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OPTIMAL ESTIMATION FOR NONLINEAR STOCHASTIC SYSTEMS

by

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ABSTRACT

The problem of optimal estimation for differentiable nonlinear systems and measurements is considered. It is seen that all knowable information concerning the state is contained in the probability density of the state conditioned on the initial conditions, measurement history, and system dynamics. This density cannot be described by a finite number of sufficient statistics, and approximation must be resorted to. This is done by representing the density as a gaussian density times a finite expansion of Hermite polynomials.

Using this formalism and the Bayes approach, second and third order filters are developed for continuous systems with continuous and discrete measurements. The optimal third order discrete measurement estimation equations are derived for the first time, and their properties investigated. These filters compute the approximate conditional mean, covariance, and third central moments.

These estimators are demonstrated numerically in three examples, yielding substantially improved accuracy compared with a linearized Kalman estimator, for very nonlinear systems. Consideration of the third moments allows the third order estimator to solve the Kalman filter divergence problem for a self contained orbital navigation system, at a substantial increase in computational complexity.

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Chapter 1

Introduction

1.1 Historical Background

Methods of obtaining the best fit in a least squares sense to a finite sample of noisy data have been known since antiquity, and were used by early astronomers for the reduction of astronomical data. With the advance of technology, interest in real time estimation for more complex situations has steadily increased.

The first significant breakthrough occurred in 1942, when Wiener solved the problem of optimal linear filtering of stationary random signals. Wiener was able to show that minimizing the variance of the steady state estimation error implied the satisfaction of an integral equation involving the noise power spectral density function. In the special case of linear systems with gaussian random inputs, the integral equation can be solved in a useful form for many applications.

Attempts to generalize these results using the integral equation - frequency domain techniques proved fruitless, and the next advance was made by reformulating the problem in the time domain and specifying the optimal filter by differential equations. The pioneer work by Kalman and Bucy (P-13,P-14) accomplished the minimization of the error variances by making the filter estimates orthogonal projections in function space; subsequent derivations of the same estimation equations have been accomplished using ordinary calculus. (B-13) The Kalman filter overcame several of the deficiencies of the Wiener Filter. The Kalman filter could make use of a priori information, could handle discrete measurements as well as continuous, and was not restricted to stationary statistics. Nevertheless it was optimal in a minimum variance sense only for linear systems and random noise with a gaussian probability density.

For the more general estimation problem, a number of approaches are possible. The least squares method is sometimes used, whereby the initial conditions and system trajectory are computed that minimize the sum of the squares of the measurement residues. This method in general is not recursive, cannot be done in real time, and is not necessarily optimal in any sense. Another approach is maximum likelihood estimation. Using this technique, a state is selected that maximizes a given probability density, called the likelihood function. One might choose $P(\underline{y}/\underline{x})$, the conditional probability density of a measurement given the state, but it is difficult to include in a meaningful way the nonlinear differential constraints which relate the states at different times. $P(\underline{x}/\underline{y})$ is in general a better choice, as this density contains all knowable information concerning the state. Nevertheless the state which maximizes this density is not in general a minimum variance estimate. The conditional mean of this density is the minimum variance estimate. For this reason, the conditional mean estimator, which can be made recursive, appears to be the method most worthy of further investigation.

Once again, further progress followed a new conceptual formulation of the problem. In 1960 Stratonovich suggested that the fundamental entity in sequential estimation is the conditional probability density for the state given the observed measurement history. It was shown by Kolmogorov that the probability density of a continuous Markov system obeys a partial differential diffusion equation. Early attempts by Stratonovich (P-32), and Wonham (P-33) to extend the Kolmogorov equation to systems with continuous noisy measurements both contained errors, and the first correct derivation was by Kushner (P-16) in 1964. This derivation was only formal (nonrigorous), and was followed by a number of other formal derivations using different techniques, by Bucy, (P-5) Schwartz, (T-5) and by Lipster and Time. (P-23)

The total knowledge concerning the state of a stochastic system can be described by the conditional probability density of the state given the measurements, or by a set of parameters describing this density function. Often the filter equations involving these parameters will be more tractable than those involving the density function. The case of a linear system driven by a Gauss-Markov process was treated extensively by Kalman and Bucy. (P-14) Since in this case the conditional density is Gaussian, the optimal filter is completely described by the temporal evolution of the first two conditional moments of the process. For a nonlinear system, the situation is substantially more complex.

The conditional density will in general be nongaussian, even if all noise sources are gaussian. On this account, the conditional density will not have a simple analytic form, and cannot be exactly described by a finite number of parameters. Since only a finite number of parameters can be computed by a practical filter, it is necessary to select a small parameter set while attempting to keep the error due to neglected parameters within acceptable bounds. The central moments of the density function have been suggested as parameters by Kushner, (P-19, P-20) while Fisher (T-2) has explored the advantages of using coefficients for the expansion of the density in terms of orthogonal polynomials.

A rigorous derivation of the differential equations for the conditional central moments has recently been published by Kushner. (P-19) The restrictions governing the rigorous validity of these equations are complex, but in general do not exclude systems of physical interest. These equations provide the conditional moments for continuous systems with continuous measurements having additive white noise. The truncation of these moment sets is discussed by Kushner (P-20) and by Schwartz (T-5). Schwartz assumes that the third and higher central moments are zero, while Kushner assumes that all nonsymmetric central moments (IE, third, fifth, etc.) are zero in his example.

The problem of continuous nonlinear systems with discrete stochastic measurements has been considered by Jazwinsky (P-10), and by Phaneuf (T-4). Jazwinsky develops the expression for the posterior density given the measurement, and considers the moment equations. Since these are difficult to solve in the general case, an approximate solution is derived with the assumption that the third and higher central moments are zero. Phaneuf develops filter equations for the minimum variance linear estimate of the first two posterior moments. These optimal linear estimates are not optimal in a broad sense, because the conditional mean estimates are in general nonlinear and depend on the higher moments. In both cases, the assumption that the third moments are zero implies that the posterior density is symmetric, and hence that the system is symmetric. For many systems of practical interest, such as the orbital navigation problem, the latter assumption is not valid.

1.2 Problem Description and Investigation Summary

True optimal nonlinear filters are physically unrealizable, since the posterior density function cannot be described by a finite number of sufficient statistics. Most of the work done to date has been concerned with only the first two moments of the conditional density function. This involves the implicit assumption that both the system and its density function are symmetric with respect to positive and negative errors. This is because only the third and higher odd moments are capable of describing asymmetries. For many nonlinear systems this symmetry assumption is not valid. For example, the system $\dot{x} = x^2$ will cause both positive and negative values to produce positive velocities, with the result that such a system starting with symmetric initial condition error will soon have an asymmetric probability density function. All systems whose Taylor series expansions about the current mean have a significant second order component fall into this nonsymmetric category.

Furthermore, the analytically convenient continuous measurement formulation seems to be of limited practical usefulness, as most physical systems of a complexity to warrant nonlinear filtering involve discrete measurements. However, the optimal nonlinear estimation problem for nonlinear systems with discrete measurements has not yet been solved.

The problem considered in this thesis is the practical development of optimal estimation techniques for nonlinear nonsymmetric systems. The mathematical development of optimal estimation equations for continuous systems with continuous and discrete measurements is undertaken. The truncation of the infinite set of filter equations retaining at least the third moments is considered, as well as the stability of the resulting coupled differential and difference equations. The performance of these filters is tested via Monte Carlo simulation, and compared with a linearized Kalman filter. The computational requirements of these more complex filters are compared.

The optimal nonlinear filters with discrete measurements are finally compared with the linearized Kalman filter for several problems of practical importance. These include the self-contained estimation of position and velocity of an orbiting spacecraft, and the estimation of imperfectly known parameters of a linear system.

Chapter 2

The Basic Principles of Linear and Nonlinear Estimation

2.1 Linear Estimation

The object of estimation is to infer the current state of a system from information concerning the state at previous times, either initial conditions or measurements, and knowledge of the system dynamics and inputs. In general, knowledge concerning the state of a system will be probabilistic. While this knowledge may be described in a number of ways, the probability density of the state conditioned on initial conditions, all previous measurements, and system dynamics, is basic; this conditional probability density contains all information knowable concerning the system.

An estimator (filter) which produces an exact description of this probability density is optimal, since it contains all knowledge which can be inferred from the available information.

For estimation purposes, the system may be thought of as a device which transforms input random variables into output random variables. Linear transformations have the remarkable property that they preserve the gaussian character of random signals. Thus linear systems with gaussian inputs and linear gaussian measurements will have a probability density of the state conditioned on all available information that is gaussian. Such a density can be exactly described by a finite number of parameters, and any filter producing a complete set of these parameters (sufficient statistics) will be an optimal filter.

The conditional mean and covariance of the state are such a set of sufficient statistics for a gaussian density, and the Kalman filter, which computes these quantities, is an optimal estimator.

2.2 Nonlinear Estimation

The tractability of linear systems is more fully appreciated after an encounter with nonlinear systems. Nonlinear transformations do not preserve the form of the density function of random signals. The conditional probability density, container of all knowable information, is nongaussian whether or not the system inputs are gaussian. In general it cannot be represented by a finite number of elementary functions, nor can it be exactly described by a finite number of sufficient statistics. A finite filter that cannot be improved is impossible to construct. Approximation is inescapable.

The well developed theory of second order has been extensively applied to nonlinear systems, resulting in various estimators which compute only second order statistics, such as the "linearized Kalman filter". This involves attempting to approximate an arbitrary density function by a gaussian density of similar mean and covariance. If the nonlinear density is close to gaussian this may work well; if not, poor accuracy or instability (Kalman filter divergence) may result.

If more accuracy is required, a model for the density function assumed by the filter must be found that provides a closer fit to the actual density function. One such method is to express the density function as a gaussian density times an expansion of Hermite Polynomials. For particular problems, another approximation might provide a better fit with fewer terms, but the Hermite Polynomial expansion seems to be the best general choice. It is, at least, a general theory of random functions not restricted to the first two moments. Though this is much less elegant and satisfying than the linear-gaussian theory, for want of a better theory of the same degree of generality its complexity must be tolerated.

The Hermite Polynomial expansion has the advantage that it is theoretically complete; that is, it can be made arbitrarily accurate by taking enough terms. Further, additional terms are orthogonal to all lower order terms, and hence their inclusion does not change lower order moments of the density. There are a number of disadvantages, however. The expansion often converges quite slowly, and there is no guarantee that inclusion of an additional term will reduce error, even though the sequence eventually converges. For multidimensional systems the third order expansion is quite complex, and higher order expansions are virtually prohibitive. The most serious disadvantage is that the expressions for the density are not positive definite, and stability problems can arise from the implicit assumption of negative probability.

Nevertheless, the inclusion of a third order Hermite Polynomial in the density function allows the description of an unsymmetric density, which can better describe an unsymmetric system. Further, this third order filter does provide increased estimation accuracy for many substantially nonlinear systems, as demonstrated by

Monte Carlo simulation tests. Filter stability can be achieved by appropriate limiting of measurement residue. The third order filter can be used to solve the problem of Kalman filter divergence, and obtain improved accuracy in the complex problem of orbital landmark navigation, with a computational complexity within the capabilities of current digital computers.

To illustrate these concepts, two probability density functions formed by a third order Hermite polynomial expansion are shown in Figure 2.1, along with a gaussian density for comparison.

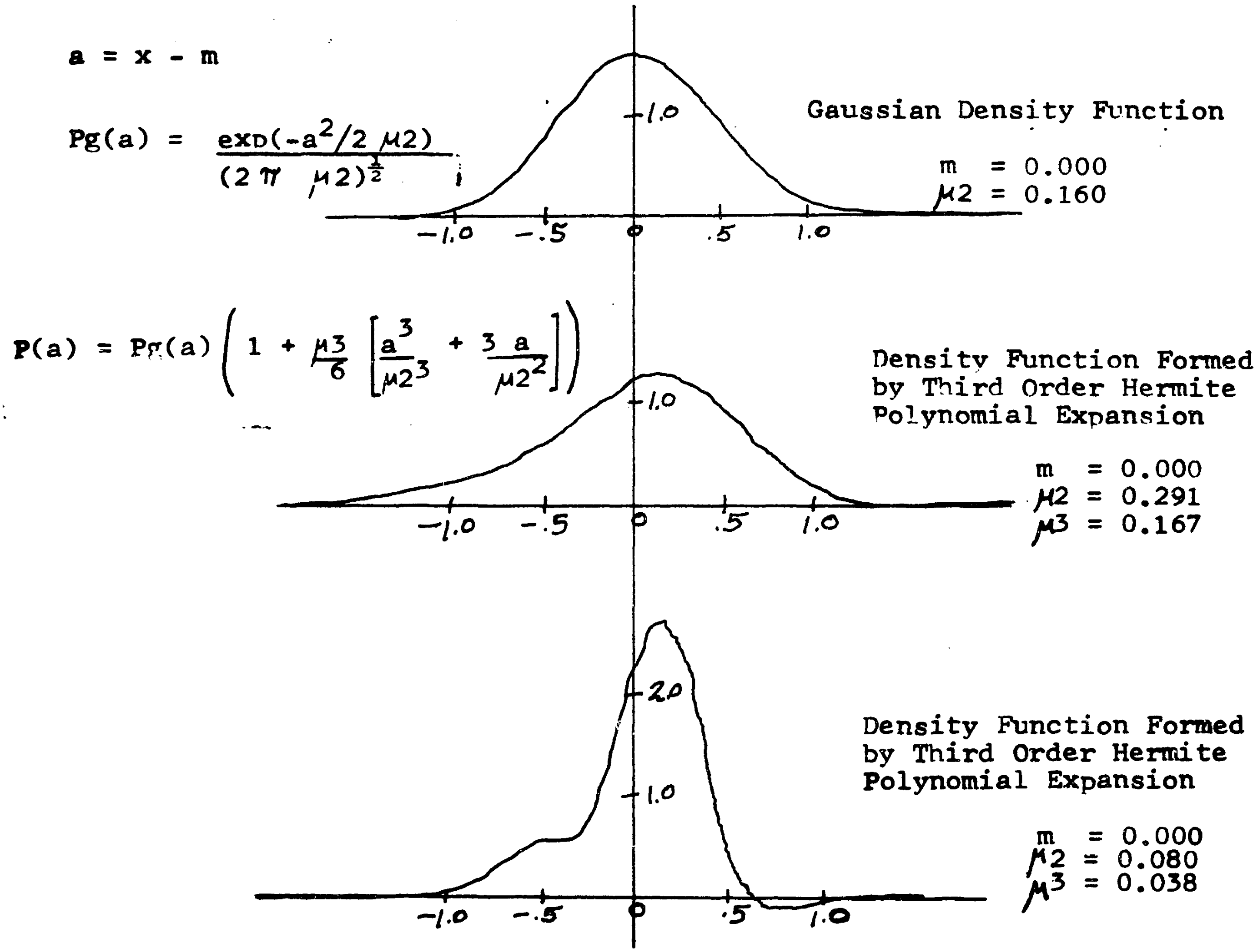

Figure 2.1 Probability Density Functions

Of course practical estimators usually compute the central moments of the density function rather than the density function itself. Nevertheless these central moments and the assumptions used in the estimation process do define a density function. To illustrate the probability distributions implied by linear and nonlinear estimators, and their relationships with the actual distribution of estimation errors, these estimated and actual distributions were computed and plotted for a scalar system. A linearized Kalman filter and a third order Hermite polynomial filter (optimal for the third order density model) were applied to linear and nonlinear systems, with and without linear measurements. After 100 Monte Carlo runs, the average error distribution implied by the filter estimates was plotted, along with a continuous density function matching the first seven central moments of the actual estimation error. Although the actual system state distribution is identical in all cases, the actual estimation error is not, as the different filters have different estimates for the mean. These density function plots are given in Figures 2.2 through 2.7 .

An ideal method of testing a nonlinear filter would be to obtain a history of its actual error distribution by Monte Carlo methods, and compare this with the true conditional density of the state given the measurements. The variance of this conditional distribution would be the minimum theoretically achievable. Comparison of the actual error density with the error density of the true optimal conditional mean filter would readily show the closeness to optimality of the estimation method being tested.

Unfortunately the true conditional density is impossible to obtain, since it cannot be produced by finite computation.

An alternative testing procedure is suggested by noting that for a true optimal filter, the estimated error density will be equal to the actual error density. For a finite filter operating on a nonlinear system, these will necessarily be different, but their closeness is an indication of how close the filter is to the theoretical optimal. Since these two densities can be readily obtained, it is a simple matter to plot them together and observe how close the filter estimates of second and third moments are to reality.

For a true optimal filter, the estimated statistics would be identical to the actual statistics to within the accuracy of the testing method. For the Kalman filter operating on a linear system, this close correspondence is seen in Figures 2.2 and 2.3 for the 100 run test. When the linearized Kalman filter operates on a nonlinear system, as is Figures 2.4 and 2.6, the assumed gaussian density can no longer closely match the actual nonsymmetric distribution. The third order filter, as seen in Figures 2.5 and 2.7 , is able to more closely approximate the true distribution and hence achieves better performance. A higher order nonlinear filter could be expected to achieve a still closer approximation, at the cost of additional computational complexity. Unlike the linear estimator, no finite nonlinear filter could ever exactly match the true error statistics. Nonlinear estimation is inherently approximate.

From these examples, it seems clear that the third order filter is a substantial improvement over the linearized Kalman filter for nonsymmetric systems. Knowledge of the prior third moments allows the third order filter to use the measurement information to substantially reduce the posterior third moments. This explains the rather surprising result that the actual estimation error is closer to gaussian for the third order filter than for the second order filter, which assumes a gaussian distribution.

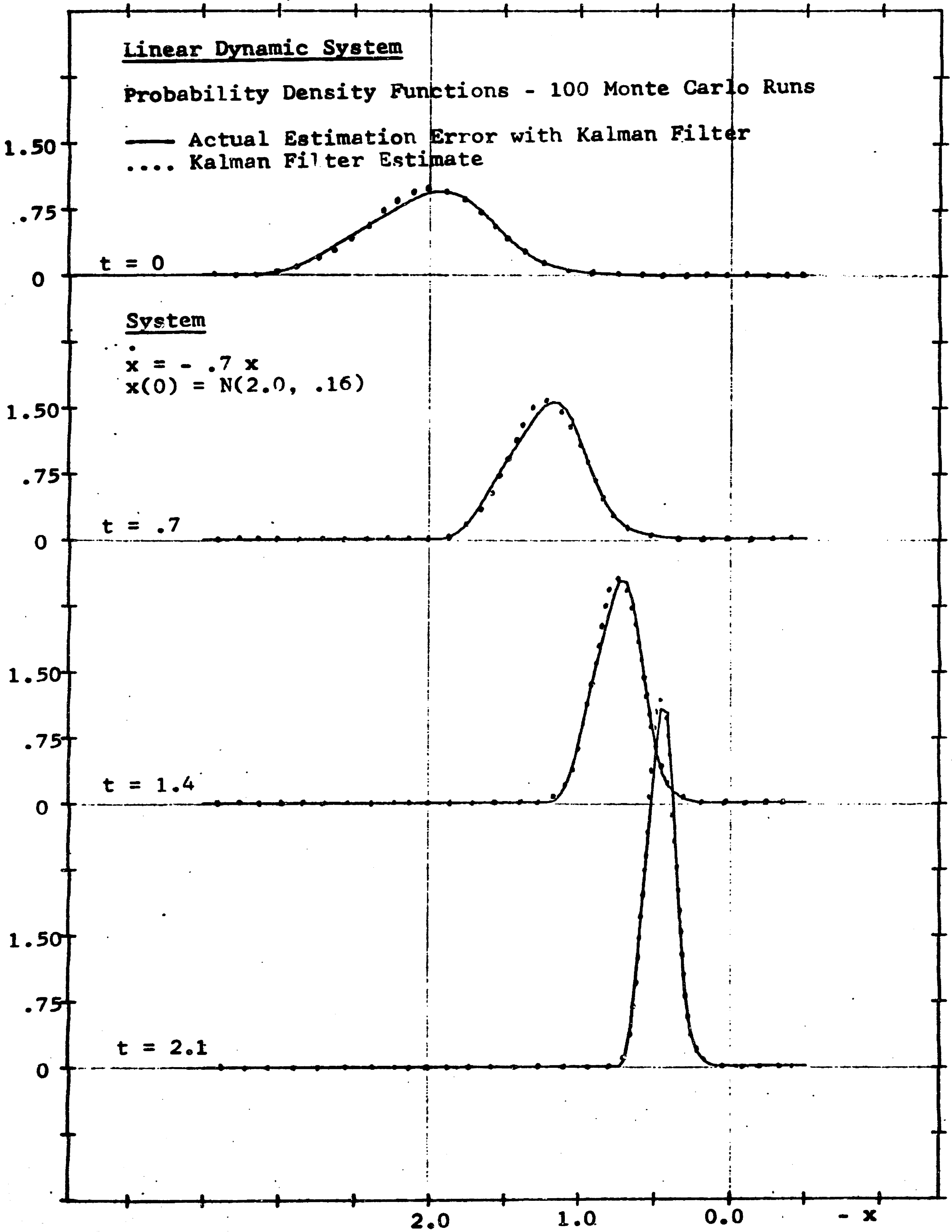

Figure 2.2 Probability Density Functions - Linear Dynamic System with Kalman Filter

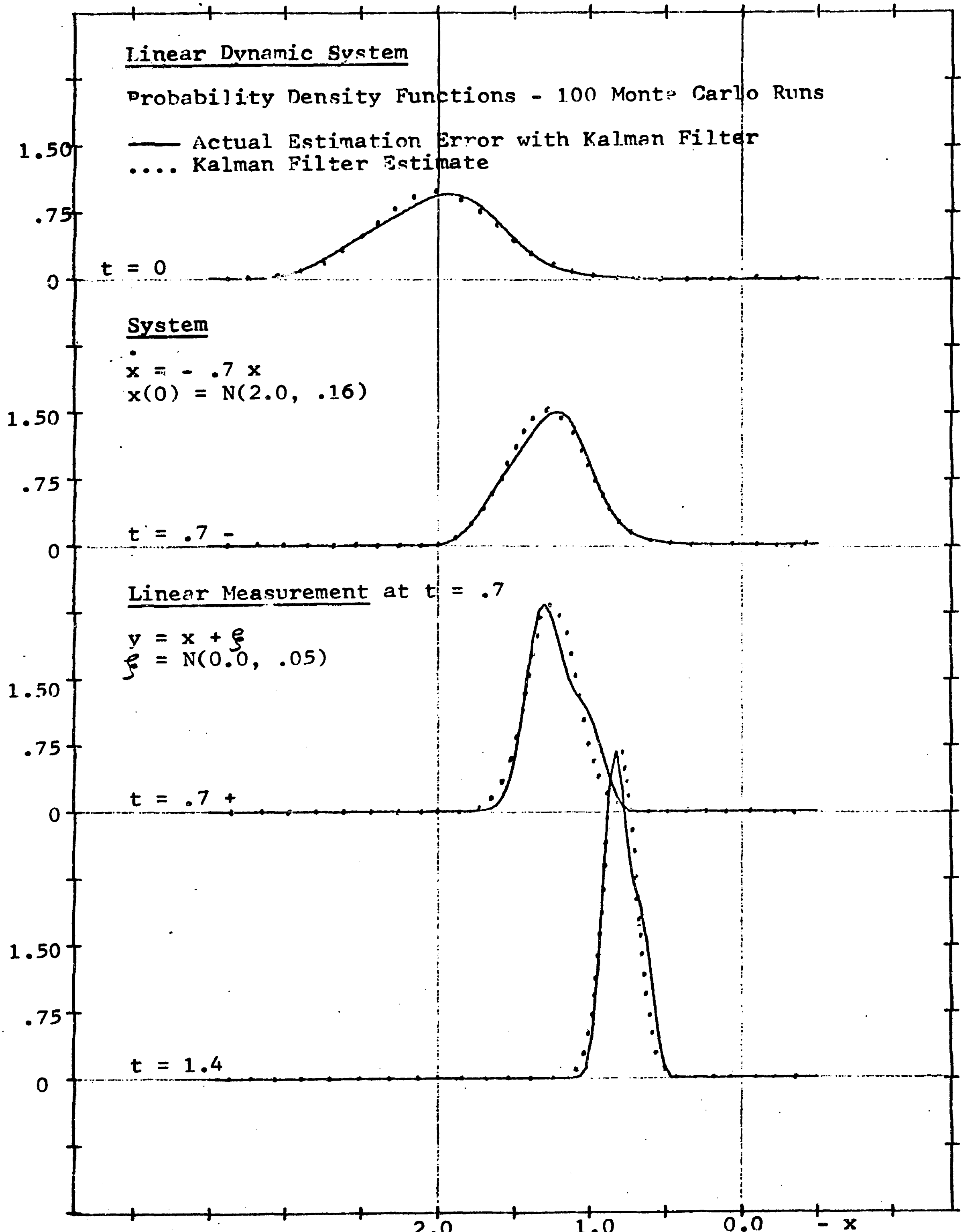

Figure 2.3 Probability Density Functions - Linear Dynamic System with Kalman Filter and Measurement

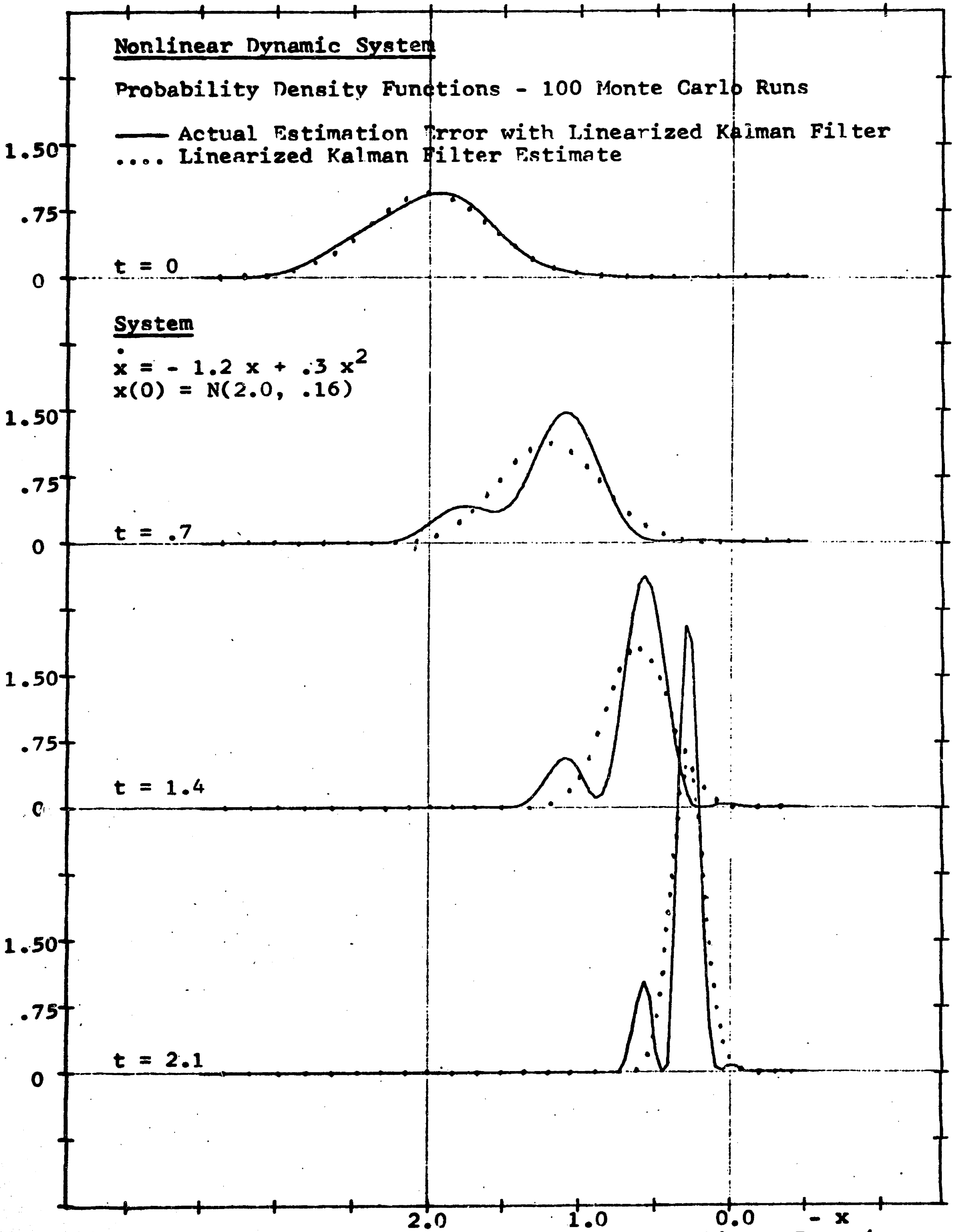

Figure 2.4 Probability Density Functions - Nonlinear Dynamic System with Linearized Kalman Filter

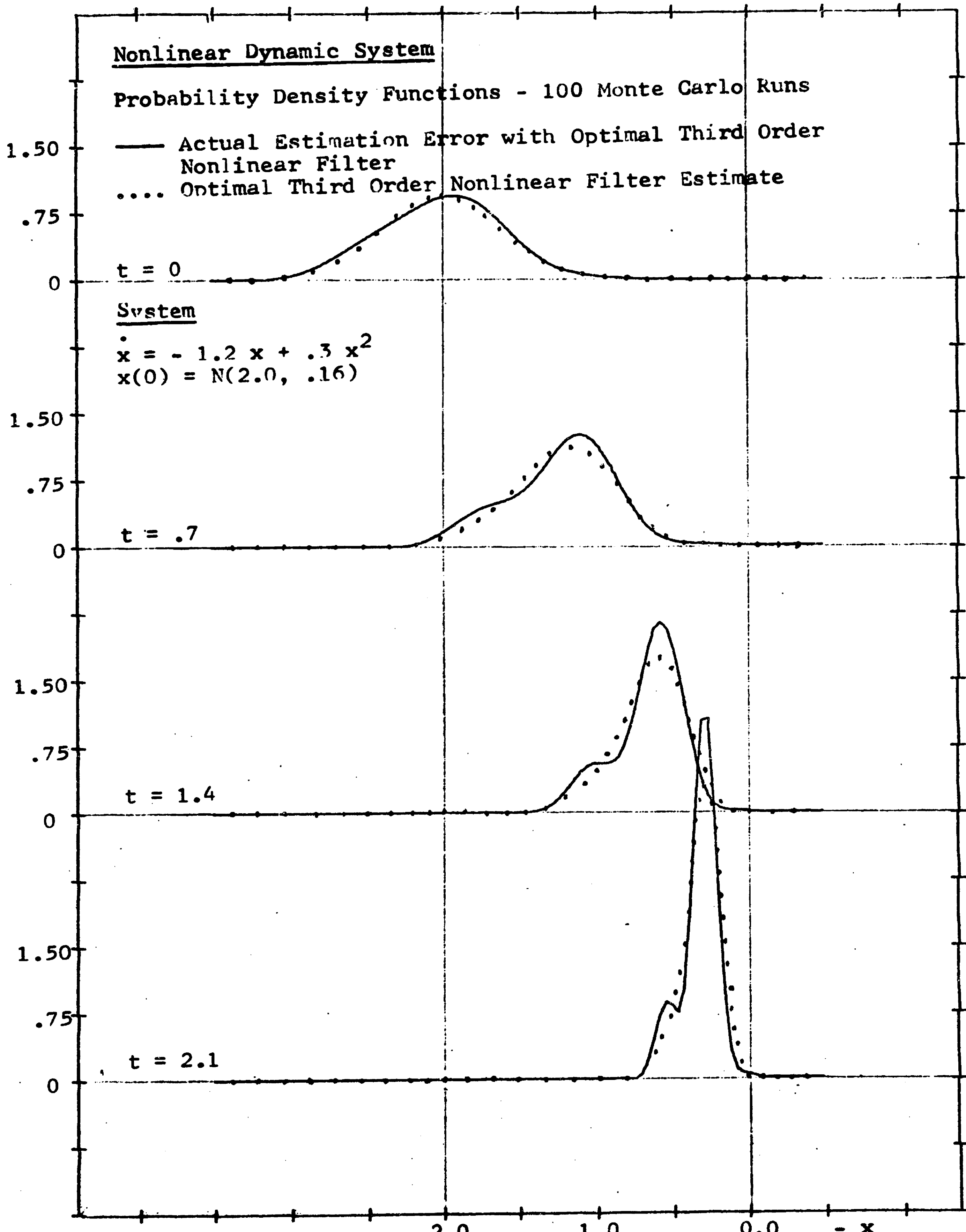

Figure 2.5 Probability Density Functions - Nonlinear Dynamic System with Optimal Third Order Nonlinear Filter

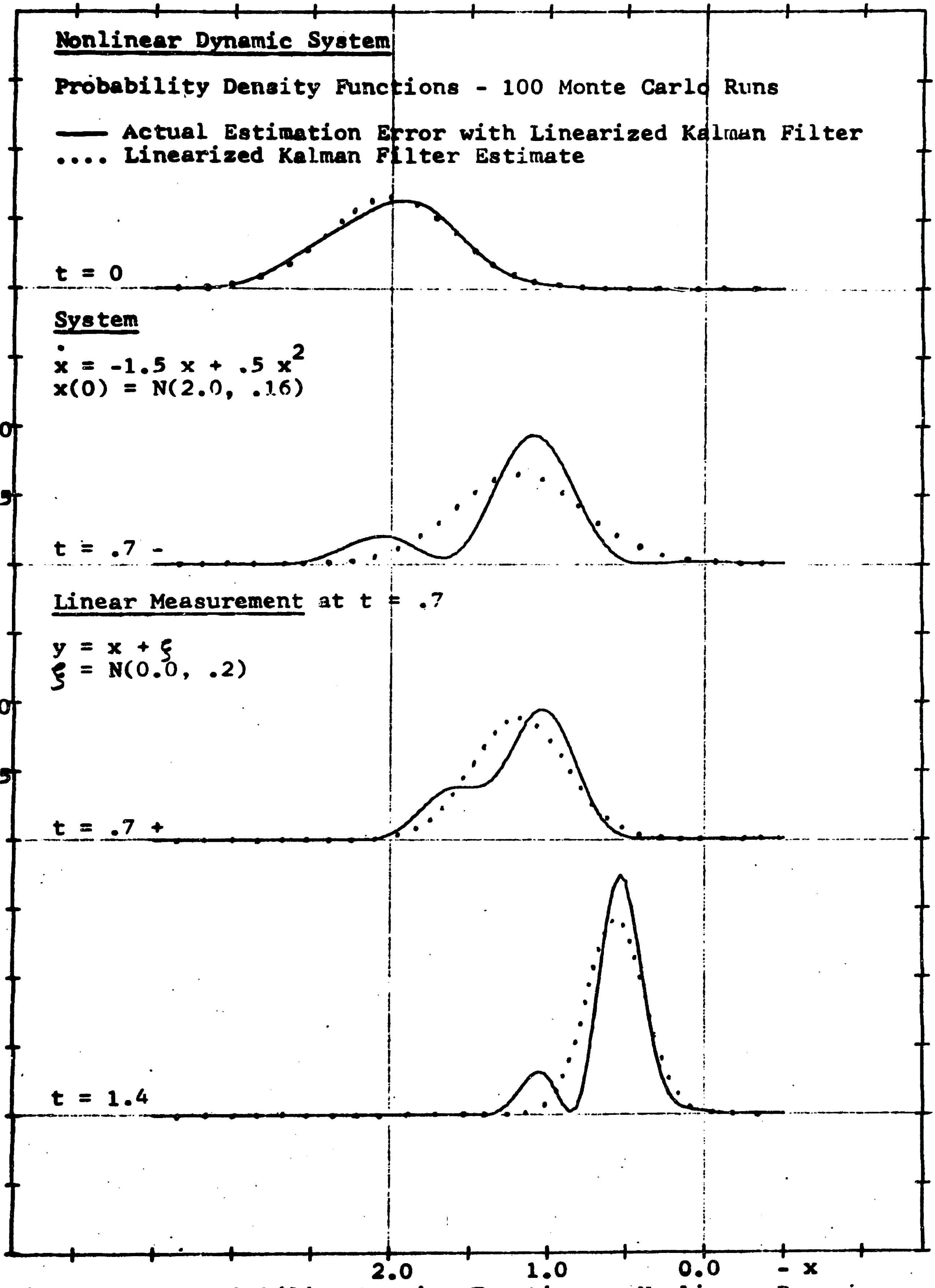

Figure 2.6 Probability Density Functions - Nonlinear Dynamic System with Linearized Kalman Filter and Measurement

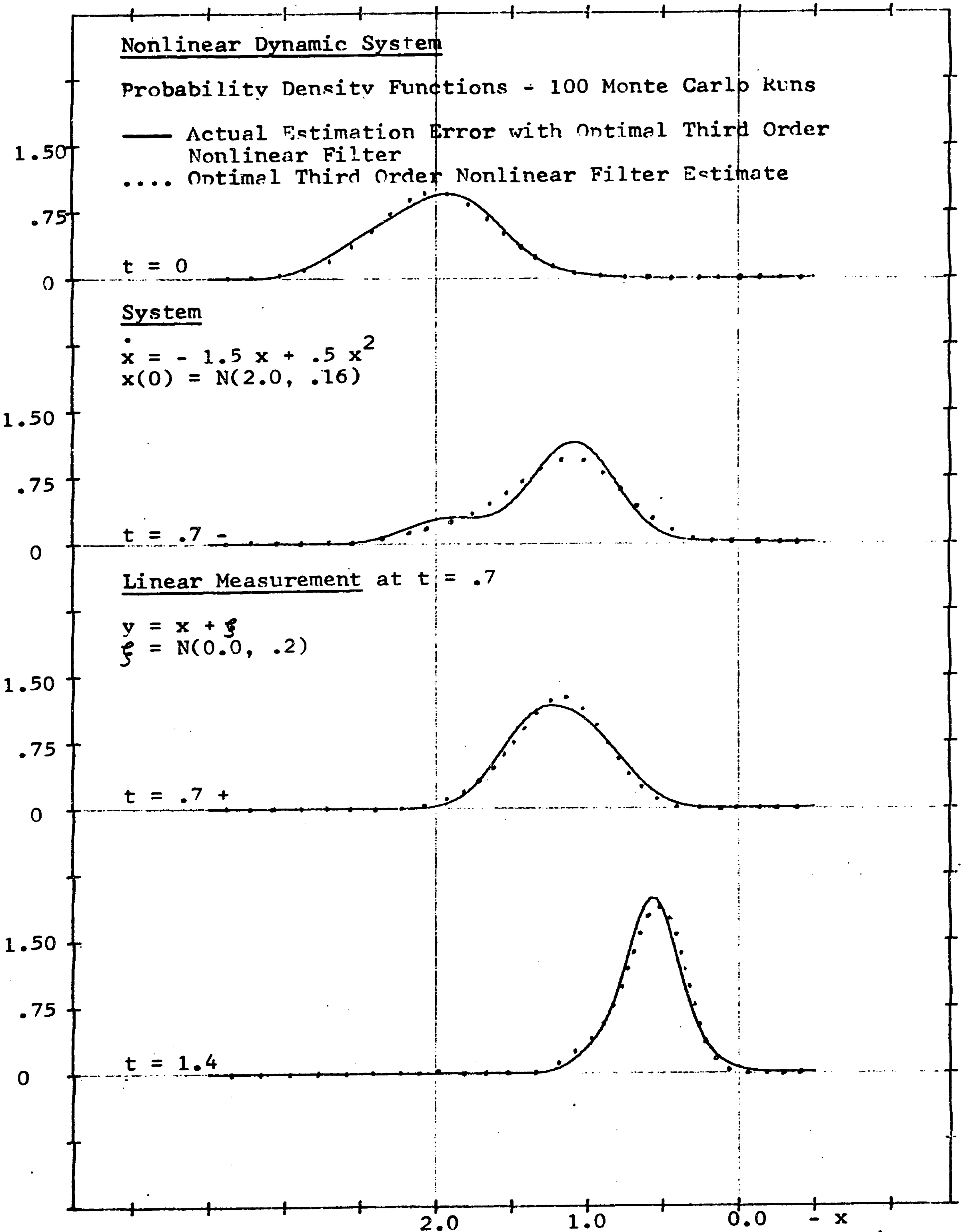

Figure 2.7 Probability Density Functions - Nonlinear Dynamic System with Optimal Third Order Nonlinear Filter and Measurement

Chapter 3

System and Measurement Model

3.1 Introduction

Continuous nonlinear systems with continuous measurements will first be considered. The following very general form will initially be used in the development of the basic theory. Subsequently a more specific form will be used to generate practical filter equations.

$$\frac{dx_i}{dt}(t) = f_i(\underline{x},t) + \eta_i(t) \qquad \text{System} \qquad (3.1)$$

$$y_r(t) = h_r(\underline{x},t) + \zeta_r(t) \qquad \text{Measurement} \qquad (3.2)$$

In the above, $\underline{x}$ is an N dimensional state vector, $\underline{y}$ is an M dimensional measurement vector, $f(\underline{x},t)$ and $h(\underline{x},t)$ are instantaneous nonlinear functions, and $\eta_i(t)$ and $\zeta_r(t)$ are correlated gaussian white noise processes. That is,

$$\mathcal{E}(\eta_i(t)\eta_j(\tau)) = V_{ij}(t) \quad \delta(t-\tau) \qquad (3.3)$$

$$\mathcal{E}(\zeta_r(t)\zeta_s(\tau)) = \Gamma_{rs}(t) \quad \delta(t-\tau)$$

$$\mathcal{E}(\eta_i(t)\zeta_s(\tau)) = \mathcal{E}(\eta_i(t)) = \mathcal{E}(\zeta_s(t)) = 0 \qquad (3.4)$$

Equations (3.1) and (3.2) can be written in a more convenient equivalent form involving only unit uncorrelated white noise processes. The equivalent equations are as follows, with n and s representing dummy subscripts which indicate summation.

$$\frac{dx_i}{dt}(t) = f_i(\underline{x},t) + G_{in}(t)\, v_n(t) \qquad (3.5)$$

$$y_r(t) = h_r(\underline{x},t) + R_{rs}(t)\,\xi_s(t) \tag{3.6}$$

Where:

$$\begin{aligned} G_{in}(t)\,G_{jn}(t) &= V_{ij}(t) \\ R_{rp}(t)\,R_{sp}(t) &= \Gamma_{rs}(t) \end{aligned} \tag{3.7}$$

$$\begin{aligned} \mathcal{E}(v_n(t)\,v_m(\tau)) &= \delta_{mn}\,\delta(t-\tau) \\ \mathcal{E}(\xi_r(t)\,\xi_s(\tau)) &= \delta_{rs}\,\delta(t-\tau) \\ \mathcal{E}(v_n(t)) = \mathcal{E}(\xi_r(t)) &= 0 \end{aligned} \tag{3.8}$$

V and Γ are the covariance matrices of the integrals of the state driving noise and the measurement noise over a unit time, respectively. They may also be thought of as power spectral density matrices for these white noises.

The matrices G and R are called square root matrices, since $G\,G^T = V$ and $R\,R^T = \Gamma$. If one is given V, G is difficult to find analytically, since the solution of N^2 simultaneous quadratic equations is required. Square root matrices can easily be found using iterative numerical techniques, however.

The tensor notation used above enjoys a number of advantages over the more usual vector notation. The order of the terms is obvious from the number of subscripts, the operations necessary for a digital computer solution of each equation are made obvious, and higher order terms and operations can be handled that cannot be described by vector notation. Principally for the last reason, this notation will be used almost exclusively in the following development. For those unfamiliar with this notation, a brief explanation is given in Appendix A.

3.2 White Noise

White noise can be described, in an engineering sense, as a random process with an autocorrelation function which is a Dirac δ function. Equivalently, it can be said to have a constant power spectral density for all frequencies. It is apparent that such a process does not exist in nature, since it would contain infinite signal power. The white noise concept is similar to the

δ function concept, in that both are limits of realizable functions. The δ function can be considered as the limit of pulses of decreasing width and unit area, while white noise can be considered as the limit of a sequence of random step functions of decreasing width and inversely increasing variance.

Both these concepts have important engineering application in dynamic systems. Any pulse of short duration compared to the fastest time constant of a linear system can be idealized as a δ function, while any wide bandwidth noise entering a narrow bandwidth system can be idealized as a white noise for purposes of computing the system response.

Nevertheless, these "functions" are mathematically intractable since they and their derivatives are unbounded. Rigorously speaking, these "functions" can only be defined in terms of their integrals. That is, the integral of a δ function is a step function, while the integral of white noise is a Brownian motion, or Wiener process.

Unfortunately these integrals cannot be treated by the ordinary theory of integration. The reason is that a Brownian motion is almost nowhere differentiable, almost surely. (That is, the Brownian motion fails to be differentiable somewhere in every interval of nonzero length, with probability one.) Thus, if b is a Brownian motion, $\frac{db}{dt}$ is unbounded and $\int g(t)\left(\frac{db}{dt}\right) dt$ is not defined. However, if db is an increment of b it has a well defined stochastic description, and the Stieltjes integral $\int g(t)\, db(t)$ is a possibly meaningful alternate form. Nevertheless b is not a function of bounded variation, and the integral can only be defined as a limit in mean. When this limit is taken the approximating sums converge in probability to a function defined as a stochastic integral. These integrals have somewhat different properties than ordinary integrals, and are discussed in detail by Skorokhod. (B-15) From an engineering standpoint, the most important result from the theory of stochastic integration is summarized by Ito's lemma, which follows in the next section.

3.3 Stochastic Differential Equations

Since the system differential equation is driven by white noise, it can rigorously only be treated incrementally, and then using the theory of stochastic integration. From (3.5), the incremental system equation becomes,

$$x_i(t) - x_i(t_o) = \int_{t_o}^{t} f_i(\underline{x},\tau)\, d\tau + \int_{t_o}^{t} G_{in}(\underline{x},\tau) dw_n(\tau) \qquad (3.9)$$

The first integral of (3.9) is an ordinary Riemann integral, while the remaining M integrals are stochastic integrals. If $w_n(t)$ is the unit Brownian motion (also called a Wiener process) corresponding to $v_n(t)$, $\left[\text{IE, } v_n(t) = \frac{dw_n}{dt}(t)\right]$, then (3.9) can be written in the simplified form,

$$dx_i = f_i(\underline{x},t)\, dt + G_{in}(\underline{x},t)\, dw_n \qquad (3.10)$$

The incremental form (3.10) is called a stochastic differential equation, and is understood to be equivalent to (3.9).

Similarly, the measurement equation (3.6) can be written as a stochastic differential equation with the formal substitutions $y_r(t) = \frac{dz_r}{dt}(t)$ and $\xi_s(t) = \frac{db_s}{dt}(t)$.

The measurement equation then becomes,

$$dz_r(t) = h_r(\underline{x},t)\,dt + R_{rs}(t)\,db_s(t) \tag{3.11}$$

The principal difference between stochastic differential equations and ordinary differential equations can be summarized by the following result, known as Ito's lemma. The scalar version of this result is proved in Skorokhod (B-15), and the vector version follows simply.

If a scalar function $\phi(\underline{x}(t),t)$ is continuous, and has a continuous partial derivative with respect to time and continuous first and second partial derivatives with respect to $\underline{x}$, then the process $y(t) = \phi(\underline{x},t)$ satisfies the following stochastic differential equation.

$$dy = \left[\frac{\partial \phi}{\partial t} + f_i \frac{\partial \phi}{\partial x_i} + \frac{1}{2} V_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j}\right] dt + G_{ij} \frac{\partial \phi}{\partial x_i}\, dw_j \tag{3.12}$$

This result is useful in the derivation of stochastic differential equations which are functions of other stochastic differential equations, and can be written in the following simpler form using the tensor notation for partial derivatives with respect to the state variables.

$$dy = \left[\frac{\partial \phi}{\partial t} + f_i\, \phi_{,i} + \frac{1}{2} V_{ij}\, \phi_{,ij}\right] dt + G_{ij}\, \phi_{,i}\, dw_j \tag{3.13}$$

Chapter 4

Optimal Estimation for Continuous Systems with Continuous Measurements

4.1 The Conditional Probability Density

It was Stratonovich who first suggested that the fundamental entity in sequential estimation is the conditional probability density of the state given the observed measurement history. This density in fact combines all past and present information, including a priori information, and represents a complete probabilistic description of the system according to an observer with knowledge of only the initial density function, the system dynamics, and the measurement history. From this density any statistic regarding the state can be computed, and an "optimal" estimate of the state, as well as the statistics of the estimate, can be prepared according to any criterion.

4.2 A Differential Equation for the Conditional Density

It was shown by Kolmogorov that the probability density of a Markov process obeys a partial differential diffusion equation. Early attempts to extend the Kolmogorov equation to systems with measurements contained errors (P-32,P-33). and the first correct formal derivation was by Kushner (P-16)in 1964. This has been followed by a number of other formal derivations, and by a rigorous derivation by the same author in 1967. (P-19)

The following derivation of the differential equations for the conditional probability density is somewhat different from those mentioned in the literature, although it follows most closely the work of Lipster and Time. (P-23) Rigor is sacrificed for clarity throughout, with the objective of making the development as independent of specialized mathematics as possible.

The system is described by the following vector stochastic differential equations.

$$dx_i = f_i(\underline{x},t)\ dt + G_{in}(t)\ dw_n(t) \tag{4.1}$$

The continuously observed measurement process is described by,

$$dz_r = h_r(\underline{x},t)\ dt + R_{rs}(t)\ db_s(t) \tag{4.2}$$

$w_n(t)$ and $b_s(t)$ are uncorrelated unit Wiener processes. That is,

$$\mathcal{E}(w_n(t)) = \mathcal{E}(b_s(t)) = 0$$
$$\mathcal{E}\left[(w_m(t)-w_m(\tau))(w_n(t)-w_n(\tau))\right] = \delta_{mn}\ |t-\tau|$$
$$\mathcal{E}\left[(b_r(t)-b_r(\tau))(b_s(t)-b_s(\tau))\right] = \delta_{rs}\ |t-\tau|$$
$$\mathcal{E}\left[(b_r(t)-b_r(\tau))(w_n(t)-w_n(\tau))\right] = 0 \tag{4.3}$$

The state driving noise power spectral density matrix V_{ij} and the measurement noise power spectral density matrix Γ_{rs} are given by,

$$V_{ij}(t) = G_{in}(t)\ G_{jn}(t) \tag{4.4}$$

$$\Gamma_{rs}(t) = R_{rp}(t)\ R_{sp}(t) \tag{4.5}$$

The system is completely described by the stochastic differential equations (4.1) and (4.2), together with the random disturbance description (4.3)-(4.5).

Let $P(\underline{a},t/Z)$ be the probability density function of the state vector $\underline{a}$ at time t, given Z, the set of all measurements $\underline{z}(\tau)$, $0 \leq \tau \leq t$. The derivation proceeds by considering the finite difference model (4.6) and subsequently taking limits.

$$\begin{aligned} \delta x_i &= f_i \, \delta t + G_{in} \, \delta w_n \\ \delta z_r &= h_r \, \delta t + R_{rs} \, \delta b_s \end{aligned} \tag{4.6}$$

A stochastic differential equation for the probability density of the state, conditioned on the measurements up to the current time, can be derived by developing an expression for the differential,

$$\delta P = P(\underline{a}, t+\delta t/Z, \delta \underline{z}) - P(\underline{a}, t/Z) \tag{4.7}$$

The above differential is composed of two distinct components; one due to the operation of the system for a time δt, and the other due to the receipt of measurement information $\delta \underline{z}$. If δP is obtained as a power series expansion of δt and $\delta \underline{z}$, then only first order terms will remain as the limit $\delta t \to o$ is taken.

An expression for $P(\underline{a}, t+\delta t/Z, \delta \underline{z})$ is difficult to obtain. This can be done, however, through the probability transition law for a Markov process, which states that;

$$P(\underline{a}, t+\delta t/Z, \delta \underline{z}) = \int_{R_N} P(\underline{a}, t+\delta t/\underline{x}, t, Z, \delta \underline{z}) \, P(\underline{x}, t/Z, \delta \underline{z}) \, d\underline{x} \tag{4.8}$$

The shorthand notation for multiple infinite integrals is as follows.

$$\int_{R_N} \phi \, d\underline{x} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi \, dx_1 \, dx_2 \cdots dx_N \tag{4.9}$$

Using the Bayes formula for conditional densities, $P(\underline{x},t/Z,\delta\underline{z})$ is found.

$$P(\underline{x},t/Z,\delta\underline{z}) = \frac{P(\underline{x},t;Z,\delta\underline{z})}{P(Z;\delta\underline{z})} = \frac{P(\delta\underline{z}/\underline{x},t;Z)\, P(\underline{x},t;Z)}{P(Z)\, P(\delta\underline{z}/Z)} \tag{4.10}$$

$$P(\underline{x},t/Z,\delta\underline{z}) = \frac{P(\delta\underline{z}/\underline{x},t;Z)\, P(\underline{x},t/Z)}{\int_{R_N} P(\delta\underline{z}/\underline{\alpha},t;Z)\, P(\underline{\alpha},t/Z)\, d\underline{\alpha}} \tag{4.11}$$

Equation (4.11) can be obtained directly from Bayes theorem. (See Rao, (B-14), p 272)

By hypothesis, $P(\delta\underline{z}/\underline{x},t;Z)$ is a multidimensional gaussian probability density function with mean $h_r(\underline{x})\delta t$ and covariance $\Gamma_{rs}\delta t$. That is,

$$P(\delta\underline{z}/\underline{x},t;Z) = \frac{\exp\left[-\tfrac{1}{2}(\delta t)^{-1}(\delta z_r - h_r(\underline{x})\delta t)\,\Gamma_{rs}^{-1}(\delta z_s - h_s(\underline{x})\delta t)\right]}{(2\pi)^{N/2}\,|\Gamma_{rs}|^{\frac{1}{2}}\,(\delta t)^{\frac{1}{2}}} \tag{4.12}$$

Substituting (4.12) into (4.11),

$$P(\underline{x},t/Z,\delta\underline{z}) = \frac{P(\underline{x},t/Z)\,\exp\left[\frac{-1}{2\delta t}(\delta z_r - h_r(\underline{x})\delta t)\,\Gamma_{rs}^{-1}(\delta z_s - h_s(\underline{x})\delta t)\right]}{\int_{R_N} P(\underline{\alpha},t/Z)\,\exp\left[\frac{-1}{2\delta t}(\delta z_r - h_r(\underline{\alpha})\delta t)\,\Gamma_{rs}^{-1}(\delta z_s - h_s(\underline{\alpha})\delta t)\right] d\underline{\alpha}} \tag{4.13}$$

Equation (4.13) can be simplified by deleting the common term $\exp\left[\frac{-1}{2\delta t}\delta z_r \Gamma_{rs}^{-1}\delta z_s\right]$ from both the numerator and denominator.

$$P(\underline{x},t/Z,\delta\underline{z}) = \frac{P(\underline{x},t/Z)\ \exp\left[\delta z_r \Gamma_{rs}^{-1} h_s(\underline{x}) - \frac{1}{2} h_r(\underline{x}) \Gamma_{rs}^{-1} h_s(\underline{x})\,\delta t\right]}{\int_{R_N} P(\underline{\alpha},t/Z)\ \exp\left[\delta z_r \Gamma_{rs}^{-1} h_s(\underline{\alpha}) - \frac{1}{2} h_r(\underline{\alpha}) \Gamma_{rs}^{-1} h_s(\underline{\alpha})\,\delta t\right] d\underline{\alpha}} \tag{4.14}$$

It is now desired to obtain an expansion of (4.14) which contains all terms of order δt or lower. Since $\mathcal{E}(\delta z_r \delta z_s) = \Gamma_{rs}\delta t$, the expansion must be carried to the second degree in the components of $\delta\underline{z}$, and to the first degree in δt. This requirement is quantitatively expressed by Ito's lemma (2.13), which in this case (considering only variations in $\underline{z}$ over the interval δt) states that;

$$\delta P = \left[\frac{\partial P}{\partial t} + \frac{1}{2}\Gamma_{rs}\frac{\partial^2 P}{\partial z_r \partial z_s}\right]\delta t + \frac{\partial P}{\partial z_r}\delta z_r \tag{4.15}$$

From (4.14),

$$\left.\frac{\partial P(\underline{x},t/Z,\delta\underline{z})}{\partial t}\right|_{\substack{\delta t=0\\ \delta\underline{z}=0}} = P(\underline{x},t/Z)\left[-\frac{1}{2}\left\{h_r(\underline{x})\Gamma_{rs}^{-1}h_s(\underline{x}) - \overline{h_r \Gamma_{rs}^{-1} h_s}\right\}\right] \tag{4.16}$$

The overhead bar refers to expectation with respect to $P(\underline{x},t/Z)$. Similarly,

$$\left.\frac{\partial P(\underline{x},t/Z,\delta\underline{z})}{\partial z_r}\right|_{\substack{\delta t=0\\ \delta\underline{z}=0}} = P(\underline{x},t/Z)\left[\Gamma_{rs}^{-1}(h_s(\underline{x}) - \overline{h_s})\right] \tag{4.17}$$

$$\left.\frac{\partial^2 P(\underline{x},t/Z,\ \underline{z})}{\partial z_r \partial z_s}\right|_{\substack{\delta t=0\\ \delta\underline{z}=0}} = P(\underline{x},t/Z)\left[\Gamma_{rs}^{-2}\left\{h_r(\underline{x})h_s(\underline{x}) - 2h_r(\underline{x})\,\overline{h_s} + 2\,\overline{h_r}\,\overline{h_s} - \overline{h_r h_s}\right\}\right] \tag{4.18}$$

Substituting (4.15)-(4.18) into (4.14), the following expression is obtained.

$$P(\underline{x},t/Z,\delta\underline{z}) = P(\underline{x},t/Z)\left[1 + (h_r(\underline{x})-\overline{h_r})\Gamma^{-1}_{rs}(\delta z_s-\overline{h_s}\delta t) + O(\delta t)\right] \tag{4.19}$$

From the formula for total probability (4.8),

$$P(\underline{a},t+\delta t/Z,\delta\underline{z}) = \int_{R_N} P(\underline{a},t+\delta t/\underline{x},t,Z,\delta\underline{z})\; P(\underline{x},t/Z)\Big[1 + (h_r(\underline{x})-\overline{h_r})\,\Gamma^{-1}_{rs}(\delta z_s-\overline{h_s}\delta t) + O(\delta t)\Big]\, d\underline{x} \tag{4.20}$$

$P(\underline{a},t+\delta t/\underline{x},t)$ can be obtained from the well known Fokker-Planck equation, which describes the temporal evolution of the probability density for a system without observations.

$$\frac{dP}{dt} = -(f_i P)_{,i} + \tfrac{1}{2}(V_{ij}P)_{,ij} \tag{4.21}$$

From the Fokker-Planck equation, considering only variations in the state over the interval δt,

$$\begin{aligned}\delta P &= P(\underline{a},t+\delta t/\underline{x},t) - P(\underline{a},t/\underline{x},t)\\ &= \Big[-(f_i P)_{,i} + \tfrac{1}{2}(V_{ij}P)_{,ij}\Big]\,\delta t + O(\delta t)\end{aligned} \tag{4.22}$$

The substitution of (4.22) into (4.20) yields the following.

$$P(\underline{a},t+\delta t/Z,\delta\underline{z}) = \int_{R_N}\Big[1 - (f_i P)_{,i}\delta t + \tfrac{1}{2}(V_{ij}P)_{,ij}\,\delta t + O(\delta t)\Big]\, P(\underline{a},t/\underline{x},t)\; P(\underline{x},t/Z)\Big[1 + (h_r(\underline{x})-\overline{h_r})\Gamma^{-1}_{rs}(\delta z_s-\overline{h_s}\,\delta t) + O(\delta t)\Big]\, d\underline{x} \tag{4.23}$$

By definition, $P(\underline{a},t/\underline{x},t)$ is the N dimensional Dirac delta function $\delta(\underline{a}-\underline{x})$. The integral in equation (4.23) thus reduces to a simple form.

$$P(\underline{a},t+\delta t/Z,\delta \underline{z}) = \Big[1+(f_iP)_{,i}\delta t + \tfrac{1}{2}(V_{ij}P)_{,ij}\delta t + O(\delta t)\Big]$$
$$P(\underline{a},t/Z)\Big[1 + (h_r(\underline{a})-\overline{h_r})\Gamma^{-1}_{rs}(\delta z_s-\overline{h_s}\ \delta t) + O(\delta t)\Big] \tag{4.24}$$

If now the limit is taken as $\delta t \to 0$, the $O(\delta t)$ terms become negligible and equation (4.24) becomes the differential equation for the conditional probability density function.

$$\frac{dP}{dt} = -(f_iP)_{,i} + \tfrac{1}{2}(V_{ij}P)_{,ij} + P(h_r-\overline{h_r})\Gamma^{-1}_{rs}(y_s-\overline{h_s}) \tag{4.25}$$

In practice, it is usually preferable to work with functions of the state, such as central moments, rather than the probability density itself. In order to derive an equation for the propagation of the moments, let $g(\underline{a})$ be an arbitrary function, having continuous partial derivatives up to third order inclusive. Then the expectation of g based on knowledge of the system and the measurements will be given by,

$$\mathcal{E}(g(\underline{a}(t+\delta t))) = \int_{R_N} g(\underline{a})P(\underline{a},t+\delta t/Z,\delta\underline{z})d\underline{a} \tag{4.26}$$

From equation (4.20),

$$\mathcal{E}(g(\underline{a}(t+\delta t))) = \int_{R_N}\int_{R_N} g(\underline{a})P(\underline{a},t+\delta t/\underline{x},t)\ P(\underline{x},t/Z)$$
$$\Big[1+(h_r(\underline{x})-\overline{h_r})\Gamma^{-1}_{rs}(\delta z_s-\overline{h_s}\delta t) + O(\delta t)\Big]\ d\underline{x}\ d\underline{a} \tag{4.27}$$

$g(\underline{a})$ can be expanded in a Taylor's series about the point $\underline{a} = \underline{x}$. The notation $\frac{\partial g}{\partial a_i} = g_{,i}$ will be used.

$$g(\underline{a}) = g(\underline{x}) + g_{,i}(a_i-x_i) + \tfrac{1}{2} g_{,ij}(a_i-x_i)(a_j-x_j) + O((a_i-x_i)^2) \tag{4.28}$$

Equation (4.28) can now be substituted into equation (4.27), and the result integrated with respect to $\underline{a}$. This yields,

$$\overline{g(t+\delta t)} = \int_{R_N}\Big[g(\underline{x}) + g_{,i}\Big\{\int_{R_N} P(\underline{a},t+\delta t/\underline{x},t)(a_i-x_i)d\underline{a}\Big\}$$
$$+\tfrac{1}{2}g_{,ij}\Big\{\int_{R_N} P(\underline{a},t+\delta t/\underline{x},t)(a_i-x_i)(a_j-x_j)d\underline{a}\Big\} + O(\delta t)\Big]$$
$$P(\underline{x},t/Z)\Big[1+(h_r(\underline{x})-\overline{h_r})\Gamma^{-1}_{rs}(\delta z_s-\overline{h_s}\delta t) + O(\delta t)\Big]\,d\underline{x} \tag{4.29}$$

The internal integrals in brackets are expectations and can be evaluated by considering the system dynamical description. Then,

$$\overline{g(t+\delta t)} = \int_{R_N}\Big[g(\underline{x}) + f_i g_{,i}\delta t + \tfrac{1}{2}V_{ij}g_{,ij}\delta t + O(\delta t)\Big]$$
$$P(\underline{x},t/Z)\Big[1+(h_r(\underline{x})-\overline{h_r})\Gamma^{-1}_{rs}(\delta z_s-\overline{h_s}\,\delta t) + O(\delta t)\Big]\,d\underline{x} \tag{4.30}$$

This is again simply an expectation integral, which simplifies to,

$$\overline{g(t+\delta t)-g(t)} = \delta\overline{g} = \Big[\overline{f_i g_{,i}} + \tfrac{1}{2}V_{ij}\overline{g_{,ij}}\Big]\delta t$$
$$+(\overline{gh_r} - \overline{g}\,\overline{h_r})\Gamma^{-1}_{rs}(\delta z_s-\overline{h_s}\,\delta t) + O(\delta t) \tag{4.31}$$

Taking the limit as $\delta t \to 0$, and neglecting the $O(\delta t)$ terms, the desired ordinary differential equation for the conditional expectation of an arbitrary differentiable function can be obtained.

$$\frac{d\bar{g}}{dt} = \overline{f_i\ g_{,i}} + \tfrac{1}{2}V_{ij}\ \overline{g_{,ij}} + (\overline{gh_r} - \bar{g}\ \overline{h_r})\Gamma^{-1}_{rs}(y_s - \overline{h_s}) \tag{4.32}$$

4.3 Representation by a Finite Parameter Set

The differential equation for the conditional density is not of direct practical importance, because its computation for a multidimensional system involves an inordinate amount of calculation, and because it is difficult to use directly even after it is computed.

It is possible, however, to represent the conditional density by equivalent sets of parameters that have a more useful form and are simpler to compute. Kushner (P-16) suggested the use of the conditional mean and central moments. That is, instead of the density function, the following parameters might be computed.

$$m_i(t) = \int_{R_N} P(\underline{x},t)\ x_i\ d\underline{x} \tag{4.33}$$

$$\mu 2_{ij}(t) = \int_{R_N} P(\underline{x},t)\ (x_i - m_i)\ (x_j - m_j)\ d\underline{x} \tag{4.34}$$

$$\mu k_{\underbrace{ij--n}_{K}} = \int_{R_N} P(\underline{x},t)\underbrace{(x_i - m_i)(x_j - m_j)\ ---\ (x_n - m_n)}_{K}\ d\underline{x} \tag{4.35}$$

The infinite set of central moments represents a complete set of sufficient statistics; that is, it contains all the information that the density function contains. Of course this infinite set cannot be calculated by a realizable filter, but then neither can the density function be computed with perfect accuracy by interpolation over a finite amount of data. It is then necessary to truncate the set (4.35) at some term k; that is, to compute only the central moments up to and including μ_k. It was suggested by Kushner and Schwartz (P-16) (T-5) that the moment set could be truncated with the assumption that all central moments of order higher than k were zero, if the filter error were sufficiently small.

In practice, the complexity of the multidimensional filter increases rapidly with increasing k. It is proved in appendix C that the k th moment of an N dimensional vector contains $\binom{N+k-1}{k}$ unique components. For this reason, it may be desirable to truncate the filter as early as possible. If the nonlinear filter were truncated after the first nonsymmetric moment (the third) this truncation method would lead to the assumption that the fourth moments are zero, which may lead to considerable inaccuracy and even instability. This truncation assumption in fact implies that the density function contains derivatives of the Dirac δ function of all orders.

In order to avoid these difficulties, Stratonovich (B-17), (P-21),(P-22) proposed that a more suitable parameter set would be the coefficients for the expansion of the density as a sum of orthogonal polynomials. Truncation to a finite number of these terms would imply a smooth and differentiable density function. Because of the orthogonality property, inclusion of additional terms would not change the optimal estimates for the preceeding terms. Multidimensional Hermite polynomials appear to be the most convenient choice from the class of orthogonal functions, since their moment integrals of all orders can be solved analytically. These functions and their properties are discussed in Appendix E.

Fisher (T-2) investigated the differential equations for the coefficients of the Hermite Polynomial expansion. These coincide with the central moments up to third order, and are called quasi-moment functions. The differential equations for the quasi-moments of order higher than three are extremely complicated, and furthermore do not have a simple physical interpretation as do the central moments.

Fortunately, the simplicity of the central moment equations can be combined with the analytic rigor of the quasi-moment truncation. If the parameter set is to be truncated after the k th quasi-moment, then the central moment equations up to k th order are computed. These always involve central moments of order greater than k for nonlinear systems. The higher order central moments, however, can be expressed in terms of the lower order central moments via the quasi-moment -- central moment relationships given in appendix E. This allows the construction of a closed form central moment filter while assuming an analytically rigorous quasi-moment truncation. This approach will be used in the following.

4.4 The Differential Equations for the Conditional Central Moments

As part of the derivation of the differential equation for the conditional density, a differential equation for the temporal evolution of an arbitrary twice differentiable function of the state was found. The differential equations for the central moments through fourth order will now be derived from this equation. The continuous state and measurement equations are,

$$\frac{dx_i}{dt}(t) = f_i(\underline{x},t) + G_{in}(t) \quad v_n(t) \tag{4.36}$$

$$y_r(t) = h_r(\underline{x},t) + R_{rs}(t) \quad \xi_s(t) \tag{4.37}$$

$$\begin{aligned} &\mathcal{E}(v_n(t)) = \mathcal{E}(\xi_s(t)) = 0 \\ &\mathcal{E}(v_m(t)\, v_n(\tau)) = \delta_{mn}\, \delta(t-\tau) \\ &\mathcal{E}(\xi_r(t)\, \xi_s(\tau)) = \delta_{rs}\, \delta(t-\tau) \\ &\mathcal{E}(v_n(t)\, \xi_r(\tau)) = 0 \end{aligned} \tag{4.38}$$

That is, $v_n(t)$ and $\xi_s(t)$ are unit gaussian white noises.

$G_{in}\, G_{jn} = V_{ij}$ = state driving noise power spectral density

$R_{rp}\, R_{sp} = \Gamma_{rs}$ = measurement noise power spectral density (4.39)

From (4.32), if $g(\underline{x})$ is an arbitrary twice differentiable function of $\underline{x}$, then,

$$\frac{d\overline{g}}{dt} = \overline{f_i\, g_{,i}} + \tfrac{1}{2}\, \overline{V_{ij}\, g_{,ij}} + (\overline{g\, h_r} - \overline{g}\, \overline{h_r})\Gamma^{-1}_{rs}(y_s - \overline{h_s}) \tag{4.40}$$

The overhead bar denotes conditional expectation. Now if x_i is taken as $g(\underline{x})$, the differential equation for the conditional mean of x_i, denoted m_i, will be obtained. Since the conditional expectation is always a mimimum variance estimate, m_i is an optimal estimate in the minimum variance sense.

From (4.40),

$$\frac{dm_i}{dt} = \frac{d\overline{x}_i}{dt} = \overline{f}_i + \overline{h_r(x_i - m_i)} \quad \Gamma^{-1}_{rs}(y_s - \overline{h_s}) \tag{4.41}$$

Since f_i is assumed to be an analytic function of $\underline{x}$, it can be expanded in a Taylor series about $\underline{m}$.

$$f_i(\underline{x},t) = f_i(\underline{m},t) + f_{i,p}(\underline{m},t)\,(x_p - m_p) + \tfrac{1}{2}\, f_{i,pq}(\underline{m},t)\,(x_p - m_p)(x_q - m_q) + \;--- \tag{4.42}$$

By definition,

$$\overline{(x_p - m_p)(x_q - m_q)} = \mu 2_{pq} \quad , \text{ etc.} \tag{4.43}$$

Therefore, from (4.42),

$$\overline{f_i} = f_i(\underline{m},t) + \tfrac{1}{2}\, f_{i,pq}\,\mu 2_{pq} + (1/6)\, f_{i,pqr}\,\mu 3_{pqr} + \;--- \tag{4.44}$$

From (4.44), it is clear that (4.41) can be written as a function of the central moments of order two and higher. Thus it is necessary to compute these moments in order to compute the conditional mean.

It should be noted that since these moments are functions of both $\underline{x}$ and $\underline{m}$, both of which are governed by stochastic differential equations, the partial differentiation implied by (4.41) must include both $\underline{x}$ and $\underline{m}$. First, the stochastic differential equation for $\underline{m}$ must be written in the standard form.

$$\frac{dm_i}{dt} = \beta_i(\underline{x},\underline{m},t) + \Lambda_{in}\xi_n \tag{4.45}$$

From (4.37) and (4.41),

$$\frac{dm_i}{dt} = \overline{f_i} + \overline{h_r(x_i-m_i)}\,\Gamma^{-1}_{rs}(h_s + R_{sn}\xi_n - \overline{h_s}) \tag{4.46}$$

$$\beta_i(\underline{x},\underline{m},t) = \overline{f_i} + \overline{h_r(x_i-m_i)}\,\Gamma^{-1}_{rs}(h_s - \overline{h_s}) \tag{4.47}$$

$$\Lambda_{in} = \overline{h_r(x_i-m_i)}\,\Gamma^{-1}_{rs}R_{sn} \tag{4.48}$$

Another power spectral density term, Ω_{ij}, can be defined as follows.

$$\Omega_{ij} = \Lambda_{in}\Lambda_{jn} \tag{4.49}$$

Now the differential equations for the conditional central moments can be derived from (4.40) remembering that the partial differentials are with respect to all the x_i and m_i.

$$\frac{d\mu 2_{ij}}{dt} = \frac{d\overline{(x_i-m_i)(x_j-m_j)}}{dt} = \overline{(f_i-\beta_i)(x_j-m_j)} + \overline{(f_j-\beta_j)(x_i-m_i)}$$
$$+ V_{ij} + \Omega_{ij} + (\overline{(x_i-m_i)(x_j-m_j)h_r} - \mu 2_{ij}\,\overline{h_r})\Gamma^{-1}_{rs}(y_s-\overline{h_s}) \tag{4.50}$$

$$\frac{d\mu 3_{ijk}}{dt} = \frac{d}{dt}\overline{(x_i-m_i)(x_j-m_j)(x_k-m_k)} = \overline{(f_i-\beta_i)(x_j-m_j)(x_k-m_k)}$$
$$+ \overline{(f_j-\beta_j)(x_i-m_i)(x_k-m_k)} + \overline{(f_k-\beta_k)(x_i-m_i)(x_j-m_j)}$$
$$+ (V_{ij}+\Omega_{ij})\overline{(x_k-m_k)} + (V_{ik}+\Omega_{ik})\overline{(x_j-m_j)} + (V_{jk}+\Omega_{jk})\overline{(x_i-m_i)}$$
$$+ (\overline{(x_i-m_i)(x_j-m_j)(x_k-m_k)h_r} - \mu 3_{ijk}\,\overline{h_r})\,\Gamma^{-1}_{rs}(y_s-\overline{h_s}) \tag{4.51}$$

Equation (4.51) can be simplified by noting that, by definition,

$$\overline{(x_k - m_k)} = 0 \quad \text{all } i,j,k$$

$$\frac{d\mu 4_{ijkl}}{dt} = \frac{d}{dt}\overline{(x_i-m_i)(x_j-m_j)(x_k-m_k)(x_l-m_l)}$$

$$= \overline{(f_i-\beta_i)(x_j-m_j)(x_k-m_k)(x_l-m_l)}$$

$$+ \overline{(f_j-\beta_j)(x_i-m_i)(x_k-m_k)(x_l-m_l)} + \overline{(f_k-\beta_k)(x_i-m_i)(x_j-m_j)(x_l-m_l)}$$

$$+ \overline{(f_l-\beta_l)(x_i-m_i)(x_j-m_j)(x_k-m_k)} + (V_{ij} + \Omega_{ij})\overline{(x_k-m_k)(x_l-m_l)}$$

$$+ (V_{ik}+ \Omega_{ik})\overline{(x_j-m_j)(x_l-m_l)} + (V_{jl}+ \Omega_{jl})\overline{(x_i-m_i)(x_k-m_k)}$$

$$+ (V_{jk}+ \Omega_{jk})\overline{(x_i-m_i)(x_l-m_l)} + (V_{jl}+ \Omega_{jl})\overline{(x_i-m_i)(x_k-m_k)}$$

$$+ \overline{(V_{kl}+\Omega_{kl})(x_i-m_i)(x_j-m_j)}$$

$$+ \left(\overline{(x_i-m_i)(x_j-m_j)(x_k-m_k)(x_l-m_l)h_r} - \mu 4_{ijkl}\, \overline{h_r}\right)\Gamma^{-1}_{rs}(y_s-\overline{h_s}) \qquad (4.52)$$

Furthermore, by definition,

$$\overline{(x_k-m_k)(x_l-m_l)} = \mu 2_{kl} \qquad (4.53)$$

The above moment equations can be written in a simpler form by taking their symmetry into account. The notation $N\{\ \}_s$ will be used, which signifies the sum of N symmetric terms of the form in the brackets. Equal symmetric terms are included only once. For example,

$$3\left\{\mu 2_{ij}\mu 2_{kl}\right\}_s = \mu 2_{ij}\,\mu 2_{kl} + \mu 2_{ik}\,\mu 2_{jl} + \mu 2_{il}\,\mu 2_{jk} \tag{4.54}$$

Using this notation, the moment equations can be written as follows.

$$\frac{dm_i}{dt} = \overline{f_i} + \overline{h_r(x_i-m_i)}\,\Gamma^{-1}_{rs}(y_s-\overline{h_s}) \tag{4.55}$$

$$\frac{d\mu 2_{ij}}{dt} = 2\left\{\overline{(f_i-\beta_i)(x_j-m_j)}\right\}_s + V_{ij} + \Omega_{ij} + \overline{h_r((x_i-m_i)(x_j-m_j) - \mu 2_{ij})}\,\Gamma^{-1}_{rs}(y_s-\overline{h_s}) \tag{4.56}$$

$$\frac{d\mu 3_{ijk}}{dt} = 3\left\{\overline{(f_i-\beta_i)(x_j-m_j)(x_k-m_k)}\right\}_s + \overline{h_r((x_i-m_i)(x_j-m_j)(x_k-m_k) - \mu 3_{ijk})}\,\Gamma^{-1}_{rs}(y_s-\overline{h_s}) \tag{4.57}$$

$$\frac{d\mu 4_{ijkl}}{dt} = 4\left\{(f_i-\beta_i)(x_j-m_j)(x_k-m_k)(x_l-m_l)\right\}_s + 6\left\{(V_{ij}+\Omega_{ij})\mu 2_{kl}\right\}_s + \overline{h_r((x_i-m_i)(x_j-m_j)(x_k-m_k)(x_l-m_l) - \mu 4_{ijkl})}\,\Gamma^{-1}_{rs}(y_s-\overline{h_s}) \tag{4.58}$$

The higher moment equations can be derived analogously. In this manner an infinite set of coupled differential equations is produced, which can be truncated as described in Section 4.3 .

It should be apparent that for the general case of nonlinear systems and measurements, the nonlinear filter equations can become quite complex. For this reason, a less general case is selected for further development.

4.5 An approximate Filter for Differentiable Nonlinearities

Nonlinear systems in which the system and measurement functions are smooth and continuous are frequently encountered in practice. Consequently this less general case will be used for more detailed development. In order to further simplify the problem, it will be assumed that both the system and measurement functions can be adequately described by a second order expansion about the conditional mean. That is, the system and measurement equations can be described as follows.

$$\frac{dx_i}{dt}(t) = f_i(\underline{m},t) + F_{ie}(\underline{m},t)(x_e-m_e) + A_{ief}(\underline{m},t)(x_e-m_e)(x_f-m_f) + G_{ie}(t)\, v_e(t) \tag{4.59}$$

$$y_r(t) = h_r(\underline{m},t) + H_{rp}(\underline{m},t)(x_p-m_p) + M_{rpq}(\underline{m},t)(x_p-m_p)(x_q-m_q) + R_{rp}(t)\,\xi_p(t) \tag{4.60}$$

$$F_{ie}(\underline{m},t) = f_{i,e}(\underline{m},t)$$

$$A_{ief}(\underline{m},t) = \tfrac{1}{2} f_{i,ef}(\underline{m},t)$$

$$H_{rp}(\underline{m},t) = h_{r,p}(\underline{m},t)$$

$$M_{rpq}(\underline{m},t) = \tfrac{1}{2}h_{r,pq}(\underline{m},t)$$

$$\mathcal{E}(v_e(t)) = \mathcal{E}(\xi_s(t)) = 0$$

$$\mathcal{E}(v_m(t)v_n(\tau)) = \delta_{mn}\,\delta(t-\tau)$$

$$\mathcal{E}(\xi_r(t)\xi_s(\tau)) = \delta_{rs}\,\delta(t-\tau)$$

$$\mathcal{E}(v_n(t)\xi_s(\tau)) = 0$$

v_e , ξ_s uncorrelated unit white noise vectors

$G_{in}G_{jn} = V_{ij}$ State Driving Noise Power Spectral Density

$R_{rp}R_{sp} = \Gamma_{rs}$ Measurement Noise Power Spectral Density

(4.61)

From (4.47) and (4.49),

$$\beta_i = f_i(\underline{m}) + A_{ief}(\underline{m})\mu2_{ef} + \left[H_{rp}(\underline{m})\,\mu2_{ip} + M_{rpq}(\underline{m})\,\mu3_{ipq}\right]$$
$$\Gamma^{-1}_{rs}\left[H_{sa}(\underline{m})(x_a-m_a) + M_{sab}(\underline{m})((x_a-m_a)(x_b-m_b) - \mu2_{ab})\right] \quad (4.62)$$

In the following, $f_i, F_{ie}, A_{ief}, h_r, H_{rp}$, and M_{rpq} will always be functions of $\underline{m}$ and t. To simplify the notation, this dependence will no longer be indicated explicitly.

$$\Omega_{ij} = (H_{rp}\,\mu2_{ip} + M_{rpq}\mu3_{ipq})\,\Gamma^{-1}_{rs}(H_{sa}\mu2_{ja} + M_{sab}\mu3_{jab}) \quad (4.63)$$

From equations (4.55)-(4.58), the differential equations for the minimum variance estimate of the central moments can be obtained.

$$\frac{dm_i}{dt} = f_i + A_{ief}\,\mu2_{ef} + (H_{rp}\,\mu2_{ip}+M_{rpq}\,\mu3_{ipq})\,\Gamma^{-1}_{rs}(y_s-h_s-M_{rpq}\mu2_{pq}) \quad (4.64)$$

$$\frac{d\mu2_{ij}}{dt} = 2\Big\{F_{ie}\mu2_{je}+A_{ief}\mu3_{jef}-(H_{rp}\mu2_{ip}+M_{rpq}\mu3_{ipq})\,\Gamma^{-1}_{rs}$$
$$(H_{sa}\mu2_{ja}+M_{sab}\mu3_{jab})\Big\}_s + V_{ij} + (H_{rp}\mu2_{ip}+M_{rpq}\mu3_{ipq})\,\Gamma^{-1}_{rs}(H_{sa}\mu2_{ja}$$
$$+M_{sab}\mu3_{jab}) + (H_{rp}\mu3_{ijp}+M_{rpq}(\mu4_{ijpq}-\mu2_{ij}\mu2_{pq}))\,\Gamma^{-1}_{rs}(y_s-h_s-M_{sab}\mu2_{ab}) \quad (4.65)$$

Since the term $(H_{rp}\mu 2_{ip} + M_{rpq}\mu 3_{ipq})\Gamma^{-1}_{rs}(H_{sa}\mu 2_{ja} + M_{sab}\mu 3_{jab})$ is symmetric in i and j, (4.65) can be simplified to the following.

$$\frac{d\mu 2}{dt}_{ij} = 2\left\{F_{ie}\mu 2_{je} + A_{ief}\mu 3_{jef}\right\}_s + V_{ij} - (H_{rp}\mu 2_{ip} + M_{rpq}\mu 3_{ipq})$$
$$\Gamma^{-1}_{rs}(H_{sa}\mu 2_{ja} + M_{sab}\mu 3_{jab})$$
$$+(H_{rp}\mu 3_{ijp} + M_{rpq}(\mu 4_{ijpq} - \mu 2_{ij}\mu 2_{pq}))\Gamma^{-1}_{rs}(y_s - h_s - M_{sab}\mu 2_{ab}) \tag{4.66}$$

$$\frac{d\mu 3}{dt}_{ijk} = 3\Big\{F_{ie}\,\mu 3_{jke} + A_{ief}(\mu 4_{jkef} - \mu 2_{jk}\mu 2_{ef})$$
$$-(H_{rp}\mu 2_{ip} + M_{rpq}\mu 3_{ipq})\Gamma^{-1}_{rs}(H_{sa}\mu 3_{ajk} + M_{sab}(\mu 4_{abjk} - \mu 2_{ab}\mu 2_{jk}))\Big\}_s$$
$$+(H_{rp}\mu 4_{ijkp} + M_{rpq}(\mu 5_{ijkpq} - \mu 3_{ijk}\mu 2_{pq}))\Gamma^{-1}_{rs}(y_s - h_s - M_{sab}\mu 2_{ab}) \tag{4.67}$$

$$\frac{d\mu 4}{dt}_{ijkl} = 4\Big\{F_{ie}\mu 4_{jkle} + A_{ief}(\mu 5_{jklef} - \mu 3_{jkl}\mu 2_{ef})$$
$$-(H_{rp}\mu 2_{ip} + M_{rpq}\mu 3_{ipq})\Gamma^{-1}_{rs}(H_{sa}\mu 4_{jkla} + M_{sab}(\mu 5_{jklab} - \mu 3_{jkl}\mu 2_{ab}))\Big\}_s$$
$$+6\left\{\left[V_{ij} + (H_{rq}\mu 2_{iq} + M_{rpq}\mu 3_{ipq})\Gamma^{-1}_{rs}(H_{sa}\mu 2_{ja} + M_{sab}\mu 3_{jab})\right]\mu 2_{kl}\right\}_s$$
$$+(H_{rp}\mu 5_{ijklp} + M_{rpq}(\mu 6_{ijklpq} - \mu 4_{ijkl}\mu 2_{pq}))\Gamma^{-1}_{rs}(y_s - h_s - M_{sab}\mu 2_{ab}) \tag{4.68}$$

In order to make use of these filter equations, a method of truncating the moment sequence must be found. A promising technique is to express the probability density function as a gaussian density multiplied by a sum of Hermite polynomials. The coefficients of these Hermite polynomials were called quasi-moment functions by Stratonovich. This method of truncating the moment sequence is described in detail in Appendix E.

Three methods of truncating the above filter equations are possible, which give rise to three filters of differing capability and complexity.

First, it is assumed that the third and higher order quasi-moment functions are zero. Then only the mean and covariance are computed, and these will have the following form.

$$\frac{dm_i}{dt} = f_i + A_{ief}\mu 2_{ef} + H_{rp}\mu 2_{ip}\Gamma^{-1}_{rs}(y_s - h_s - M_{rpq}\mu 2_{pq}) \quad (4.69)$$

$$\frac{d\mu 2_{ij}}{dt} = F_{ie}\mu 2_{je} + F_{je}\mu 2_{ie} + V_{ij} - H_{rp}\mu 2_{ip}\Gamma^{-1}_{rs}H_{sa}\mu 2_{ja}$$

$$+ M_{rpq}(\mu 2_{ip}\mu 2_{jq} + \mu 2_{iq}\mu 2_{jp})\Gamma^{-1}_{rs}(y_s - h_s - M_{sab}\mu 2_{ab}) \quad (4.70)$$

If it is now assumed that the system and measurement functions are linear, then the above equations can be written in the vector form,

$$\frac{d\underline{m}}{dt} = F\,\underline{m} + \mu 2\, H^T \Gamma^{-1}(\underline{y} - \underline{h}) \quad (4.71)$$

$$\frac{d\mu 2}{dt} = F\,\mu 2 + \mu 2\, F^T - \mu 2\, H^T \Gamma^{-1} H\, \mu 2 + V \quad (4.72)$$

These are the familiar Kalman filter equations, while (4.69) and (4.70) are the optimal second order nonlinear filter equations. The latter are similar to the "linearized Kalman filter" equations, but have some additional terms.

Now the filter capability can be increased to handle nonsymmetric density functions, by calculating the third moments and assuming that the fourth and higher order quasi-moments are zero. The third order optimal filter equations will then be given by (4.64), (4.66), and (4.67) with the following substitutions obtained from Appendix E.

$$\mu 4_{ijkl} = 3\left\{\mu 2_{ij}\ \mu 2_{kl}\right\}_s$$

$$\mu 5_{ijklm} = 10\left\{\mu 2_{ij}\ \mu 3_{klm}\right\}_s \qquad (4.73)$$

Similarly the fourth moments can be included and the sequence truncated by assuming that the fifth and higher order quasi-moment functions are zero. This is equivalent to representing the density function by a fourth order Hermite polynomial expansion. In this case the filter equations are given by (4.64), (4.66), (4.67), and (4.68) with the substitutions,

$$k4_{ijkl} = \mu 4_{ijkl} - 3\left\{\mu 2_{ij}\ \mu 2_{kl}\right\}_s$$

$$\mu 5_{ijklm} = 10\left\{\mu 2_{ij}\ \mu 3_{klm}\right\}_s$$

$$\mu 6_{ijklmn} = 15\left\{\mu 2_{ij}\ k4_{klmn}\right\}_s + 15\left\{\mu 2_{ij}\ \mu 2_{kl}\ \mu 2_{mn}\right\}_s + 10\left\{\mu 3_{ijk}\ \mu 3_{lmn}\right\}_s \qquad (4.74)$$

The three estimators described above will be called second, third, and fourth order nonlinear filters, and represent increasingly sophisticated approximations to the infinite dimensional true minimum variance estimator.

It should be apparent that the computation of the higher moments becomes progressively more lengthy. In addition, the number of components of these moments grows rapidly with increasing order for systems of large dimension. The k th moment of an N dimensional vector is a k th order tensor of N^k components. Fortunately this tensor is symmetric in all indices, with the result (proved in appendix C) that it has only $\binom{N+k-1}{k}$ unique components. For example, the fourth moment of a five dimensional system has 625 components, of which only 70 are unique. In order to make the moment computation practical for systems of high order, a method must be found of addressing the unique components in a dense array, so that storage is not wasted and unique components are not computed more than once. Two methods suitable for digital computers are derived in appendix D.

These methods are sufficiently complex that it may be advantageous to compute and store all the moments for systems of low dimension, in order to avoid having to compute the addresses.

The problem remains of how many moments need to be computed in order to obtain a desired accuracy. This complex question depends on the degree of nonlinearity of the system, and can probably best be answered by empirical means.

4.6 Stability

The third and fourth order nonlinear filters are sets of coupled nonlinear differential equations driven by white noise. Nothing in their derivation gives any indication of their stability. If they are to be of practical importance, they must be sufficiently stable that they can be integrated on a digital computer with a reasonable step size.

The stability of these coupled equations is a very complex subject. To illustrate this, consider the very simple case of a scalar system with no dynamics and a linear measurement. The third order filter is described by,

$$\frac{dm}{dt} = \mu 2\ H\ \Gamma^{-1}(y-Hm)$$

$$\frac{d\mu 2}{dt} = \mu 3\ H\ \Gamma^{-1}(y-Hm) \quad - \mu 2^2\ H^2\ \Gamma^{-1}$$

$$\frac{d\mu 3}{dt} = 3\ \mu 2^2\ H\ \Gamma^{-1}(y-Hm) - 3\ \mu 2\ \mu 3\ H^2\ \Gamma^{-1} \qquad (4.75)$$

Let $y = H\ x + R\,\xi$, where ξ is a unit white noise. Since x is constant and arbitrary, let it be zero. If the substitutions

$\delta = H\ \Gamma^{-1} R$

and $\bar{a}^2 = H^2 \Gamma^{-1}$ are used, then the filter equations can be written in the equivalent form,

$$\frac{dm}{dt} = -\mu 2\, \bar{a}^2 m \qquad\qquad + \mu 2\, \bar{a}\, \zeta$$

$$\frac{d\mu 2}{dt} = -\mu 3\, \bar{a}^2 m \;-\; \mu 2^2\, \bar{a}^2 \qquad + \mu 3\, \bar{a}\, \zeta$$

$$\frac{d\mu 3}{dt} = -3\, \mu 2^2\, \bar{a}^2\, m - 3\, \mu 2\, \mu 3\, \bar{a}^2 + 3\, \mu 2^2\, \bar{a}\, \zeta \qquad (4.76)$$

The stability of these equations even without the white noise terms is a formidable problem. The direct method of Lyapunov appears to be the most useful method, but no V function could be found which would prove stability or instability. Furthermore, since the equations are nonlinear, they could easily be convergent for one set of initial conditions and divergent for another.

The only feasable method of determining the stability of these nonlinear filters appears to be their simulation using the expected range of inputs and initial conditions. This method will also yield data on relative performance of the second, third, and fourth order filters.

4.7 Practical Application

The practical application of the truncated optimal filter for continuous systems and continuous measurements, or Kushner filter, is severely limited by its large computation requirements. This is primarily brought on not by complexity, but by poor stability characteristics which require very small step sizes for the integration of the moment differential equations. The equations without the measurement terms are characterized by low damping, and the addition of the random measurement errors further undermines stability.

In order to integrate the continuous equations with a step Δt, the white noise of covariance $\Gamma\delta(t)$ must be approximated by a sequence of random pulses of width Δt and variance $\Gamma/\Delta t$. As $\Delta t \to 0$, the triangular autocorrelation function of this approximation will approach a Dirac δ function of the proper magnitude. From the scalar measurement equations (4.75), the moment corrections for rectangular integration and neglecting system dynamics will be approximately the following, for small Δt,

$$\Delta m \approx \mathcal{E}(\mu 2 \ H \ \Gamma^{-1} \int_0^{\Delta t} R \xi \, dt)$$

$$\Delta \mu 2 \approx \mathcal{E}(\mu 3 \ H \ \Gamma^{-1} \int_0^{\Delta t} R \xi \, dt) \tag{4.77}$$

$$\Delta \mu 3 \approx \mathcal{E}(3 \ \mu 2^2 \ H \ \Gamma^{-1} \int_0^{\Delta t} R \xi \, dt)$$

Since $\mathcal{E}\left[\left(\int_0^{\Delta t} R \xi \, dt\right)^2\right] = \Gamma \ \Delta t$,

$$\mathrm{RMS}(\Delta m) \approx \mu 2 \ H \sqrt{\Delta t/\Gamma}$$

$$\mathrm{RMS}(\Delta \mu 2) \approx \mu 3 \ H \sqrt{\Delta t/\Gamma}$$

$$\mathrm{RMS}(\Delta \mu 3) \approx 3 \ \mu 2^2 \ H \sqrt{\Delta t/\Gamma} \tag{4.78}$$

As a rough rule, the stable integration of the set (4.75) requires that,

$$\mathcal{E}(\Delta m^2) < \frac{1}{100} \mathcal{E}(m^2) \tag{4.79}$$

Since m is assumed to be zero mean in this simple example,

$$\mathcal{E}(\Delta m^2) < \frac{1}{100} \mu 2 \tag{4.80}$$

$$\Delta t < \frac{\Gamma}{100\, H^2 \mu^2_{max}} \qquad (4.81)$$

From the above, it appears that the third order Kushner filter requires an integration step size at least an order of magnitude smaller than the corresponding Kalman filter. This result has been verified by simulation. The basic reason for this is that each equation is coupled and driven by the measurement noise, while the Kalman filter equations are uncoupled and only the equation for the mean is driven by measurement noise.

Chapter 5

Optimal Estimation for Continuous systems with Discrete Measurements

5.1 Introduction

For many problems of engineering significance, the model of continuous observations is not useful. The case of discrete measurements occurs frequently, particularly in complex systems with many sensors which are controlled by a digital computer.

For continuous measurements, the optimal filtering equations were obtained by considering the Bayes equation for the conditional density, and taking a formal limit as the time increment approached zero. In this limiting process, only the first order terms in δt and δz, and second order terms in δz (Ito's lemma) in the series expansions remain significant. Optimal estimation with discrete observations is necessarily more difficult, since the higher order terms cannot be ignored. In fact, the discrete solution will have the continuous solution as a special limiting case.

A simplified problem of this form was investigated by Jazwinski, (P-10) who considered only symmetric systems and second order filters. The following analysis will consider more general continuous nonlinear systems, with nonlinear discrete measurements.

This discrete measurement nonlinear estimation problem is of formidable complexity, and initially only linear measurements were considered. Two approximate methods of solving the Bayes equation for higher moments were derived, both of which were found to be of limited usefulness. Subsequently an exact third order solution was discovered which overcame these difficulties. Following this, a two step method of dealing with nonlinear measurements was derived.

The principal advantage of discrete measurement filters is that they have much better stability characteristics than the corresponding continuous measurement filters. The total computational requirements are substantially reduced, overcoming the principal limitation of the continuous measurement filters.

5.2 An Equation for the Posterior Density

The following continuous nonlinear system and discrete nonlinear measurement process are considered.

$$\frac{dx_i}{dt} = f_i(\underline{x},t) + G_{in}(\underline{x},t)\; v_n(t) \tag{5.1}$$

$$y_r(n\Delta t) = h_r(\underline{x},n\Delta t) + R_{rs}(n\Delta t)\,\xi_s(n\Delta t) \tag{5.2}$$

$$\begin{aligned}
&G_{ik}\, G_{jk} = V_{ij} \\
&R_{rq}\, R_{sq} = \Sigma_{rs} \\
&\mathcal{E}(v_n(t)) = \mathcal{E}(\xi_r(n\Delta t)) = 0 \\
&\mathcal{E}(v_n(t)\; v_m(\tau)) = \delta_{mn}\; \delta(t-\tau) \\
&\mathcal{E}(\xi_r(n\Delta t)\,\xi_s(m\Delta t)) = \delta_{rs}\; \delta_{mn} \\
&\mathcal{E}(v_n(t)\,\xi_s(m\Delta t)) = 0
\end{aligned} \tag{5.3}$$

For $t \neq n\Delta t$, no measurement information is received by the estimator, and the differential equations for the probability density P and the conditional expectation of an arbitrary twice differentiable function of the state $g(\underline{x})$ are given by the previously derived equations (4.32) and (4.25) with continuous measurements of infinite variance. That is, for $t \neq n\Delta t$,

$$\frac{dP}{dt} = -(f_i\; P)_{,i} + \tfrac{1}{2}\,(V_{ij}\; P)_{,ij} \tag{5.4}$$

$$\frac{d\overline{g}}{dt} = \overline{f_i\; g_{,i}} + \tfrac{1}{2}\, V_{ij}\, \overline{g_{,ij}} \tag{5.5}$$

Equation (5.4) is Kolmogorov's forward equation, or the Fokker-Planck equation, and is also used to describe diffusion phenomena.

In like manner, the moment equations for $t \neq n\Delta t$ are given by (4.55) - (4.58) with $\Gamma_{ys}^{-1} = 0$. If the nonlinear system can be adequately described by a second order expansion, then the moment equations (4.64) - (4.68) with $\Gamma_{ys}^{-1} = 0$ can be used. The resulting set of coupled nonlinear differential equations is more stable than the original set, because the white noise driving terms due to the measurement have been eliminated.

Just before $t = n\Delta t$, the conditional expectation of $P(\underline{x})$ is available from integration of (5.4), and is designated as $P(\underline{x}, n\Delta t^-) = P(\underline{x}, t_n^-)$. An instant later, the measurement has been received, and the probability density for the state conditioned on the new measurement is given by Baye's equation.

$$P(\underline{x},t_n^-) = P(\underline{x},t_n/\underline{y}_1,\underline{y}_2 --- \underline{y}_{n-1}) = P(\underline{x},t_n/Y)$$

$$P(\underline{x},t_n^+) = P(\underline{x},t_n/\underline{y}_1,\underline{y}_2, --- \underline{y}_n) = P(\underline{x},t_n/Y,\underline{y}_n) \qquad (5.6)$$

Y is the set of all measurements up to t_{n-1}.

$$P(\underline{x},t_n^+) = \frac{P(\underline{x},t_n;\ Y,\underline{y}_n)}{P(\underline{y}_n;Y)} \qquad (5.7)$$

$$= \frac{P(\underline{y}_n/\underline{x},t_n;Y)\ P(\underline{x},t_n;Y)}{P(Y)\ P(\underline{y}_n/Y)} \qquad (5.8)$$

$$= \frac{P(\underline{y}_n/\underline{x},t_n;Y)\ P(\underline{x},t_n/Y)}{\int_{R_N} P(\underline{y}_n/\underline{a},t_n;Y)\ P(\underline{a},t_n/Y)\ d\underline{a}} \qquad (5.9)$$

From the measurement equation (5.2),

$$P(\underline{y}_n/\underline{x},t_n;Y) = P(\underline{y}_n/\underline{x},t_n) \qquad (5.10)$$

Equations (5.9) and (5.10) together yield the relationship,

$$P(\underline{x},t_n^+) = \frac{P(\underline{y}_n/\underline{x},t_n)\; P(\underline{x},t_n)}{\int_{R_N} P(\underline{y}_n/\underline{a},t_n)\; P(\underline{a},t_n)\; d\underline{a}} \tag{5.11}$$

By hypothesis, $P(\underline{y}_n/\underline{x},t_n)$ is a multidimensional gaussian probability density function with mean $h_r(\underline{x})$ and covariance Σ_{rs}. That is,

$$P(\underline{y}_n/\underline{x},t_n) = \frac{\exp(-\frac{1}{2}(y_r-h_r(\underline{x}))\,\Sigma_{rs}^{-1}(y_s-h_s(\underline{x}))\;)}{(2\pi)^{\frac{n}{2}}\,|\Sigma_{rs}|^{\frac{1}{2}}} \tag{5.12}$$

Equation (5.11) then becomes the desired equation for the posterior conditional density.

$$P(\underline{x},t_n^+) = \frac{P(\underline{x},t_n)\;\exp(-\frac{1}{2}(y_r-h_r(\underline{x}))\,\Sigma_{rs}^{-1}(y_s-h_s(\underline{x}))\;)}{\int_{R_N} P(\underline{a},t_n)\;\exp(-\frac{1}{2}(y_r-h_r(\underline{a}))\,\Sigma_{rs}^{-1}(y_s-h_s(\underline{a}))\;)\;d\underline{a}} \tag{5.13}$$

The function $P(\underline{x},t_n^+)$ can be obtained from (5.13) by integration. But for nonlinear systems, $P(\underline{x},t_n)$ will not in general be gaussian and the integration cannot be done analytically in a simple way.

As in the continuous case, it it usually computationally advantageous to work with the moments rather than the density. What is desired is a method of computing the posterior moments from the prior moments, at the instant the measurement is received, t_n. The prior and posterior moments are defined as follows.

$$m_i = \int_{R_N} P(\underline{a},t_n)\; a_i\; d\underline{a} \tag{5.14}$$

$$\mu 2_{ij} = \int_{R_N} P(\underline{a},t_n)(a_i-m_i)(a_j-m_j)\; d\underline{a} \tag{5.15}$$

etc.

$$m_i^+ = \int_{R_N} p(\underline{a}, t_n^+)\ a_i\ d\underline{a} \qquad (5.16)$$

$$\mu 2_{ij}^+ = \int_{R_N} p(\underline{a}, t_n^+)\ (a_i - m_i)(a_j - m_j)\ d\underline{a} \qquad (5.17)$$

etc.

5.3 Two Approximate Solutions for the Posterior Moments

Initially, an attempt was made to solve the Bayes equation for linear measurements by a series expansion technique. That is, expressions for the posterior central moments were obtained by expanding the exponential function in a power series about the prior mean. This yielded a rather complex series solution for the posterior moments, which was found to converge slowly for measurements of variance as small as the prior covariance. Although this problem limits the general usefulness of this method, it may be effectively applied to systems having closely spaced measurements of large variance. A derivation and description of this approximate series solution is given in Appendix F.

Following this, an attempt was made to obtain a solution for the posterior moments by approximating the discrete measurement as a short duration continuous measurement. This yields a set of differential equations involving the moments and measurement value, which can be integrated to give an approximate solution for the posterior moments. This integration updating method requires substantial amounts of computation, and gives moments which result in an unstable extrapolation if the system is unstable. Nevertheless it has given good results in Monte Carlo tests involving stable systems. A derivation and description are given in Appendix G.

After these two approximate methods were developed and tested, an exact solution for the Bayes equation was found, by modeling the density function as a gaussian density times a sum of Hermite polynomials. This exact solution overcame many of the shortcomings of the approximate solutions, while providing greater theoretical insight into the nonlinear estimation problem.

It should be pointed out that in the linear gaussian case, all three of these solutions reduce to the Kalman estimation formulas. The third order solutions represent different methods of incorporating the third moment information into the estimation process.

5.4 An Exact Solution for Linear Measurements

It has been shown by Kuznetsov, Stratonovich, and Tikhonov (P-21) that an arbitrary continuous probability density function can be expressed with arbitrary accuracy in an integrated mean square error sense by a normal probability density multiplied by a sum of multidimensional Hermite polynomials. This fact was used in Appendix F to obtain an analytically meaningful truncation of the moment equations, which were then solved by an approximate series expansion technique.

These equations will now be solved exactly, using the convenient analytic properties of the Hermite polynomials. This will eliminate the convergence difficulties of the series expansion solution, without a significant increase in computational complexity.

From Appendix E, an arbitrary density function can be expressed in terms of a gaussian density function, quasi-moment functions, and Hermite polynomials.

$$P(\underline{x},t) = P_g(\underline{x},t)\left\{ 1 + \sum_{N=3}^{\infty} \frac{1}{N!} \; kN_{jk--l}(t) \; HN_{jk--l}(\underline{x}(t)-\underline{m}(t),t) \right\} \quad (5.18)$$

$P_g(\underline{x},t)$ is the gaussian density having the same mean and covariance as the density in question, kN_{jk--l} are Nth order quasi-moment functions, which can be expressed in terms of the central moments, and HN_{jk--l} are Nth order multidimensional Hermite polynomials, of the same dimension as the state.

Alternatively, this can be written,

$$P(\underline{x},t) = P_g(\underline{x},t) + \sum_{N=3}^{\infty} \frac{(-1)^N}{N!} \; KN_{\underbrace{jk---l}_{N}}(t) \underbrace{\frac{\partial^N P_g(\underline{x},t)}{\partial x_j \partial x_k --- \partial x_l}}_{N} \tag{5.19}$$

The simple case is considered in which the density function is approximated as gaussian plus a third order quasi-moment term. All higher quasi-moments are assumed to be zero. This is the simplest model that can account for asymmetries in the density function. By definition,

$$P_g(\underline{x},t) = \frac{e^{-\frac{1}{2}(x_r(t)-m_r(t))\mu 2^{-1}_{rs}(x_s(t)-m_s(t))}}{(2\pi)^{\frac{N}{2}} |\mu 2|^{\frac{1}{2}}} \tag{5.20}$$

$$\frac{\partial P_g}{\partial x_i} = - P_g \, \mu 2^{-1}_{is} \, (x_s - m_s)$$

$$\frac{\partial^2 P_g}{\partial x_i \partial x_j} = P_g \left[\mu 2^{-1}_{ir} \, \mu 2^{-1}_{js} (x_r - m_r)(x_s - m_s) - \mu 2^{-1}_{ij} \right]$$

$$\frac{\partial^3 P_g}{\partial x_i \partial x_j \partial x_k} = -P_g \left[\mu 2^{-1}_{ir} \, \mu 2^{-1}_{js} \, \mu 2^{-1}_{kt} (x_r - m_r)(x_s - m_s)(x_t - m_t) \right.$$

$$\left. -(\mu 2^{-1}_{ij} \, \mu 2^{-1}_{kt} + \mu 2^{-1}_{jk} \, \mu 2^{-1}_{it} + \mu 2^{-1}_{ik} \, \mu 2^{-1}_{jt})(x_t - m_t) \right] \tag{5.21}$$

Let $e_i = x_i - m_i$. Then (5.19) becomes,

$$P(\underline{x},t) \approx P_g(\underline{x},t) \left\{ 1 + \tfrac{1}{6} \mu 3_{ijk} \left[\mu 2^{-1}_{ir} \, \mu 2^{-1}_{js} \, \mu 2^{-1}_{kt} \, e_r e_s e_t \right. \right.$$

$$\left. \left. - (\mu 2^{-1}_{jk} \, \mu 2^{-1}_{ir} + \mu 2^{-1}_{ik} \, \mu 2^{-1}_{jr} + \mu 2^{-1}_{ij} \, \mu 2^{-1}_{kr}) \, e_r \right] \right\} \tag{5.22}$$

The above relationship can now be used to obtain an analytic solution to the Bayes equation, (5.13), in the case of a linear measurement. The denominator integral can be written;

$$D = \int_{R_N} \frac{e^{-\frac{1}{2}(z_r-H_{rp}e_p)\Sigma^{-1}_{rs}(z_s-H_{sq}e_q)} \; e^{-\frac{1}{2}e_r \mu 2^{-1}_{rs} e_s} \Big\{1 + \frac{1}{6}\mu 3_{abc}\Big[\mu 2^{-1}_{ar}\,\mu 2^{-1}_{bs}\,\mu 2^{-1}_{ct}\, e_r e_s e_t - (\mu 2^{-1}_{bc}\,\mu 2^{-1}_{ar} + \mu 2^{-1}_{ac}\,\mu 2^{-1}_{br} + \mu 2^{-1}_{ab}\,\mu 2^{-1}_{cr})\, e_r\Big]\Big\}}{(2\pi)^{\frac{N}{2}}\,|\mu 2|^{\frac{1}{2}}}\, d\underline{e} \quad (5.23)$$

From equation (F-20), (5.23) can be written as;

$$D = \frac{e^{-\frac{1}{2}\omega}\, e^{\theta}}{(2\pi)^{\frac{N}{2}}|\mu 2|^{\frac{1}{2}}} \int_{R_N} e^{-\frac{1}{2}(e_i-d_i)\Omega^{-1}_{ij}(e_j-d_j)} \Big\{1 + \frac{1}{6}[\quad]\Big\} d\underline{e} \quad (5.24)$$

$$e_r e_s e_t = (e_r-d_r)(e_s-d_s)(e_t-d_t) + d_r(e_s-d_s)(e_t-d_t)$$
$$+ d_s(e_r-d_r)(e_t-d_t) + d_t(e_r-d_r)(e_s-d_s) + d_s d_t(e_r-d_r)$$
$$+ d_r d_t(e_s-d_s) + d_r d_s(e_t-d_t) + d_r d_s d_t \quad (5.25)$$

Using (5.25) and the integrals (F.21)-(F.25), D can be expressed as follows.

$$D = \frac{e^{-\frac{1}{2}\omega}\, e^{\theta}}{\alpha}\Big\{1 + \frac{1}{6}\mu 3_{abc}\Big[\mu 2^{-1}_{ar}\,\mu 2^{-1}_{bs}\,\mu 2^{-1}_{ct}(d_r\Omega_{st} + d_s\Omega_{rt} + d_t\Omega_{rs} + d_r d_s d_t) - (\mu 2^{-1}_{bc}\,\mu 2^{-1}_{ar} + \mu 2^{-1}_{ca}\,\mu 2^{-1}_{br} + \mu 2^{-1}_{ab}\,\mu 2^{-1}_{cr})d_r\Big]\Big\} \quad (5.26)$$

This can be simplified by use of the following substitutions.

$$\eta_i = \mu 2^{-1}_{ir} d_r$$

$$G_{ij} = \Omega_{ie} \mu 2^{-1}_{ej}$$

$$Q_{ij} = \mu 2^{-1}_{ie} G_{ej} \tag{5.27}$$

Then,

$$D = \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{\alpha} \Big\{ 1 + \frac{1}{6} \mu 3_{abc} (\eta_a Q_{bc} + \eta_b Q_{ac} + \eta_c Q_{ab}$$

$$+ \eta_a \eta_b \eta_c - \mu 2^{-1}_{bc} \eta_a - \mu 2^{-1}_{ac} \eta_b - \mu 2^{-1}_{ab} \eta_c) \Big\} \tag{5.28}$$

Let $E_{ij} = Q_{ij} - \mu 2^{-1}_{ij} + (1/3) \eta_i \eta_j$

Then (5.28) becomes,

$$D = \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{\alpha} \Big\{ 1 + \frac{1}{6} \mu 3_{abc} \Big[3 \{ \eta_a E_{bc} \}_s \Big] \Big\} \tag{5.29}$$

Since $\mu 3_{abc}$ is symmetric in all indices,

$$\mu 3_{abc} \eta_a E_{bc} = \mu 3_{abc} \eta_b E_{ac} = \mu 3_{abc} \eta_c E_{ab} \tag{5.30}$$

Therefore,

$$D = \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{\alpha} \Big\{ 1 + \nu \Big\}$$

$$\nu = \frac{1}{2} \mu 3_{abc} \eta_a E_{bc} \tag{5.31}$$

Computation of the Posterior Moments

By definition,

$$m_i^+ = \int_{R_N} P(\underline{x}, t_n^+) x_i \, d\underline{x} = \frac{1}{D} \int_{R_N} P(\underline{x}, t_n) \, e^{-\frac{1}{2}\gamma} \, x_i \, d\underline{x} \tag{5.32}$$

From (F.20) and (5.32),

$$m_i^+ = \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{D\,(2\pi)^{\frac{N}{2}} |\mu 2|^{\frac{1}{2}}} \int_{R_N} e^{-\frac{1}{2}(e_r - d_r)\Omega_{rs}^{-1}(e_s - d_s)} \left\{ 1 + \frac{1}{6}[\;] \right\} ((e_i - d_i) + d_i + m_i) d\underline{e} \tag{5.33}$$

Now using the integrals (F.21) - (F.25),

$$m_i^+ = m_i + d_i + \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{D\,\alpha} \Big\{ \frac{1}{6} \mu 3_{abc} \Big[\mu 2_{ar}^{-1} \mu 2_{bs}^{-1} \mu 2_{ct}^{-1} (\Omega_{ir} \Omega_{st}$$

$$+ \Omega_{is} \Omega_{rt} + \Omega_{it} \Omega_{rs} + d_s d_t \Omega_{ir} + d_r d_t \Omega_{is} + d_r d_s \Omega_{it})$$

$$- (\mu 2_{ar}^{-1} \mu 2_{bc}^{-1} + \mu 2_{br}^{-1} \mu 2_{ac}^{-1} + \mu 2_{cr}^{-1} \mu 2_{ab}^{-1}) \Omega_{ir} \Big] \Big\} \tag{5.34}$$

Using the substitutions (5.27) , the above simplifies to the following.

$$m_i^+ = m_i + d_i + \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{D\,\alpha} \Big\{ \frac{1}{6} \mu 3_{abc} \Big[G_{ia} Q_{bc} + G_{ib} Q_{ac} + G_{ic} Q_{ab}$$

$$+ G_{ia} \eta_b \eta_c + G_{ib} \eta_a \eta_c + G_{ic} \eta_a \eta_b - G_{ia} \mu 2_{bc}^{-1} - G_{ib} \mu 2_{ac}^{-1} - G_{ic} \mu 2_{ab}^{-1} \Big] \Big\} \tag{5.35}$$

Let

$$O_{ij} = Q_{ij} - \mu 2_{ij}^{-1} + \eta_i \eta_j \tag{5.36}$$

Taking the symmetry of $\mu 3_{abc}$ into account, (5.35) becomes,

$$m_i^+ = m_i + d_i + \phi_i \tag{5.37}$$

$$\phi_i = \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{D\,\alpha} \; \frac{1}{2} \mu 3_{abc} \, G_{ia} \, O_{bc}$$

$$= \frac{1}{(1+\nu)} \; \frac{1}{2} \mu 3_{abc} \, G_{ia} \, O_{bc} \tag{5.38}$$

Second Posterior Moment

The second posterior moment is obtained in a similar way. The expression for the posterior moment about the posterior mean is,

$$\mu 2^{+}_{ij} = \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{D(2\pi)^{\frac{N}{2}} |\mu 2|^{\frac{1}{2}}} \int_{R_N} e^{-\frac{1}{2}(e_r - d_r)\Omega^{-1}_{rs}(e_s - d_s)} \left\{ 1 + \frac{1}{6}[\ ] \right\} (x_i - m^{+}_i)(x_j - m^{+}_j) d\underline{e}$$

(5.39)

From (5.37),

$$(x_i - m^{+}_i) = x_i - m_i - d_i - \phi_i = e_i - d_i - \phi_i = (b_i - \phi_i)$$
$$e_i = x_i - m_i$$
$$b_i = e_i - d_i \qquad (5.40)$$

Therefore,

$$(x_i - m^{+}_i)(x_j - m^{+}_j) = b_i b_j - \phi_i b_j - \phi_j b_i + \phi_i \phi_j \qquad (5.41)$$

By definition,

$$e_r = b_r - d_r$$
$$e_r e_s e_t = b_r b_s b_t + 3\left\{d_r b_s b_t\right\}_s + 3\left\{d_r d_s b_t\right\}_s + d_r d_s d_t \qquad (5.42)$$

Application of the integrals (F.21) - (F.25) to equation (5.39) now yields,

$$\mu 2^{+}_{ij} = \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{D\,\alpha} \Big\{ \Omega_{ij} + \phi_i \phi_j + \frac{1}{6}\mu 3_{abc} \Big[\mu 2^{-1}_{ar}\, \mu 2^{-1}_{bs}\, \mu 2^{-1}_{ct} ($$

$$-\phi_i\, 3\left\{\Omega_{jr}\Omega_{st}\right\}_s - \phi_j\, 3\left\{\Omega_{ir}\Omega_{st}\right\}_s + 9\left\{d_r \Omega_{ij}\Omega_{st}\right\}_s +$$

$$+ \phi_i\phi_j \; 3\left\{d_r\Omega_{st}\right\}_s - \phi_i \; 3\left\{d_r d_s \Omega_{jt}\right\}_s - \phi_j \; 3\left\{d_r d_s \Omega_{it}\right\}_s$$

$$+ d_r d_s d_t \Omega_{ij} + \phi_i\phi_j d_r d_s d_t) - (\mu 2^{-1}_{bc}\mu 2^{-1}_{ar} + \mu 2^{-1}_{ca}\,\mu 2^{-1}_{rb} + \mu 2^{-1}_{cr}\,\mu 2^{-1}_{ab})$$

$$(-\phi_i\,\Omega_{jr} - \phi_j\Omega_{ir} + d_r\Omega_{ij} + d_r\phi_i\phi_j)\Big]\Big\} \qquad (5.43)$$

Using the substitutions (5.27), the above simplifies to,

$$\mu 2^{+}_{ij} = \frac{e^{-\frac{1}{2}\omega}e^{\Theta}}{D\,\alpha}\Big\{\Omega_{ij} + \phi_i\phi_j + \tfrac{1}{6}\,\mu 3_{abc}\Big[-\phi_i \; 3\left\{G_{ja}Q_{bc}\right\}_s$$

$$-\phi_j \; 3\left\{G_{ia}Q_{bc}\right\}_s + 3\left\{\eta_a(-\Omega_{ij}Q_{bc} + G_{ib}G_{jc} + G_{ic}G_{jb})\right\}_s$$

$$+ \phi_i\phi_j \; 3\left\{\eta_a Q_{bc}\right\}_s - \phi_i 3\left\{\eta_a\, {}_bG_{jc}\right\}_s - \phi_j \; 3\left\{\eta_a\eta_b G_{ic}\right\}_s$$

$$+(\Omega_{ij} + \phi_i\phi_j)\eta_a\eta_b\eta_c + \phi_i \; 3\left\{G_{ja}\,\mu 2^{-1}_{bc}\right\}_s + \phi_j \; 3\left\{G_{ia}\mu 2^{-1}_{bc}\right\}_s$$

$$-(\Omega_{ij} + \phi_i\phi_j) \; 3\left\{\eta_a\mu 2^{-1}_{bc}\right\}\Big]\Big\} \qquad (5.44)$$

Grouping terms and taking symmetry into account, the above simplifies to the following.

$$\mu 2^{+}_{ij} = \frac{e^{-\frac{1}{2}\omega}e^{\Theta}}{D\,\alpha}\Big\{\Omega_{ij} + \phi_i\phi_j + \tfrac{1}{2}\,\mu 3_{abc}\Big[-\phi_i G_{ja}(Q_{bc} - \mu 2^{-1}_{bc} + \eta_b\eta_c)$$

$$- \phi_j \, G_{ia}(Q_{bc} - \mu 2^{-1}_{bc} + \eta_b\eta_c) + (\Omega_{ij} + \phi_i\phi_j)\eta_a\,(Q_{bc} - \mu 2^{-1}_{bc} + \tfrac{1}{3}\,\eta_b\eta_c)$$

$$+ 2\,\eta_a\, G_{ib}G_{jc}\Big]\Big\} \qquad (5.45)$$

Using the substitutions for ν, D, E_{ij}, O_{ij}, and ϕ_i previously defined, (5.45) can be written as follows;

$$\mu 2^{+}_{ij} = \frac{1}{1+\nu}\Big\{(\Omega_{ij} + \phi_i\phi_j)(1+\nu) - 2\,\phi_i\phi_j(1+\nu) + \mu 3_{abc}\eta_a G_{ib}G_{jc}\Big\} \qquad (5.46)$$

$$\mu 2^{+}_{ij} = \Omega_{ij} + \Psi_{ij} - \phi_i \phi_j$$

$$\Psi_{ij} = \frac{1}{(1+\nu)} \mu 3_{abc}\, \eta_a G_{ib} G_{jc} \tag{5.47}$$

Third Posterior Moment

The third posterior central moment about the posterior mean can be written as follows.

$$\mu 3^{+}_{ijk} = \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{D(2\pi)^{\frac{N}{2}} |\mu 2|^{\frac{1}{2}}} \int_{R_N} e^{-\frac{1}{2}(e_r - d_r)\Omega^{-1}_{rs}(e_s - d_s)} \left\{ 1 + \frac{1}{6}[\;] \right\} (x_i - m^{+}_i)$$

$$(x_j - m^{+}_j)(x_k - m^{+}_k)\; d\underline{e} \tag{5.48}$$

From (5.40),

$$(x_i - m^{+}_i)(x_j - m^{+}_j)(x_k - m^{+}_k) = (b_i - \phi_i)(b_j - \phi_j)(b_k - \phi_k)$$

$$= b_i b_j b_k - 3\left\{\phi_i b_j b_k\right\}_s + 3\left\{\phi_i \phi_j b_k\right\}_s - \phi_i \phi_j \phi_k \tag{5.49}$$

Using (5.49) and (5.42) together with the integrals (F.21) - (F.25), the third posterior moment equation can be rewritten;

$$\mu 3^{+}_{ijk} = \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{D\alpha} \Big\{ - \phi_i \Omega_{jk} - \phi_j \Omega_{ik} - \phi_k \Omega_{ij} - \phi_i \phi_j \phi_k$$

$$+ \frac{1}{6} \mu 3_{abc} \Big[\mu 2^{-1}_{ar} \mu 2^{-1}_{bs} \mu 2^{-1}_{ct} \;(15\left\{\Omega_{rs} \Omega_{ti} \Omega_{jk}\right\}_s$$

$$+ \phi_i \phi_j\, 3\left\{\Omega_{kr} \Omega_{st}\right\}_s + \phi_i \phi_k\, 3\left\{\Omega_{jr} \Omega_{st}\right\}_s + \phi_j \phi_k\, 3\left\{\Omega_{ir} \Omega_{st}\right\}_s$$

$$- \phi_i\, 9\left\{d_r \Omega_{jk} \Omega_{st}\right\}_s - \phi_j\, 9\left\{d_r \Omega_{ik} \Omega_{st}\right\}_s - \phi_k\, 9\left\{d_r \Omega_{ij} \Omega_{st}\right\}_s$$

$$- \phi_i\phi_j\phi_k \; 3\{d_r \Omega_{st}\}_s + 9\{d_r d_s \Omega_{it} \Omega_{jk}\}_s + \phi_i\phi_j \; 3\{d_r d_s \Omega_{kt}\}_s$$

$$+ \phi_i\phi_k \; 3\{d_r d_s \Omega_{jt}\}_s + \phi_j\phi_k \; 3\{d_r d_s \Omega_{it}\}_s - d_r d_s d_t(\phi_i \Omega_{jk}$$

$$+ \phi_j \Omega_{ik} + \phi_k \Omega_{ij}) - d_r d_s d_t \phi_i \phi_j \phi_k) - (\mu 2^{-1}_{bc} \, \mu 2^{-1}_{ar} + \mu 2^{-1}_{ca} \, \mu 2^{-1}_{rb}$$

$$+ \mu 2^{-1}_{cr} \, \mu 2^{-1}_{ab})(\; 3\{\Omega_{ir} \Omega_{jk}\}_s + \phi_i\phi_j \Omega_{rk} + \phi_i\phi_k \Omega_{rj} + \phi_j\phi_k \Omega_{ir}$$

$$- \phi_i d_r \Omega_{jk} - \phi_j d_r \Omega_{ik} - \phi_k d_r \Omega_{ij} - d_r \phi_i \phi_j \phi_k)\Big]\Big\} \qquad (5.50)$$

Using the substitutions (5.27), the third posterior moment becomes,

$$\mu 3^{+}_{ijk} = \frac{e^{-\frac{1}{2}\omega} e^{\theta}}{D \; \alpha} \Big\{ - \phi_i \Omega_{jk} - \phi_j \Omega_{ik} - \phi_k \Omega_{ij} - \phi_i \phi_j \phi_k$$

$$+ \frac{1}{6} \mu 3_{abc} \Big[\Omega_{ij} \; 3\{G_{ka} Q_{bc}\}_s + \Omega_{ik} \; 3\{G_{ja} Q_{bc}\}_s + \Omega_{jk} \; 3\{G_{ia} Q_{bc}\}_s$$

$$+ 6\{G_{ia} G_{jb} G_{kc}\}_s + \phi_i\phi_j \; 3\{G_{ka} Q_{bc}\}_s + \phi_i\phi_k \; 3\{G_{ja} Q_{bc}\}_s$$

$$+ \phi_j\phi_k \; 3\{G_{ia} Q_{bc}\}_s - \phi_i \; 3\{\eta_a(\Omega_{jk} Q_{bc} + G_{jb} G_{kc} + G_{kb} G_{jc})\}_s$$

$$- \phi_j \; 3\{\eta_a(\Omega_{ik} Q_{bc} + G_{ib} G_{kc} + G_{kb} G_{ic})\}_s - \phi_k \; 3\{\eta_a(\Omega_{ij} Q_{bc}$$

$$+ G_{ib} G_{jc} + G_{jb} G_{ic})\}_s - \phi_i\phi_j\phi_k \; 3\{\eta_a Q_{bc}\}_s + 3\{\eta_a \eta_b (G_{ic} \Omega_{jk}$$

$$+ G_{jc} \Omega_{ik} + G_{kc} \Omega_{ij})\}_s + \phi_i\phi_j \; 3\{\eta_a \eta_b G_{kc}\}_s + \phi_i\phi_k \; 3\{\eta_a \eta_b G_{jc}\}_s$$

$$+ \phi_j\phi_k \; 3\{\eta_a \eta_b G_{ic}\}_s - \eta_a \eta_b \eta_c(\phi_i \Omega_{jk} + \phi_j \Omega_{ik} + \phi_k \Omega_{ij} + \phi_i \phi_j \phi_k)$$

$$- \Omega_{jk} \; 3\{G_{ia} \, \mu 2^{-1}_{bc}\}_s - \Omega_{ik} \; 3\{G_{ja} \, \mu 2^{-1}_{bc}\}_s - \Omega_{ij} \; 3\{G_{ka} \, \mu 2^{-1}_{bc}\}_s$$

$$- \phi_i\phi_k \; 3\{G_{ja} \, \mu 2^{-1}_{bc}\}_s - \phi_j\phi_k \; 3\{G_{ia} \, \mu 2^{-1}_{bc}\}_s - \phi_i\phi_j \; 3\{G_{ka} \, \mu 2^{-1}_{bc}\}_s$$

$$+ (\phi_i \Omega_{jk} + \phi_j \Omega_{ik} + \phi_k \Omega_{ij} + \phi_i \phi_j \phi_k) \; 3\{\eta_a \, \mu 2^{-1}_{bc}\}_s \Big] \Big\} \qquad (5.51)$$

Taking symmetry into account and grouping terms, (5.51) becomes,

$$\mu 3^{+}_{ijk} = \frac{e^{-\frac{1}{2}\omega} e^{\Theta}}{D\,\alpha} \Big\{ - \phi_i \Omega_{jk} - \phi_j \Omega_{ik} - \phi_k \Omega_{ij} - \phi_i\phi_j\phi_k$$

$$+ \tfrac{1}{2}\mu 3_{abc}\Big[(\Omega_{ij} + \phi_i\phi_j) G_{ka}(Q_{bc} - \mu 2^{-1}_{bc} + \eta_b\eta_c) + (\Omega_{jk} + \phi_j\phi_k)$$

$$G_{ia}(Q_{bc} - \mu 2^{-1}_{bc} + \eta_b\eta_c) + (\Omega_{ik} + \phi_i\phi_k) G_{ja}(Q_{bc} - \mu 2^{-1}_{bc} + \eta_b\eta_c)$$

$$+ 2\, G_{ia}G_{jb}G_{kc} - \phi_i\Omega_{jk}\eta_a(Q_{bc} - \mu 2^{-1}_{bc} + \tfrac{1}{3}\eta_b\eta_c) - \phi_j\Omega_{ik}\eta_a(Q_{bc}$$

$$- \mu 2^{-1}_{bc} + \tfrac{1}{3}\eta_b\eta_c) - \phi_k\Omega_{ij}\eta_a(Q_{bc} - \mu 2^{-1}_{bc} + \tfrac{1}{3}\eta_b\eta_c)$$

$$- \phi_i\phi_j\phi_k\eta_a(Q_{bc} - \mu 2^{-1}_{bc} + \tfrac{1}{3}\eta_b\eta_c) - 2\,\phi_i\eta_a\, G_{jb}G_{kc}$$

$$- 2\,\phi_j\eta_a\, G_{ib}G_{kc} - 2\,\phi_k\eta_a\, G_{ib}G_{jc}\Big]\Big\} \tag{5.52}$$

Substituting the previously defined variables ν, E_{ij}, O_{ij}, and ψ_{ij}, the desired simplified expression for the third posterior moment is obtained.

$$\mu 3^{+}_{ijk} = \frac{1}{(1+\nu)} \Big\{ (-\phi_i\Omega_{jk} - \phi_j\Omega_{ik} - \phi_k\Omega_{ij} - \phi_i\phi_k\phi_k)(1+\nu)$$

$$+ (\Omega_{ij} + \phi_i\phi_j)\phi_k(1+\nu) + (\Omega_{ik} + \phi_i\phi_k)\phi_j(1+\nu)$$

$$+ (\Omega_{jk} + \phi_j\phi_k)\phi_i(1+\nu) + \mu 3_{abc} G_{ia}G_{jb}G_{kc} - (\phi_i\psi_{jk} + \phi_j\psi_{ik}$$

$$+ \phi_k\psi_{ij})(1+\nu)\Big\} \tag{5.53}$$

Therefore,

$$\mu 3^{+}_{ijk} = N_{ijk} - \phi_i\psi_{jk} - \phi_j\psi_{ik} - \phi_k\psi_{ij} + 2\,\phi_i\phi_j\phi_k$$

$$N_{ijk} = \frac{1}{(1+\nu)}\mu 3_{abc}\, G_{ia}G_{jb}G_{kc} \tag{5.54}$$

The above equations for the posterior moments about the posterior mean were derived by substituting the expression for the posterior mean into the moment equations. Originally, these equations were derived by computing the posterior moments about the prior mean, and using another set of relationships between moments about different origins. Both methods resulted in the same equations, providing a check on the above results.

Summary of the Optimal Discrete Updating Equations

The central moments m_i, $\mu2_{ij}$, and $\mu3_{ijk}$ are available prior to the measurement from the autonomous filter equations, which extrapolate the moments between measurements from a knowledge of the system dynamics. At some point in time a discrete measurement of the following form is received.

$$v_r = h_r(\underline{m}) + H_{re}(x_e - m_e) + \xi_r \qquad (5.55)$$

ξ_r is a normally distributed random vector such that,

$$\mathcal{E}(\xi_r) = 0$$

$$\mathcal{E}(\xi_r \xi_s) = \Sigma_{rs}$$

In more concise notation,

$$\xi_r = N(0, \Sigma_{rs})$$

Then, from the foregoing derivation, the first three central moments of the state conditioned on the measurement can be obtained exactly from the following equations, subject to the assumption that the prior density function of the vector x_i can be expressed exactly as the product of a gaussian density times a third order expansion of multidimensional Hermite polynomials.

Table 5.1 Exact Third Order Updating Equations

Tensor Notation	Vector Notation
$z_r = v_r - h_r$	$z = v - h$
$\Lambda_{ij} = H_{qi}\Sigma^{-1}_{qr} H_{rj}$	$\Lambda = H^T\Sigma^{-1}H$
$\Omega_{ij} = (\mu 2^{-1} + \Lambda)^{-1}_{ij}$	$\Omega = (\mu 2^{-1} + \Lambda)^{-1}$
$d_i = \Omega_{ij}H_{kj}\Sigma^{-1}_{ke}z_e$	$d = \Omega H^T\Sigma^{-1}z$
$\eta_i = \mu 2^{-1}_{ie} d_e$	$\eta = \mu 2^{-1}d$
$G_{ij} = \Omega_{ie}\mu 2^{-1}_{ej}$	$G = \Omega\mu 2^{-1}$
$Q_{ij} = \mu 2^{-1}_{ie} G_{ej}$	$Q = \mu 2^{-1} G$
$E_{ij} = Q_{ij} - \mu 2^{-1}_{ij} + \frac{1}{3}\eta_i\eta_j$	$E = Q - \mu 2^{-1} + \frac{1}{3}\eta\eta^T$
$O_{ij} = Q_{ij} - \mu 2^{-1}_{ij} + \eta_i\eta_j$	$O = Q - \mu 2^{-1} + \eta\eta^T$

$\nu = \frac{1}{2}\mu 3_{abc}\eta_a E_{bc}$

$K = 1/(1+\nu)$

$\phi_i = \frac{1}{2} K \mu 3_{abc} G_{ia} O_{bc}$

$\psi_{ij} = K \mu 3_{abc} \eta_a G_{ib} G_{jc}$

$N_{ijk} = K \mu 3_{abc} G_{ia} G_{jb} G_{kc}$

These third order tensor contractions cannot be written in vector notation.

$m^+_i = m_i + d_i + \phi_i$

$\mu 2^+_{ij} = \Omega_{ij} + \psi_{ij} - \phi_i\phi_j$

$\mu 3^+_{ijk} = N_{ijk} - \phi_i \psi_{jk} - \phi_j \psi_{ik} - \phi_k \psi_{ij} + 2\phi_i\phi_j\phi_k$

5.5 Properties of the Hermite Polynomial Solution

For the special case in which all the prior third moments $\mu 3_{abc}$ are zero, the exact third order measurement updating equations in Table 5.1 reduce to the familiar Kalman updating equations.

$$
\begin{aligned}
m_i^+ &= m_i + d_i \\
\mu 2_{ij}^+ &= \Omega_{ij}
\end{aligned}
\qquad (5.56)
$$

For a scalar system, the third order tensor contractions in Table 5.1 reduce to single terms. Hence the scalar updating equations are particularly simple. The computed posterior moments are not linear in the measurement y as in the case of the Kalman filter, but instead are functions of higher powers of y.

$$
\begin{aligned}
m^+ &= f(m,\ y,\ y^2) \\
\mu 2^+ &= f(\Omega, y, y^4) \\
\mu 3^+ &= f(\mu 3,\ y^3,\ y^6)
\end{aligned}
\qquad (5.57)
$$

Since the measurement noise is assumed to be zero mean, the a priori expectations for the terms found in the updating equations can be calculated. From Table 5.1, in vector notation,

$$0 = \mu 2^{-1}\Omega\,\mu 2^{-1} - \mu 2^{-1} + \mu 2^{-1}\Omega\, H^T \Sigma^{-1} z\ z^T \Sigma^{-1} H \Omega\, \mu 2^{-1} \qquad (5.58)$$

$$\mathcal{E}(0) = \mu 2^{-1}\Omega\,\mu 2^{-1} - \mu 2^{-1} + \mu 2^{-1}\Omega H^T \Sigma^{-1}(\Sigma + H\mu 2\ H^T)\Sigma^{-1} H \Omega\, \mu 2^{-1} \qquad (5.59)$$

$$\mathcal{E}(0) = \mu 2^{-1}\Omega(\Omega^{-1} - \Omega^{-1}\mu 2\,\Omega^{-1} + H^T \Sigma^{-1}(\Sigma + H\,\mu 2\ \ H^T)\Sigma^{-1} H)\Omega\,\mu 2^{-1} \qquad (5.60)$$

From the definition of Ω, this becomes,

$$\mathcal{E}(0) = \mu 2^{-1}\Omega\left[\mu 2^{-1} + H^T\Sigma^{-1}H - (\mu 2^{-1} + H^T\Sigma^{-1}H)\mu 2(\mu 2^{-1} + H^T\Sigma^{-1}H)\right.$$

$$\left. + H^T\Sigma^{-1}H + H^T\Sigma^{-1}H\mu 2\, H^T\Sigma^{-1}H\right]\Omega\,\mu 2^{-1} \qquad (5.61)$$

The terms in the center brackets all cancel, with the result that;

$$\mathcal{E}(0) = 0 \qquad (5.62)$$

From Table 5.1,

$$\phi_i = \tfrac{1}{2} K\, \mu 3_{abc}\, G_{ia}\, O_{bc} \qquad (5.63)$$

If the third moments are small, (so that the Hermite polynomial expansion is a good approximation) then the terms $\mathcal{V}$, ϕ_i, and ψ_{ij} will all be small. Thus K will be near one, and even though K is a function of z, from the form of (5.63) it is apparent that,

$$\mathcal{E}(\phi_i) \approx 0 \qquad (5.64)$$

By similar reasoning,

$$\mathcal{E}(\psi_{ij}) \approx 0 \qquad (5.65)$$

Since the expectations of these terms are small, they can be dropped from the updating equations without changing the mean of the estimates. And, since the terms themselves are not large, it appears reasonable also that the expectations of their products, being second order, will also be small. For the assumed density function, the estimates were optimal. For this reason , neglecting any terms should increase the resulting variance of the estimates. This provides an interesting method of testing the validity of the previously derived nonlinear estimation theory. That is, the following four filters have progressively higher complexity and should have progressively better performance.

1 Optimal Second Order Nonlinear Filter

a) Second order autonomous filter equations used between measurements.

b) At the measurements,

$$m_i^+ = m_i + d_i$$

$$\mu 2_{ij}^+ = \Omega_{ij} \qquad (5.66)$$

2 First Approximate Third Order Nonlinear Filter

a) Third order autonomous filter equations used between measurements.

b) At the measurements,

$$m_i^+ = m_i + d_i$$

$$\mu 2_{ij}^+ = \Omega_{ij}$$

$$\mu 3_{ijk}^+ = N_{ijk} \qquad (5.67)$$

3 Second Approximate Third Order Nonlinear Filter

a) Third order autonomous filter equations used between measurements.

b) At the measurements,

$$m_i^+ = m_i + d_i + \phi_i$$

$$\mu 2_{ij}^+ = \Omega_{ij}$$

$$\mu 3_{ijk}^+ = N_{ijk} \qquad (5.68)$$

4 Optimal Third Order Nonlinear Filter

a) Third order autonomous filter equations used between measurements.

b) At the measurements,

$$m_i^+ = m_i + d_i + \phi_i$$

$$\mu 2_{ij}^+ = \Omega_{ij} + \Psi_{ij} - \phi_i \phi_j$$

$$\mu 3_{ijk}^+ = N_{ijk} - \phi_i \Psi_{jk} - \phi_j \Psi_{ik} - \phi_k \Psi_{ij} + 2\,\phi_i \phi_j \phi_k \qquad (5.69)$$

These four filters are tested by Monte Carlo simulation in a later section.

The techniques used to derive equations (5.69) could be used to develop an optimal fourth order estimate. Since the Hermite Polynomials are orthogonal, the second and third order terms for this fourth order estimate would be the same as those in (5.69) However, an equation for $\mu 4_{ijkl}^+$ would be added, and additional terms involving fourth order contractions of $\mu 4_{ijkl}$ would be added to the optimal estimates for m_i^+ , $\mu 2_{ij}^+$, and $\mu 3_{ijk}^+$. It should be apparent that the optimal fourth order filter would be substantially more complex than the third order one, and hence of very limited practical importance.

In like manner, the optimal fifth and higher moments could conceptually be computed, forming fifth and higher order contraction terms. The true optimal nonlinear filter is infinite dimensional, and consists of an infinite number of these contraction terms. Nevertheless, one can refer to optimal second and third order filters, as has been done here. The reason why a third order filter seems a significant gain over a second order one is that this allows asymmetry in the modeled probability distribution. For most physical problems, the magnitude of the higher order terms falls off rather sharply; in fact a system must be highly nonlinear in order to make the third order contraction terms significant, and to make the third order filter significantly more accurate than the second order one. A number of systems of practical interest do fall into this category, however, as will be seen in later chapters.

5.6 Nonlinear Measurements

The discrete measurement updating methods derived thus far have been restricted to linear measurements. If the measurement function is nonlinear but differentiable, then it can be expanded in a Taylor's series about the conditional mean. In order to keep the problem tractable, it will be assumed that the measurement function can be adequately described by a second order expansion.

$$h_r(\underline{x}) \approx h_r(\underline{m}) + H_{re}(x_e - m_e) + M_{ref}(x_e - m_e)(x_f - m_f)$$

$$H_{re} = h_{r,e}(\underline{m})$$

$$M_{ref} = \tfrac{1}{2}\, h_{r,ef}(\underline{m}) \tag{5.70}$$

The Bayes equation for the posterior probability density is;

$$P(\underline{x}, t_n^+) = \frac{P(\underline{x}, t_n)\ \exp(-\frac{1}{2}(y_r - h_r(\underline{x}))\Sigma^{-1}_{rs}(y_s - h_s(\underline{x})))}{\int_{R_N} P(\underline{a}, t_n)\ \exp(-\frac{1}{2}(y_r - h_r(\underline{a}))\Sigma^{-1}_{rs}(y_s - h_s(\underline{a})))\ d\underline{a}} \tag{5.71}$$

Following the derivation of the Hermite polynomial solution to the Bayes equation, the relationship (5.22) can be used to write the denominator integral of (5.71) in the following form.

Let $z_r = y_r - h_r(\underline{m})$

$e_p = x_p - m_p$

$$D = \int_{R_N} \exp\Big[-\tfrac{1}{2}(z_r - H_{rp}e_p - M_{rpq}e_pe_q)\,\Sigma^{-1}_{rs}(z_s - H_{sa}e_a - M_{sab}e_ae_b)$$

$$-\tfrac{1}{2}\,e_r \mu 2^{-1}_{rs} e_s\Big]\Big\{1 + \frac{\mu 3_{ijk}}{6}\Big[\mu 2^{-1}_{ir}\,\mu 2^{-1}_{js}\,\mu 2^{-1}_{kt}\,e_re_se_t$$

$$-(\mu 2^{-1}_{jk}\,\mu 2^{-1}_{ir} + \mu 2^{-1}_{ik}\,\mu 2^{-1}_{jr} + \mu 2^{-1}_{ij}\,\mu 2^{-1}_{kr})\,e_r\Big]\Big\}\,d\underline{e} \qquad (5.72)$$

In order to solve the above equation, an analytic solution for integrals of the following form is required.

$$\int_{-\infty}^{+\infty} \exp(-a(b - c\,x - d\,x^2)^2)\quad (x-\beta)^P\,dx \qquad (5.73)$$

The solution of integrals of this type could not be found in published tables of definite integrals, nor could any of the usual methods of definite integration, such as residues or Gamma function substitution, be sucessfully applied.

An alternative solution of (5.72) can be obtained by noting that,

$$D(M_{rpq} + \Delta M_{rpq}) \approx D(M_{rpq}) + \left.\frac{\partial D}{\partial M_{rpq}}\right|_{M_{rpq}} \Delta M_{rpq} \qquad (5.74)$$

And hence,

$$D(M_{rpq}) = D(M_{rpq}=0) + \int_0^{M_{rpq}} \left.\frac{\partial D}{\partial m_{rpq}}\right|_{m_{rpq}} dm_{rpq} \qquad (5.75)$$

From (5.72),

$$\frac{\partial D}{\partial m_{rpq}} = \int_{R_N} \exp(-\tfrac{1}{2}(\quad))\Big\{1+\tfrac{1}{6}[\;]\Big\}\Big[\Sigma^{-1}_{rs}(z_se_pe_q - H_{sa}e_ae_pe_q - m_{sab}e_ae_be_pe_q)\Big]\,d\underline{e}$$

$$(5.76)$$

The above equation can be expressed in terms of the posterior moments about the posterior mean as follows.

$$g_i = m_i^+ - m_i$$
$$b_i = x_i - m_i^+$$
$$e_i = x_i - m_i = b_i + g_i$$

$$e_i e_j = b_i b_j + 2\{g_i b_j\}_s + g_i g_j$$

$$e_i e_j e_k = b_i b_j b_k + 3\{g_i b_j b_k\}_s + 3\{g_i g_j b_k\}_s + g_i g_j g_k$$

$$e_i e_j e_k e_l = b_i b_j b_k b_l + 4\{g_i b_j b_k b_l\}_s + 6\{g_i g_j b_k b_l\}_s$$
$$+ 4\{g_i g_j g_k b_l\}_s + g_i g_j g_k g_l \qquad (5.77)$$

Using (5.76) and (5.77), the partial derivatives of the denominator integral and the moments can be expressed in terms of known quantities.

$$\frac{\partial D}{\partial m_{rpq}} = D\Sigma^{-1}_{rs}\Big[z_s(\mu 2^+_{pq} + g_p g_q) - H_{sa}(\mu 3^+_{apq} + 3\{g_a \mu 2^+_{pq}\}_s + g_a g_p g_q)$$
$$-m_{sab}(\mu 4^+_{abpq} + 4\{g_a \mu 3^+_{bpq}\}_s + 6\{g_a g_b \mu 2^+_{pq}\}_s + g_a g_b g_p g_q)\Big]$$
$$(5.78)$$

$$\frac{\partial m_i^+}{\partial m_{rpq}} = \Sigma^{-1}_{rs}\Big[z_s(\mu 3^+_{ipq} + 2\{g_p \mu 2^+_{iq}\}_s + m_i^+(\mu 2^+_{pq} + g_p g_q))$$
$$-H_{sa}(\mu 4^+_{iapq} + 3\{g_a \mu 3^+_{ipq}\}_s + 3\{g_a g_p \mu 2^+_{iq}\}_s + m_i^+(\mu 3^+_{apq} + 3\{g_a \mu 2^+_{pq}\}_s$$
$$+ g_a g_p g_q)) - m_{sab}(\mu 5^+_{iabpq} + 4\{g_a \mu 4^+_{ibpq}\}_s + 6\{g_a g_b \mu 3^+_{ipq}\}_s$$
$$+ 4\{g_a g_b g_p \mu 2^+_{iq}\}_s + m_i^+(\mu 4^+_{abpq} + 4\{g_a \mu 3^+_{bpq}\}_s + 6\{g_a g_b \mu 2^+_{pq}\}_s$$

$$+ g_a g_b g_p g_q))\Big] - \frac{m_i^+}{D} \frac{\partial D}{\partial m_{rpq}} \tag{5.79}$$

$$\frac{\partial \mu 2_{ij}^+}{\partial m_{rpq}} = \Sigma_{rs}^{-1}\Big[z_s(\mu 4_{ijpq}^+ + 2\{g_p \mu 3_{ijq}^+\}_s + g_p g_q \mu 2_{ij}^+)$$

$$-H_{sa}(\mu 5_{ijapq}^+ + 3\{g_a \mu 4_{ijpq}^+\}_s + 3\{g_a g_p \mu 3_{ijq}^+\}_s + g_a g_p g_q \mu 2_{ij}^+)$$

$$-m_{sab}(\mu 6_{ijabpq}^+ + 4\{g_a \mu 5_{ijbpq}^+\}_s + 6\{g_a g_b \mu 4_{ijpq}^+\}_s$$

$$+4\{g_a g_b g_p \mu 3_{ijq}^+\}_s + g_a g_b g_p g_q \mu 2_{ij}^+)\Big] - \frac{\mu 2_{ij}^+}{D} \frac{\partial D}{\partial m_{rpq}} \tag{5.80}$$

$$\frac{\partial \mu 3_{ijk}^+}{\partial m_{rpq}} = \Sigma_{rs}^{-1}\Big[z_s(\mu 5_{ijkpq}^+ + 2\{g_p \mu 4_{ijkq}^+\}_s + g_p g_q \mu 3_{ijk}^+)$$

$$-H_{sa}(\mu 6_{ijkapq}^+ + 3\{g_a \mu 5_{ijkpq}^+\}_s + 3\{g_a g_p \mu 4_{ijkq}^+\}_s + g_a g_p g_q \mu 3_{ijk}^+)$$

$$-m_{sab}(\mu 7_{ijkabpq}^+ + 4\{g_a \mu 6_{ijkbpq}^+\}_s + 6\{g_a g_b \mu 5_{ijkpq}^+\}_s$$

$$+4\{g_a g_b g_p \mu 4_{ijkq}^+\}_s + g_a g_b g_p g_q \mu 3_{ijk}^+)\Big] - \frac{\mu 3_{ijk}^+}{D} \frac{\partial D}{\partial m_{rpq}} \tag{5.81}$$

For a moment expansion truncated after the third quasi-moment,

$$\mu 4_{ijkl} = 3\{\mu 2_{ij}\, \mu 2_{kl}\}_s$$

$$\mu 5_{ijklm} = 10\{\mu 2_{ij}\, \mu 3_{klm}\}_s$$

$$\mu 6_{ijklmn} = 15\{\mu 2_{ij}\, \mu 2_{kl}\, \mu 2_{mn}\}_s + 10\{\mu 3_{ijk}\, \mu 3_{lmn}\}_s$$

$$\mu 7_{ijklmno} = 105\{\mu 2_{ij}\, \mu 2_{kl}\, \mu 3_{mno}\}_s \tag{5.82}$$

With these substitutions, the moment equations become,

$$\frac{\partial m_i^+}{\partial m_{rpq}} = \Sigma_{rs}^{-1}\Big[z_s(\mu3^+_{ipq} + 2\{g_p\mu2^+_{iq}\}_s) - H_{sa}(3\{\mu2^+_{ia}\mu2^+_{pq}\}_s + 3\{g_a\mu3^+_{ipq}\}_s + 3\{g_a g_p\mu2^+_{iq}\}_s) - m_{sab}(10\{\mu2^+_{ia}\mu3^+_{bpq}\}_s + 12\{g_a\mu2^+_{ib}\mu2^+_{pq}\}_s + 6\{g_a g_b\mu3^+_{ipq}\}_s + 4\{g_a g_b g_p\mu2^+_{iq}\}_s)\Big] \tag{5.83}$$

$$\frac{\partial \mu2^+_{ij}}{\partial m_{rpq}} = \Sigma_{rs}^{-1}\Big[z_s(3\{\mu2^+_{ij}\mu2^+_{pq}\}_s + 2\{g_p\mu3^+_{ijk}\}_s - \mu2^+_{ij}\mu2^+_{pq}) - H_{sa}(10\{\mu2^+_{ij}\mu3^+_{apq}\}_s + 9\{g_a\mu2^+_{ij}\mu2^+_{pq}\}_s + 3\{g_a g_p\mu3^+_{ijq}\}_s - \mu2^+_{ij}\mu3^+_{apq} - \mu2^+_{ij}3\{g_a\mu2^+_{pq}\}_s) - m_{sab}(15\{\mu2^+_{ij}\mu2^+_{ab}\mu2^+_{pq}\}_s + 10\{\mu3^+_{ija}\mu3^+_{bpq}\}_s + 40\{g_a\mu2^+_{ij}\mu3^+_{bpq}\}_s + 18\{g_a g_b\mu2^+_{ij}\mu2^+_{pq}\}_s + 4\{g_a g_b g_p\mu3^+_{ijq}\}_s - \mu2^+_{ij}(3\{\mu2^+_{ab}\mu2^+_{pq}\}_s + 4\{g_a\mu3^+_{bpq}\}_s + 6\{g_a g_b\mu2^+_{pq}))\Big] \tag{5.84}$$

$$\frac{\partial \mu3^+_{ijk}}{\partial m_{rpq}} = \Sigma_{rs}^{-1}\Big[z_s(10\{\mu2^+_{ij}\mu3^+_{kpq}\}_s + 6\{g_p\mu2^+_{ij}\mu2^+_{kq}\}_s - \mu2^+_{pq}\mu3^+_{ijk}) - H_{sa}(15\{\mu2^+_{ij}\mu2^+_{ka}\mu2^+_{pq}\}_s + 10\{\mu3^+_{ijk}\mu3^+_{apq}\}_s + 30\{g_a\mu2^+_{ij}\mu3^+_{kpq}\}_s + 9\{g_a g_p\mu2^+_{ij}\mu2^+_{kq}\}_s - \mu3^+_{ijk}\mu3^+_{apq} - \mu3^+_{ijk}3\{g_a\mu2^+_{pq}\}_s) - m_{sab}(105\{\mu2^+_{ij}\mu2^+_{ka}\mu3^+_{bpq}\}_s + 60\{g_a\mu2^+_{ij}\mu2^+_{kb}\mu2^+_{pq}\}_s + 40\{g_a\mu3^+_{ijk}\mu3^+_{bpq}\}_s + 60\{g_a g_b\mu2^+_{ij}\mu3^+_{kpq}\}_s$$

$$+12\left\{g_a g_b g_p \mu 2^+_{ij} \mu 2^+_{kq}\right\}_s - \mu 3^+_{ijk} 3\left\{\mu 2^+_{ab} \mu 2^+_{pq}\right\}_s - \mu 3^+_{ijk} 4\left\{g_a \mu 3^+_{bpq}\right\}_s$$

$$- \mu 3^+_{ijk} 6\left\{g_a g_b \mu 2^+_{pq}\right\}_s\Big)\Big] \qquad (5.85)$$

The nonlinear measurement solution to the Bayes equation can be obtained by integrating (5.83), (5.84), and (5.85) over $m_{rpq} = 0$ to $m_{rpq} = M_{rpq}$. The initial values for the integrals at $m_{rpq} = 0$ are simply the Hermite polynomial solutions for a linear measurement. For a scalar system, the solution is obtained as follows.

$$z = y - h(m)$$

$$H = \left.\frac{\partial h}{\partial x}\right|_m$$

$$M = \frac{1}{2}\left.\frac{\partial^2 h}{\partial x^2}\right|_m \qquad (5.86)$$

Calculate the optimal third order filter solution, m^+, $\mu 2^+$, and $\mu 3^+$ for a linear measurement from the equations in Table 5.1 . Now integrate the following equations from E =0 to E = M.

$$g = m^+ - m$$

$$\frac{\partial m^+}{\partial E} = \Sigma^{-1}\Big[z(\mu 3^+ + 2\, g\, \mu 2^+) - 3\, H(\mu 2^{+2} + g\, \mu 3^+ + g^2\, \mu 2^+)$$

$$-E(10\, \mu 2^+ \mu 3^+ + 12\, g\, \mu 2^{+2} + 6\, g^2\, \mu 3^+ + 4\, g^3\, \mu 2^+)\Big] \qquad (5.87)$$

$$\frac{\partial \mu 2^{+}}{\partial E} = \Sigma^{-1}\Big[z(2\,\mu 2^{+2} + 2\,g\,\mu 3^{+}) - H(9\,\mu 2^{+}\,\mu 3^{+} + 6\,g\,\mu 2^{+2} + 3\,g^{2}\,\mu 3^{+}) - E(12\,\mu 2^{+3} + 10\,\mu 3^{+2} + 36\,g\,\mu 2^{+}\mu 3^{+} + 12\,g^{2}\,\mu 2^{+2} + 4\,g^{3}\,\mu 3^{+})\Big] \qquad (5.88)$$

$$\frac{\partial \mu 3^{+}}{\partial E} = \Sigma^{-1}\Big[z(9\,\mu 2^{+}\,\mu 3^{+} + 6\,g\,\mu 2^{+2}) - H(15\,\mu 2^{+3} + 9\,\mu 3^{+2} + 27\,g\,\mu 2^{+}\mu 3^{+} + 9\,g^{2}\,\mu 2^{+2}) - E(102\,\mu 2^{+2}\,\mu 3^{+} + 60\,g\,\mu 2^{+3} + 36\,g\,\mu 3^{+2} + 54\,g^{2}\,\mu 2^{+}\mu 3^{+} + 12\,g^{3}\,\mu 2^{+2})\Big] \qquad (5.89)$$

The above procedure provides a solution to the Bayes equation for a nonlinear measurement function. Note that this is really a numerical solution to the necessary integrals. Alternatively, one might do numerical integration of the entire density function. In the one dimensional simulation tests, both techniques were tried. They produced substantially identical results, but the two step method described above was much faster than direct integration, since the required numerical integrals are over a better defined range.

While the general vector equations derived above are very complex, simpler approximate forms may be useful for special cases. For example, if the measurement nonlinearity is sufficiently small, the integrals may be replaced by first order differential corrections. These can be easily obtained from (5.83) - (5.85). Their complexity is still substantial, and it should be clear that many other approximate solutions are possible.

The results of this section show the tremendous complexity of the nonlinear measurement problem, and the high order of the terms involved. If an analytic solution to the Bayes integral could be found, it might or might not be simpler than the procedures described. This work is only the barest introduction, and the problem of optimal estimation using discrete nonlinear measurements must be left as a very open area for future investigation.

Chapter 6

Practical Implementation of the Nonlinear Estimation Equations

6.1 Introduction

The nonlinear estimation techniques discussed in this thesis are best suited to digital computation. In order to assess the practicality of these methods for a given system, it is necessary to know the performance, memory storage requirements, and computational requirements for the second, third, and fourth order nonlinear filters. These truncated optimal nonlinear filters can then be compared to the linearized Kalman filter, and to other types of suboptimal, approximate nonlinear estimators.

The storage requirements for each filter can be determined in a straightforward fashion, and in general are not a critical factor. Computational requirements, on the other hand, depend on both the calculation required per step, and the step size. Stability of the filter equations primarily determines the step size, and appears to be both the most complex and critical factor in the application of nonlinear estimation techniques.

6.2 Storage of the Moments

The linearized Kalman filter and second order nonlinear filter are very similar, since both involve estimation of the same number of parameters. For the same reason, the continuous and discrete measurement filters have the same storage requirements.

The D th moment of an N dimensional vector is a D th order tensor of N^D components. Since this tensor is symmetric in all indices, only $\binom{N+D-1}{D}$ of these components are unique. In order to avoid computing and storing these unique components more than once, a method of enumerating the unique components and storing them in a dense array is required. Two methods of accomplishing efficient storage and addressing are derived in Appendix D.

The address computation is somewhat complex and time consuming, so that in some cases it may be preferable to use full arrays. In particular, if memory volume is available and computation speed is critical, it will be advantageous to compute only the unique moments, while storing them in a full symmetric array. However, in computing contractions on these arrays, all symmetries should be utilized to avoid repeated identical multiplications. In this case, the time consuming address computation can be avoided. The moment storage required for second, third, and fourth order filters is summarized below.

Filter	Memory Words Required for Moment Storage	
All Moments	N = System Order	Example N=5
2 nd Order Filter	$N + N^2$	30
3 rd Order Filter	$N + N^2 + N^3$	155
4 th Order Filter	$N + N^2 + N^3 + N^4$	780
Unique Moments		
2 nd Order Filter	$N + \binom{N+1}{2}$	20
3 rd Order Filter	$N + \binom{N+1}{2} + \binom{N+2}{3}$	55
4 th Order Filter	$N + \binom{N+1}{2} + \binom{N+2}{3} + \binom{N+3}{4}$	125

Table 6.1 Moment Storage Requirements

6.3 The Third Order Tensor Contraction

The execution of the exact third order updating equations of Table 5.1 is a formidable task, since these equations contain third order tensor contractions of the following form.

$$N_{ijk} = K\, \mu 3_{abc}\, G_{ia} G_{jb} G_{kc} \qquad (6.1)$$

That is, for all i, j, k an N_{ijk} must be computed that is a triple sum as indicated in (6.1). This would indicate that the required computation is proportional to the system order to the sixth power! Actually the symmetry of N_{ijk} and $\mu 3_{abc}$ can be taken into account to substantially reduce the required computation, and the execution of the equations of Table 5.1 on an electronic digital computer is quite feasible for systems of low order.

These equations do, however, become prohibitively lengthy quite rapidly as system order increases.

The following is a listing of a computer program similar to the one used to simulate the optimal third order estimator, which illustrates an efficient method of performing the third order tensor contraction. Initially, $\mu 2_{ij}$ and $\mu 3_{ijk}$ are stored in compact arrays. Subsequently $\mu 2_{ij}$ is brought into a full array for convenience. The third moment is always used in a compact array. The addressing and manipulation of compact arrays is discussed in Appendix D.

The program is optimized for execution speed. This is done by computing only the unique second and third posterior moments, computing the compact third moment addresses by the fastest method (address table), and by using all symmetries to eliminate repeated multiplications in computing only the unique terms in the symmetric tensor contractions. The resulting program is still quite complex, and has an execution time roughly proportional to the fifth power of the state dimension.

```
R                 THIS IS A SAMPLE MAC PROGRAM TO EFFICIENTLY PERFORM THE OPTIMAL
R                 THIRD ORDER UPDATING EQUATIONS OF TABLE 5.1, INCLUDING THE THIRD
R                 ORDER TENSOR CONTRACTIONS. IN THIS EXAMPLE, N, THE SYSTEM
R                 ORDER, IS THREE.

M                 DIMENSION (Z,3),(CLAMDA,3X3),(OMEGA,3X3),(D,3),(ETA,3),(G,3X3),
M                 (Q,3X3),(E,3X3),(O,3X3),(PHI,3),(PSI,3X3),(PSIC,6),(EPC,10),
M                 (M,3),(M2,3X3),(M2C,6),(M3C,10)

R                 M2, M3, AND PSI ARE STORED IN COMPACT VECTORS M2C, M3C,
R                 AND PSIC.
R                 IT IS ASSUMED THAT THE AUTONOMOUS FILTER HAS CURRENT
R                 ESTIMATES OF THE FIRST THREE MOMENTS STORED IN THE COMPACT
R                 ARRAYS M(3), M2C(6), AND M3C(10).  THE POSTERIOR
R                 ESTIMATES WILL ALSO BE IN THESE ARRAYS.

E                 - - * -
M                 Z=Y-H M

E                 *       *T *    -1 *
M                 CLAMDA=H  SIGMA    H

R                 PUT COMPACT M2C INTO FULL ARRAY M2

M                 R=-1

M                 DO TO 60 FOR I=0(1)(N-1)

M                 DO TO 60 FOR J=I(1)(N-1)

M                 R=R+1

M                 M2       =M2C
S                   N I+J      R

M        60       M2       =M2
S                   N J+I    N I+J

E                 *      * -1 *       -1
M                 OMEGA=(M2   +CLAMDA)

E                 - *      *T *    -1 -
M                 D=OMEGA H  SIGMA    Z

E                 -    * -1 -
M                 ETA=M2    D

E                 * *      * -1
M                 G=OMEGA M2

E                 * * -1 *
M                 Q=M2    G

E                 * * * -1       -    -
M                 E=Q-M2   +(1/3)ETA ETA
```

```
E                  * * * -1       -   -
M                  D=D-M2   +     ETA ETA

R                                              -1
R                  IN AN OPERATIONAL PROGRAM, M2   WOULD BE COMPUTED ONLY ONCE.
R                  THIS PROGRAM IS ILLUSTRATIVE ONLY.
F                  -      -      -
M                  PHI=0,PSC=0,ERC=0,NU=0

R                  BEGIN CONTRACTION

M                  DO TO 100 FOR A=0(1)(N-1)

M                  DO TO 100 FOR B=0(1)(N-1)

M                  DO TO 100 FOR C=0(1)(N-1)

R                  F IS THE COMPACT ADDRESS OF THE ABC COMPONENT OF THE THIRD
R                  MOMENT.

M                  F=ADDRESS
S                           N N A+N B+C

M                  R=-1,S=-1

M                  DO TO 90 FOR I=0(1)(N-1)

M                  DO TO 80 FOR J=I(1)(N-1)

M                  DO TO 70 FOR K=J(1)(N-1)

M                  R=R+1

M        70        ERC =ERC +M3C  G       G       G
S                     R    R     F  N I+A  N J+B  N K+C

M                  S=S+1

M        80        PSIC =PSIC +M3C  ETA  G       G
S                        S     S     F    A  N I+B  N J+C

M        90        PHI =PHI +M3C  G       O
S                     I    I     F  N I+A  N B+C

M        100       NU=NU+M3C  ETA  E
S                             F    A  N B+C

M                  K1=1/(1+.5 NU)

E                  -         -
M                  PHI=.5 K1 PHI

R                  FORM FULL PSI ARRAY FROM COMPACT ARRAY PSIC.

M                  R=-1

M                  DO TO 110 FOR I=0(1)(N-1)
```

```
M                  DO TO 110 FOR J=I(1)(N-1)

                   R=R+1

M                  PSI     =K1 PSIC
S                     N I+J       R

M        110       PSI     =PSI
S                     N J+I    N I+J

R                  EXECUTE MOMENT UPDATE

M                  R=-1,S=-1

M                  DO TO 120 FOR I=0(1)(N-1)

M                  M =M +D +PHI
S                   I  I  I    I

M                  DO TO 120 FOR J=I(1)(N-1)

M                  R=R+1

M                  M2C =OMEGA      +PSI      -PHI   PHI
S                     R        N I+J    N I+J     I     J

M                  DO TO 120 FOR K=J(1)(N-1)

                   S=S+1

M        120       M3C = K1 ERC -PHI  PSI       -PHI  PSI       -PHI  PSI
S                     S         S    I     N J+K     J    N I+K     K    N I+J

M                  +2 PHI  PHI  PHI
S                         I    J    K

R                  THIS COMPLETES THE OPTIMAL THIRD ORDER MOMENT UPDATE.
R                  THE NEW ESTIMATES ARE STORED IN THE COMPACT ARRAYS M(3),
R                  M2C(6), AND M3C(10).
```

6.4 Stability

The chief drawback to the practical application of the continuous measurement Kushner filter is the large amount of computation required. This is brought on primarily not by the complexity of the derivative calculations, but by the poor stability of the set of coupled differential filter equations. The autonomous filter equations are poorly behaved numerically, and the inclusion of "white" measurement noise causes stable integration to be achieved only with very small step sizes.

A large part of this stability problem is eliminated by the discrete measurement formulation. Similar nonlinear coupled differential equations are solved for the moment propagation between measurements, but these are not forced by measurement noise and hence can be integrated using a larger step size. For typical systems, twenty to fifty times as large a step size can be tolerated by the autonomous filter.

At the measurement times, the moments of the density function are discontinuously updated. The series update, the integration update, and the Hermite polynomial solution all involve approximations. All three methods approximate the probability density as a gaussian density times an expansion of Hermite polynomials. In addition, the series and integration methods use approximations in the solution of the Bayes integrals.

As a consequence of these approximations, all three methods are found to be unstable under certain conditions. The series update is unstable for sufficiently accurate measurements, since the series approximation converges poorly in this case, and the integration update method is found to embark upon divergent oscillations when used in conjunction with a marginally stable or unstable system.

The Hermite polynomial solution has the best stability properties of the three, which is logical since it involves the fewest approximations. Nevertheless it will occasionally diverge for certain sequences of random noise, especially with marginally stable plants, such as an undamped oscillator. An insight into this occasional divergence problem can be gained by examining the updating equations of Table 5.1. From these equations, it can be readily seen that the posterior third moments are a function of the sixth power of the measurement residue! A true optimal nonlinear estimator would involve all moments of the density, and all powers of the measurement residue. It seems apparent that the truncation error can become considerable for large measurement residues.

This mode of instability can be eliminated by limiting the maximum and minumum residues used in the estimator at some multiple of their estimated standard deviation. Limiting the residues at plus or minus one to three estimated standard deviations was found to completely eliminate the occasional divergence problems encountered in the simulation tests. This limiter is further described in Chapter 8.

Another possible source of instability is the integration of the moment equations used between measurements, after a measurement update is completed. Since the third order density function model is not positive definite, it is possible that under certain (unusual) conditions, a negative variance estimate may result. Although this would cause a second order matrix Riccati equation to diverge, the third order autonomous filter equations are stable in such a situation, and will quickly restore the variance to a positive value. This low probability event has been observed during Monte Carlo simulation tests in which the average performance of the third order filter was substantially better than that of an optimal second order filter. The possible appearance of negative variances is a direct result of using a density function model which is not positive definite. It is quite remarkable that this assumed existence of negative probability causes almost no problems in a practical filter.

In general, the integration step required for the autonomous filter is dependent on the system, the initial conditions, and the particular noise sequence encountered. For a given step size, The filter may be stable for some noise sequences and unstable for others, making generalizations difficult. Nevertheless satisfactory performance can be obtained for most systems through simulation testing and parameter adjustment based on performance. Stability is a much more difficult problem for nonlinear filters than linear filters, because the equations are coupled, and because superposition does not apply.

6.5 Computational Requirements

The computation required per derivative evaluation is very similar for both the continuous measurement filter and the discrete measurement filter between measurements. In the discrete case, however, no white noise is added to the derivatives and integration step sizes twenty to fifty times larger can be used.

In addition, the discrete filter must perform an update computation each time a measurement is received. The total volume of this computation depends on the frequency of the measurements, and the update method used. Approximate computation requirements for the various methods of discrete measurement updating are given below. These are given in number of multiplications per update, and do not consider symmetry of the moments. Taking symmetry into account will substantially reduce the computation required when N, the system order, is large.

Third Order Filter - Approximate Number of Multiplications/Update

Series Method	$30\ N^2 + 14\ N^3 + 18\ N^4 + 28\ N^5$
Integration Update (5 steps)	$55\ N^2 + 30\ N^3 + 55\ N^4$
Hermite Polynomial	$30\ N^2 + 2\ N^3 + 2\ N^4 + 3\ N^5 + 3\ N^6$

Fourth Order Filter - Approximate Number of Multiplications/Update

Series Method	$30\ N^2 + 14\ N^3 + 18\ N^4 + 28\ N^5 + 46\ N^6$
Integration Update	Analyzed only to third order
Hermite Polynomial	Very complex

It is apparent that the integration updating technique is the least complex for systems of high order. For low order systems, the Hermite Polynomial method is not only the least complex, but also promises the best performance as it involves the least approximations.

It is also apparent that a practical vector filter must take the symmetry of the moments into account, not only in computing and storing the moments, but also in the calculation of the tensor contractions, particularly for the exact Hermite Polynomial filter.

6.6 Practical Filter Selection

The most important remaining question is whether to use a second, third, or fourth order nonlinear filter in a given situation. These are progressively more complex filters of progressively greater accuracy. In most cases, however, the complexity grows at an increasing rate while the accuracy grows at a decreasing rate. A point will then occur where the complexity of a higher order filter is not warranted by the improved accuracy. For most engineering applications, this will be for either a second or third order filter. It should be pointed out that the third order filter is <u>much</u> more complex than the second order filter. However, the second order filter cannot account for asymmetries, and for nonsymmetric systems where accuracy is at a premium the complexity of the third order nonlinear filter may be justified.

The relative performance and stability, and hence computational requirements, of these filters can be determined by Monte Carlo simulation. In consequence of their nonlinearity and complexity, more general methods appear prohibitively difficult.

Chapter 7

Scalar Simulation Tests

7.1 Introduction

A number of scalar simulation tests were performed in order to verify the continuous and discrete estimation equations, and to compare the performance of the various filters. A one dimensional system was chosen for these initial tests, because the relative simplicity of the scalar estimation equations allowed extensive Monte Carlo tests to be run at a reasonable expenditure of computer time.

7.2 Simulation Description

The following system and measurements were used to test the scalar nonlinear estimation equations.

$$\frac{dx(t)}{dt} = -2\,x(t) + .7\,x^2(t)$$

$$x(0) = N(2.0,\ .16) \qquad (7.1)$$

Continuous Measurement

$$y(t) = x(t) + \zeta(t)$$

$$\mathcal{E}(\zeta(t)) = 0$$

$$\mathcal{E}(\zeta(t)\zeta(\tau)) = \Gamma\,\delta(t-\tau)$$

$$\zeta(t) = N(0.0,\ \Gamma/\Delta t) \quad \text{(digital simulation)} \qquad (7.2)$$

Discrete Measurement

$$y(n\Delta t_m) = x(n\Delta t_m) + \zeta(n\Delta t_m)$$

$$\mathcal{E}(\xi(n\Delta t_m)) = 0$$

$$\mathcal{E}(\xi(n\Delta t_m)\zeta(l\Delta t_m)) = \Sigma\,\delta_{nl}$$

$$\zeta(n\Delta t_m) = N(0.0, \Sigma) \qquad (7.3)$$

The system (7.1) and filters were simulated on an IBM 360-75 digital computer using a program written in the MAC compiler language. The Monte carlo tests were conducted by simulating the system and each filter from 20 to 80 times from t = 0 to t = 2.0 seconds. The filters were started with m = 2.0, $\mu 2$ = .16, $\mu 3$ = 0.0, and $\mu 4 = 3\,(.16)^2$, while the system was started with normally distributed random initial conditions as specified in (7.1).

The normally distributed random measurement and initial condition noises were generated using the MAC compiler's pseudo random number generator. This operates by taking the sum of twelve rectangularly distributed random numbers, which is approximately normal by the central limit theorem.

In order to simulate the continuous measurement (7.2), it is necessary to approximate a zero mean "white" noise $\zeta(t)$ of power spectral density Γ by a sequence of pulses of width Δt and variance σ^2. Such a pulse train would have a triangular autocorrelation function of height σ^2 and width $2\Delta t$. The area of the autocorrelation function is then $\sigma^2 \Delta t$. This would appear to closely approximate a white noise whose autocorrelation function is an impulse of weight $\sigma^2 \Delta t$, if Δt is sufficiently small. This conclusion can also be implied for more general situations by considering the limit of a sequence of Markov processes. (See Skorokhod (B-15), Chapter 6)

An approximation to a zero mean white noise of power spectral density Γ is then a pulse train with width Δt and variance $\Gamma/\Delta t$. This method was used to generate the "white" noise for the continuous measurements.

In the course of the simulation tests, it was discovered that the third and fourth order filters are somewhat sensitive to initial condition biases, especially during the first second or so of operation. That is, if the initial condition bias was the same sign as the computed third moment, the third order filter would perform much better than the second order filter, and if the bias were opposite in sign to the computed third moment, the third order filter might perform worse. With a small number of Monte Carlo runs, the random bias of the small sample of random initial conditions made the results difficult to interpret. For this reason, subsequent tests were made by generating the random initial conditions at the beginning of the simulation, and then removing the bias and correcting the sample variance to the desired value. This procedure would be unnecessary if a large enough number of Monte Carlo runs were made, because in that case the mean and variance error of the sample could be made adequately small.

Continuous measurement filters of second, third, and fourth order were simulated. The scalar filter equations are;

$$F = -2 + 1.4\,m$$

$$A = .7$$

$$H = 1.0$$

$\Gamma = 0.16$ = measurement noise covariance

$$z = y - m$$

Second Order Continuous Optimal Nonlinear Filter

$$\dot{m} = -2\,m + .7\,m^2 + A\,\mu 2 + \frac{H\,\mu 2}{\Gamma}\,z$$

$$\dot{\mu}2 = 2\,F\,\mu 2 - \frac{H^2\,\mu 2^2}{\Gamma} \tag{7.4}$$

Third Order Continuous Optimal Nonlinear Filter

$$\dot{m} = -2\,m + .7\,m^2 + A\,\mu 2 + \frac{H\,\mu 2}{\Gamma}\,z$$

$$\dot{\mu}2 = 2(F\,\mu 2 + A\,\mu 3) - \frac{H^2\,\mu 2^2}{\Gamma} + \frac{\mu 3\,H}{\Gamma}\,z$$

$$\dot{\mu}3 = 3(F\,\mu 3 + 2\,A\,\mu 2^2 - \frac{H^2\,\mu 2\,\mu 3}{\Gamma}) \tag{7.5}$$

Fourth Order Continuous Optimal Nonlinear Filter

$$\dot{m} = -2\,m + .7\,m^2 + A\,\mu 2 + \frac{H\,\mu 2}{\Gamma}\,z$$

$$\dot{\mu}2 = 2(F\,\mu 2 + A\,\mu 3) - \frac{H^2\,\mu 2^2}{\Gamma} + \frac{\mu 3\,H}{\Gamma}\,z$$

$$\dot{\mu}3 = 3(F\,\mu 3 + A(\mu 4 - \mu 2^2) - \frac{H^2\mu 2\mu 3}{\Gamma}) + \frac{\mu 4\,H}{\Gamma}\,z$$

$$\dot{\mu}4 = 4(F\,\mu 4 + 9\,A\,\mu 2\,\mu 3 - \frac{H^2\,\mu 2\,\mu 4}{\Gamma}) \tag{7.6}$$

A rectangular integration rule was used, with a number of different step sizes. A Runge-Kutta integration algorithm was tried, but this proved unstable even when a single noise value was used over each complete integration cycle. Apparently the assumptions regarding continuity and differentiability implicit in using a higher order integration algorithm are not valid in this case.

Series expansion filters of second, third, and fourth order were simulated. The measurements were assumed to be discrete, and the moments were propagated between measurements using equations (7.4) - (7.6) with $\Gamma = \infty$. Since no noise drives these equations, a Runge-Kutta integration could be used with a much larger step size than in the continuous measurement case. The scalar updating equations (F.42) - (F.47) were used at the measurement times.

Integration update filters of second and third order were simulated. The autonomous filter equations used between measurements were the same as for the series filter, and at the measurement times equations (G.21) were used to update the filter estimates.

Next, second and third order discrete measurement filters were simulated using the optimal Hermite polynomial solution. The same autonomous filters were used as in the previous simulations, and the exact updating equations of Table 5.1 were used at the measurement times. The second order filter is equivalent to a linearized Kalman filter with one additional term, as in the series and integration update solutions. In order to properly verify the third order theory, two approximate "suboptimal" filters, as described in section (5.5), were simulated. The updating equations for the three third order filters simulated are as follows;

First Suboptimal Third Order Filter Equations

$$m^+ = m + d$$
$$\mu 2^+ = \Omega$$
$$\mu 3^+ = N \qquad (7.7)$$

Second Suboptimal Third Order Filter Equations

$$m^+ = m + d + \phi$$
$$\mu 2^+ = \Omega$$
$$\mu 3^+ = N \qquad (7.8)$$

Optimal Third Order Filter Equations

$$m^+ = m + d + \phi$$
$$\mu 2^+ = \Omega + \psi - \phi^2$$
$$\mu 3^+ = N - 3\,\phi\psi + 2\,\phi^3 \qquad (7.9)$$

Finally, nonlinear discrete measurements were used. The measurement function tested was:

$$y(n\Delta t) = 3.0\, x(n\Delta t) - 0.5\, x^2(n\Delta t) + \xi(n\Delta t)$$
$$\xi(n\Delta t) = N(0.0,\ 1.6) \qquad (7.10)$$

The two step procedure described by equations (5.86) - (5.89) in Section 5.6 was used in conjunction with optimal second and third order Hermite polynomial filters. For comparison, similar results were obtained by direct numerical integration of the Bayes equation for the nonlinear measurement.

7.3 Results

For each simulation test, the RMS estimation error, mean estimation error, average estimated second and third moments, and true second and third moments for the Monte Carlo sequence were plotted against time. The linearized Kalman filter and optimal second and third order filter results are plotted together for each test. These can be directly compared because the same pseudo-random initial condition error and measurement error sequence was used for each test.

For a Monte Carlo test of n trials, the RMS estimation error is an estimate of the standard deviation of the filter error. The standard deviation of this estimate (the standard deviation of an ensemble of n run determinations of the RMS error) can be obtained from the following.

V = estimate of variance

S = estimate of standard deviation

$$\sigma_V^2 = \frac{1}{n}\left[\mu_{4x} - \frac{n-3}{n-1}\sigma_x^4\right] \qquad n > 1 \tag{7.11}$$

$$S = \sqrt{V} + \frac{1}{8}\left(\frac{\sigma_V^2}{\sigma_x^4}\right)\sigma_x + \cdots \tag{7.12}$$

$$\sigma_S = \frac{1}{2}\frac{\sigma_V}{\sigma_x^2}\sqrt{1 - \frac{1}{16}\left(\frac{\sigma_V}{\sigma_x^2}\right)^2}\;\sigma_x \tag{7.13}$$

In the case of sampling a nearly normal distribution,

$$\sigma_S \approx \frac{1}{\sqrt{2(n-1)}}\sqrt{1 - \frac{1}{8(n-1)}}\;\sigma_x \tag{7.14}$$

This yields the following results.

$$\begin{aligned} n &= 10 \qquad \sigma_S = .235\,\sigma_x \\ n &= 20 \qquad \sigma_S = .165\,\sigma_x \\ n &= 80 \qquad \sigma_S = .079\,\sigma_x \end{aligned} \tag{7.15}$$

This standard deviation of the Monte Carlo test is shown on each plot of the RMS filter error, and represents a lower bound on the significance of the test results. The results are actually more significant since the RMS error histories of the filters are generated using identical noise histories, a procedure which will detect small differences in filters. The statistical significance of these differences is hard to determine, however, due to the nonlinearity of the filter equations and the variations in particular noise sequences.

Figures 7.1 through 7.6 show the computer plotted results of a linearized Kalman filter, an optimal second order filter, and an optimal third order filter operating on a nonlinear system with no measurements. This test of the autonomous filter equations derived in Chapter 4 shows the optimal second and third order filters to have similar state estimation errors. The third order filter has substantially better estimates of the second and third moments, a property that promises to be of great value when measurements are taken.

Figures 7.7 - 7.9 show the results of the continuous measurement filter tests. Good results were obtained from the third order filter if the integration step was taken small enough, but accuracy deteriorated rapidly with increasing step size. It is interesting to note that even with very small step sizes, the third order filter outperformed its fourth order counterpart. The source of this difficulty may have been properly diagnosed by Stratonovich, (B-17) who observed that, while the Hermite polynomial expansion always eventually converges, it often converges slowly. There is no assurance that the addition of another term will reduce the error; it may even cause it to increase, particularly with extreme nonlinearities. It is clear that, for this example, an integration step size of .001 seconds or less must be used. An attempt to use Runge-Kutta integration for the filter caused divergent oscillations for step sizes of .01 or less, even if the same noise value was used for one complete integration step. The random noise present in the filter equations apparently caused the fourth order integration algorithm to become unstable in some complex fashion.

This illustrates the fundamental limitation of the continuous nonlinear filter; its marginal stability necessitates use of extremely small integration steps, which make the filter computationally burdensome.

If the discrete measurement formulation is used, then the autonomous filter equations used between measurements are quite stable, and can be integrated using a Runge-Kutta algorithm with a step size at least 20 times as large as for the continuous filter. Figures 7.10 - 7.12 show the results of such a filter test, using the approximate series solution derived in Appendix F at the measurement times. These simulation tests show that this filter is unstable for short measurement times because the Runge-Kutta algorithm used for the autonomous filter reacts to the noisy measurement corrections, and is unstable for large measurement times and accurate measurements because the series solution converges poorly. Nevertheless, it enables the use of a measurement processing interval twenty times as large as the integration interval required to integrate the continuous filter equations.

The approximate integration update method derived in Appendix G does not have this convergence problem, and shows that consideration of the third moments can lead to much improved estimation procedures for nonlinear systems. Figures 7.13 - 7.15 show simulation results for this method.

Next, Figures 7.16 - 7.25 show results from the Hermite polynomial estimation formulas derived in Section 5.4. The results clearly show the first suboptimal third order filter to be superior to the second order filter, the second suboptimal filter to be superior to the first, and the optimal third order filter to surpass all the others. These results are as predicted by the theory.

The question naturally arises of how the third order filter compares with a globally optimal infinite order nonlinear filter. This is difficult to answer directly, because it is impossible to demonstrate an infinite order filter. However it is known that a globally optimal filter will also have an optimal estimate for the higher moments. Figures 7.21 - 7.23 show that for the example given, the third order filter is tremendously more accurate in estimating the second and third moments than the second order filter. In fact, these estimates are so close that it appears unlikely that a higher order filter could obtain any significant gain in performance.

Finally, Figures 7.26 through 7.30 show results from the nonlinear measurement theory of Section 5.6 . The two step filters of equations (5.86) - (5.89) are compared with corresponding linearized measurement filters in Figure 7.26, for a linear system with nonlinear measurements. These results show a close correspondence with results obtained by direct numerical integration of the Bayes equation, given in Figure 7.27. The RMS error of the two step filter for a nonlinear system with nonlinear measurements is given in Figure 7.28. Figures 7.29 and 7.30 show the actual and estimated third moments for this test. These results show that the two step procedure of Section 5.6 is a valid, though complex estimation method. The Monte Carlo tests for nonlinear measurement filtering showed a larger variation than those for linear measurements, due to greater dependence on the higher moments of the measurement noise sequence.

These scalar simulation tests substantially verify the theory of chapters 4 and 5, and further show that consideration of third moments can lead to substantially improved estimation procedures for nonlinear systems.

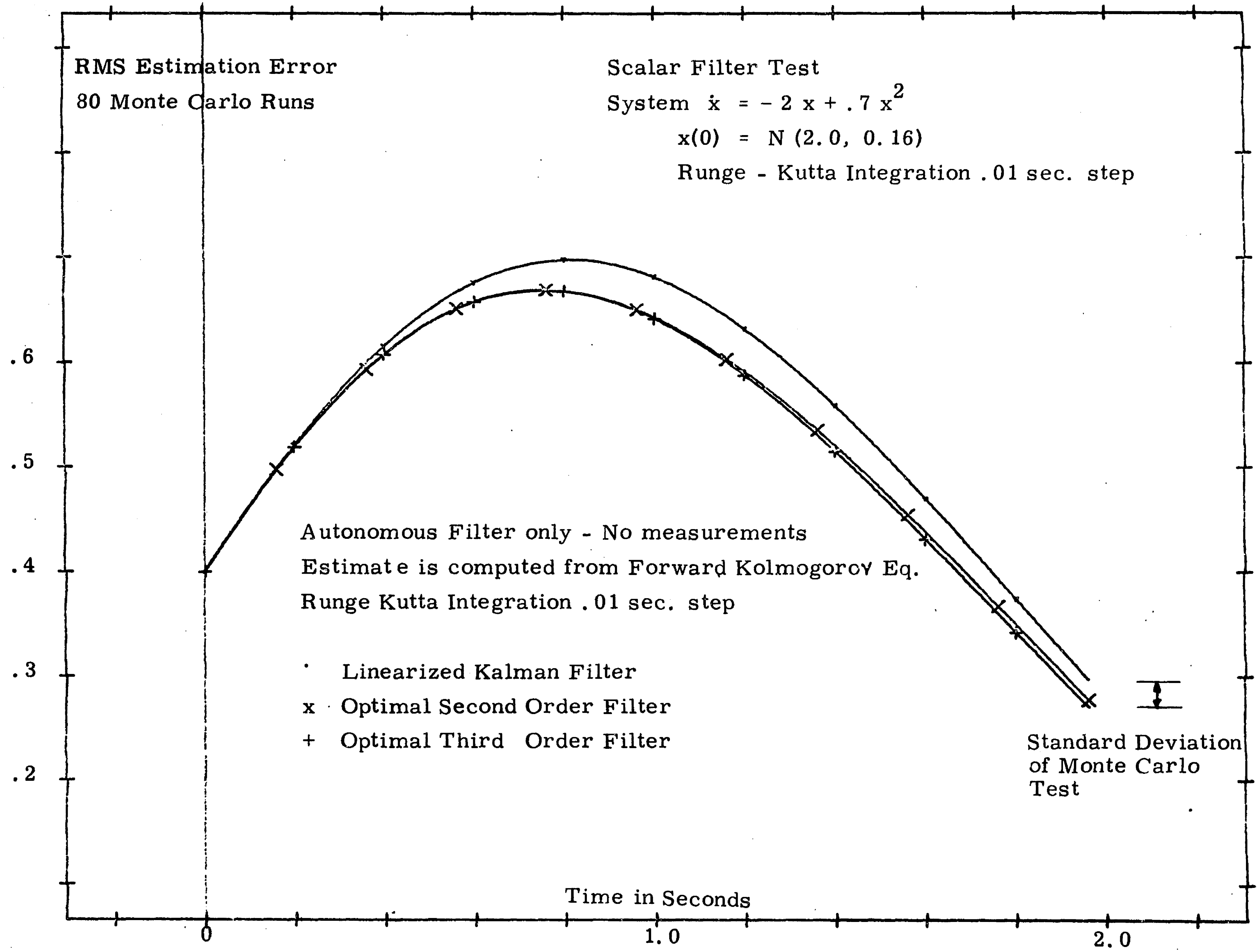

Figure 7. 1 Continuous Filter Test with No Measurements - RMS Estimation Error

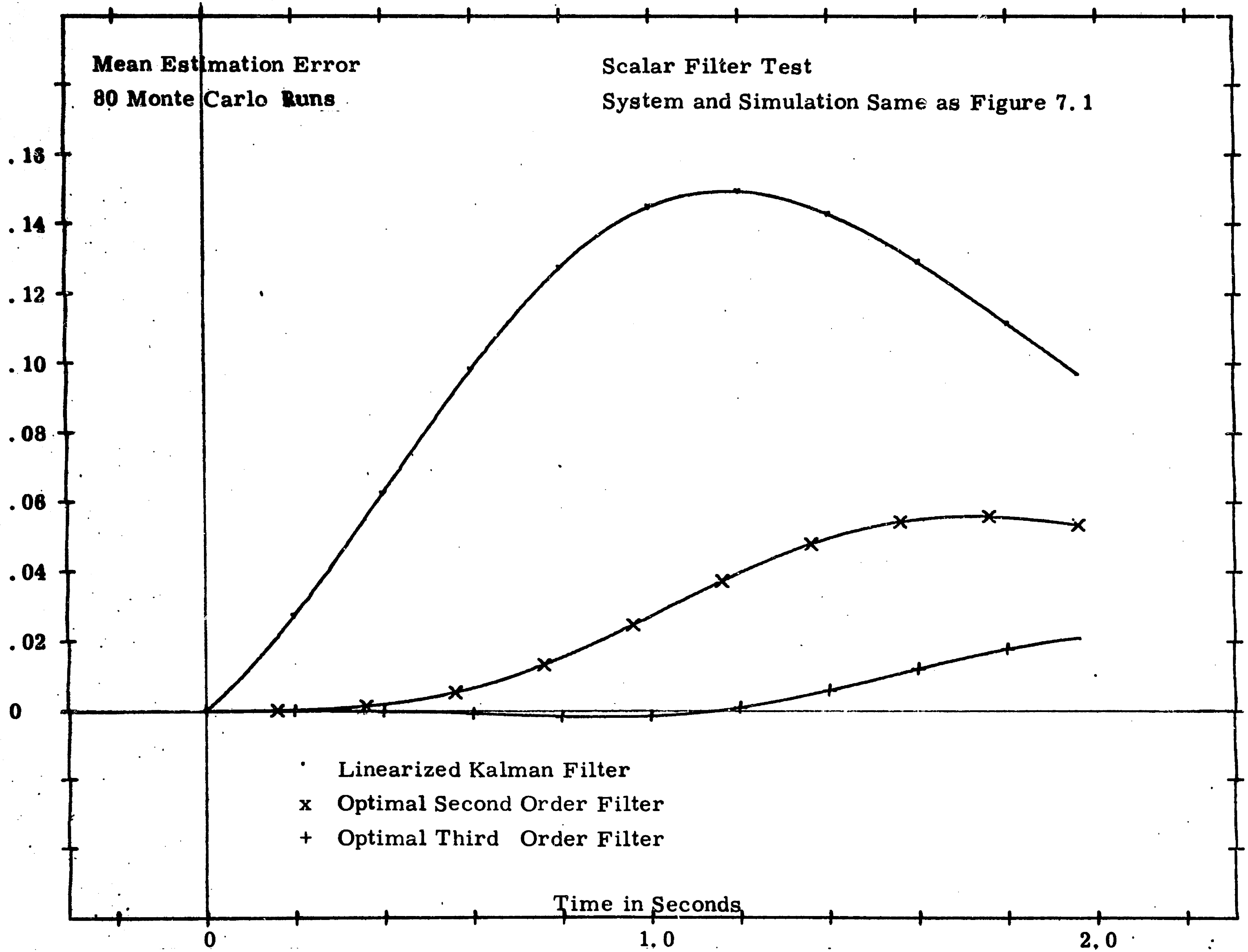

Figure 7.2 Continuous Filter Test with No Measurements - Mean Estimation Error

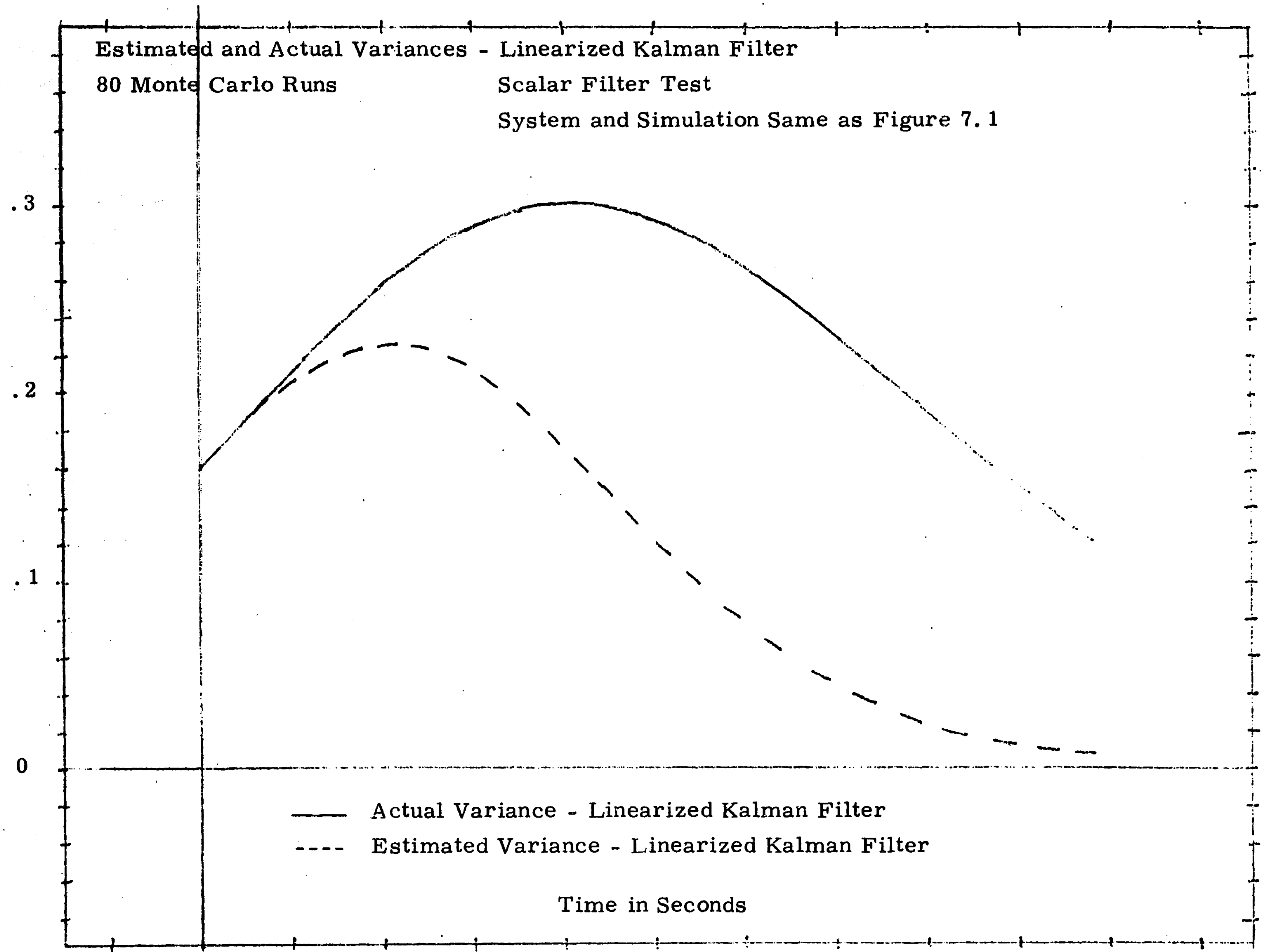

Figure 7.3 Continuous Filter Test with No Measurements
Estimated and Actual Variance - Linearized Kalman Filter

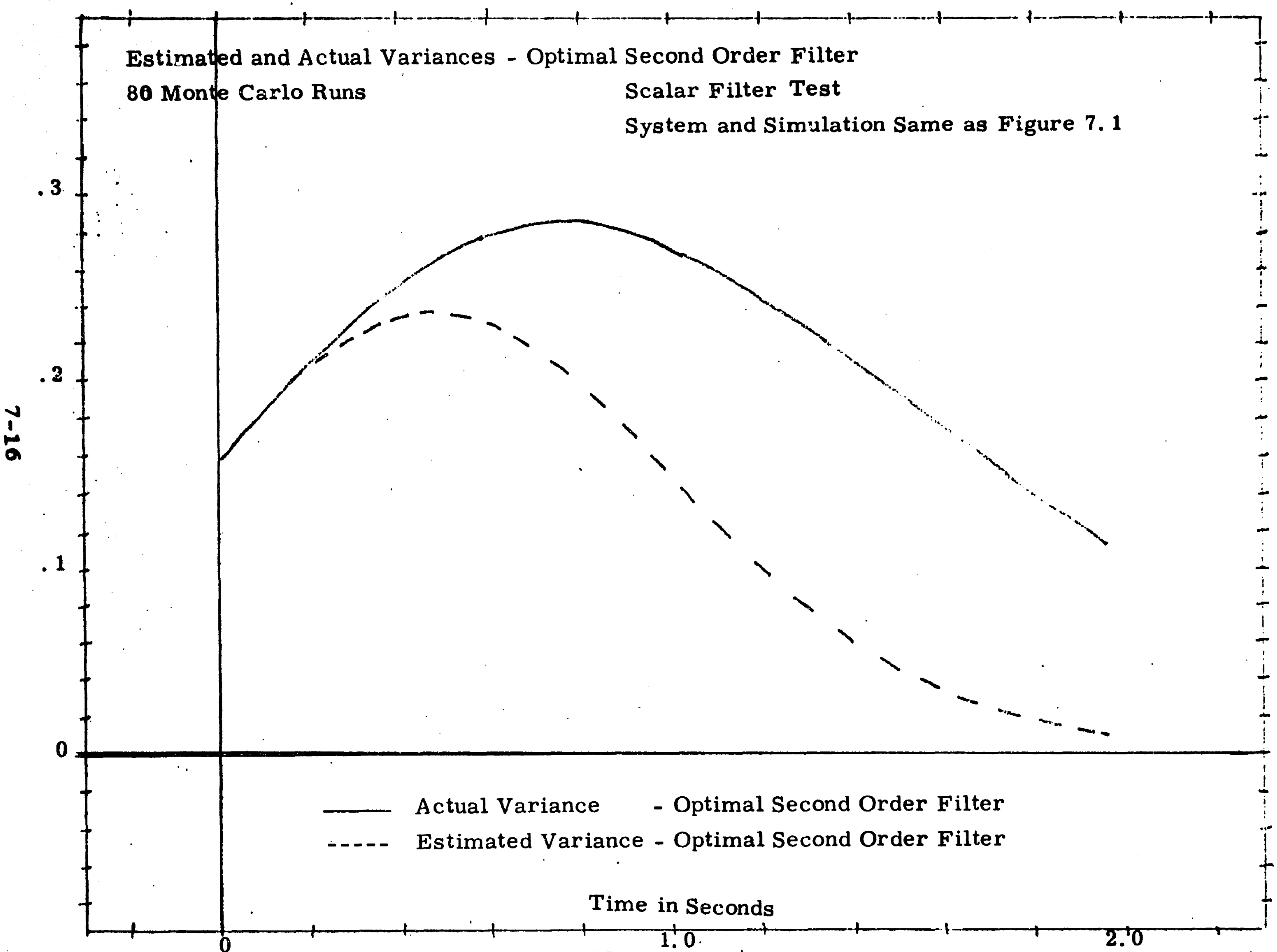

Figure 7.4 Continuous Filter Test with No Measurements
Estimated and Actual Variances - Optimal Second Order Filter

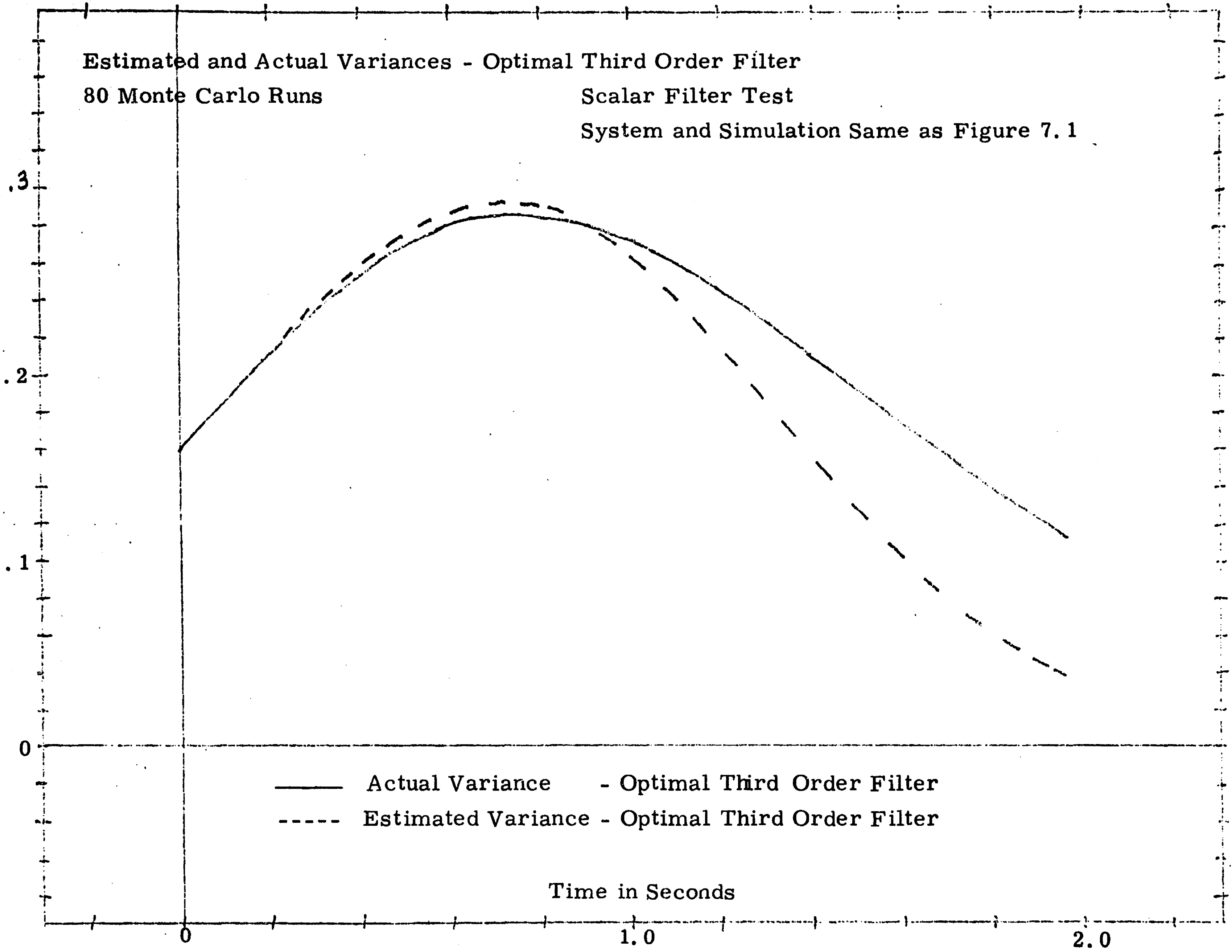

Figure 7.5 Continuous Filter Test with No Measurements
Estimated and Actual Variances - Optimal Third Order Filter

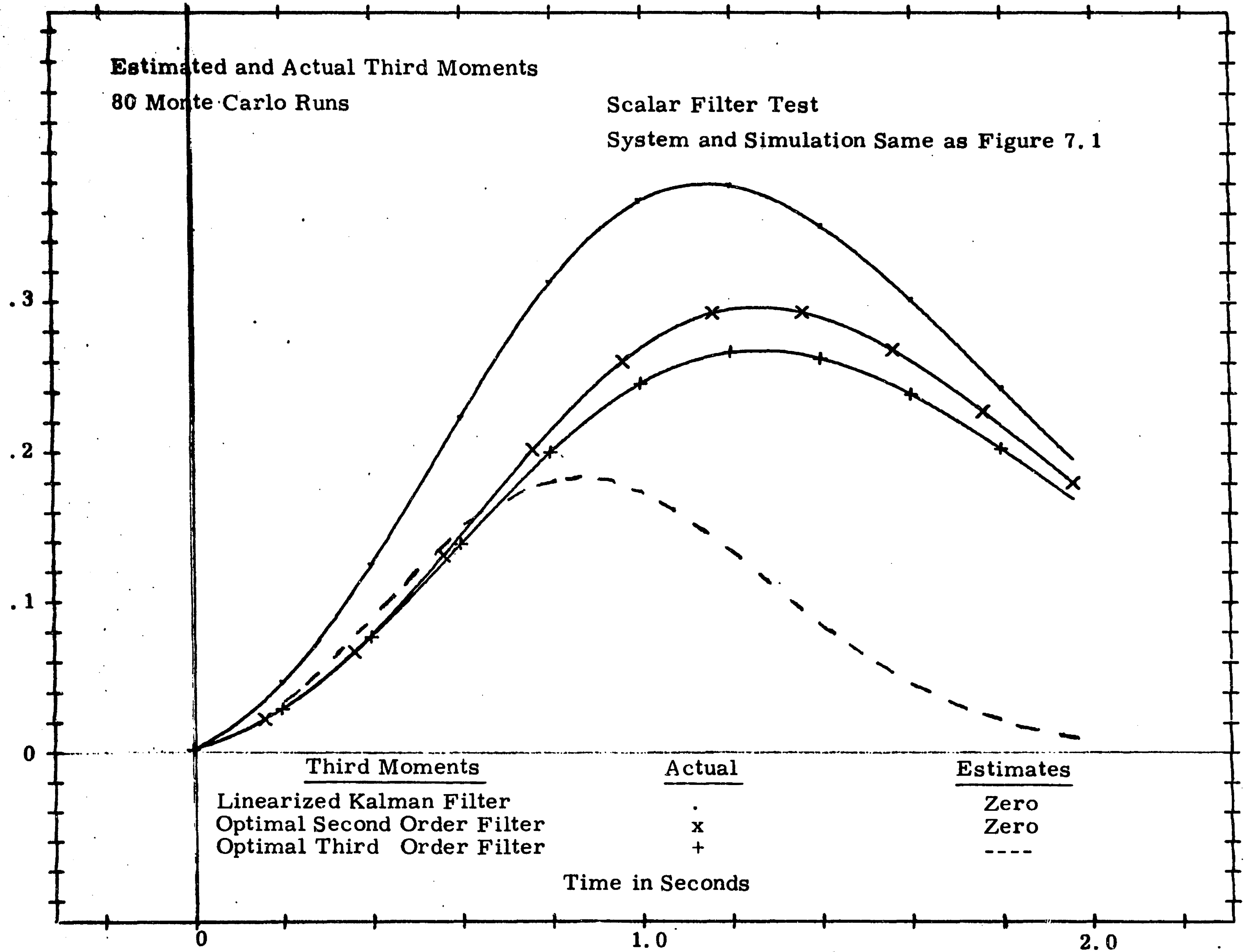

Figure 7.6 Continuous Filter Test with no Measurements
Estimated and Actual Third Moments

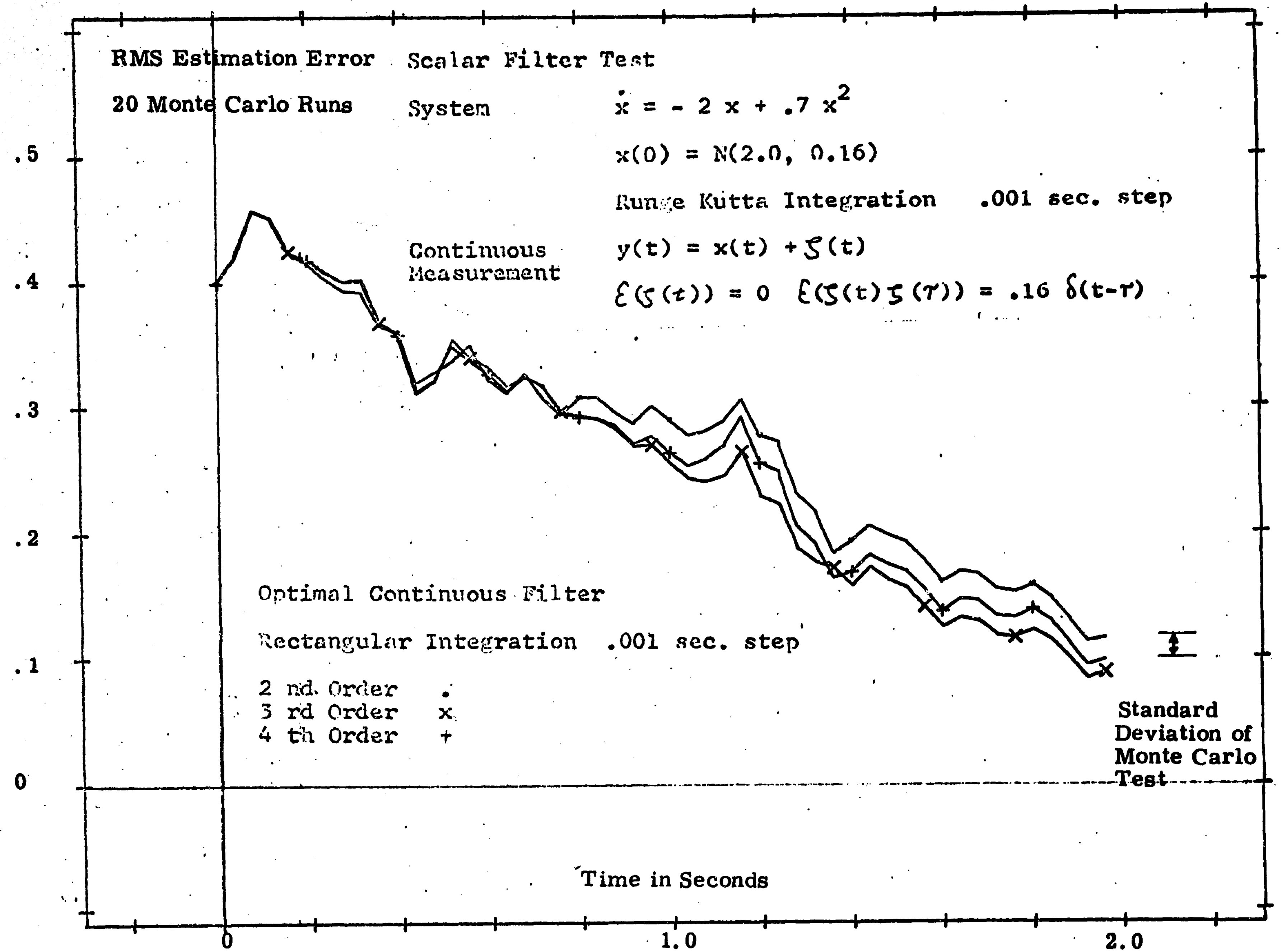

Figure 7.7 Optimal Continuous Filter Test - Δt = .001

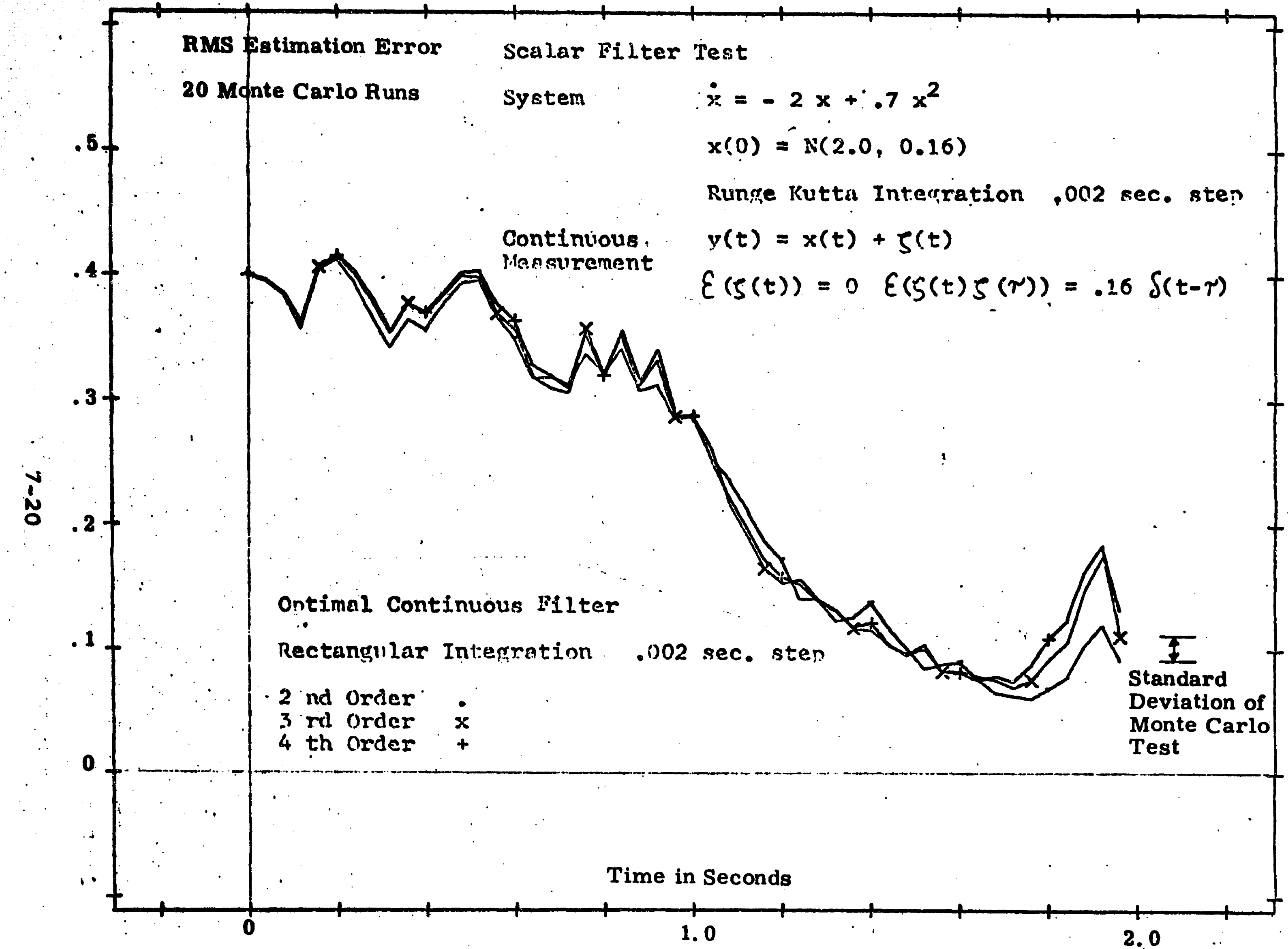

Figure 7.8 Optimal Continuous Filter Test - $\Delta t = .002$

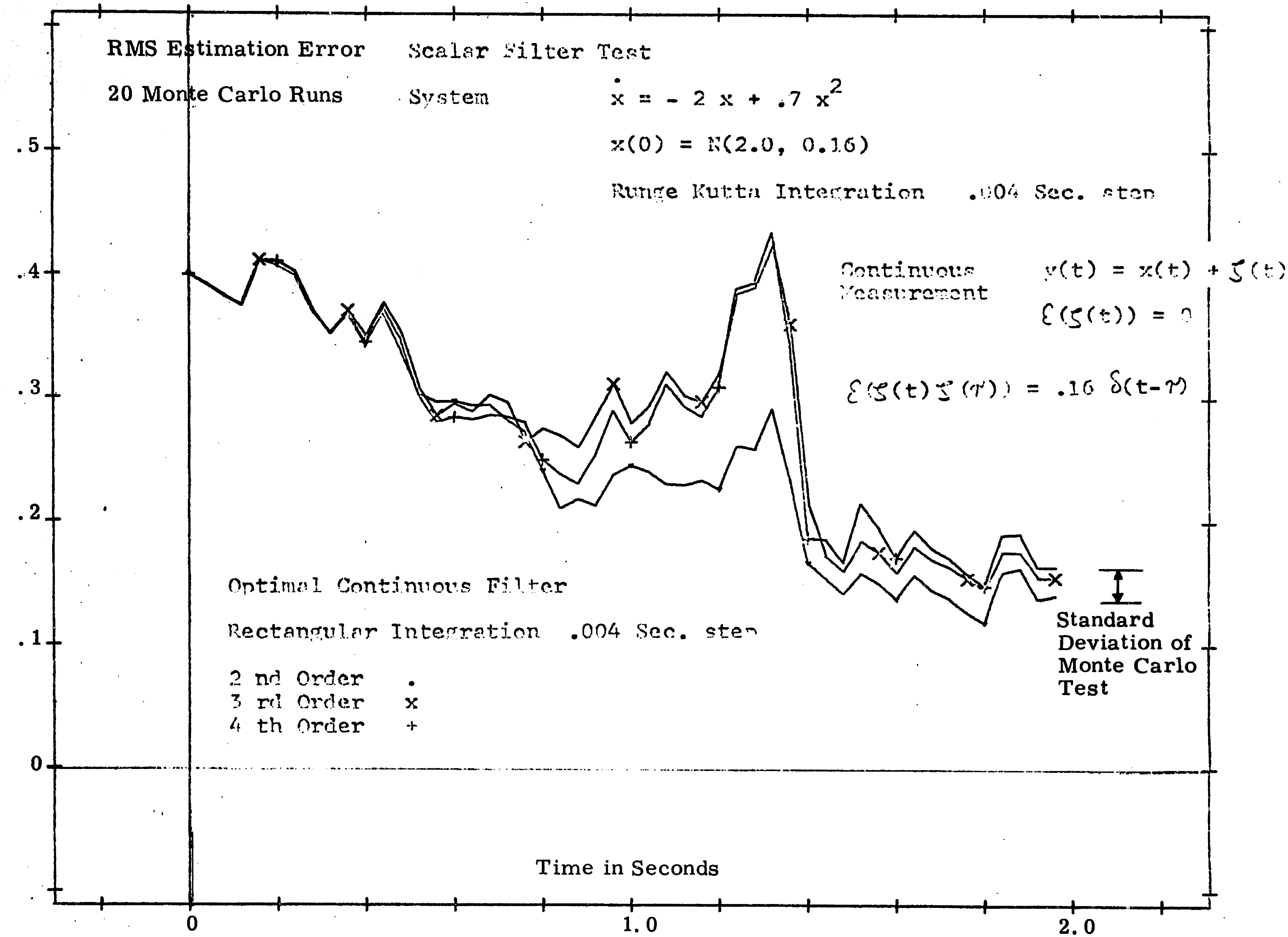

Figure 7.9 Optimal Continuous Filter Test - Δt = .004

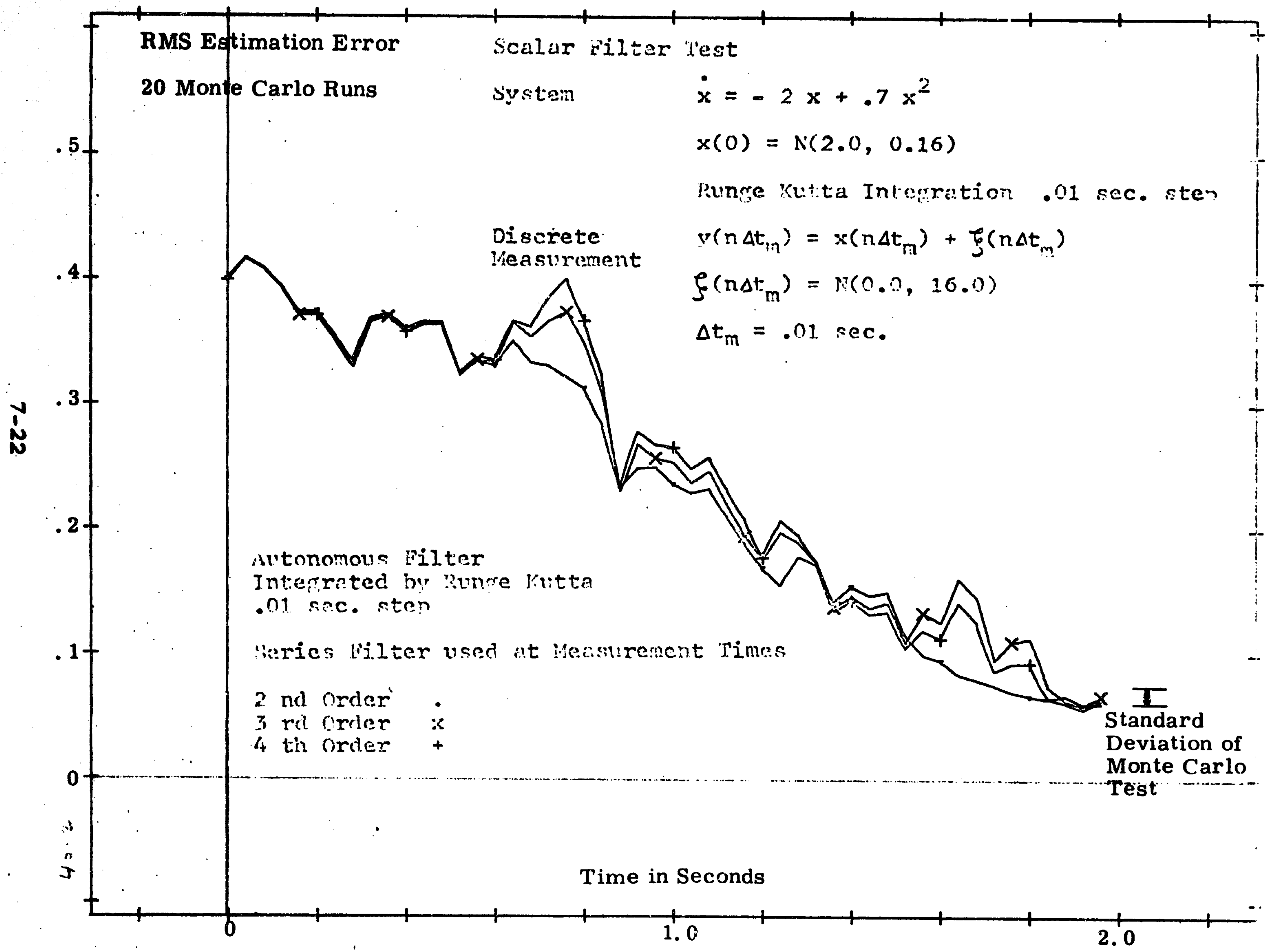

Figure 7. 10 Series Filter Test - .01 Sec Measurement Interval

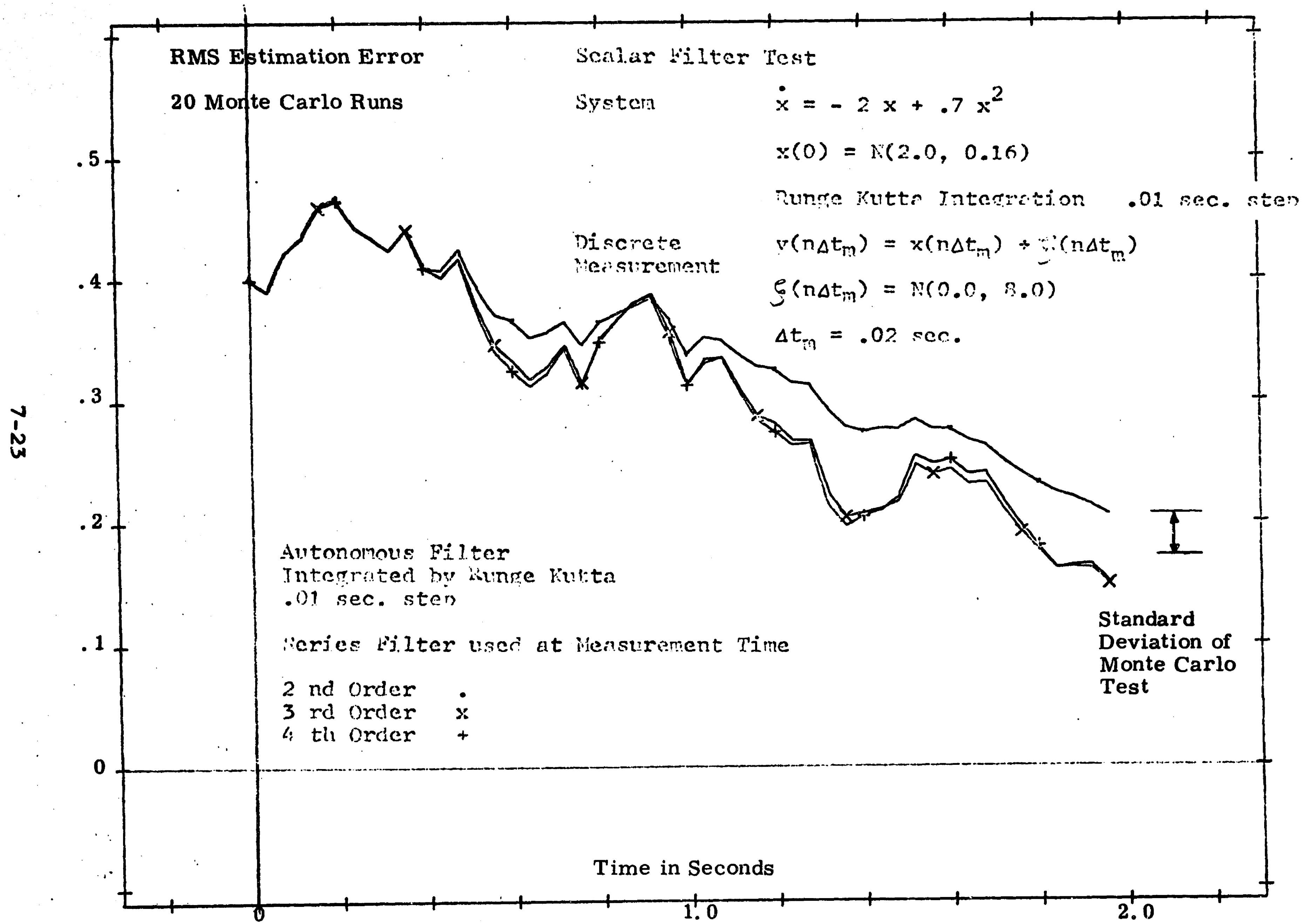

Figure 7.11 Series Filter Test - .02 Sec. Measurement Interval

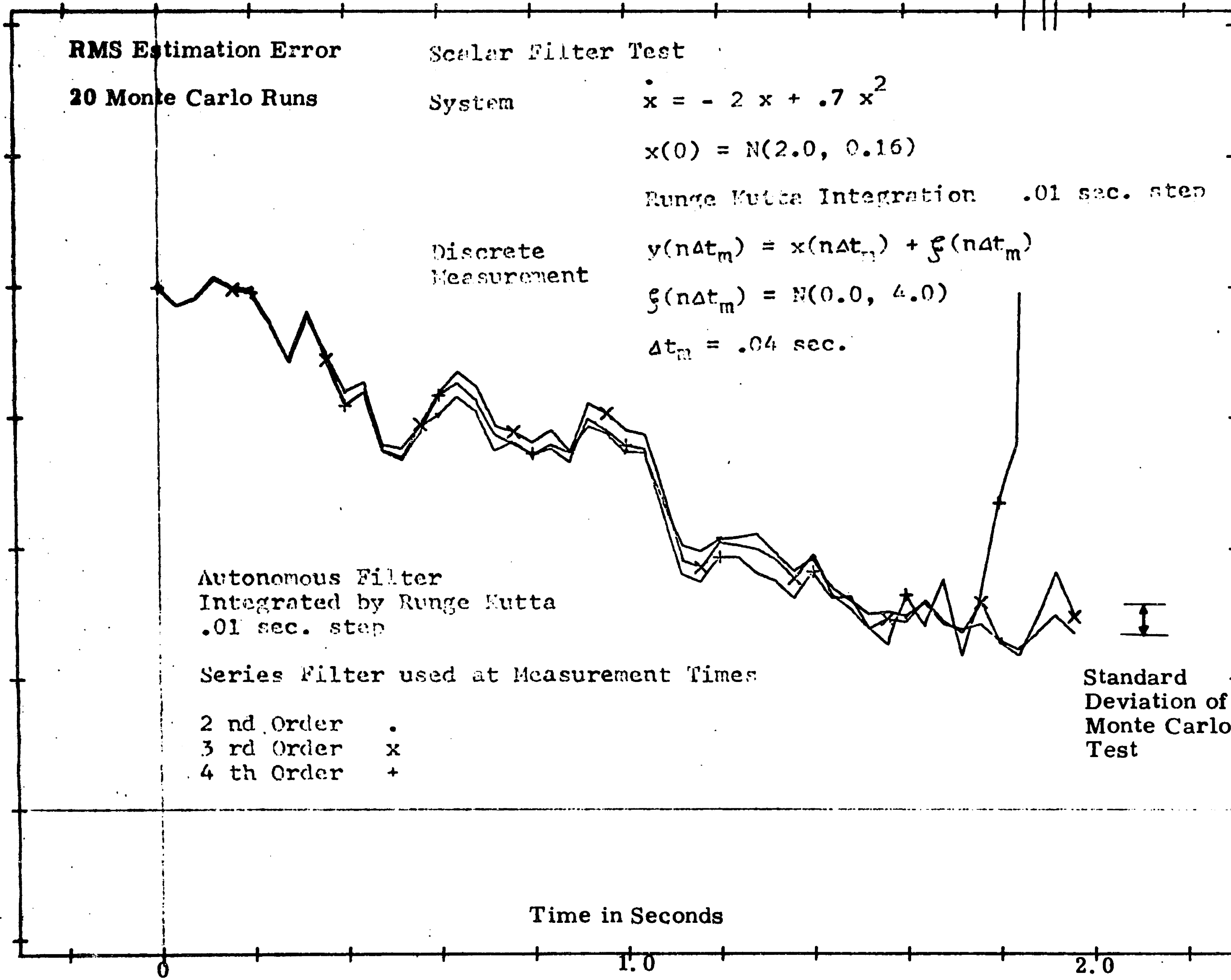

Figure 7. 12 Series Filter Test - .04 Sec. Measurement Interval

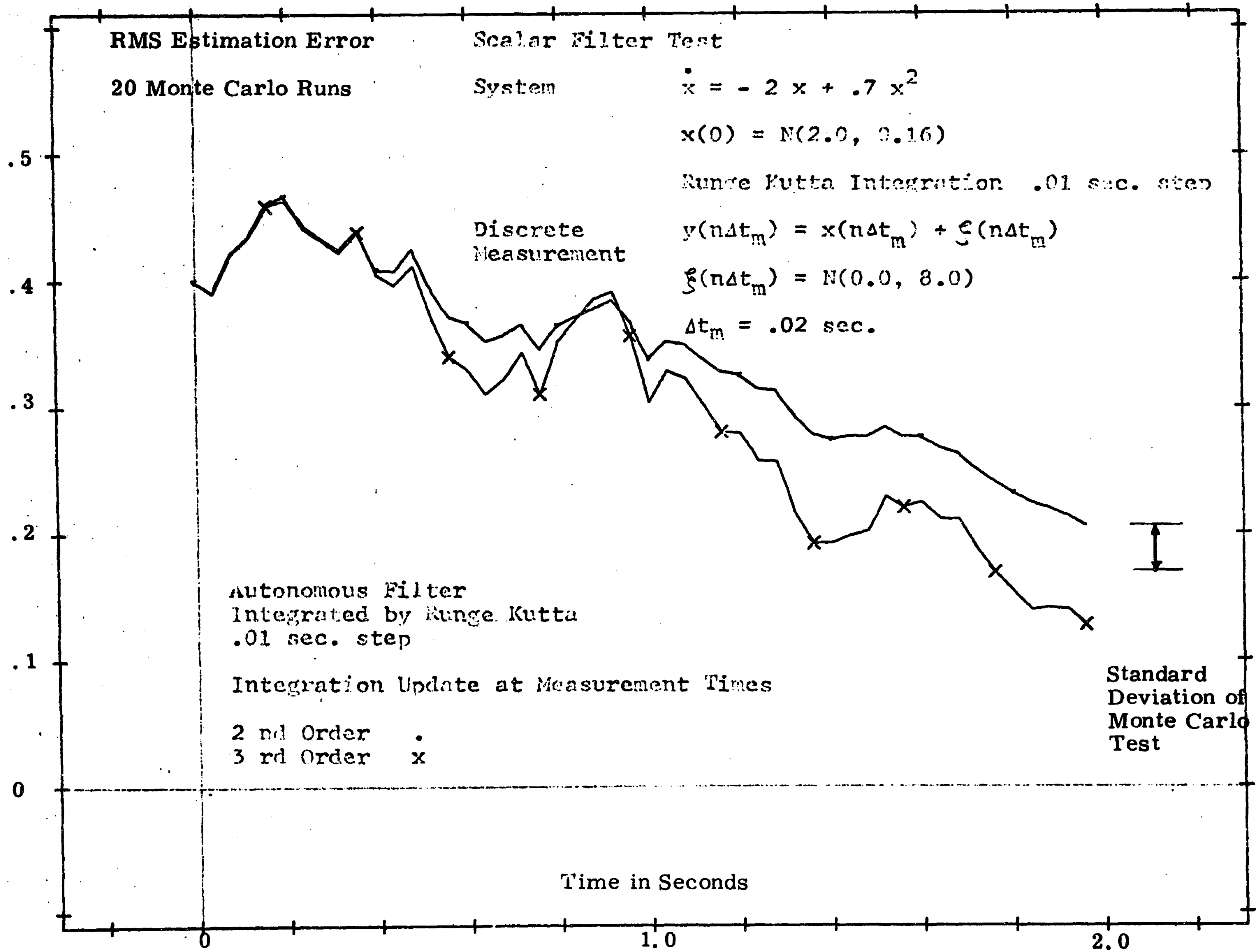

Figure 7.13 Integration Update Filter Test - .02 Sec. Measurement Interval

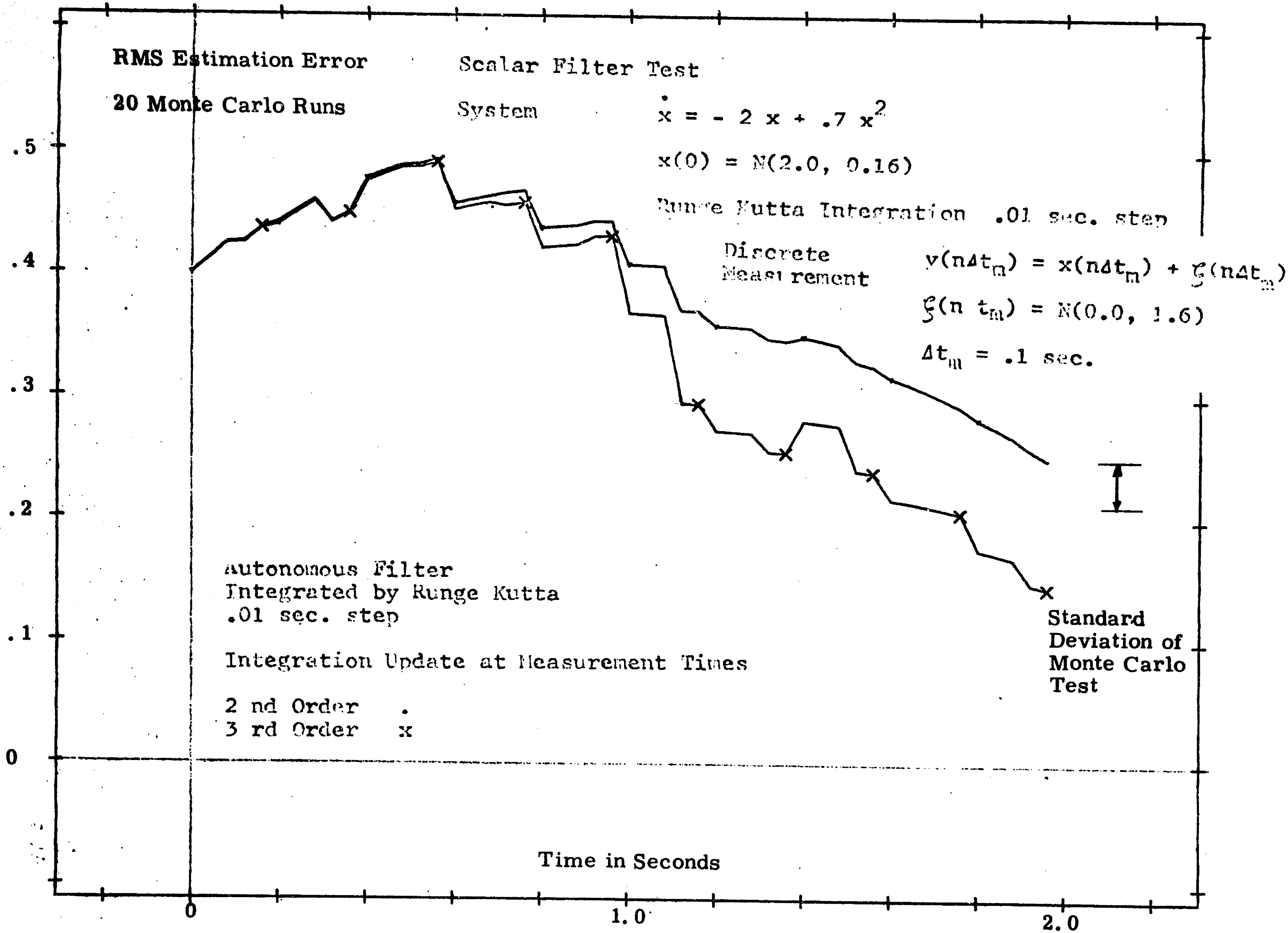

Figure 7.14 Integration Update Filter Test - .1 Sec. Measurement Interval

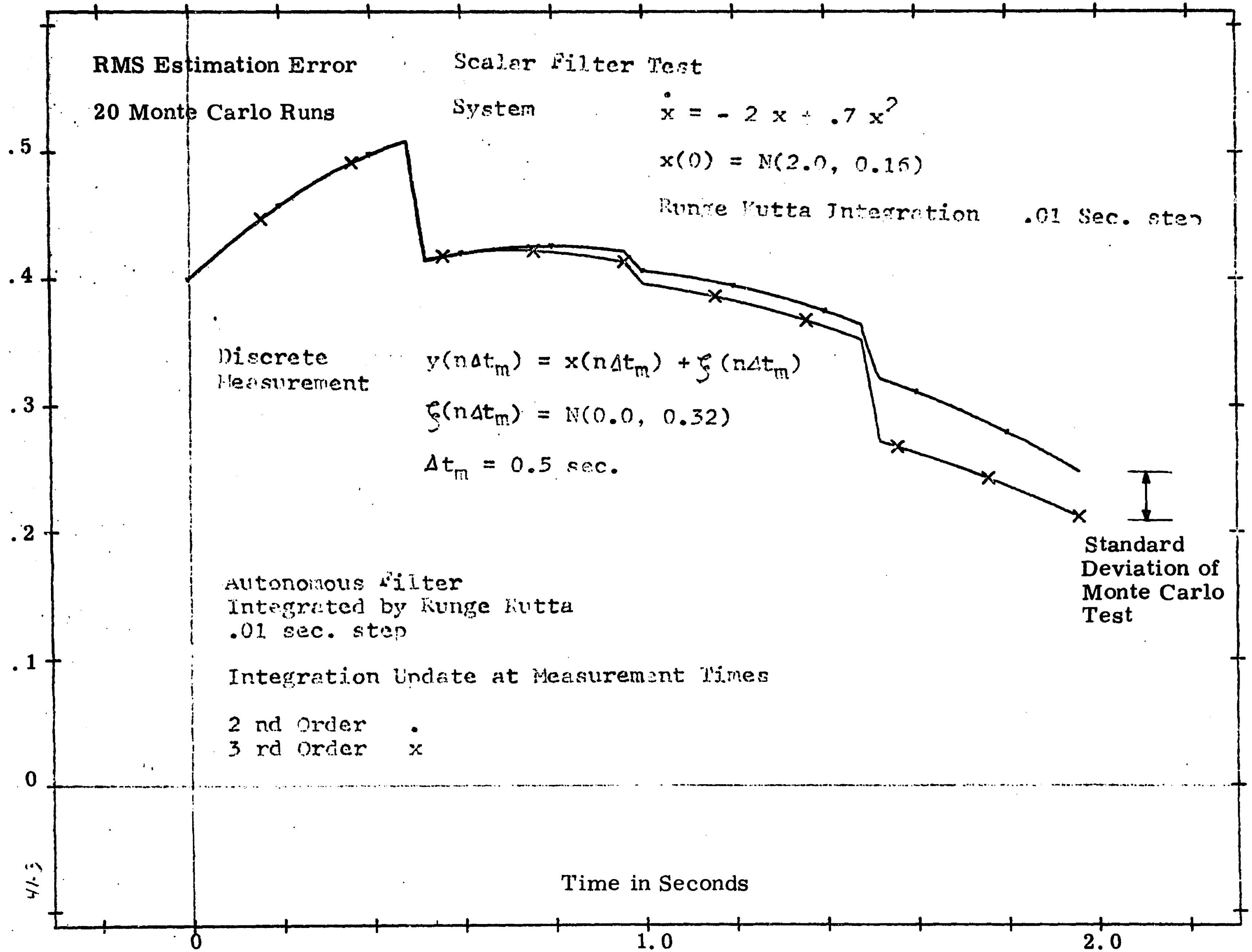

Figure 7.15 Integration Update Filter Test - .5 Sec. Measurement Interval

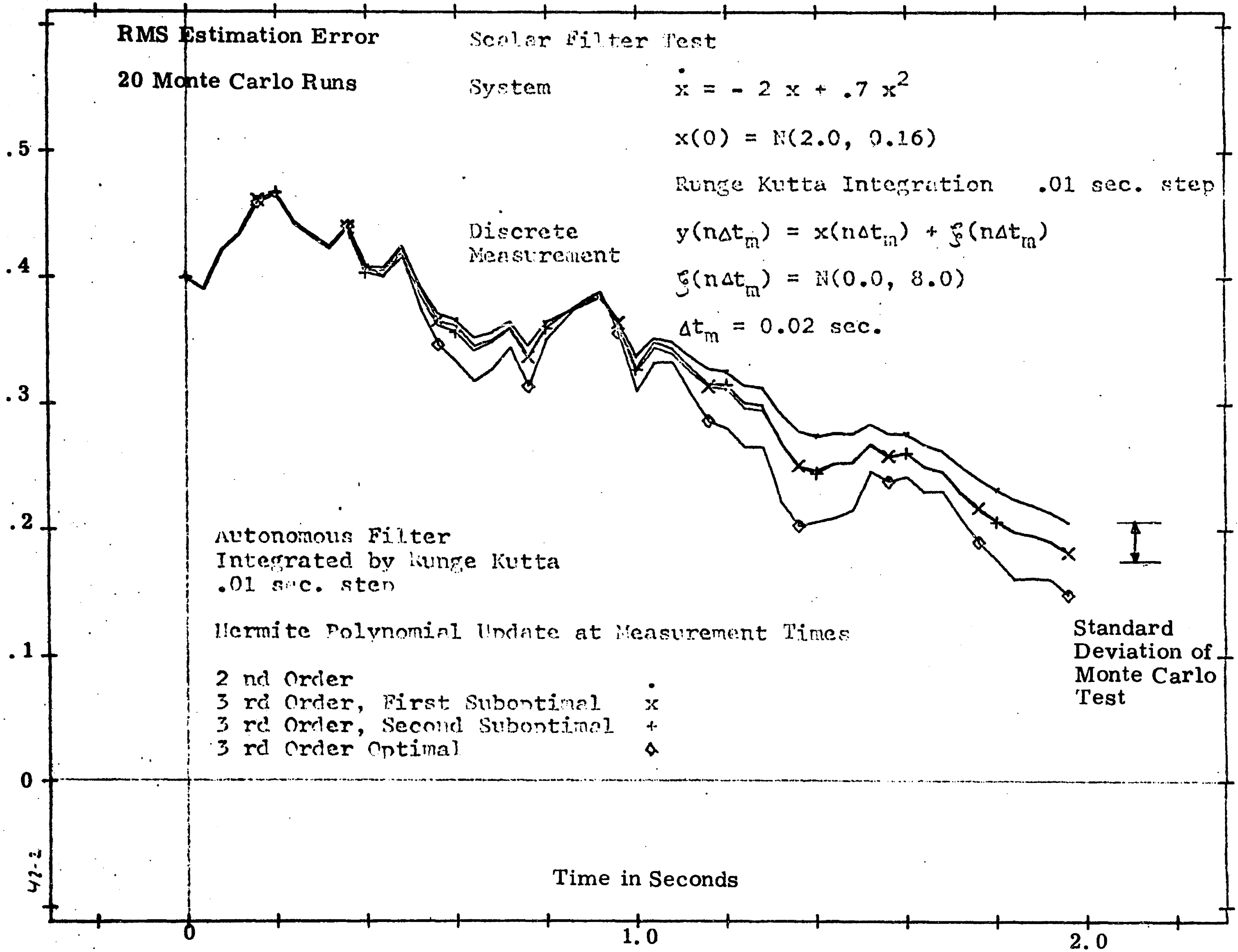

Figure 7.16 Hermite Polynomial Filter Test - .02 Sec. Measurement Interval

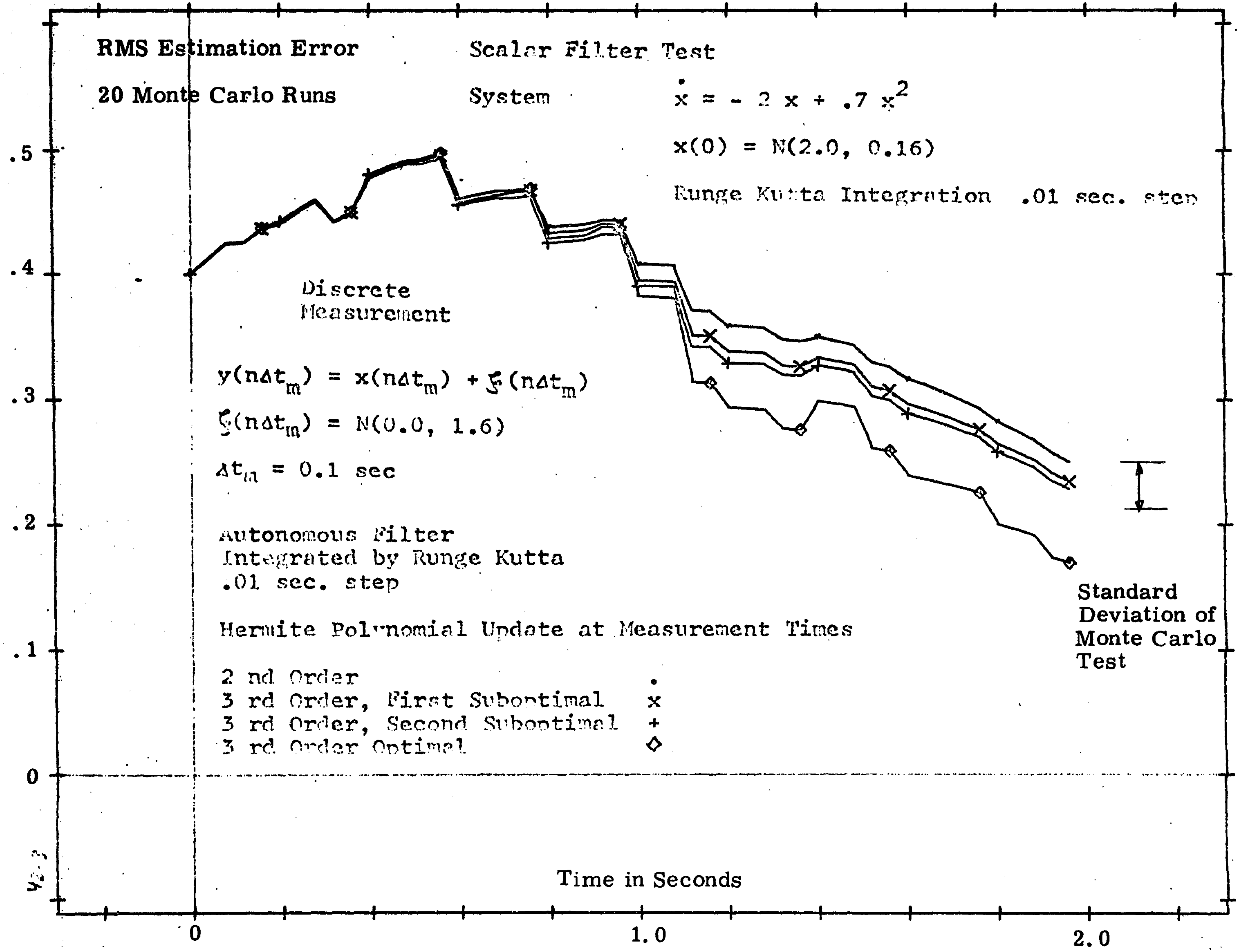

Figure 7.17 Hermite Polynomial Filter Test - .1 Sec. Measurement Interval

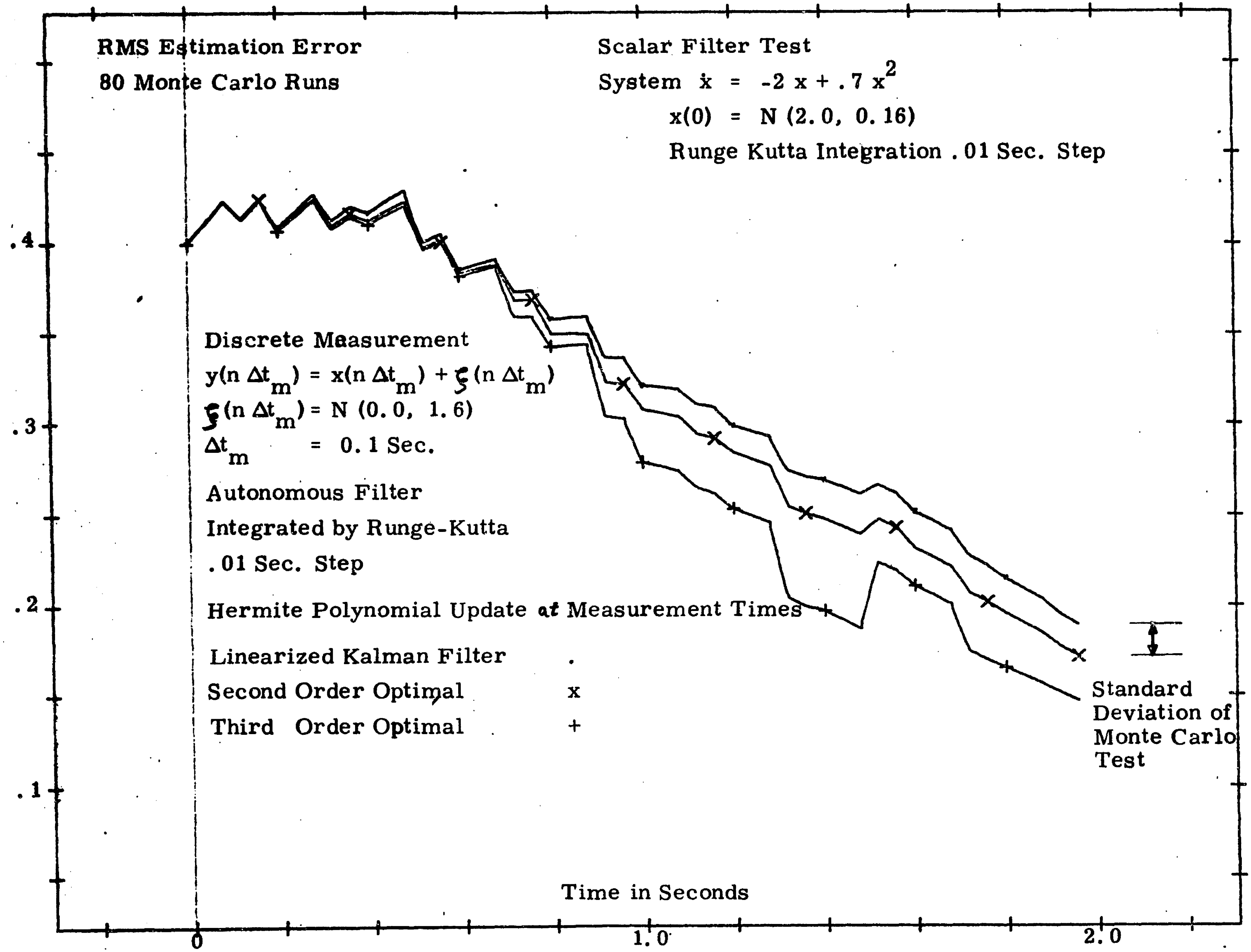

Figure 7.18 Hermite Polynomial Filter Test - RMS Estimation Error

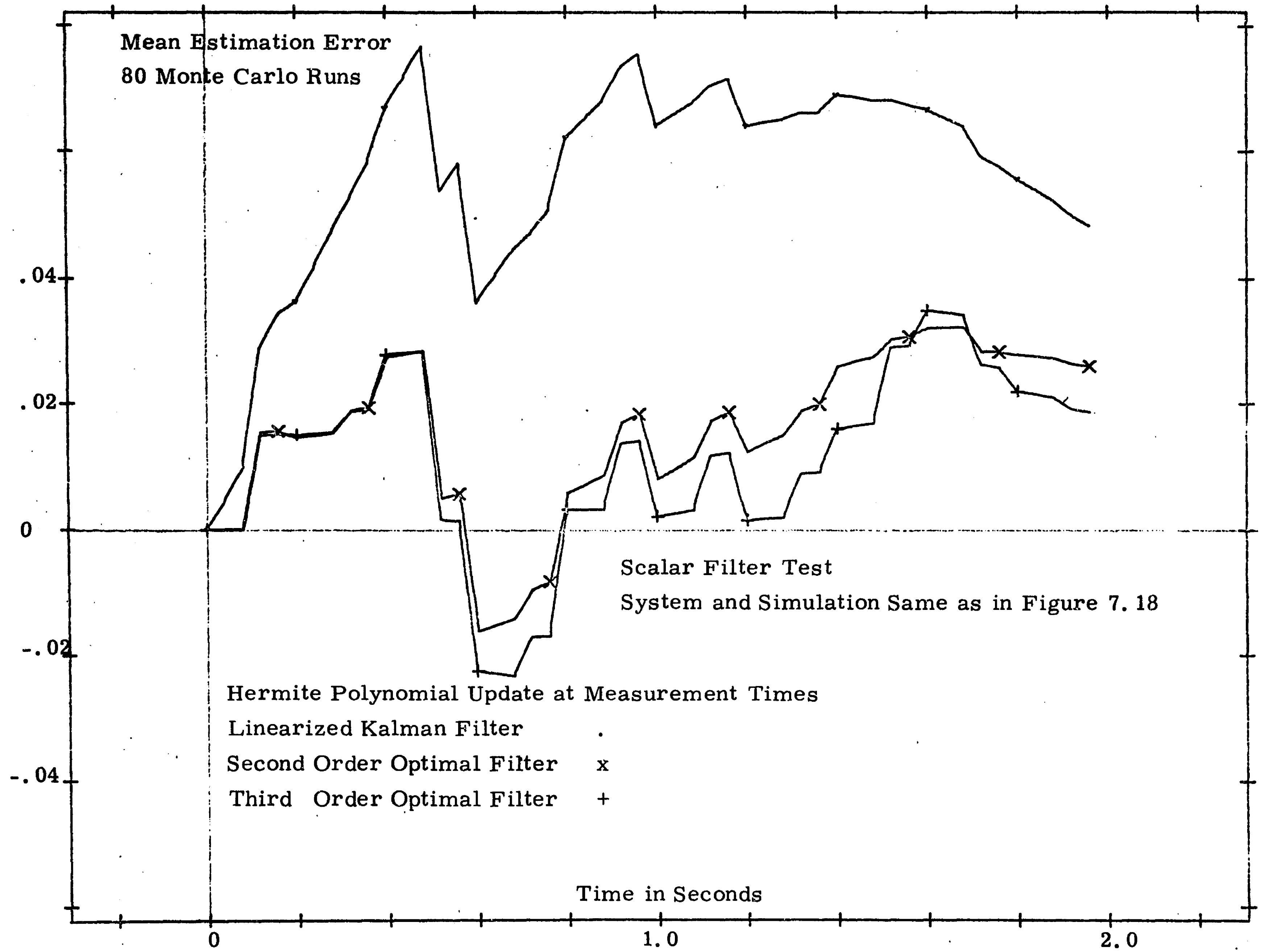

Figure 7.19 Hermite Polynomial Filter Test - Mean Estimation Error

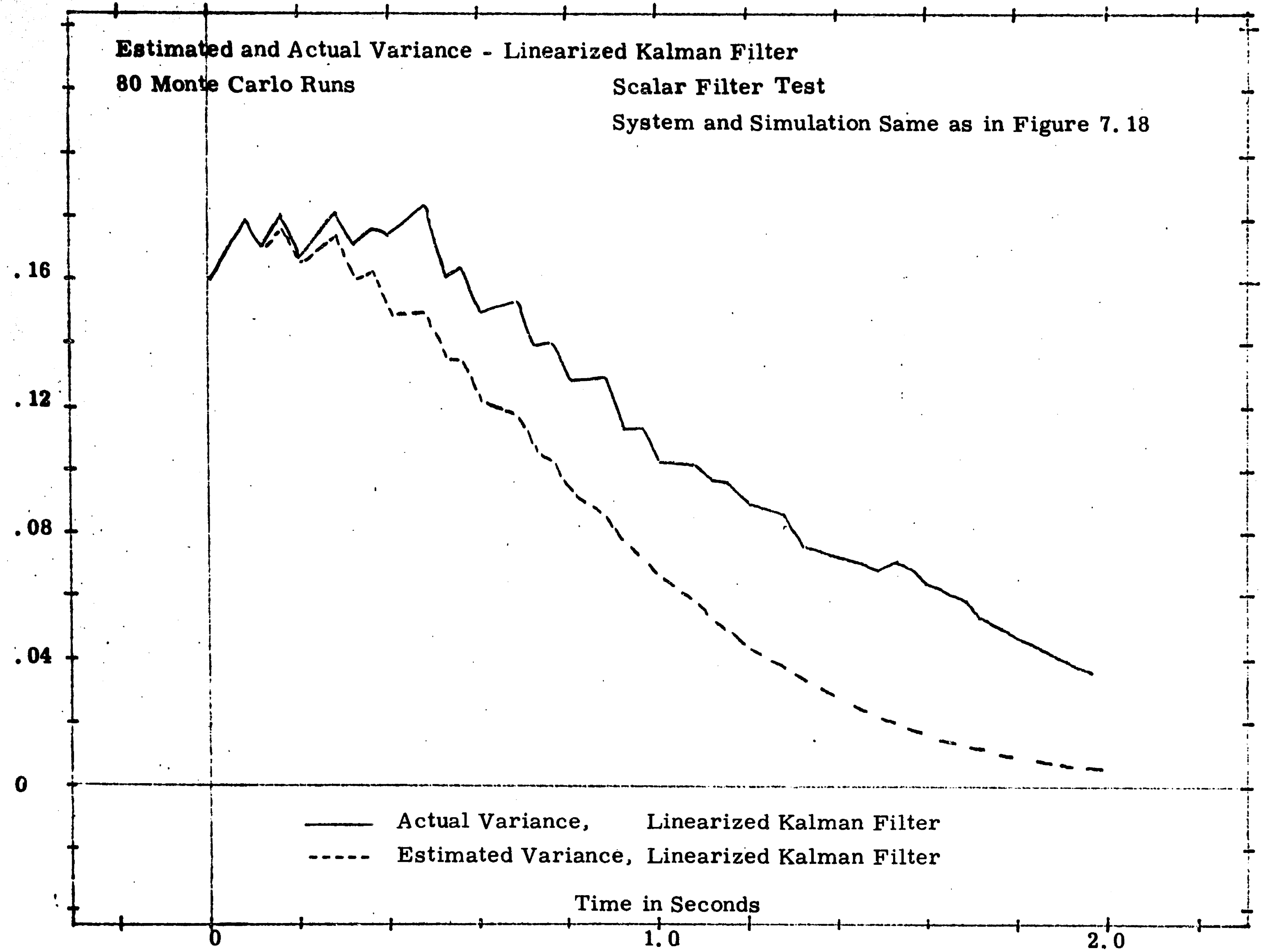

Figure 7.20 Hermite Polynomial Filter Test
Estimated and Actual Variance - Linearized Kalman Filter

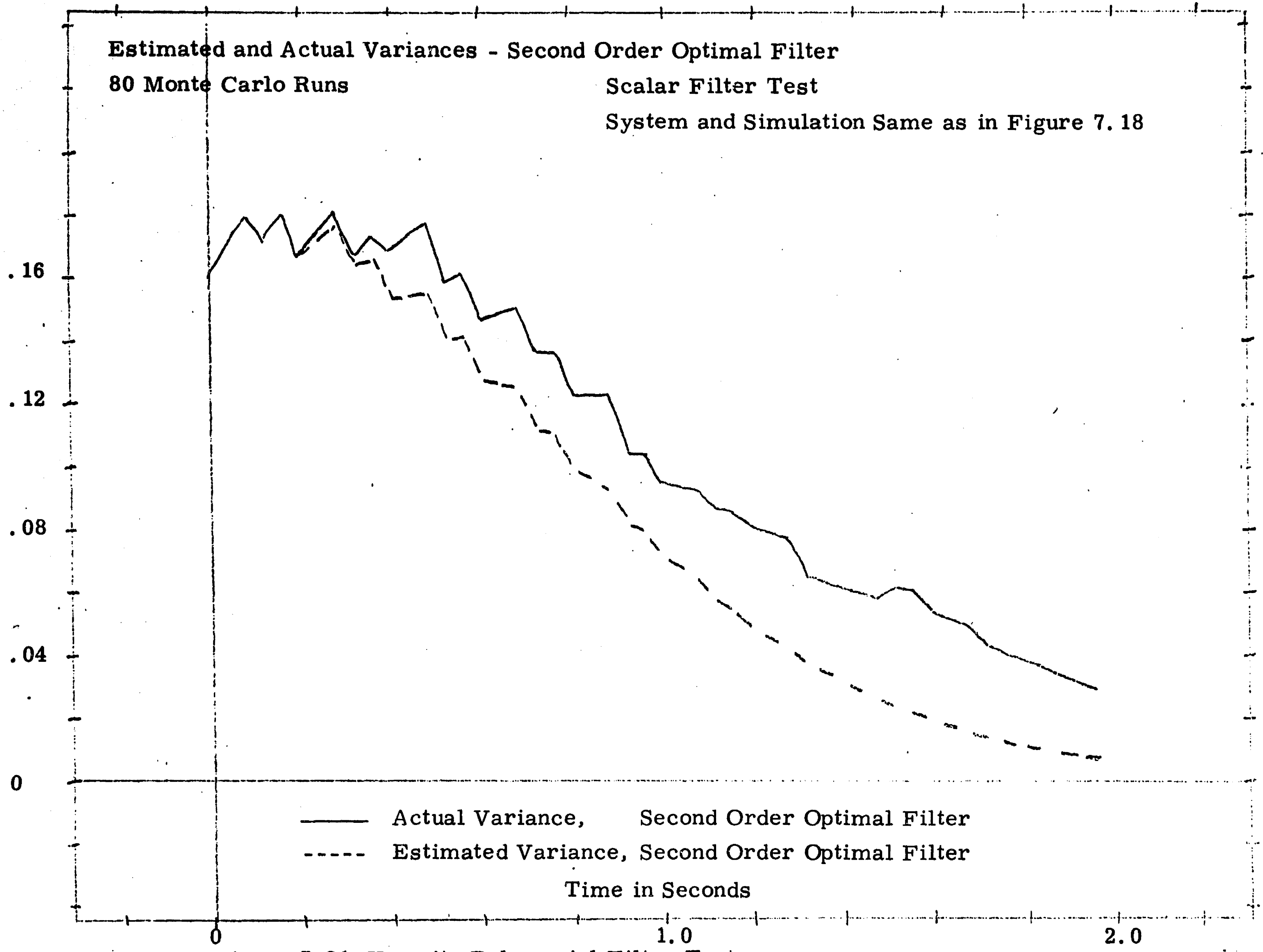

Figure 7.21 Hermite Polynomial Filter Test
Estimated and Actual Variances - Second Order Optimal Filter

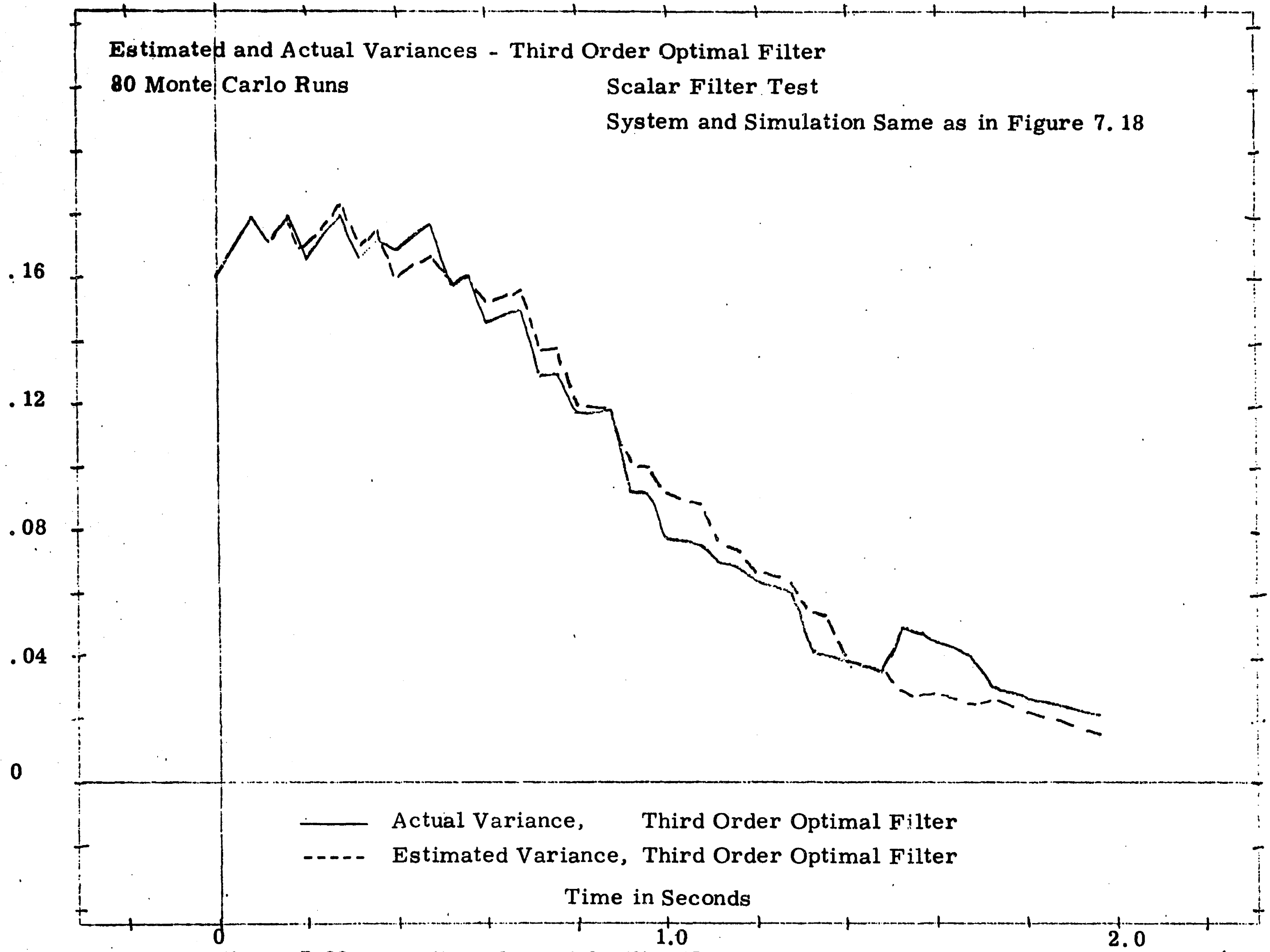

Figure 7.22 Hermite Polynomial Filter Test
Estimated and Actual Variances - Third Order Optimal Filter

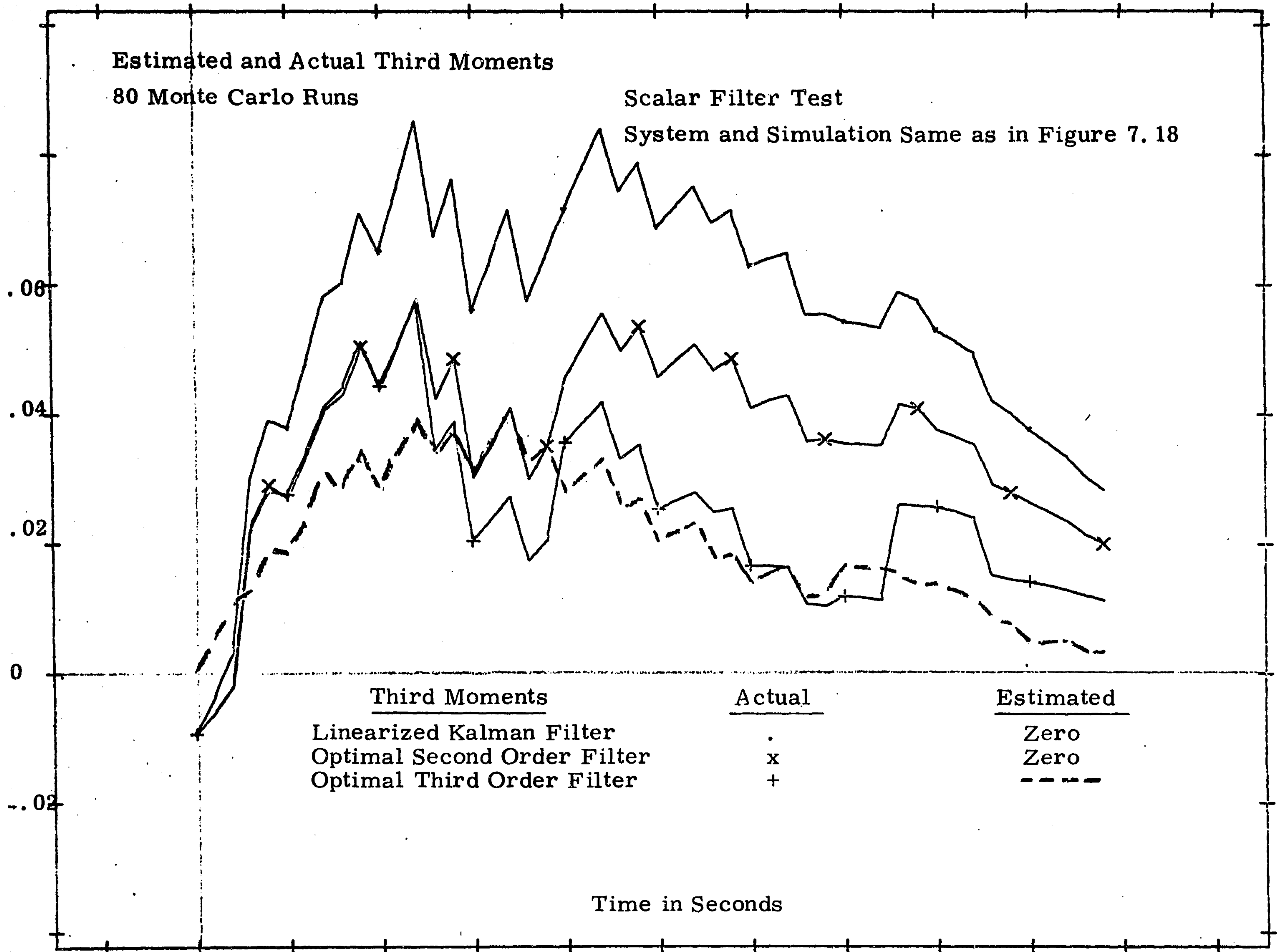

Figure 7.23 Hermite Polynomial Filter Test
Estimated and Actual Third Moments

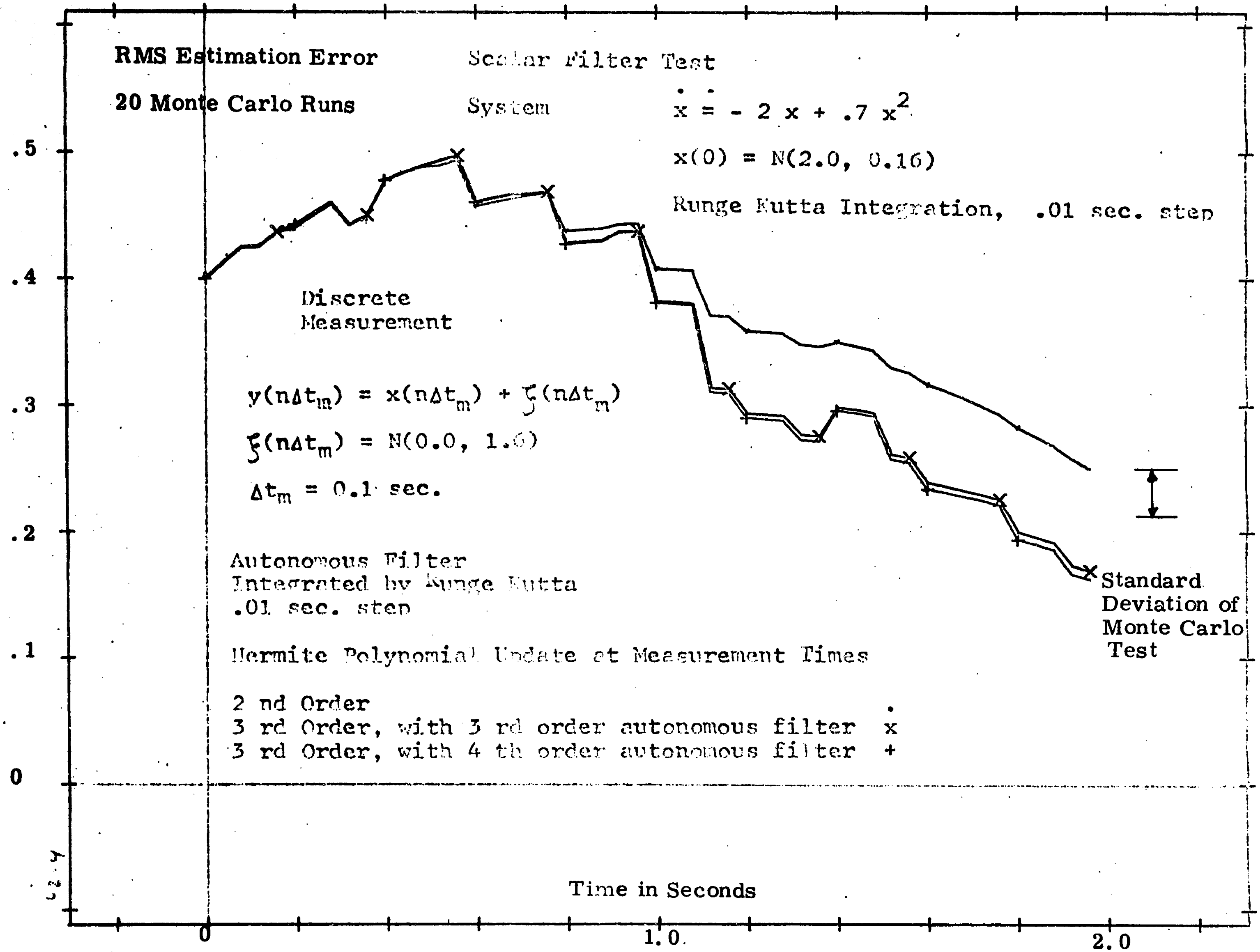

Figure 7.24 Hermite Polynomial Filter Test - Third and Fourth Order Autonomous Filters

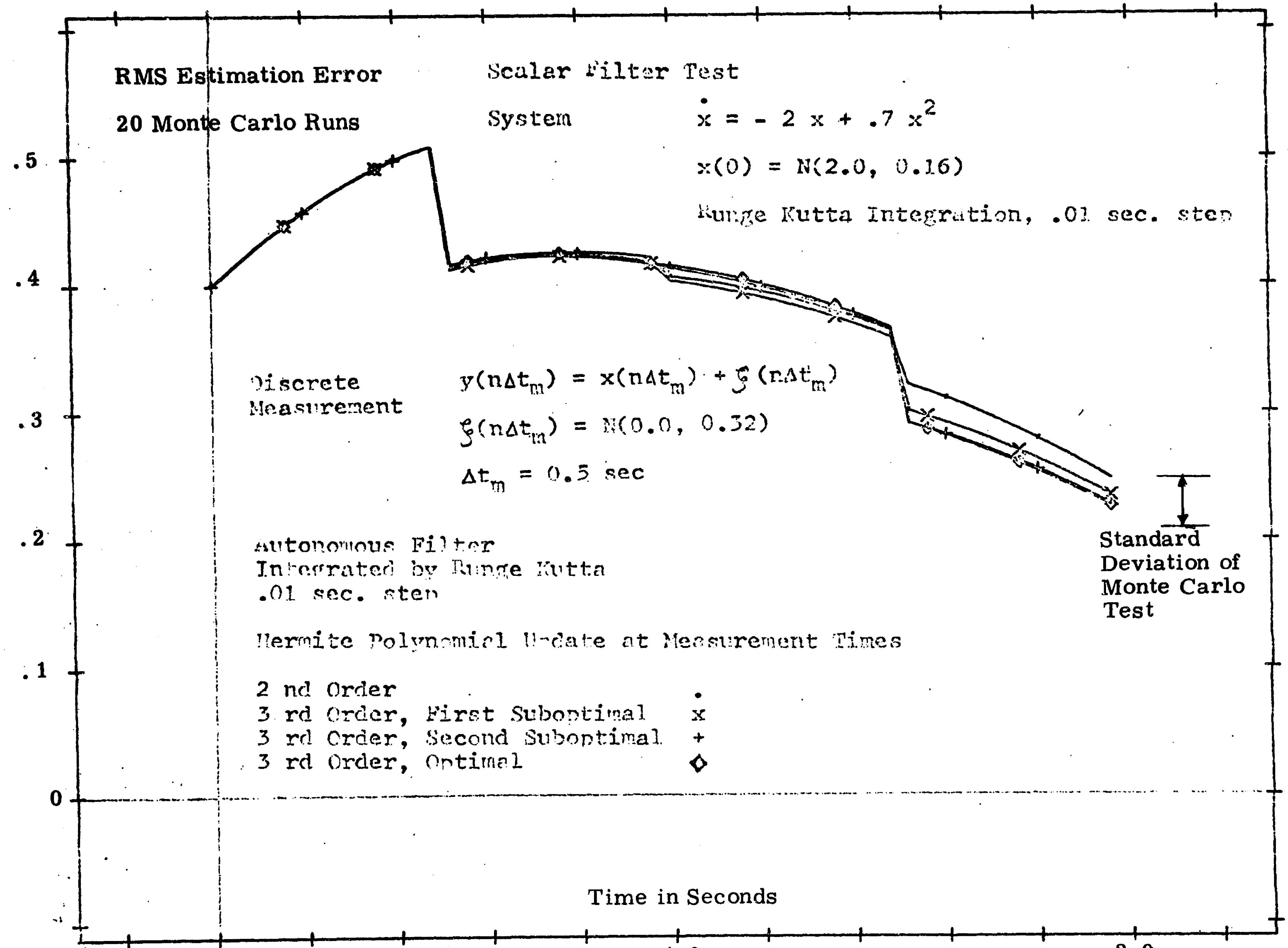

Figure 7.25 Hermite Polynomial Filter Test - .5 Sec. Measurement Interval

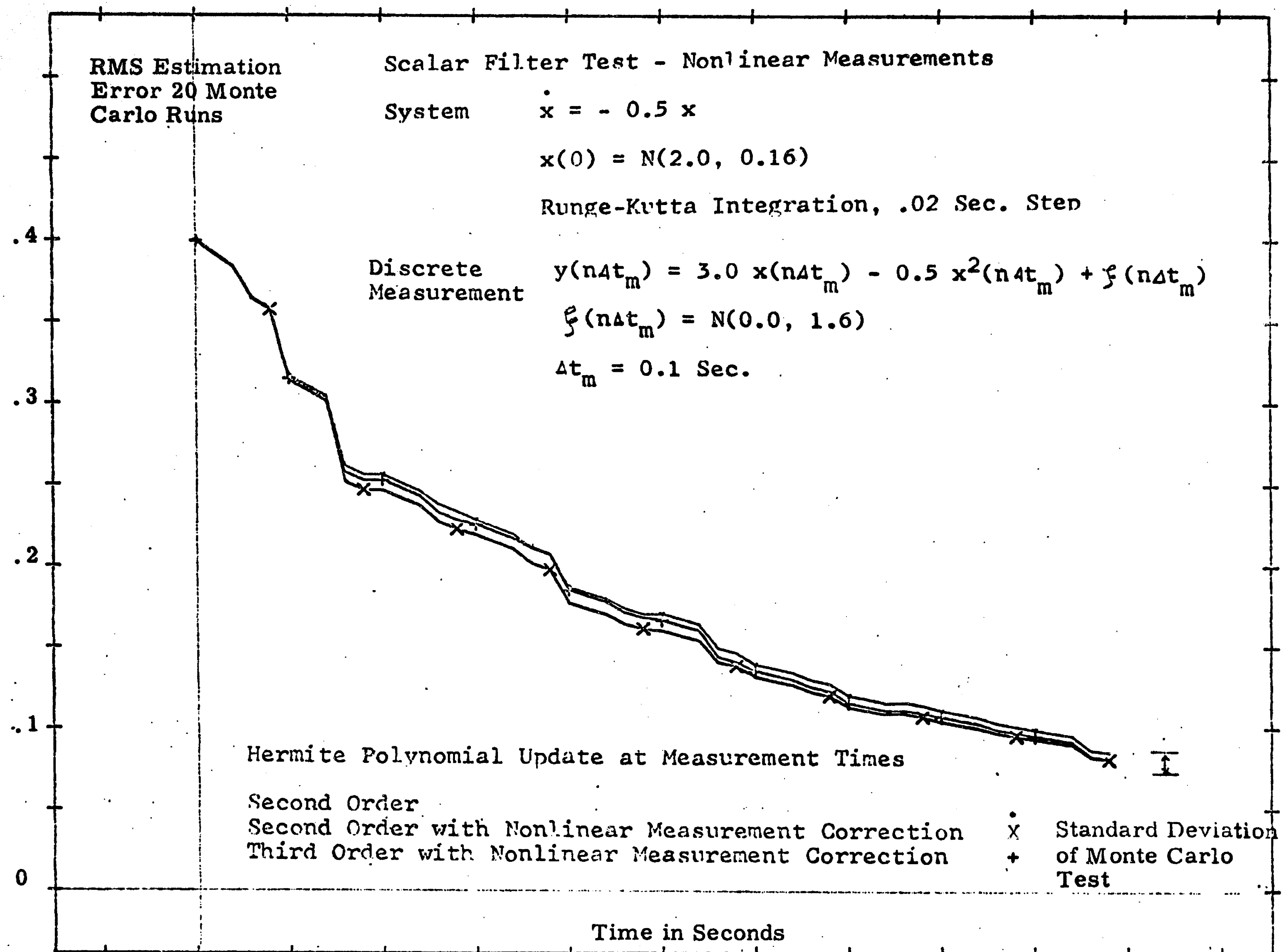

Figure 7.26 Nonlinear Measurement Test - Linear System

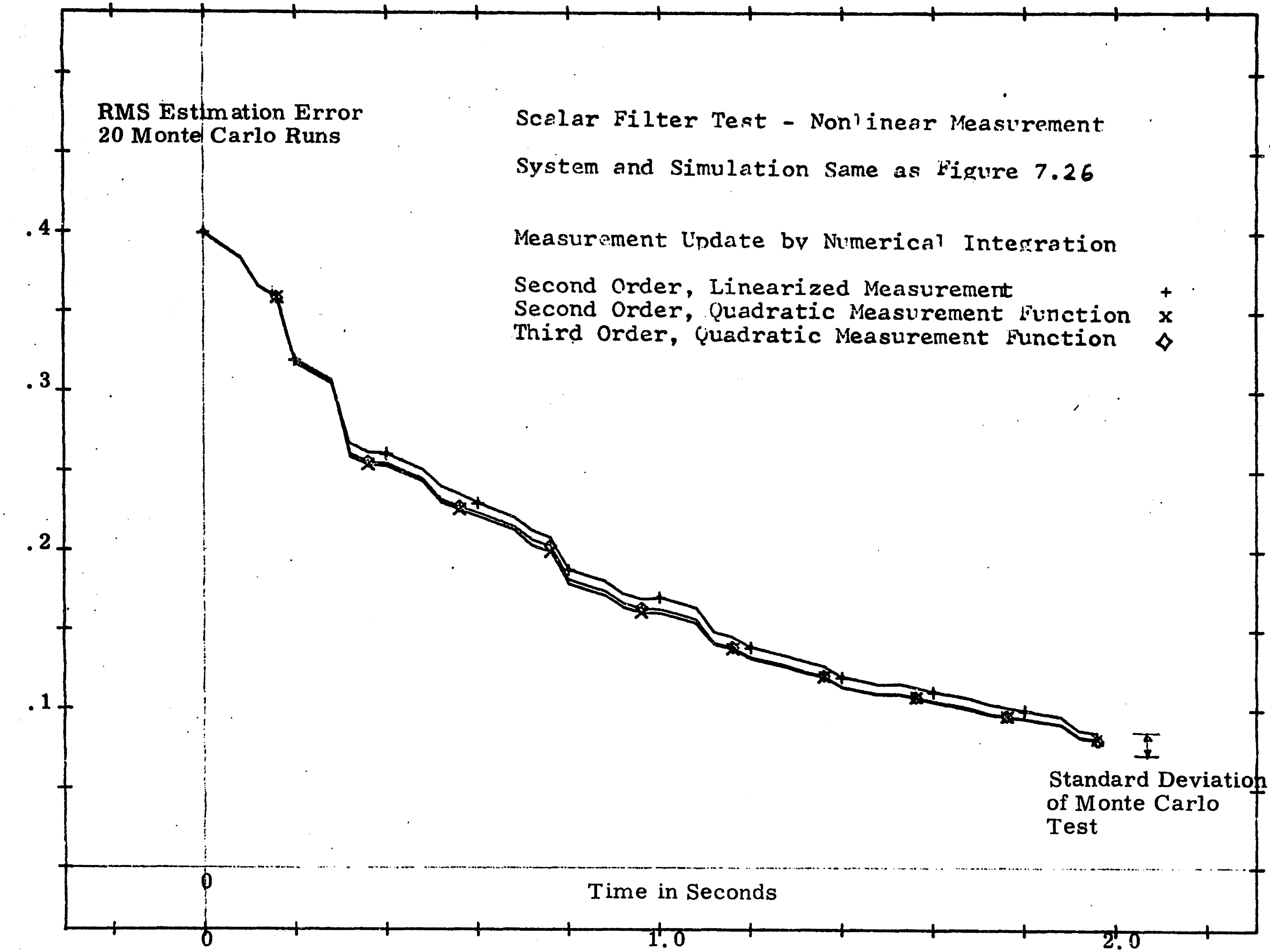

Figure 7.27 Nonlinear Measurement Test - Numerical Integration Solution

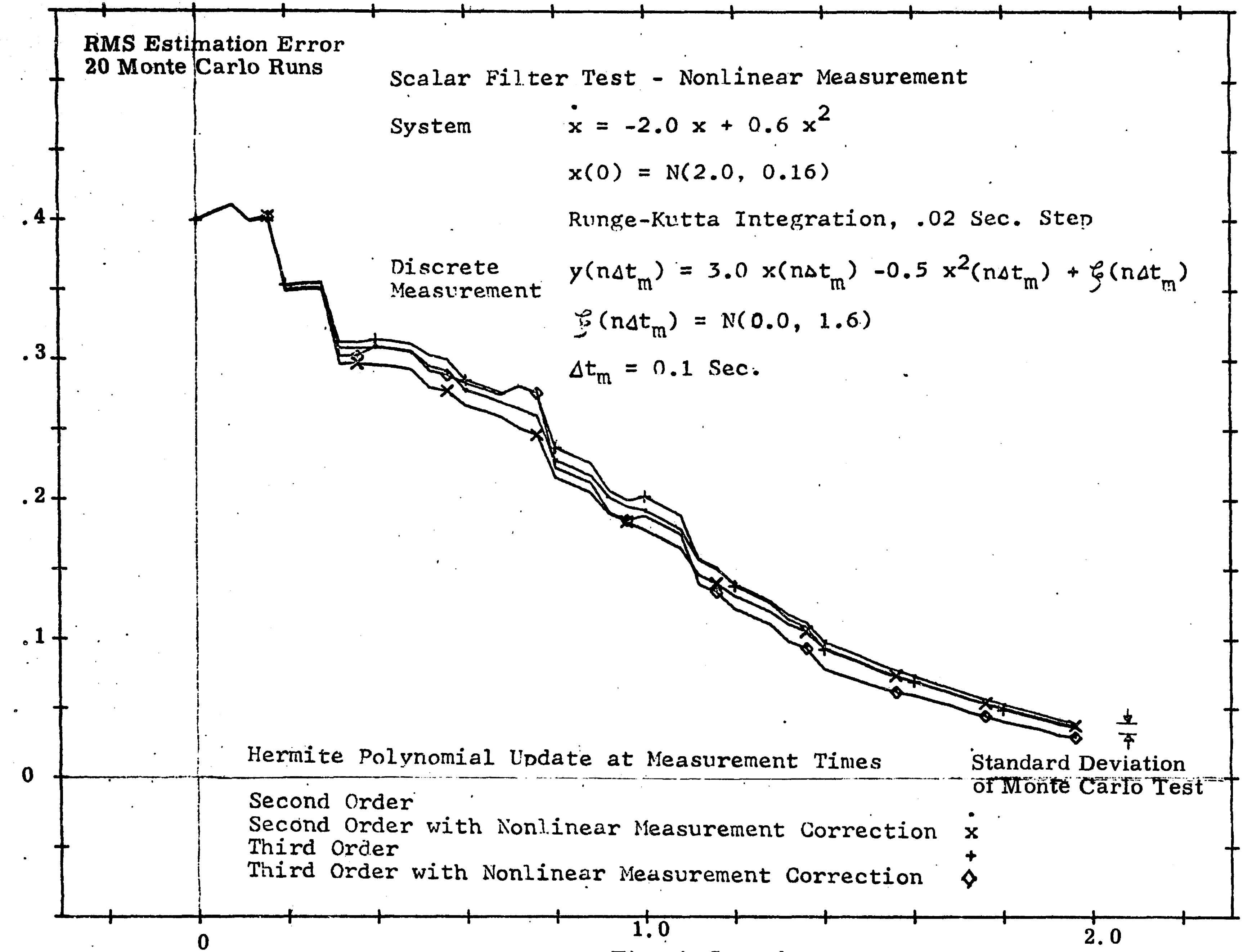

Figure 7.28 Nonlinear Measurement Test - Nonlinear System

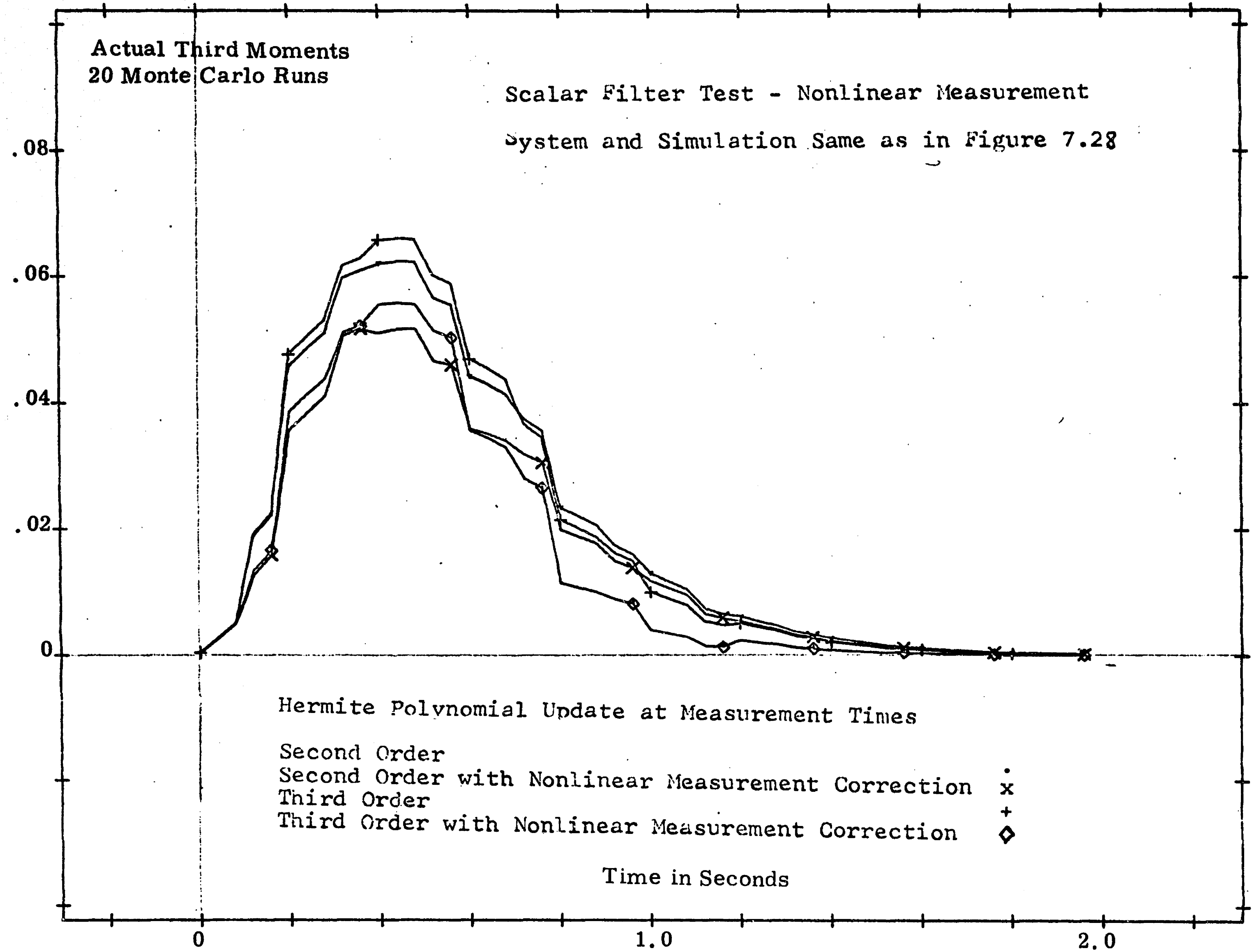

Figure 7.29 Nonlinear Measurement Test - Actual Third Moments

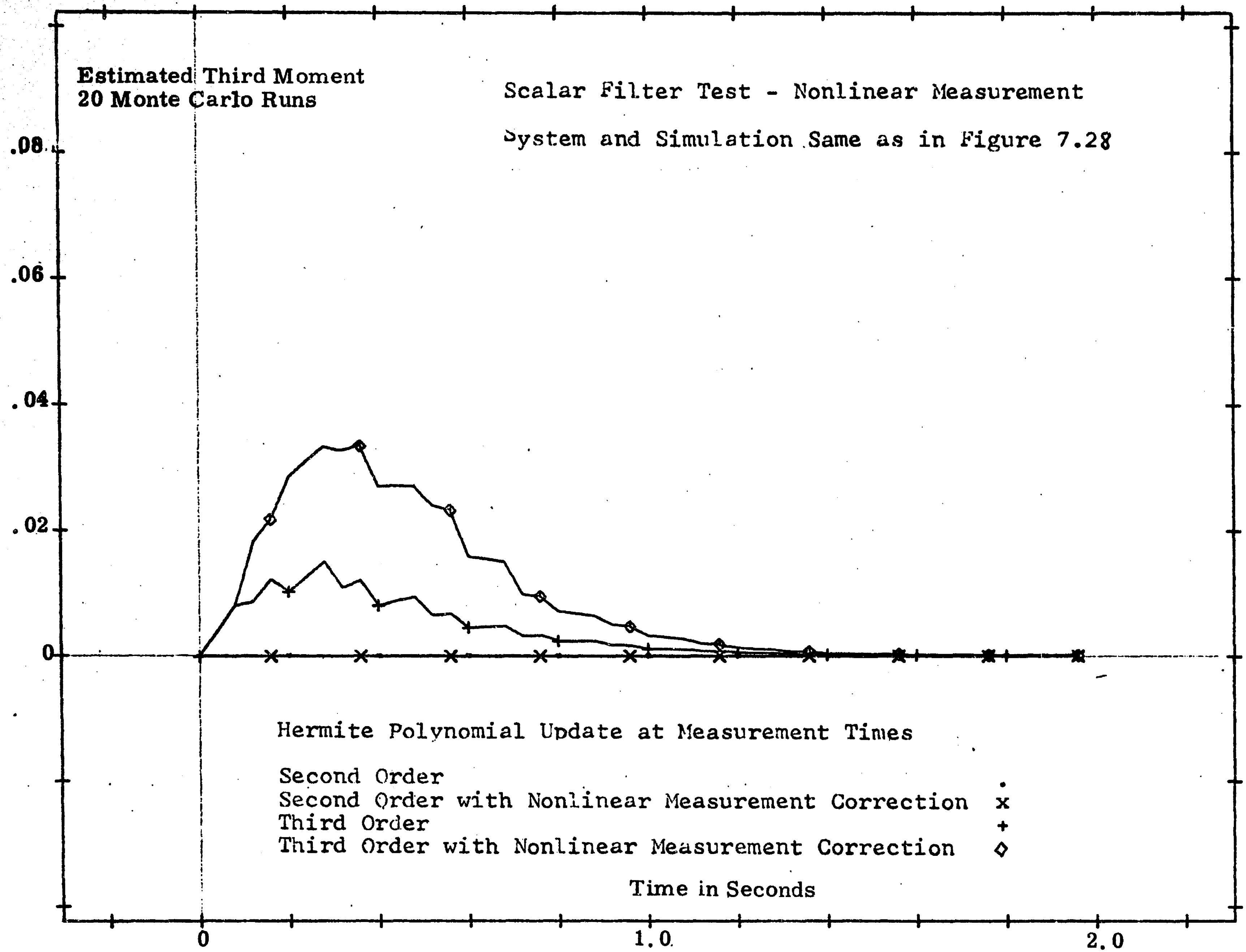

Figure 7.30 Nonlinear Measurement Test - Computed Third Moments

Chapter 8

Vector Simulation Tests

8.1 The linear System Identification Problem

A linear dynamic system can be described by the first order vector linear differential equation;

$$\dot{x}_i = F_{ie} x_e + G_{ie} v_e \tag{8.1}$$

An important class of problems arises when some of the elements of F are imperfectly known. If a (noisy) measurement of the state $\underline{x}$ were available, one might attempt to infer both the state and the imperfectly known F elements from the observations. This would be a quadratic estimation problem, because the state equations involve quadratic products of the variables being estimated. That is, (8.1) could always be written in the following form with known F' and A'.

$$\dot{x}'_i = F'_{ie} x'_e + A'_{ief} x'_e x'_f + G_{ie} v_e \tag{8.2}$$

$\underline{x}'$ is the augmented state; that is, the original state with the uncertain F elements added.

For example, a simple harmonic oscillator with uncertain damping ratio driven by white noise is considered. The system equation is,

$$\frac{d^2z}{dt^2} + 2\zeta\omega_n \frac{dz}{dt} + \omega_n^2 z = G v \tag{8.3}$$

v is a unit "white" noise.

The system can be written in first order form by choosing the following state variables;

$$x_1 = z$$
$$x_2 = \frac{dz}{dt}$$
$$x_3 = \zeta \tag{8.4}$$

Then,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\omega_n^2 x_1 - 2\omega_n x_2 x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ G v \\ 0 \end{bmatrix} \tag{8.5}$$

If the initial probability distribution for the state (8.4) is known, and discrete linear measurements are available, then this nonlinear estimation problem can be conveniently treated by the methods of Chapter 5. Between measurements, the estimated moments can be propagated by an autonomous filter derived from the power series expansion of the state equations. At the measurement times the moments can be updated by the approximate series or integration update methods, or by the optimal Hermite polynomial method. It is clear that two or more moments can be computed, and that additional moments beyond the second should yield improved accuracy, since the state equations are nonlinear.

The optimal second and third order filters, as well as the suboptimal third order filters previously discussed were programmed and tested via Monte Carlo simulation for the above example. The second order expansion about the estimated mean for the system is as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} m_2 \\ -\omega_n^2 m_1 - 2\omega_n m_2 m_3 \\ 0 \end{bmatrix} + \begin{bmatrix} \delta x_2 \\ -\omega_n^2 \delta x_1 - 2\omega_n (m_2 \delta x_3 + m_3 \delta x_2) \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -2\omega_n \delta x_2 \delta x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ G v \\ 0 \end{bmatrix}$$

$$\delta \underline{x} = \underline{x} - \underline{m} \tag{8.6}$$

The optimal third order autonomous filter equations can now be written from the general equations (4.64) - (4.67).

Autonomous Filter Equations (Third Order)

$F_{12} = 1$ $\qquad\qquad$ $A_{223} = -2\omega_n$

$F_{21} = -\omega_n^2$

$F_{22} = -2\omega_n m_3$ $\qquad\qquad$ All other F_{ij}, $A_{ijk} = 0$

$F_{23} = -2\omega_n m_2$ $\qquad\qquad$ (8.7)

$$\dot{m}_1 = m_2$$

$$\dot{m}_2 = -\omega_n^2 m_1 - 2\omega_n m_2 m_3 + A_{223}\,\mu_{223}$$

$$\dot{m}_3 = 0 \tag{8.8}$$

$$\dot{\mu}2_{11} = 2\ F_{12}\,\mu2_{12}$$

$$\dot{\mu}2_{12} = F_{12}\,\mu2_{22} + F_{21}\,\mu2_{11} + F_{22}\mu2_{12} + F_{23}\mu2_{13} + A_{223}\mu3_{123}$$

$$\dot{\mu}2_{13} = F_{12}\mu2_{23}$$

$$\dot{\mu}2_{22} = G^2 + 2(F_{21}\mu2_{12} + F_{22}\mu2_{22} + F_{23}\mu2_{23} + A_{223}\mu3_{223})$$

$$\dot{\mu}2_{23} = F_{21}\mu2_{13} + F_{22}\mu2_{23} + F_{23}\mu2_{33} + A_{223}\mu3_{233}$$

$$\dot{\mu}2_{33} = 0 \qquad (8.9)$$

$$\dot{\mu}3_{111} = 3\ F_{12}\mu3_{112}$$

$$\dot{\mu}3_{112} = 2\ F_{12}\mu3_{122} + F_{21}\mu3_{111} + F_{22}\mu3_{112} + F_{23}\mu3_{113}$$

$$+ 2\ A_{223}\mu2_{12}\mu2_{13}$$

$$\dot{\mu}3_{113} = 2\ F_{12}\mu3_{123}$$

$$\dot{\mu}3_{122} = F_{12}\mu3_{222} + 2(F_{21}\mu3_{112} + F_{22}\mu3_{122} + F_{23}\mu3_{123}$$

$$+ A_{223}(\mu2_{13}\mu2_{22} + \mu2_{12}\mu2_{23}))$$

$$\dot{\mu}3_{123} = F_{12}\mu3_{223} + F_{21}\mu3_{113} + F_{22}\mu3_{123} + F_{23}\mu3_{133}$$

$$+ A_{223}(\mu2_{13}\mu2_{23} + \mu2_{12}\mu2_{33})$$

$$\dot{\mu}3_{133} = F_{12}\mu3_{233}$$

$$\dot{\mu}3_{222} = 3(F_{21}\mu3_{122} + F_{22}\mu3_{222} + F_{23}\mu3_{223} + 2\ A_{223}\mu2_{22}\mu2_{23})$$

$$\dot{\mu}3_{223} = 2(F_{21}\mu 3_{123} + F_{22}\mu 3_{223} + F_{23}\mu 3_{233} + A_{223}(\mu 2_{23}\mu 2_{23} + \mu 2_{22}\mu 2_{33}))$$

$$\dot{\mu}3_{233} = F_{21}\mu 3_{133} + F_{22}\mu 3_{233} + F_{23}\mu 3_{333} + 2\ A_{223}\mu 2_{23}\mu 2_{33}$$

$$\dot{\mu}3_{333} = 0 \tag{8.10}$$

The optimal second order autonomous filter equations are given by (8.8) and (8.9) with the assumption that $\mu 3_{ijk} = 0$.

8.2 Simulation Description and Results

The harmonic oscillator (8.5) was used as an example for simulation testing. Periodic discrete measurements of the following form were assumed.

$$y_r(n\Delta t) = H_{re}\ x_e(n\Delta t) + R_{rs}\xi_s(n\Delta t) \tag{8.11}$$

$R_{rp}\ R_{sp} = \Sigma_{rs}$ = discrete measurement covariance

$\xi_s(n\Delta t)$ = unit uncorrelated Gaussian noise vector $= N(0, \delta_{st})$

The natural frequency of the oscillator was chosen as one, and the initial conditions (IE, position, velocity, and damping ratio) were chosen from a normal distribution.

The system, autonomous filters, and discrete measurement updating techniques were simulated on an IBM 360-75 digital computer. The system equations (8.5) and autonomous filter equations (8.8), (8.9), and (8.10) were integrated using a Runge-Kutta algorithm with a step size of .05 second. Pseudo random numbers with an approximately normal distribution were generated using a standard routine, which adds twelve rectangularly distributed random numbers. By the central limit theorem, this sum is approximately normal.

The system was simulated by choosing random initial conditions and integrating the equations of motion. The autonomous filter was started at the expected value of the initial conditions. The system and filter equations, including discrete measurement computations, were then simulated repeatedly over the interval 0 - 10 seconds, and the RMS error computed for that filter and random noise sequence. The procedure was then repeated using the same random noise sequence and a different filter. This Monte Carlo testing procedure allows a direct comparison of the two filters, with an accuracy dependent on the number of trials in the Monte Carlo sequence.

To improve the accuracy of this procedure for a limited number of runs, the initial condition noise sequence was generated from the random noise generator, and subsequently modified to give the sample exactly the desired mean and main diagonal covariances. For a large number of trials, this procedure would be unnecessary because then the sample statistics would be close to those used to program the random noise generator.

First the system and three types of autonomous filter equations were simulated with no measurements. These were the linearized Kalman filter, and second and third order optimal nonlinear filters. Figures 8.1, 8.2, and 8.3 show the estimation error history for the position, velocity, and damping ratio. The damping ratio estimation error can only be changed by measurement data, as is clear from figure 8.3.

Figures 8.4, 8.5, and 8.6 show the effect of incorporating discrete measurements every .2 seconds. For this test, the linearized Kalman updating equations and the optimal second and third order discrete updating equations of Table 5.1 were used. For this and subsequent tests,

$$H = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix} \qquad (8.12)$$

In early tests of this type, a most baffling type of instability was found. For some random noise sequences good performance would be obtained, while poor performance and even divergence would result from other noise sequences with the same statistics. This problem was also noted in the nonlinear filters of Schwartz and Stear. (P-29)

This difficulty results from the fact that the optimal second and third order filters are actually truncations of the optimal infinite order filter. As such, they are only locally optimal for nonlinearities which are not too severe. The derivation of the updating equations assumes a zero mean prior estimate of the moments, a situation which may not be strictly correct for very nonlinear systems, or due to computational limitations. Further note that the optimal third order update for the third moment involves the sixth power of the measurement residue. Hence if the filter estimate should become sufficiently inaccurate for a certain random measurement sequence, large measurement residues will appear. The sixth power of these improperly modeled residues will be fed back into the third moment estimates, a situation which can clearly lead to instability.

A possible solution is to limit the maximum and minimum values of the measurement residues fed into the measurement update. The measurement residues should be approximately normal, with a covariance that can be computed from the filter estimates.

$$\underline{z} = H\,\underline{x} + R\,\underline{\xi} \tag{8.13}$$

$$R\,R^T = \Sigma$$

$$\Gamma = \text{covariance of } \underline{z}$$

$$\Gamma = H\,\mu 2\,H^T + \Sigma \tag{8.14}$$

The measurement residues z_i were limited at $\pm 3\sqrt{\Gamma_{ii}}$, which would limit only .3 % of the measurements received if the prior estimates were zero mean and the covariance estimate $\mu 2$ were correct. However, if a "bad noise sequence" were received and the filter means were substantially in error, this would prevent large erroneous corrections from being added to the moments. This residue limitation at $3\,\sigma_z$ was tried, and found to completely solve the occasional divergence problem. The above type of limiter was used in all the vector simulations.

To further verify the optimal nonlinear updating theory, the suboptimal third order updating equations (5.67) and (5.68) were simulated. These were predicted to have accuracies between the optimal second and third order methods. This was well borne out by simulation testing, as can be seen from Figures 8.7, 8.8, and 8.9.

The final test shows the effects of a longer measurement interval on the optimal second and third order nonlinear filters. The RMS error histories in Figures 8.10, 8.11, and 8.12 show more clearly the better performance of the third order autonomous filter between measurements.

These simulation tests clearly show that the optimal third order filter can achieve sizeable error reductions, provided that the system is sufficiently nonlinear to develop a large third moment of the probability density of the variables being estimated.

Both the autonomous and discrete updating equations are substantially more complicated than for the second order filter, however. For this three dimensional problem, the second order filter operated approximately 5.2 times as fast as the third order filter, for a measurement interval of .2 seconds. For the scalar example the second order filter was approximately twice as fast, which shows that the third order filter becomes relatively more complex as system order increases.

The series expansion updating technique was not tested on this problem, as its poor convergence in the scalar example seemed too serious a fault to warrant further investigation.

The integration update method was tested, and gave good performance on those simulations with positive damping ratios. In those cases where the (random) damping ratio was negative, this updating procedure proved unstable even with the measurement residue limiter in effect.

Of the three third order updating methods derived, only the optimal Hermite Polynomial technique possesses both good stability and convergence characteristics. In addition, it is the least complex computationally for systems of low order (three or less).

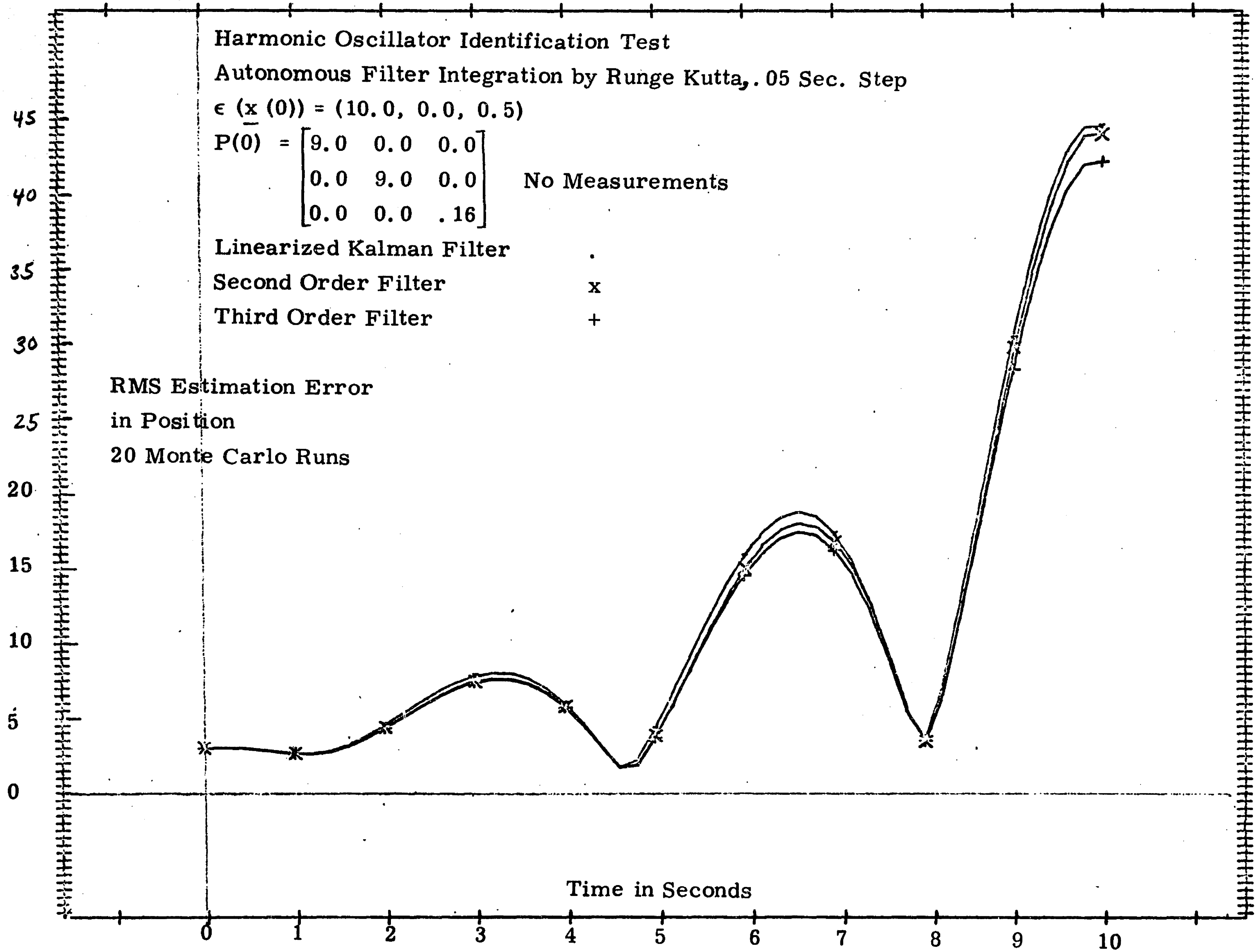

Figure 8.1 Autonomous Filter Test with No Measurements - Position Error

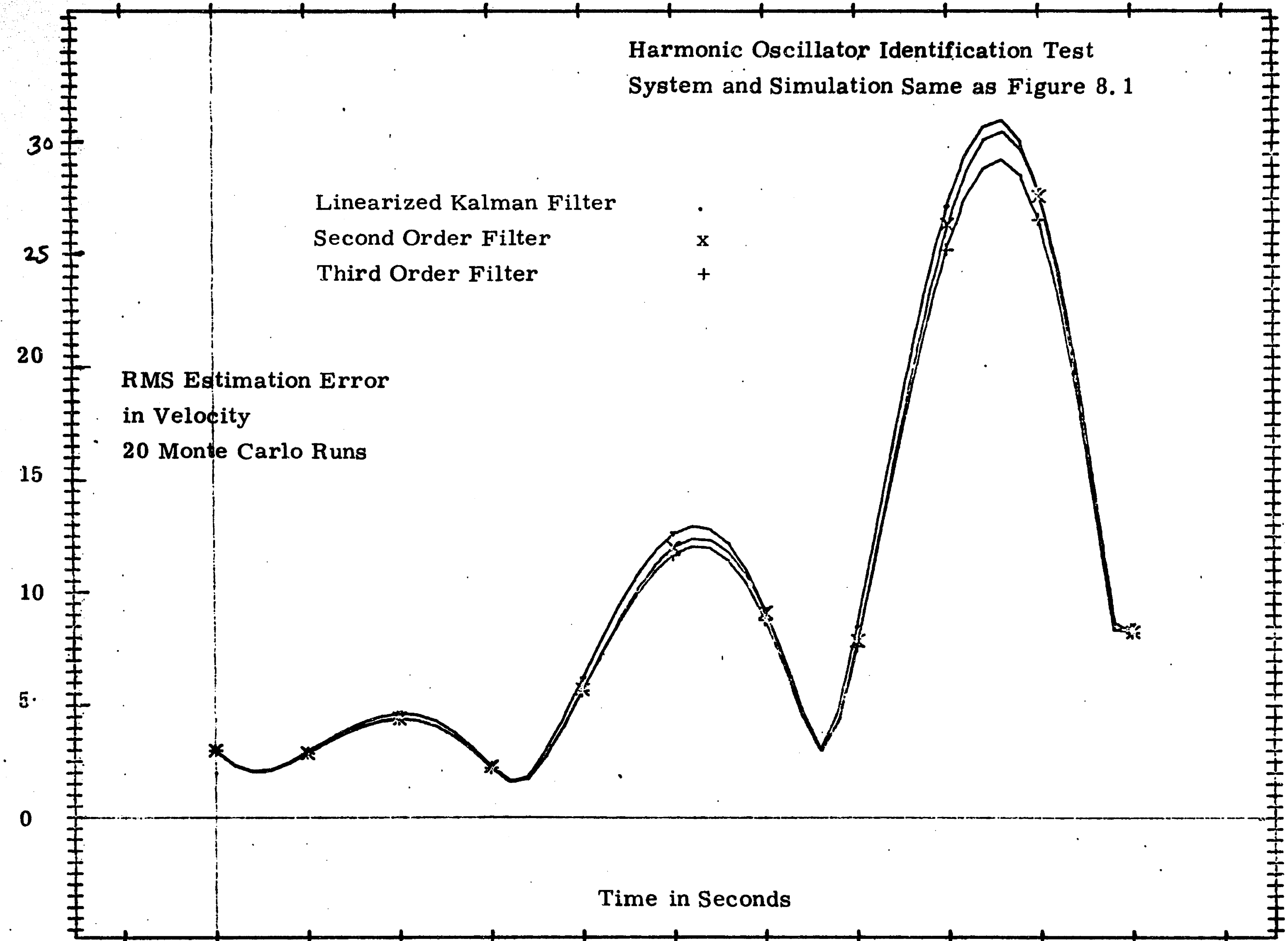

Figure 8.2 Autonomous Filter Test with No Measurements - Velocity Error

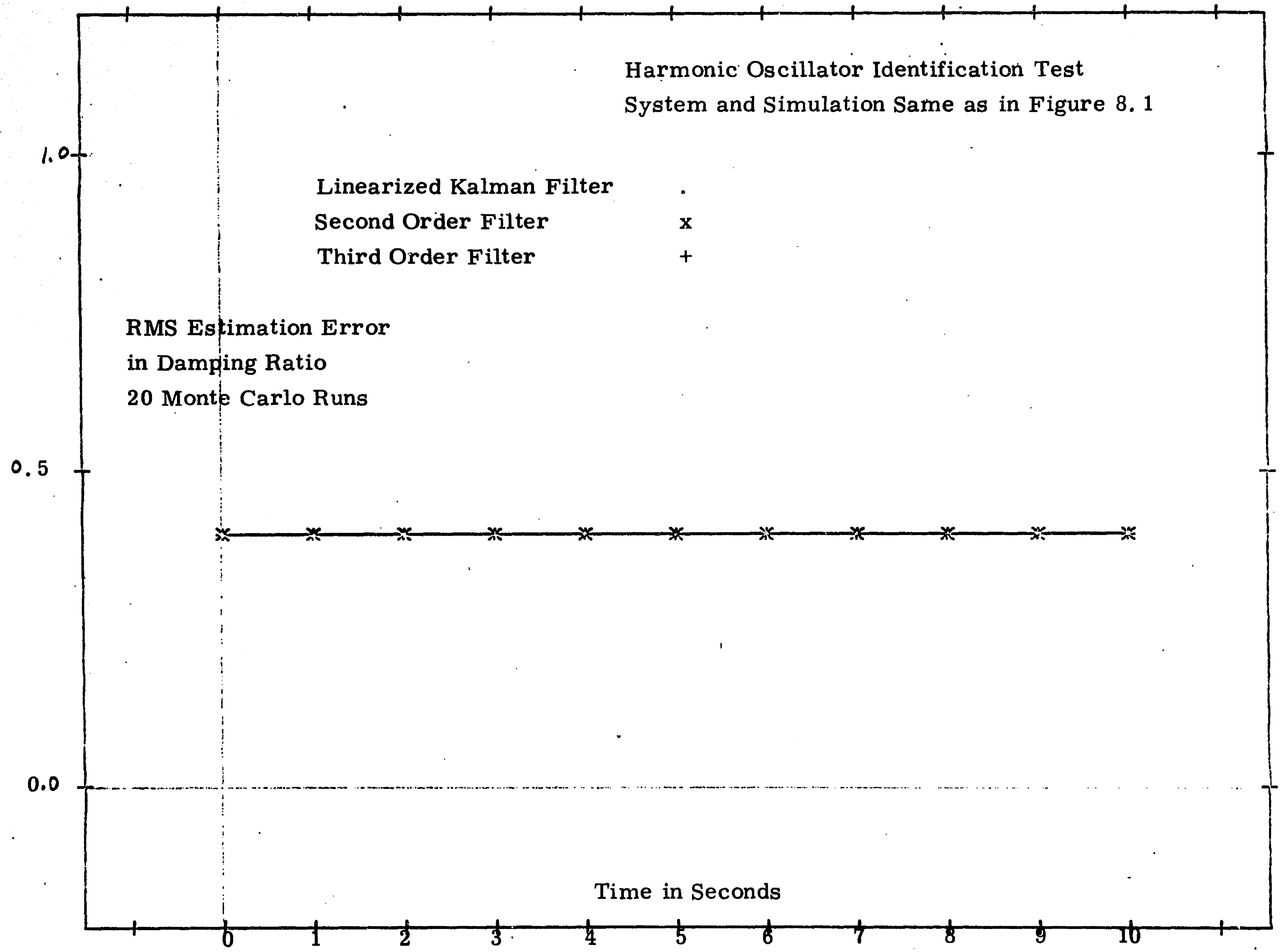

Figure 8.3 Autonomous Filter Test with No Measurements - Damping Ratio Error

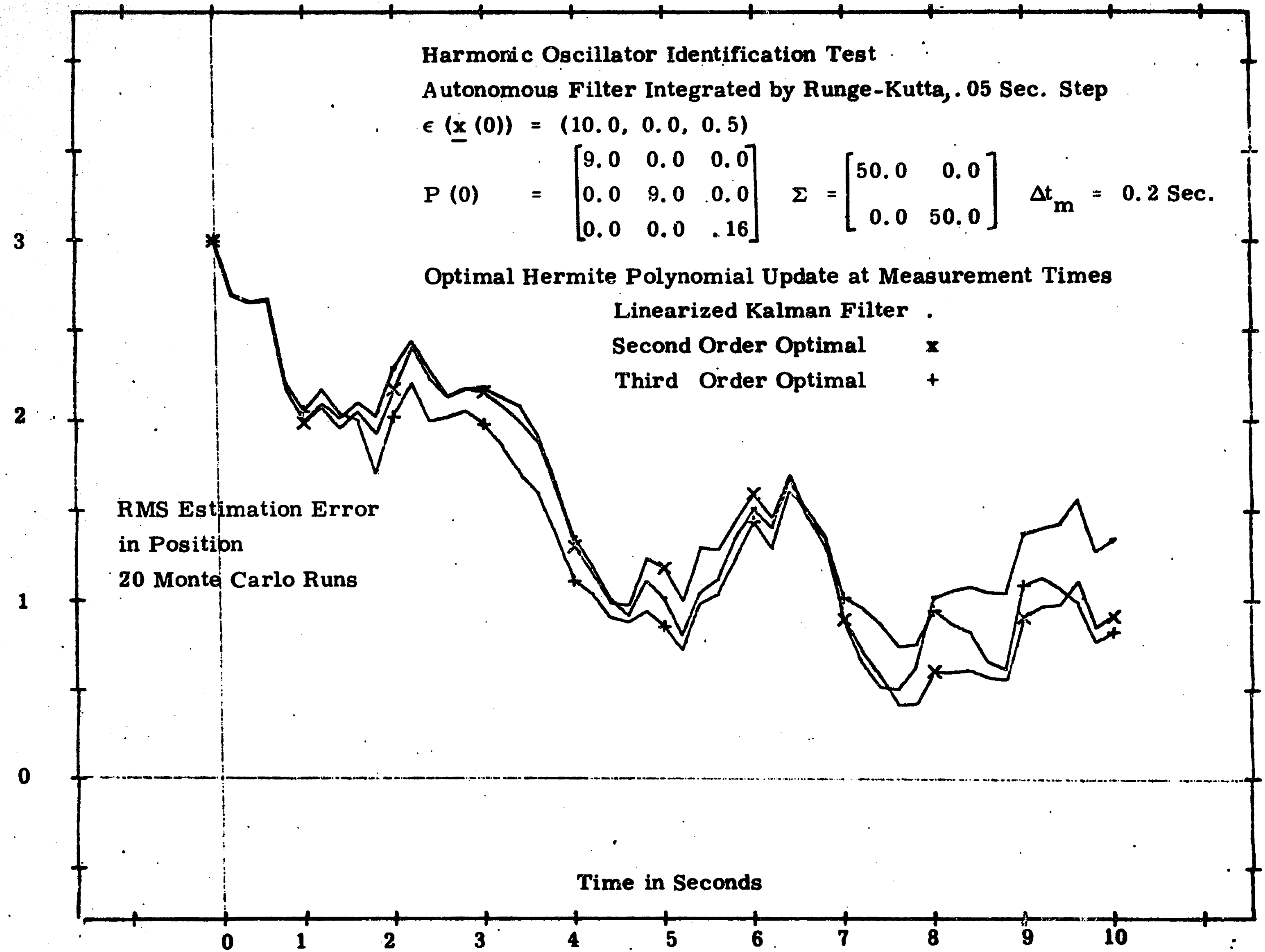

Figure 8.4 Optimal Nonlinear Filter Test - Position Error

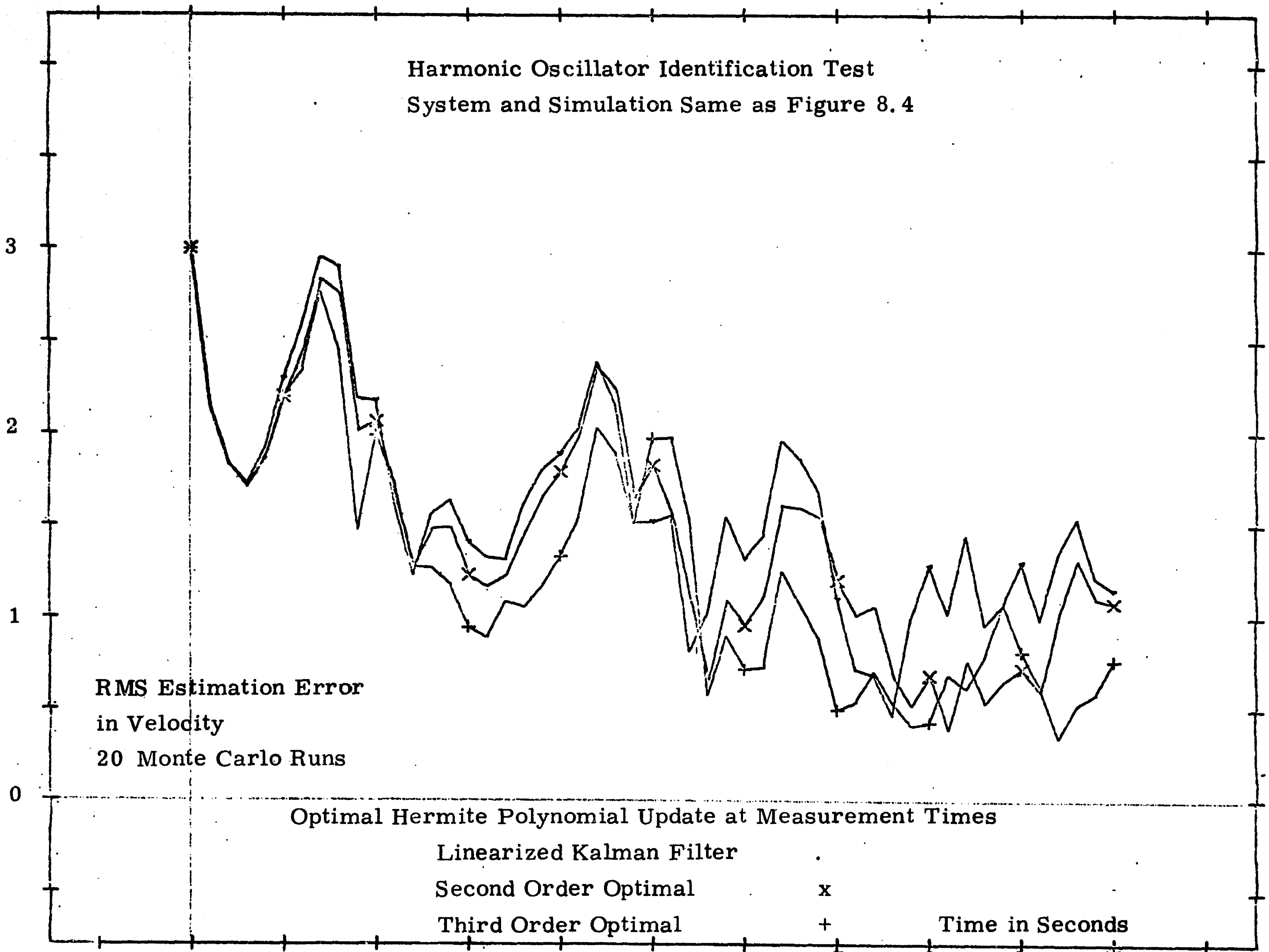

Figure 8.5 Optimal Non-linear Filter Test - Velocity Error

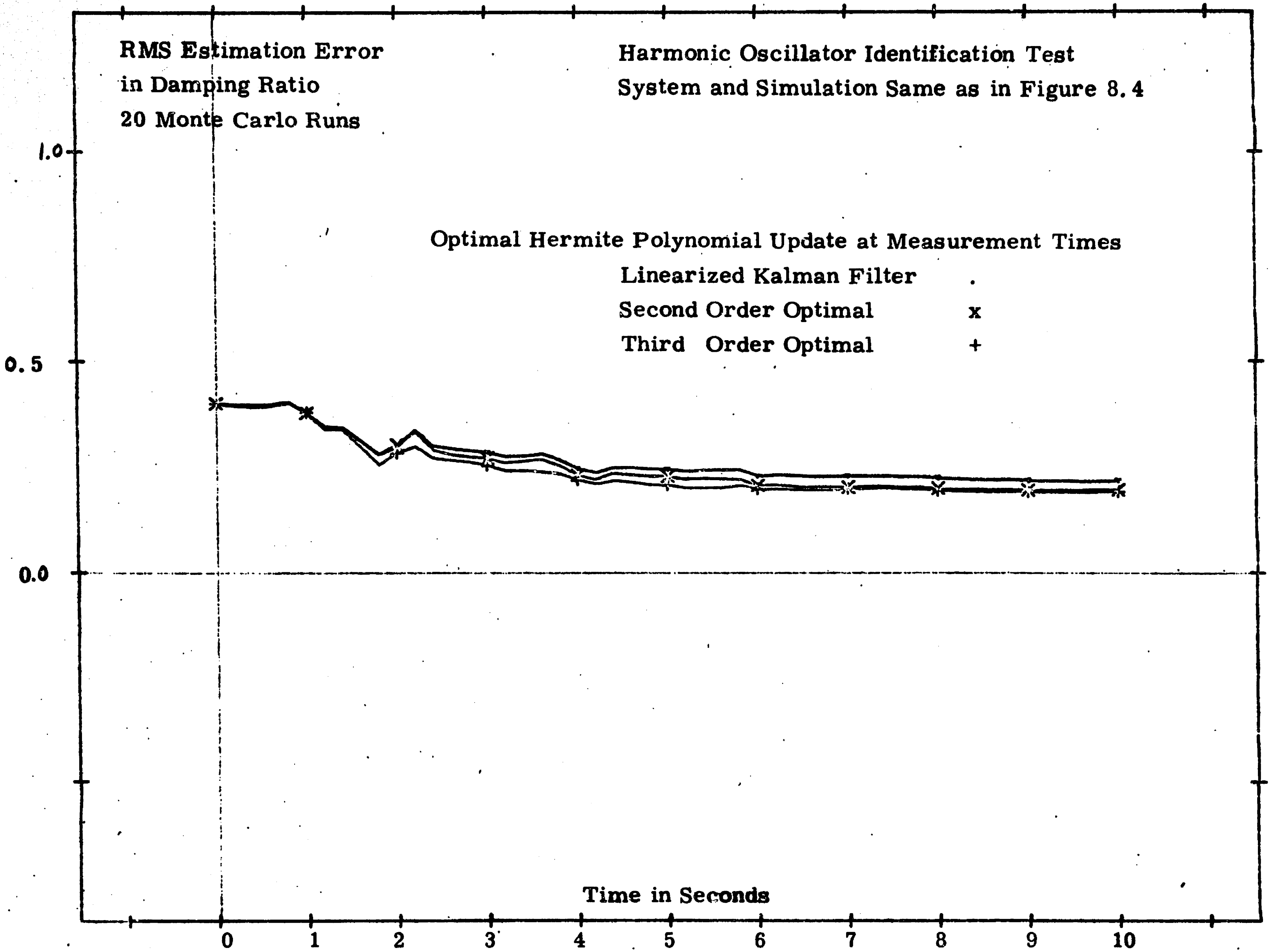

Figure 8.6 Optimal Nonlinear Filter Test - Damping Ratio Error

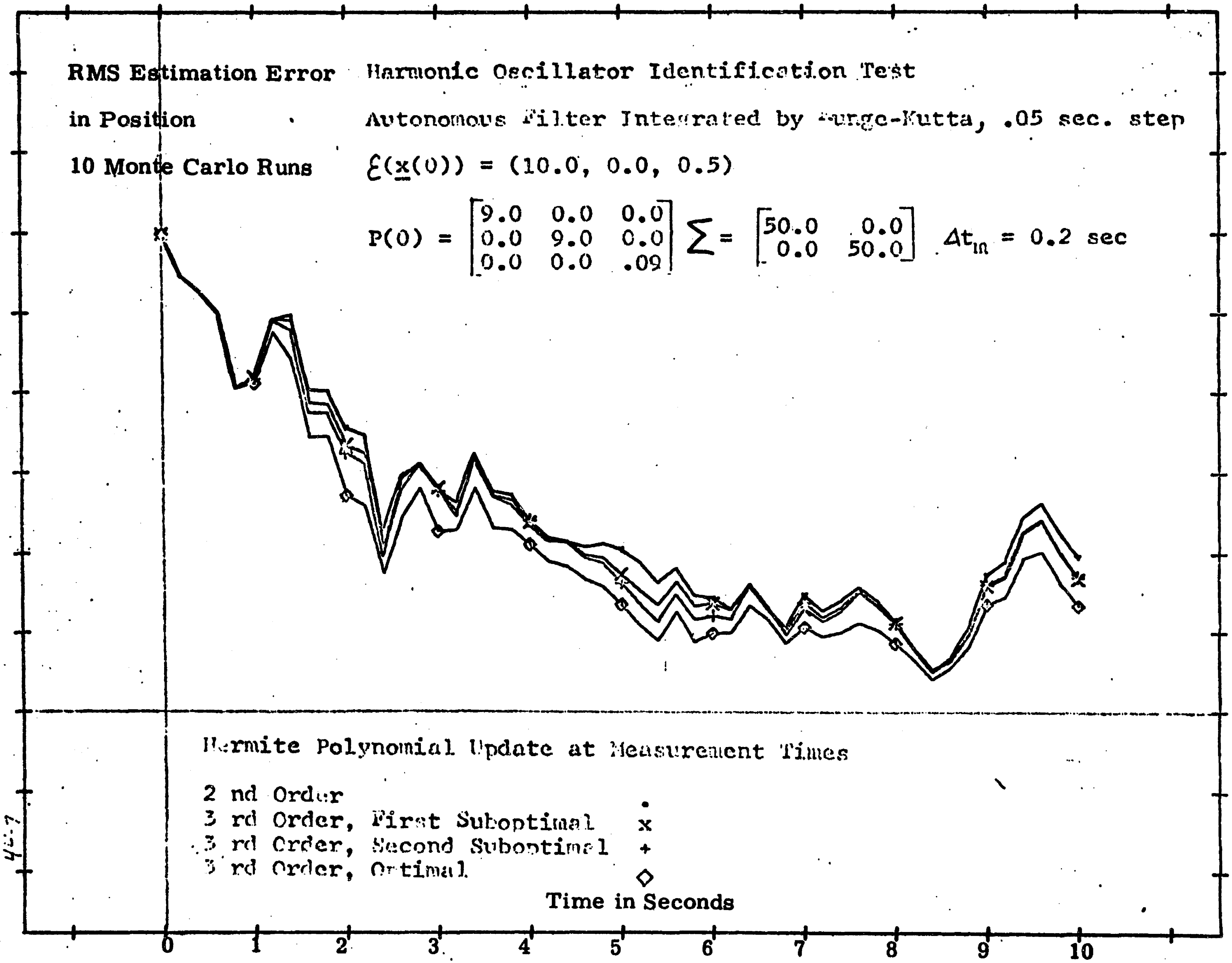

Figure 8.7 Suboptimal Nonlinear Filter Test - Position Error

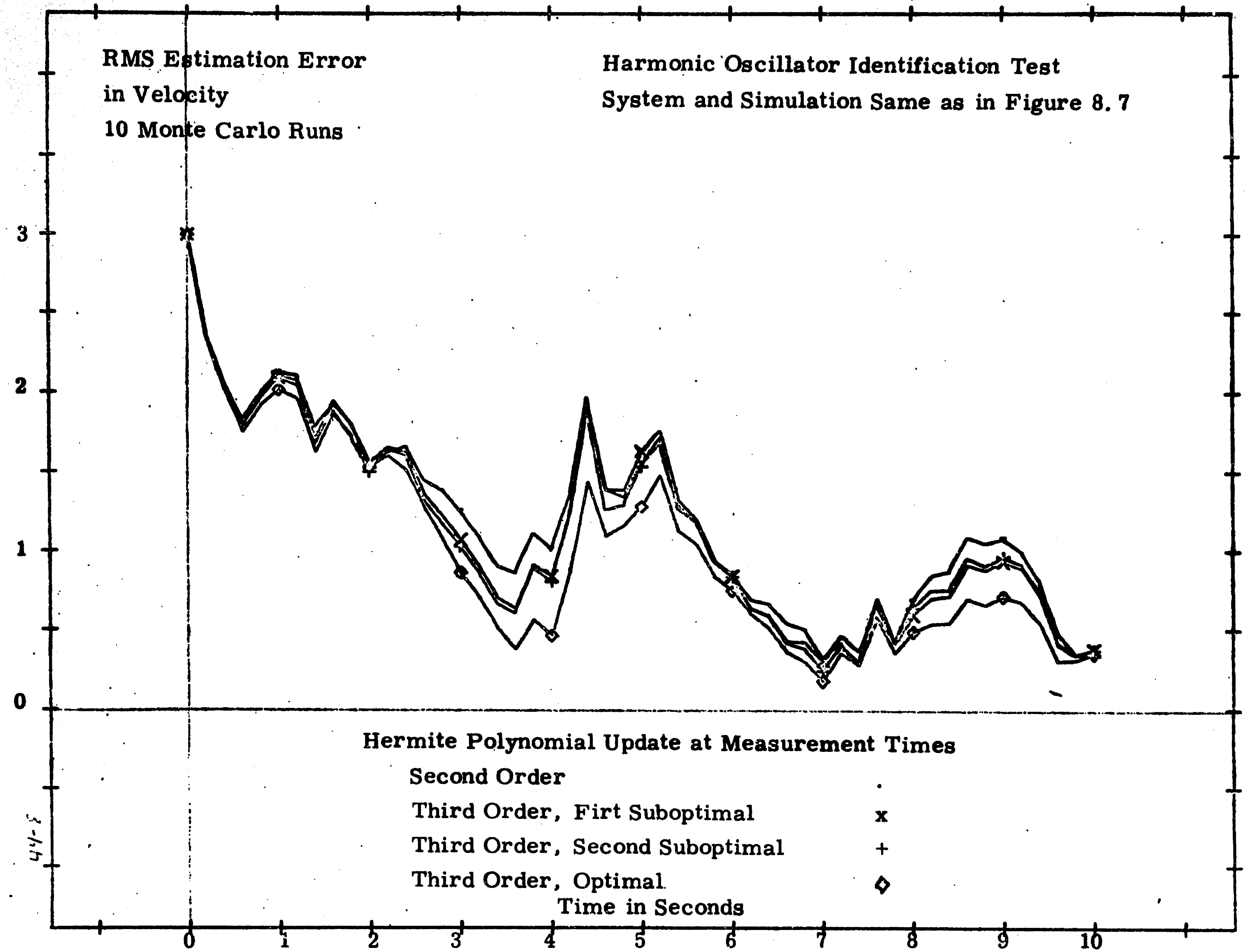

Figure 8.8 Suboptimal Nonlinear Filter Test - Velocity Error

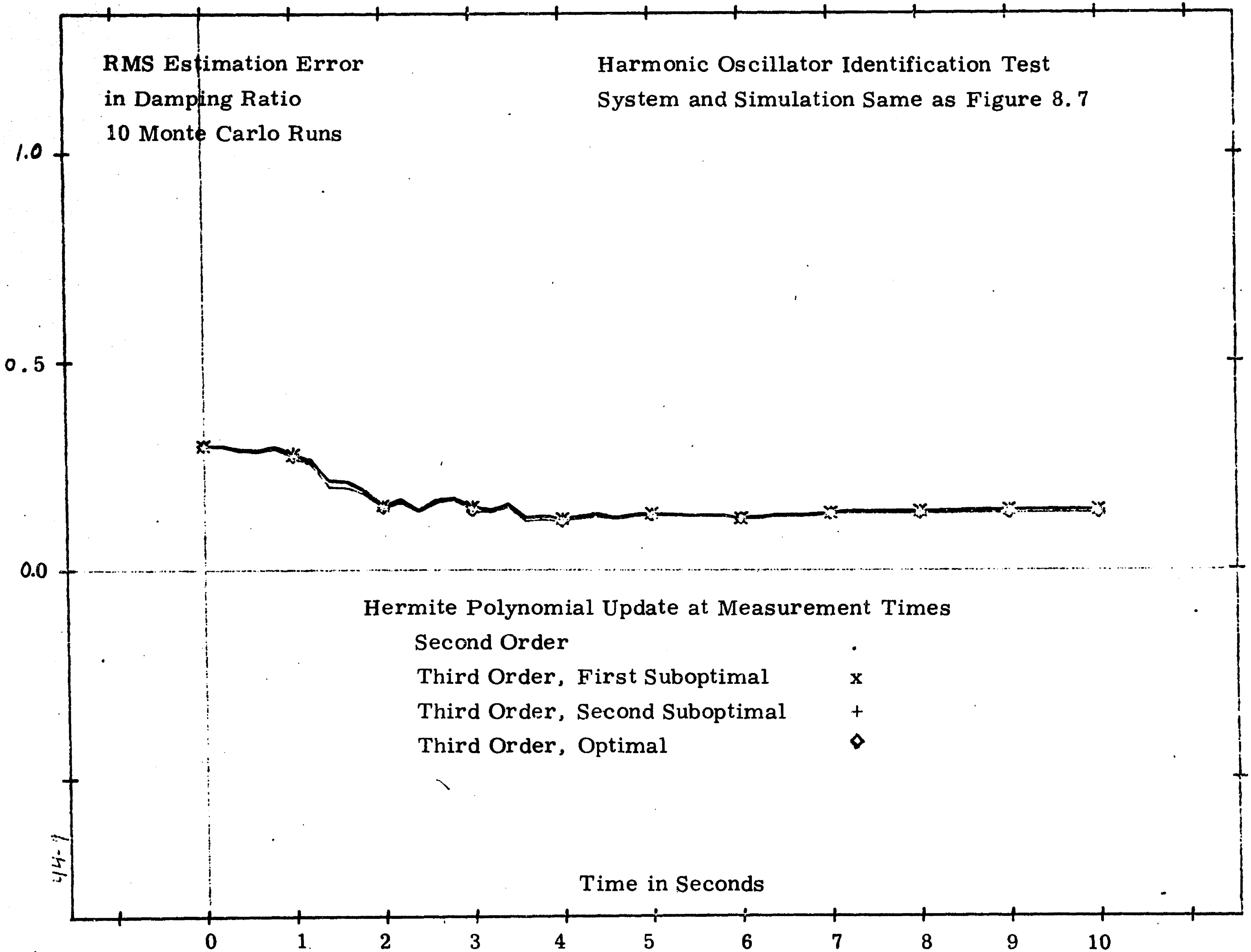

Figure 8.9 Suboptimal Non-Linear Filter Test - Damping Ratio Error

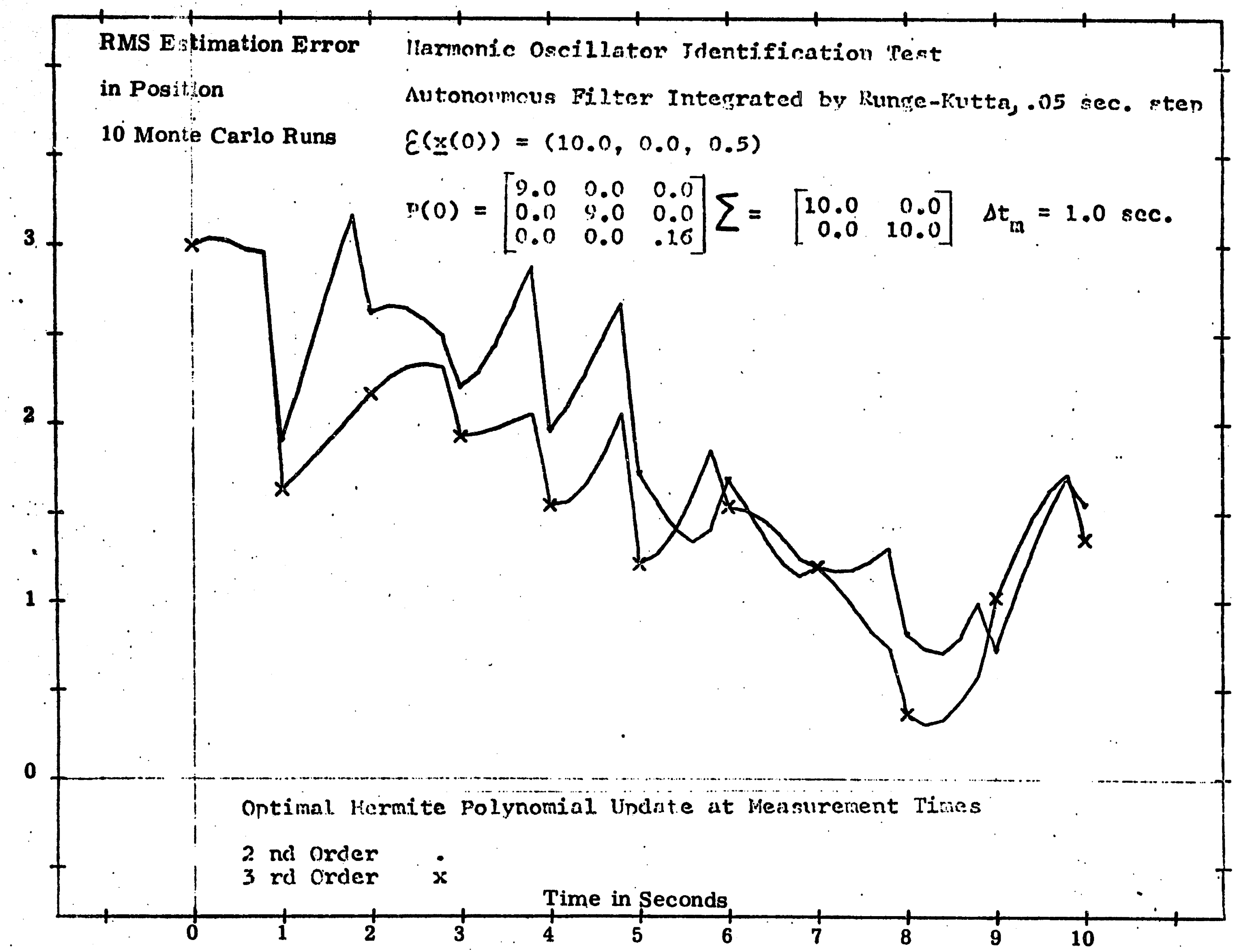

Figure 8. 10 Optimal Nonlinear Filter Test - Position Error

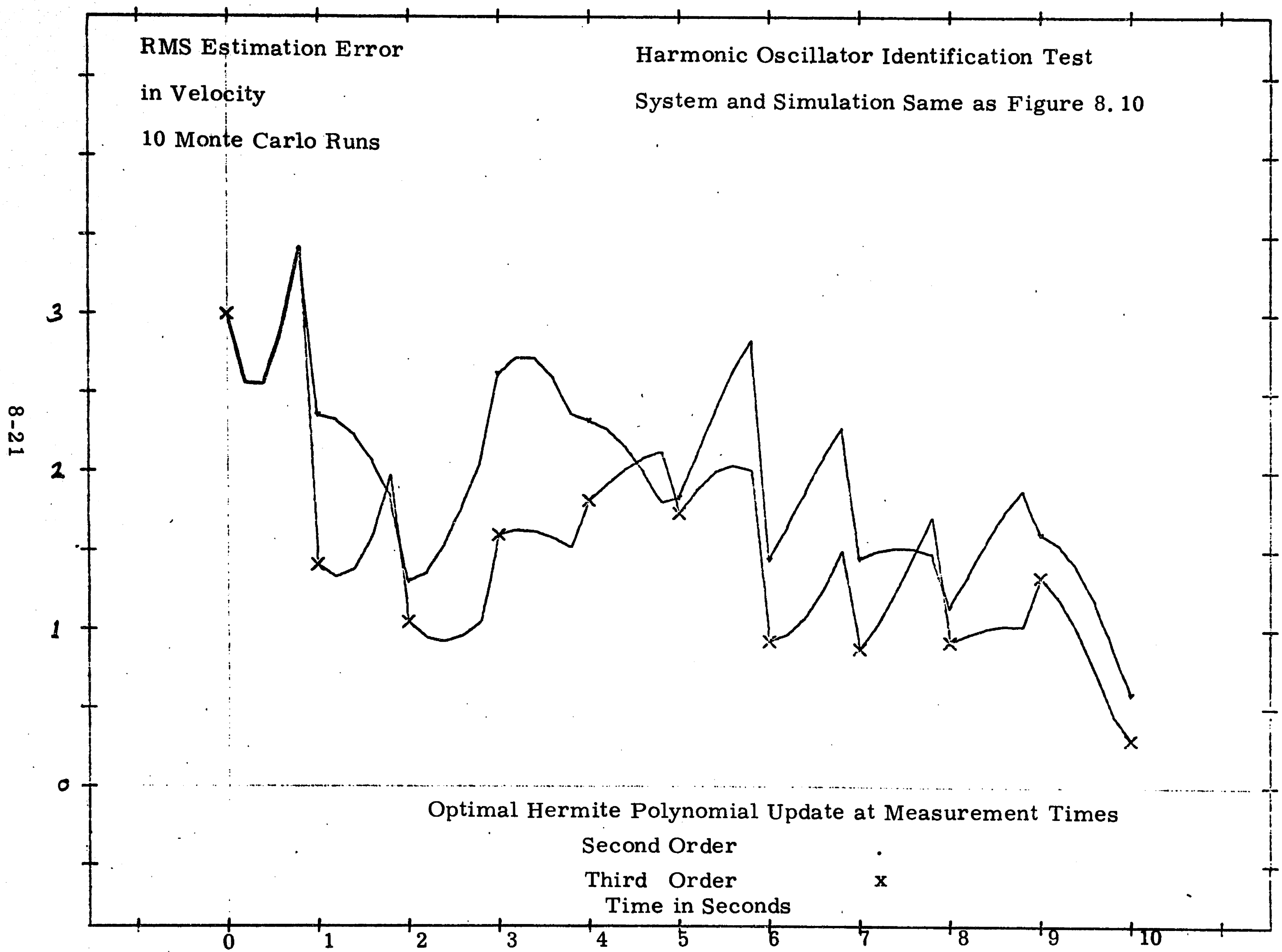

Figure 8. 11 Optimal Nonlinear Filter Test - Velocity Error

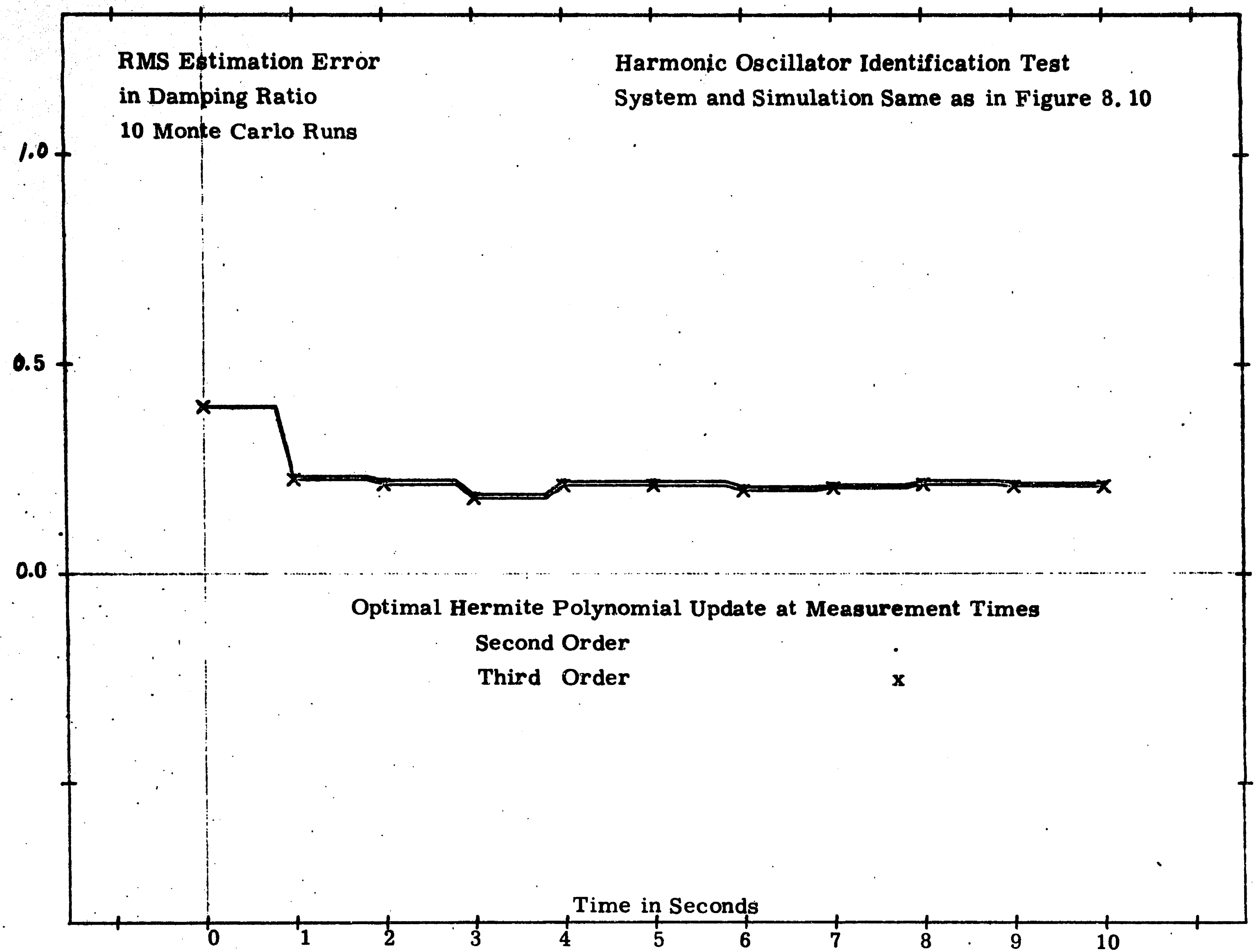

Figure 8. 12 Optimal Nonlinear Filter Test. Damping Ratio Error

8.3 The Orbital Landmark Navigation Problem

In the lunar landing mission, the Apollo spacecraft is first placed in a low altitude circular orbit about the moon. Subsequently the lunar module will separate, and make a soft landing on the surface of the moon by firing a rocket engine in a direction opposite to the velocity of the spacecraft. Prior to the descent phase of the mission, it is important to accurately determine the position and velocity of the spacecraft in lunar orbit.

One convenient method of obtaining information useful in orbit determination is to optically measure the angles between lunar surface landmarks and the stars at various times. The propagation of estimates of the position and velocity from the initial orbit injection data, and the updating of these estimates from discrete angular measurements are nonlinear estimation problems. Both the nonlinear system and measurement equations can be written as second order expansions, so that the filters derived in previous chapters can be directly applied.

For simplicity, the orbit will be confined to a plane.

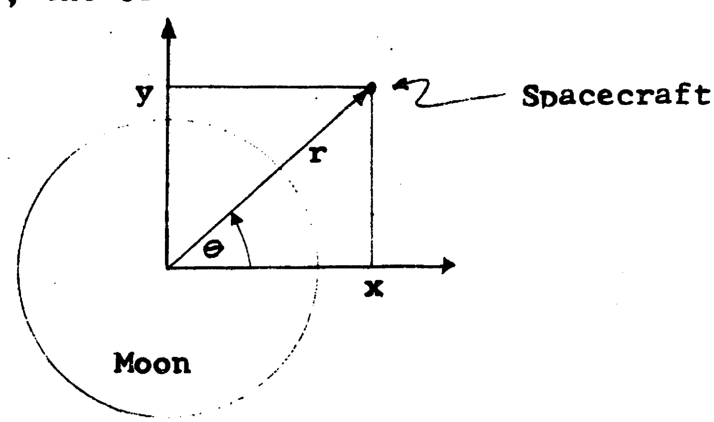

Figure 8.13 Lunar Coordinate Systems

The system equations are;

$$\frac{d^2r}{dt^2} = -\frac{\mu}{r^2} + \left(\frac{d\theta}{dt}\right)^2 r \tag{8.15}$$

$$\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0 \tag{8.16}$$

Alternatively, the system equations can be written in Cartesian coordinates.

$$\frac{d^2x}{dt^2} = \frac{-\mu x}{(x^2+y^2)^{3/2}} \tag{8.17}$$

$$\frac{d^2y}{dt^2} = \frac{-\mu y}{(x^2+y^2)^{3/2}} \tag{8.18}$$

Which coordinate system is most favorable (from the standpoint of estimation) depends on the nature of the probability distributions to be encountered. For example, if the error ellipsoid tends to be shaped like a cigar, as shown below, then the Cartesian system can best describe the density function with a small number of moments.

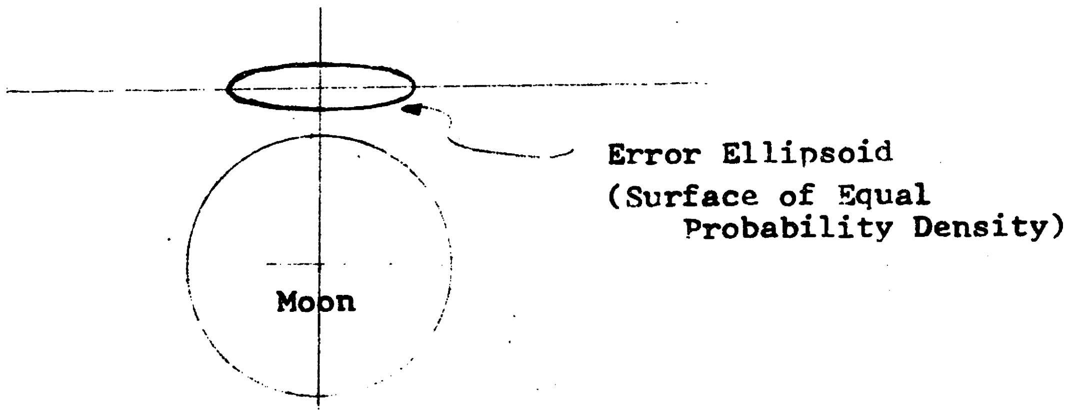

Figure 8.14 Cartesian Error Ellipsoid

On the other hand, if the error ellipsoid is sausage-shaped, then a polar coordinate description may yield a close fit with only first and second order terms.

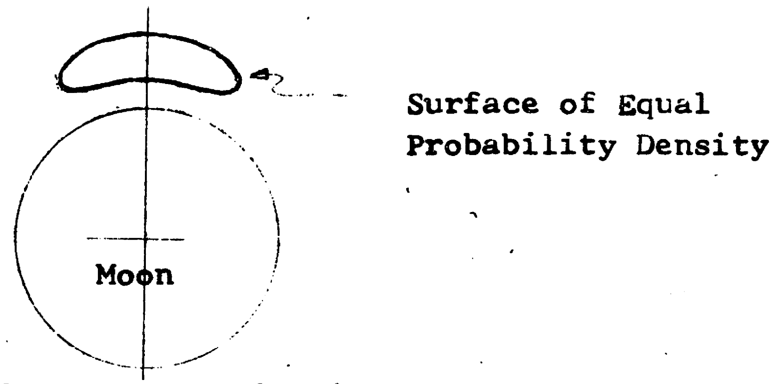

Figure 8.15 Polar Error Ellipsoid

The shape of the expected probability density, and the choice of the coordinate system together determine how good a fit can be obtained with second order gaussian statistics, and hence whether or not a third order nonlinear filter will be of substantial benefit.

For the orbital landmark navigation problem, the probability density of the estimation error turns out to be sausage-shaped as in Figure 8.15. For other reasons, however, a Cartesian coordinate frame is preferred for Apollo. Thus it is reasonable to suspect that a third order nonlinear filter may offer substantially improved performance.

In order to describe the planar orbital navigation problem as a set of first order differential equations, the following four dimensional state vector is defined.

$$x_1 = x \qquad x_3 = dx/dt$$
$$x_2 = y \qquad x_4 = dy/dt \qquad (8.19)$$

From (8.17) and (8.18), the equations of state are as follows.

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 = -\mu x_1/(x_1^2 + x_2^2)^{3/2}$$
$$\dot{x}_4 = -\mu x_2/(x_1^2 + x_2^2)^{3/2} \qquad (8.20)$$

These state equations can be approximated by a second order expansion about $\underline{m}$, the current estimate of $\underline{x}$, as follows.

$$\dot{x}_i \approx f_i(\underline{m}) + F_{ie}(\underline{m})\,(x_e - m_e) + A_{ief}(\underline{m})\,(x_e - m_e)(x_f - m_f) \qquad (8.21)$$

$$F_{ie}(\underline{m}) = \left.\frac{\partial f_i(\underline{x})}{\partial x_e}\right|_{\underline{x}=\underline{m}}$$

$$A_{ief}(\underline{m}) = \frac{1}{2}\left.\frac{\partial^2 f_i(\underline{x})}{\partial x_e \partial x_f}\right|_{\underline{x}=\underline{m}} \qquad (8.22)$$

Since $(x_e - m_e)(x_f - m_f)$ is symmetric in e and f, A_{ief} may be combined with A_{ife}.

Let $r = (m_1^2 + m_2^2)^{\frac{1}{2}}$ (8.23)

From (8.22), the terms in (8.21) are as follows.

$$f_1 = m_3$$
$$f_2 = m_4$$
$$f_3 = -\mu m_1/r^3$$
$$f_4 = -\mu m_2/r^3 \qquad (8.24)$$

$$F_{13} = 1$$
$$F_{24} = 1$$
$$F_{31} = -\mu(r^2 - 3\ m_1^2)/r^5$$
$$F_{32} = 3\mu m_1 m_2/r^5$$
$$F_{41} = F_{32}$$
$$F_{42} = -\mu(r^2 - 3\ m_2^2)/r^5 \qquad (8.25)$$

$$A_{311} = -.5\,\mu(-9m_1 r^2 + 15\ m_1^3)/r^7$$
$$A_{312} = -\ \mu(-3m_2 r^2 + 15\ m_1^2\ m_2)/r^7$$
$$A_{322} = -.5\,\mu(-3m_1 r^2 + 15\ m_1\ m_2^2)/r^7$$
$$A_{411} = .5\ A_{312}$$
$$A_{412} = 2\ A_{322}$$
$$A_{422} = -.5\,\mu(-9\ m_2\ r^2 + 15\ m_2^3)/r^7$$

All other F_{ie}, $A_{ief} = 0.$ (8.26)

The third order autonomous filter equations for this example are,

$$\frac{dm_i}{dt} = f_i + A_{ief}\,\mu 2_{ef}$$
$$\frac{d\mu 2_{ij}}{dt} = 2\left\{F_{ie}\mu 2_{je} + A_{ief}\,\mu 3_{jef}\right\}_s$$
$$\frac{d\mu 3_{ijk}}{dt} = 3\left\{F_{ie}\,\mu 3_{jke} + A_{ief}(\mu 2_{je}\,\mu 2_{kf} + \mu 2_{jf}\,\mu 2_{ke})\right\}_s \qquad (8.27)$$

The above are differential equations for four first moments, ten second moments, and twenty third moments. These could be solved on a digital computer by putting the F_{ie} and A_{ief} into arrays, and performing the indicated summations by repetitive

DO loops. This method is very easy to program, but executes slowly because of all the zero elements that are computed. Alternatively, (8.27) could be solved by programming these equations explicitly, omitting all the zero elements. This results in a longer program which is very difficult to write, because all the numerical addresses must be specified by the programmer and must be absolutely correct. This program will execute about twenty times as fast as the array type program.

Angular measurements from the stars to surface landmarks are taken as follows. After the spacecraft passes a specified central angle θ_m , according to its own guidance system, a landmark at θ_1 is sighted and the angle ϕ from the landmark to the reference coordinate system is measured. These relationships are shown below.

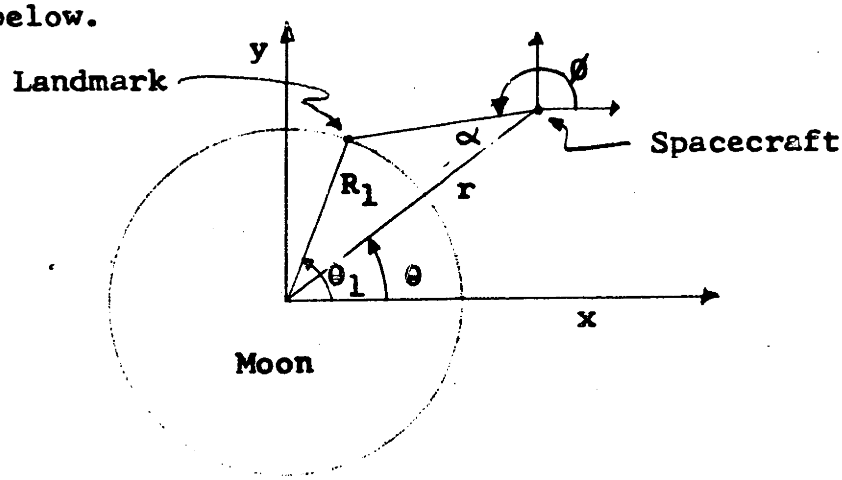

Figure 8.16 Landmark Measurement Geometry

The measurement ϕ can be expressed as follows;

$$\phi = \pi + \theta - \alpha$$
$$= \pi + \theta - \sin^{-1}\left(\frac{R_1 \sin(\theta_1 - \theta)}{\sqrt{R_1^2 + r^2 - 2 R_1 r \cos(\theta_1 - \theta)}}\right) \qquad (8.28)$$

Using the quantities available in a Cartesian coordinate navigation system, the above becomes,

$$r = \sqrt{x^2 + y^2}$$

$$\sin(\theta) = y/r$$

$$\cos(\theta) = x/r$$

$$\theta = \sin^{-1}(y/r) \quad \text{or} \quad \cos^{-1}(x/r)$$

$$\phi = \pi + \theta - \sin^{-1}\left(\frac{R_1(\sin(\theta_1)\cos(\theta)-\cos(\theta_1)\sin(\theta))}{\sqrt{R_1^2 + r^2 - 2R_1 r(\cos(\theta_1)\cos(\theta)+\sin(\theta_1)\sin(\theta))}} \right) \tag{8.29}$$

Equation (8.29) is in a useful form, because α will always be between $\pm \pi/2$, and the only ambiguity is in the angle θ, which may be $\pm 2n\pi$.

The measurement function may be expanded in a two term Taylor's series.

$$\phi \approx \phi(\underline{m}) + H_a(x_a - m_a) + M_{ab}(x_a - m_a)(x_b - m_b) \tag{8.30}$$

$$H = \begin{bmatrix} \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & 0 & 0 \end{bmatrix} \tag{8.31}$$

$$M = \begin{bmatrix} \frac{1}{2}\frac{\partial^2\phi}{\partial x^2} & \frac{\partial^2\phi}{\partial x \partial y} & 0 & 0 \\ 0 & \frac{1}{2}\frac{\partial^2\phi}{\partial y^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{8.32}$$

The partial derivatives in (8.31) and (8.32) can be computed analytically from (8.29), but a numerical solution is much simpler and easier to perform on a digital computer. Using (8.29), the following can be computed.

$$\underline{\phi}(x,y)$$

$$\Delta\underline{\phi} = \begin{bmatrix} \phi(x+\Delta x, y) - \phi(x,y) \\ \phi(x, y+\Delta y) - \phi(x,y) \\ \phi(x+\Delta x, y+\Delta y) - \phi(x,y) \\ \phi(x+2\Delta x, y) - \phi(x,y) \\ \phi(x, y+2\Delta y) - \phi(x,y) \end{bmatrix} \qquad (8.33)$$

For small Δx and Δy,

$$\Delta\underline{\phi} \approx \begin{bmatrix} \Delta x & 0 & 0 & \frac{1}{2}(\Delta x)^2 & 0 \\ 0 & \Delta y & 0 & 0 & \frac{1}{2}(\Delta y)^2 \\ \Delta x & \Delta y & \Delta x\Delta y & \frac{1}{2}(\Delta x)^2 & \frac{1}{2}(\Delta y)^2 \\ 2\Delta x & 0 & 0 & 2(\Delta x)^2 & 0 \\ 0 & 2\Delta y & 0 & 0 & 2(\Delta y)^2 \end{bmatrix} \begin{bmatrix} d\phi/dx \\ d\phi/dy \\ d^2\phi/dxdy \\ d^2\phi/dx^2 \\ d^2\phi/dy^2 \end{bmatrix} \qquad (8.34)$$

The above equation is written in symbolic notation as,

$$\Delta\underline{\phi} = Q\ \underline{\delta} \qquad (8.35)$$

Therefore,

$$\underline{\delta} = Q^{-1}\ \Delta\underline{\phi} \qquad (8.36)$$

The matrix Q^{-1} need be computed only once, for a given Δx and Δy. The above method of computing the partial derivatives was used in the simulation tests, with Δx and Δy = 100 feet. Although both the measurement first and second partial derivatives were computed, the second derivatives were too small to have an appreciable effect for the orbital position uncertainties encountered. For this reason, linearized measurement updating methods were used in the simulation examples.

The actual measurement is modeled as having an additive gaussian random error. That is,

$$\phi = \phi(\underline{x}) + \xi$$
$$\xi = N(0.0, \Sigma) \tag{8.37}$$

8.4 Simulation Description and Results

The system (8.15),(8.16), autonomous filter, (8.27), and angular measurements (8.37) were simulated on an IBM 360-75 digital computer. Linearized Kalman filters, and optimal second and third order filters as described in Table 5.1 were tested on the system, using both single runs and Monte Carlo procedures. The autonomous filter was programmed in both the slow simple indexed form, and the fast complex explicit form. The explicit form, which consists of about 400 compiler statements, was programmed and punched by another computer program, which made easy the task of getting all the perhaps 3000 numerical addresses absolutely correct.

The initial example simulated was that of a vehicle in a 100 mile circular orbit, with two landmarks 180 degrees apart on the surface of the moon. Measurements were taken every other orbit, since the astronauts' time is at a premium in this part of the flight, and because this makes the filter test more rigorous. Three measurements were taken at each landmark pass, as shown below in Figure 8.17.

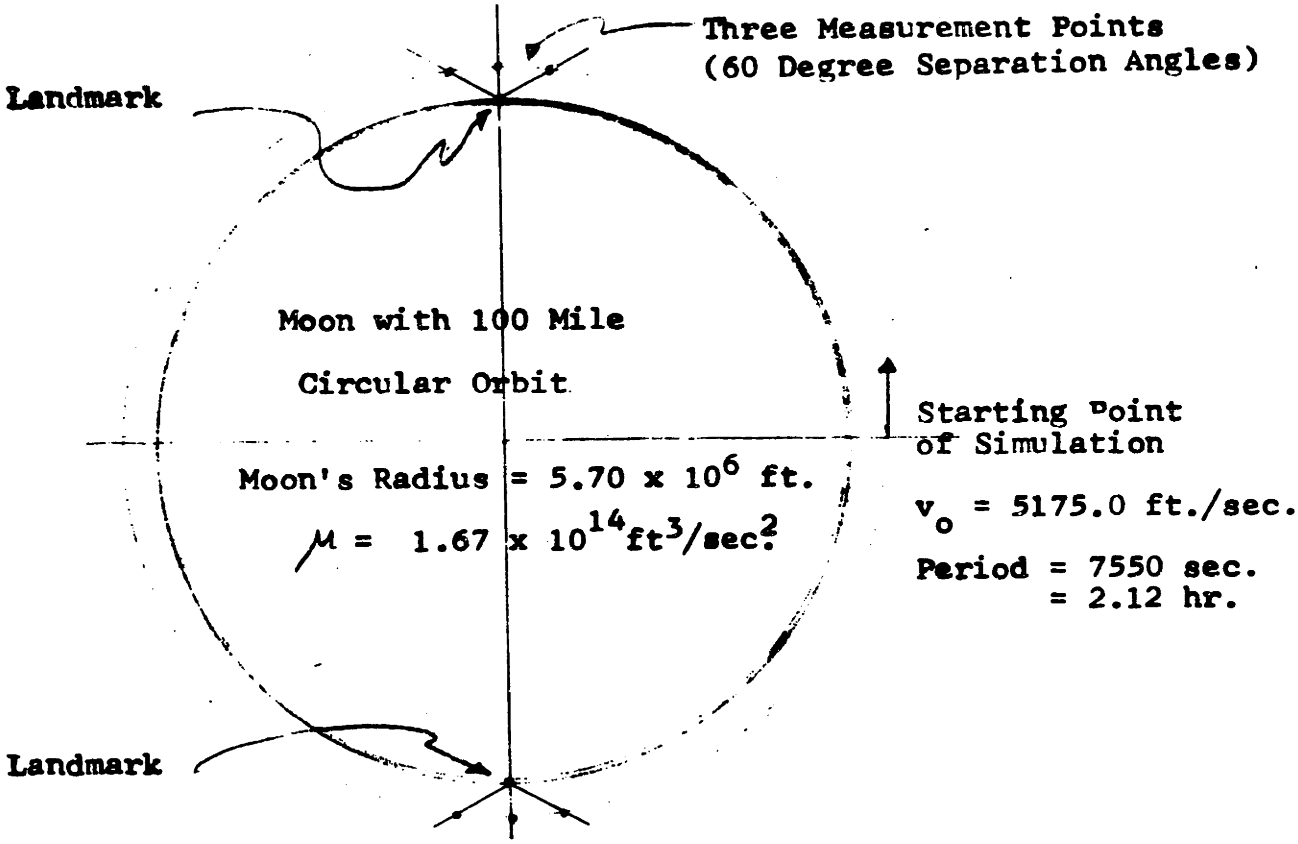

Figure 8.17 100 Mile Lunar Orbit

The filter initial conditions were as follows.

$$m_i = \begin{bmatrix} 6.228 \times 10^6 \text{ ft.} \\ 0.0 \text{ ft.} \\ 0.0 \text{ ft./sec.} \\ 5.175 \times 10^3 \text{ ft./sec.} \end{bmatrix} \qquad (8.38)$$

$$\mu 2_{ij} = \begin{bmatrix} 1.0 \times 10^6 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 \times 10^6 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \qquad (8.39)$$

$$\mu 3_{ijk} = 0 \quad \text{all } i,j,k. \qquad (8.40)$$

The measurement variance was .000016 rad^2. The true initial states used for the Monte Carlo runs were generated from the mean (8.38) plus a set of normal random numbers with zero mean and covariance approximately given by (8.39). These numbers were generated by the pseudo-random number generator previously described, and adjusted to zero mean and variance as described by the main diagonal of (8.39).

Figures (8.19) - (8.22) show the RMS estimation error for a ten run Monte Carlo sequence. The linearized Kalman filter, and optimal second and third order filters were simulated without measurements. This autonomous filter test shows the oscillatory nature of the system, and shows the slight improvement in accuracy in the optimal second and third order autonomous filters. These and subsequent error plots have been resolved into local vertical coordinates; that is, into altitude and range position and velocity errors.

Figures (8.23) - (8.26) show the results of a similar simulation with a set of measurements as described in Figure 8.17 performed every other orbit. The measurement residues used in the nonlinear part of the optimal third order update were limited at ± 1.5 times their expected standard deviations, as described by equation (8.14). The maximum value of the measurement first derivative was about 2.0×10^{-6} rad./ft. , while the maximum value of the second derivative was about 4.0×10^{-12} rad./ft.^{2}. For the position errors encountered (up to 10,000 feet), the second order terms were less than one per cent of the first order terms. For this reason, all measurement updating was done using a linearized measurement model.

These results show the nonlinear filters to have a slight improvement in performance over the linearized Kalman filter for this example.

The above problem was also simulated using a set of filters operating in polar coordinates. The results of this simulation, given in Figures (8.27) - (8.30), are directly comparable to

Figures (8.23) - (8.26), since the simulation data, random noise sequences, and output coordinates are identical. These results show that the polar coordinate filters have somewhat better accuracy, and that the nonlinear filters do not show any advantage in the polar coordinate frame for this example.

Tables 8.1 - 8.4 show why this is so. These tables show the average estimated and actual moments for the ten run sequence for all three filters in the Cartesian frame, and the third order filter in the polar frame. The estimated third moments in the polar frame are substantially smaller than in the Cartesian frame. The actual third moments show a less clear trend, but a close correspondence should not necessarily be expected from a ten run test. A much larger number of runs was required in the scalar filter tests to get a close correspondence between actual and estimated second and third moments. Such a large number of runs would be very expensive for this much more complex multidimensional filter, as the ten run tests required about eight minutes on the 360-75 computer.

Since the third moments are apparently much smaller in the polar frame, it is logical that their inclusion in the estimation process should lead to a lesser improvement than in the Cartesian frame.

From these results, it appears that use of a third order filter is most advantageous when the third moments of the density function are large. This will happen in the orbital estimation problem when the error ellipsoid becomes large, so that a linear

```
ESTIMATED SECOND MOMENTS AT T = 40000.000

11  2.6826 E  6   12  6.3836 E  5   13 -8.1345 E  2   14  1.9417 E  3   22  5.8361 E  5
23  1.2872 E  2   24  5.9413 E  2   33  4.9503 E- 1   34 -5.1254 E- 1   44  1.5106 E  0


MEAN ESTIMATION ERROR AT T = 40000.0000

   -7.604 E  2 -8.648 E  2 -2.079 E- 1 -8.645 E- 1


ACTUAL SECOND MOMENTS AT T = 40000.000

11  2.8723 E  6   12  2.1390 E  6   13  3.3063 E  1   14  2.5788 E  3   22  2.4695 E  6
23  6.1627 E  2   24  2.1966 E  3   33  4.0897 E- 1   34  2.1573 E- 1   44  2.4229 E  0
```

Table 8.1 Estimated and Actual Moments, Linearized Kalman Filter Cartesian Coordinate Frame

```
ESTIMATED SECOND MOMENTS AT T = 40000.000

11  2.6999 E  6   12  6.4539 E  5   13 -8.1722 E  2   14  1.9557 E  3   22  5.8669 E  5
23  1.2739 E  2   24  5.9999 E  2   33  4.9599 E- 1   34 -5.1545 E- 1   44  1.5220 E  0


MEAN ESTIMATION ERROR AT T = 40000.0000

   -4.251 E  2 -5.487 E  2 -1.638 E- 1 -5.342 E- 1


ACTUAL SECOND MOMENTS AT T = 40000.000

11  2.4073 E  6   12  1.6864 E  6   13 -5.0347 E  1   14  2.0938 E  3   22  1.9897 E  6
23  5.2261 E  2   24  1.7134 E  3   33  3.9331 E- 1   34  1.2568 E- 1   44  1.9150 E  0
```

Table 8.2 Estimated and Actual Moments, Optimal Second Order Filter Cartesian Coordinate Frame

ESTIMATED SECOND MOMENTS AT T = 40000.000

11	2.6097 E 6	12	6.4239 E 5	13	-7.7357 E 2	14	1.8911 E 3	22	6.4075 E 5
23	1.6883 E 2	24	6.1320 E 2	33	5.0468 E- 1	34	-4.7320 E- 1	44	1.4805 E 0

MEAN ESTIMATION ERROR AT T = 40000.0000

-3.673 E 2 -5.515 E 2 -1.892 E- 1 -4.948 E- 1

ACTUAL SECOND MOMENTS AT T = 40000.000

11	1.7817 E 6	12	1.2006 E 6	13	-7.8546 E 1	14	1.5154 E 3	22	1.5600 E 6
23	4.7747 E 2	24	1.2583 E 3	33	3.8273 E- 1	34	9.8311 E- 2	44	1.3815 E 0

ESTIMATED THIRD MOMENTS AT T = 40000.000

111	-1.2323 E 7	112	-2.5115 E 6	113	6.1582 E 3	114	-8.9380 E 3	122	3.6575 E 6
123	4.5499 E 3	124	-8.2967 E 2	133	-1.4828 E- 1	134	5.2785 E 0	144	-6.2827 E 0
222	7.0841 E 6	223	3.5714 E 3	224	4.4079 E 3	233	2.0909 E- 1	234	4.1163 E 0
244	5.4113 E- 1	333	3.1603 E- 4	334	6.3556 E- 5	344	4.6301 E- 3	444	-4.1624 E- 3

ACTUAL THIRD MOMENTS AT T = 40000.000

111	-1.0516 E 9	112	-1.6589 E 9	113	-5.6854 E 5	114	-1.3286 E 6	122	-2.3738 E 9
123	-7.5378 E 5	124	-1.8541 E 6	133	-2.2783 E 2	134	-5.6697 E 2	144	-1.5353 E 3
222	-3.4521 E 9	223	-1.1239 E 6	224	-2.6310 E 6	233	-3.8320 E 2	234	-8.3247 E 2
244	-2.0548 E 3	333	-1.4582 E- 1	334	-2.8148 E- 1	344	-6.0275 E- 1	444	-1.7124 E 0

Table 8.3 Estimated and Actual Moments - Third Order Optimal Filter Cartesian Coordinate Frame

ESTIMATED SECOND MOMENTS AT T = 40000.00

11	3.2000 E 5	12	2.0867 E 4	13	-4.2098 E 1	14	-5.3722 E 2	22	2.6127 E 6
23	1.5613 E 2	24	-5.6362 E 1	33	7.9318 E- 2	34	7.1128 E- 2	44	9.0214 E- 1

MEAN ESTIMATION ERROR AT T = 40000.000

-7.311 E 1 1.636 E 2 1.093 E- 1 1.268 E- 1

ACTUAL SECOND MOMENTS AT T = 40000.000

11	4.4448 E 5	12	-1.2811 E 5	13	-2.0028 E 2	14	-7.5245 E 2	22	2.9322 E 6
23	4.1679 E 2	24	1.9687 E 2	33	1.8946 E- 1	34	3.3790 E- 1	44	1.2740 E 0

ESTIMATED THIRD MOMENTS AT T = 40000.00

111	1.5505 E 5	112	-4.4485 E 5	113	-8.9833 E 1	114	-1.8519 E 2	122	-4.0090 E 4
123	2.0203 E 1	124	7.4776 E 2	133	1.0183 E- 2	134	1.4126 E- 1	144	1.8494 E- 1
222	-3.3355 E 5	223	-7.6317 E 1	224	6.5490 E 1	233	-3.4262 E- 3	234	-3.4782 E- 2
244	-1.2569 E 0	333	-1.3295 E- 6	334	-1.6843 E- 5	344	-2.2118 E- 4	444	-9.9280 E- 5

ACTUAL THIRD MOMENTS AT T = 40000.000

111	2.3561 E 8	112	-6.1144 E 8	113	-2.0751 E 5	114	-3.9456 E 5	122	7.0377 E 8
123	4.7188 E 5	124	1.0307 E 6	133	1.5438 E 2	134	3.4770 E 2	144	6.6068 E 2
222	-2.5008 E 9	223	-4.0237 E 5	224	-1.1709 E 6	233	-2.5567 E 2	234	-7.9566 E 2
244	-1.7378 E 3	333	-9.4535 E- 2	334	-2.5921 E- 1	344	-5.8255 E- 1	444	-1.1061 E 0

Table 8.4 Estimated and Actual Moments - Third Order Optimal Filter Polar Coordinate Frame

approximation to the gravity field inside the error ellipsoid is no longer a good approximation to reality.

To test this hypothesis, another example using a spacecraft in a 500 mile lunar orbit was simulated. Again, two landmarks 180 degrees apart were used, with measurements taken every other orbit. Two measurements of variance .000025 rad.^{2} were taken with each pass over the landmark, as shown in Figure 8.18.

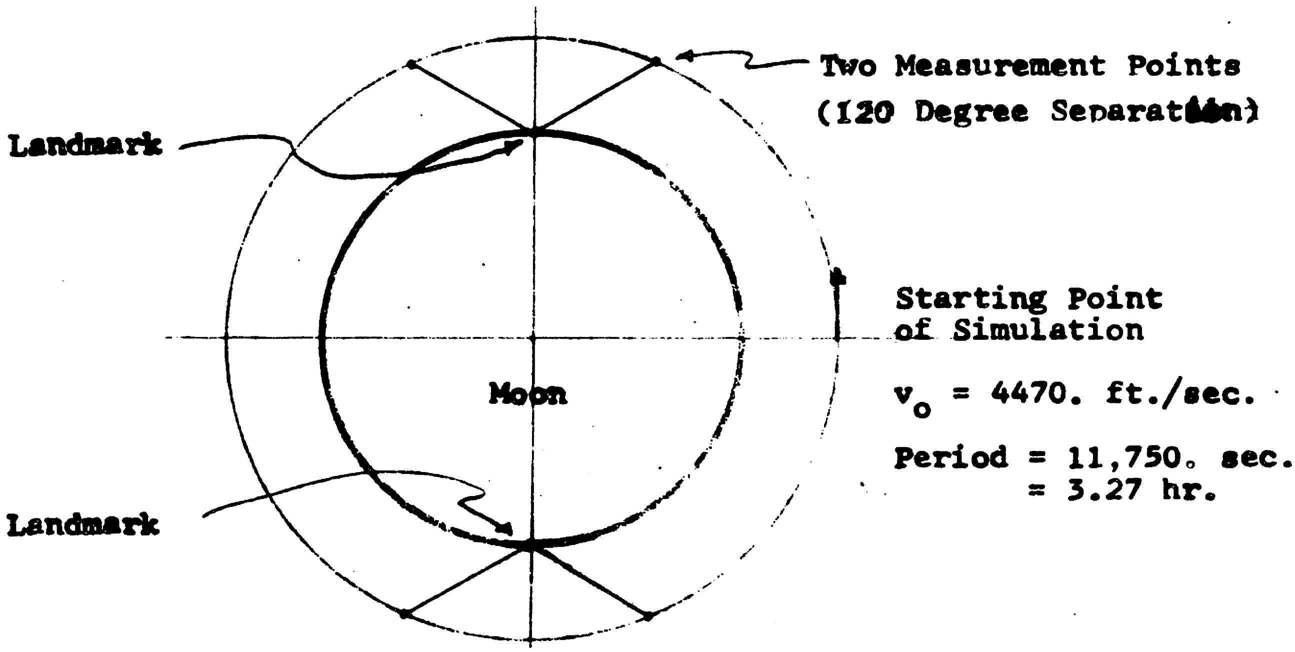

Figure 8.18 500 Mile Lunar Orbit

The filter initial conditions were as follows.

$$\underline{\underline{x}} = \begin{bmatrix} 8.34 \times 10^6 & \text{ft.} \\ 0.0 & \text{ft.} \\ 0.0 & \text{ft./sec.} \\ 4.47 \times 10^3 & \text{ft./sec.} \end{bmatrix} \qquad (8.41)$$

$$\mu 2' = \begin{bmatrix} 2.5 \times 10^7 & 0.0 & 0.0 & 0.0 \\ 0.0 & 2.5 \times 10^7 & 0.0 & 0.0 \\ 0.0 & 0.0 & 4.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 4.0 \end{bmatrix} \qquad (8.42)$$

$$\mu^3{}_{ijk} = 0 \quad \text{All } i,j,k. \tag{8.43}$$

First, a single run test was conducted for 300,000. seconds, starting with the true state at a one sigma error from the estimate. The altitude and range error for the linearized Kalman, second and third order optimal nonlinear filters are shown in Figures (8.31) - (8.36). For this example, the linearized Kalman filter diverges, while the second and third order nonlinear filters converge. Here the third order filter has a clear advantage in accuracy.

The same example was rerun starting with a two sigma initial error. The altitude and range error histories for the linearized Kalman filter, second order optimal filter, and third order optimal filters with measurement residues limited at .5, 1.0, and 1.5 sigma are given in Figures (8.37) - (8.46). In this case only the third order filter converges to near the true orbit. Limiting the measurement residues at one sigma appears to give the best performance, with under and overlimiting leading to bias errors and erratic behavior.

It should be emphasized that these are one run tests, and therefore their results are not conclusive. Nevertheless, they show the optimal third order nonlinear filter to have substantially better stability and accuracy than the second order filters, for this very nonlinear example.

Although the computer time required (418 sec. of 360-75 for one run of third order filter test for this example) for a Monte Carlo test of this example is prohibitive, the computation does not appear to be prohibitive for a real time spaceborne computer. The current unoptimized 360-75 program simulates the third order filter at about 700 times real time, so it does not appear unreasonable that future navigation computers will have sufficient capacity for computations of this type.

In summary, a linearized Kalman filter eventually diverges when used for orbital landmark navigation. This divergence is aggravated by large initial errors and a low data rate. The optimal third order filter eliminated the divergence for all cases tested.

It should be pointed out that the possibility exists of making the Kalman filter stable by adding fictitious noise driving the state to the system model. This would keep the covariance matrix larger, and can be used to prevent Kalman filter divergence. The addition of random noise into the model helps to account for modeling errors. Since the second order expansion used for the third order filter is not an exact system description either, addition of fictitious noise driving the state might improve this filter also. Since the latter filter uses a more precise system description, one would expect the "best" fictitious noise level to be smaller than for the linearized Kalman filter.

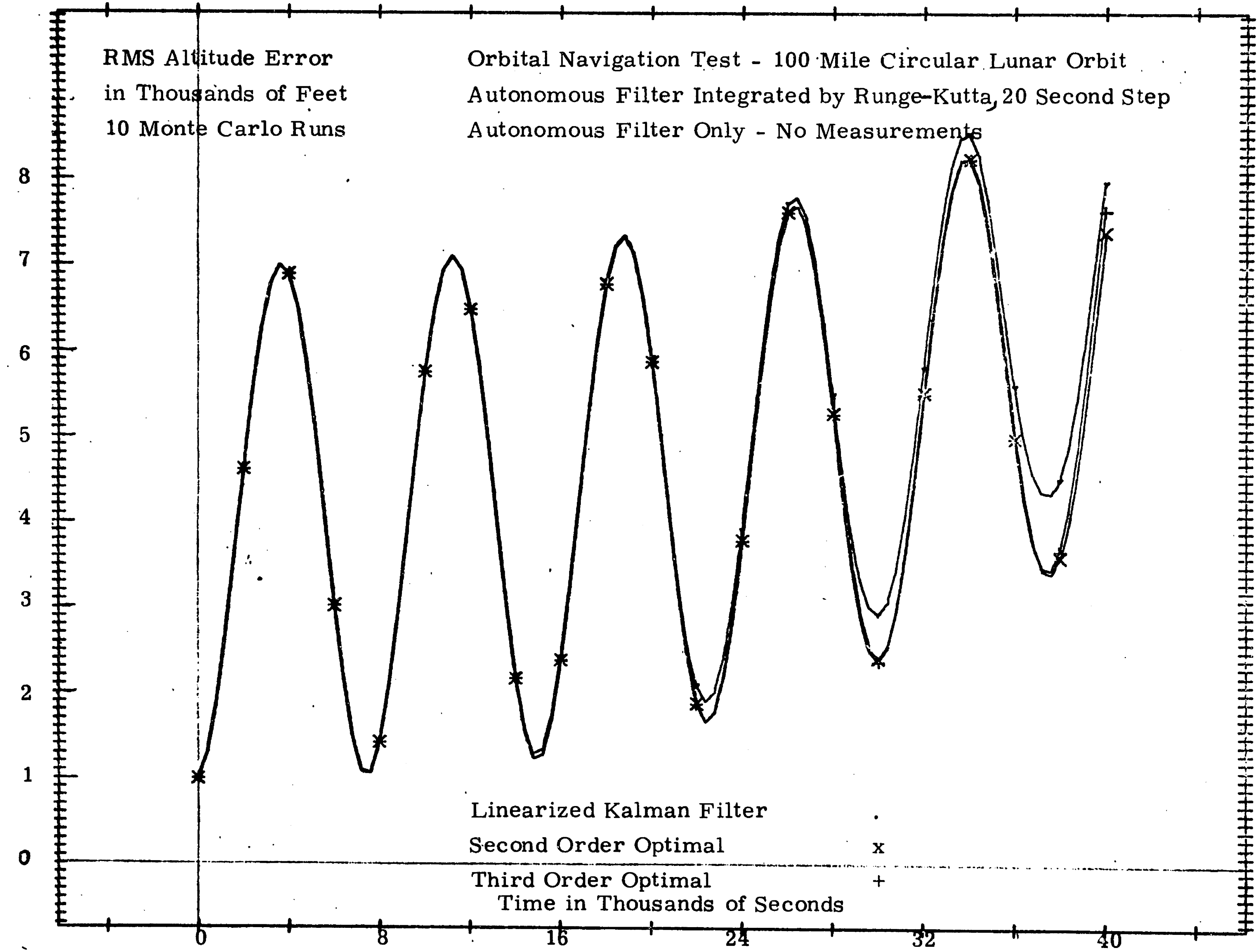

Figure 8. 19 Orbital Navigation Test - Altitude Error

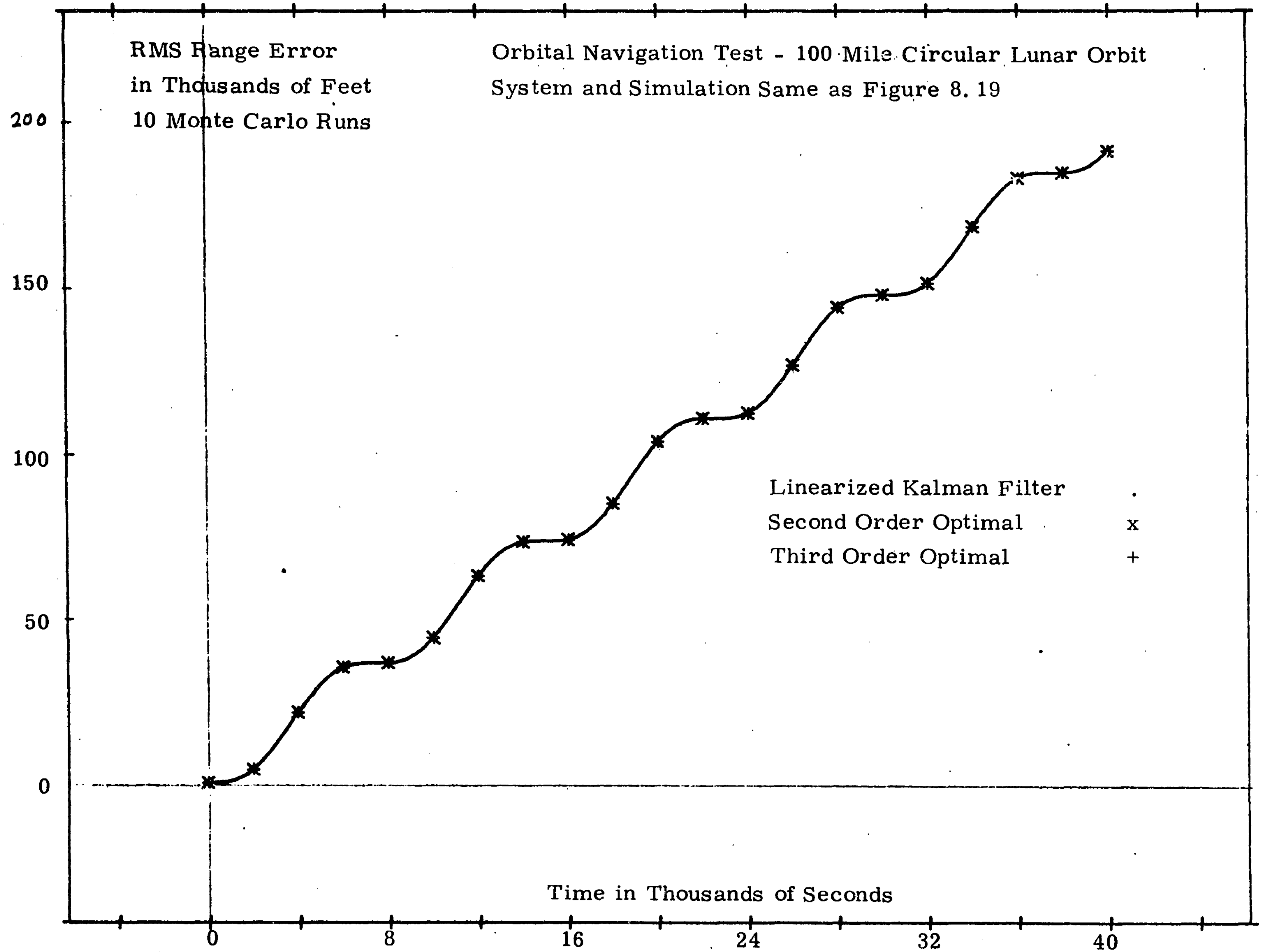

Figure 8. 20 Orbital Navigation Test - Range Error

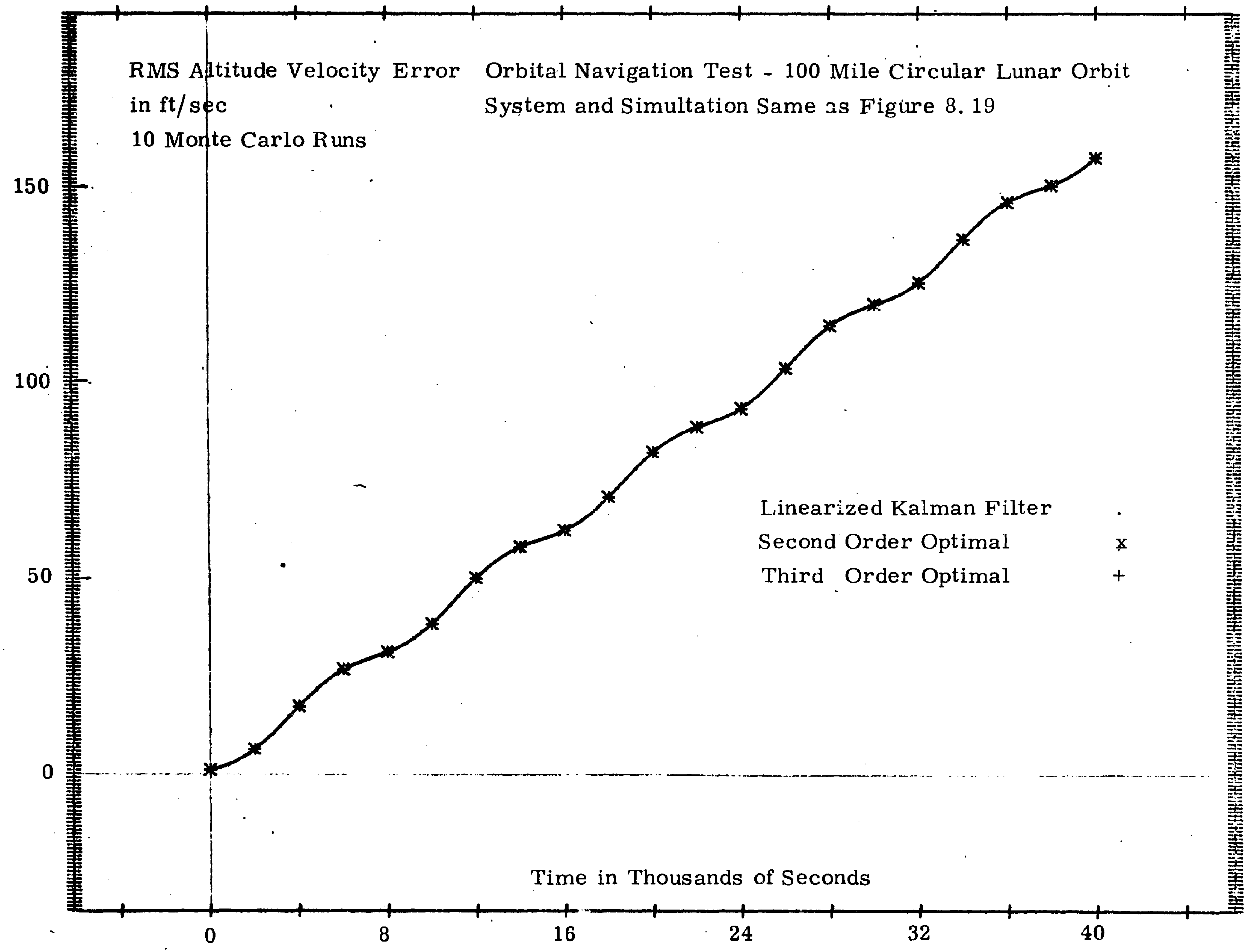

Figure 8.21 Orbital Navigation Test - Altitude Velocity Error

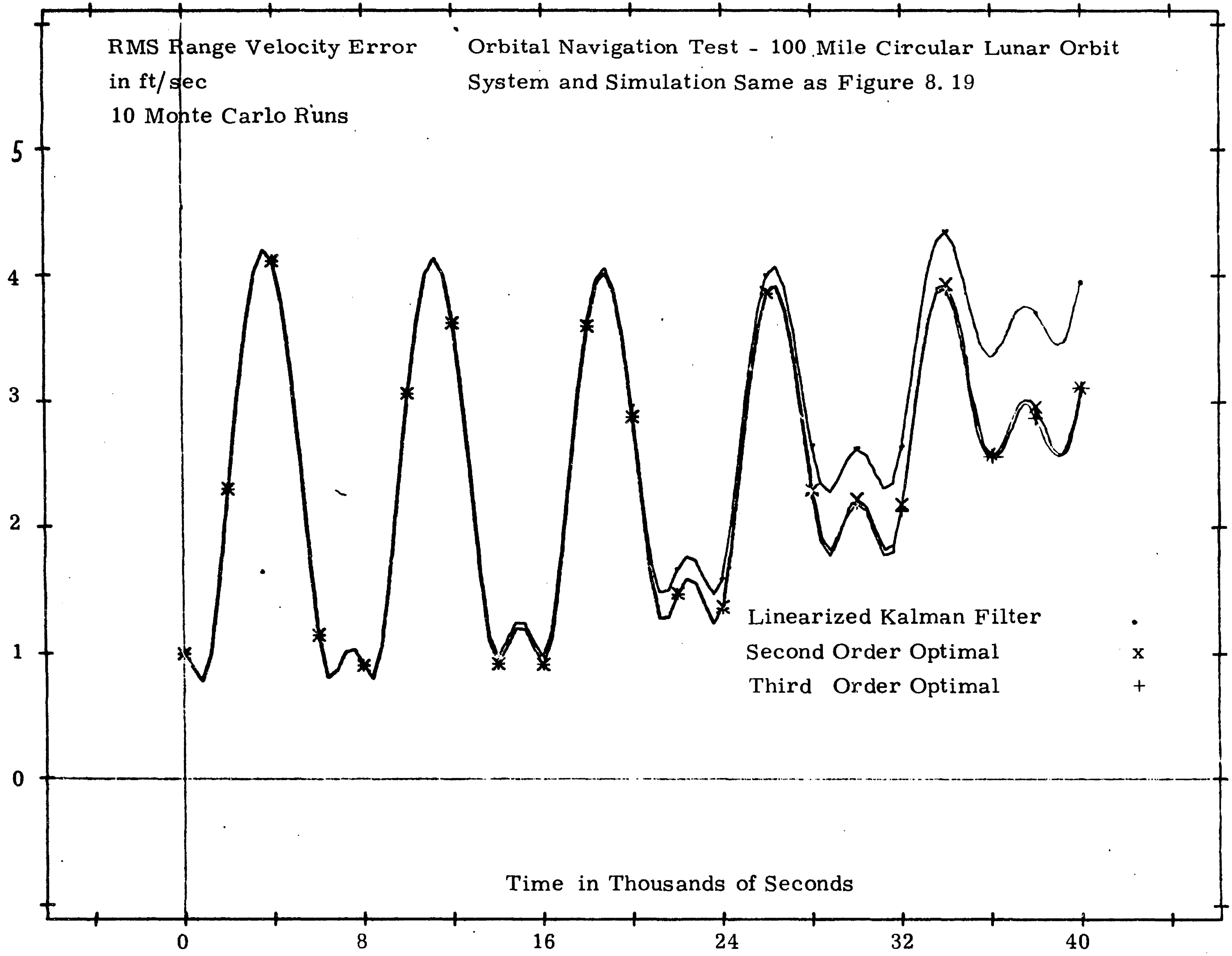

Figure 8. 22 Orbital Navigation Test - Range Velocity Error

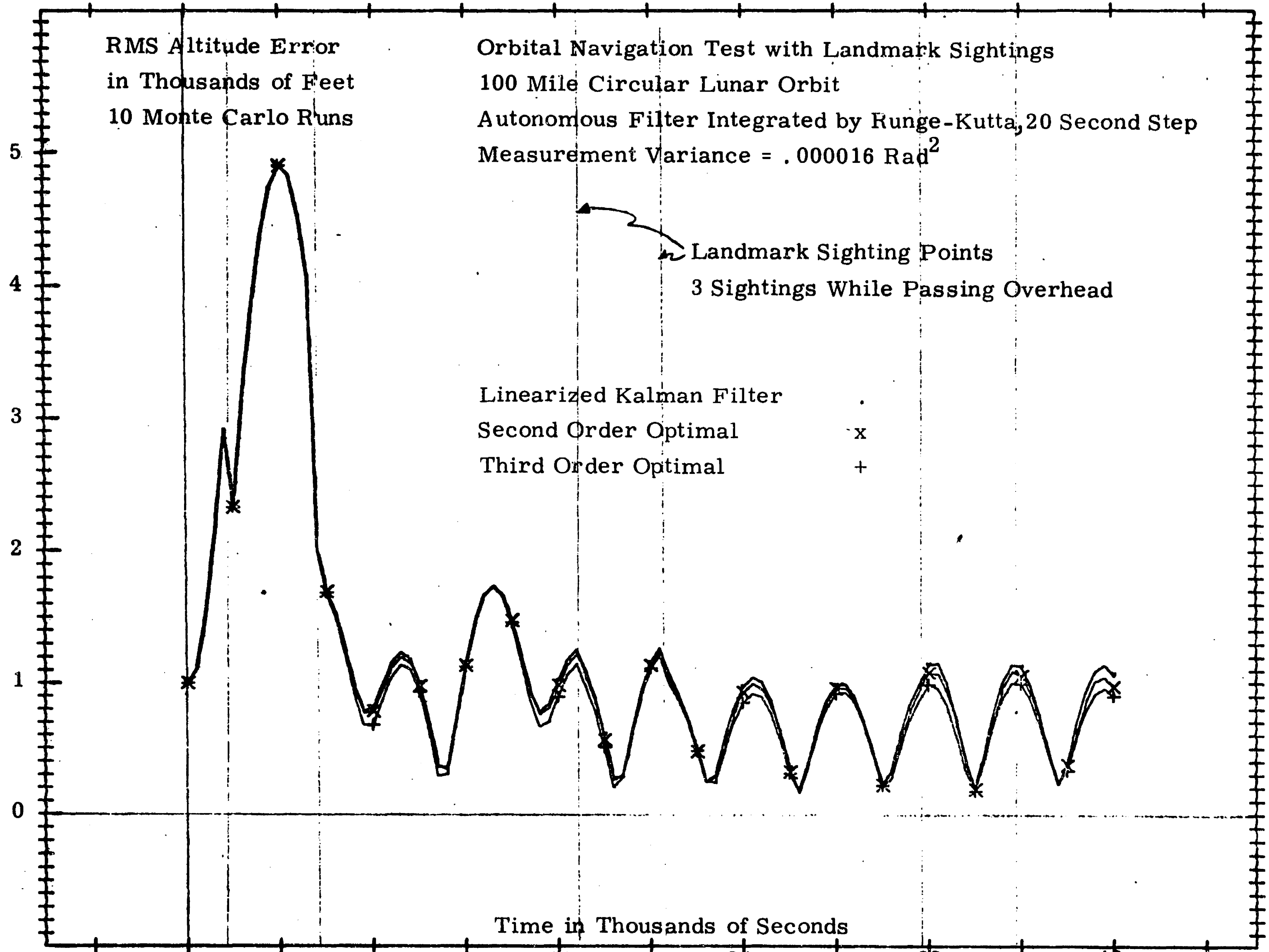

Figure 8.23 Orbital Landmark Navigation Test - Altitude Error

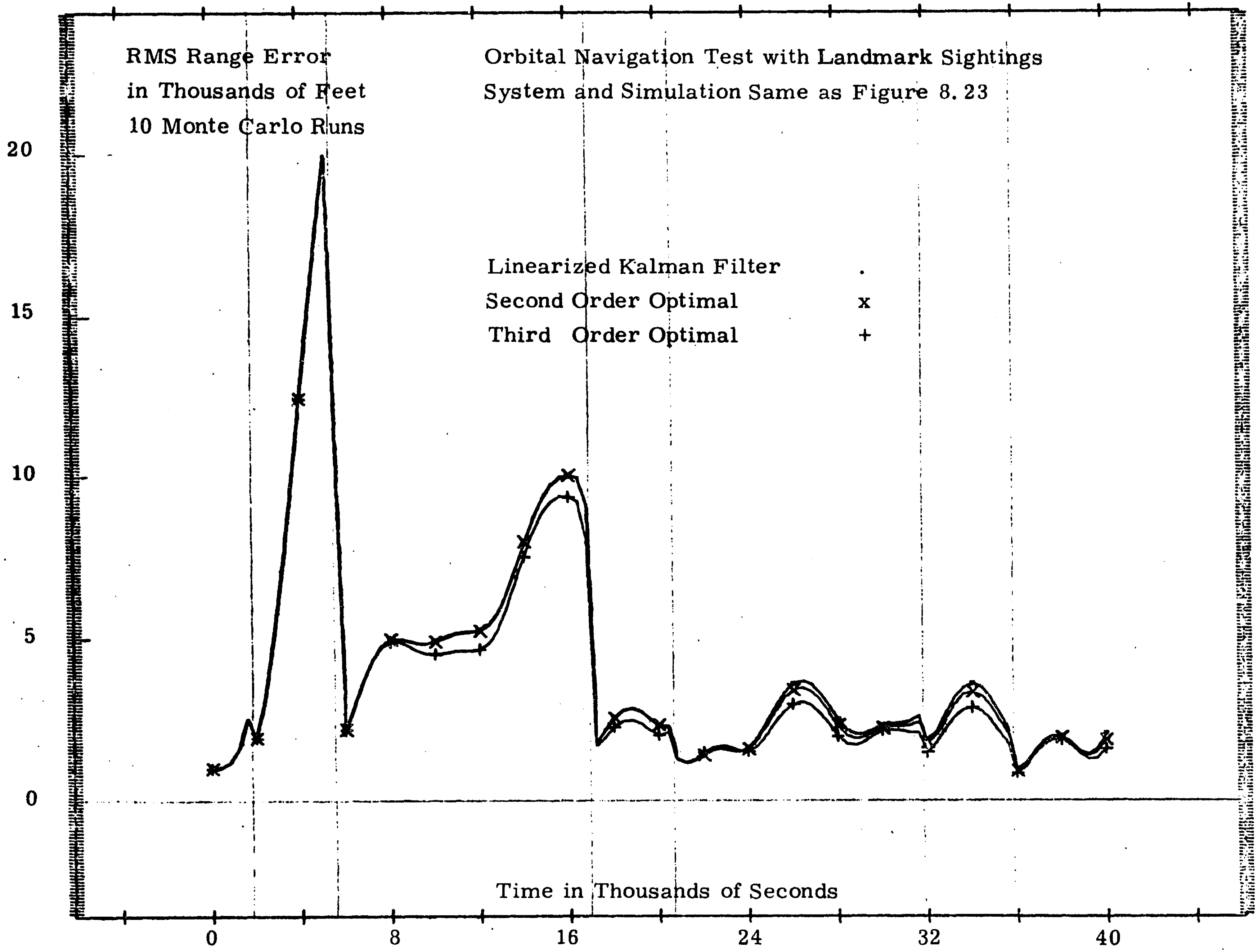

Figure 8.24 Orbital Landmark Navigation in Polar Coordinates - Range Error

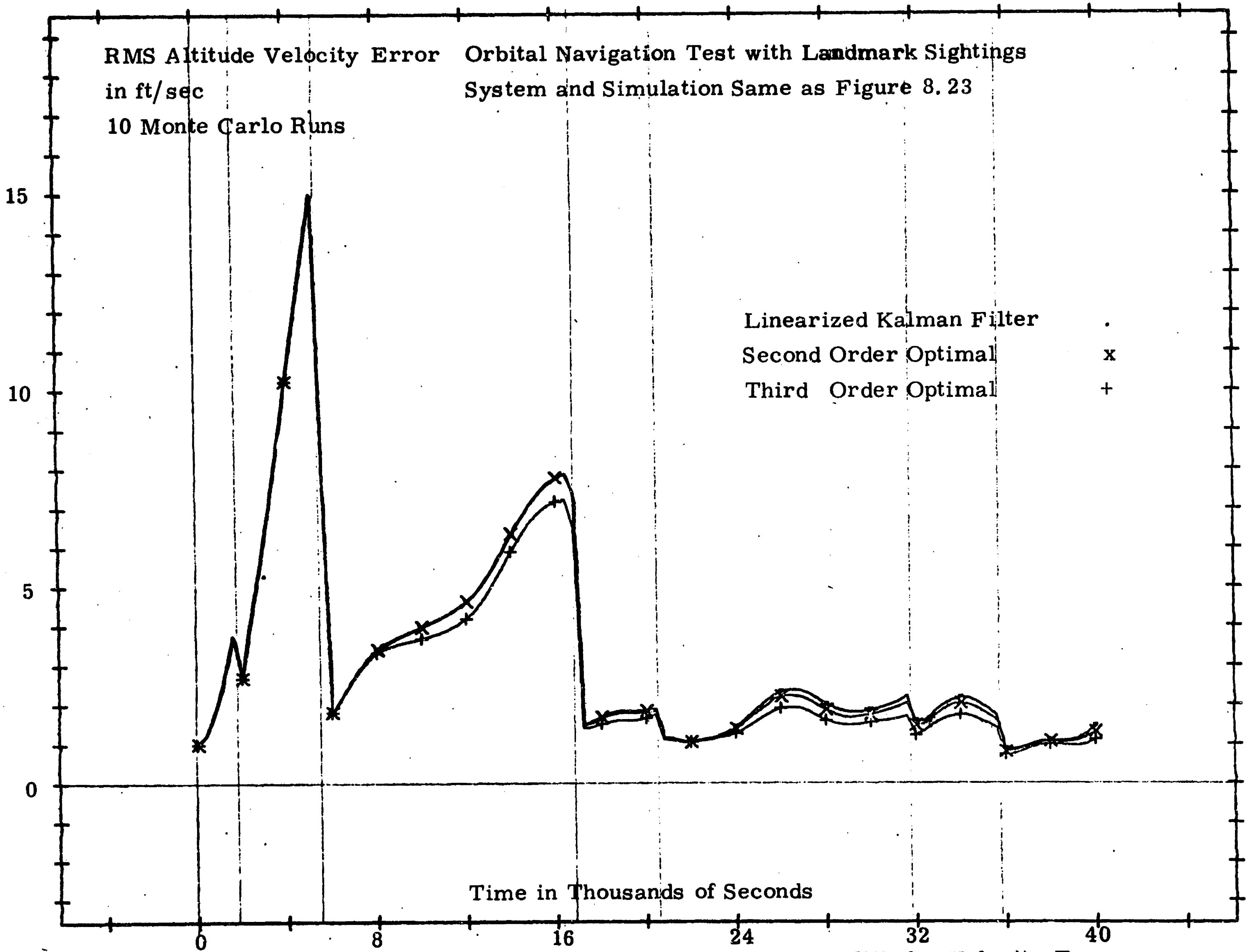

Figure 8.25 Orbital Landmark Navigation Test - Altitude Velocity Error

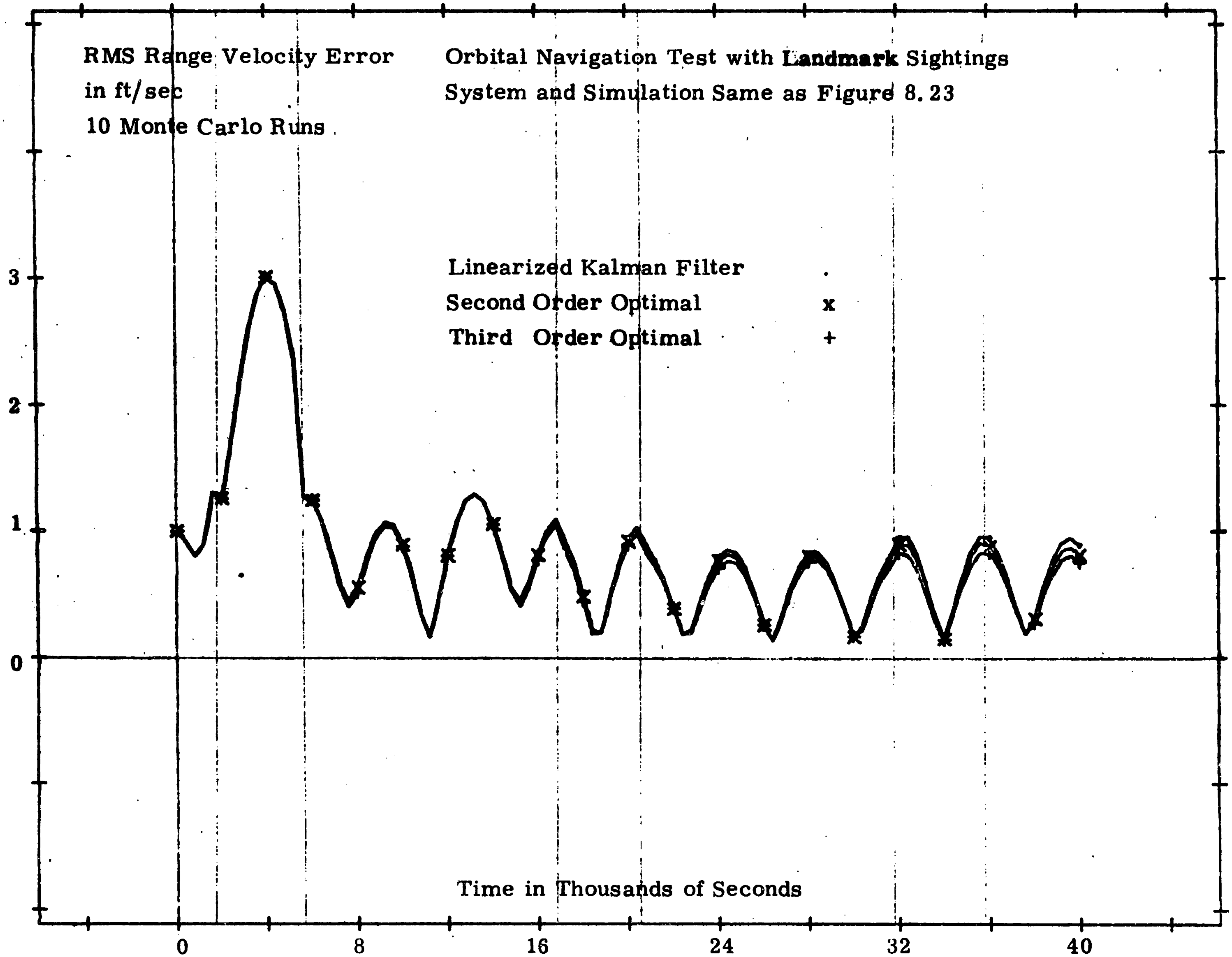

Figure 8.26 Orbital Landmark Navigation Test - Range Velocity Error

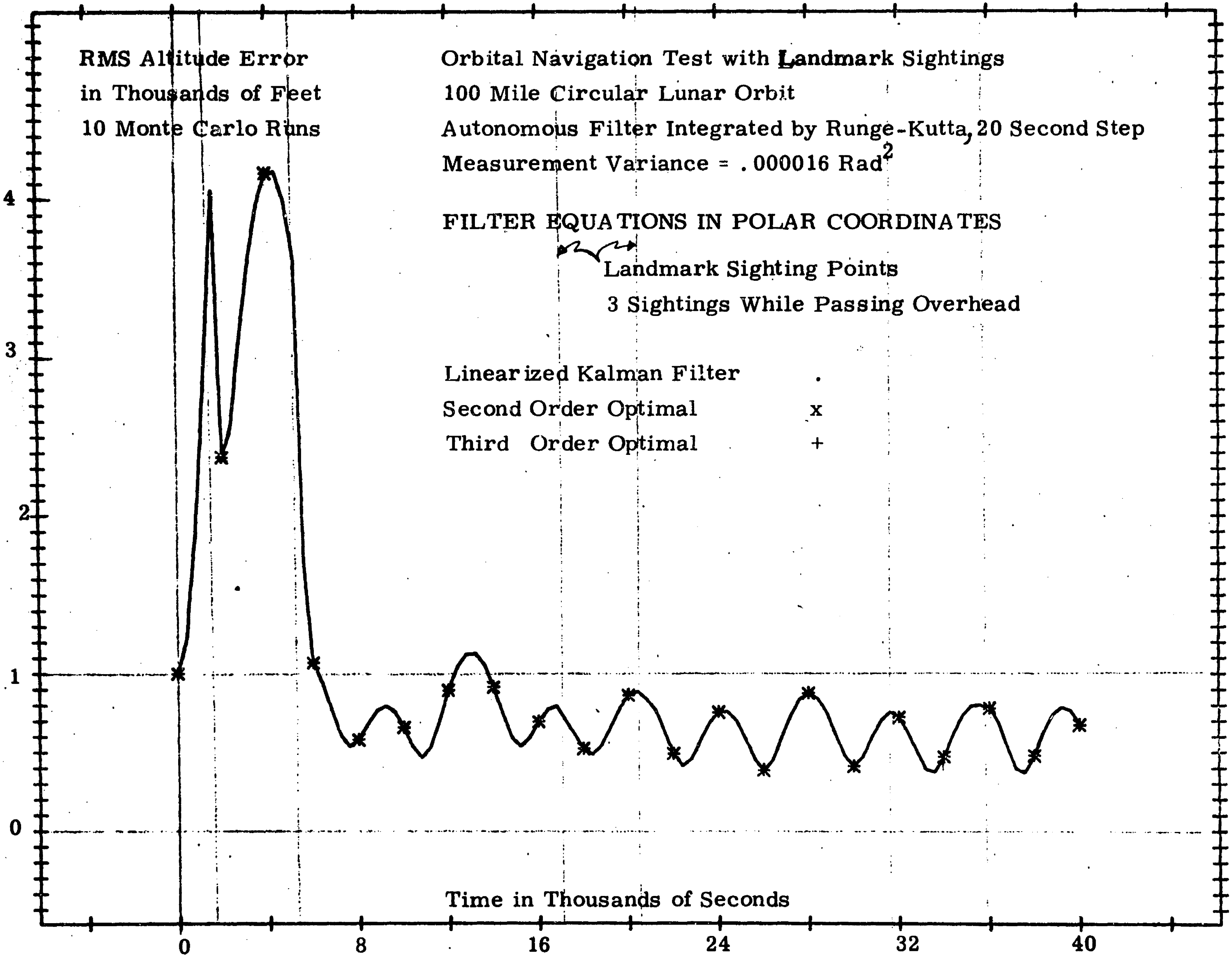

Figure 8.27 Orbital Landmark Navigation in Polar Coordinates - Altitude Error

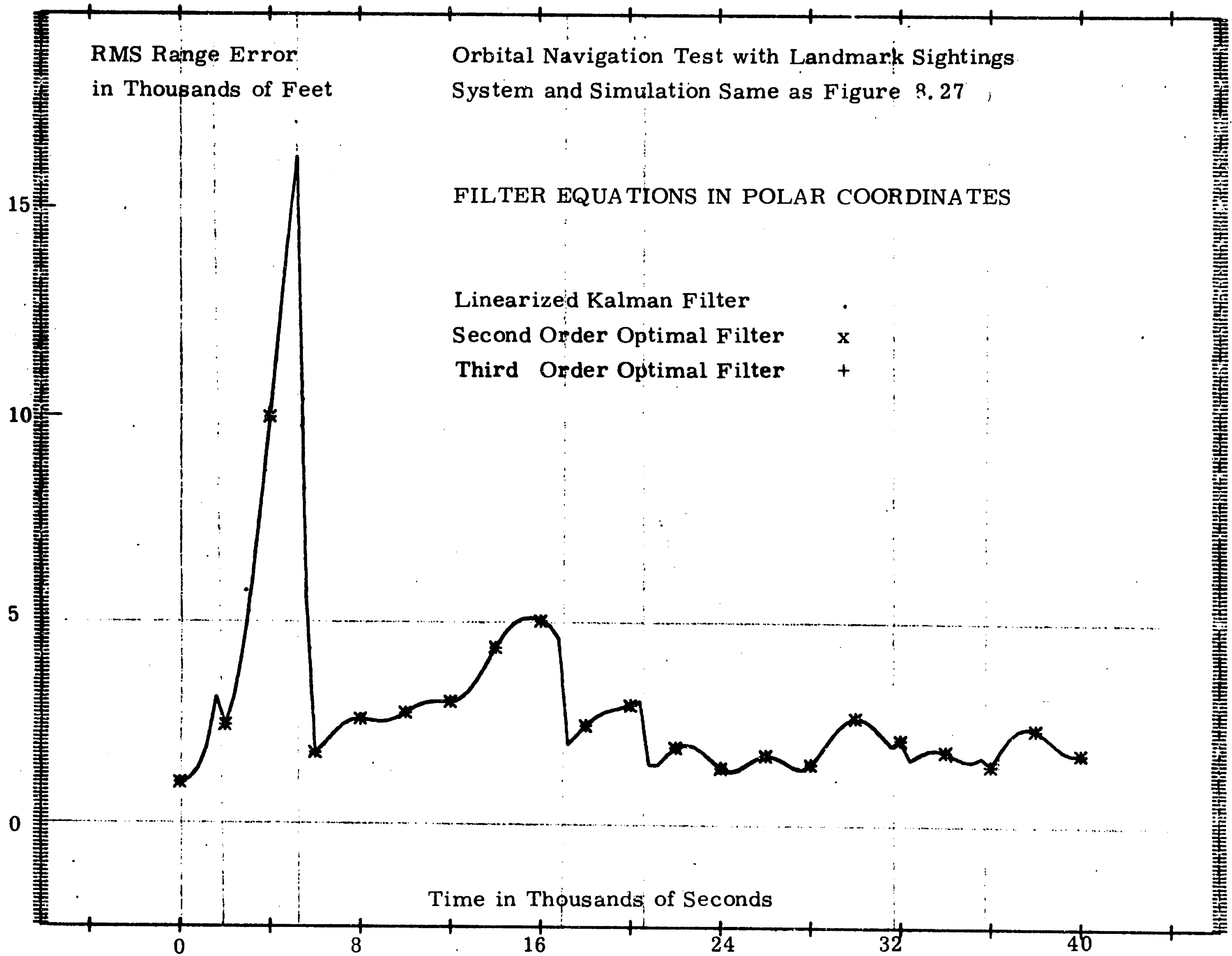

Figure 8.28 Orbital Landmark Navigation in Polar Coordinates - Range Error

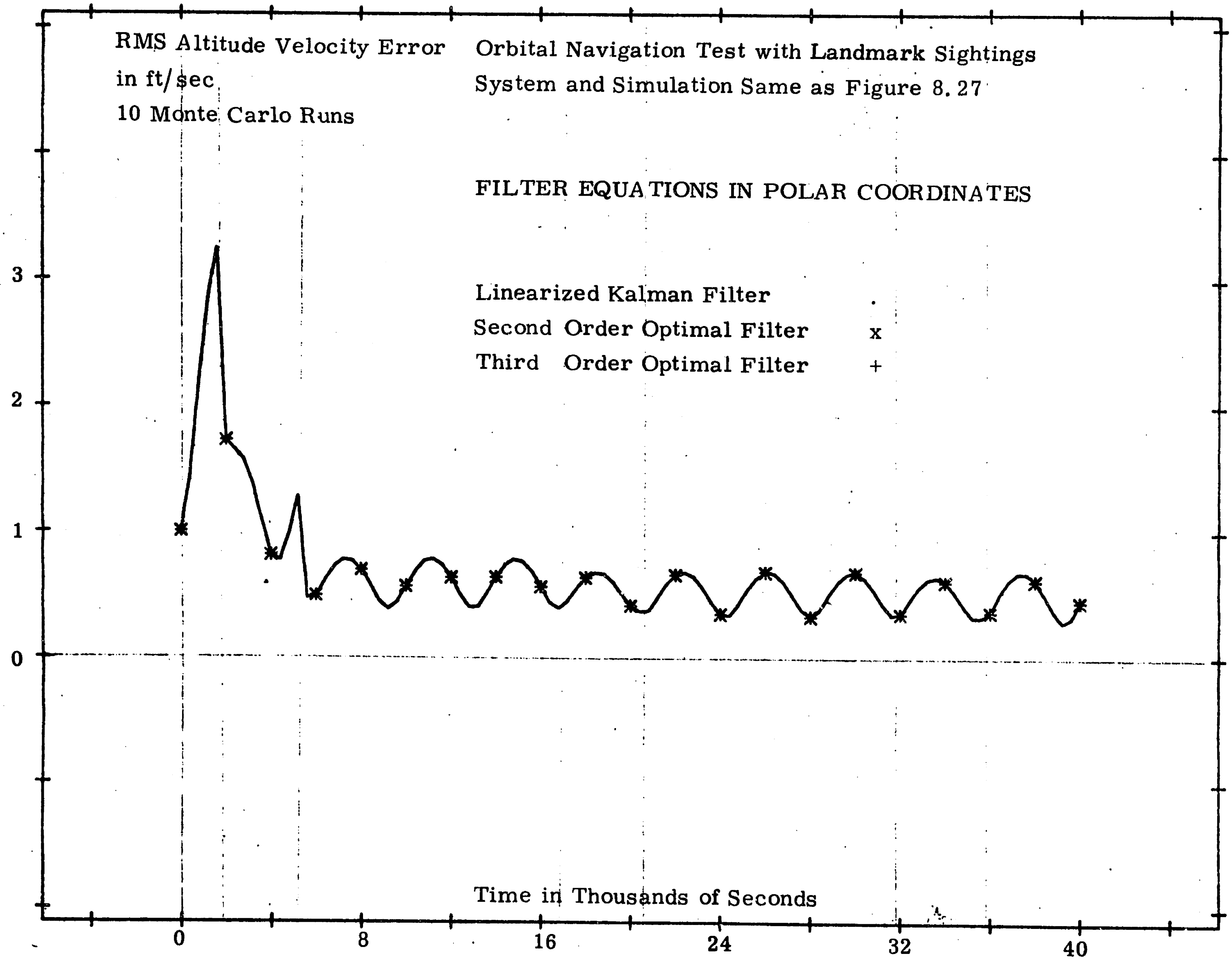

Figure 8.29 Orbital Landmark Navigation in Polar Coordinates - Altitude Velocity Error

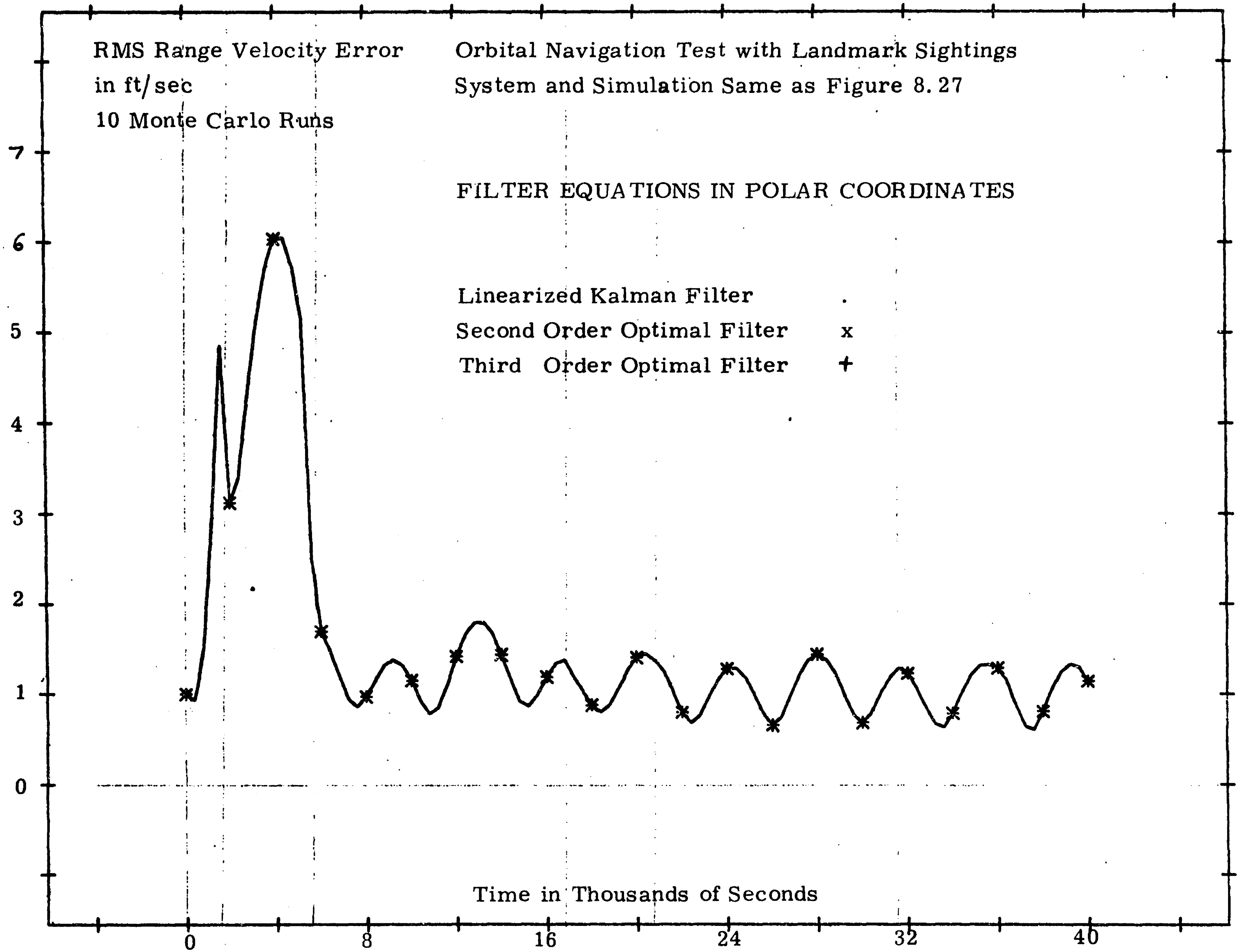

Figure 8.30 Orbital Landmark Navigation in Polar Coordinates - Range Velocity Error

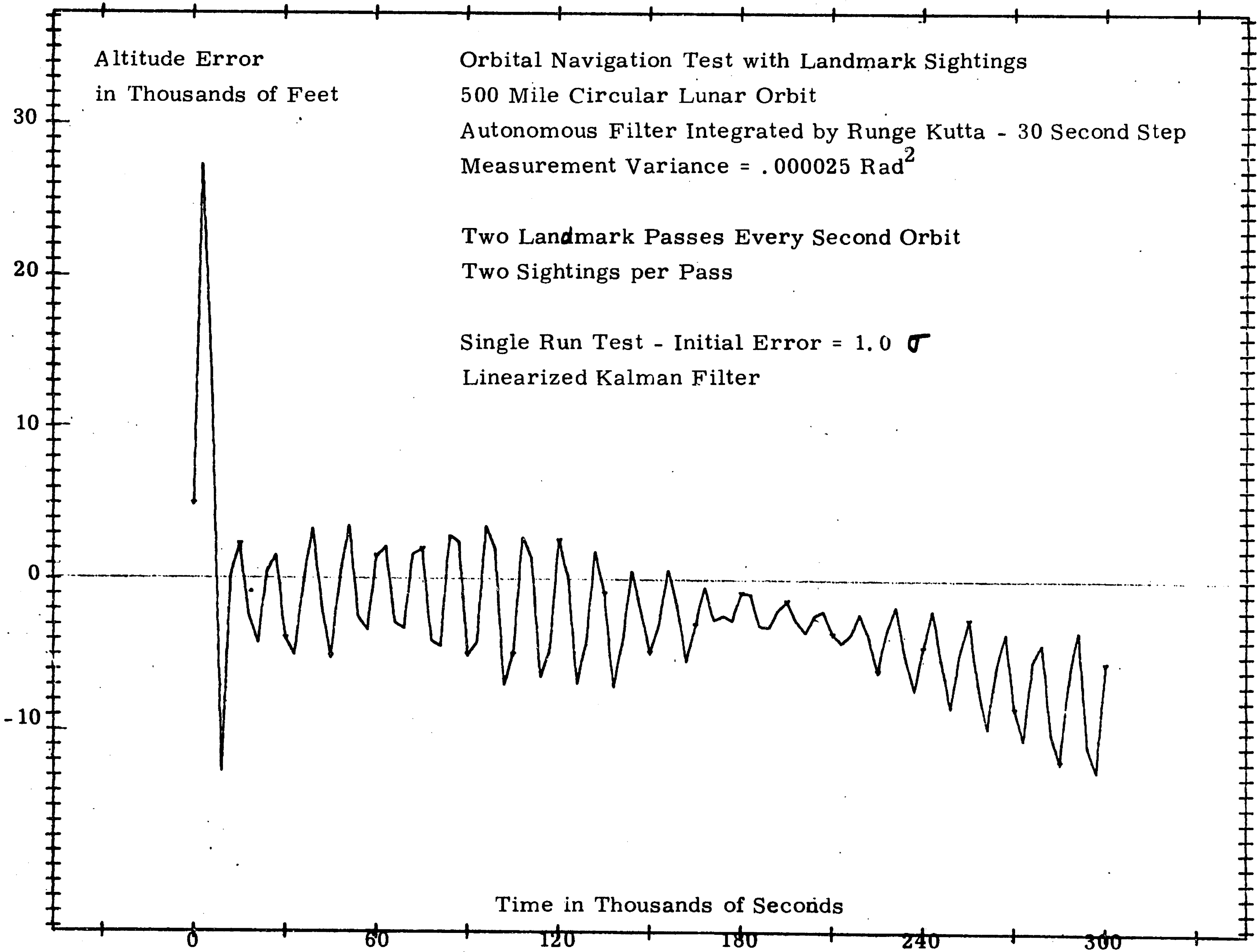

Figure 8.31 Altitude Error - Linearized Kalman Filter

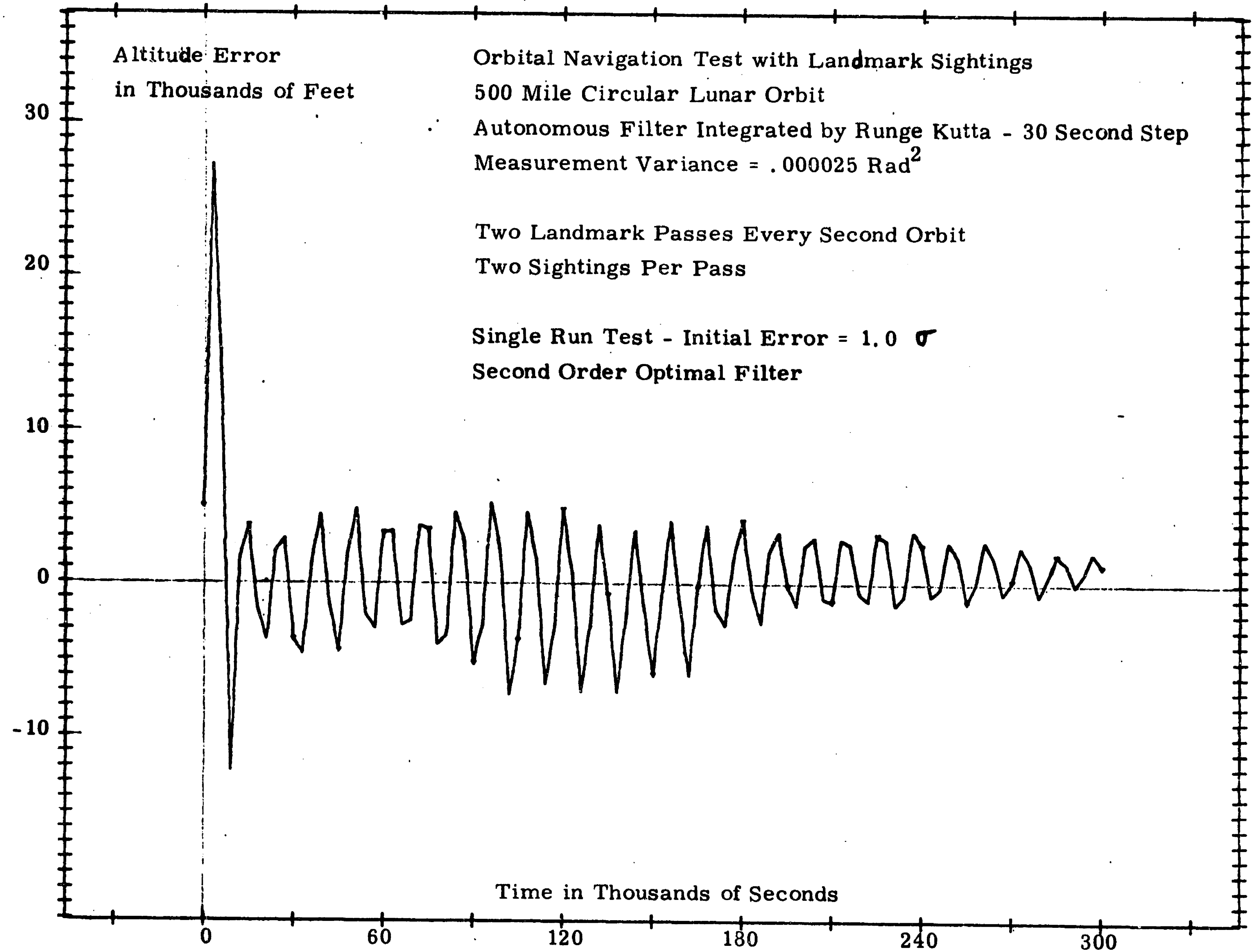

Figure 8.32 Altitude Error - Second Order Optimal Filter

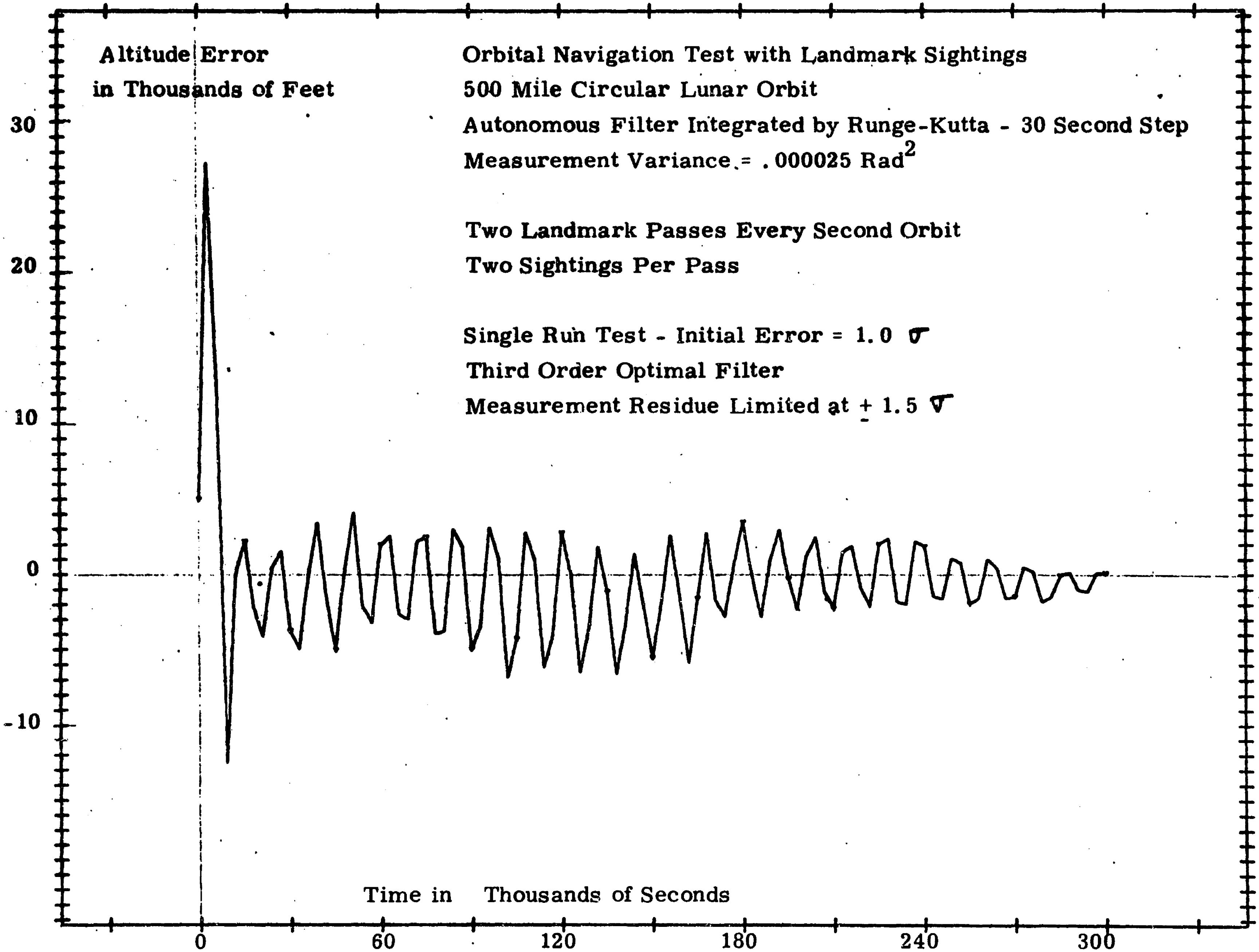

Figure 8.33 Altitude Error - Third Order Optimal Filter

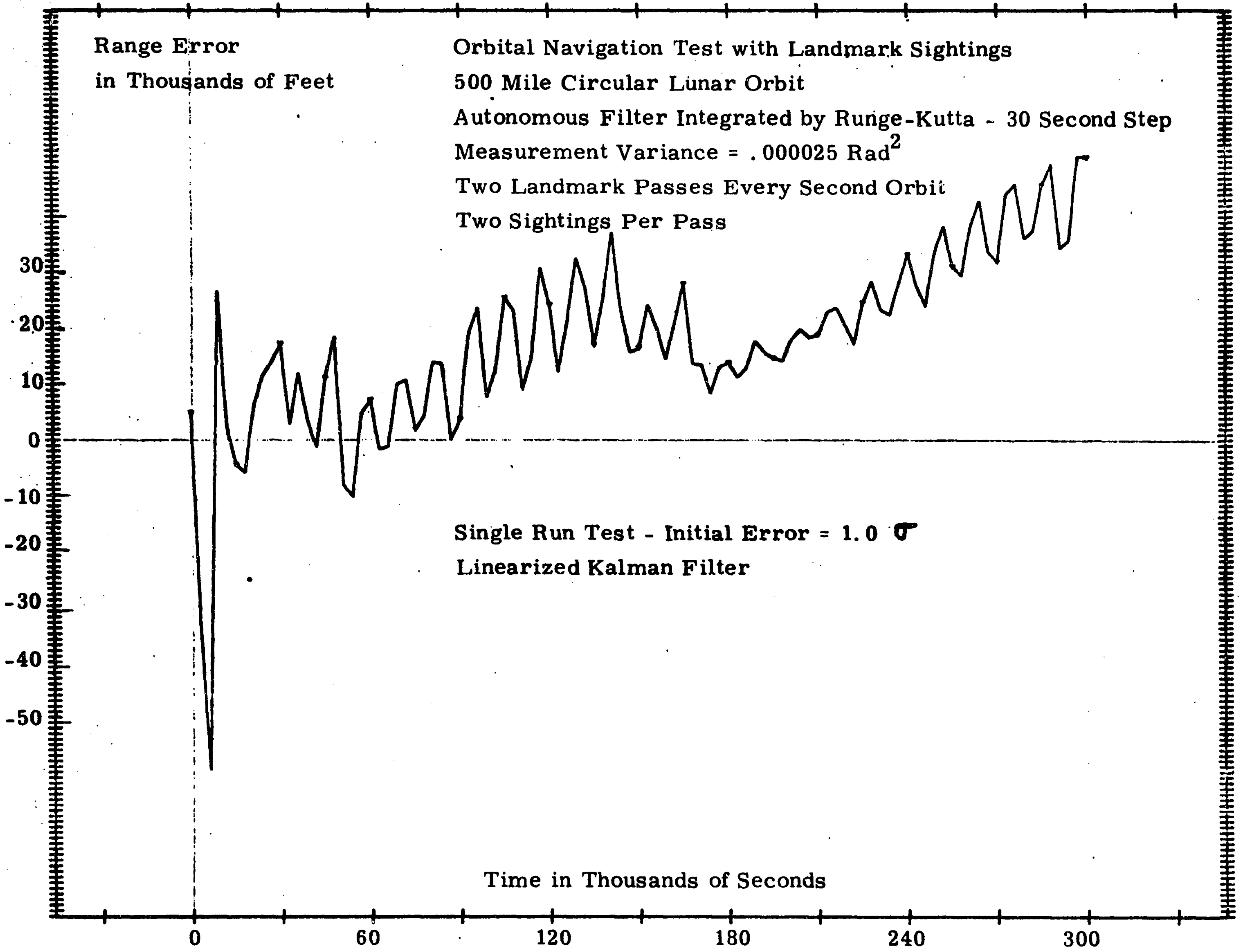

Figure 8.34 Range Error - Linearized Kalman Filter

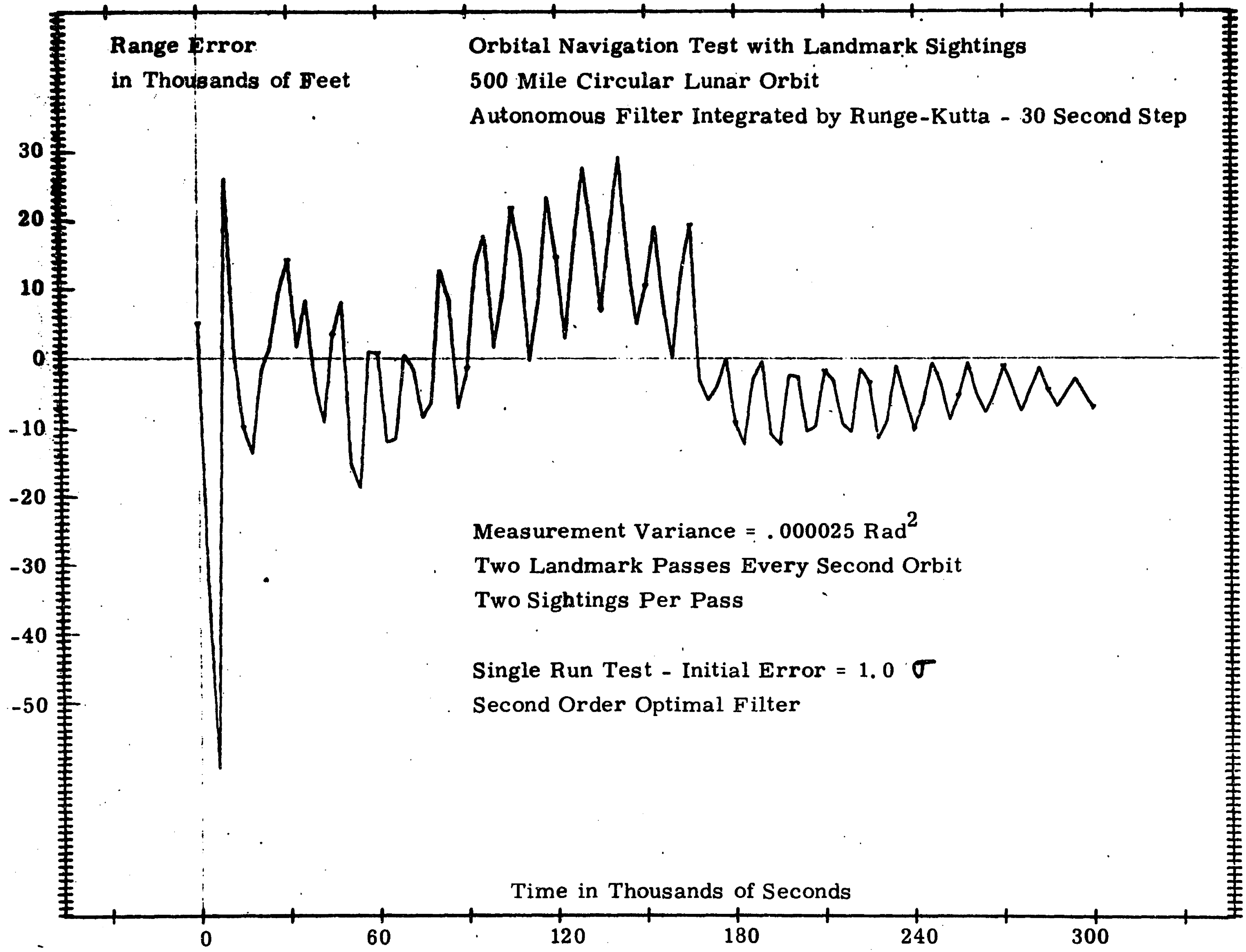

Figure 8.35 Range Error - Second Order Optimal Filter

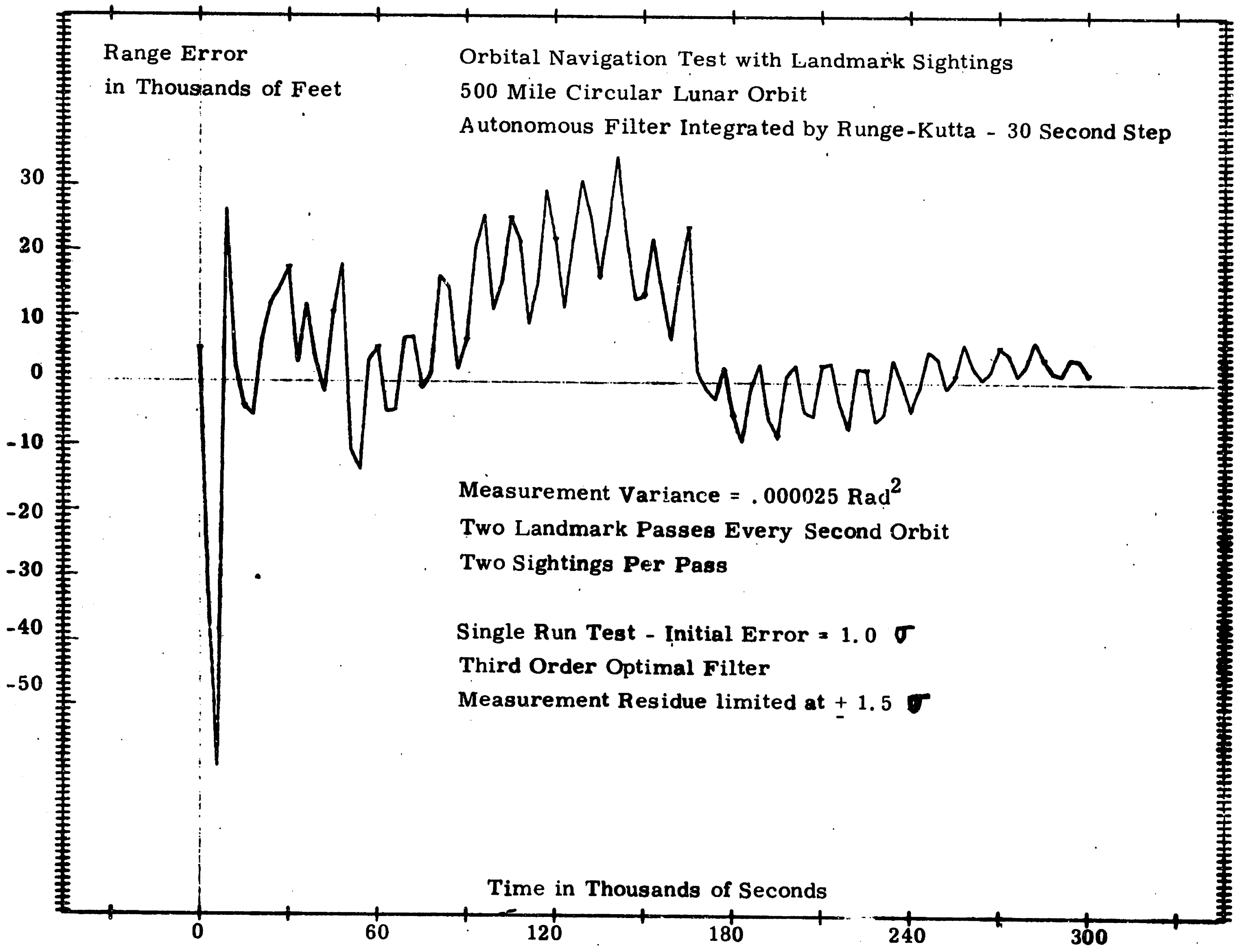

Figure 8.36 Range Error - Third Order Optimal Filter

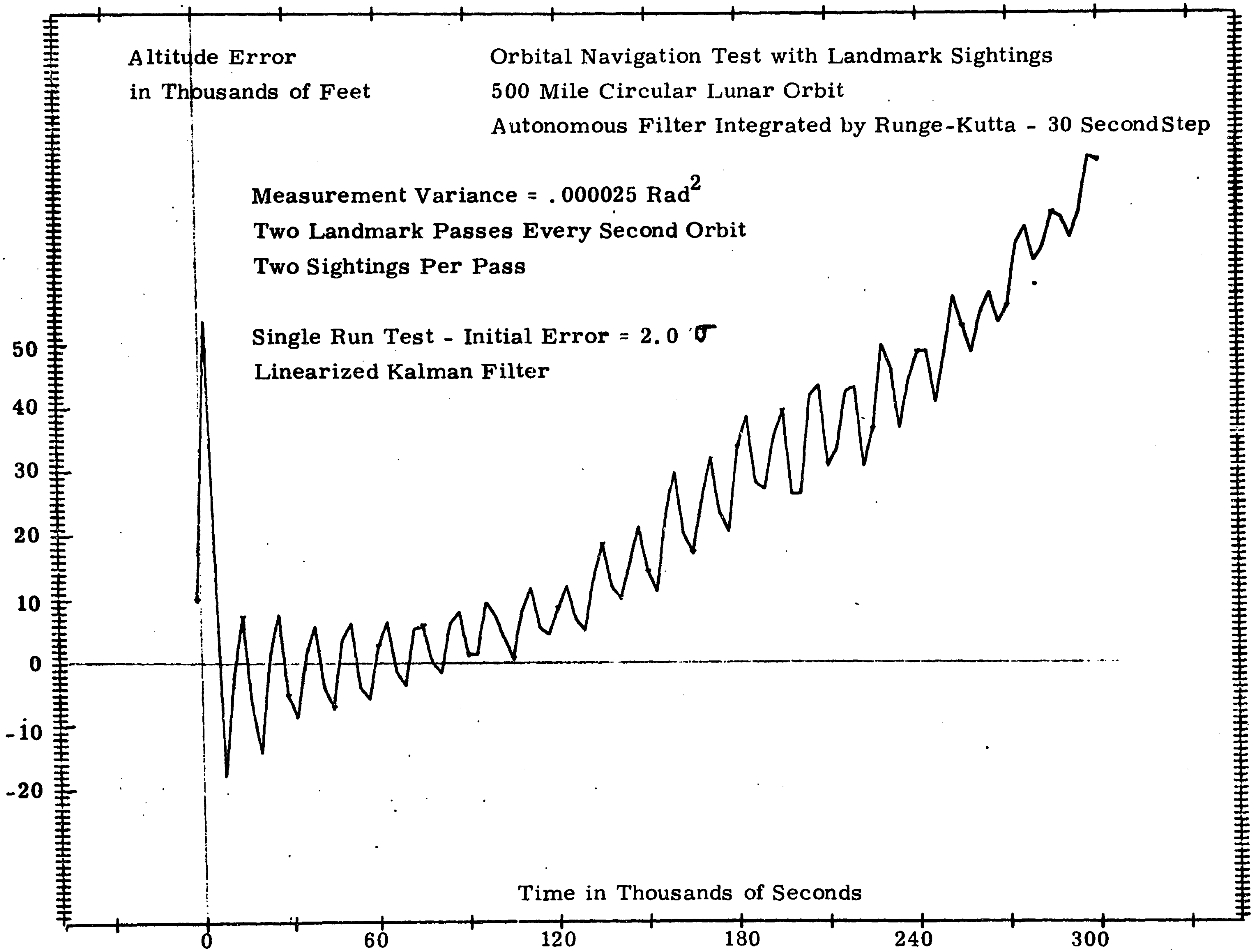

Figure 8.37 Altitude Error - Linearized Kalman Filter

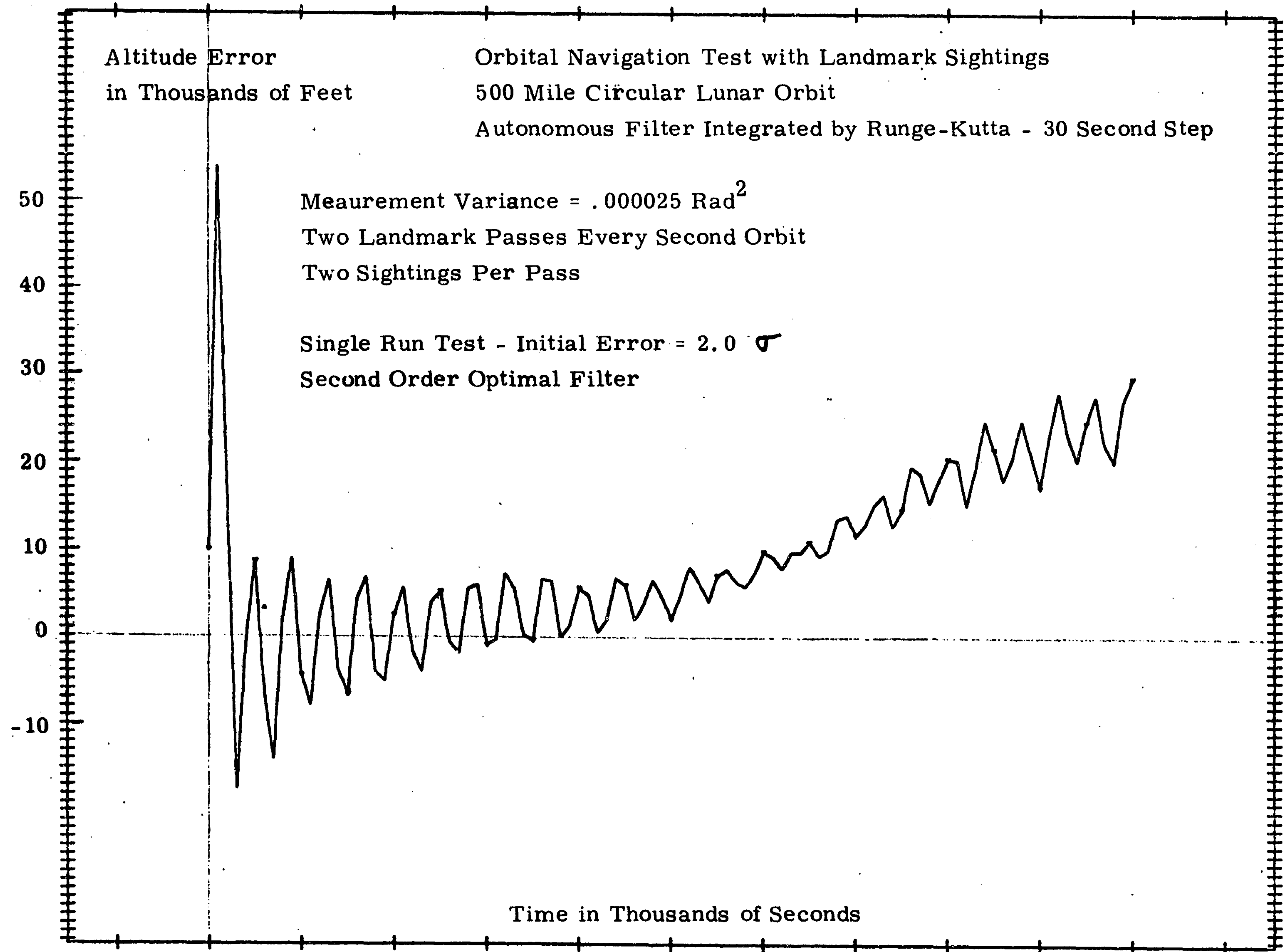

Figure 8.38 Altitude Error - Second Order Optimal Filter

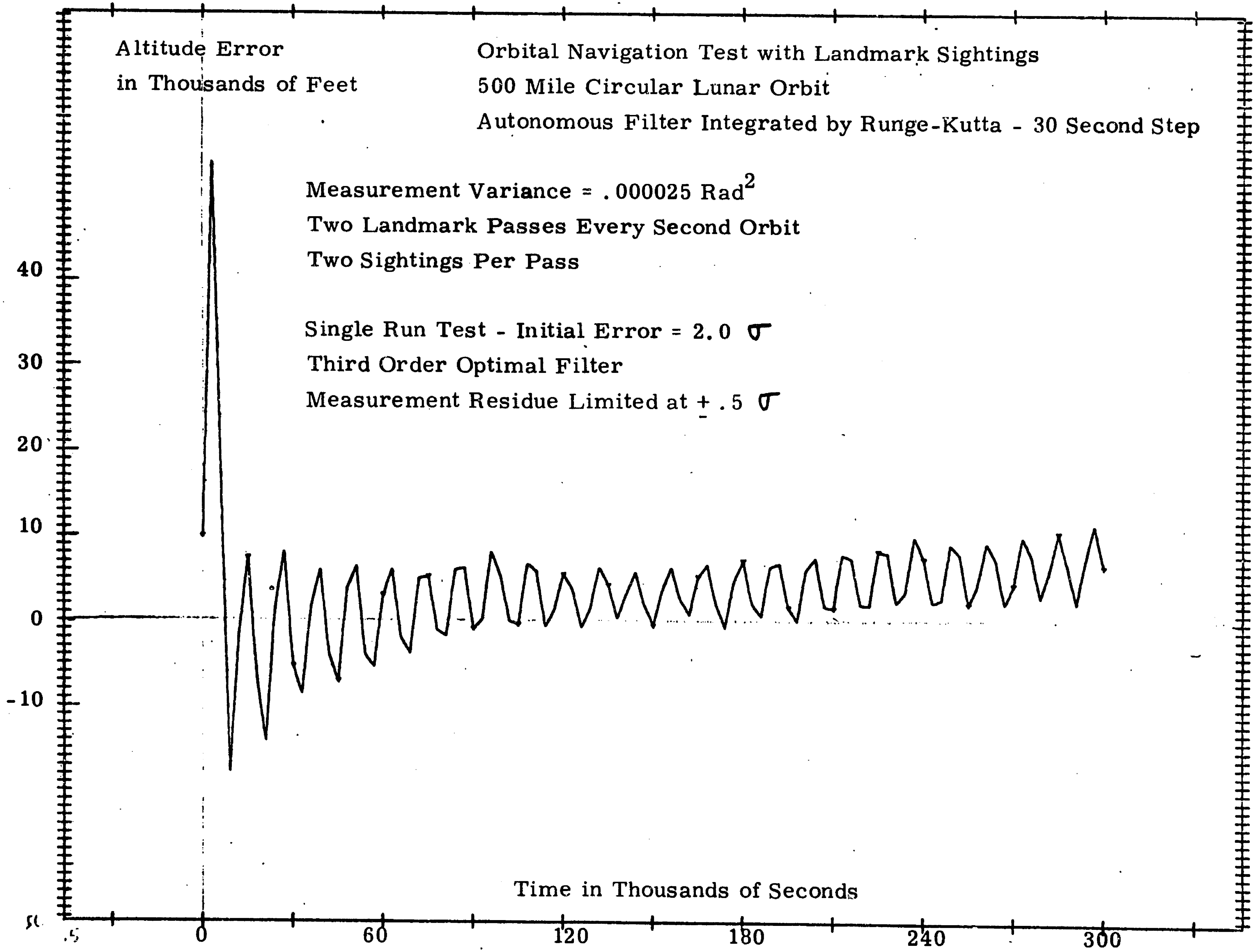

Figure 8.39 Altitude Error - Third Order Optimal Filter - Z Limited at .5 σ

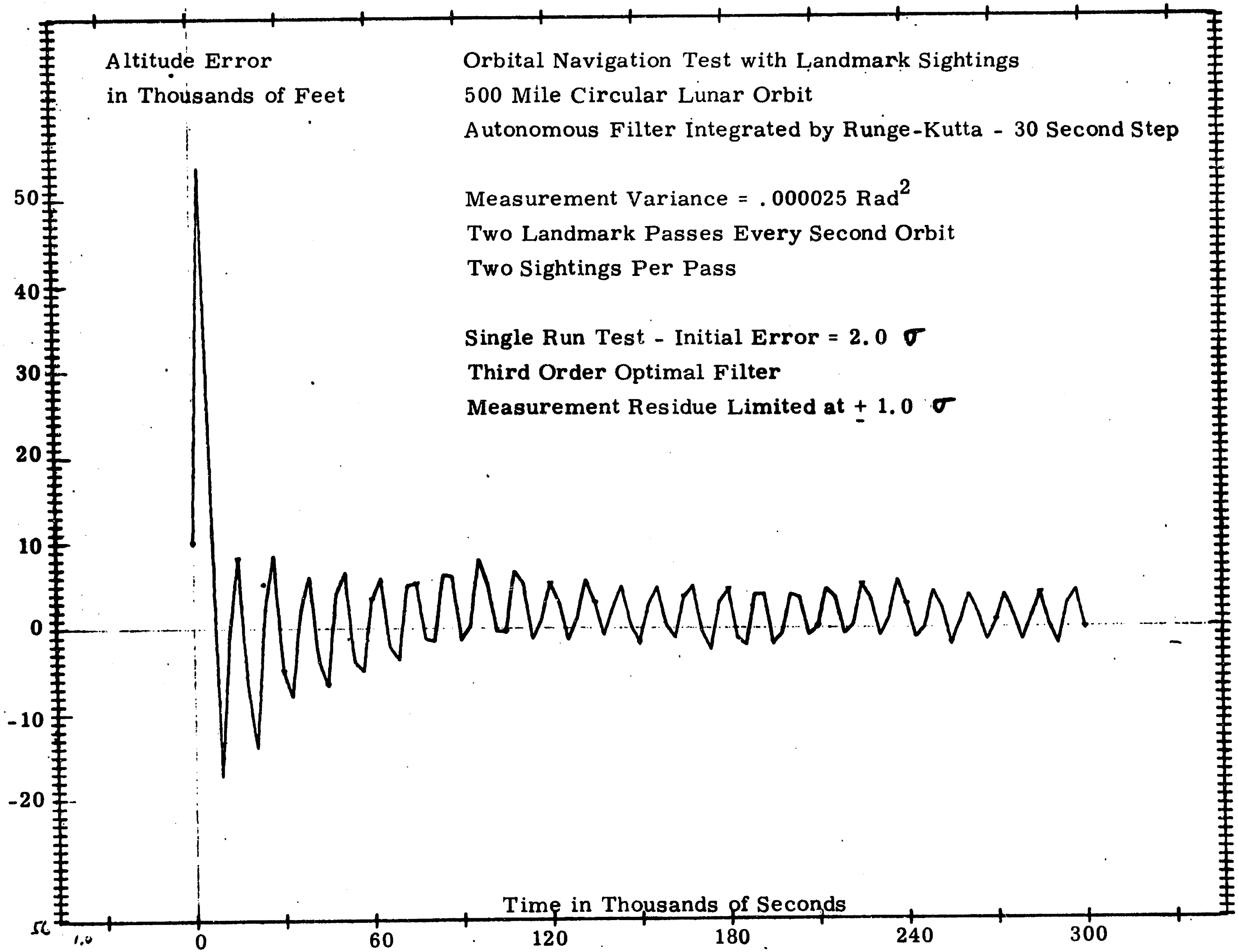

Figure 8.40 Altitude Error - Third Order Optimal Filter - Z Limited at 1.0 σ

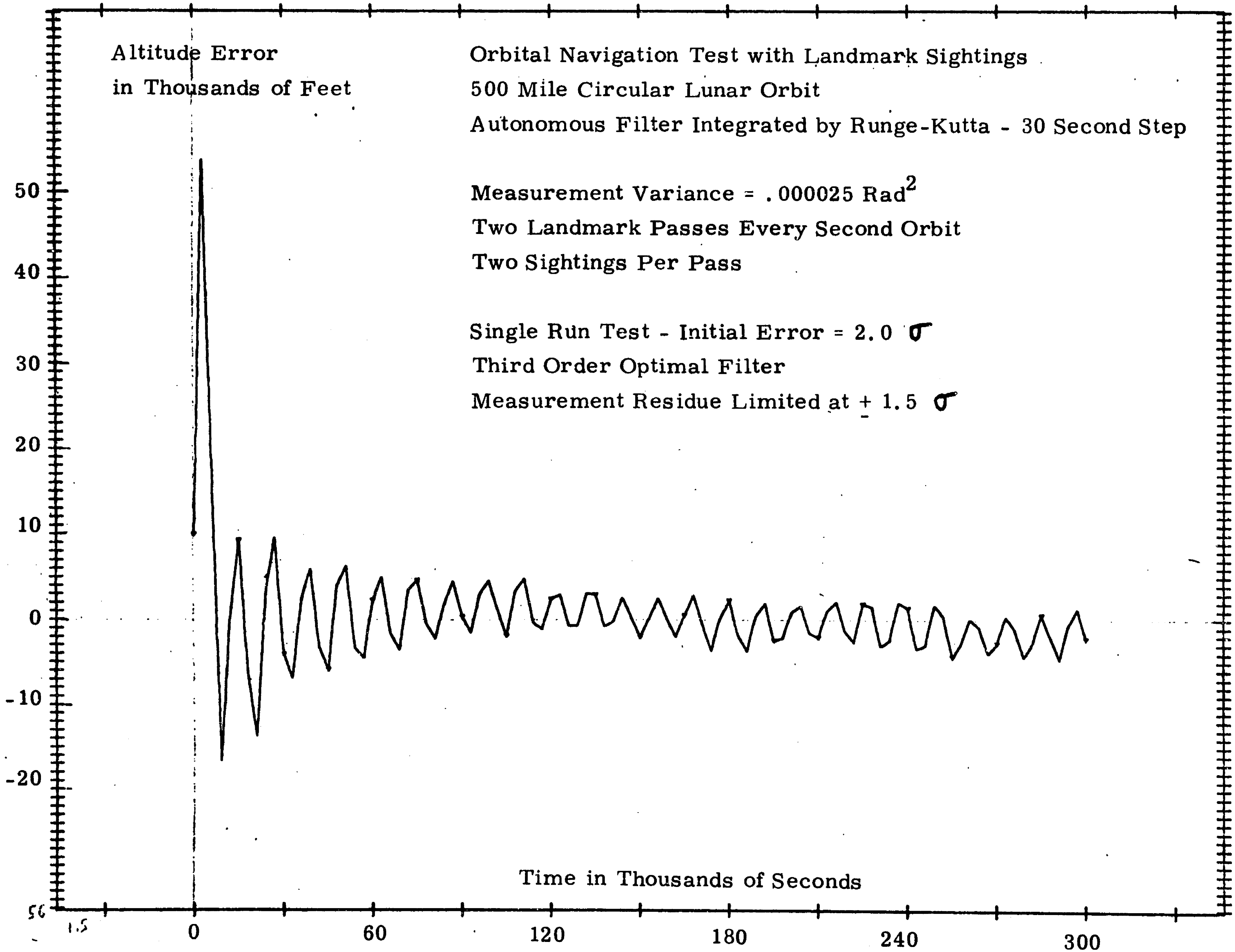

Figure 8.41 Altitude Error - Third Order Optimal Filter - Z Limited at 1.5 σ

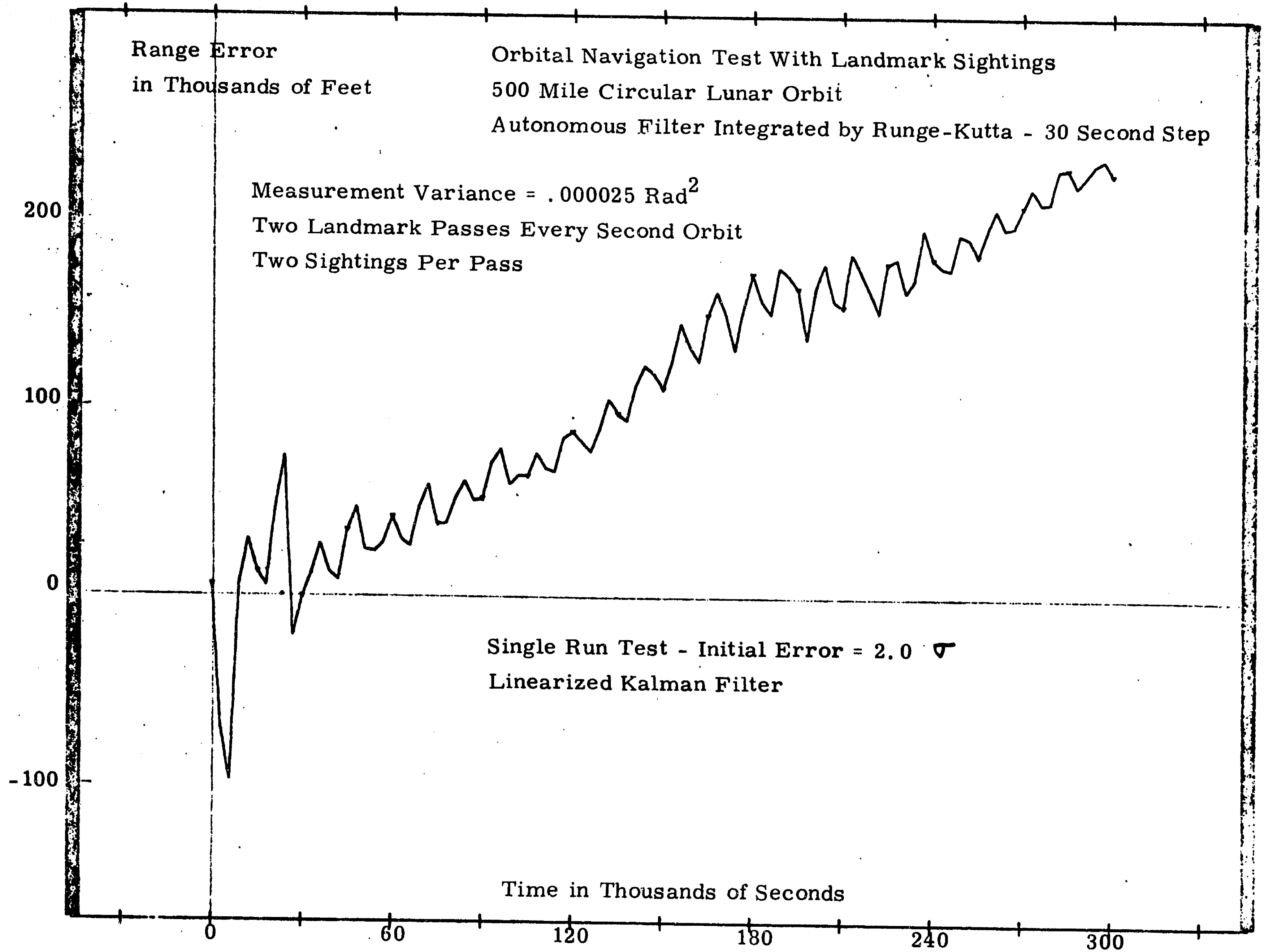

Figure 8.42 Range Error - Linearized Kalman Filter

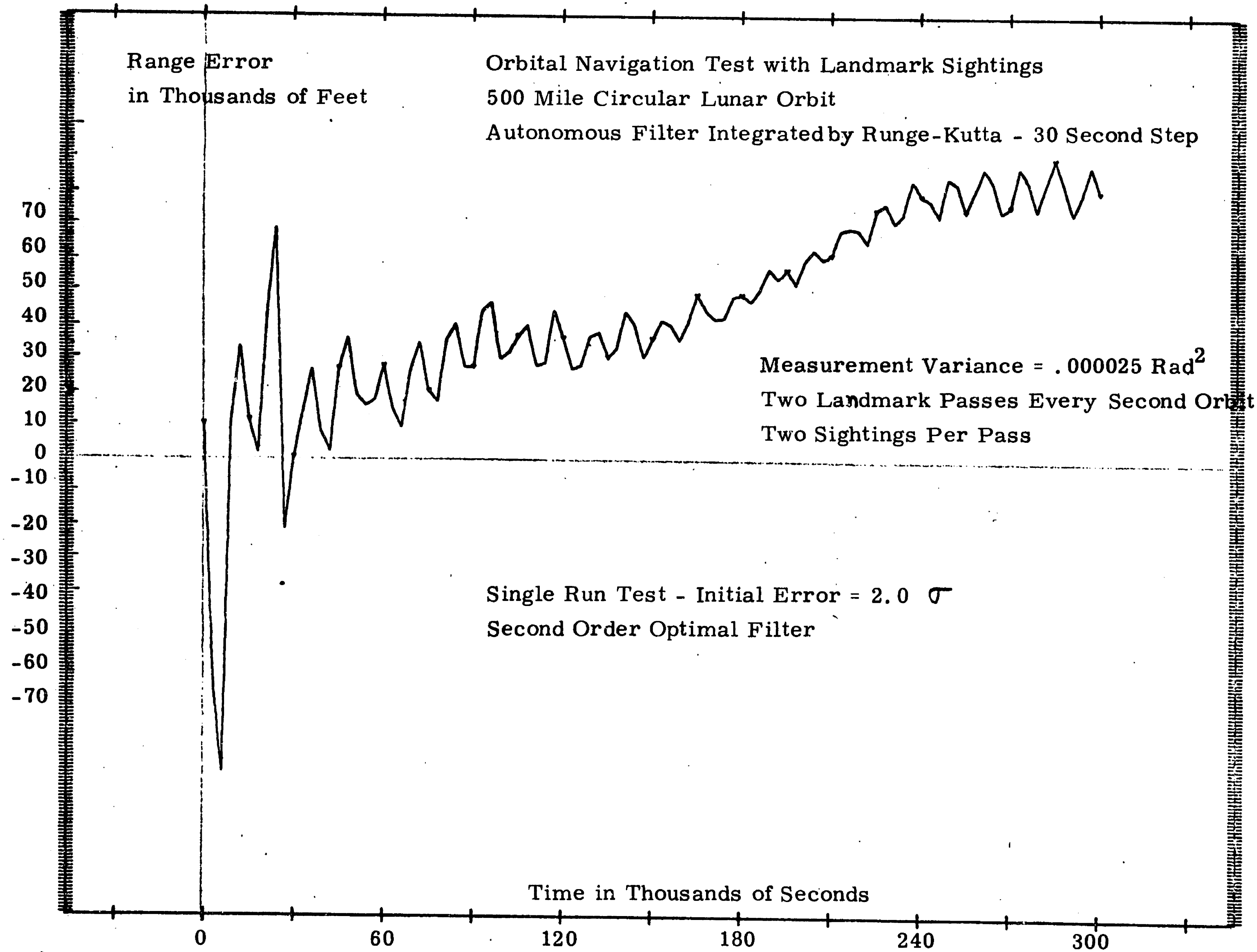

Figure 8.43 Range Error - Optimal Seccnd Order Filter

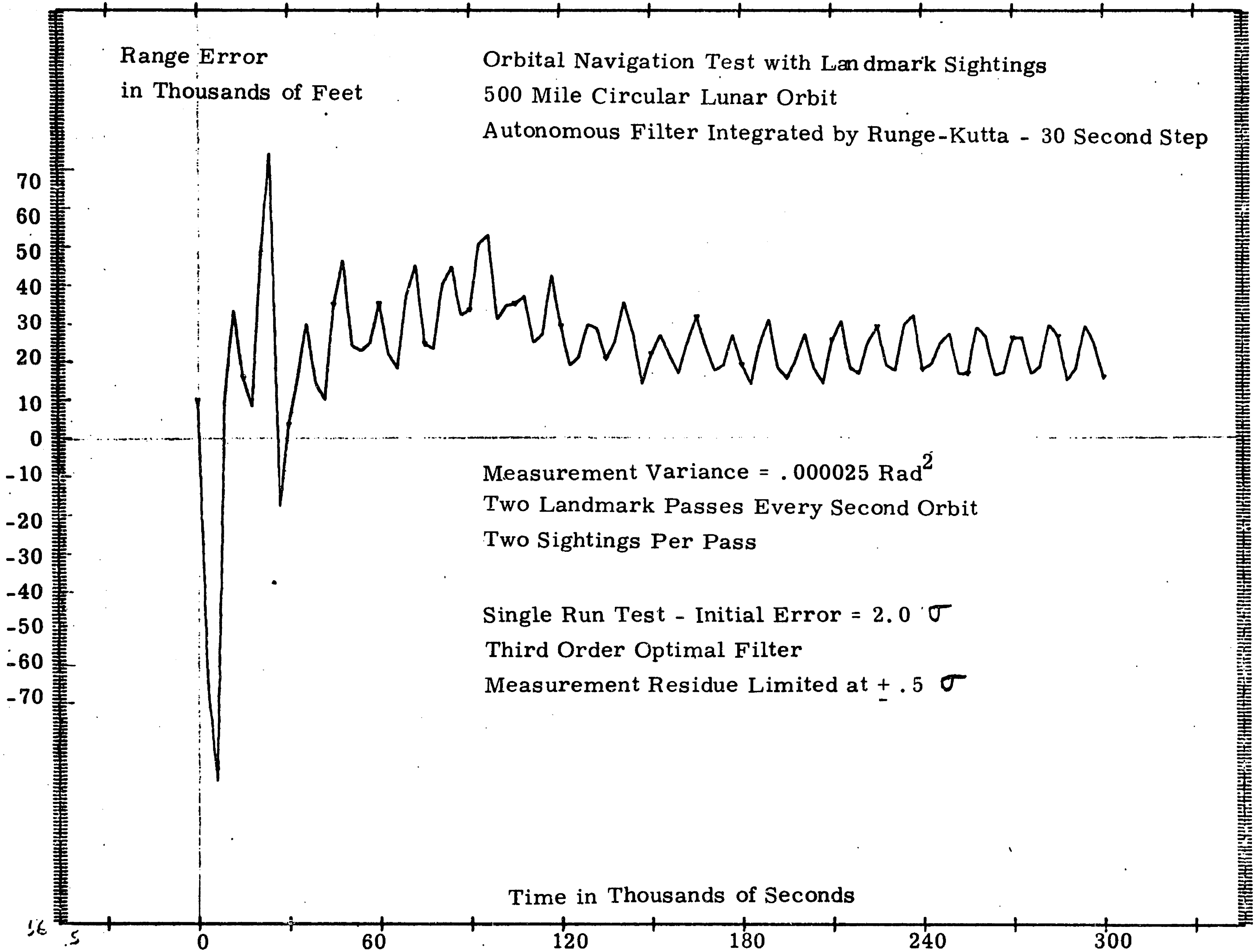

Figure 8.44 Range Error - Optimal Third Order Filter - Z Limited at .5 σ

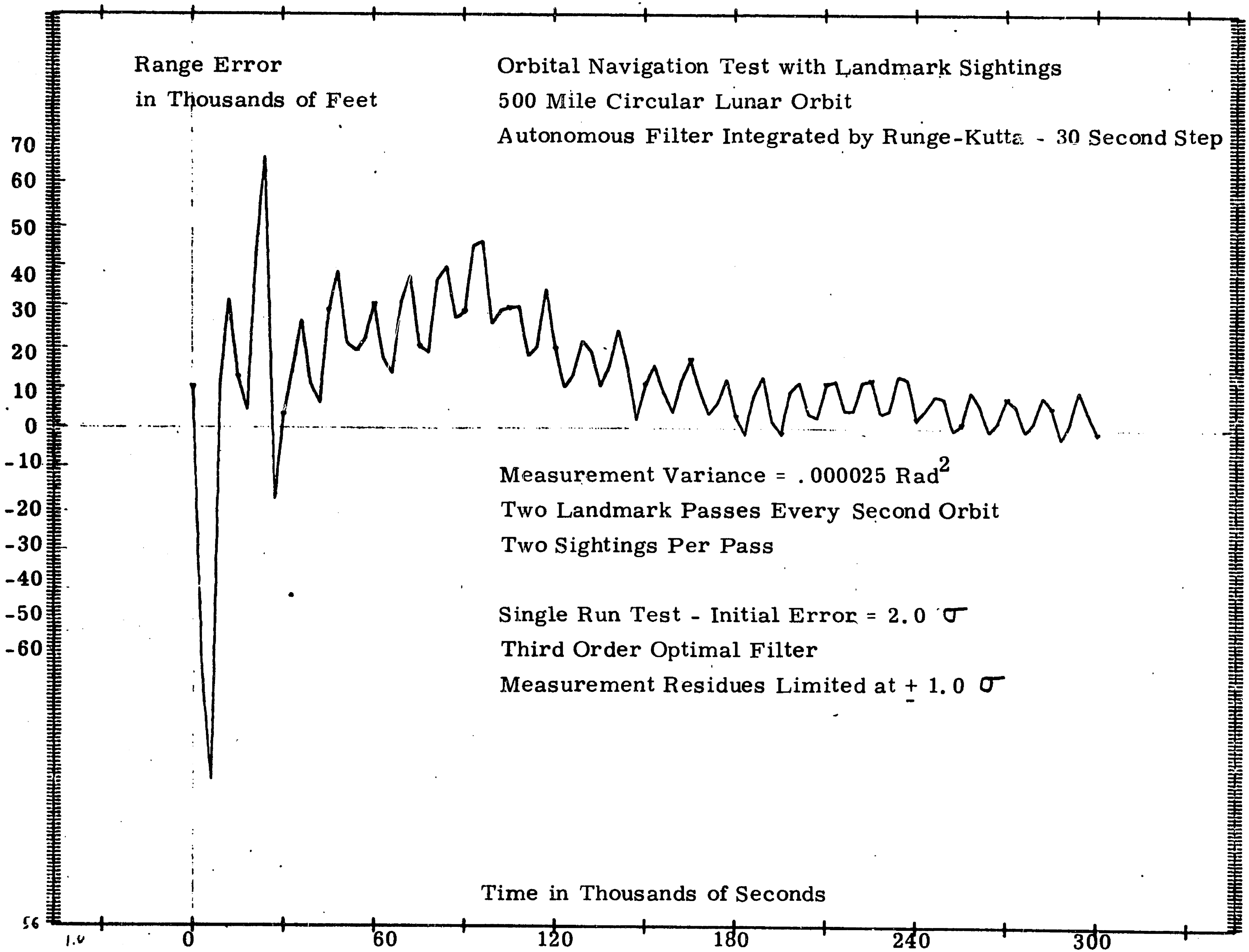

Figure 8.45 Range Error - Optimal Third Order Filter - Z Limited at 1.0 σ

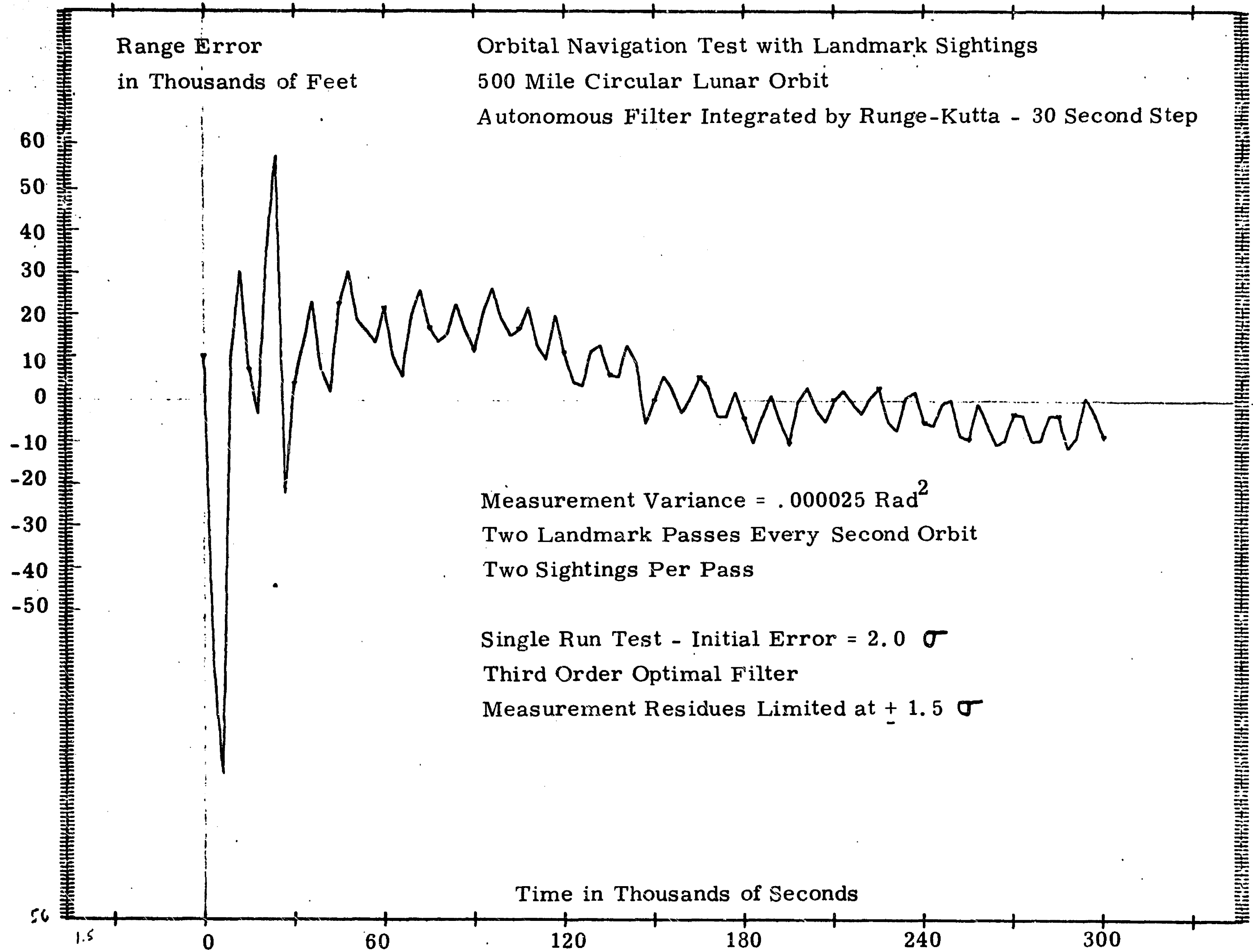

Figure 8.46 Range Error -Optimal Third Order Filter - Z Limited at 1.5 σ

Chapter 9

Conclusion

9.1 Conclusions

In order to deal effectively with nonlinear nonsymmetric systems, an estimator must be able to model random processes whose probability density functions are nonsymmetric. The simplest general estimator capable of doing this must explicitly or implicitly compute at least the means, covariances, and third central moments of the process.

A general theory of nonlinear estimation has been developed which models the posterior probability density function, container of all knowable information concerning the system, as a gaussian density times an expansion of multidimensional Hermite polynomials. This theory indicates the possibility of constructing a sequential estimator which computes the central moments up to any order. The moment sequence must be truncated at some point in a practical filter, a procedure which necessarily introduces approximation. In addition, an analytic solution for the propagation of the truncated moment sequence with time may not be possible in all cases.

This theory was applied to differentiable nonlinear systems with differentiable nonlinear measurements containing additive gaussian noise. Estimators computing the means, covariances, and third central moments were derived for continuous systems with continuous measurements, and continuous systems with discrete measurements. These estimators compute the conditional expectation of the state (the minimum variance estimate), based on the assumed third order density model.

The third order continuous measurement filter is analytically simple for both linear and nonlinear measurements. This filter, however, appears to be of very limited usefulness because of its poor stability characteristics and large computational requirements. The optimal third order discrete measurement filter, on the other hand, is analytically difficult. A somewhat complex analytic solution was obtained for linear measurements, but an analytic solution for nonlinear measurements appears impossible. A numerical technique for solving the nonlinear measurement filter equations was developed, however. In spite of these difficulties, the third order discrete linear measurement filter appears very promising. This filter was stable in all cases tested if the measurement residues were properly limited, and gave substantially better results than second order filters for very nonlinear systems. This filter solved the problem of Kalman filter divergence in the orbital navigation problem, and gave improved accuracy in the other nonlinear system simulation tests.

The third order discrete measurement filter consists of a set of coupled nonlinear differential equations that must be integrated between measurements, and a complex set of updating equations involving triple summations that are to be solved at the measurement times. These are much more complex than the Kalman filter equations, but are tractable for low order systems if efficient solution techniques, such as computing only the unique terms, are used. For high order systems, the complexity becomes staggering.

An optimal second order nonlinear filter was also derived. This is only slightly more complex than a Kalman filter, and results in a moderate improvement when very nonlinear systems are encountered.

The Hermite polynomial expansion - discrete measurement theory gives a complete and workable, though complex, theory of optimal estimation for nonlinear differentiable systems. This theory is still incompletely developed for nonlinear measurement functions.

9.2 Suggestions for Further Study

Much research remains to be done in the difficult and challenging field of nonlinear estimation. A number of possibilities follow from the approach taken in this thesis.

Extension of the optimal discrete updating equations for scalar systems with linear measurements to fourth and fifth order would be of academic and possibly practical interest. These derivations would be exceedingly complex for a vector system, but should be relatively simple for a scalar system. It would be interesting to know how the higher order quantities enter into the optimal estimation equations, and what powers of the measurement residues are used.

Further work could also be done with differentiable nonlinear measurements. The method described in this thesis seems too complex to be practical. Other methods of direct solution of the Bayes equation might be investigated, as might approximate suboptimal methods.

An analytic method of determining the importance of neglected terms in a truncated nonlinear estimator would be useful. Although this might be nearly impossible, methods or guidelines for selecting nonlinear estimators for given problems might result. In particular, methods of discerning whether or not a third order estimator will give an appreciable improvement over a second order one would be very useful.

Another intriguing possibility is the transformation of coordinates so that the statistics could be described by the theory of second order. In the orbital navigation problem, it was seen that a transformation from Cartesian to polar coordinates enabled the second order filters to achieve nearly optimal results. Perhaps a more general class of coordinate transformations could be found that would map an arbitrary density function into a gaussian density.

A major deficiency of the theory presented in this thesis is that it cannot be applied to nondifferentiable "hard" nonlinearities, such as quantizers. It might be possible, however, to develop a parallel Bayesian theory for these devices using different methods of treating the expectation integrals and diffusion equations. This would represent a major advance in the theory of nonlinear systems, if discontinuous devices could be included.

Finally, much useful work remains to be done in the field of suboptimal computation limited nonlinear estimation. In particular, one could look for simple nonsymmetric functions for use as density functions. The object would be to develop simple suboptimal filters with small numbers of parameters, that might fit particular estimation problems with good accuracy. In return for simpler computation, one would forgo the great generality of the Hermite polynomial filter.

Appendix A

Tensor Notation

Tensor notation is very useful in writing multidimensional arrays of high order. While vector-matrix notation cannot be used for arrays of order greater than two, tensor notation is not so limited. In addition this notation facilitates the computer programming of complex equations, as it indicates the elementary computations (multiplications and additions of scalars) to be performed, and in fact resembles a compiler language. The main features are given below.

1) Indices

Latin indices, used as subscripts, will take all values in a specified range 1 -- N.

$x_i = (x_1, x_2, ---x_N)$

= vector (definition as row or column vector meaningless)

= first order tensor of N components

$x_{ij} = (x_{11}, x_{12}, -----x_{NN})$

= second order tensor of N^2 components

$x_{ijk} = (x_{111}, x_{112}, -----x_{NNN})$

= third order tensor of N^3 components.

2) Summation

If a latin index is repeated in a term, then it is understood that a summation with respect to that index over the specified range of the index is implied.

I.E., $a_i x_i = \sum_{i=1}^{N} a_i x_i$

3) Products

The product of two tensors is a tensor whose order is the sum of the orders of the two, provided all the indices are unique.

$$A_{ij}B_k = C_{ijk}$$

4) Contraction

Identical latin indices in a term (indicating a summation) do not count in the order of that term.

$$A_{ijk}B_{jk} = C_i = \text{first order tensor}$$

5) Partial Derivatives

Partial derivatives with respect to the state vector x_i will be indicated as follows;

$$\frac{\partial A_i}{\partial x_j} = A_{i,j}$$

$$\frac{\partial^2 A_{ijk}}{\partial x_l \partial x_m} = A_{ijk,lm}$$

Example of the use of Tensor Notation

Consider a scalar function ϕ in N dimensional state space x_i. For numerical optimization, it is often desirable to expand the function near a point in terms of its derivatives.

Vector Notation

$$\phi(\underline{x}) \approx \phi(0) + \left(\frac{\partial \phi}{\partial \underline{x}}\right) \underline{x} + \frac{1}{2} \underline{x}^T \left(\frac{\partial^2 \phi}{\partial \underline{x}^2}\right) \underline{x}$$ + higher order terms that cannot be expressed in vector notation.

Tensor Notation

$$\phi(\underline{x}) \approx \phi(0) + \phi_{,i}\, x_i + \tfrac{1}{2}\phi_{,ij}\, x_i x_j + 1/6\ \phi_{,ijk}\, x_i x_j x_k$$

$$+ 1/24\ \phi_{,ijkl}\, x_i x_j x_k x_l$$ plus higher order terms.

By the contraction principle, it is easy to see that all products in the above expression for $\Phi(\underline{x})$ are scalars.

Comparison of Vector-Matrix and Tensor Notation

B is an N x M matrix

x is an N x 1 vector

Vector-Matrix Notation	Dimension	Tensor Notation	
x	N x 1	x_i	i,j = 1 -- N
B	N x M	B_{ik}	k,l = 1 -- M
x^T	1 x N	Not defined or required	
B^T	M x N	with tensor notation. *	
B x	N x 1	$B_{ik}\ x_k$	conformable only if M = N
$B^T x$	M x 1	$B_{ik}\ x_i$	
B B	N x N	$B_{ik}B_{kl}$	conformable only if M = N
$B\ B^T$	N x N	$B_{ik}B_{jk}$	
$B^T B$	M x M	$B_{ik}B_{il}$	
$x^T x$	1 x 1	$x_i x_i$	
$x\ x^T$	N x N	$x_i x_j$	

* Transposition can of course be accomplished by the equation $A_{ji} = B_{ij}$. Taken alone, however, B_{ji} has the some meaning as B_{ij}, as they both stand for the second order tensor B. That is, $B_{ij}x_j = B_{ji}x_i$.

Appendix B

Moments of a Multidimensional Gaussian Distribution

The probability density function of an N dimensional Gaussian distribution is given by;

$$P(\underline{x}) = \frac{\exp(-\frac{1}{2}(x_i-\mu_i)P^{-1}_{ij}(x_j-\mu_j))}{(2\pi)^{\frac{1}{2}N}|P|^{\frac{1}{2}}} \tag{B.1}$$

$$\mu_i = \mathcal{E}(x_i)$$

$$P_{ij} = \mathcal{E}((x_i-\mu_i)(x_j-\mu_j))$$

The distribution for x_i has rank $M \leq N$, if and only if x_i can be represented as follows;

$$x_i = \mu_i + B_{im}\, g_m \qquad \begin{matrix} i = 1 \text{ -- } N \\ m = 1 \text{ -- } M \end{matrix} \tag{B.2}$$

$$B_{im}B_{jm} = P_{ij} \tag{B.3}$$

B is an N x M matrix, while g is an M x 1 vector of random variables of Gaussian distribution with zero mean and a unit covariance matrix P. (See Rao, (B-14), p 440)

Moments of the Distribution

Let μ represent the mean of the distribution, and $\mu 2, \mu 3, \mu 4$,.. represent the second, third, and fourth central moments about the mean. Let $z_i = x_i - \mu_i$. Then,

$$\mu 2_{ij} = \mathcal{E}(z_i z_j) = \mathcal{E}(B_{ik}g_k B_{jl} g_l) = B_{ik}B_{jl}\,\mathcal{E}(g_k g_l)$$

$$\mu 2_{ij} = B_{ik}B_{jl}\,\delta_{kl} = B_{il}B_{jl} = P_{ij} \tag{B.4}$$

$$\mu 3_{ijk} = \mathcal{E}(z_i z_j z_k) = B_{il} B_{jm} B_{kn}\, \mathcal{E}(g_l g_m g_n) = 0 \qquad (B.5)$$

$$\mu 4_{ijkl} = \mathcal{E}(z_i z_j z_k z_l) = B_{im} B_{jn} B_{ko} B_{lp}\, \mathcal{E}(g_m g_n g_o g_p)$$

$$= B_{im} B_{jn} B_{ko} B_{lp}\, (3\, \delta_{mn}\, \delta_{no}\, \delta_{op}) \quad \text{(See Cramer (B-3), p 17.1)}$$

$$= 3\, B_{in} B_{jn} B_{kn} B_{ln} = 4 \text{ th order tensor of } N^4 \text{ components}$$

(B.6)

The 2n th central moments of the g_i, which have zero mean and unit covariance, are given by;

$$(1/\sqrt{2\pi}) \int_{-\infty}^{\infty} a^{2n}\, e^{-\frac{1}{2}a^2} da = 1 \cdot 3 \cdot 5 \;---\; (2n-1) \qquad (B.7)$$

This gives the relationships,

$$\mu 2_{ij} = B_{iq} B_{jq} = P_{ij} \qquad N^2 \text{ components}$$

$$\mu 3_{ijk} = 0 \qquad N^3 \text{ components}$$

$$\mu 4_{ijkl} = 3\, B_{iq} B_{jq} B_{kq} B_{lq}$$

$$= 3 \left\{ P_{ij} P_{kl} \right\}_s \qquad N^4 \text{ components}$$

$$\mu 5_{ijklm} = 0 \qquad N^5 \text{ components}$$

$$\mu 6_{ijklmn} = 15\, B_{iq} B_{jq} B_{kq} B_{lq} B_{mq} B_{nq}$$

$$= 15 \left\{ P_{ij} P_{kl} P_{mn} \right\}_s \qquad N^6 \text{ components}$$

(B.8)

Appendix C

The Number of Unique D th Moments of an N Dimensional Vector

Consider as an example the third moment of a four dimensional vector, $\mathcal{E}(x_i x_j x_k)$, i,j,k, = 1 --4. This moment has 4^3 or 64 components, but many of them are not unique. For example, it is clear that,

$$\mathcal{E}(x_1 x_2 x_2) = \mathcal{E}(x_2 x_1 x_2) \tag{C.1}$$

The unique D th moments of an N dimensional vector can be enumerated as follows. Start with all indices at 1, and permute the rightmost index through all possible increasing values. Then increase the next index on the left by one, initializing all indices to the right at this number. Continuing this process up to NNNN---N will give a set of indices of all the unique components. For the third moment of a four dimensional vector, this procedure yields,

111	222	
112	223	
113	224	
114	233	
122	234	
123	244	
124	333	
133	334	
134	344	
144	444	20 unique components

Let n(N,D) represent the number of unique Dth moments of an N dimensional vector. Then from the enumeration strategy described above, it can be seen that;

$$n(N,D) = n(N,D-1) + n(N-1,D-1) + n(N-2,D-1) + \text{---} + n(1,D-1) \tag{C.2}$$

The terms on the right hand side of the above equation represent the number of unique moments for which the leftmost index equals 1, 2, --- N.

It is clear from the definitions, that,

$$n(1,D) = 1 \tag{C.3}$$

$$n(N,1) = N \tag{C.4}$$

It follows that,

$$n(N,2) = n(N,1) + n(N-1,1) + \text{---} + 1$$

$$= N + (N-1) + (N-2) + \text{---} + 1$$

$$= \sum_{j=1}^{N} j = N(N+1)/2 = \binom{N+1}{2} \tag{C.5}$$

$$n(N,3) = n(N,2) + n(N-1,2) + \text{---} + n(1,2)$$

$$= \binom{N+1}{2} + \binom{N}{2} + \binom{N-1}{2} + \text{---} + 1$$

$$= \frac{(N+1)N + N(N-1) + (N-1)(N-2) + \text{---} + 2}{2}$$

$$= \frac{2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + \text{---} + N(N+1)}{2}$$

$$= \tfrac{1}{2}(1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + \text{---} + N \cdot N)$$

$$+ \tfrac{1}{2}(1 + 2 + 3 + \text{---} + N) \tag{C.6}$$

From the theory of numbers,

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2} \qquad \sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6} \tag{C.7}$$

Combining (C.6) and (C.7), (C.8) is obtained.

$$n(N,3) = \tfrac{1}{2}\left[\frac{2(N^3 + 3N^2 + 2N)}{6}\right]$$

$$= \frac{(N+2)(N+1)N}{6} = \binom{N+2}{3} \tag{C.8}$$

Repeating the above process, it is seen that,

$$n(N,4) = n(N,3) + n(N-1,3) + \text{---} + n(1,3)$$

$$= \binom{N+2}{3} + \binom{N+1}{3} + \binom{N}{3} + \text{---} + 1$$

$$n(N,4) = \frac{(N+2)(N+1)N + (N+1)N(N-1) + \cdots + 6}{6}$$

$$= \frac{3\cdot2\cdot1 + 4\cdot3\cdot2 + \cdots + (N+2)(N+1)N}{6}$$

$$= \sum_{j=1}^{N}\left(j^3 + 3j^2 + 2j\right) \qquad (C.9)$$

From the theory of numbers,

$$\sum_{j=1}^{N} j^3 = \frac{N^2(N+1)^2}{4} \qquad (C.10)$$

$$n(N,4) = \frac{1}{6}\left[\frac{N^2(N+1)^2}{4} + \frac{N(N+1)(2N+1)}{2} + N(N+1)\right]$$

$$= \frac{1}{6}\left[\frac{N^4 + 6N^3 + 11N^2 + 6N}{4}\right]$$

$$= \binom{N+3}{4} \qquad (C.11)$$

The desired result is obtained by induction.

$$n(N,D) = \binom{N+D-1}{D} \qquad (C.12)$$

Appendix D

Compact Storage of the Unique Moments of an N Dimensional Vector

As was shown in Appendix C, the complete symmetry of the moments implies that the higher moments have a surprisingly small number of unique components. For example, the fourth moment of an eight dimensional vector, which has 4096 components if symmetry is not considered, has only 330 unique components.

The addressing of the components in a digital computer is simple when all 4096 components are computed. In this case the addressing scheme,

$$A_{ijk\ell} = 512(i-1) + 64(j-1) + 8(k-1) + \ell \qquad \text{(D.1)}$$

will assign consecutive addresses from 1 to 4096 to the components of $\mu 4_{ijk\ell}$ as i,j,k, and ℓ take on all values from one to eight. This is, however, wasteful of both computation and storage space as most of the components will be computed many times and stored in many locations.

It is desirable, then, to find a computation and addressing scheme that will allow computation of each unique component but once, and the storage of these components with consecutive addresses from 1 to 330.

A suitable computation method is indicated by the counting technique described in Appendix C. For example, a set of unique fourth moments are computed by the following program;

```
      DO TO 4 FOR I = 1,1,N
      DO TO 3 FOR J = I,1,N
      DO TO 2 FOR K = J,1,N
      DO TO 1 FOR L = K.1,N
1     DM4(I,J,K,L) = F(μ4ijkℓ,μ3ijk,μ2ij, mi, yi, etc.)
2     CONTINUE
3     CONTINUE
4     CONTINUE
```

The above computation method enumerates the components so that the indices are always non-decreasing from I through L. A suitable addressing method can be derived from the observation that each index value forms a base to which the bases of all indices to the right can be added to get the component address. That is, there exist a set of functions $f1(N,\ell)$, $f2(N,k)$, $f3(N,j)$, and $f4(N,i)$, such that the address of $\mu 4_{ijk\ell}$ will be given by,

$$A_{ijk\ell} = f4(N,i) + f3(N,j) + f2(N,k) + f1(N,\ell)$$

As i,j,k, and ℓ vary from 1 to N in nondecreasing order, then $A_{ijk\ell}$ will vary from 1 to $\binom{N+4-1}{4}$. From this, it is clear that,

$$\begin{aligned} f1(N,1) &= 1 \\ f2(N,1) &= 0 \\ f3(N,1) &= 0 \\ f4(N,1) &= 0 \end{aligned} \qquad \text{(D.3)}$$

From the counting procedure of Appendix C and equation (D.2), the filling of consecutive addresses as the indices vary from right to left implies that the following difference equations must be satisfied;

$$\begin{aligned} f1(N,\ell) &= f1(N,\ell-1) + 1 \\ f2(N,k) &= f2(N,k-1) + f1(N,N) - f1(N,k) + 1 \\ f3(N,j) &= f3(N,j-1) + f2(N,N) + f1(N,N) - f2(N,j) - f1(N,j) + 1 \\ f4(N,i) &= f4(N,i-1) + f3(N,N) + f2(N,N) + f1(N,N) - f3(N,i) \\ &\quad - f2(N,i) - f1(N,i) + 1 \end{aligned} \qquad \text{(D.4)}$$

The functions f1 through f4 can be computed from the difference equations (D.4) with the initial conditions (D.3). Furthermore, if $f2(N,k)$, $f3(N,j)$, and $f4(N, i)$ are expanded as power series in N, then the coefficients of the powers of N will be found to obey difference equations with known initial conditions. Using a variety of series formulas, these difference equations can be solved to yield the coefficients in terms of i,j,k, and ℓ. The first few terms and the closed form solutions for f1 through f4 are given below.

ℓ	f1(N,ℓ)
1	1
2	2
3	3
ℓ	ℓ

(D.5)

k	f2(N,k)
1	0
2	N-1
3	2N-3
4	3N-6
k	(k-1)(N-k)

(D.6)

j	f3(N,j)
1	0
2	$\frac{1}{2}(N^2-N)$
3	$\frac{1}{2}(2N^2-4N)+1$
4	$\frac{1}{2}(3N^2-9N)+4$
5	$\frac{1}{2}(4N^2-16N)+10$
6	$\frac{1}{2}(5N^2-25N)+20$
j	$\frac{(j-1)}{2}N^2-\frac{(j-1)^2}{2}N+\frac{j(j-1)(j-2)}{6}$

(D.7)

i	f4(N,i)
1	0
2	$\frac{1}{6}(N^3-N)$
3	$\frac{1}{6}(2N^3-3N^2+N)$
4	$\frac{1}{6}(3N^3-9N^2+12N)-1$
5	$\frac{1}{6}(4N^3-18N^2+38N)-5$
6	$\frac{1}{6}(5N^3-30N^2+85N)-15$

$$i \quad \frac{(i-1)}{6} N^3 - \frac{(i-1)(i-2)}{4} N^2 + \left[\frac{i(i-1)(2i-1)}{12} - \frac{i(i-1)}{2} + \frac{(i-1)}{3}\right] N$$
$$- \frac{i(i-1)(i-2)(i-3)}{24} \qquad (D.8)$$

The desired addressing scheme for the fourth moment with the indices nondecreasing from i to ℓ is given by,

$$A_{ijk\ell} = \frac{(i-1)}{6} N^3 - \left[\frac{(i-1)(i-2)}{4} - \frac{(j-1)}{2}\right] N^2$$
$$+ \left[\frac{i(i-1)(2i-1)}{12} - \frac{i(i-1)}{2} + \frac{(i-1)}{3} - \frac{(j-1)^2}{2} + (k-1)\right] N$$
$$- \left[\frac{i(i-1)(i-2)(i-3)}{24} - \frac{j(j-1)(j-2)}{6} + k(k-1) - \ell\right] \qquad (D.9)$$

Note that $A_{1111} = 1$, and $A_{NNNN} = \binom{N+3}{4}$ (D.10)

Computation and consecutive storage of the third and second moments.

The derivation of the addressing method for the third moments is a special case of the above derivation. That is, f4(N,i) does not exist and equations (D.2) and (D.9) hold with all terms involving i deleted. Similarly, the equations for the second moment are obtained by deleting all terms involving i and j.

Address Tables

In practice, the solution of (D.9) for the address of $\mu 4_{ijk\ell}$ would require a significant amount of computation, as (D.9) would have to be solved each time a component of $\mu 4_{ijk\ell}$ is stored or retrieved. An alternative method better suited to digital computers is the use of address tables.

That is, in order to manipulate the D th moment of an N vector, define D N vectors f1(ℓ), f2(k), f3(j), etc.. Load these vectors with the solutions of the difference equations (D.4) with initial conditions (D.3). This computation need be done only once. Then the address function will be simply;

$$A_{ij -- cd} = f1(d) + f2(c) + --- + fD(i) \qquad (D.11)$$

A total of DN storage locations will be required for the address tables.

Retrieval

In order to retrieve a moment component, it will be necessary to first rearrange the specified indices in nondecreasing order, and then solve (D.11) for the address. The proper component can then be retrieved at that address plus the address of the beginning of the moment storage area - 1.

Appendix E

Approximation of the Density Function using Orthogonal Polynomials

A fundamental problem in nonlinear estimation theory is that, in general, the probability density function is nongaussian and cannot be described exactly by a finite number of parameters. It is desirable, then, to describe the density function by a sequence of parameters of diminishing importance, so that the sequence can be truncated at some point to obtain the desired accuracy using as few parameters as possible.

It has been suggested by Kushner and others (P-16),(T-5) that the central moments of the probability density form such a parameter set, and that these can be approximated as zero for moments above a certain order. It is evident, however, that this assumption is of limited physical validity, as the even central moments are in general nonzero. It is pointed out by Fisher (T-2) that this assumption implies that the density function contains derivatives of all orders of the Dirac delta function.

To avoid these difficulties, it has been suggested by Stratonovich and others (B-17),(B-4),(B-11) that the density function be approximated as a gaussian density multiplied by a sum of orthogonal polynomials. The orthogonality property would ensure that the addition of a higher polynomial to improve the fit would not change the "best fit" coefficients for the lower polynomials. The Legendre, Ultra Spherical, and Hermite polynomials have been suggested for this purpose. Of these, the Hermite polynomials appear to be the best choice, because they give a distribution that can be integrated analytically to yield the moments of all orders, and because their coefficients have a simple relationship to the central moments. The properties of this type of Hermite polynomial expansion have been extensively investigated by Stratonovich, (P-31),(P-32), who called the coefficients for the expansion quasi-moment functions on account of their similarity to the central moments.

Let $P(\underline{a},t)$ be the arbitrary N dimensional density function under consideration, and $P_g(\underline{a},t)$ be the N dimensional gaussian density function having the same mean and variance. Let $\phi(\underline{\alpha},t)$ and $\phi_g(\underline{\alpha},t)$ be the corresponding characteristic functions. Then, by definition,

$$P_g(\underline{a},t) = \frac{e^{-\frac{1}{2}(a_r-m_r(t))\mu 2^{-1}_{rs}(t)(a_s-m_s(t))}}{(2\pi)^{\frac{N}{2}}|\mu 2|^{\frac{1}{2}}} \quad \text{(E.1)}$$

$$\phi_g(\underline{\alpha},t) = e^{i\alpha_r m_r(t)-\frac{1}{2}\alpha_r \mu 2_{rs}(t)\alpha_s} \quad \text{(E.2)}$$

$m_r(t)$ and $\mu 2_{rs}(t)$ are the mean and covariance of the distributions. It can be shown that the quasi-moment functions are the coefficients of the expansion of the ratio $P(\underline{a},t)/P_g(\underline{a},t)$ in a series of multidimensional Hermite polynomials.

With this motivation, define,

$$\gamma(\underline{\alpha},t) = \frac{\phi(\underline{\alpha},t)}{\phi_g(\underline{\alpha},t)}$$

$$= \phi(\underline{\alpha},t)e^{-i\alpha_r m_r(t)+\frac{1}{2}\alpha_r \mu 2_{rs}(t)\alpha_s} \quad \text{(E.3)}$$

$\gamma(t)$ can now be expanded in an N dimensional Maclaurin series.

$$\gamma(\underline{\alpha},t) = 1 + i\, k1_j(t)\alpha_j + \frac{i^2}{2}\, k2_{jk}(t)\alpha_j\alpha_k + \cdots$$

$$\frac{i^n}{n!}\, kn_{\underbrace{jk\cdots l}_{N}}(t)\,\underbrace{\alpha_j\alpha_k\cdots\alpha_l}_{N} \qquad \text{(E.4)}$$

$$kn_{\underbrace{jk\cdots l}_{N}} = \frac{1}{i^n}\, \frac{\partial^n \gamma(\underline{\alpha},t)}{\partial\alpha_j\,\partial\alpha_k\cdots\partial\alpha_l}\Bigg|_{\underline{\alpha}=\underline{0}} \qquad \text{(E.5)}$$

The kn's are the quasi-moment functions of order n.

From the definitions of mean and covariance,

$$\frac{\partial\phi(\underline{\alpha},t)}{\partial\alpha_j}\Bigg|_{\underline{\alpha}=\underline{0}} = i\, m_j(t) \qquad \text{(E.6)}$$

$$\frac{\partial^2\phi(\underline{\alpha},t)}{\partial\alpha_j\,\partial\alpha_k}\Bigg|_{\underline{\alpha}=\underline{0}} = -(\mu 2_{jk}(t) + m_j(t)\, m_k(t)) \qquad \text{(E.7)}$$

Therefore, from (E.5), (E.6), and (E.7), it is apparent that,

$$k1_j(t) = 0 \qquad j = 1,2, \cdots N \qquad \text{(E.8)}$$

$$k2_{jk}(t) = 0 \qquad j,k = 1,2 \cdots N \qquad \text{(E.9)}$$

All first and second order quasi-moment functions are zero. That is, the quasi-moment functions have no effect on the mean or covariance of the distribution. Therefore the characteristic function can be written as follows.

$$\phi(\underline{\alpha},t) = e^{i\alpha_r m_r(t) - \frac{1}{2}\alpha_r \mu 2_{rs}(t)\alpha_s} \left\{ 1 + \sum_{n=3}^{\infty} \frac{i^n}{n!} kn_{\underbrace{jk--l}_{n}} \alpha_j \alpha_k -- \alpha_l \right. \tag{E.10}$$

The probability density is obtained by taking the N-dimensional Fourier transform of the characteristic function.

$$P(\underline{a},t) = \frac{1}{(2\pi)^N} \int_{R_N} \phi(\underline{\alpha},t)\, e^{-i\alpha_r a_r}\, d\underline{\alpha} \tag{E.11}$$

Similarly, $P_g(\underline{a},t)$ is the Fourier transform of $\phi_g(\underline{a},t)$.

$$P_g(\underline{a},t) = \frac{1}{(2\pi)^N} \int_{R_N} e^{-i\alpha_r(a_r - m_r(t)) - \frac{1}{2}\alpha_r \mu 2_{rs}(t)\alpha_s}\, d\underline{\alpha} \tag{E.12}$$

Substitution of (E.10) into (E.11) and using (E.12) yields,

$$P(\underline{a},t) = P_g(\underline{a},t) + \sum_{n=3}^{\infty} \frac{(-1)^n}{n!} kn_{\underbrace{jk--l}_{n}}(t) \frac{\partial^n P_g(\underline{a},t)}{\partial a_j \partial a_k --- \partial a_l} \tag{E.13}$$

The arbitrary distribution $P(\underline{a},t)$ can be written as the product of a gaussian density times a sum of n th order N dimensional Hermite polynomials, Hn_{jk--l}.

$$Hn_{jk--l}(\underline{a},t) = (-1)^n e^{\frac{1}{2}a_r \mu 2^{-1}_{rs}(t) a_s} \frac{\partial^n \left(e^{-\frac{1}{2}a_r \mu 2^{-1}_{rs} a_s}\right)}{\partial a_j \partial a_k --- \partial a_l} \tag{E.14}$$

From (E.14), the Hermite polynomials can be written in terms of the gaussian distribution and its derivatives.

$$Hn_{jk--l}(\underline{a}-\underline{m}(t),t) = \frac{(-1)^n}{P_g(\underline{a},t)} \underbrace{\frac{\partial^n P_g(\underline{a},t)}{\partial a_j \partial a_k --- \partial a_l}}_{n} \tag{E.15}$$

Equation (E.13) can now be rewritten using (E.15).

$$P(\underline{a},t) = P_g(\underline{a},t)\left\{1 + \sum_{n=3}^{\infty} \frac{1}{n!} \; kn_{jk--1} \; Hn_{jk--1}(\underline{a}-\underline{m}(t),t)\right\} \qquad (E.16)$$

From either (E.13) or (E.16), it is clear that $P(\underline{a},t)$ can be easily expressed in terms of the mean, covariance, and third and higher order quasi-moment functions. Furthermore, these terms have a relatively simple physical interpretation. The mean specifies the "center" of the distribution, the covariance specifies the first order "spread" of the distribution, the third order quasi-moment specifies the first order asymmetry or "skewness", and so forth. In addition, the assumption that all k's higher than a certain order are zero is perfectly reasonable, resulting in a smooth density function.

The parameters m_j, $\mu 2_{jk}$, $k3_{jkl}$, etc. represent an ideal sequence of parameters of diminishing importance. Any distribution function can be approximated to any desired degree of accuracy in the integrated square error sense by a sufficient number of these parameters. If the distribution is close to gaussian, as many distributions of practical interest are, a small number of parameters will give an accurate representation. Further, the addition of a higher order term will not affect the optimal choice of any lower order terms, because of the orthogonality property of the Hermite polynomials.

It is convenient to derive the optimal estimation equations in terms of the central moments. This yields an infinite set of coupled nonlinear differential equations. This sequence may be truncated by the assumption that all quasi-moments higher than a certain order are zero, provided that the relationships between the central moments and quasi-moments are known. These can be obtained by expanding $\phi(\underline{a},t)\, e^{-i\, a_r\, m_r(t)}$ in a Maclaurin series and comparing the resulting series term by term with the series in (E.10). This was done incorrectly by Fisher. ((T-2), Appendix B) Fisher omitted a number of terms, giving the correlation functions instead of the quasi-moment functions. The correct relations are given below with μn_{jk--l} representing the n th central moment with n subscripts.

$$\mu 3_{ijk}(t) = k3_{ijk}(t) \tag{E.17}$$

$$\mu 4_{ijkl}(t) = k4_{ijkl}(t) + 3\left\{\mu 2_{ij}(t)\, \mu 2_{kl}(t)\right\}_s \tag{E.18}$$

$$\mu 5_{ijklm}(t) = k5_{ijklm}(t) + 10\left\{\mu 2_{ij}(t)\, \mu 3_{klm}(t)\right\}_s \tag{E.19}$$

$$\mu 6_{ijklmn}(t) = k6_{ijklmn}(t) + 15\left\{\mu 2_{ij}(t)\, k4_{klmn}(t)\right\}_s + 15\left\{\mu 2_{ij}(t)\mu 2_{kl}(t)\mu 2_{mn}(t)\right\}_s + 10\left\{\mu 3_{ijk}(t)\mu 3_{lmn}(t)\right\}_s \tag{E.20}$$

$$\mu 7_{ijklmno}(t) = k7_{ijklmno}(t) + 21\left\{\mu 2_{ij}(t) k5_{klmno}(t)\right\}_s + 105\left\{\mu 2_{ij}(t)\mu 2_{kl}(t)\mu 3_{mno}(t)\right\}_s + 35\left\{\mu 3_{ijk}(t)\, k4_{lmno}(t)\right\}_s \tag{E.21}$$

The expression $N\{\ \}_s$ is used to signify the operation of symmetrizing the expression inside the brackets with respect to all of the subscripts. The number N is the number of terms in the expression. Equal symmetric terms, such as $\mu 2_{ij}\, \mu 2_{kl}$

and $\mu 2_{lk}\ \mu 2_{ji}$, are included only once. For example,

$$3\left\{\mu 2_{ij}\ \mu 2_{kl}\right\}_s = \mu 2_{ij}\mu 2_{kl} + \mu 2_{ik}\mu 2_{jl} + \mu 2_{il}\mu 2_{jk} \qquad \text{(E.22)}$$

Equations (E.17) - (E.21) can be used to find the central moments for a distribution described by a truncated sequence of quasi-moments. For example, if m_i, $\mu 2_{ij}$, and $k3_{ijk}$ are computed by the filter and all higher order quasi-moments are assumed to be zero, then the estimated central moments will be given by,

$$m_i(t) = m_i(t)$$

$$\mu 2_{ij}(t) = \mu 2_{ij}(t)$$

$$\mu 3_{ijk}(t) = k3_{ijk}(t)$$

$$\mu 4_{ijkl}(t) = 3\left\{\mu 2_{ij}(t)\mu 2_{kl}(t)\right\}_s$$

$$\mu 5_{ijklm}(t) = 10\left\{\mu 2_{ij}(t)\ k3_{klm}(t)\right\}_s \qquad \text{(E.23)}$$

etc.

Note that <u>all</u> the central moments will in general be nonzero, although the density is approximately described by a small number of parameters. Estimation of $m_i(t)$, $\mu 2_{ij}(t)$, and $k3_{ijk}(t)$, $i,j,k = 1,2,3,$ --- N is the task of the truncated third order nonlinear filter.

Appendix F

An Approximate Series Expansion Solution for the Posterior Moments

Following the receipt of a discrete gaussian measurement, the posterior probability density function is given by equation (5.13), derived from Bayes' equation.

$$P(\underline{x},t_n^+) = \frac{P(\underline{x},t_n)\ \exp(-\frac{1}{2}(y_r-h_r(\underline{x}))\Sigma_{rs}^{-1}(y_s-h_s(\underline{x})))}{\int_{R_N} P(\underline{a},t_n)\ \exp(-\frac{1}{2}(y_r-h_r(\underline{a}))\Sigma_{rs}^{-1}(y_s-h_s(\underline{a})))\ d\underline{a}} \quad (F.1)$$

The general solution of this equation in terms of the prior and posterior moments is very difficult. For the somewhat restricted case of linear measurements, however, an approximate solution can be generated by a series expansion of the exponential function. For a linear (or linearized) measurement,

$$h_r(\underline{x}) = h_r(\underline{m}) + H_{rs}(x_s-m_s) \quad (F.2)$$

The problem of truncating the moment sequence is exactly the same as in the continuous measurement case. In the following development, the density $P(\underline{x},t_n)$ will be described as a gaussian density times a sum of Hermite polynomials. A fourth order sum will be used, and all higher order terms in the quasi-moment expansion are assumed zero. Then, from Appendix E,

$$\mu 3_{ijk} = k3_{ijk}$$

$$\mu 4_{ijkl} = k4_{ijkl} + 3\{\mu 2_{ij}\mu 2_{kl}\}_s = k4_{ijkl} + \mu 4g_{ijkl}$$

k3,k4 = quasi-moment functions = coefficients in the Hermite polynomial expansion.

$\mu 4g$ = fourth moment of a gaussian distribution of covariance $\mu 2$.

$$\mu 5_{ijklm} = 10\{\mu 2_{ij}\, \mu 3_{klm}\}_s$$

$$\mu 6_{ijklmn} = 15\{\mu 2_{ij}\, k4_{klmn}\}_s + 10\{\mu 3_{ijk}\, \mu 3_{lmn}\}_s + 15\{\mu 2_{ij}\mu 2_{kl}\mu 2_{mn}\}_s \tag{F.3}$$

The first step in the evaluation of the posterior moments is the solution of the denominator integral of equation (F.1).

$$D = \int_{R_N} P(\underline{a},t_n)\, \exp(-\tfrac{1}{2}(y_r-h_r(\underline{m})-H_{rp}(a_p-m_p))\Sigma^{-1}_{rs}(y_s-h_s(\underline{m}) -H_{sq}(a_q-m_q)))\, d\underline{a} \tag{F.4}$$

The arbitrary density $P(\underline{a},t_n)$ can always be written as the sum of a gaussian and a nongaussian part. The gaussian part, $Pg(\underline{a},t_n)$ is simply a gaussian distribution with the same mean and covariance as $P(\underline{a},t_n)$. The nongaussian part is what is left over; IE,

$$Png(\underline{a},t_n) \doteq P(\underline{a},t_n) - Pg(\underline{a},t_n) \tag{F.5}$$

Consequently, the denominator integral D can be split into a gaussian and nongaussian part. The gaussian part can be obtained analytically, while the nongaussian portion can be obtained approximately by solving (F.4) by a series expansion technique, and then deleting all terms involving the moments of a gaussian distribution. This is better than using the series expansion alone, since it includes all of the infinite series terms for the gaussian part.

To obtain the series expansion solution, it is necessary to write (F.4) in an expansion of the prior moments. To facilitate the algebra, the following definitions are made.

$$\gamma = \text{scalar} = (y_r - h_r(\underline{m}) - H_{rp}(a_p - m_p))\Sigma_{rs}^{-1}(y_s - h_s(\underline{m}) - H_{sq}(a_q - m_q))$$

$$e_r = a_r - m_r$$
$$z_r = y_r - h_r(\underline{m})$$
$$\omega = z_r \Sigma_{rs}^{-1} z_s$$
$$\lambda_p = -2 H_{rp} \Sigma_{rs}^{-1} z_s$$
$$\Lambda_{qr} = H_{iq} \Sigma_{ij}^{-1} H_{jr} \qquad \text{(F.6)}$$

From the above,

$$\gamma = (z_r - H_{rp} e_p)\Sigma_{rs}^{-1}(z_s - H_{sq} e_q)$$
$$= z_r \Sigma_{rs}^{-1} z_s - 2 H_{rp} \Sigma_{rs}^{-1} z_s e_p + H_{rp} \Sigma_{rs}^{-1} H_{sq} e_p e_q$$
$$= \omega + \lambda_p e_p + \Lambda_{qr} e_q e_r \qquad \text{(F.7)}$$

In the following analysis, z_r is considered to be a number derived from a given measurement, and not a function of the state.

The exponential function can be expanded in a power series that converges for all values of the argument.

$$e^{-\frac{1}{2}\gamma} = \sum_{n=0}^{\infty} \frac{1}{(-2)^n n!} \gamma^n \tag{F.8}$$

Combining (F.7) and (F.8), the following relations are obtained

$$e^{-\frac{1}{2}\gamma} = e^{-\frac{1}{2}\omega}\sum_{n=0}^{\infty}\frac{1}{(-2)^n n!}(\lambda_p e_p + \Lambda_{qr}\, e_q\, e_r)^n$$

$$= e^{-\frac{1}{2}\omega}\sum_{n=0}^{\infty}\frac{1}{(-2)^n n!}\sum_{b=0}^{n}\binom{n}{b}(\lambda_p e_p)^{n-b}(\Lambda_{qr}e_q e_r)^b \tag{F.9}$$

By definition,

$$(\lambda_p e_p)^n = (\lambda_1 e_1 + \lambda_2 e_2 + \cdots \lambda_N e_N)^n \tag{F.10}$$

Therefore,

$$(\lambda_p e_p)^2 = \lambda_i \lambda_j e_i e_j$$

$$(\lambda_p e_p)^3 = \lambda_i \lambda_j \lambda_k\, e_i e_j e_k \quad \text{etc.} \tag{F.11}$$

From (F.4) and (F.9),

$$D = e^{-\frac{1}{2}\omega}\sum_{n=0}^{\infty}\frac{1}{(-2)^n n!}\sum_{b=0}^{n}\binom{n}{b}\int_{R_N} P(\underline{a}, t_n)\left[(\lambda_p e_p)^{n-b}(\Lambda_{qr}e_q e_r)^b\right] d\underline{a} \tag{F.12}$$

Since z_r is a number and not a function, λ_p and Λ_{qr} are independent of the state. The above equation for D can thus be expressed in terms of the prior central moments.

$$D = e^{-\frac{1}{2}\omega}\Big\{1 - \tfrac{1}{2}(\Lambda_{qr}\,\mu 2_{qr}) + \tfrac{1}{8}\,(\lambda_a \lambda_b \mu 2_{ab} + 2\,\lambda_a \Lambda_{bc}\,\mu 3_{abc}$$

$$+ \Lambda_{ab}\Lambda_{cd}\,\mu 4_{abcd}) - \tfrac{1}{48}\,(\lambda_a \lambda_b \lambda_c\,\mu 3_{abc}$$

$$+ 3\lambda_a\lambda_b\Lambda_{cd}\mu 4_{abcd} + 3\lambda_a\Lambda_{bc}\Lambda_{de}\mu 5_{abcde} + \Lambda_{ab}\Lambda_{cd}$$

$$\Lambda_{ef}\mu 6_{abcdef}) + \frac{1}{384}(\lambda_a\lambda_b\lambda_c\lambda_d\mu 4_{abcd} + 4\lambda_a\lambda_b\lambda_c\Lambda_{de}\mu 5_{abcde}$$

$$+ 6\lambda_a\lambda_b\Lambda_{cd}\Lambda_{ef}\mu 6_{abcdef} + ---) - \frac{1}{3840}(\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e\mu 5_{abcde}$$

$$+ 5\lambda_a\lambda_b\lambda_c\lambda_d\Lambda_{ef}\mu 6_{abcdef} + ---)$$

$$+ \frac{1}{46{,}080}(\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e\lambda_f\ \mu 6_{abcdef} + ---)\}$$

$$+ \text{terms involving } \mu 7, \mu 8, \text{ etc.} \qquad (F.13)$$

Collecting the coefficients of each moment up to sixth order, (F.13) becomes,

$$D = e^{-\frac{1}{2}\omega}\Big\{1 + \frac{1}{8}(\lambda_a\lambda_b - 4\Lambda_{ab})\mu 2_{ab} - \frac{1}{48}(\lambda_a\lambda_b\lambda_c - 12\lambda_a\Lambda_{bc})\mu 3_{abc}$$

$$+ \frac{1}{384}(\lambda_a\lambda_b\lambda_c\lambda_d - 24\lambda_a\lambda_b\Lambda_{cd} + 48\Lambda_{ab}\Lambda_{cd})\mu 4_{abcd}$$

$$- \frac{1}{3840}(\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e - 40\lambda_a\lambda_b\lambda_c\Lambda_{de} + 240\lambda_a\Lambda_{bc}\Lambda_{de})\mu 5_{abcde}$$

$$+ \frac{1}{46{,}080}(\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e\lambda_f - 60\lambda_a\lambda_b\lambda_c\lambda_d\Lambda_{ef} + 720\lambda_a\lambda_b\Lambda_{cd}\Lambda_{ef}$$

$$-960\Lambda_{ab}\Lambda_{cd}\Lambda_{ef})\mu 6_{abcdef}\Big\} + \text{higher order terms} \qquad (F.14)$$

The integral D can be thought of as the sum of the contributions of a normal distribution, plus third and fourth order quasi-moment functions. The gaussian part can be evaluated analytically as follows.

$$G(D) = \int_{R_N} \frac{e^{-\frac{1}{2}(x_r - m_r)\mu 2^{-1}_{rs}(x_s - m_s) - \frac{1}{2}(y_r - H_{rp}x_p)\Sigma^{-1}_{rs}(y_s - H_{sq}x_q)}}{(2\pi)^{\frac{N}{2}}|\mu 2|^{\frac{1}{2}}} d\underline{x} \qquad (F.15)$$

This derivation is somewhat more convenient using vector notation. Equation (F.15) in vector form is,

$$G(D) = \int_{R_N} \frac{e^{-\frac{1}{2}((\underline{x}-\underline{m})^T \mu 2^{-1}(\underline{x}-\underline{m})+(\underline{y}-H\underline{x})^T \Sigma^{-1}(\underline{y}-H\underline{x}))}}{(2\pi)^{\frac{N}{2}}|\mu 2|^{\frac{1}{2}}} d\underline{x} \tag{F.16}$$

Let $\underline{e} = \underline{x}-\underline{m}$, $\underline{z} = \underline{y}-H\underline{m}$. Then the exponent in (F.16) can be written as follows.

$$\beta = \underline{e}^T \mu 2^{-1} \underline{e} + (\underline{z}-H\underline{e})^T \Sigma^{-1} (\underline{z}-H\underline{e})$$

$$\beta = \underline{e}^T \mu 2^{-1} \underline{e} + \underline{z}^T \Sigma^{-1} \underline{z} - 2\, \underline{z}^T \Sigma^{-1} H\, \underline{e} + \underline{e}^T H^T \Sigma^{-1} H\underline{e}$$

$$\beta = \underline{e}^T (\mu 2^{-1} + H^T \Sigma^{-1} H) \underline{e} - 2\, \underline{z}^T \Sigma^{-1} H\underline{e} + \underline{z}^T \Sigma^{-1} \underline{z} \tag{F.17}$$

Define Ω, $\underline{d}$, and K such that,

$$\beta = (\underline{e}-\underline{d})^T \Omega^{-1} (\underline{e}-\underline{d}) - 2K$$

$$\beta = \underline{e}^T \Omega^{-1} \underline{e} - 2\, \underline{d}^T \Omega^{-1} \underline{e} + \underline{d}^T \Omega^{-1} \underline{d} - 2K \tag{F.18}$$

From (F.17),

$$\Omega^{-1} = (\mu 2^{-1} + H^T \Sigma^{-1} H)$$

$$\underline{d} = \Omega H^T \Sigma^{-1} \underline{z}$$

$$K = \frac{1}{2} \underline{z}^T (\Sigma^{-1} H \Omega H^T \Sigma^{-1} - \Sigma^{-1}) \underline{z} \tag{F.19}$$

With these substitutions, (F.16) becomes,

$$G(D) = \frac{e^K}{(2\pi)^{\frac{N}{2}}|\mu 2|^{\frac{1}{2}}} \int_{R_N} e^{-\frac{1}{2}(\underline{e}-\underline{d})^T \Omega^{-1} (\underline{e}-\underline{d})} d\underline{e} \tag{F.20}$$

For the solution of (F.20) and other integrals to be encountered in this section, the following results are required.

$$I_0 = \int_{R_N} e^{-\frac{1}{2}(x_r-b_r)\Omega^{-1}_{rs}(x_s-b_s)} d\underline{x} \qquad =(2\pi)^{\frac{N}{2}}|\Omega_{ij}|^{\frac{1}{2}} \quad (F.21)$$

$$I_1 = \int_{R_N} e^{-\frac{1}{2}(x_r-b_r)\Omega^{-1}_{rs}(x_s-b_s)} (x_i-b_i)d\underline{x} = 0 \qquad (F.22)$$

$$I_2 = \int_{R_N} e^{-\frac{1}{2}(x_r-b_r)\Omega^{-1}_{rs}(x_s-b_s)} (x_i-b_i)(x_j-b_j)d\underline{x}$$

$$= (2\pi)^{\frac{N}{2}}|\Omega_{ij}|^{\frac{1}{2}}\Omega_{ij} \qquad (F.23)$$

$$I_3 = \int_{R_N} e^{-\frac{1}{2}(x_r-b_r)\Omega^{-1}_{rs}(x_s-b_s)} (x_i-b_i)(x_j-b_j)(x_k-b_k)d\underline{x} = 0 \qquad (F.24)$$

$$I_4 = \int_{R_N} e^{-\frac{1}{2}(x_r-b_r)\Omega^{-1}_{rs}(x_s-b_s)} (x_i-b_i)(x_j-b_j)(x_k-b_k)(x_l-b_l)d\underline{x}$$

$$= (2\pi)^{\frac{N}{2}}|\Omega_{ij}|^{\frac{1}{2}} (\Omega_{ij}\Omega_{kl} + \Omega_{ik}\Omega_{jl} + \Omega_{il}\Omega_{jk}) \qquad (F.25)$$

Combining (F.20) and (F.21), the gaussian part of the integral is given by,

$$G(D) = \frac{e^K|\Omega|^{\frac{1}{2}}}{|\mu 2|^{\frac{1}{2}}} = \frac{e^K|\mu 2^{-1}\,\Omega|^{\frac{1}{2}}}{|\mu 2^{-1}\,\mu 2|^{\frac{1}{2}}} = e^K|\Omega^{-1}\mu 2|^{-\frac{1}{2}}$$

$$G(D) = \frac{e^{-\frac{1}{2}\,\underline{z}^T(\Sigma^{-1}-\Sigma^{-1}H\Omega H^T\Sigma^{-1})\underline{z}}}{|I + H^T\Sigma^{-1}H\,\mu 2|^{\frac{1}{2}}} \qquad (F.26)$$

Since (F.14) is linear in the moments, and since (F.26) expresses the gaussian part of the integral, then (F.26) can be substituted for the gaussian part of (F.14). Using the quasi-moment truncation described by (F.3), the denominator integral can be written in the desired form as follows.

$$D = e^{-\frac{1}{2}\omega}\left\{\frac{e^{\theta}}{\alpha} - \frac{1}{48}(\lambda_a\lambda_b\lambda_c-12\,\lambda_a\Lambda_{bc})\mu 3_{abc}\right.$$

$$+\frac{1}{384}(\lambda_a\lambda_b\lambda_c\lambda_d-24\,\lambda_a\lambda_b\Lambda_{cd}+48\,\Lambda_{ab}\Lambda_{cd})E4_{abcd} +$$

$$- \frac{1}{3840}(\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e - 40\,\lambda_a\lambda_b\lambda_c\Lambda_{de} + 240\,\lambda_a\Lambda_{bc}\Lambda_{de})E5_{abcde}$$

$$+ \frac{1}{46{,}080}(\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e\lambda_f - 60\,\lambda_a\lambda_b\lambda_c\lambda_d\Lambda_{ef} + 720\,\lambda_a\lambda_b\Lambda_{cd}\Lambda_{ef}$$

$$- 960\,\Lambda_{ab}\Lambda_{cd}\Lambda_{ef})E6_{abcdef}\Big\} + \text{higher order terms} \qquad (F.27)$$

$$z_r = y_r - h_r(\underline{m})$$

$$\omega = z_r \Sigma^{-1}_{rs}\, z_s$$

$$\Lambda_{ij} = H_{qi}\, \Sigma^{-1}_{qr}\, H_{rj}$$

$$\Omega_{ij} = (\mu 2^{-1}_{ij} + \Lambda_{ij})^{-1}$$

$$\theta = \tfrac{1}{2}\, z_r\, \Sigma^{-1}_{rj}\, H_{jk}\Omega_{kl}\, H_{ml}\Sigma^{-1}_{ms}\, z_s$$

$$\lambda_i = -\, 2\, H_{ri}\, \Sigma^{-1}_{rs}\, z_s$$

$$\alpha = \left|\delta_{ij} + \Lambda_{ik}\,\mu 2_{kj}\right|^{\frac{1}{2}}$$

$$E4_{abcd} = \mu 4_{abcd} - 3\left\{\mu 2_{ab}\,\mu 2_{cd}\right\}_s$$

$$E5_{abcde} = 10\left\{\mu 2_{ab}\,\mu 3_{cde}\right\}_s$$

$$E6_{abcdef} = 15\left\{\mu 2_{ab}\, E4_{cdef}\right\}_s + 10\left\{\mu 3_{abc}\,\mu 3_{def}\right\}_s \qquad (F.28)$$

The posterior density function is now given by,

$$P(\underline{x},t_n^+) = \frac{P(\underline{x},t_n)\; e^{-\frac{1}{2}(z_r - H_{rp}e_p)\,\Sigma^{-1}_{rs}(z_s - H_{sq}e_q)}}{D} \qquad (F.29)$$

The conditional posterior mean (a minumum variance estimate) can now be obtained from the expression,

$$m_i^+ = \int_{R_N} P(\underline{x},t_n^+)x_i \, d\underline{x} = \frac{1}{D}\int_{R_N} P(\underline{x},t_n)\, e^{-\frac{1}{2}\gamma}((x_i-m_i)+m_i)\, d\underline{x}$$

$$m_i^+ = m_i + \frac{1}{D}\int_{R_N} P(\underline{x},t_n)\, e^{-\frac{1}{2}\gamma}(x_i-m_i)\, d\underline{x} \qquad (F.30)$$

From equation (F.7), the above yields,

$$m_i^+ = m_i + \frac{e^{-\frac{1}{2}\omega}}{D}\int_{R_N} P(\underline{x},t_n)\, e^{-\frac{1}{2}(\lambda_p e_p + \Lambda_{qr}e_q e_r)} e_i \, d\underline{x} \qquad (F.31)$$

Following the results of (F.9) and (F.12), (F.31) can be written as,

$$m_i^+ = m_i + \frac{e^{-\frac{1}{2}\omega}}{D}\sum_{n=0}^{\infty}\frac{1}{(-2)^n n!}\sum_{b=0}^{n}\binom{n}{b}\overline{(\lambda_p e_p)^{n-b}(\Lambda_{qr}e_q e_r)^b e_i} \qquad (F.32)$$

$$m_i^+ = m_i + \frac{e^{-\frac{1}{2}\omega}}{D}\Big\{-\tfrac{1}{2}(\lambda_a\mu 2_{ia} + \Lambda_{ab}\mu 3_{iab}) + \tfrac{1}{8}(\lambda_a\lambda_b\mu 3_{iab}$$

$$+ 2\,\lambda_a\Lambda_{bc}\mu 4_{abci} + \Lambda_{ab}\Lambda_{cd}\mu 5_{abcdi}) - \tfrac{1}{48}(\lambda_a\lambda_b\lambda_c\mu 4_{abci}$$

$$+ 3\,\lambda_a\lambda_b\Lambda_{cd}\mu 5_{abcdi} + 3\,\lambda_a\Lambda_{bc}\Lambda_{de}\mu 6_{abcdei} + \cdots)$$

$$+\tfrac{1}{384}(\lambda_a\lambda_b\lambda_c\lambda_d\mu 5_{abcdi} + 4\,\lambda_a\lambda_b\lambda_c\Lambda_{de}\mu 6_{abcdei} + \cdots)$$

$$-\tfrac{1}{3840}(\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e\mu 6_{abcdei} + \cdots)\Big\} + \text{higher order terms} \qquad (F.33)$$

The gaussian part of m_i^+ can be computed as before.

From (F.20),

$$G(m_i^+) = \frac{e^{-\frac{1}{2}\omega}e^{\Theta}}{D(2\pi)^{\frac{N}{2}}|\mu 2|^{\frac{1}{2}}}\int_{R_N} e^{-\frac{1}{2}(e_i-d_i)\Omega^{-1}_{ij}(e_j-d_j)}((e_i-d_i)+m_i+d_i)\, d\underline{e} \qquad (F.34)$$

$$G(m_i^+) = \frac{e^{-\frac{1}{2}\omega} e^{\theta}}{D\,\alpha}(m_i + d_i) \qquad \text{(F.35)}$$

$$d_i = \Omega_{ij} H_{kj} \Sigma^{-1}_{kl} z_l \qquad \text{(F.36)}$$

Substituting (F.35) for the gaussian part of (F.33), and using the quasi-moment truncation relations, the desired equation for the conditional posterior mean is obtained.

$$m_i^+ = \frac{e^{-\frac{1}{2}\omega}}{D}\Big\{\frac{e^{\theta}}{\alpha}(m_i + d_i) + \frac{1}{8}(\lambda_a\lambda_b - .4\Lambda_{ab})\mu 3_{abi}$$
$$-\frac{1}{48}(\lambda_a\lambda_b\lambda_c - 12\,\lambda_a\Lambda_{bc})E4_{abci} + \frac{1}{384}(\lambda_a\lambda_b\lambda_c\lambda_d - 24\,\lambda_a\lambda_b\Lambda_{cd}$$
$$+48\Lambda_{ab}\Lambda_{cd})E5_{abcdi} - \frac{1}{3840}(\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e - 40\,\lambda_a\lambda_b\lambda_c\Lambda_{de}$$
$$+240\,\lambda_a\Lambda_{bc}\Lambda_{de})E6_{abcdei}\Big\} + \text{higher order terms} \qquad \text{(F.37)}$$

Higher Posterior Moments

The higher posterior moments about the prior mean can be derived in the same manner as (F.37). In order to find the posterior moments about the posterior mean, the following relationships are necessary. These follow directly from the definition of the central moments.

$$\epsilon_i = m_i^+ - m_i$$

$$\mu 2^+_{ij} = \mu 2^{+-}_{ij} - \epsilon_i\,\epsilon_j$$

$$\mu 3^+_{ijk} = \mu 3^{+-}_{ijk} - 3\{\epsilon_i \mu 2^+_{jk}\}_s - \epsilon_i\,\epsilon_j\,\epsilon_k$$

$$\mu4^{+}_{ijkl} = \mu4^{+-}_{ijkl} - 6\left\{\epsilon_i\ \epsilon_j\ \mu2^{+}_{kl}\right\}_s - 3\left\{\epsilon_i\ \mu3^{+}_{jkl}\right\}_s - \epsilon_i\ \epsilon_j\ \epsilon_k\ \epsilon_l \tag{F.38}$$

Using these relationships, the higher posterior moments about the posterior mean are found to be as follows.

$$\mu2^{+}_{ij} = \frac{e^{-\frac{1}{2}\omega}}{D}\left\{\frac{e^{\Theta}}{\alpha}(\Omega_{ij} + d_i d_j) - \tfrac{1}{2}\lambda_a \mu3_{aij} + \tfrac{1}{8}(\lambda_a\lambda_b - 4\Lambda_{ab})E4_{abij}\right.$$

$$- \tfrac{1}{48}(\lambda_a\lambda_b\lambda_c - 12\ \lambda_a\Lambda_{bc})E5_{abcij} + \tfrac{1}{384}(\lambda_a\lambda_b\lambda_c\lambda_d - 24\ \lambda_a\lambda_b\Lambda_{cd}$$

$$\left. + 48\Lambda_{ab}\Lambda_{cd})E6_{abcdij}\right\} - \epsilon_i\,\epsilon_j \ + \text{ higher order terms} \tag{F.39}$$

$$\mu3^{+}_{ijk} = \frac{e^{-\frac{1}{2}\omega}}{D}\left\{\frac{e^{\Theta}}{\alpha}(d_i\,\Omega_{jk} + d_j\Omega_{ik} + d_k\Omega_{ij} + d_i d_j d_k)\right.$$

$$+ \mu3_{ijk} - \tfrac{1}{2}\lambda_a E4_{aijk} + \tfrac{1}{8}(\lambda_a\lambda_b - 4\Lambda_{ab})E5_{abijk} - \tfrac{1}{48}(\lambda_a\lambda_b\lambda_c$$

$$\left. - 12\ \lambda_a\Lambda_{bc})E6_{abcijk}\right\} - \epsilon_i\mu2^{+}_{jk} - \epsilon_j\mu2^{+}_{ik} - \epsilon_k\mu2^{+}_{ij} - \epsilon_i\,\epsilon_j\epsilon_k$$

$$+ \text{ higher order terms} \tag{F.40}$$

$$\mu4^{+}_{ijkl} = \frac{e^{-\frac{1}{2}\omega}}{D}\left\{\frac{e^{\Theta}}{\alpha}\Big[\Omega_{ij}\Omega_{kl} + \Omega_{ik}\Omega_{jl} + \Omega_{il}\Omega_{jk} + d_i d_j\Omega_{kl}\right.$$

$$+ d_i d_k\Omega_{jl} + d_i d_l\Omega_{jk} + d_j d_k\Omega_{il} + d_j d_l\Omega_{ik} + d_k d_l\Omega_{ij}$$

$$\left. + d_i d_j d_k d_l\Big] + E4_{ijkl} - \tfrac{1}{2}\lambda_a E5_{aijkl} + \tfrac{1}{8}(\lambda_a\lambda_b - 4\Lambda_{ab})E6_{abijkl}\right\}$$

$$- \epsilon_i \epsilon_j \mu 2^+_{kl} - \epsilon_i \epsilon_k \mu 2^+_{jl} - \epsilon_i \epsilon_l \mu 2^+_{jk} - \epsilon_j \epsilon_k \mu 2^+_{il} - \epsilon_j \epsilon_l \mu 2^+_{ik}$$

$$- \epsilon_k \epsilon_l \mu 2^+_{ij} - \epsilon_i \mu 3^+_{jkl} - \epsilon_j \mu 3^+_{ikl} - \epsilon_k \mu 3^+_{ijl} - \epsilon_l \mu 3^+_{ijk}$$

$$- \epsilon_i \epsilon_j \epsilon_k \epsilon_l \quad + \text{higher order terms} \tag{F.41}$$

The equations for higher posterior moments can be derived analogously. Equations (F.37) and (F.39) - (F.41) are the equations for updating the vector moments of a multidimensional nonlinear density function given a discrete linear measurement. As a simple illustration, the computation required to update the moments of a scalar system is given below.

$$E4 = \mu 4 - 3\,\mu 2^2$$

$$E5 = 10\,\mu 2\,\mu 3$$

$$E6 = 15\,\mu 2\,E4 + 10\,\mu 3^2$$

$$z = y - h(m)$$

$$\Lambda = H^2/\Sigma$$

$$\Omega = \mu 2/(1 + \Lambda \mu 2)$$

$$\theta = \tfrac{1}{2}\Omega (Hz/\Sigma)^2$$

$$\lambda = -2\,H\,z/\Sigma$$

$$\alpha = (1 + \Lambda \mu 2)^{\frac{1}{2}}$$

$$d = \Omega H\,z/\Sigma$$

$$G_1 = -\tfrac{1}{2}\lambda$$

$$G_2 = \tfrac{1}{8}(\lambda^2 - 4\Lambda)$$

$$G_3 = -\tfrac{1}{48}(\lambda^3 - 12\lambda\Lambda)$$

$$G_4 = \tfrac{1}{384}(\lambda^4 - 24\lambda^2\Lambda + 48\Lambda^2)$$

$$G_5 = -\frac{1}{3840}(\lambda^5 - 40\lambda^3\Lambda + 240\lambda\Lambda^2)$$

$$G_6 = \frac{1}{46{,}080}(\lambda^6 - 60\lambda^4\Lambda + 720\lambda^2\Lambda^2 - 960\Lambda^3) \qquad (F.42)$$

$$K = \frac{e^\theta}{\alpha} + G_3\mu 3 + G_4 E4 + G_5 E5 + G_6 E6 \qquad (F.43)$$

$$m^+ = \frac{1}{K}\left\{\frac{e^\theta}{\alpha}(m+d) + G_2\mu 3 + G_3 E4 + G_4 E5 + G_5 E6\right\} \qquad (F.44)$$

$$\epsilon = m^+ - m$$

$$\mu 2^+ = \frac{1}{K}\left\{\frac{e^\theta}{\alpha}(\Omega + d^2) + G_1\mu 3 + G_2 E4 + G_3 E5 + G_4 E6\right\} - \epsilon^2 \qquad (F.45)$$

$$\mu 3^+ = \frac{1}{K}\left\{\frac{e^\theta}{\alpha}(3\,d\Omega + d^3) + \mu 3 + G_1 E4 + G_2 E5 + G_3 E6\right\} - 3\epsilon\,\mu 2^+ - \epsilon^3 \qquad (F.46)$$

$$\mu 4^+ = \frac{1}{K}\left\{\frac{e^\theta}{\alpha}(3\Omega^2 + 6\,d^2\Omega + d^4) + E4 + G_1 E5 + G_2 E6\right\} - 4\epsilon\,\mu 3^+ - 6\epsilon^2\mu 2^+ - \epsilon^4 \qquad (F.47)$$

Equations (F.42) - (F.47) are the scalar update equations for a discrete measurement, based on a solution of the Bayes equation by a series expansion of the exponential function. The analytic (gaussian) part of the series has been replaced by its exact value, however. That is, the terms in the series corresponding to gaussian moments $(m, \mu 2, 3\,\mu 2^2, 15\,\mu 2^3$, etc.) were removed, and replaced with an analytic value that is the limit of the gaussian part of the infinite series. For this reason, if $\mu 3$, E4, E5, and E6 are all assumed zero, these equations are equivalent to the discrete Kalman updating formulas.

$$m^+ = m + d = m + \Omega H z/\Sigma$$

$$\mu 2^+ = \Omega = \mu 2/(1 + (H^2/\Sigma)\mu 2) \quad \text{(F.48)}$$

If the initial density function is nongaussian, however, successively better approximations to the posterior moments can be generated by including the $\mu 3$, E4, E5, and E6 terms in the equations. It should be emphasized that it is not absolutely necessary to include **all of these terms**; the higher terms simply represent better and better approximations to the exact solution.

Convergence

The series solution for the posterior moments derived above will always converge if enough terms are taken. Under certain conditions, however, the number of terms shown will be insufficient for an accurate solution. This will happen when the sequence $G_3\mu 3$, G_4 E4, G_5 E5, etc., does not approach zero with sufficient rapidity. This convergence is difficult to evaluate analytically, but it can be done very approximately as follows. Since for $\mu 2 \approx \Sigma$, $\mathcal{E}(e^{\Theta}/\alpha) \approx 1$, from (F.43) is seems reasonable to require that,

$$\mathcal{E}(G_6 \text{ E6}) < .1 \quad \text{(F.49)}$$

G_6 is dominated by its first term. This yields the very approximate relationship,

$$\mathcal{E}(g_6) \quad \frac{\mathcal{E}(\lambda^6)}{46{,}080.} \cdots = \frac{2^6 H^6 \mathcal{E}(z^6)}{46{,}080.\ \Sigma^6} \quad \text{(F.50)}$$

Since z is approximately gaussian, it follows that,

$$\mathcal{E}(G_6) \approx \frac{2^6 H^6 (\mu 2 + \Sigma)^3}{46,080. \Sigma^6} \tag{F.51}$$

if $\mu 6$ is that assumed for a gaussian distribution, the condition (F.49) becomes, very approximately,

$$\frac{2^6 H^6 (\mu 2 + \Sigma)^3 (15 \mu 2^3)}{46,080. \Sigma^6} < .1 \tag{F.52}$$

$$\Sigma > \frac{H \mu 2}{2} \tag{F.53}$$

This implies that the series expansion method converges rather slowly, and cannot handle measurements as accurate as the prior covariance with a reasonable number of terms. This result was verified by simulation tests, which show that the series expansion indeed converges very slowly for many cases of practical interest. In fact, the inclusion of additional terms often causes the error to increase, even though the series eventually converges.

The series expansion method also requires quite lengthy computation, particularly for systems of high order. For these reasons, the series expansion method appeared to be of quite limited usefulness, and other methods of solving the Bayes equation were investigated.

Appendix G

An Approximate Solution by Integration for the Posterior Moments

After receiving a discrete gaussian measurement, the posterior probability density function is given by Bayes equation, (F.1). Suppose, however, that instead of a discrete measurement of covariance Σ, a continuous measurement of power spectral density Γ had been received over a short time interval Δt. The central moments at the end of the interval could then be found by integrating the continuous estimation equations (4.66) - (4.70). If Δt were sufficiently small, then the system dynamics could be neglected and the continuous third order estimation equations would simplify to the following.

$$z_s(t) = y_s(t) - h(\underline{m}(t)) \tag{G.1}$$

$$\frac{dm_i}{dt} = \mu 2_{ie} H_{re} \Gamma^{-1}_{rs} z_s \tag{G.2}$$

$$\frac{d\mu 2_{ij}}{dt} = -\mu 2_{ie} H_{re} \Gamma^{-1}_{rs} H_{sn} \mu 2_{jn} + \mu 3_{ije} H_{re} \Gamma^{-1}_{rs} z_s \tag{G.3}$$

$$\frac{d\mu 3_{ijk}}{dt} = -3\left\{\mu 2_{ie} H_{re} \Gamma^{-1}_{rs} H_{sf} \mu 3_{jkf}\right\}_s + \mu 4_{ijke} H_{re} \Gamma^{-1}_{rs} z_s \tag{G.4}$$

It is apparent that the central moments could be updated for a discrete measurement by integration of the above equations over the interval $[0, \Delta t]$, if a continuous measurement over this interval could be found that was equivalent to the discrete measurement.

For a second order filter, $\mu 3_{ijk}(t) = 0$. Then the above equations for $m_i(t)$ and $\mu 2_{ij}(t)$ become uncoupled, and can be solved analytically. From (G.3), the solution for a scalar system is as follows.

$$\frac{d\mu 2(\tau)}{d\tau} = - H^2 \Gamma^{-1} \mu 2^2(\tau) \tag{G.5}$$

The solution of the variance equation is,

$$\mu 2(\tau) = \frac{\mu 2(0)}{(1+ H^2 \Gamma^{-1} \mu 2(0) \tau)} \tag{G.6}$$

Since this must be equal to the Kalman filter estimate at $\tau = \Delta t$, it is necessary to choose Γ as follows.

$$\Gamma = \Sigma \Delta t \tag{G.7}$$

Applying (G.1) and (G.2) to a linear measurement yields the following;

$$\frac{dm(\tau)}{d\tau} = \mu 2(\tau) H \Gamma^{-1} (y(\tau) - H m(\tau)) \tag{G.8}$$

If now $y(\tau)$ is taken to be a constant, and (G.6) is used, the above becomes,

$$\frac{d(y-Hm(\tau))}{d\tau} = \frac{- H^2 \Gamma^{-1} \mu 2(0) (y-Hm(\tau))}{(1+ H^2 \Gamma^{-1} \mu 2(0) \tau)} \tag{G.9}$$

$$(y-Hm(\tau)) = \frac{(y-Hm(0))}{(1+ H^2 \Gamma^{-1} \mu 2(0) \tau)} \tag{G.10}$$

$$m(\tau) = m(0) + \frac{H\,\tau\,(v - H\,m(0))}{\Gamma\left(\frac{1}{\mu^2(0)} + \frac{H^2\tau}{\Gamma}\right)} \qquad \text{(G.11)}$$

If v is equal to the discrete measurement, and $\Gamma = \Sigma\,\Delta t$, then (G.11) is equivalent to the Kalman updating formula.

For a third order filter, the coupled equations (G.1) - (G.4) can be integrated over the interval $[0,\ \Delta t]$ with the substitutions,

$y_s = y_s(n\,\delta t)$ (the discrete measurement)

$$\Gamma_{ij} = \Sigma_{ij}\,\Delta t \qquad \text{(G.12)}$$

In order for this updating method to be optimal, the integrals of the continuous and discrete measurement residues must be equivalent. Furthermore, it is required that the second and higher integrals be equivalent as well, since the estimation equations are coupled. If these conditions are met, the short duration continuous measurement would be equivalent to the discrete measurement, and the continuous updating equations could be used.

The continuous and discrete measurement processes are described by,

$$y(t) = H\,x + R\,\xi(t)$$

$$\mathcal{E}(\xi(t)) = 0$$

$$\mathcal{E}(\xi(t)\,\xi(\tau)) = \delta(t-\tau)$$

$$R^2 = \Gamma = \Sigma\,\Delta t \qquad \text{(G.13)}$$

$$y(n\delta t) = H\,x + B\,\nu(n\delta t) \qquad B^2 = \Sigma$$

$$\mathcal{E}(\nu(n\delta t)) = 0$$

$$\mathcal{E}(\nu(m\delta t)\nu(n\delta t)) = \delta_{mn} \tag{G.14}$$

The integrals of the continuous and discrete measurement residues can now be written as follows.

$$I_c = \frac{H}{\Sigma \Delta t} \int_0^{\Delta t} (H\,x + R\xi(\tau) - H\,m(\tau))\,d\tau \tag{G.15}$$

$$I_d = \frac{H}{\Sigma \Delta t} \int_0^{\Delta t} (H\,x + B\nu(n\delta t) - H\,m(\tau))\,d\tau \tag{G.16}$$

Consequently,

$$I_c = \frac{H^2}{\Sigma\,\Delta t} \int_0^{\Delta t} (x-m(\tau))\,d\tau + \frac{H\,R}{\Sigma\,\Delta t} \int_0^{\Delta t} \xi(\tau)\,d\tau \tag{G.17}$$

$$I_d = \frac{H^2}{\Sigma\,\Delta t} \int_0^{\Delta t} (x-m(\tau))\,d\tau + \frac{H\,B}{\Sigma\,\Delta t} \int_0^{\Delta t} \nu(n\delta t)\,d\tau \tag{G.18}$$

From the above, it is clear that the means of the two integrals are identical. The variances are found to be given by,

$$\mathrm{Var}(I_c) = (H\,R/\Sigma\Delta t)^2\,\Delta t \quad = H^2/\Sigma \tag{G.19}$$

$$\mathrm{Var}(I_d) = (H\,B/\Sigma\Delta t)^2\,\Delta t^2 \quad = H^2/\Sigma \tag{G.20}$$

The integrals of the continuous and discrete measurement processes are therefore identical. Since the estimation equations are coupled, it is required that the double integrals and nth integrals be equivalent, also.

Computation of the double integrals will show that they are not equivalent, because the correlation properties for higher integrals of the white noise cause a difference.

This method can therefore not be used to solve equations coupled in the measurements. It can be used to generate an approximate solution by ignoring the measurement contribution to the third moment. That is, the third moment inferred from the system operation between measurements can be used to more accurately estimate the mean and variance, but the measurement value is not used to update the third moment. This approximate integration updating method is not as accurate as the exact solution of the Bayes equation, but may be more stable or simpler in some cases.

The integration update is accomplished by integrating the following equations over the interval $\tau = [0, 1]$ each time a discrete measurement of covariance Σ is received.

$$\zeta_e = H_{re} \Sigma^{-1}_{rs} (y_s - H_{sn} m_n(\tau))$$

$$\beta_{ab} = \mu 2_{ae} H_{re} \Sigma^{-1}_{rs} H_{sb}$$

$$\gamma_{ac} = \beta_{ab} \mu 2_{cb}$$

$$\frac{dm_i}{d\tau} = \mu 2_{ie} \zeta_e(\tau)$$

$$\frac{d\mu 2_{ij}}{d\tau} = - \gamma_{ij} + \mu 3_{ije} \zeta_e(\tau)$$

$$\frac{d\mu 3_{ijk}}{d\tau} = - 3\left\{\beta_{ie} \mu 3_{jke}\right\}_s \qquad (G.21)$$

The integration over $[0, \Delta t]$ with $\Gamma = \Sigma \Delta t$ can be replaced by an integration over $[0, 1]$ with $\Gamma = \Sigma$, since the differential equations are all linear in Γ.

This approximate updating method has been simulation tested for a scalar system, and gave good results. It appears to be of limited usefulness, however, since it is more complex than the exact solution for low order systems, and can react unstably with divergent systems. The exact solution of the Bayes equation, which was discovered after this approximate method, appears to be more promising.

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BIOGRAPHY

Calvin O. Culver was born December 6, 1941 in Montour Falls, New York. His parents live near Trumansburg, New York where he attended Trumansburg Central School, graduating in 1959.

He entered M. I. T., and received his Bachelor of Science and Master of Science degrees from the Department of Aeronautics and Astronautics in June, 1964. As an undergraduate, he was manager and Captain of the M.I.T. pistol team, and was elected to Tau Beta Pi and Sigma Gamma Tau honorary fraternities. As a senior and graduate student, he was elected chapter president of Sigma Gamma Tau, and was awarded the Douglas Aircraft Co. fellowship.

Following graduation in June 1964, Mr. Culver was employed by Litton Systems, Inc. in Woodland Hills, California. He worked as an analyst in the Advanced Systems Engineering Department until his return to M.I.T. in September, 1966. At that time he joined the SABRE group of the M.I.T. Instrumentation Laboratory, and worked on optimal calibration methods for inertial navigation systems under the supervision of Mr. Kenneth Fertig. The thesis research was continued in this group, with support from a National Science Foundation Traineeship.

Mr. Culver has been resident tutor at Theta Chi fraternity since his return to M.I.T. in 1966. In 1968 he became an honorary member of the fraternity.