritical focusing[4] NLS also on the hyperbolic space. In the critical setting, in [34] the authors proved global well-posedness and scattering of the energy-critical NLS in $\mathbb{H}^3$. This result uses an ad hoc profile decomposition technique to transfer the already available result in $\mathbb{R}^3$ [15] into the $\mathbb{H}^3$ setting. A similar technique was used also in $\mathbb{T}^3$ for the same energy critical problem [33]. We do not think that this method, which is well suited for critical settings, could work in our subcritical setting, when the initial data are in $H^s(\mathbb{H}^2)$, $0 < s < 1$, but it may work to transfer in $\mathbb{H}^2$ the result that Dodson proved for mass critical in $\mathbb{R}^2$ [26]. To the best of the authors' knowledge, there are no known subcritical global well-posedness and scattering results with initial data not at the conservation law level in hyperbolic spaces.

We now state the main result of this work for the initial value problem (1.1). Later in Section 6 we state a similar result for the more general version (1.6) with $p > 3$.

Theorem 1.1. *The initial value problem* (1.1) *with radial initial data* $\phi \in H^s(\mathbb{H}^2)$ *with* $s > \frac{3}{4}$ *is globally-well-posed and scattering holds, that is there exists* $u_\pm \in H^s(\mathbb{H}^2)$ *such that*

$$\lim_{t \to \pm\infty} \left\| u(t) - e^{it\Delta_{\mathbb{H}^2}} u_\pm \right\|_{H_x^s(\mathbb{H}^2)} = 0. \tag{1.7}$$

Remark 1.2. Here we conduct a discussion on the indices of regularity for global well-posedness and we make a comparison with other results.

As discussed above, the equivalent case we consider here but in $\mathbb{R}^2$ was treated by Bourgain without redial symmetry using the $X^{s,b}$ space. Since we cannot use the same approach in $\mathbb{H}^2$, we decided first to rework this case using different tools such as Kato smoothing effect, maximal function estimates and better Sobolev embedding. We did this because in $\mathbb{R}^2$ we have a Littlewood-Paley decomposition that works very well. Using these tools in the implementation of the high-low method, we obtained that the cubic radial NLS is globally well posed when $s > \frac{4}{5}$, that is $g^3_{\mathbb{R}^2} = \frac{4}{5}$ (see Theorem A.5 in the Appendix). Recall that Bourgain's result gives $g^3_{\mathbb{R}^2} = \frac{3}{5}$, which is better than what we can do in $\mathbb{R}^2$, and it is for general data. But what we achieved in this first step is a blue print that is generalizable to the $\mathbb{H}^2$ space.

[4]An NLS is called focusing when the nonlinearity in (1.4) has a negative sigh, that is, $i\partial_t u + \Delta_{\mathbb{H}^2} u = - |u|^{p-1} u$.

One notes that the index $s^3_{\mathbb{H}^2} = \frac{3}{4}$ that we attained in Theorem 1.1 is smaller than the one we obtained in $\mathbb{R}^2$, where we worked out only the global well-posedness, not the scattering. This is because of the better radial Sobolev embedding in $\mathbb{H}^2$ and of the help coming from the strong Morawetz estimate used in the local theory.

Now a little bit of history concerning the indices of regularity for global well-posedness. In Bourgain's paper, where the high-low method was introduced [5], the global existence index is $g^3_{\mathbb{R}^2} = \frac{3}{5}$. Later in [12], where the I-method was used, the index $g^3_{\mathbb{R}^2}$ was improved to $\frac{4}{7}$, and later in [14], thanks to a sophisticated treatment of the Fourier multiplier involved in the I-mehtod, the global existence index $g^3_{\mathbb{R}^2}$ was lowered further to $\frac{1}{2}$. Global well-posedness of cubic NLS in two dimensions with $H^{\frac{1}{2}}$ data was proved in [28]. Also $g^3_{\mathbb{R}^2}$ was improved to $\frac{1}{3}$ in [16] and to $\frac{1}{4}$ in [21]. In [22, 25, 26], Dodson proved global well-posedness and scattering for the mass-critical NLS in any dimension. Also as a consequence, via the persistence of regularity property, mass-critical NLS equations with any subcritical initial data are globally well-posed as well.

1.1. **Blue print of the proof.** In this subsection we summarize the main three of the proof of the main Theorem 1.1. In general terms we combine the high-low method with a Morawetz type estimate that gives a bound for the spacetime L^4 norm.

The first part of the proof deals with the analysis of the *energy increment*. Following Bourgain's high-low method, we first decompose the initial datum into a high and a low frequency part. Then we write the solution u as the sum of the linear evolution of the high frequency part and a reminder ζ that solve a difference equation that evolves from the low frequency part of the original initial datum. In this first step we assume that in an interval $[0, \tau]$, where τ could be infinity, the L^4 spacetime norm of the solution is small. We then prove an estimate for the energy increment of ζ. This is the content of Proposition 3.1. To prove this energy increment estimate we further decompose $\zeta = \zeta_1 + \zeta_2$, where ζ_1 is the nonlinear solution starting from the low frequency part of the datum and ζ_2 solves a difference equation with zero datum. This part is similar to the high-low method of Bourgain, but here the interval of time is not small, the smallness comes from the L^4 norm. The *miracle* step is then to be able to show that ζ_2 is smoother and small in the appropriate norms. In the second part of the proof we assume that the total L^4 spacetime norm of the solution is bounded and we subdivide the time line into finitely many intervals in which this norm is small. Here we apply the first part describe above and we prove a global energy increment for ζ, this is Proposition 3.2. In the last part we use a bootstrapping argument to show that indeed the L^4 norms of the solution u is bounded. This part requires a modification of the Morawetz estimate in [35], see Proposition 4.1, and it uses the global energy increment proved in Proposition 3.2.

To summarize, the rest of this paper is organized as follows. In Section 2, we discuss the geometry of the domain $\mathbb{H}^2$ and collect the useful analysis tools in $\mathbb{H}^2$. In Section 3, we present the calculation of the energy increment of the smoother part of the solution. Next, in Section 4, we prove a modified Morawetz estimate, which will be used in Section 5. Finally, in Section 5, we run a bootstrapping argument based on the estimates derived from Sections 3 and 4 and complete the proof of Theorem 1.1.

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2. Preliminaries

2.1. Notations.
We define

$$\|f\|_{L_t^q L_x^r(I\times\mathbb{H}^2)} := \left[\int_I \left(\int_{\mathbb{H}^2} |f(t,x)|^r \, dx\right)^{\frac{q}{r}} dt\right]^{\frac{1}{q}},$$

where I is a time interval.

We use the Japanese bracket notation in the following sense:

$$\| \langle \Omega \rangle f\|_X = \|f\|_X + \|\Omega f\|_X ,$$

where X is one of the normed spaces we use below.

We adopt the usual notation that $A \lesssim B$ or $B \gtrsim A$ to denote an estimate of the form $A \leq CB$, for some constant $0 < C < \infty$ depending only on the a priori fixed constants of the problem.

2.2. Geometry of the domain $\mathbb{H}^2$.
We consider the Minkowshi space $\mathbb{R}^{2+1}$ with the standard Minkowski metric

$$-(dx^0)^2 + (dx^1)^2 + (dx^2)^2$$

and we define the bilinear form on $\mathbb{R}^{2+1} \times \mathbb{R}^{2+1}$,

$$[x, y] = x^0y^0 - x^1y^1 - x^2y^2.$$

The hyperbolic space $\mathbb{H}^2$ is defined as

$$\mathbb{H}^2 = \{x \in \mathbb{R}^{2+1} : [x, x] = 1 \text{ and } x^0 > 0\}.$$

An alternative definition for the hyperbolic space is

$$\mathbb{H}^2 = \{x = (t, s) \in \mathbb{R}^{2+1}, (t, s) = (\cosh r, \sinh r\omega), r \geq 0, \omega \in \mathbb{S}^1\}.$$

One has

$$dt = \sinh r \, dr, \quad ds = \cosh r\omega \, dr + \sinh r \, d\omega$$

and the metric induced on $\mathbb{H}^2$ is

$$dr^2 + \sinh^2 r \, d\omega^2,$$

where $d\omega^2$ is the metric on the sphere $\mathbb{S}^1$.

Then one can rewrite integrals as

$$\int_{\mathbb{H}^2} f(x) \, dx = \int_0^\infty \int_{\mathbb{S}^1} f(r, \omega) \sinh r \, dr d\omega.$$

The length of a curve

$$\gamma(t) = (\cosh r(t), \sinh r(t)\omega(t)),$$

with t varying from a to b, is defined

$$L(\gamma) = \int_a^b \sqrt{|\gamma'(t)|^2 + |\sinh r(t)|^2 \; |\omega'(t)|^2}\, dt.$$

Let $\mathbf{0} = \{(1,0,0)\}$ denote the origin of $\mathbb{H}^2$. The distance of a point to $\mathbf{0}$ is

$$d((\cosh r, \sinh r\omega), \mathbf{0}) = r.$$

More generally, the distance between two arbitrary points is

$$d(x, x') = \cosh^{-1}([x, x']).$$

The general definition of the Laplace-Beltrami operator is given by

$$\Delta_{\mathbb{H}^2} = \partial_r^2 + \frac{\cosh r}{\sinh r}\partial_r + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^1}.$$

Remark 2.1. The form of the Laplace-Beltrami operator implies that there will be no scaling symmetry in $\mathbb{H}^2$ as we usually have in the $\mathbb{R}^d$ setting.

2.3. **Tools on $\mathbb{H}^2$.** In this subsection we recall some important and classical analysis developed for the hyperbolic spaces.

2.3.1. *Fourier Transform on $\mathbb{H}^d$.* For $\theta \in \mathbb{S}^{d-1}$ and λ a real number, the functions of the type

$$h_{\lambda,\theta}(x) = [x, \Lambda(\theta)]^{i\lambda - \frac{d-1}{2}},$$

where $\Lambda(\theta)$ denotes the point of $\mathbb{R}^{d+1}$ given by $(1,\theta)$, are generalized eigenfunctions of the Laplacian-Beltrami operator. Indeed, we have

$$-\Delta_{\mathbb{H}^d} h_{\lambda,\theta} = \left(\lambda^2 + \frac{(d-1)^2}{4}\right) h_{\lambda,\theta}.$$

The Fourier transform on $\mathbb{H}^d$ is defined as

$$\hat{f}(\lambda,\theta) := \int_{\mathbb{H}^d} h_{\lambda,\theta}(x) f(x)\, dx,$$

and the Fourier inversion formula on $\mathbb{H}^d$ takes the form of

$$f(x) = \int_{-\infty}^{\infty}\int_{\mathbb{S}^{d-1}} \bar{h}_{\lambda,\theta}(x)\hat{f}(\lambda,\theta)\frac{d\theta d\lambda}{|c(\lambda)|^2},$$

where $c(\lambda)$ is the Harish-Chandra coefficient

$$\frac{1}{|c(\lambda)|^2} = \frac{1}{2(2\pi)^d}\frac{\left|\Gamma(i\lambda + \frac{d-1}{2})\right|^2}{|\Gamma(i\lambda)|^2}.$$

2.3.2. *Strichartz Estimates.* In this subsection we recall the Strichartz estimates proved in the hyperbolic space. We say that a couple (q,r) is admissible if $(\frac{1}{q},\frac{1}{r})$ belong to the triangle $T_d = \{(\frac{1}{q},\frac{1}{r}) \in (0,\frac{1}{2}] \times (0,\frac{1}{2}) \mid \frac{2}{q}+\frac{d}{r} \geq \frac{d}{2}\} \cup \{(0,\frac{1}{2})\}$. We have the following theorem.

Theorem 2.2 (Strichartz estimates in [1, 35])**.** *Assume u is the solution to the inhomogeneous initial value problem*

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^d} u = F, \quad t \in \mathbb{R}, \quad x \in \mathbb{H}^d, \\ u(0,x) = \phi(x). \end{cases} \tag{2.1}$$

Then, for any admissible exponents (q,r) and $(\tilde{q},\tilde{r})$ we have the Strichartz estimates:

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{H}^d)} \lesssim \|\phi\|_{L_x^2(\mathbb{H}^d)} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{H}^d)}.$$

Remark 2.3. Strichartz estimates are better in $\mathbb{H}^d$ in the sense that the set T_d of admissible pairs for $\mathbb{H}^d$ is much wider than the corresponding set I_d for $\mathbb{R}^d$ in (A.1) (which is just the lower edge of the triangle). See also Figure 1 below.

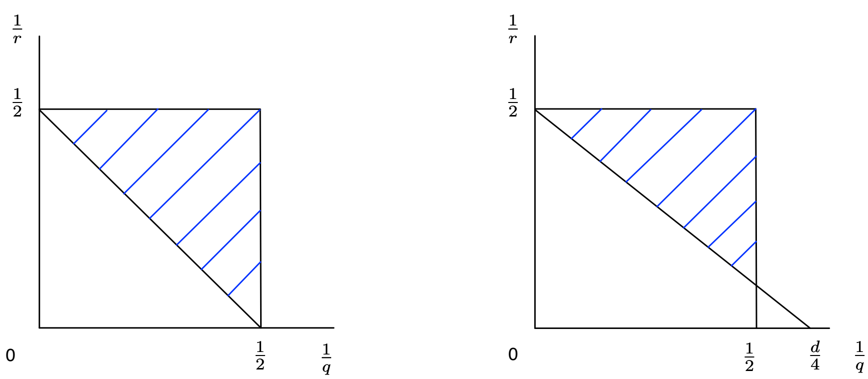

FIGURE 1. Strichartz admissible pair regions ($d = 2$ and $d \geq 3$) for the hyperbolic space.

Definition 2.4 (Strichartz spaces)**.** We define the Banach space

$$S^0(I) = \left\{ f \in C(I : L^2(\mathbb{H}^2)) : \|f\|_{S^0(I)} = \sup_{(q,r) \text{ admissible}} \|f\|_{L_t^q L_x^r(I\times\mathbb{H}^2)} < \infty \right\}.$$

Also we define the Banach space $S^\sigma(I)$, where $\sigma > 0$,

$$S^\sigma(I) = \left\{ f \in C(I : H^\sigma(\mathbb{H}^2)) : \|f\|_{S^\sigma(I)} = \left\| (-\Delta)^{\frac{\sigma}{2}} f \right\|_{S^0(I)} < \infty \right\}.$$

2.3.3. *Local Smoothing Estimates in the Hypebolic Space.*

Theorem 2.5 (Theorem 1.2 in [36]: Local Smoothing Estimates in $\mathbb{H}^2$)**.** *For any $\varepsilon > 0$,*

$$\left\| \langle x \rangle^{-\frac{1}{2}-\varepsilon} |\nabla|^{\frac{1}{2}} e^{it\Delta} f \right\|_{L_{t,x}^2(\mathbb{R}\times\mathbb{H}^2)} \lesssim \|f\|_{L_x^2(\mathbb{H}^2)},$$

$$\left\| \langle x \rangle^{-\frac{1}{2}-\varepsilon} \nabla \int_0^t e^{i(t-s)\Delta} F(s,x)\, ds \right\|_{L_{t,x}^2(\mathbb{R}\times\mathbb{H}^2)} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+\varepsilon} F \right\|_{L_{t,x}^2(\mathbb{R}\times\mathbb{H}^2)}.$$

Remark 2.6. In [36], the author considered more general manifolds that there are denoted with X. To obtain the theorem above one needs to take $p(\lambda) = |\lambda|^2$, $p(D) = -\Delta_X - |\rho|^2$ and $m = 2$.

2.3.4. *Heat-Flow-Based Littlewood-Paley Projections and Functional Inequalities on* $\mathbb{H}^2$*.* The Littlewood-Paley projections on $\mathbb{H}^2$ that we use in this paper are based on the linear heat equation $e^{s\Delta}$. It turned out in fact that for us this is a great substitute for the standard Littlewood-Paley decomposition used in $\mathbb{R}^d$, since in $\mathbb{H}^d$ one cannot localize in frequencies efficiently. We report below several results that first appeared in [39].

Definition 2.7 (Section 2.7.1 in [39]: Heat-flow-based Littlewood-Paley projections)**.** For any $s > 0$, we define

$$P_{\geq s} f = e^{s\Delta} f, \quad P_s f = s(-\Delta) e^{s\Delta} f.$$

By the fundamental theorem of calculus, it is straightforward to verify that

$$P_{\geq s} f = \int_s^\infty P_{s'} f \, \frac{ds'}{s'} \qquad \text{for } s > 0.$$

In particular, we have

$$f = \int_0^\infty P_{s'} f \, \frac{ds'}{s'},$$

which is the basic identity that relates f with its Littlewood-Paley resolution $\{P_s f\}_{s \in (0,\infty)}$. We also have

$$P_{\leq s} f = \int_0^s P_{s'} f \, \frac{ds'}{s'}.$$

Remark 2.8. Intuitively, $P_s f$ may be interpreted as a projection of f to frequencies comparable to $s^{-\frac{1}{2}}$. $P_{\geq s}$ and $P_{\leq s}$ can be viewed as the projections into low and high frequencies, respectively.

Lemma 2.9 (Lemma 2.5 in [39])**.** *Let* $1 < p < \infty$ *and* $p \leq q \leq \infty$*. Let* ρ_0 *satisfy*

$$0 < \rho_0^2 < \frac{1}{2} \min\{\frac{1}{p}, 1 - \frac{1}{p}\}.$$

For $f \in L^p_x(\mathbb{H}^2)$ *and* $s > 0$*, we have*

$$\left\| e^{s\Delta} f \right\|_{L^q_x(\mathbb{H}^2)} + \left\| s\Delta e^{s\Delta} f \right\|_{L^q_x(\mathbb{H}^2)} \lesssim s^{-(\frac{1}{p} - \frac{1}{q})} e^{-\rho_0^2 s} \left\| f \right\|_{L^p_x(\mathbb{H}^2)}.$$

Remark 2.10. In particular, if $p = q$ in Lemma 2.9,

$$\left\| e^{s\Delta} f \right\|_{L^p_x(\mathbb{H}^2)} + \left\| s\Delta e^{s\Delta} f \right\|_{L^p_x(\mathbb{H}^2)} \lesssim e^{-\rho_0^2 s} \left\| f \right\|_{L^p_x(\mathbb{H}^2)}.$$

That is,

$$\left\| P_{\geq s} f \right\|_{L^p_x(\mathbb{H}^2)} + \left\| P_s f \right\|_{L^p_x(\mathbb{H}^2)} \lesssim \left\| f \right\|_{L^p_x(\mathbb{H}^2)}.$$

Lemma 2.11 (Corollary 2.7 in [39])**.** *Let* $0 < \alpha < 1$ *and* $1 < p < \infty$*. For* $f \in L^p(\mathbb{H}^2)$*, we have*

$$\left\| s^\alpha (-\Delta)^\alpha e^{s\Delta} f \right\|_{L^p_x(\mathbb{H}^2)} \lesssim \left\| f \right\|_{L^p_x(\mathbb{H}^2)}.$$

Lemma 2.12 (Lemma 2.9 in [39]: Boundedness of Riesz transform)**.** *Let $f \in C_0^\infty(\mathbb{H}^2)$. Then for $1 < p < \infty$ we have*

$$\|\nabla f\|_{L_x^p(\mathbb{H}^2)} \simeq \left\|(-\Delta)^{\frac{1}{2}} f\right\|_{L_x^p(\mathbb{H}^2)} .$$

Lemma 2.13 (Lemma 2.10 in [39]: L^p interpolation inequalities)**.** *Let $f \in C_0^\infty(\mathbb{H}^2)$. Then for any $0 \leq \beta \leq \alpha$ and $1 < p < \infty$ we have*

$$\left\|(-\Delta)^\beta f\right\|_{L_x^p(\mathbb{H}^2)} \lesssim \|f\|_{L_x^p(\mathbb{H}^2)}^{1-\frac{\beta}{\alpha}} \|(-\Delta)^\alpha f\|_{L_x^p(\mathbb{H}^2)}^{\frac{\beta}{\alpha}} .$$

Moreover, for $1 < p < \infty$, $p \leq q \leq \infty$ and $0 < \theta = \frac{1}{\alpha}(\frac{1}{p} - \frac{1}{q}) < 1$, we have

$$\|f\|_{L_x^q(\mathbb{H}^2)} \lesssim \|f\|_{L_x^p(\mathbb{H}^2)}^{1-\theta} \|(-\Delta)^\alpha f\|_{L_x^p(\mathbb{H}^2)}^{\theta} .$$

Lemma 2.14 (Sobolev embedding)**.**

$$W^{s,p}(\mathbb{H}^d) \hookrightarrow L^p(\mathbb{H}^d), \quad \text{if } 1 < p \leq q < \infty \text{ and } \frac{1}{p} - \frac{1}{q} = \frac{s}{d}.$$

Lemma 2.15 (Lemma 2.12 in [39]: Gagliardo-Nirenberg inequality)**.** *Let $f \in C_0^\infty(\mathbb{H}^2)$. Then for any $1 < p < \infty$, $p \leq q \leq \infty$ and $0 < \theta < 1$ such that $\frac{1}{q} = \frac{1}{p} - \frac{\theta}{2}$, we have*

$$\|f\|_{L_x^q(\mathbb{H}^2)} \lesssim \|f\|_{L_x^p(\mathbb{H}^2)}^{1-\theta} \|\nabla f\|_{L_x^p(\mathbb{H}^2)}^{\theta} .$$

In particular, for any $s > 0$

$$\|f\|_{L_x^\infty(\mathbb{H}^2)} \lesssim \|f\|_{L_x^4(\mathbb{H}^2)}^{\frac{1}{2}} \|\nabla f\|_{L_x^4(\mathbb{H}^2)}^{\frac{1}{2}} \lesssim \|f\|_{L_x^2(\mathbb{H}^2)}^{\frac{1}{4}} \|\nabla f\|_{L_x^2(\mathbb{H}^2)}^{\frac{1}{2}} \|\Delta f\|_{L_x^2(\mathbb{H}^2)}^{\frac{1}{4}} \lesssim s^{-\frac{1}{2}} \|\langle s\Delta\rangle f\|_{L_x^2(\mathbb{H}^2)} .$$

Lemma 2.16 (Proposition 2.14 in [39]: Sobolev product rule)**.** *For $\sigma > 0$, we have*

$$\|fg\|_{H_x^\sigma(\mathbb{H}^2)} \lesssim \|f\|_{L_x^\infty(\mathbb{H}^2)} \|g\|_{H_x^\sigma(\mathbb{H}^2)} + \|f\|_{H_x^\sigma(\mathbb{H}^2)} \|g\|_{L_x^\infty(\mathbb{H}^2)} .$$

Lemma 2.17 (General Sobolev product rule)**.** *For $\sigma > 0$, we have*

$$\|fg\|_{W_x^{\sigma,r}(\mathbb{H}^2)} \lesssim \|f\|_{W_x^{\sigma,p_1}(\mathbb{H}^2)} \|g\|_{L_x^{p_2}(\mathbb{H}^2)} + \|f\|_{L_x^{q_1}(\mathbb{H}^2)} \|g\|_{W_x^{\sigma,q_2}(\mathbb{H}^2)} .$$

Remark 2.18. Lemma 2.17 allows more possible L^p norms than Lemma 2.16 in the product rule. The proof of Lemma 2.17 is using Triebel's argument in [45] (see for example Sections 7.2.2 and 7.2.4), and also can be found in Proposition 2.14 of [39]. This proof relies on a localization lemma (see for example Lemma 2.16 in [39]) to reduce to the standard Sobolev product rule.

Lemma 2.19 (Bernstein inequalities)**.** *For $0 \leq \beta < \alpha < \beta + 1$*

$$\left\|(-\Delta)^\beta P_{\leq s} f\right\|_{L_x^2(\mathbb{H}^2)} \lesssim s^{\alpha-\beta} \|(-\Delta)^\alpha f\|_{L_x^2(\mathbb{H}^2)} ,$$
$$\|(-\Delta)^\alpha P_{\geq s} f\|_{L_x^2(\mathbb{H}^2)} \lesssim s^{\beta-\alpha} \left\|(-\Delta)^\beta f\right\|_{L_x^2(\mathbb{H}^2)} .$$

Proof. Using Definition 2.7, Lemma 2.11 and the formal property $(-\Delta)^a(-\Delta)^b = (-\Delta)^{a+b}$,

$$\begin{aligned}\left\|(-\Delta)^\beta P_{\leq s} f\right\|_{L_x^2(\mathbb{H}^2)} &= \left\|\int_0^s (-\Delta)^\beta t(-\Delta) e^{t\Delta} f \,\frac{dt}{t}\right\|_{L_x^2(\mathbb{H}^2)} \lesssim \int_0^s \left\|(-\Delta)^{\beta+1} e^{t\Delta} f\right\|_{L_x^2(\mathbb{H}^2)} dt \\ &= \int_0^s t^{\alpha-\beta-1} \left\|t^{1+\beta-\alpha}(-\Delta)^{1+\beta-\alpha} e^{t\Delta}(-\Delta)^\alpha f\right\|_{L_x^2(\mathbb{H}^2)} dt\end{aligned}$$

$$\lesssim \int_0^s t^{\alpha-\beta-1} \|(-\Delta)^\alpha f\|_{L^2_x(\mathbb{H}^2)} \, dt = \int_0^s t^{\alpha-\beta-1} \, dt \, \|(-\Delta)^\alpha f\|_{L^2_x(\mathbb{H}^2)}$$
$$= s^{\alpha-\beta} \|(-\Delta)^\alpha f\|_{L^2_x(\mathbb{H}^2)} .$$

Similarly,

$$\|(-\Delta)^\alpha P_{\geq s} f\|_{L^2_x(\mathbb{H}^2)} = s^{\beta-\alpha} \left\| s^{\alpha-\beta} (-\Delta)^{\alpha-\beta} e^{s\Delta} (-\Delta)^\beta f \right\|_{L^2_x(\mathbb{H}^2)} \lesssim s^{\beta-\alpha} \left\| (-\Delta)^\beta f \right\|_{L^2_x(\mathbb{H}^2)} .$$

□

2.3.5. *Radial Sobolev Embeddings.*

Lemma 2.20 (Lemma 2.13 in [39]: Radial Sobolev embeddings in $\mathbb{H}^2$)**.** *For any $s > 0$ and any function f radial,*

$$\left\| \sinh^{\frac{1}{2}}(r) f \right\|_{L^\infty_x(\mathbb{H}^2)} \lesssim \|f\|^{\frac{1}{2}}_{L^2_x(\mathbb{H}^2)} \|\nabla f\|^{\frac{1}{2}}_{L^2_x(\mathbb{H}^2)} \lesssim s^{-\frac{1}{4}} \left\| \langle s\Delta \rangle^{\frac{1}{2}} f \right\|_{L^2_x(\mathbb{H}^2)} .$$

Corollary 2.21 (Frequency localized radial Sobolev embeddings in $\mathbb{H}^2$)**.** *For any $s > 0$ and f radial,*

$$\left\| \sinh^{\frac{1}{2}}(r) P_s f \right\|_{L^\infty_x(\mathbb{H}^2)} \lesssim s^{-\frac{1}{4}} \left\| P_{\frac{s}{2}} f \right\|_{L^2_x(\mathbb{H}^2)} .$$

Proof. Taking $f = P_s f$ in Lemma 2.20, we have

$$\left\| \sinh^{\frac{1}{2}}(r) P_s f \right\|_{L^\infty_x(\mathbb{H}^2)} \lesssim s^{-\frac{1}{4}} \left\| \langle s\Delta \rangle^{\frac{1}{2}} P_s f \right\|_{L^2_x(\mathbb{H}^2)} = s^{-\frac{1}{4}} \left(\|P_s f\|_{L^2_x(\mathbb{H}^2)} + \left\| (s\Delta)^{\frac{1}{2}} P_s f \right\|_{L^2_x(\mathbb{H}^2)} \right).$$

Both terms in parentheses are bounded by $\left\| P_{\frac{s}{2}} f \right\|_{L^2_x(\mathbb{H}^2)}$. In fact, by Definition 2.7, Remark 2.10 and Lemma 2.11,

$$\|P_s f\|_{L^2_x(\mathbb{H}^2)} = \left\| (s\Delta) e^{s\Delta} f \right\|_{L^2_x(\mathbb{H}^2)} = \left\| e^{\frac{s}{2}\Delta} (s\Delta e^{\frac{s}{2}\Delta} f) \right\|_{L^2_x(\mathbb{H}^2)} \lesssim \left\| P_{\frac{s}{2}} f \right\|_{L^2_x(\mathbb{H}^2)} ,$$
$$\left\| (s\Delta)^{\frac{1}{2}} P_s f \right\|_{L^2_x(\mathbb{H}^2)} = \left\| (s\Delta)^{\frac{1}{2}} (s\Delta) e^{s\Delta} f \right\|_{L^2_x(\mathbb{H}^2)} = \left\| (s\Delta)^{\frac{1}{2}} e^{\frac{s}{2}\Delta} (s\Delta e^{\frac{s}{2}\Delta} f) \right\|_{L^2_x(\mathbb{H}^2)} \lesssim \left\| P_{\frac{s}{2}} f \right\|_{L^2_x(\mathbb{H}^2)} .$$

□

Corollary 2.22. *For any $s > 0$, $\frac{1}{4} < \alpha < 1$ and f radial,*

$$\left\| \sinh^{\frac{1}{2}}(r) f \right\|_{L^\infty_x(\mathbb{H}^2)} \lesssim \|f\|^{1-\frac{1}{4\alpha}}_{L^2_x(\mathbb{H}^2)} \|(-\Delta)^\alpha f\|^{\frac{1}{4\alpha}}_{L^2_x(\mathbb{H}^2)} .$$

Proof. We write f into its Littlewood-Paley decomposition and use the frequency localized radial Sobolev embedding Corollary 2.21, then we have

$$\left| \sinh^{\frac{1}{2}}(r) f \right| = \left| \sinh^{\frac{1}{2}}(r) \int_0^\infty P_s f \, \frac{ds}{s} \right| = \left| \int_0^\infty \sinh^{\frac{1}{2}}(r) P_s f \, \frac{ds}{s} \right|$$
$$\lesssim \int_0^\infty s^{-\frac{1}{4}} \left\| P_{\frac{s}{2}} f \right\|_{L^2_x(\mathbb{H}^2)} \frac{ds}{s}$$
$$= \int_0^T s^{-\frac{1}{4}} \left\| P_{\frac{s}{2}} f \right\|_{L^2_x(\mathbb{H}^2)} \frac{ds}{s} + \int_T^\infty s^{-\frac{1}{4}} \left\| P_{\frac{s}{2}} f \right\|_{L^2_x(\mathbb{H}^2)} \frac{ds}{s} := I + II.$$

Here T is a constant that will be chosen later. Before calculating I and II, let us recall the following two estimates for $\|P_s f\|_{L^p_x(\mathbb{H}^2)}$: for $1 < p < \infty$ and $0 < \theta < 1$,

$$\|P_s f\|_{L^p_x(\mathbb{H}^2)} = \left\|s\Delta e^{s\Delta} f\right\|_{L^p_x(\mathbb{H}^2)} \lesssim \|f\|_{L^p_x(\mathbb{H}^2)}, \tag{2.2}$$

$$\|P_s f\|_{L^p_x(\mathbb{H}^2)} = \left\|s\Delta e^{s\Delta} f\right\|_{L^p_x(\mathbb{H}^2)} = \left\|s^\theta s^{1-\theta}(-\Delta)^{1-\theta} e^{s\Delta}(-\Delta)^\theta f\right\|_{L^p_x(\mathbb{H}^2)} \lesssim s^\theta \left\|(-\Delta)^\theta f\right\|_{L^p_x(\mathbb{H}^2)}. \tag{2.3}$$

Now by (2.3), for $\frac{1}{4} < \alpha < 1$

$$\begin{aligned} I &= \int_0^T s^{-\frac{1}{4}} \left\|P_{\frac{s}{2}} f\right\|_{L^2_x(\mathbb{H}^2)} \frac{ds}{s} \lesssim \int_0^T s^{-\frac{1}{4}} s^\alpha \left\|(-\Delta)^\alpha f\right\|_{L^2_x(\mathbb{H}^2)} \frac{ds}{s} \\ &= \left(\int_0^T s^{\alpha-\frac{1}{4}} \frac{ds}{s}\right) \left\|(-\Delta)^\alpha f\right\|_{L^2_x(\mathbb{H}^2)} = T^{\alpha-\frac{1}{4}} \left\|(-\Delta)^\alpha f\right\|_{L^2_x(\mathbb{H}^2)}. \end{aligned}$$

By (2.2)

$$II = \int_T^\infty s^{-\frac{1}{4}} \left\|P_{\frac{s}{2}} f\right\|_{L^2_x(\mathbb{H}^2)} \frac{ds}{s} \lesssim \left(\int_T^\infty s^{-\frac{1}{4}} \frac{ds}{s}\right) \|f\|_{L^2_x(\mathbb{H}^2)} = T^{-\frac{1}{4}} \|f\|_{L^2_x(\mathbb{H}^2)}.$$

Therefore, for any $T > 0$

$$\left\|\sinh^{\frac{1}{2}}(r) f\right\|_{L^\infty_x(\mathbb{H}^2)} \lesssim T^{\alpha-\frac{1}{4}} \left\|(-\Delta)^\alpha f\right\|_{L^2_x(\mathbb{H}^2)} + T^{-\frac{1}{4}} \|f\|_{L^2_x(\mathbb{H}^2)}.$$

Optimizing the choice of T, we obtain

$$\left\|\sinh^{\frac{1}{2}}(r) f\right\|_{L^\infty_x(\mathbb{H}^2)} \lesssim \left\|(-\Delta)^\alpha f\right\|^{\frac{1}{4\alpha}}_{L^2_x(\mathbb{H}^2)} \|f\|^{1-\frac{1}{4\alpha}}_{L^2_x(\mathbb{H}^2)}.$$

□

3. Energy increment on $\mathbb{H}^2$

In this section we analyze a certain energy increment. As mentioned in the introduction we present a modified Morawetz type estimate in Section 5, and in Section 6 we conclude the global well-posedness and scattering proof by showing that the space-time L^4 norm of the solution is uniformly bounded.

Let us recall schematically below the heat-flow-based Littlewood-Paley projections:

$$\begin{aligned} P_s f = s(-\Delta)e^{s\Delta} f &\rightsquigarrow \text{ a projection to frequencies comparable to } s^{-\frac{1}{2}}, \\ P_{\geq s} f = e^{s\Delta} f = \int_s^\infty P_{s'} f \frac{ds'}{s'} &\rightsquigarrow \text{ a projection to frequencies lower than } \sim s^{-\frac{1}{2}}, \\ P_{\leq s} f = \int_0^s P_{s'} f \frac{ds'}{s'} &\rightsquigarrow \text{ a projection to frequencies higher than } \sim s^{-\frac{1}{2}}. \end{aligned}$$

Now we decompose the initial data ϕ into a low frequency component $\eta_0 = P_{>s_0}\phi$ and a high frequency component $\psi_0 = P_{\leq s_0}\phi$, where s_0^{-1} is a fixed large frequency and will be determined later in the proof. Note that $s_0^{-\frac{1}{2}}$ plays the same role as N_0 in [5].

Using the decomposition above, we would like to write u into the sum of the following two solutions ψ and ζ, where $\psi = e^{it\Delta} P_{\leq s_0}\phi$ solves the linear Schrödinger with high frequency data

$$\begin{cases} i\partial_t \psi + \Delta_{\mathbb{H}^2}\psi = 0, \\ \psi(0,x) = \psi_0 = P_{\leq s_0}\phi, \end{cases} \tag{3.1}$$

and ζ solves the difference equation with low frequency data

$$\begin{cases} i\partial_t \zeta + \Delta_{\mathbb{H}^2}\zeta = |u|^2 u = G(\zeta,\psi), \\ \zeta(0,x) = \eta_0 = P_{> s_0}\phi, \end{cases} \tag{3.2}$$

here $G(\zeta,\psi) = |\zeta+\psi|^2(\zeta+\psi) = |\zeta|^2\zeta + \mathcal{O}(\zeta^2\psi) + \mathcal{O}(\zeta\psi^2) + \mathcal{O}(\psi^3)$.

3.1. **Main results in the section.** The main results in this section are a local energy increment (Proposition 3.1) and a conditional global energy increment (Proposition 3.2) for the solution ζ.

Proposition 3.1 (Local energy increment)**.** *Consider u as in* (1.1) *defined on $I \times \mathbb{H}^2$ where $I = [0,\tau]$, such that*

$$\|u\|^4_{L^4_{t,x}(I\times\mathbb{H}^2)} = \varepsilon \tag{3.3}$$

for some universal constant ε. Then for $s > \frac{3}{4}$ and sufficiently small s_0, the solution ζ, under the decomposition $u = \psi + \zeta$ defined as in (3.1) *and* (3.2)*, satisfies the following energy increment*

$$E(\zeta(\tau)) \leq E(\zeta(0)) + C s_0^{\frac{3}{2}s - \frac{5}{4}}.$$

Proposition 3.2 (Conditional global energy increment)**.** *Consider u as in* (1.1) *defined on $[0,T] \times \mathbb{H}^2$ where*

$$\|u\|^4_{L^4_{t,x}([0,T]\times\mathbb{H}^2)} \leq M$$

for some constant M. Then for $s > \frac{3}{4}$ and sufficiently small s_0, the energy of ζ satisfies the following energy increment

$$E(\zeta(T)) \leq E(\zeta(0)) + C\frac{M}{\varepsilon} s_0^{\frac{3}{2}s - \frac{5}{4}}.$$

where ε is the small constant in Proposition 3.1.

Remark 3.3. T could be infinity. In fact, the ultimate goal of this paper is to show that the spacetime L^4 norm is bounded for all time intervals, which implies scattering.

3.2. **Proof of Proposition 3.1.** To analyze the behavior of the solution ζ more carefully, we first make a further decomposition. That is, we would like to separate the differential equation (3.2) into a cubic NLS with low frequency data,

$$\begin{cases} i\partial_t \zeta_1 + \Delta_{\mathbb{H}^2}\zeta_1 = |\zeta_1|^2\zeta_1, \\ \zeta_1(0,x) = \eta_0 = P_{> s_0}\phi, \end{cases} \tag{3.4}$$

and a difference equation with zero initial value,

$$\begin{cases} i\partial_t \zeta_2 + \Delta_{\mathbb{H}^2}\zeta_2 = |u|^2 u - |\zeta_1|^2\zeta_1, \\ \zeta_2(0,x) = 0. \end{cases} \tag{3.5}$$

Hence $\zeta = \zeta_1 + \zeta_2$ and the full solution u is the sum of these three solutions:

$$u = \zeta_1 + \zeta_2 + \psi. \tag{3.6}$$

It is worth mentioning that the decomposition in Bourgain's work [5] is a cubic NLS with low frequency data,

$$\begin{cases} i\partial_t u_0 + \Delta u_0 = \left|u_0\right|^2 u_0, \\ u_0(0,x) = \phi_0(x) = P_{<N_0}\phi. \end{cases} \tag{3.7}$$

and a difference equation with high frequency data

$$\begin{cases} i\partial_t v + \Delta v = F(u_0, v), \\ v(0,x) = \psi_0(x) = P_{\geq N_0}\phi, \end{cases} \tag{3.8}$$

where $F(u_0,v) = \left|v\right|^2 v + 2u_0 \left|v\right|^2 + \bar{u}_0 v^2 + u_0^2 \bar{v} + 2\left|u_0\right|^2 v$. Then the full solution u is $u = u_0 + v$. In our work we need to be more careful.

Notice that $s_0^{-\frac{1}{2}}$ plays the same role as N_0 in [5]. When comparing these two decomposition, we can relate them in the following sense: ζ_1 is the same as u_0 and $\psi + \zeta_2$ is in fact v, where ψ is the linear solution in (3.8) and ζ_2 is the Duhamel term w in (3.8).

$$\zeta_1 \leftrightsquigarrow u_0 \qquad \zeta_2 \leftrightsquigarrow w \qquad \psi \leftrightsquigarrow e^{it\Delta}P_{\leq s_0}\phi.$$

Step 1: Understanding the decomposed initial data. Recall that we decomposed the initial data $\phi = \eta_0 + \psi_0$, where $\eta_0 = P_{>s_0}\phi$ and $\psi_0 = P_{\leq s_0}\phi$. Here we list several facts of the decomposed initial data η_0 and ψ_0.

Fact 3.4. For the low frequency data η_0,

(1) $\eta_0 \in H^1$, and $\left\|\eta_0\right\|_{H_x^1(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-1)}$,
(2) $\left\|\eta_0\right\|_{H_x^\sigma(\mathbb{H}^2)} \lesssim s_0^{\frac{\sigma}{2}(s-1)}$ for $0 < \sigma < 1$,
(3) $E(\eta_0) \lesssim s_0^{s-1}$.

In fact, by Bernstein inequality (Lemma 2.19),

$$\left\|\eta_0\right\|_{H_x^1(\mathbb{H}^2)} = \left\|P_{>s_0}\phi\right\|_{H_x^1(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-1)} \left\|P_{>s_0}\phi\right\|_{H_x^s(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-1)}.$$

This gives *(1)*. *(2)* follows from Lemma 2.13 by interpolating L^2 and H^1 norms,

$$\left\|\eta_0\right\|_{\dot{H}_x^\sigma(\mathbb{H}^2)} \lesssim \left\|\eta_0\right\|_{L_x^2(\mathbb{H}^2)}^{1-\sigma} \left\|\eta_0\right\|_{H_x^1(\mathbb{H}^2)}^{\sigma} \lesssim s_0^{\frac{\sigma}{2}(s-1)}.$$

Then by Sobolev embedding (Lemma 2.14) and *(2)*, we see that

$$E(\eta_0) \lesssim \left\|\eta_0\right\|_{\dot{H}_x^1(\mathbb{H}^2)}^2 + \left\|\eta_0\right\|_{L_x^4(\mathbb{H}^2)}^4 \lesssim \left\|\eta_0\right\|_{\dot{H}_x^1(\mathbb{H}^2)}^2 + \left\|\eta_0\right\|_{H_x^{\frac{1}{2}}(\mathbb{H}^2)}^4 \lesssim s_0^{s-1}.$$

Fact 3.5. For the high frequency data ψ_0,

(1) $\left\|\psi_0\right\|_{L_x^2(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s}$,
(2) $\left\|\psi_0\right\|_{H_x^s(\mathbb{H}^2)} \lesssim 1$.

Here *(1)* follows from Bernstein inequality (Lemma 2.19),

$$\|\psi_0\|_{L^2_x(\mathbb{H}^2)} = \|P_{\leq s_0}\phi\|_{L^2_x(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s} \|\phi\|_{H^s_x(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s},$$

while *(2)* is due to the fact that ϕ being in H^s.

Step 2: Estimation on the solution ψ of (3.1).

In fact, the solution ψ of the linear equation (3.1) is global, although it lives in a rough space H^s. Moreover, from the linear Strichartz estimates, Lemma 2.19 and *(1)* in Fact 3.5 one has

$$\|\psi\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{H}^2)} = \left\|e^{it\Delta}\psi_0\right\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{H}^2)} \lesssim \|\psi_0\|_{L^2_x(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s}. \tag{3.9}$$

More generally,

$$\|\psi\|_{S^0(\mathbb{R})} \lesssim s_0^{\frac{1}{2}s} \text{ and } \|\psi\|_{S^\sigma(\mathbb{R})} \lesssim s_0^{\frac{1}{2}(s-\sigma)} \text{ for } 0 \leq \sigma \leq s. \tag{3.10}$$

Step 3: Estimation on the solution ζ_1 of (3.4).

Lemma 3.6. *Due to the low frequency component $\eta_0(x)$ of ϕ being in H^1, $\zeta_1(t)$ is a global solution and $\|\zeta_1(t)\|_{H^1_x(\mathbb{H}^2)}$ is conserved. More precisely,*

(1) $\zeta_1(t)$ exists globally, and $E(\zeta_1)(t) = E(\eta_0) \lesssim s_0^{s-1}$,
(2) $\|\zeta_1\|^4_{L^4_{t,x}(I\times\mathbb{H}^2)} \lesssim \varepsilon + s_0^{2s} + \|\zeta_2\|^4_{L^4_{t,x}(I\times\mathbb{H}^2)}$

Remark 3.7. Ultimately, we will show $\|\zeta_1\|^4_{L^4_{t,x}(I\times\mathbb{H}^2)} \lesssim \varepsilon$ in Corollary 3.9, and here *(2)* is an intermediate step.

Proof of Lemma 3.6. First, with the conservation of $E(\zeta_1)$ and *(3)* in Fact 3.4, it is easy to see that

$$E(\zeta_1(t)) \equiv E(\eta_0) \lesssim s_0^{s-1}.$$

With the initial data η_0 being in H^1, thanks to [35], ζ_1 is globally well-posed, which proves *(1)*.

For *(2)*, recall the decomposition of u in (3.6), then we simply use the triangle inequality, the assumption (3.3), and (3.9) and obtain

$$\|\zeta_1\|^4_{L^4_{t,x}(I\times\mathbb{H}^2)} \lesssim \|u\|^4_{L^4_{t,x}(I\times\mathbb{H}^2)} + \|\psi\|^4_{L^4_{t,x}(I\times\mathbb{H}^2)} + \|\zeta_2\|^4_{L^4_{t,x}(I\times\mathbb{H}^2)} \lesssim \varepsilon + s_0^{2s} + \|\zeta_2\|^4_{L^4_{t,x}(I\times\mathbb{H}^2)}.$$

□

Step 4: Estimation on the solution ζ_2 of (3.5) and extra estimates on ζ_1.

Recall (3.5)

$$\begin{cases} i\partial_t\zeta_2 + \Delta_{\mathbb{H}^2}\zeta_2 = F(\zeta_1,\zeta_2,\psi), \\ \zeta_2(0,x) = 0, \end{cases}$$

where

$$F(\zeta_1,\zeta_2,\psi) = |u|^2 u - |\zeta_1|^2\zeta_1 = \mathcal{O}((\zeta_1+\zeta_2+\psi)^3) - |\zeta_1|^2\zeta_1$$

$$= \mathcal{O}(\zeta_2^3) + \mathcal{O}(\psi^3) + \mathcal{O}(\zeta_2^2\zeta_1) + \mathcal{O}(\psi^2\zeta_1) + \mathcal{O}(\zeta_2\zeta_1^2) + \mathcal{O}(\psi\zeta_1^2).$$

Lemma 3.8. *The solution ζ_2 satisfies the following estimates on I:*

(1) $\|\zeta_2\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s}$,
(2) $\|\zeta_2\|_{L^\infty_t L^2_x(I\times\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s}$,
(3) $\|\zeta_2\|_{L^\infty_t H^1_x(I\times\mathbb{H}^2)} \lesssim s_0^{s-\frac{3}{4}}$.

Proof of Lemma 3.8. Noticing that (3.5) has zero initial value, we write out the integral equation using its Duhamel formula

$$\zeta_2(t) = i\int_0^t e^{i(t-s)\Delta_{\mathbb{H}^2}} F\, ds.$$

By Strichartz estimates and Hölder inequality, we have

$$\begin{aligned}\|\zeta_2\|_{L^4_{t,x}(I\times\mathbb{H}^2)} &\lesssim \|F\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} \lesssim \|\zeta_2\|^3_{L^4_{t,x}(I\times\mathbb{H}^2)} + \|\psi\|^3_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ &\quad + \|\zeta_2\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \|\zeta_1\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)} + \|\psi\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \|\zeta_1\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)}.\end{aligned} \tag{3.11}$$

Note that here there should be two more nonlinear terms in F that contribute to $\|F\|_{L^{\frac{4}{3}}_{t,x}}$ in (3.11), which are $\mathcal{O}(\zeta_2^2\zeta_1)$ and $\mathcal{O}(\psi^2\zeta_1)$. But we dropped them, since their contributions are controlled by a multiple of those of the four nonlinear terms that are written in (3.11). We will also drop them in the rest of the paper.

Using (3.9) and *(2)* in Lemma 3.6, we write (3.11) into

$$\begin{aligned}\|\zeta_2\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \lesssim \|\zeta_2\|^3_{L^4_{t,x}(I\times\mathbb{H}^2)} + s_0^{\frac{3}{2}s} + \|\zeta_2\|_{L^4_{t,x}(I\times\mathbb{H}^2)} (\varepsilon^{\frac{1}{2}} + s_0^s + \|\zeta_2\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)}) \\ + s_0^{\frac{1}{2}s}(\varepsilon^{\frac{1}{2}} + s_0^s + \|\zeta_2\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)}).\end{aligned}$$

Noticing that initially $\zeta_2(0) = 0$, then by a continuity argument, we obtain

$$\|\zeta_2\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s},$$

which proves *(1)*. The estimate in (3.11) also works for $\|\zeta_2\|_{L^\infty_t L^2_x(I\times\mathbb{H}^2)}$, hence *(2)* holds.

We postpone the proof of *(3)* to Step 6.

□

With enough estimates on ζ_2 in hand, as we promised in Lemma 3.6, we will finish the analysis of ζ_1.

Corollary 3.9. *As a consequence of (1) in Lemma 3.8,*

(1) we improve the bound in Lemma 3.6 by $\|\zeta_1\|^4_{L^4_{t,x}(I\times\mathbb{H}^2)} \lesssim \varepsilon + s_0^{2s} \lesssim \varepsilon$,
(2) and obtain $\|\zeta_1\|_{S^\sigma(I)} \lesssim s_0^{\frac{\sigma}{2}(s-1)}$ *where* $0 \leq \sigma \leq 1$.

Proof of Corollary 3.9. Combining *(2)* in Lemma 3.6 and *(1)* in Lemma 3.8, it is easy to see that *(1)* holds.

For *(2)*, we use the integral equation corresponding to the initial value problem via the Duhamel principle, Strichartz estimates and *(1)*, and we obtain

$$\|\zeta_1\|_{S^0(I)} \lesssim \|\eta_0\|_{L^2_x(\mathbb{H}^2)} + \left\| |\zeta_1|^2 \zeta_1 \right\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} \lesssim 1 + \|\zeta_1\|^3_{L^4_{t,x}(I\times\mathbb{H}^2)} \lesssim 1 + \varepsilon^{\frac{3}{4}},$$

$$\|\zeta_1\|_{S^1(I)} \lesssim \|\nabla\eta_0\|_{L^2_x(\mathbb{H}^2)} + \left\|\nabla\, |\zeta_1|^2 \zeta_1 \right\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-1)} + \|\zeta_1\|_{S^1(I)}\, \varepsilon^{\frac{1}{2}}.$$

Therefore

$$\|\zeta_1\|_{S^0(I)} \lesssim 1, \tag{3.12}$$

and

$$\|\zeta_1\|_{S^1(I)} \lesssim s_0^{\frac{1}{2}(s-1)}. \tag{3.13}$$

Then interpolating (3.12) and (3.13), we have that for $0 < \sigma < 1$

$$\|\zeta_1\|_{S^\sigma(I)} \lesssim s_0^{\frac{\sigma}{2}(s-1)}.$$

□

Step 5: Local energy increment.

Now we are ready to compute the energy increment from 0 to τ and show such increment is as described in Proposition 3.1. That is, we will show

$$\begin{aligned} E(\zeta(\tau)) &= E(\zeta_1(\tau) + \zeta_2(\tau)) \\ &= E(\zeta_1(\tau)) + \left(E(\zeta_1(\tau) + \zeta_2(\tau)) - E(\zeta_1(\tau))\right) \\ &\le E(\eta_0) + C s_0^{\frac{3}{2}s - \frac{5}{4}}. \end{aligned}$$

In fact, a direct computation of the difference of the energy gives

$$\begin{aligned} &|E(\zeta_1(\tau) + \zeta_2(\tau)) - E(\zeta_1(\tau))| \\ \lesssim\ & \left(\|\zeta_1(\tau)\|_{H^1_x(\mathbb{H}^2)} + \|\zeta_2(\tau)\|_{H^1_x(\mathbb{H}^2)} \right) \|\zeta_2(\tau)\|_{H^1_x(\mathbb{H}^2)} + \left\| \left(|\zeta_1(\tau)| + |\zeta_2(\tau)| \right)^3 |\zeta_2(\tau)| \right\|_{L^1_x(\mathbb{H}^2)} \\ :=\ & I + II. \end{aligned}$$

By the energy conservation of ζ_1, *(1)* in Lemma 3.6 and *(3)* in Lemma 3.8

$$\begin{aligned} I &= \left(\|\zeta_1(\tau)\|_{H^1_x(\mathbb{H}^2)} + \|\zeta_2(\tau)\|_{H^1_x(\mathbb{H}^2)} \right) \|\zeta_2(\tau)\|_{H^1_x(\mathbb{H}^2)} \\ &\le E(\eta_0)^{\frac{1}{2}} \|\zeta_2(\tau)\|_{H^1_x(\mathbb{H}^2)} + \|\zeta_2(\tau)\|^2_{H^1_x(\mathbb{H}^2)} \\ &\lesssim s_0^{\frac{1}{2}(s-1)} \cdot s_0^{s-\frac{3}{4}} + s_0^{2(s-\frac{3}{4})} \\ &\lesssim s_0^{\frac{3}{2}s-\frac{5}{4}}, \end{aligned}$$

for $s > \frac{1}{2}$. ($\|\zeta_1(\tau)\|_{H^1_x} \|\zeta_2(\tau)\|_{H^1_x}$ dominates in I.)

Using Sobolev embedding, Lemma 2.13, *(2), (3)* in Lemma 3.8 and *(1)* in Lemma 3.6, we have the following L^4 norm estimates for ζ_1 and ζ_2

$$\|\zeta_2(\tau)\|_{L^4_x(\mathbb{H}^2)} \lesssim \|\zeta_2(\tau)\|_{H^{\frac{1}{2}}_x(\mathbb{H}^2)} \lesssim \|\zeta_2(\tau)\|^{\frac{1}{2}}_{L^2_x(\mathbb{H}^2)} \|\zeta_2(\tau)\|^{\frac{1}{2}}_{H^1_x(\mathbb{H}^2)} \lesssim s_0^{\frac{3}{4}s-\frac{3}{8}},$$
$$\|\zeta_1(\tau)\|_{L^4_x(\mathbb{H}^2)} \lesssim E(\zeta_1)^{\frac{1}{4}} \lesssim s_0^{\frac{1}{4}(s-1)}.$$

Combining with Hölder inequality, we compute

$$\begin{aligned} II &= \left\| \left(|\zeta_1(\tau)| + |\zeta_2(\tau)| \right)^3 |\zeta_2(\tau)| \right\|_{L^1_x(\mathbb{H}^2)} \\ &\lesssim \left(\|\zeta_1(\tau)\|_{L^4_x(\mathbb{H}^2)} + \|\zeta_2(\tau)\|_{L^4_x(\mathbb{H}^2)} \right)^3 \|\zeta_2(\tau)\|_{L^4_x(\mathbb{H}^2)} \\ &\lesssim (s_0^{\frac{1}{4}(s-1)} + s_0^{\frac{3}{4}s-\frac{3}{8}})^3 \cdot s_0^{\frac{3}{4}s-\frac{3}{8}} \\ &\lesssim s_0^{\frac{3}{2}s-\frac{9}{8}}, \end{aligned}$$

for $s > \frac{1}{4}$. ($\|\zeta_1(\tau)\|^3_{L^4_x(\mathbb{H}^2)} \|\zeta_2(\tau)\|_{L^4_x(\mathbb{H}^2)}$ dominates in II, and I dominates in $E(\zeta(\tau)) - E(\zeta(0))$.)

Now we finish the calculation of the analysis of the energy increment in Proposition 3.1.

Step 6: Proof of *(3)* in Lemma 3.8.

Before proving *(3)* in Lemma 3.8, we first state the following lemma,

Lemma 3.10. *For $t \in I$ defined in* (3.3) *we have for* $0 \leq \sigma \leq s$

$$\|\zeta_2\|_{S^\sigma(I)} \lesssim s_0^{\frac{1}{2}(\sigma s+s-\sigma)}. \tag{3.14}$$

Proof of Lemma 3.10. For $0 < \sigma < s$, by the integral equation, Bernstein inequality (Lemma 2.19), Lemma 2.17, Strichartz inequalities and (3.3)

$$\begin{aligned} \|\zeta_2\|_{S^\sigma(I)} \lesssim & \left\| \langle -\Delta \rangle^{\frac{\sigma}{2}} \mathcal{O}(\zeta_2^3) \right\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} + \left\| \langle -\Delta \rangle^{\frac{\sigma}{2}} \mathcal{O}(\psi^3) \right\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} \\ & + \left\| \langle -\Delta \rangle^{\frac{\sigma}{2}} \mathcal{O}(\zeta_2\zeta_1^2) \right\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} + \left\| \langle -\Delta \rangle^{\frac{\sigma}{2}} \mathcal{O}(\psi\zeta_1^2) \right\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} \\ \lesssim & \|\zeta_2\|_{S^\sigma(I)} \|\zeta_2\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)} + \|\psi\|_{S^\sigma(I)} \|\psi\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ & + \|\zeta_2\|_{S^\sigma(I)} \|\zeta_1\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)} + \|\zeta_1\|_{S^\sigma(I)} \|\zeta_1\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \|\zeta_2\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ & + \|\zeta_2\|_{S^\sigma(I)} \|\zeta_1\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)} + \|\zeta_1\|_{S^\sigma(I)} \|\zeta_1\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \|\psi\|_{L^4_{t,x}(I\times\mathbb{H}^2)}. \end{aligned} \tag{3.15}$$

Using *(1)* in Lemma 3.8, (3.9), (3.10) and Corollary 3.9, we write (3.15) into

$$\|\zeta_2\|_{S^\sigma(I)} \lesssim \|\zeta_2\|_{S^\sigma(I)} s_0^s + s_0^{\frac{1}{2}(s-\sigma)} s_0^s + \|\zeta_2\|_{S^\sigma(I)} \varepsilon^{\frac{1}{2}} + s_0^{\frac{\sigma}{2}(s-1)} \varepsilon^{\frac{1}{4}} s_0^{\frac{1}{2}s} + \|\zeta_2\|_{S^\sigma(I)} \varepsilon^{\frac{1}{2}} + s_0^{\frac{\sigma}{2}(s-1)} \varepsilon^{\frac{1}{4}} s_0^{\frac{1}{2}s}.$$

Then we have

$$\|\zeta_2\|_{S^\sigma(I)} \lesssim s_0^{\frac{\sigma}{2}(s-1)} s_0^{\frac{1}{2}s} = s_0^{\frac{1}{2}(\sigma s+s-\sigma)}.$$

□

Finally, we arrive at the proof of *(3)* in Lemma 3.8.

Proof of (3) in Lemma 3.8. In this step, we prove the smoothness of the solution ζ_2 using the local smoothing estimate and the radial assumption of the initial data. In fact, this is the only place where the radial assumption is used, and all other steps work for all general data.

First, by Strichartz inequalities, we write

$$\begin{aligned}\|\nabla\zeta_2\|_{L_t^\infty L_x^2(I\times\mathbb{H}^2)} &\lesssim \left\|\nabla\mathcal{O}(\zeta_2^3)\right\|_{L_{t,x}^{\frac{4}{3}}(I\times\mathbb{H}^2)} + \left\|\nabla\mathcal{O}(\psi^3)\right\|_{L_{t,x}^{\frac{4}{3}}(I\times\mathbb{H}^2)} \\ &\quad + \left\|\nabla\mathcal{O}(\zeta_2\zeta_1^2)\right\|_{L_{t,x}^{\frac{4}{3}}(I\times\mathbb{H}^2)} + \left\|\nabla\mathcal{O}(\psi\zeta_1^2)\right\|_{L_{t,x}^{\frac{4}{3}}(I\times\mathbb{H}^2)} \\ &:= I + II + III + IV.\end{aligned} \tag{3.16}$$

Before estimating I, II, III and IV, we compute the following norms that are needed in the rest of the proof:

Claim 3.11. For $r = |x|$,

$$\begin{aligned}\left\|\sinh^{\frac{1}{2}}(r)\psi\right\|_{L_{t,x}^\infty(I\times\mathbb{H}^2)} &\lesssim s_0^{\frac{1}{2}s-\frac{1}{4}}, \\ \left\|\sinh^{\frac{1}{2}}(r)\zeta_1\right\|_{L_{t,x}^\infty(I\times\mathbb{H}^2)} &\lesssim s_0^{\frac{1}{4}(s-1)}, \\ \left\|\sinh^{\frac{1}{2}}(r)\zeta_2\right\|_{L_{t,x}^\infty(I\times\mathbb{H}^2)} &\lesssim s_0^{\frac{1}{2}s-\frac{1}{4}}.\end{aligned} \tag{3.17}$$

Proof of Claim 3.11. Using the radial Sobolev embedding (Corollary 2.22), Strichartz estimates and Fact 3.5

$$\begin{aligned}\left\|\sinh^{\frac{1}{2}}(r)\psi\right\|_{L_{t,x}^\infty(I\times\mathbb{H}^2)} &\lesssim \|\psi\|_{L_t^\infty L_x^2(I\times\mathbb{H}^2)}^{1-\frac{1}{4\alpha}} \|(-\Delta)^\alpha\psi\|_{L_t^\infty L_x^2(I\times\mathbb{H}^2)}^{\frac{1}{4\alpha}} \\ &\lesssim \|\psi_0\|_{L_x^2(\mathbb{H}^2)}^{1-\frac{1}{4\alpha}} \|\psi_0\|_{H_x^{2\alpha}(\mathbb{H}^2)}^{\frac{1}{4\alpha}} \lesssim s_0^{\frac{1}{2}s\times(1-\frac{1}{4\alpha})} s_0^{\frac{1}{2}(s-2\alpha)\times\frac{1}{4\alpha}} = s_0^{\frac{1}{2}s-\frac{1}{4}}.\end{aligned}$$

By Corollary 2.22 and Corollary 3.9

$$\left\|\sinh^{\frac{1}{2}}(r)\zeta_1\right\|_{L_{t,x}^\infty(I\times\mathbb{H}^2)} \lesssim \|\zeta_1\|_{L_t^\infty L_x^2(I\times\mathbb{H}^2)}^{\frac{1}{2}} \|\nabla\zeta_1\|_{L_t^\infty L_x^2(I\times\mathbb{H}^2)}^{\frac{1}{2}} \lesssim s_0^{\frac{1}{4}(s-1)}.$$

By Corollary 2.22 and Lemma 3.10

$$\begin{aligned}\left\|\sinh^{\frac{1}{2}}(r)\zeta_2\right\|_{L_{t,x}^\infty(I\times\mathbb{H}^2)} &\lesssim \|\zeta_2\|_{L_t^\infty L_x^2(I\times\mathbb{H}^2)}^{1-\frac{1}{4\alpha}} \|(-\Delta)^\alpha\zeta_2\|_{L_t^\infty L_x^2(I\times\mathbb{H}^2)}^{\frac{1}{4\alpha}} \\ &\lesssim s_0^{\frac{1}{2}s\times(1-\frac{1}{4\alpha})} s_0^{\frac{1}{2}(2\alpha s+s-2\alpha)\times\frac{1}{4\alpha}} = s_0^{\frac{3}{4}s-\frac{1}{4}}.\end{aligned}$$

□

Let us also recall some estimates from previous subsections ((3.10), (3.9), Corollary 3.9, Lemma 3.10 and Lemma 3.8):

$$\begin{aligned}&\|\psi\|_{S^\sigma(\mathbb{R})} \lesssim s_0^{\frac{1}{2}(s-\sigma)} \text{ for } 0 \le \sigma \le s, && \|\psi\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s}, \\ &\|\zeta_1\|_{S^\sigma(I)} \lesssim s_0^{\frac{\sigma}{2}(s-1)} \text{ for } 0 \le \sigma \le 1, && \|\zeta_1\|_{L_{t,x}^4(I\times\mathbb{H}^2)}^4 \lesssim \varepsilon,\end{aligned} \tag{3.18}$$

$$\|\zeta_2\|_{S^\sigma(I)} \lesssim s_0^{\frac{1}{2}(s-\sigma)} \text{ for } 0 \le \sigma \le s, \qquad \|\zeta_2\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s}.$$

Now we continue working on (3.16). By Lemma 3.10

$$I = \left\|\nabla \mathcal{O}(\zeta_2^3)\right\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} \lesssim \|\nabla\zeta_2\|_{L^\infty_t L^2_x(I\times\mathbb{H}^2)} \|\zeta_2\|^2_{L_t^{\frac{8}{3}} L_x^8(I\times\mathbb{H}^2)} \lesssim s_0^s \|\nabla\zeta_2\|_{L^\infty_t L^2_x(I\times\mathbb{H}^2)}.$$

This term will be absorbed by the left hand side of (3.16).

For II, we employ the local smoothing estimate. Since ψ is a linear solution, the linear version should be enough for this term. To implement the local smoothing estimate, we would like to introduce the weight $\langle x\rangle^{-\frac{1}{2}-\varepsilon_1}$, where ε_1 is a small positive number, and split out half derivative from the full gradient. Then by chain rule and Hölder inequality, we write

$$\left\|\nabla\mathcal{O}(\psi^3)\right\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} \lesssim \left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon_1} |\nabla|^{\frac{1}{2}} \left(|\nabla|^{\frac{1}{2}}\psi\right)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\psi^2)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \tag{3.19}$$

Now we compute the two factors above separately. Using the linear local smoothing estimate (Lemma 2.3.3) and Lemma 2.19, we write the first factor into

$$\left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon_1} |\nabla|^{\frac{1}{2}} \left(|\nabla|^{\frac{1}{2}}\psi\right)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} \lesssim \left\| |\nabla|^{\frac{1}{2}} P_{\le s_0}\phi\right\|_{L^2_x(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-\frac{1}{2})}. \tag{3.20}$$

To estimate the second factor in (3.19), by Hölder inequality, Sobelev embedding, (3.10), (3.9) and Claim 3.11, we have

$$\begin{aligned}
\left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\psi^2)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)} &\lesssim \left\|\chi_{\{|x|\le 1\}}\mathcal{O}(\psi^2)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)} + \left\|\chi_{\{|x|>1\}} |x|^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\psi^2)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \\
&\lesssim \|\psi\|^2_{L^8_{t,x}(I\times\mathbb{H}^2)} + \left\|\sinh^{\frac{1}{2}}(r)\psi\right\|_{L^\infty_{t,x}(I\times\mathbb{H}^2)} \|\psi\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \\
&\lesssim s_0^{s-\frac{1}{2}} + s_0^{\frac{1}{2}s-\frac{1}{4}} s_0^{\frac{1}{2}s} \lesssim s_0^{s-\frac{1}{2}},
\end{aligned} \tag{3.21}$$

for all s. Then combining (3.20) and (3.21), we obtain

$$II \lesssim (3.19) \lesssim s_0^{\frac{1}{2}(s-\frac{1}{2})} s_0^{s-\frac{1}{2}} = s_0^{\frac{3}{2}s-\frac{3}{4}}.$$

For III and IV, we write them in a similar way as in (3.19).

$$\begin{aligned}
III &\lesssim \left\|\nabla\mathcal{O}(\zeta_2\zeta_1^2)\right\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} \\
&\lesssim \left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon_1} |\nabla|^{\frac{1}{2}} \left(|\nabla|^{\frac{1}{2}}\zeta_2\right)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_1^2)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \\
&\quad + \left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon_1} |\nabla|^{\frac{1}{2}} \left(|\nabla|^{\frac{1}{2}}\zeta_1\right)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_1\zeta_2)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)},
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
IV &\lesssim \left\|\nabla\mathcal{O}(\psi\zeta_1^2)\right\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{H}^2)} \\
&\lesssim \left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon_1} |\nabla|^{\frac{1}{2}} \left(|\nabla|^{\frac{1}{2}}\psi\right)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_1^2)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \\
&\quad + \left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon_1} |\nabla|^{\frac{1}{2}} \left(|\nabla|^{\frac{1}{2}}\zeta_1\right)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_1\psi)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)}.
\end{aligned} \tag{3.23}$$

Due to the mixed terms in III and IV, the calculation will be more technical. Noting that there are some common terms in III and IV, we will preform the estimation of III and IV at the same time. Our goal here is to prove that III is bounded by $s_0^{s-\frac{3}{4}}$ and IV is also by $s_0^{s-\frac{3}{4}}$.

We first start with the terms with no derivative and positive weights. By Hölder inequality, Sobolev embedding, Corollary 3.9 and (3.17)

$$\begin{aligned}\left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_1^2)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)} &\lesssim \|\zeta_1\|^2_{L^8_{t,x}(I\times\mathbb{H}^2)} + \left\|\sinh^{\frac{1}{2}}(r)\zeta_1\right\|_{L^\infty_{t,x}(I\times\mathbb{H}^2)} \|\zeta_1\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ &\lesssim \left\| \langle -\Delta\rangle^{\frac{1}{4}} \zeta_1\right\|^2_{L^8_t L^{\frac{8}{3}}_x(I\times\mathbb{H}^2)} + s_0^{\frac{1}{4}(s-1)}\varepsilon^{\frac{1}{4}} \lesssim s_0^{\frac{1}{2}(s-1)} + s_0^{\frac{1}{4}(s-1)} \lesssim s_0^{\frac{1}{2}(s-1)},\end{aligned}$$

for $s<1$.

By Hölder inequality, Sobolev embedding, Corollary 3.9, Lemma 3.8, Lemma 3.10 and (3.17)

$$\begin{aligned}\left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_1\zeta_2)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)} &\lesssim \|\zeta_1\|_{L^8_{t,x}(I\times\mathbb{H}^2)} \|\zeta_2\|_{L^8_{t,x}(I\times\mathbb{H}^2)} + \left\|\sinh^{\frac{1}{2}}(r)\zeta_2\right\|_{L^\infty_{t,x}(I\times\mathbb{H}^2)} \|\zeta_1\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ &\lesssim \left\| \langle -\Delta\rangle^{\frac{1}{4}} \zeta_1\right\|_{L^8_t L^{\frac{8}{3}}_x(I\times\mathbb{H}^2)} \left\| \langle -\Delta\rangle^{\frac{1}{4}} \zeta_2\right\|_{L^8_t L^{\frac{8}{3}}_x(I\times\mathbb{H}^2)} + s_0^{\frac{3}{4}s-\frac{1}{4}}\varepsilon^{\frac{1}{4}} \\ &\lesssim s_0^{\frac{1}{4}(s-1)} s_0^{\frac{1}{2}(\frac{3}{2}s-\frac{1}{2})} + s_0^{\frac{3}{4}s-\frac{1}{4}}\varepsilon^{\frac{1}{4}} \lesssim s_0^{s-\frac{1}{2}},\end{aligned}$$

for $s<1$.

By Hölder inequality, Sobolev embedding, Corollary 3.9, (3.10) and (3.17)

$$\begin{aligned}\left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_1\psi)\right\|_{L^4_{t,x}(I\times\mathbb{H}^2)} &\lesssim \|\zeta_1\|_{L^{12}_{t,x}(I\times\mathbb{H}^2)} \|\psi\|_{L^6_{t,x}(I\times\mathbb{H}^2)} + \left\|\sinh^{\frac{1}{2}}(r)\psi\right\|_{L^\infty_{t,x}(I\times\mathbb{H}^2)} \|\zeta_1\|_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ &\lesssim \left\| \langle -\Delta\rangle^{\frac{1}{3}} \zeta_1\right\|_{L^{12}_t L^{\frac{12}{5}}_x(I\times\mathbb{H}^2)} \left\| \langle -\Delta\rangle^{\frac{1}{6}} \psi\right\|_{L^6_t L^3_x(I\times\mathbb{H}^2)} + s_0^{\frac{1}{2}s-\frac{1}{4}}\varepsilon^{\frac{1}{4}} \\ &\lesssim s_0^{\frac{1}{3}(s-1)} s_0^{\frac{1}{2}(s-\frac{1}{3})} + s_0^{\frac{1}{2}s-\frac{1}{4}}\varepsilon^{\frac{1}{4}} \lesssim s_0^{\frac{1}{2}s-\frac{1}{4}},\end{aligned}$$

for $s>\frac{3}{4}$.

Then we focus on the terms with derivatives and negative weights.

Notice that we have treated $\left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon_1} |\nabla|^{\frac{1}{2}} (|\nabla|^{\frac{1}{2}} \psi)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)}$ in (3.20).

By Hölder inequality, Fact 3.4, Sobolev embedding, Corollary 3.9 and (3.17)

$$\begin{aligned}\left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon_1} |\nabla|^{\frac{1}{2}} (|\nabla|^{\frac{1}{2}} \zeta_1)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} &\lesssim \left\| |\nabla|^{\frac{1}{2}} \eta_0\right\|_{L^2_x(\mathbb{H}^2)} + \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} |\zeta_1|^2\zeta_1\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} \\ &\lesssim s_0^{\frac{1}{4}(s-1)} + \|\zeta_1\|^3_{L^6_{t,x}(I\times\mathbb{H}^2)} + \left\|\sinh^{\frac{1}{2}}(r)\zeta_1\right\|_{L^\infty_{t,x}(I\times\mathbb{H}^2)} \|\zeta_1\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ &\lesssim s_0^{\frac{1}{4}(s-1)} + \left\| \langle -\Delta\rangle^{\frac{1}{6}} \zeta_1\right\|^3_{L^6_t L^3_x(I\times\mathbb{H}^2)} + s_0^{\frac{1}{4}(s-1)}\varepsilon^{\frac{1}{2}} \\ &\lesssim s_0^{\frac{1}{4}(s-1)} + s_0^{\frac{1}{6}(s-1)\times 3} + s_0^{\frac{1}{4}(s-1)}\varepsilon^{\frac{1}{2}} \lesssim s_0^{\frac{1}{2}(s-1)},\end{aligned}$$

for $s<1$.

The last term in this category needs more work. By Corollary 2.22 and triangle inequality, we have

$$\begin{aligned}\left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon_1} |\nabla|^{\frac{1}{2}} (|\nabla|^{\frac{1}{2}} \zeta_2)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} &\lesssim \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} F\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} \\ &\lesssim \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_2^3)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} + \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\psi^3)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} \\ &\quad + \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_2\zeta_1^2)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} + \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\psi\zeta_1^2)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} .\end{aligned} \tag{3.24}$$

Next we will estimate these four terms above separately. By Hölder inequality, Sobolev embedding, Corollary 3.9, Lemma 3.8, Lemma 3.10, (3.10), (3.9) and (3.17)

$$\begin{aligned}\left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_2^3)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} &\lesssim \|\zeta_2\|^3_{L^6_{t,x}(I\times\mathbb{H}^2)} + \left\|\sinh^{\frac{1}{2}}(r)\zeta_2\right\|_{L^\infty_{t,x}(I\times\mathbb{H}^2)} \|\zeta_2\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ &\lesssim \left\| \langle -\Delta\rangle^{\frac{1}{6}} \zeta_2\right\|^3_{L^6_tL^3_x(I\times\mathbb{H}^2)} + s_0^{\frac{3}{4}s-\frac{1}{4}} s_0^s \lesssim s_0^{\frac{1}{2}(\frac{4}{3}s-\frac{1}{3})\times 3} + s_0^{\frac{3}{4}s-\frac{1}{4}} s_0^s \lesssim s_0^{2s-\frac{1}{2}},\end{aligned}$$

for $s<1$.

$$\begin{aligned}\left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\psi^3)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} &\lesssim \|\psi\|^3_{L^6_{t,x}(I\times\mathbb{H}^2)} + \left\|\sinh^{\frac{1}{2}}(r)\psi\right\|_{L^\infty_{t,x}(I\times\mathbb{H}^2)} \|\psi\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ &\lesssim \left\| \langle -\Delta\rangle^{\frac{1}{6}} \psi\right\|^3_{L^6_tL^3_x(I\times\mathbb{H}^2)} + s_0^{\frac{1}{2}s-\frac{1}{4}} s_0^s \lesssim s_0^{\frac{1}{2}(s-\frac{1}{3})\times 3} + s_0^{\frac{1}{2}s-\frac{1}{4}} s_0^s \lesssim s_0^{\frac{3}{2}s-\frac{1}{2}},\end{aligned}$$

for all s.

$$\begin{aligned}\left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\zeta_2\zeta_1^2)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} &\lesssim \|\zeta_2\|_{L^6_{t,x}(I\times\mathbb{H}^2)} \|\zeta_1\|^2_{L^6_{t,x}(I\times\mathbb{H}^2)} + \left\|\sinh^{\frac{1}{2}}(r)\zeta_2\right\|_{L^\infty_{t,x}(I\times\mathbb{H}^2)} \|\zeta_1\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ &\lesssim \left\| \langle -\Delta\rangle^{\frac{1}{6}} \zeta_2\right\|_{L^6_tL^3_x(I\times\mathbb{H}^2)} \left\| \langle -\Delta\rangle^{\frac{1}{6}} \zeta_1\right\|^2_{L^6_tL^3_x(I\times\mathbb{H}^2)} + s_0^{\frac{3}{4}s-\frac{1}{4}}\varepsilon^{\frac{1}{2}} \\ &\lesssim s_0^{\frac{1}{2}(\frac{4}{3}s-\frac{1}{3})} s_0^{\frac{1}{6}(s-1)\times 2} + s_0^{\frac{3}{4}s-\frac{1}{4}}\varepsilon^{\frac{1}{2}} \lesssim s_0^{s-\frac{1}{2}},\end{aligned}$$

for $s<1$.

$$\begin{aligned}\left\| \langle x\rangle^{\frac{1}{2}+\varepsilon_1} \mathcal{O}(\psi\zeta_1^2)\right\|_{L^2_{t,x}(I\times\mathbb{H}^2)} &\lesssim \|\psi\|_{L^6_{t,x}(I\times\mathbb{H}^2)} \|\zeta_1\|^2_{L^6_{t,x}(I\times\mathbb{H}^2)} + \left\|\sinh^{\frac{1}{2}}(r)\psi\right\|_{L^\infty_{t,x}(I\times\mathbb{H}^2)} \|\zeta_1\|^2_{L^4_{t,x}(I\times\mathbb{H}^2)} \\ &\lesssim \left\| \langle -\Delta\rangle^{\frac{1}{6}} \psi\right\|_{L^6_tL^3_x(I\times\mathbb{H}^2)} \left\| \langle -\Delta\rangle^{\frac{1}{6}} \zeta_1\right\|^2_{L^6_tL^3_x(I\times\mathbb{H}^2)} + s_0^{\frac{1}{2}s-\frac{1}{4}}\varepsilon^{\frac{1}{2}} \\ &\lesssim s_0^{\frac{1}{2}(s-\frac{1}{3})} s_0^{\frac{1}{6}(s-1)\times 2} + s_0^{\frac{1}{2}s-\frac{1}{4}}\varepsilon^{\frac{1}{2}} \lesssim s_0^{\frac{1}{2}s-\frac{1}{4}},\end{aligned}$$

for $s>\frac{3}{4}$.

Then summing up all these four terms gives

$$(3.24) \lesssim s_0^{2s-\frac{1}{2}} + s_0^{\frac{3}{2}s-\frac{1}{2}} + s_0^{s-\frac{1}{2}} + s_0^{\frac{1}{2}s-\frac{1}{4}} \lesssim s_0^{\frac{1}{2}s-\frac{1}{4}},$$

for $s>\frac{1}{4}$.

Therefore, continue from (3.22) and (3.23)

$$III \lesssim s_0^{\frac{1}{2}s-\frac{1}{4}} s_0^{\frac{1}{2}(s-1)} + s_0^{\frac{1}{2}(s-1)} s_0^{s-\frac{1}{2}} \lesssim s_0^{s-\frac{3}{4}}, \quad \text{for } s > \frac{1}{4},$$
$$IV \lesssim s_0^{\frac{1}{2}(s-\frac{1}{2})} s_0^{\frac{1}{2}(s-1)} + s_0^{\frac{1}{2}(s-1)} s_0^{\frac{1}{2}s-\frac{1}{4}} \approx s_0^{s-\frac{3}{4}}, \quad \text{for all } s.$$

Combining all the terms I, II, III and IV, we write (3.16) into

$$\|\nabla\zeta_2\|_{L_t^\infty L_x^2(I\times\mathbb{H}^2)} \lesssim I + II + III + IV \lesssim s_0^s \|\nabla\zeta_2\|_{L_t^\infty L_x^2(I\times\mathbb{H}^2)} + s_0^{\frac{3}{2}s-\frac{3}{4}} + s_0^{s-\frac{3}{4}} + s_0^{s-\frac{3}{4}}.$$

Then we have

$$\|\nabla\zeta_2\|_{L_t^\infty L_x^2(I\times\mathbb{H}^2)} \lesssim s_0^{s-\frac{3}{4}}.$$

This concludes the proof of *(3)* in Lemma 3.8 and complete the analysis of the energy increment. □

3.3. **Proof of Proposition 3.2.** We divide the time interval $[0,T]$ into $[0,T] = \cup_i I_i = \cup_i [a_i, a_{i+1}]$, such that on each I_i, $\|u\|^4_{L^4_{t,x}(I_i\times\mathbb{H}^2)} = \varepsilon$. Hence

$$\# I_i \sim \frac{M}{\varepsilon}.$$

Let us remark that the length of such small intervals could be very long, and if some of them is an infinite interval, say $[a_k,\infty)$, then we just call $a_{k+1} = \infty$.

On the first interval $I_1 = [0, a_1]$, we can apply Proposition 3.1 and have the local energy increment

$$E(\zeta(a_1)) \leq E(\zeta(0)) + C s_0^{\frac{3}{2}s-\frac{5}{4}}.$$

On the second time interval $I_2 = [a_1, a_2]$, we solve ζ by solving a cubic NLS with smoother data,

$$\begin{cases} i\partial_t \zeta_1^{(1)} + \Delta_{\mathbb{H}^2}\zeta_1^{(1)} = \left|\zeta_1^{(1)}\right|^2 \zeta_1^{(1)}, \\ \zeta_1^{(1)}(a_1, x) = \zeta(a_1) = \zeta_1(a_1) + \zeta_2(a_1), \end{cases} \tag{3.25}$$

and a difference equation with zero initial value,

$$\begin{cases} i\partial_t \zeta_2^{(1)} + \Delta_{\mathbb{H}^2}\zeta_2^{(1)} = |u|^2 u - \left|\zeta_1^{(1)}\right|^2 \zeta_1^{(1)}, \\ \zeta_2^{(1)}(a_1, x) = 0. \end{cases} \tag{3.26}$$

Hence $\zeta = \zeta_1^{(1)} + \zeta_2^{(1)}$ and the full solution will be $u = \zeta + \psi = \zeta_1^{(1)} + \zeta_2^{(1)} + \psi$. Applying Proposition 3.1 again, we see that

$$E(\zeta(a_2)) \leq E(\zeta(a_1)) + C s_0^{\frac{3}{2}s-\frac{5}{4}}.$$

The reason why we are safe to apply Proposition 3.1 on I_2 is that new decomposed initial data $\zeta(a_1)$ and $\psi(a_1)$ satisfy all the facts in Facts 3.4 and 3.5, and all the calculations that we did in Proposition 3.1 will apply to the new systems (3.25) and (3.26). In particular, the size of new initial data $\zeta(a_1)$ in energy is the size of $\zeta(0)$ in (3.2) plus a small error from $\zeta_2(a_1)$, which can be seen from Proposition 3.1

$$E(\zeta(a_1)) \leq E(\zeta(0)) + \underbrace{C s_0^{\frac{3}{2}s-\frac{5}{4}}}_{\text{small error}} \sim E(\zeta(0)) \lesssim s_0^{s-1}.$$

Also the H^1 norm of $\zeta(a_1)$ can be thought as the H^1 norm of $\zeta(0)$ plus a small error,

$$\|\zeta(a_1)\|_{H^1_x(\mathbb{H}^2)} \leq \|\zeta_1(a_1)\|_{H^1_x(\mathbb{H}^2)} + \|\zeta_2(a_1)\|_{H^1_x(\mathbb{H}^2)} \lesssim \underbrace{s_0^{\frac{1}{2}(s-1)}}_{\text{size of } H^1 \text{ norm of } \zeta(0)} + \underbrace{s_0^{s-\frac{3}{4}}}_{\text{small error}} \sim s_0^{\frac{1}{2}(s-1)}.$$

Then we can continue this iteration as long as the accumulated energy increment does not surpass the size of the initial energy of $\zeta(0)$, which guarantees that the setup for the smoother component remains the same size in the next iteration. That is,

$$C\frac{M}{\varepsilon} s_0^{\frac{3}{2}s-\frac{5}{4}} \leq E(\zeta(0)) \sim s_0^{s-1},$$

which gives

$$M \sim s_0^{-\frac{1}{2}s+\frac{1}{4}}. \tag{3.27}$$

And the total energy increment is

$$E(\zeta(T)) \leq E(\zeta(0)) + C\frac{M}{\varepsilon} s_0^{\frac{3}{2}s-\frac{5}{4}}. \tag{3.28}$$

For now, we finish the proof of Proposition 3.2 and give the choice of s_0.

Remark 3.12 (boundedness of H^s norm of u)**.** As a consequence of Proposition 3.2, we conclude that the H^s norm of u has the following bound

$$\|u(T)\|_{H^s_x(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-1)s}.$$

In fact, (3.28) implies the boundedness of the H^1 norm of $\zeta(T)$,

$$\|\zeta(T)\|^2_{H^1_x(\mathbb{H}^2)} \leq E(\zeta(T)) \leq E(\zeta(0)) + C\frac{M}{\varepsilon} s_0^{\frac{3}{2}s-\frac{5}{4}} \lesssim s_0^{s-1}. \tag{3.29}$$

And triangle inequality and the mass conservation laws of u and ψ with (3.10) give the boundedness of L^2 norm of $\zeta(T)$

$$\begin{aligned} \|\zeta(T)\|_{L^2_x(\mathbb{H}^2)} &\leq \|u(T)\|_{L^2_x(\mathbb{H}^2)} + \|\psi(T)\|_{L^2_x(\mathbb{H}^2)} = \|u(0)\|_{L^2_x(\mathbb{H}^2)} + \|\psi(0)\|_{L^2_x(\mathbb{H}^2)} \\ &\lesssim \|u(0)\|_{L^2_x(\mathbb{H}^2)} + s_0^{\frac{1}{2}s} \leq 2\|u(0)\|_{L^2_x(\mathbb{H}^2)}. \end{aligned} \tag{3.30}$$

Then the H^s bound $\zeta(T)$ follows from the interpolation (3.30) and (3.29)

$$\|\zeta(T)\|_{H^s_x(\mathbb{H}^2)} \lesssim \|\zeta(T)\|^{1-s}_{L^2_x(\mathbb{H}^2)} \|\zeta(T)\|^s_{H^1_x(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-1)s}. \tag{3.31}$$

Therefore the H^s norm of $u(T)$ is bounded due (3.31) and the fact $\psi \in H^s(\mathbb{H}^2)$,

$$\|u(T)\|_{H^s_x(\mathbb{H}^2)} \leq \|\zeta(T)\|_{H^s_x(\mathbb{H}^2)} + \|\psi(T)\|_{H^s_x(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-1)s} + 1 \lesssim s_0^{\frac{1}{2}(s-1)s}.$$

Consequently, we also have the bound of the $H^\sigma (0 < \sigma < s)$ norm by interpolating the H^s with L^2 norms

$$\|u(T)\|_{H^\sigma_x(\mathbb{H}^2)} \lesssim \|u(T)\|^{\frac{\sigma}{s}}_{H^s_x(\mathbb{H}^2)} \|u(T)\|^{1-\frac{\sigma}{2}}_{L^2_x(\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-1)\sigma}.$$

4. Morawetz estimates on $\mathbb{H}^2$

Recall that the Morawetz estimate of the cubic NLS on $\mathbb{H}^2$ in [35], when u is the solution to the cubic NLS equation $i\partial_t u + \Delta_{\mathbb{H}^2} u = |u|^2 u$, reads as

$$\|u\|^4_{L^4_{t,x}([t_1,t_2]\times\mathbb{H}^2)} \lesssim \|u\|_{L^\infty_t L^2_x([t_1,t_2]\times\mathbb{H}^2)} \|u\|_{L^\infty_t H^1_x([t_1,t_2]\times\mathbb{H}^2)} .$$

Proposition 4.1. *If we modify the NLS equation, that is u solves*

$$i\partial_t u + \Delta_{\mathbb{H}^2} u = |u|^2 u + \mathcal{N},$$

then the modified Morawetz estimate becomes

$$\begin{aligned}\|u\|^4_{L^4_{t,x}([t_1,t_2]\times\mathbb{H}^2)} \lesssim{}& \|u\|_{L^\infty_t L^2_x([t_1,t_2]\times\mathbb{H}^2)} \|u\|_{L^\infty_t H^1_x([t_1,t_2]\times\mathbb{H}^2)} \\ &+ \|\mathcal{N}\bar{u}\|_{L^1_{t,x}([t_1,t_2]\times\mathbb{H}^2)} + \|\mathcal{N}\nabla\bar{u}\|_{L^1_{t,x}([t_1,t_2]\times\mathbb{H}^2)} .\end{aligned} \tag{4.1}$$

Remark 4.2. The proof of Proposition 4.1 is very similar as the proof in [35]. We report it below for the convenience of the reader. The difference is that we consider a more general nonlinear term, which mainly gives two extra terms that account for the two extra terms in (4.1).

Proof of Proposition 4.1. It is possible to construct a function $a(x)$ satisfying the following requirements.

Lemma 4.3. [35] *There is a smooth function $a : \mathbb{H}^2 \to [0,\infty)$ with the following properties:*

$$\begin{aligned}&\Delta a = 1 \text{ on } \mathbb{H}^2\\ &|\nabla a| = |\mathbf{D}^\alpha a \mathbf{D}_\alpha a|^{\frac{1}{2}} \leq C \text{ on } \mathbb{H}^2\\ &\mathbf{D}^2 a \geq 0 \text{ on } \mathbb{H}^2.\end{aligned}$$

For $\varepsilon \in (0, \frac{1}{10}]$ let $u_\varepsilon = P_\varepsilon u$, where P_ε is the smoothing operator defined by the Fourier multiplier $\lambda \to e^{-\varepsilon^2\lambda^2}$. We fix a smooth even function $\eta_0 : \mathbb{R} \to [0,1]$ supported in $[-2,-\frac{1}{2}] \cup [\frac{1}{2}, 2]$ with the property that

$$\sum_{j\in\mathbb{Z}} \eta_0(\frac{\lambda}{2^j}) = 1 \text{ for any } \lambda \in \mathbb{R}\setminus\{0\}.$$

For any $j \in \mathbb{Z}$, let $\eta_j(\lambda) = \eta_0(\frac{\lambda}{2^j})$ and $\eta_{\leq j} = \sum_{j' \leq j} \eta_{j'}$. Let $\psi_\varepsilon : \mathbb{H}^2 \to [0,1]$, $\psi_\varepsilon(x) = \eta_{\leq 0}(\varepsilon r)$.

With a as in Lemma 4.3, we define the Morawetz action $M_a : \mathbb{R} \to \mathbb{R}$,

$$M_a(t) = 2\,\mathrm{Im} \int_{\mathbb{H}^2} \psi_\varepsilon \mathbf{D}^\alpha a(x) \cdot \bar{u}_\varepsilon(x) \mathbf{D}_\alpha u_\varepsilon(x)\, d\mu(x).$$

Let

$$f_\varepsilon = P_\varepsilon(\,|u|^2 u + \mathcal{N}),$$

thus

$$\partial_t u_\varepsilon = i\Delta u_\varepsilon - i f_\varepsilon \text{ and } \partial_t \bar{u}_\varepsilon = -i\Delta\bar{u}_\varepsilon + i\bar{f}_\varepsilon.$$

We compute

$$\begin{aligned}
\partial_t M_a(t) &= 2\,\mathrm{Im}\int_{\mathbb{H}^2} \psi_\varepsilon \mathbf{D}^\alpha a \cdot (\partial_t \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha u_\varepsilon + \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha \partial_t u_\varepsilon)\, d\mu \\
&= 2\,\mathrm{Im}\int_{\mathbb{H}^2} \psi_\varepsilon \mathbf{D}^\alpha a \cdot ((-i\Delta \bar{u}_\varepsilon + i\bar{f}_\varepsilon) \cdot \mathbf{D}_\alpha u_\varepsilon + \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha (i\Delta u_\varepsilon - i f_\varepsilon))\, d\mu \\
&= 2\,\mathrm{Re}\int_{\mathbb{H}^2} \psi_\varepsilon \mathbf{D}^\alpha a \cdot ((-\Delta \bar{u}_\varepsilon + \bar{f}_\varepsilon) \cdot \mathbf{D}_\alpha u_\varepsilon + \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha (\Delta u_\varepsilon - f_\varepsilon))\, d\mu \\
&= \int_{\mathbb{H}^2} \psi_\varepsilon \mathbf{D}^\alpha a \cdot (\bar{u}_\varepsilon \cdot \mathbf{D}_\alpha \Delta u_\varepsilon + u_\varepsilon \cdot \mathbf{D}_\alpha \Delta \bar{u}_\varepsilon - \Delta \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha u_\varepsilon - \Delta u_\varepsilon \cdot \mathbf{D}_\alpha \bar{u}_\varepsilon)\, d\mu \\
&\quad + \int_{\mathbb{H}^2} \psi_\varepsilon \mathbf{D}^\alpha a \cdot (\bar{f}_\varepsilon \cdot \mathbf{D}_\alpha u_\varepsilon + f_\varepsilon \cdot \mathbf{D}_\alpha \bar{u}_\varepsilon - \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha f_\varepsilon - u_\varepsilon \cdot \mathbf{D}_\alpha \bar{f}_\varepsilon)\, d\mu \\
&= I + II
\end{aligned}$$

By integration by parts and using $\mathbf{D}_\alpha \mathbf{D}_\beta v = \mathbf{D}_\beta \mathbf{D}_\alpha v$ for any scalar v, we compute

$$\begin{aligned}
I &= \int_{\mathbb{H}^2} -[\mathbf{D}_\alpha(\psi_\varepsilon \mathbf{D}^\alpha a](\bar{u}_\varepsilon \Delta u_\varepsilon + u_\varepsilon \Delta \bar{u}_\varepsilon) - 2\psi_\varepsilon \mathbf{D}^\alpha a(\Delta \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha u_\varepsilon + \Delta u_\varepsilon \cdot \mathbf{D}_\alpha \bar{u}_\varepsilon)\, d\mu \\
&= \int_{\mathbb{H}^2} -(\psi_\varepsilon \Delta a + \mathbf{D}_\alpha \psi_\varepsilon \mathbf{D}^\alpha a)[\Delta(u_\varepsilon \bar{u}_\varepsilon) - 2\mathbf{D}_\beta u_\varepsilon \mathbf{D}^\beta \bar{u}_\varepsilon]\, d\mu \\
&\quad - 2\int_{\mathbb{H}^2} \psi_\varepsilon \mathbf{D}^\alpha a(\mathbf{D}^\beta \mathbf{D}_\beta \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha u_\varepsilon + \mathbf{D}^\beta \mathbf{D}_\beta u_\varepsilon \cdot \mathbf{D}_\alpha \bar{u}_\varepsilon)\, d\mu \\
&= 2\int_{\mathbb{H}^2} \mathbf{D}^\beta(\psi_\varepsilon \mathbf{D}^\alpha a) \cdot (\mathbf{D}_\beta \bar{u}_\varepsilon \mathbf{D}_\alpha u_\varepsilon + \mathbf{D}_\beta u_\varepsilon \mathbf{D}_\alpha \bar{u}_\varepsilon) \\
&\quad + \int_{\mathbb{H}^2} -\Delta(\psi_\varepsilon \Delta a + \mathbf{D}_\alpha \psi_\varepsilon \mathbf{D}^\alpha a) \cdot (u_\varepsilon \bar{u}_\varepsilon)\, d\mu \\
&\quad + \int_{\mathbb{H}^2} 2(\psi_\varepsilon \Delta a + \mathbf{D}_\alpha \psi_\varepsilon \mathbf{D}^\alpha a) \cdot \mathbf{D}_\beta u_\varepsilon \mathbf{D}^\beta \bar{u}_\varepsilon + 2\psi_\varepsilon \mathbf{D}^\alpha a(\mathbf{D}_\beta \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha \mathbf{D}^\beta u_\varepsilon + \mathbf{D}_\beta u_\varepsilon \cdot \mathbf{D}_\alpha \mathbf{D}^\beta \bar{u}_\varepsilon)\, d\mu \\
&= 2\int_{\mathbb{H}^2} (\psi_\varepsilon \mathbf{D}^\beta \mathbf{D}^\alpha a + \mathbf{D}^\beta \psi_\varepsilon \mathbf{D}^\alpha a)(\mathbf{D}_\beta \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha u_\varepsilon + \mathbf{D}_\beta u_\varepsilon \cdot \mathbf{D}_\alpha \bar{u}_\varepsilon)\, d\mu \\
&\quad + \int_{\mathbb{H}^2} -\Delta(\psi_\varepsilon \Delta a + \mathbf{D}_\alpha \psi_\varepsilon \mathbf{D}^\alpha a) \cdot (u_\varepsilon \bar{u}_\varepsilon)\, d\mu = A + B
\end{aligned}$$

since $\mathbf{D}_\beta \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha \mathbf{D}^\beta u_\varepsilon + \mathbf{D}_\beta u_\varepsilon \cdot \mathbf{D}_\alpha \mathbf{D}^\beta \bar{u}_\varepsilon = \mathbf{D}_\alpha(\mathbf{D}_\beta u_\varepsilon \mathbf{D}^\beta \bar{u}_\varepsilon)$. We write $f_\varepsilon = |u_\varepsilon|^2 u_\varepsilon + G_\varepsilon = |u_\varepsilon|^2 u_\varepsilon + g_\varepsilon + \mathcal{N}_\varepsilon$ and use the identity $|u_\varepsilon|^2 \bar{u}_\varepsilon \cdot \mathbf{D}_\alpha u_\varepsilon + |u_\varepsilon|^2 u_\varepsilon \cdot \mathbf{D}_\alpha \bar{u}_\varepsilon = \frac{1}{2}\mathbf{D}_\alpha(|u_\varepsilon|^4)$ to compute

$$\begin{aligned}
II &= 2\int_{\mathbb{H}^2} \psi_\varepsilon \mathbf{D}^\alpha a(\bar{f}_\varepsilon \cdot \mathbf{D}_\alpha u_\varepsilon + f_\varepsilon \cdot \mathbf{D}_\alpha \bar{u}_\varepsilon) + \mathbf{D}_\alpha(\psi_\varepsilon \mathbf{D}^\alpha a) \cdot (\bar{f}_\varepsilon u_\varepsilon + f_\varepsilon \bar{u}_\varepsilon)\, d\mu \\
&= 2\int_{\mathbb{H}^2} \psi_\varepsilon \mathbf{D}^\alpha a(\bar{g}_\varepsilon \cdot \mathbf{D}_\alpha u_\varepsilon + g_\varepsilon \cdot \mathbf{D}_\alpha \bar{u}_\varepsilon) + \mathbf{D}_\alpha(\psi_\varepsilon \mathbf{D}^\alpha a) \cdot (\bar{g}_\varepsilon u_\varepsilon + g_\varepsilon \bar{u}_\varepsilon)\, d\mu \\
&+ \quad 2\int_{\mathbb{H}^2} \psi_\varepsilon \mathbf{D}^\alpha a(\bar{\mathcal{N}}_\varepsilon \cdot \mathbf{D}_\alpha u_\varepsilon + \mathcal{N}_\varepsilon \cdot \mathbf{D}_\alpha \bar{u}_\varepsilon) + \mathbf{D}_\alpha(\psi_\varepsilon \mathbf{D}^\alpha a) \cdot (\bar{\mathcal{N}}_\varepsilon u_\varepsilon + \mathcal{N}_\varepsilon \bar{u}_\varepsilon)\, d\mu \\
&\quad + \int_{\mathbb{H}^2} \mathbf{D}_\alpha(\psi_\varepsilon \mathbf{D}^\alpha a) \cdot |u_\varepsilon|^4\, d\mu = C + E + D
\end{aligned}$$

We integrate these identities on the interval $[t_1, t_2]$ to conclude that

$$M_a(t_2) - M_a(t_1) = \int_{t_1}^{t_2} (A + B + C + E + D)\, dt.$$

We use now that $u \in S^1_{p_\sigma}(-T, T)$, thus $\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{S^1_{p_\sigma}(-T',T')} = 0$ and, using

$$\left\|(-\Delta)^{\frac{1}{2}} f\right\|_{L^{p_1}(I\times\mathbb{H}^d)} + \|f\|_{L^{p_2}(I\times\mathbb{H}^d)} \lesssim \|f\|_{S^1_q(I)},$$

for any $f \in S^1_q(I)$, $p_1 \in [q, \frac{2d+4}{d}$, and $p_2 \in [q, \frac{2d+4}{d-2})$, we have that

$$\lim_{\varepsilon\to 0} \|g_\varepsilon\|_{L^{p'_\sigma}((-T',T')\times\mathbb{H}^d)} = 0$$

for any $T' < T$. We let $\varepsilon \to 0$, using Lemma 4.3 to conclude that

$$\lim_{\varepsilon\to 0}\int_{t_1}^{t_2} A\, dt = 2\int_{[t_1,t_2]\times\mathbb{H}^2} \mathbf{D}^\beta\mathbf{D}^\alpha a \cdot (\mathbf{D}_\beta \bar{u}\cdot \mathbf{D}_\alpha u + \mathbf{D}_\beta u \cdot \mathbf{D}_\alpha \bar{u})\, d\mu dt$$

$$\lim_{\varepsilon\to 0}\int_{t_1}^{t_2} B\, dt = \lim_{\varepsilon\to 0}\int_{t_1}^{t_2} C\, dt = 0,$$

$$\lim_{\varepsilon\to 0}\int_{t_1}^{t_2} D\, dt = \int_{[t_1,t_2]\times\mathbb{H}^2} |u|^4\, d\mu dt$$

$$\lim_{\varepsilon\to 0}\int_{t_1}^{t_2} E\, dt = 2\int_{[t_1,t_2]\times\mathbb{H}^2} \mathbf{D}^\alpha a(\bar{\mathcal{N}}\cdot \mathbf{D}_\alpha u + \mathcal{N}\cdot \mathbf{D}_\alpha \bar{u}) + \mathbf{D}_\alpha\mathbf{D}^\alpha a \cdot (\bar{\mathcal{N}} u + \mathcal{N}\bar{u})\, d\mu$$

Since $|M_a(t)| \le C \sup_{t\in[t_1,t_2]} \|u(t)\|_{L^2_x(\mathbb{H}^2)} \|u(t)\|_{H^1_x(\mathbb{H}^2)}$ using Lemma 4.3, it follows that

$$\begin{aligned}
&2\int_{[t_1,t_2]\times\mathbb{H}^2} \mathbf{D}^\beta\mathbf{D}^\alpha a \cdot (\mathbf{D}_\beta \bar{u}\cdot \mathbf{D}_\alpha u + \mathbf{D}_\beta u \cdot \mathbf{D}_\alpha \bar{u})\, d\mu dt + \int_{[t_1,t_2]\times\mathbb{H}^2} |u|^4\, d\mu dt \\
&\le C \sup_{t\in[t_1,t_2]} \|u(t)\|_{L^2_x(\mathbb{H}^2)} \|u(t)\|_{H^1_x(\mathbb{H}^2)} + \|\mathcal{N}\nabla\bar{u}\|_{L^1_{t,x}([t_1,t_2]\times\mathbb{H}^2)} + \|\mathcal{N}\bar{u}\|_{L^1_{t,x}([t_1,t_2]\times\mathbb{H}^2)}
\end{aligned}$$

□

5. Global well-posedness and scattering on $\mathbb{H}^2$

In this section, we use a bootstrapping argument to finally show the global well-posedness and scattering results stated in Theorem 1.1.

5.1. **Step 1: set-up of the open-close argument.** Define

$$W := \left\{T : \|u\|^4_{L^4_{t,x}([0,T]\times\mathbb{H}^2)} \le M\right\},$$

where $M > 0$ is a constant. W is closed and non-empty. Now we want to show that W is open. If $T_1 \in W$, then due to the local well-posedness theory and Remark 3.12, for some $T_0 > T_1$ and T_0 sufficiently close to T_1 we have

$$\|u\|^4_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)} \le 2M.$$

In fact, the H^s norm of $u(T_1)$ is bounded, then using a standard local well-posedness argument, we can continue the solution u from time T_1 at least for a short time. Within such short time period, due to the sub-criticality, the spacetime L^4 norm of u will be bounded by twice the H^σ, for σ arbitrarily small, norm at T_1, which is of order $s_0^{\frac{1}{2}(s-1)\sigma}$ (see Remark

3.12). Hence we want to ensure that $s_0^{\frac{1}{2}(s-1)\sigma} < M^{\frac{1}{4}} \sim s_0^{\frac{1}{4}(-\frac{1}{2}s+\frac{1}{4})}$, which is achieved for any $s > \frac{3}{4}$ by taking σ small enough (say, $\sigma = \frac{1}{4}$). This guarantees the existence of such T_0.

Now we show that $T_0 \in W$, that is

$$\|u\|_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}^4 \leq M. \tag{5.1}$$

Recall the decomposition of the solution u. That is, we can think of $u = \psi + \zeta$, where $\psi = e^{it\Delta}P_{\leq s_0}\phi$ solves a linear Schrödinger equation with high frequency data

$$\begin{cases} i\partial_t\psi + \Delta_{\mathbb{H}^2}\psi = 0, \\ \psi(0,x) = P_{\leq s_0}\phi, \end{cases}$$

and ζ solves a difference equation with low frequency data

$$\begin{cases} i\partial_t\zeta + \Delta_{\mathbb{H}^2}\zeta = |u|^2 u = G(\zeta,\psi), \\ \zeta(0,x) = P_{>s_0}\phi, \end{cases}$$

where $G(\zeta,\psi) = |\zeta+\psi|^2(\zeta+\psi) = |\zeta|^2\zeta + \mathcal{O}(\zeta^2\psi) + \mathcal{O}(\zeta\psi^2) + \mathcal{O}(\psi^3)$. From Proposition 3.2, we learned that we can divide $[0,T_0] = \cup_i I_i = \cup_i[a_i, a_{i+1}]$, such that on each I_i,

$$\|u\|_{L^4_{t,x}(I_i\times\mathbb{H}^2)}^4 = \varepsilon, \tag{5.2}$$

and

$$\# I_i \sim \frac{2M}{\varepsilon}.$$

The total energy increment on $[0,T_0]$ is

$$E(\zeta(T_0)) \leq E(\zeta(0)) + C\frac{2M}{\varepsilon}s_0^{\frac{3}{2}s-\frac{5}{4}},$$

and the choice of s_0 is based on M

$$M \sim s_0^{-\frac{1}{2}s+\frac{1}{4}}.$$

Using the smallness of L^4 norm of ψ in (3.9)

$$\|\psi\|_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s},$$

one can reduce (5.1) to

$$\|\zeta\|_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}^4 \leq \frac{1}{2}M. \tag{5.3}$$

Now we will prove the improved bound of $L^4_{t,x}$ in (5.3) in Steps 2 and 3.

5.2. **Step 2: improving the bound for $L^4_{t,x}$.** Recall the modified Morawetz estimate in (4.1) that now gives

$$\begin{aligned} \|\zeta\|_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}^4 \lesssim \|\zeta\|_{L_t^\infty L_x^2([0,T_0]\times\mathbb{H}^2)} \|\zeta\|_{L_t^\infty H_x^1([0,T_0]\times\mathbb{H}^2)} \\ + \left\|\mathcal{N}\bar{\zeta}\right\|_{L^1_{t,x}([0,T_0]\times\mathbb{H}^2)} + \left\|\mathcal{N}\nabla\bar{\zeta}\right\|_{L_t^1L_x^1([0,T_0]\times\mathbb{H}^2)}. \end{aligned} \tag{5.4}$$

In our case $\mathcal{N}$ is given by

$$\mathcal{N} = |u|^2 u - |\zeta|^2\zeta$$

$$= |\psi+\zeta|^2(\psi+\zeta) - |\zeta|^2\zeta = |\psi|^2\psi + (2|\psi|^2\zeta + \psi^2\bar{\zeta}) + (2\psi|\zeta|^2 + \bar{\psi}\zeta^2)$$
$$= |\psi|^2\psi + \mathcal{O}(\psi^2\zeta) + \mathcal{O}(\psi\zeta^2).$$

Now we estimate the right hand side terms in (5.4).

For the second term in (5.4), by Hölder inequality, (3.9) and (5.3), we have

$$\begin{aligned}\left\|\mathcal{N}\bar{\zeta}\right\|_{L^1_{t,x}([0,T_0]\times\mathbb{H}^2)} &\lesssim \|\psi\|^3_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}\|\zeta\|_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)} + \|\psi\|_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}\|\zeta\|^3_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}\\ &\lesssim s_0^{\frac{3}{2}s}\|\zeta\|_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)} + s_0^{\frac{1}{2}s}\|\zeta\|^3_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)} \lesssim s_0^{\frac{3}{2}s}M^{\frac14} + s_0^{\frac12 s}M^{\frac34}.\end{aligned}$$

We write the last term in (5.4) as

$$\begin{aligned}\left\|\mathcal{N}\nabla\bar{\zeta}\right\|_{L^1_{t,x}([0,T_0]\times\mathbb{H}^2)} &\lesssim \left\|\mathcal{O}(\zeta^2\psi)\nabla\bar{\zeta}\right\|_{L^1_{t,x}([0,T_0]\times\mathbb{H}^2)} + \left\|\mathcal{O}(\psi^3)\nabla\bar{\zeta}\right\|_{L^1_{t,x}([0,T_0]\times\mathbb{H}^2)}\\ &\lesssim \|\zeta\|^2_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}\|\psi\|_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}\|\nabla\zeta\|_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}\\ &\quad + \|\psi\|^3_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}\|\nabla\zeta\|_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)}.\end{aligned} \tag{5.5}$$

Claim 5.1. We claim that

(1) $\|\nabla\zeta\|^4_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)} \lesssim M s_0^{2(s-1)}$,

(2) $\|\nabla\zeta\|_{L^\infty_t L^2_x([0,T_0]\times\mathbb{H}^2)} \lesssim s_0^{\frac12(s-1)}$.

Assuming Claim 5.1, we continue the estimation of (5.5),

$$(5.5) \lesssim M^{\frac12}s_0^{\frac12 s}(Ms_0^{2(s-1)})^{\frac14} + s_0^{\frac32 s}(Ms_0^{2(s-1)})^{\frac14} = M^{\frac34}s_0^{s-\frac12} + M^{\frac14}s_0^{2s-\frac12}.$$

Using the same calculation as in (3.30), we have

$$\|\zeta(t)\|_{L^2_x(\mathbb{H}^2)} \leq \|u(t)\|_{L^2_x(\mathbb{H}^2)} + \|\psi(t)\|_{L^2_x(\mathbb{H}^2)} \lesssim \|u(0)\|_{L^2_x(\mathbb{H}^2)} + s_0^{\frac12 s} \leq 2\|u(0)\|_{L^2_x(\mathbb{H}^2)},$$

hence the first term in (5.4) is bounded by

$$\|\zeta\|_{L^\infty_t L^2_x([0,T_0]\times\mathbb{H}^2)}\|\zeta\|_{L^\infty_t H^1_x([0,T_0]\times\mathbb{H}^2)} \lesssim \|u(0)\|_{L^2_x(\mathbb{H}^2)} s_0^{\frac12(s-1)}.$$

Now (5.4) becomes

$$\|\zeta\|^4_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)} \lesssim s_0^{\frac12(s-1)} + (s_0^{\frac32 s}M^{\frac14} + s_0^{\frac12 s}M^{\frac34}) + (M^{\frac34}s_0^{s-\frac12} + M^{\frac14}s_0^{2s-\frac12}).$$

To close the argument, we need the following inequality holds for $M \sim s_0^{-\frac12 s+\frac14}$

$$s_0^{\frac12(s-1)} + (s_0^{\frac32 s}M^{\frac14} + s_0^{\frac12 s}M^{\frac34}) + (M^{\frac34}s_0^{s-\frac12} + M^{\frac14}s_0^{2s-\frac12}) < \frac12 M. \tag{5.6}$$

This requirement of (5.6) can be achieved for

$$s > \frac34. \tag{5.7}$$

Now we are left to prove Claim 5.1.

5.3. **Step 3: Proof of Claim 5.1.**

Proof of Claim 5.1. We start with *(2)*. Recall the the total energy increment,

$$E(\zeta(t)) \leq E(\zeta(0)) + \frac{M}{\varepsilon} s_0^{\frac{3}{2}s-\frac{5}{4}} \sim s_0^{s-1},$$

for all $t \in [0, T_0]$. This yields *(2)*

$$\|\nabla \zeta\|^2_{L_t^\infty L_x^2([0,T_0]\times \mathbb{H}^2)} \leq \sup_t E(\zeta(t)) \lesssim s_0^{s-1}.$$

To estimate $\|\nabla\zeta\|_{L_t^4 L_x^4([0,T_0]\times\mathbb{H}^2)}$, we consider the subintervals I_i's. We claim that

$$\|\nabla \zeta\|_{L^4_{t,x}(I_i\times\mathbb{H}^2)} \lesssim \|\nabla\zeta(a_i)\|_{L^2_x(\mathbb{H}^2)}. \tag{5.8}$$

In fact, by Strichartz estimates

$$\begin{aligned}\|\nabla \zeta\|_{L^4_{t,x}(I_i\times\mathbb{H}^2)} &\lesssim \|\nabla\zeta(a_i)\|_{L^2_x(\mathbb{H}^2)} + \left\|\nabla(\zeta+\psi)^3\right\|_{L^{\frac{4}{3}}_{t,x}(I_i\times\mathbb{H}^2)} \\ &\lesssim \|\nabla\zeta(a_i)\|_{L^2_x(\mathbb{H}^2)} + \left\|\nabla\mathcal{O}(\zeta^3)\right\|_{L^{\frac{4}{3}}_{t,x}(I_i\times\mathbb{H}^2)} + \left\|\nabla\mathcal{O}(\psi^3)\right\|_{L^{\frac{4}{3}}_{t,x}(I_i\times\mathbb{H}^2)}.\end{aligned} \tag{5.9}$$

The second term in (5.9) will be absorbed by the left hand side of (5.9)

$$\left\|\nabla\mathcal{O}(\zeta^3)\right\|_{L^{\frac{4}{3}}_{t,x}(I_i\times\mathbb{H}^2)} \lesssim \|\nabla\zeta\|_{L^4_{t,x}(I_i\times\mathbb{H}^2)} \|\zeta\|^2_{L^4_{t,x}(I_i\times\mathbb{H}^2)} \lesssim \|\nabla\zeta\|_{L^4_{t,x}(I_i\times\mathbb{H}^2)}\varepsilon^{\frac{1}{2}}.$$

The last inequality above is due to (3.9) and (5.2)

$$\|\zeta\|_{L^4_{t,x}(I_i\times\mathbb{H}^2)} \leq \|\psi\|_{L^4_{t,x}(I_i\times\mathbb{H}^2)} + \|u\|_{L^4_{t,x}(I_i\times\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}s} + \varepsilon^{\frac{1}{4}} \lesssim \varepsilon^{\frac{1}{4}}. \tag{5.10}$$

For the last term in (5.9), the same calculation as in (3.19) gives

$$\left\|\nabla\mathcal{O}(\psi^3)\right\|_{L^{\frac{4}{3}}_{t,x}(I_i\times\mathbb{H}^2)} \lesssim s_0^{\frac{3}{2}s-\frac{3}{4}}.$$

Then (5.9) becomes

$$\|\nabla \zeta\|_{L^4_{t,x}(I_i\times\mathbb{H}^2)} \lesssim \|\nabla\zeta(a_i)\|_{L^2_x(\mathbb{H}^2)} + \|\nabla\zeta\|_{L^4_{t,x}(I_i\times\mathbb{H}^2)}\varepsilon^{\frac{1}{2}} + s_0^{\frac{3}{2}s-\frac{3}{4}}.$$

Therefore the claim (5.8) follows.

Putting all the small intervals together and using *(2)* we get

$$\|\nabla \zeta\|^4_{L^4_{t,x}([0,T_0]\times\mathbb{H}^2)} \lesssim \frac{M}{\varepsilon}\sup_{I_i} \|\nabla\zeta(a_i)\|^4_{L^2_x(\mathbb{H}^2)} \leq \frac{M}{\varepsilon}\|\nabla\zeta\|^4_{L_t^\infty L_x^2([0,T_0]\times\mathbb{H}^2)} \lesssim \frac{M}{\varepsilon} s_0^{2(s-1)}.$$

Now we finish the proof of Claim 5.1.

∎

5.4. **Step 4: proof of scattering.** Recall the definition of scattering: given a global solution $u \in H^s$ to (1.1), we say that u scatters to $u_\pm \in H^s$ if

$$\lim_{t\to\pm\infty} \left\| u(t) - e^{it\Delta_{\mathbb{H}^2}} u_\pm \right\|_{H^s_x(\mathbb{H}^2)} = 0.$$

It is clear that scattering is equivalent to showing that the improper time integral

$$\int_0^\infty e^{-it'\Delta} \, |u|^2 \, u(t') \, dt'$$

converges in H^s and in particular this will give the formula for u_+, that is

$$u_+ = u_0 - i \int_0^\infty e^{-it'\Delta} \, |u|^2 \, u(t') \, dt'.$$

By Strichartz and Lemma 2.17, we have that

$$\left\| \int_0^\infty e^{-it'\Delta} \, |u|^2 \, u(t') \, dt' \right\|_{H^s_x(\mathbb{H}^2)} \lesssim \left\| \langle -\Delta \rangle^{\frac{s}{2}} \left(|u|^2 \, u \right) \right\|_{L^{\frac{4}{3}}_{t,x}(\mathbb{R}\times\mathbb{H}^2)} \lesssim \left\| \langle -\Delta \rangle^{\frac{s}{2}} \, u \right\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{H}^2)} \|u\|^2_{L^4_{t,x}(\mathbb{R}\times\mathbb{H}^2)} .$$

It is clear that the scattering follows once we show that

$$\|u\|_{S^s(\mathbb{R})} \leq C.$$

Moreover, we can reduce to prove

$$\|\zeta\|_{S^s(\mathbb{R})} \leq C,$$

since we learned in (3.10) that $\|\psi\|_{S^s(\mathbb{R})} \lesssim 1$.

We proved in Step 3 that $\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{H}^2)} \leq C$, so we divide the time interval $(-\infty, \infty)$ into $\cup I_i = \cup[a_i, a_{i+1}]$, $i = 1, \cdots, K < \infty$ such that

$$\|u\|^4_{L^4_{t,x}(I_i\times\mathbb{H}^2)} = \varepsilon$$

for all $i = 1, \cdots, K$.

On each $I_i = [a_i, a_{i+1}]$, by the same calculation as in (5.9)

$$\begin{aligned} \|\nabla\zeta\|_{S^0(I_i)} &\lesssim \|\zeta(a_i)\|_{H^1_x(\mathbb{H}^2)} + \left\|\nabla\mathcal{O}(\zeta^3)\right\|_{L^{\frac{4}{3}}_{t,x}(I_i\times\mathbb{H}^2)} + \left\|\nabla\mathcal{O}(\psi^3)\right\|_{L^{\frac{4}{3}}_{t,x}(I_i\times\mathbb{H}^2)} \\ &\lesssim \|\zeta(a_i)\|_{H^1_x(\mathbb{H}^2)} + \varepsilon^{\frac{1}{2}} \, \|\nabla\zeta\|_{S^0(I_i)} + s_0^{\frac{3}{2}s-\frac{3}{4}}. \end{aligned}$$

Then we have

$$\|\nabla\zeta\|_{S^0(I_i)} \lesssim \|\zeta(a_i)\|_{H^1_x(\mathbb{H}^2)} .$$

Therefore, due to the finiteness of number of I_i intervals,

$$\|\nabla\zeta\|_{S^0(\mathbb{R})} \leq C.$$

Using the integral equation and Strichartz estimates again with (5.10) and (3.9), we have

$$\begin{aligned} \|\zeta\|_{S^0(I_i)} &\lesssim \|\zeta(a_i)\|_{L^2_x(\mathbb{H}^2)} + \left\|\mathcal{O}(\zeta^3)\right\|_{L^{\frac{4}{3}}_{t,x}(I_i\times\mathbb{H}^2)} + \left\|\mathcal{O}(\psi^3)\right\|_{L^{\frac{4}{3}}_{t,x}(I_i\times\mathbb{H}^2)} \\ &\lesssim \|\zeta(a_i)\|_{L^2_x(\mathbb{H}^2)} + \|\zeta\|^3_{L^4_{t,x}(I_i\times\mathbb{H}^2)} + \|\psi\|^3_{L^4_{t,x}(I_i\times\mathbb{H}^2)} \lesssim \|\zeta(a_i)\|_{L^2_x(\mathbb{H}^2)} + \varepsilon^{\frac{3}{4}} + s_0^{\frac{3}{2}s}. \end{aligned}$$

Then

$$\|\zeta\|_{S^0(I_i)} \lesssim \|\zeta(a_i)\|_{L^2_x(\mathbb{H}^2)} ,$$

and due to the finiteness of number of I_i intervals,

$$\|\zeta\|_{S^0(\mathbb{R})} \leq C.$$

Therefore, interpolating S^0 with S^1 gives

$$\|\zeta\|_{S^s(\mathbb{R})} \leq C.$$

We finish the proof of scattering.

6. General nonlinearities

Our result for the cubic NLS in Theorem 1.1 can be generalized to a larger class of nonlinearities. In fact we have the following result.

Theorem 6.1. *The initial value problem* (1.6) *with radial initial data* $\phi \in H^s(\mathbb{H}^2)$ *is globally-well-posed and scatters in* $H^s(\mathbb{H}^2)$ *when* $s > \frac{3p-6}{3p-5}$.

Note that the scaling of (1.6) is $s_c = 1 - \frac{2}{p-1}$.

6.1. Sketch of the proof.

We will briefly present how the method presented above for the cubic NLS can be generalized to nonlinearities of order p.

(1) As in Section 3, we decompose u in ψ and ζ, where $\psi = e^{it\Delta} P_{\leq s_0}\phi$ solves the linear Schrödinger with high frequency data and ζ solves the difference equation with low frequency data

$$\begin{cases} i\partial_t \psi + \Delta_{\mathbb{H}^2}\psi = 0, \\ \psi(0,x) = \psi_0 = P_{\leq s_0}\phi, \end{cases} \qquad \begin{cases} i\partial_t \zeta + \Delta_{\mathbb{H}^2}\zeta = |u|^{p-1} u, \\ \zeta(0,x) = \eta_0 = P_{>s_0}\phi. \end{cases} \tag{6.1}$$

Then using similar analysis we obtain the global energy increment given the boundedness of u in the critical spacetime $L_{t,x}^{2(p-1)}$, (see Proposition 6.2 and Proposition 6.3 below). The analogues of all the estimates that we used in the cubic case can be found in (6.3).

(2) Similarly, a bootstrapping argument on the $L_{t,x}^{2(p-1)}$ norm gives both the global existence and scattering.

Note that the only difference in the general case is that the spacetime $L_{t,x}^{2(p-1)}$ in the local theory (Proposition 6.2) is different from the Morawetz norm. Hence in the bootstrapping argument an intermediate step is needed. In fact, in this step, we first obtain and improve the estimates on the Morawetz norm, then we bootstrap the $L_{t,x}^{2(p-1)}$ norm with the better Morawetz bound. Notice that $L_{t,x}^{2(p-1)}$ agrees with the Morawetz norm when $p = 3$, hence such step is not needed in Section 5.

6.2. Analogues of the main propositions.

We now present the analogues of Proposition 3.1 and Proposition 3.2 on the energy increment.

Proposition 6.2 (Local energy increment)**.** *Consider u as in* (1.6) *defined on $I \times \mathbb{H}^2$ where $I = [0, \tau]$, such that*

$$\|u\|^{2(p-1)}_{L^{2(p-1)}_{t,x}(I\times\mathbb{H}^2)} = \varepsilon$$

for some universal constant ε. Then for $s > \frac{p}{p+1}$ and sufficiently small s_0, the solution ζ, under the decomposition $u = \psi + \zeta$ defined as in (6.1)*, satisfies the following energy increment*

$$E(\zeta(\tau)) \leq E(\zeta(0)) + C s_0^{\frac{p+3}{4}s - \frac{p+2}{4}}.$$

Proposition 6.3 (Conditional global energy increment)**.** *Consider u as in* (1.6) *defined on $[0, T] \times \mathbb{H}^2$ where*

$$\|u\|^{2(p-1)}_{L^{2(p-1)}_{t,x}([0,T]\times\mathbb{H}^2)} \leq M$$

for some constant M. Then for $s > \frac{p}{p+1}$ and sufficiently small s_0, the energy of ζ satisfies the following energy increment

$$E(\zeta(T)) \leq E(\zeta(0)) + C\frac{M}{\varepsilon} s_0^{\frac{p+3}{4}s - \frac{p+2}{4}}.$$

where ε is the small constant in Proposition 6.2.

6.3. **Analogues of the main estimates.** Within the proofs of two propositions above, we need a further decomposition for ζ as is (3.4), (3.5) and (3.6)

$$\begin{cases} i\partial_t\zeta_1 + \Delta_{\mathbb{H}^2}\zeta_1 = |\zeta_1|^{p-1}\zeta_1, \\ \zeta_1(0,x) = \eta_0 = P_{>s_0}\phi, \end{cases} \qquad \begin{cases} i\partial_t\zeta_2 + \Delta_{\mathbb{H}^2}\zeta_2 = |u|^{p-1}u - |\zeta_1|^{p-1}\zeta_1, \\ \zeta_2(0,x) = 0. \end{cases} \tag{6.2}$$

Hence the full solution u is the sum of these three solutions $u = \zeta_1 + \zeta_2 + \psi$.

The analogue of (3.18) can be computed similarly as follows

$$\begin{aligned} &\|\psi\|_{S^\sigma(\mathbb{R})} \lesssim s_0^{\frac{1}{2}(s-\sigma)} \text{ for } 0 \leq \sigma \leq s, && \|\psi\|_{L^{2(p-1)}_{t,x}(\mathbb{R}\times\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-s_c)}, \\ &\|\zeta_1\|_{S^\sigma(I)} \lesssim_{\|\phi\|_{H^{s_c}_x}} \begin{cases} 1 & \text{for } 0 \leq \sigma \leq s_c, \\ s_0^{\frac{\sigma-s_c}{2(1-s_c)}(s-1)} & \text{for } s_c \leq \sigma \leq 1, \end{cases} && \|\zeta_1\|^{2(p-1)}_{L^{2(p-1)}_{t,x}(I\times\mathbb{H}^2)} \lesssim_{\|\phi\|_{H^{s_c}_x}} \varepsilon, \\ &\|\zeta_2\|_{S^\sigma(I)} \lesssim_{\|\phi\|_{H^{s_c}_x}} s_0^{\frac{1}{2}(s-\sigma)} \text{ for } 0 \leq \sigma \leq s, && \|\zeta_2\|_{L^{2(p-1)}_{t,x}(I\times\mathbb{H}^2)} \lesssim_{\|\phi\|_{H^{s_c}_x}} s_0^{\frac{1}{2}(s-s_c)}. \end{aligned} \tag{6.3}$$

Claim 3.11 will be the same in the general setting. Most importantly, the analogue of *(3)* in Lemma 3.8 is

$$\|\zeta_2\|_{L^\infty_t H^1_x(I\times\mathbb{H}^2)} \lesssim s_0^{\frac{p+1}{4}s - \frac{p}{4}}.$$

The choice of s_0 in Proposition 6.3 is given by

$$M \sim s_0^{\frac{1}{2}(\frac{1-s}{1-s_c} - \frac{1}{2})}. \tag{6.4}$$

It is also worth mentioning that the hidden constant in (6.3) at the second iteration is bounded by the H^{s_c} norm of ϕ plus a small error

$$\|(\zeta_1+\zeta_2)(a_1)\|_{H^{s_c}_x(\mathbb{H}^2)} \leq \|\zeta_1(a_1)\|_{H^{s_c}_x(\mathbb{H}^2)} + \|\zeta_2(a_1)\|_{H^{s_c}_x(\mathbb{H}^2)} \lesssim \|\phi\|_{H^{s_c}_x(\mathbb{H}^2)} + s_0^{\frac{1}{2}s} s_0^{-\frac{1}{2}s_c(\frac{1}{2}+\frac{1-s}{1-s_c})},$$

for $s > \frac{p}{p+1}$. Then the accumulated gain of this hidden constant will be dominated by the size of the H^{s_c} norm of ϕ, hence not growing.

6.4. **A different bootstrapping argument.** We consider

$$W := \left\{ T : \|u\|^{2(p-1)}_{L^{2(p-1)}_{t,x}([0,T]\times\mathbb{H}^2)} \leq M \right\},$$

We are then reduced to showing that for T_0 chosen in the same manner as in Step 1 in Section 5

$$\|\zeta\|^{2(p-1)}_{L^{2(p-1)}_{t,x}([0,T_0]\times\mathbb{H}^2)} \leq \frac{1}{2}M.$$

Things are different here. First, interpolating $\|\zeta\|^{2+}_{L^{2+}_{t,x}([0,T_0]\times\mathbb{H}^2)} \lesssim M$ with the bound of the $L^{2(p-1)}_{t,x}$ norm in the assumption gives an estimate on the Morawetz norm

$$\|\zeta\|^{p+1}_{L^{p+1}_{t,x}([0,T_0]\times\mathbb{H}^2)} \lesssim M. \tag{6.5}$$

Using (6.5) and the modified Morawetz estimate (4.1), we obtain as before

$$\|\zeta\|^{p+1}_{L^{p+1}_{t,x}([0,T_0]\times\mathbb{H}^2)} \lesssim s_0^{\frac{1}{2}(s-1)}.$$

If we simply require $s_0^{\frac{1}{2}(s-1)} < M$ here, there will be no room to improve the $L^{2(p-1)}_{t,x}$ norm at all. So to this end, we demand it to be much smaller than M, that is for $\alpha \in (0,1)$

$$s_0^{\frac{1}{2}(s-1)} \leq M^{\alpha} \ll M, \tag{6.6}$$

hence recalling (6.4), we get the first restriction on s

$$s > 1 - \frac{\alpha}{2 + \alpha(p-1)}.$$

With this better Morawetz bound, we can improve the $L^{2(p-1)}_{t,x}$ norm by making it smaller than M by Hölder inequality, (6.6) and Proposition 6.3 with the choice of M as in (6.4),

$$\begin{aligned}\|\zeta\|^{2(p-1)}_{L^{2(p-1)}_{t,x}([0,T_0]\times\mathbb{H}^2)} &\lesssim \|\zeta\|^{p+1-}_{L^{p+1}_{t,x}([0,T_0]\times\mathbb{H}^2)} \|\zeta\|^{p-3+}_{L^{\infty-}_{t,x}([0,T_0]\times\mathbb{H}^2)} \lesssim M^{\alpha-} \left\| \langle\nabla\rangle^{1-} \zeta \right\|^{p-3+}_{L^{\infty-}_t L^{2+}_x([0,T_0]\times\mathbb{H}^2)} \\ &\lesssim M^{\alpha} s_0^{\frac{1}{2}(s-1)(p-3)} \ll M.\end{aligned}$$

This with (6.4) implies the second restriction on s

$$s > 1 - \frac{1-\alpha}{2(p-3) + (1-\alpha)(p-1)}.$$

Therefore combining both conditions on s, we choose $s > s^p_{\mathbb{H}^2}$ to be the best possible scattering index, where

$$s^p_{\mathbb{H}^2} = \min_{\alpha\in(0,1)} \max\left\{1 - \frac{\alpha}{2+\alpha(p-1)}, 1 - \frac{1-\alpha}{2(p-3)+(1-\alpha)(p-1)}\right\} = 1 - \frac{1}{3p-5} = \frac{3p-6}{3p-5}.$$

Appendix A. Global well-posedness result in $\mathbb{R}^2$

A.1. **Tools used in the proof on $\mathbb{R}^2$.** In this subsection we recall known estimates for the Schrödinger operator in $\mathbb{R}^2$. We start by recalling that a couple (q, r) of exponents is admissible if $(\frac{1}{q}, \frac{1}{r})$ belongs to the line

$$I_d = \{(\frac{1}{q}, \frac{1}{r}) \in [0, \frac{1}{2}] \times (0, \frac{1}{2}] \,|\, \frac{2}{q} + \frac{d}{r} = \frac{d}{2}\}. \tag{A.1}$$

Then we have the following

Theorem A.1 (Strichartz Estimates [29, 48, 37])**.** *Assume u is the solution to the inhomogeneous initial value problem*

$$\begin{cases} i\partial_t u + \Delta u = F, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d, \\ u(0, x) = f(x), \end{cases} \tag{A.2}$$

For any admissible exponents (q, r) and $(\tilde{q}, \tilde{r})$ we have the Strichartz estimates:

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L_x^2(\mathbb{R}^d)} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}.$$

Definition A.2 (Strichartz Spaces)**.** We define the Banach space

$$S^0(I) = \left\{ f \in C(I : L^2(\mathbb{R}^2)) : \|f\|_{S^0(I)} = \sup_{(q,r) \text{ admissible}} \|f\|_{L_t^q L_x^r(I\times\mathbb{R}^2)} < \infty \right\}.$$

Also we define the Banach space $S^\sigma(I)$, where $\sigma > 0$,

$$S^\sigma(I) = \left\{ f \in C(I : H^\sigma(\mathbb{R}^2)) : \|f\|_{S^\sigma(I)} = \|\langle\nabla^\sigma\rangle f\|_{S^0(I)} < \infty \right\}.$$

Theorem A.3 (Local Smoothing Estimates in $\mathbb{R}^2$ [17, 41, 47])**.** *For any $\varepsilon > 0$,*

$$\left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon} \, |\nabla|^{\frac{1}{2}} \, e^{it\Delta} f \right\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \lesssim \|f\|_{L_x^2(\mathbb{R}^2)},$$

$$\left\| \langle x\rangle^{-\frac{1}{2}-\varepsilon} \, \nabla \int_0^t e^{i(t-s)\Delta} F(s, x)\, ds \right\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \lesssim \left\| \langle x\rangle^{\frac{1}{2}+\varepsilon} \, F \right\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^2)}.$$

Proposition A.4 (Radial Sobolev Embeddings in $\mathbb{R}^d$ in [44])**.** *Let $d \geq 1$, $1 \leq q \leq \infty$, $0 < s < d$ and $\beta \in \mathbb{R}$ obey the conditions*

$$\beta > -\frac{d}{q}, \quad 0 \leq \frac{1}{p} - \frac{1}{q} \leq s$$

and the scaling condition

$$\beta + s = \frac{d}{p} - \frac{d}{q}$$

with at most one of the equalities

$$p = 1, \quad p = \infty, \quad q = 1, \quad q = \infty, \quad \frac{1}{p} - \frac{1}{q} = s$$

holding. Then for any spherically symmetric function $f \in \dot{W}^{s,p}(\mathbb{R}^d)$, we have

$$\left\| \, |x|^\beta \, f \right\|_{L^q(\mathbb{R}^d)} \lesssim \| \, |\nabla|^s \, f \|_{L^p(\mathbb{R}^d)}.$$

A.2. **Theorem.** If we follow the same blue print set up in the hyperbolic setting, we can prove the following global well-posedness result in $\mathbb{R}^2$.

Theorem A.5. *The initial value problem*

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u, & t \in \mathbb{R}, x \in \mathbb{R}^2, \\ u(0,x) = \phi(x), \end{cases} \tag{A.3}$$

is globally well-posed for radial data $\phi \in H^s(\mathbb{R}^2)$ when $s > \frac{4}{5}$.

Remark A.6. We did not prove the scattering part in $\mathbb{R}^2$ setting, since the Morawetz estimate and the Strichartz estimates are less favorable in two dimensional Euclidean space. More precisely, the Morawetz estimate is significantly different from the one in higher dimensions $\mathbb{R}^d$ $(d \geq 3)$, also from the one that we used in $\mathbb{H}^d$. So it is not straightforward to employ the Morawetz in $\mathbb{R}^2$ setting. Also the range of Strichartz admission pairs are limited comparing to that in the hyperbolic space.

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