ANALYSIS OF FREE-EDGE EFFECTS IN COMPOSITE LAMINATES BY AN ASSUMED-STRESS METHOD

by

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ABSTRACT

An approximate, second-order theory of anisotropic laminated structures which permits the calculation of stresses in the vicinity of free edges is proposed. The theory is based on the assumption of an exponential subsidence of the edge-effect stresses which is characterized by a "boundary-layer thickness" parameter. The parameter is incorporated into an initially-assumed stress field which identically satisfies equilibrium and the free-boundary conditions and is asymptotic to the stress field predicted by first-order lamination theory in the laminate interior. Satisfaction of compatibility in the mean is then assured by determining the thickness parameter according to the principle of minimum complementary energy.

The theory is applied to the analysis of simple, balanced angle-ply and cross-ply laminates in NARMCO 5505 boron/epoxy and Morgenite II graphite/epoxy filamentary composites under uniaxial loading, and the results checked against finite-difference and finite-element computer solutions. Accuracy is found to be good. The behavior of the solutions under variations in ply orientation (angle-ply cases) and lamina thickness ratio (cross-ply cases) is investigated. The intensity and depth of penetration of the free-edge effects are discussed, and design recommendations are made. Extension of the theory to more general free-edge problems involving composite laminates is considered, and the relative merits as regards convenience of application of the approximate second-order theory and finite-element methods are discussed.

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Title: Professor of Aeronautics
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SYMBOLS

Symbol	Meaning
a	laminate half-width
$\mathcal{a}$	finite element y_2 dimension
dA	element of area
b	lamina thickness ratio
$\mathcal{b}$	finite element y_3 dimension
C_i $(i = 1,\ldots,3)$	cross-ply biquadratic coefficients
E_{ii} $(i = 1,\ldots,3)$	Young's moduli
$\mathcal{F}_i$ $(i = 1,\ldots,4)$	functions of b used in cross-ply integrations
G_{ij} $(i,j = 1,\ldots,3;\ i \neq j)$	shear moduli
$[H]$	strain energy generating matrix
k	boundary-layer thickness parameter
$[k]$	element stiffness matrix
$[L]$	surface displacement generating matrix
$\ell_{\tilde{i}j}$ $(i,j = 1,\ldots,3)$	direction cosines
n_m	unit normal vector in the y_m direction
$[P]$	stress generating matrix
$\{q\}$	corner displacement matrix
$[R]$	surface traction generating matrix
$[S]$	compliance matrix

Symbol	Meaning
ℓ_i $(i = 1,\ldots,7)$	functions of compliances
S_{ijkl} $(i,j,k,l = 1,\ldots,3)$	general tensor compliances
S_{ij} $(i,j = 1,\ldots,6)$	contracted tensor compliances
$S_{ij}^{(o)}$ $(i,j = 1,\ldots,6)$	contracted tensor compliances for uniaxial material
$[T]$	surface work generating matrix
t	lamina thickness
$\vec{u}$	surface displacement
$\{u\}$	surface displacement matrix
y_i $(i = 1,\ldots,3)$	Cartesian coordinates
dv	element of volume
$\{\beta\}$	stress coefficient matrix
ϵ_{ij} $(i,j = 1,\ldots,3)$	general tensor strains
ϵ_i $(i = 1,\ldots,6)$	contracted tensor strains
ϵ^*	lamination-theory strain in general lamina
ν_{ij} $(i,j = 1,\ldots,3;\ i \neq j)$	Poisson's ratios
Π_c	complementary energy
σ_{ij} $(i,j = 1,\ldots,3)$	general tensor stresses
σ_i $(i = 1,\ldots,6)$	contracted tensor stresses
$\{\sigma\}$	stress matrix
$\vec{\sigma}_m$	resolved surface traction in the y_m direction
$\{\vec{\sigma}\}$	surface traction matrix
$\sigma^{(+)}$	lamination-theory stress in $+\theta$ lamina
$\sigma^{(-)}$	lamination-theory stress in $-\theta$ lamina

Symbol	Meaning
$\sigma^{(0)}$	lamination-theory stress in 0° lamina
$\sigma^{(90)}$	lamination-theory stress in 90° lamina
σ^{*}	lamination-theory stress in general lamina
θ	lamina angle

CHAPTER 1

INTRODUCTION

1.1 Filamentary Composites: Their Fabrication and Analysis

That class of composite materials which includes those made from continuous, high-strength and high-modulus filaments imbedded in a relatively low-strength and compliant matrix material is presently showing great promise for future application to aerospace and other structures where high strength-to-weight and stiffness-to-weight ratios are required. Materials of this type go by the generic name of "filamentary composites". Many such materials are presently under study or development, but work thus far is most advanced on those which use boron or graphite fibers in a polymeric matrix which may be either an epoxy or a polyimide.

Filamentary composites are produced and used in ways fundamentally different from those characteristic of the metals for which, in many applications, they may eventually be substituted. The boron filament is made by vapor-deposition from boron trichloride gas onto a heated tungsten core; the graphite by high-temperature decomposition of synthetic fibers similar to those used in the textile industry. The filaments are then coated with uncured (fluid) matrix material and made into tape of standard width three inches by laying them on a thin "scrim cloth", also resin-impregnated. This "prepreg tape", wound on reels with "parting paper" inserted between the wraps to prevent the tape (whose resin is now maintained in the "B-stage", or partially cured, by refrigeration) from adhering to itself, is the raw material from which the finished structure is made. To fabricate a structural element from the uniaxial tape, the manufacturer places a number of layers, or "laminae", of tape, one atop the other, at various orientations. Each lamina may be composed of any number of "plies" -- single thicknesses of tape. A ply thickness ranges from half a thousandth of an inch for graphite/epoxy to about five thousandths for boron/epoxy.

The tape -- and, consequently, structures produced from it -- is highly anisotropic in elastic and strength properties, being extremely stiff and strong in the fiber-axis direction but rather weak and compliant in shear and in either of the transverse directions. For this reason the filamentary composites offer significant advantages in specific strength and modulus over metals only in applications involving uniaxial or biaxial loading, or where otherwise three-dimensional loading environments can be broken down into uniaxial or biaxial loading paths by replacing one component with several discrete elements.

And also for this reason, the analytical treatment of filamentary composite structures is considerably more difficult than the analysis of corresponding metal structures.

The mathematical analysis of filamentary composites today can be broken down into three fundamental levels. The first of these may appropriately be called macromechanics, although it is more commonly known as the first-order theory of laminated structures or simply "lamination theory". Lamination theory first assumes the composite laminate to be anisotropic but deformationally homogeneous over its cross-section (by applying a simplified interlaminar compatibility condition) for the purpose of calculating the internal strain state resulting from a given loading. The elastic non-homogeneity of the laminate is then taken into account by calculating the stresses produced in each lamina by the strains thus computed. Lamination theory is therefore useful for predicting the overall stiffness properties and natural frequencies of vibration of composite structures, and the stresses in the interior regions of each lamina produced by a given loading environment.

In the regions near the edges and surfaces of laminated components, however, the stress and strain fields predicted by lamination theory are inaccurate. A level of analysis one might appropriately call mesomechanics is required to produce an accurate determination of the elastic fields in these regions by taking into account the mechanism of load transfer between the individual laminae.

By such a determination mesomechanical analysis makes possible the prediction of stress concentrations near the edges and surfaces of the material which are not identifiable by lamination theory and which may, in certain cases, account for failure of the structure by delamination, lamina buckling, or weakening of the bond between laminae.

Finally there is the field of micromechanics, which concerns itself with the properties and technology of the constituent materials from which the uniaxial composite tape is fabricated, and with the mechanism of load transfer from the matrix material to the fibers contained therein. At the "fine" end of its spectrum of interests micromechanics shades into polymer technology, crystallography, solid-state physics and other aspects of materials science, while at the "coarse" end it includes the failure mechanisms of uniaxial composite, such as fiber buckling in compression and fiber fracture and fiber-matrix debonding in tension and shear.

1.2 Free-Edge Effects and Their Importance to Laminate Performance

An edge or surface of an elastic body upon which there exists no normal or shear traction, or stress resultant, is said to be a free edge or free boundary. Free edges exist, for example, on the lateral surfaces of beams and torsion members under most loading conditions, on the unattached edges of aircraft control surfaces, and along the edges of holes and cutouts in plate and membrane structures.

Lamination theory, as has been mentioned, is inadequate to predict the stress distributions near the boundaries of composite laminates. This is especially true of free edges normal to the plane of the laminae, since some of the stresses predicted by lamination theory to be constant throughout each lamina must in fact go to zero at such edges. Lamination theory itself accounts for this by postulating interlaminar shear impulses which bring the transverse extensional stress and/or the in-plane shear stress to zero at any free lateral edge, but such a model does not provide an accurate picture of the actual stress state near the edge -- and a relatively good knowledge of the actual stress in these regions is necessary to a proper understanding of some of the failure mechanisms of composite laminates. There exist, for instance, in longitudinally-loaded structural elements made of laminated filamentary composites, interlaminar shear and tension distributions near the free edges which cannot be accurately identified by lamination theory. As these interlaminar stresses can have a significant effect on failure modes such as delamination and can significantly lower fatigue life, an accurate knowledge of the interlaminar stress fields and of the maximum values of the interlaminar stresses is required for the rational design of such structural components.

1.3 Study Objectives

The objectives of this investigation were twofold.

First, it was desired to develop a second-order (meso-mechanical) theory of laminated structures, one step in complexity beyond lamination theory, which would nevertheless allow the determination of stress distributions near the free edges of such structures with sufficient accuracy for design purposes. Secondly, it was desired to apply the theory to the formulation and solution of some simple problems in order to provide physical insight into the basic behavior of elastic fields near the free edges of composite laminates, and to check the accuracy of the solutions thus obtained against solutions obtained by finite-difference and/or finite-element methods which are known to converge to the exact elasticity solutions of these problems as the size of the differences or elements becomes infinitesimal.

CHAPTER 2

TENSOR NOTATION AND MATERIAL PROPERTIES

2.1 Contracted Tensor Notation

A contracted tensor notation has been adopted in the composite materials field, replacing the full tensor notation of the general mathematical theory of elasticity in order to effect a reduction in the number of subscripts required to identify a given stress, strain, or elastic constant, and also to simplify the equations required to write out the stress-strain relations. The compliance form of the stress-strain relations was used throughout the work of this study, as the developed form of the proposed second-order theory is based upon assumed functional forms for the stresses in the vicinity of the free edge.

The correspondence of the contracted tensor notation as given in reference [1] to the full, general elasticity tensor notation, using the compliance form of the stress-strain relations, is as follows:

$$\sigma_1 = \sigma_{11} \tag{2.1}$$

$$\sigma_2 = \sigma_{22} \tag{2.2}$$

$$\sigma_3 = \sigma_{33} \tag{2.3}$$

$$\sigma_4 = \sigma_{23} \tag{2.4}$$

$$\sigma_5 = \sigma_{13} \tag{2.5}$$

$$\sigma_6 = \sigma_{12} \tag{2.6}$$

$$\epsilon_1 = \epsilon_{11} \tag{2.7}$$

$$\epsilon_2 = \epsilon_{22} \tag{2.8}$$

$$\epsilon_3 = \epsilon_{33} \tag{2.9}$$

$$\epsilon_4 = 2\epsilon_{23} \tag{2.10}$$

$$\epsilon_5 = 2\epsilon_{13} \tag{2.11}$$

$$\epsilon_6 = 2\epsilon_{12} \tag{2.12}$$

$$S_{11} = S_{1111} \tag{2.13}$$

$$S_{22} = S_{2222} \tag{2.14}$$

$$S_{33} = S_{3333} \tag{2.15}$$

$$S_{12} = S_{1122} \tag{2.16}$$

$$S_{13} = S_{1133} \tag{2.17}$$

$$S_{23} = S_{2233} \tag{2.18}$$

$$S_{44} = 4S_{2323} \tag{2.19}$$

$$S_{55} = 4S_{1313} \tag{2.20}$$

$$S_{66} = 4S_{1212} \tag{2.21}$$

$$S_{45} = 4S_{2313} \tag{2.22}$$

$$S_{16} = 2S_{1112} \tag{2.23}$$

$$S_{26} = 2S_{2212} \tag{2.24}$$

$$S_{36} = 2S_{3312} \tag{2.25}$$

2.2 Uniaxial Material Properties

The contracted tensor compliance constants are computable from a set of engineering constants which can be measured directly by tension and torsion tests. Listed below are the equations for determining the compliances of uniaxial, or "zero-degree", material, in cartesian coordinates for which the y_1 direction is parallel to the filaments and the $y_1 y_2$ plane lies in the plane of the tape [1].

$$S_{11} = \frac{1}{E_{11}} \tag{2.26}$$

$$S_{22} = \frac{1}{E_{22}} \tag{2.27}$$

$$S_{33} = \frac{1}{E_{33}} \tag{2.28}$$

$$S_{12} = -\frac{\nu_{12}}{E_{11}} \tag{2.29}$$

$$S_{13} = -\frac{\nu_{13}}{E_{11}} \tag{2.30}$$

$$S_{23} = -\frac{\nu_{23}}{E_{22}} \tag{2.31}$$

$$S_{44} = \frac{1}{G_{23}} \tag{2.32}$$

$$S_{55} = \frac{1}{G_{13}} \tag{2.33}$$

$$S_{66} = \frac{1}{G_{12}} \tag{2.34}$$

where

$$E_{ii} = \frac{\sigma_{ii}}{\epsilon_{ii}} \quad (i = 1,\ldots,3) \tag{2.35}$$

$$G_{ij} = \frac{\sigma_{ij}}{\epsilon_{ij}} \quad (i,j = 1,\ldots,3;\ i \neq j) \tag{2.36}$$

$$\nu_{ij} = -\frac{\epsilon_{jj}}{\epsilon_{ii}} \text{ due to } \sigma_{ii} \quad (i,j = 1,\ldots,3;\ i \neq j) \tag{2.37}$$

The E_{ii} thus represent Young's moduli, the G_{ij} shear moduli, and the ν_{ij} Poisson's ratios for an anisotropic material. As there is no shear coupling in uniaxial material, the compliances S_{45}, S_{16}, S_{26}, and S_{36} are zero.

Two of the most common filamentary composites presently in use, NARMCO 5505 and Morgenite II, were chosen to supply actual material properties for use in calculation. NARMCO 5505 is a boron/epoxy material with the following engineering constants:

$$E_{11} = 31.8 \times 10^6 \text{ lb./in.}^2 \tag{2.38}$$

$$E_{22} = E_{33} = 3.14 \times 10^6 \text{ lb./in.}^2 \tag{2.39}$$

$$G_{12} = G_{13} = 1.0 \times 10^6 \text{ lb./in.}^2 \tag{2.40}$$

$$G_{23} = 0.32 \times 10^6 \text{ lb./in.}^2 \tag{2.41}$$

$$\nu_{12} = \nu_{13} = 0.21 \tag{2.42}$$

$$\nu_{23} = 0.25 \tag{2.43}$$

Substituting these values in (2.26) through (2.34) gives

$$S_{11} = 3.14 \times 10^{-8} \text{ in.}^2/\text{lb.} \tag{2.44}$$

$$S_{22} = S_{33} = 3.18 \times 10^{-7} \text{ in.}^2/\text{lb.} \tag{2.45}$$

$$S_{12} = S_{13} = -6.61 \times 10^{-9} \text{ in.}^2/\text{lb.} \tag{2.46}$$

$$S_{23} = -7.95 \times 10^{-8} \text{ in.}^2/\text{lb.} \tag{2.47}$$

$$S_{44} = 3.13 \times 10^{-6} \text{ in.}^2/\text{lb.} \tag{2.48}$$

$$S_{55} = S_{66} = 1.0 \times 10^{-6} \text{ in.}^2/\text{lb.} \tag{2.49}$$

Morgenite II, a graphite/epoxy composite, has uniaxial engineering constants as follows:

$$E_{11} = 20.0 \times 10^{6} \text{ lb./in.}^2 \tag{2.50}$$

$$E_{22} = E_{33} = 2.1 \times 10^{6} \text{ lb./in.}^2 \tag{2.51}$$

$$G_{12} = G_{13} = 0.85 \times 10^{6} \text{ lb./in.}^2 \tag{2.52}$$

$$G_{23} = 0.67 \times 10^{6} \text{ lb./in.}^2 \tag{2.53}$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.21 \tag{2.54}$$

from which one obtains

$$S_{11} = 5.0 \times 10^{-8} \text{ in.}^2/\text{lb.} \tag{2.55}$$

$$S_{22} = S_{33} = 4.77 \times 10^{-7} \text{ in.}^2/\text{lb.} \tag{2.56}$$

$$S_{12} = S_{13} = -1.05 \times 10^{-8} \text{ in.}^2/\text{lb.} \tag{2.57}$$

$$S_{23} = -1.0 \times 10^{-7} \text{ in.}^2/\text{lb.} \tag{2.58}$$

$$S_{44} = 1.49 \times 10^{-6} \text{ in.}^2/\text{lb.} \tag{2.59}$$

$$S_{55} = S_{66} = 1.18 \times 10^{-6} \text{ in.}^2/\text{lb.} \tag{2.60}$$

It must be noted here that there is by no means universal agreement on the values of the engineering constants of either NARMCO 5505 or Morgenite II. Different experimenters have measured different values, and even among identical tests performed by the same experimenter there is more scatter in the data than is the case with metals. Every effort was made to obtain the best available data for use in this study. Some of the values used were taken from [1], while others (more recent) were obtained in personal consultation with Mr. R. Byron Pipes of General Dynamics/Fort Worth. In cases where a given Young's modulus had a slightly different value in tension from that measured in compression, the average of the two values was taken.

CHAPTER 3

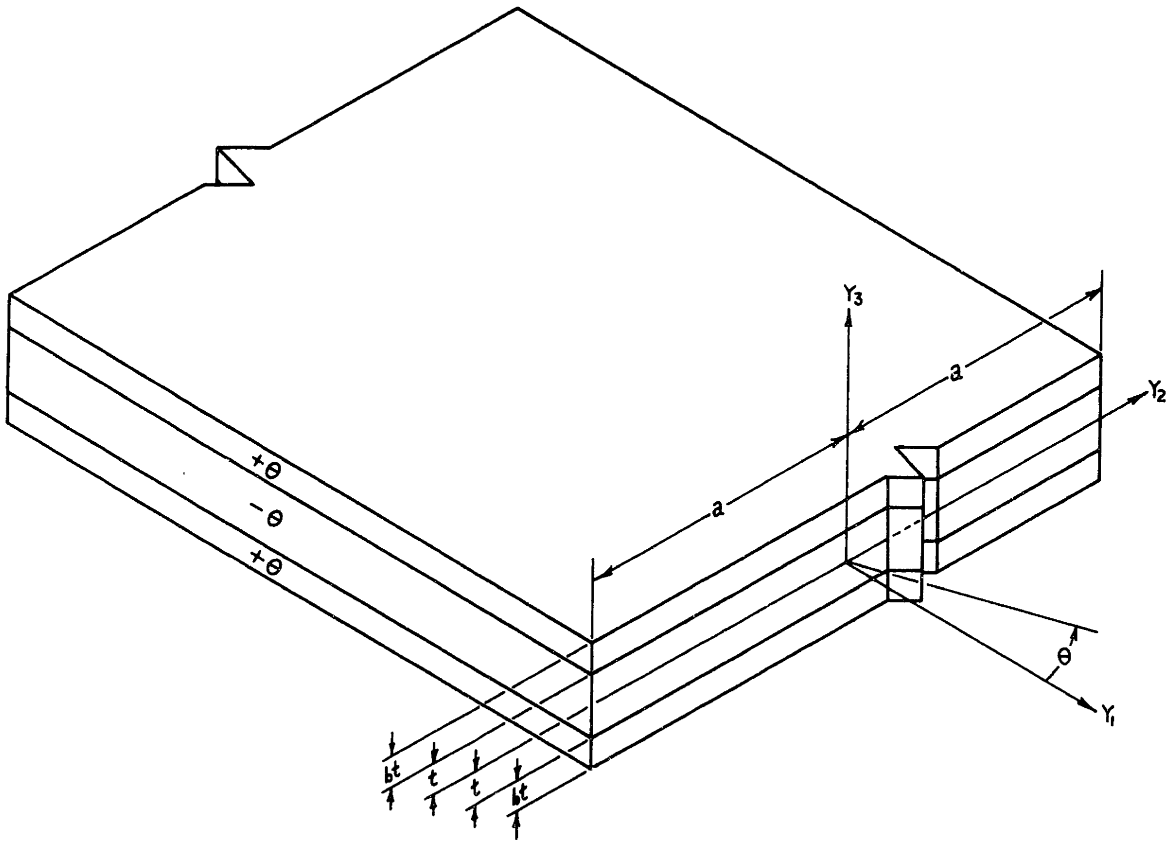
FORMULATION OF ANGLE-PLY PROBLEM

3.1 Laminate Configuration

The first problem chosen for application of the second-order theory was that of the laminate shown in Figure 1, which is essentially an idealized tensile specimen, or "coupon". If the laminae are numbered (1) through (3), top to bottom, laminae (1) and (3) are seen to be oriented at an angle $+\theta$ to the y_1 axis while lamina (2) is oriented at $-\theta$ to that axis. This configuration produces a laminate that will not deflect vertically under axial loading and is therefore said to be balanced about its midplane, the y_1y_2 plane. For the general case where θ is neither 0° or 90°, such a configuration is referred to as an angle-ply laminate. The length (y_1 dimension) of the laminate is large compared to a, and the laminate is loaded at its ends in longitudinal tension or compression. By Saint Venant's principle, all effects characteristic of the particular means by which this loading is introduced have subsided in the regions away from the ends, permitting the loading state to be characterized by two parameters: a uniform

FIGURE 1

ANGLE-PLY LAMINATE CONFIGURATION



longitudinal strain ϵ_1 and an average composite longitudinal stress $\bar{\sigma}_1$. It should be noted here that, for an angle-ply problem of this type, the condition of uniform ϵ_1 does not guarantee that plane cross sections remain plane. Conversely, however, the assumption of plane cross sections in the deformed state need not be made to insure the existence of a uniform ϵ_1. The problem may thus be described as one of "modified" plane strain. A final -- and very valuable -- condition insured by the absence of end effects in the central region of the laminate is that, within this region, all derivatives of stress and strain with respect to y_1 are zero.

3.2 Angular Dependence of Compliances

It was noted in Section 2.2 that the compliances S_{45}, S_{16}, S_{26}, and S_{36} are zero for uniaxial material because there exists no shear coupling for the 0° orientation. To state this in terms of the mathematical theory of elasticity is to say that uniaxial material is orthotropic; i.e., that it is characterized by nine independent compliances. Strictly speaking, filamentary composites are orthotropic no matter what their orientation since there always exists some coordinate system in which plies of a given orientation will exhibit no shear coupling. As a practical matter, however, problems involving laminates containing plies of several different orientations must be solved with respect to a single coordinate system. Plies oriented at angles other than 0° or 90° to the y_1 axis <u>do</u> exhibit shear coupling and therefore have nonzero values of s_{45}, s_{16}, s_{26}, and s_{36}. The existence of thirteen independent compliances is characteristic of <u>monoclinic</u> materials. In the case of the composite angle ply, however, there are thirteen nonzero compliances but all thirteen can be expressed in terms of the nine independent compliances of the uniaxial material. For the sake of analytical accuracy, therefore, it would be most appropriate to refer to the elastic behavior of angle plies as <u>quasi-monoclinic</u>.

The elastic compliances of composite laminae oriented at general values of θ to the y_1 axis must be determined by reverting to the full, general elasticity tensor notation for compliances. In this system of notation [2],

$$\tilde{S}_{klst} = S_{mnpr}\, l_{\tilde{k}m}\, l_{\tilde{l}n}\, l_{\tilde{s}p}\, l_{\tilde{t}r} \qquad (3.1)$$

describes the coordinate transformation of the compliance tensor, where the compliances on the left-hand side of (3.1) are those observed with respect to the coordinate system of Figure 1 and those on the right are the compliances of the uniaxial material. In this case,

$$l_{\tilde{1}1} = \cos\,\theta \qquad (3.2)$$

$$l_{\tilde{1}2} = -\sin\,\theta \qquad (3.3)$$

$$l_{\tilde{2}1} = \sin\,\theta \qquad (3.4)$$

$$\ell_{\tilde{2}2} = \cos\ \theta \tag{3.5}$$

$$\ell_{\tilde{1}3} = 0 \tag{3.6}$$

$$\ell_{\tilde{2}3} = 0 \tag{3.7}$$

$$\ell_{\tilde{3}2} = 0 \tag{3.8}$$

$$\ell_{\tilde{3}3} = 1 \tag{3.9}$$

$$\ell_{\tilde{3}1} = 0 \tag{3.10}$$

Performing the indicated calculations and reverting to the contracted tensor notation, taking into account the multiplicative factors involved in equations (2.19) through (2.25), one obtains

$$S_{11} = S_{11}^{(0)}\cos^4\theta + \left(2S_{12}^{(0)} + S_{66}^{(0)}\right)\sin^2\theta\cos^2\theta + S_{22}^{(0)}\sin^4\theta \tag{3.11}$$

$$S_{12} = \left(S_{11}^{(0)} + S_{22}^{(0)} - S_{66}^{(0)}\right)\sin^2\theta\ \cos^2\theta + S_{12}^{(0)}\left(\sin^4\theta + \cos^4\theta\right) \tag{3.12}$$

$$S_{13} = S_{13}^{(0)}\cos^2\theta + S_{23}^{(0)}\sin^2\theta \tag{3.13}$$

$$S_{22} = S_{11}^{(0)}\sin^4\theta + \left(2S_{12}^{(0)} + S_{66}^{(0)}\right)\sin^2\theta\cos^2\theta + S_{22}^{(0)}\cos^4\theta \tag{3.14}$$

$$S_{23} = S_{13}^{(0)}\sin^2\theta + S_{23}^{(0)}\cos^2\theta \tag{3.15}$$

$$S_{33} = S_{33}^{(0)} \tag{3.16}$$

$$S_{16} = 2S_{11}^{(0)}\cos^3\theta\sin\theta - 2S_{22}^{(0)}\sin^3\theta\cos\theta + (2S_{12}^{(0)} + S_{66}^{(0)})(\sin^3\theta\cos\theta - \cos^3\theta\sin\theta) \quad (3.17)$$

$$S_{26} = 2S_{11}^{(0)}\sin^3\theta\cos\theta - 2S_{22}^{(0)}\cos^3\theta\sin\theta + (2S_{12}^{(0)} + S_{66}^{(0)})(\cos^3\theta\sin\theta - \sin^3\theta\cos\theta) \quad (3.18)$$

$$S_{36} = 2(S_{13}^{(0)} - S_{23}^{(0)})\sin\theta\cos\theta \quad (3.19)$$

$$S_{44} = S_{55}^{(0)}\sin^2\theta + S_{44}^{(0)}\cos^2\theta \quad (3.20)$$

$$S_{45} = (S_{55}^{(0)} - S_{44}^{(0)})\sin\theta\cos\theta \quad (3.21)$$

$$S_{55} = S_{55}^{(0)}\cos^2\theta + S_{44}^{(0)}\sin^2\theta \quad (3.22)$$

$$S_{66} = 4(S_{11}^{(0)} + S_{22}^{(0)} - 2S_{12}^{(0)})\sin^2\theta\cos^2\theta + S_{66}^{(0)}(\cos^4\theta + \sin^4\theta - 2\sin^2\theta\cos^2\theta) \quad (3.23)$$

where the compliances identified with a superscript (0) denote those of the uniaxial composite. When the uniaxial compliances given by equations (2.44) through (2.49) for NARMCO 5505, and by equations (2.55) through (2.60) for Morgenite II, are substituted into (3.11) through (3.23) one obtains the angular dependences illustrated in Figures 2 through 8.

3.3 Lamination-Theory Solution for $b \doteq 1$

According to lamination theory, the specimen of Figure 1 will deform under longitudinal loading such that the

FIGURE 2

S_{11} AND S_{12} VS. θ

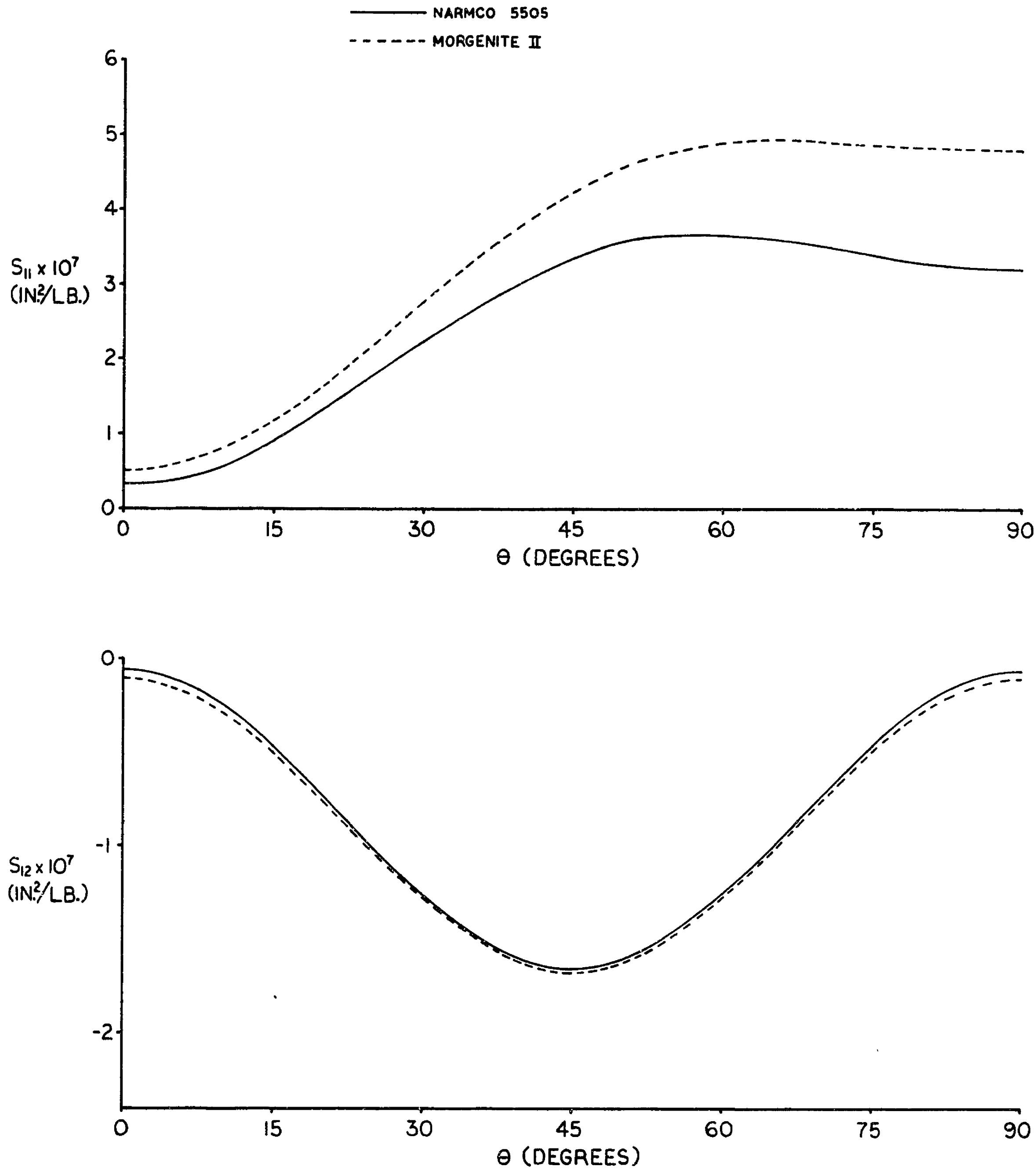


FIGURE 3

S_{13} AND S_{22} VS. θ

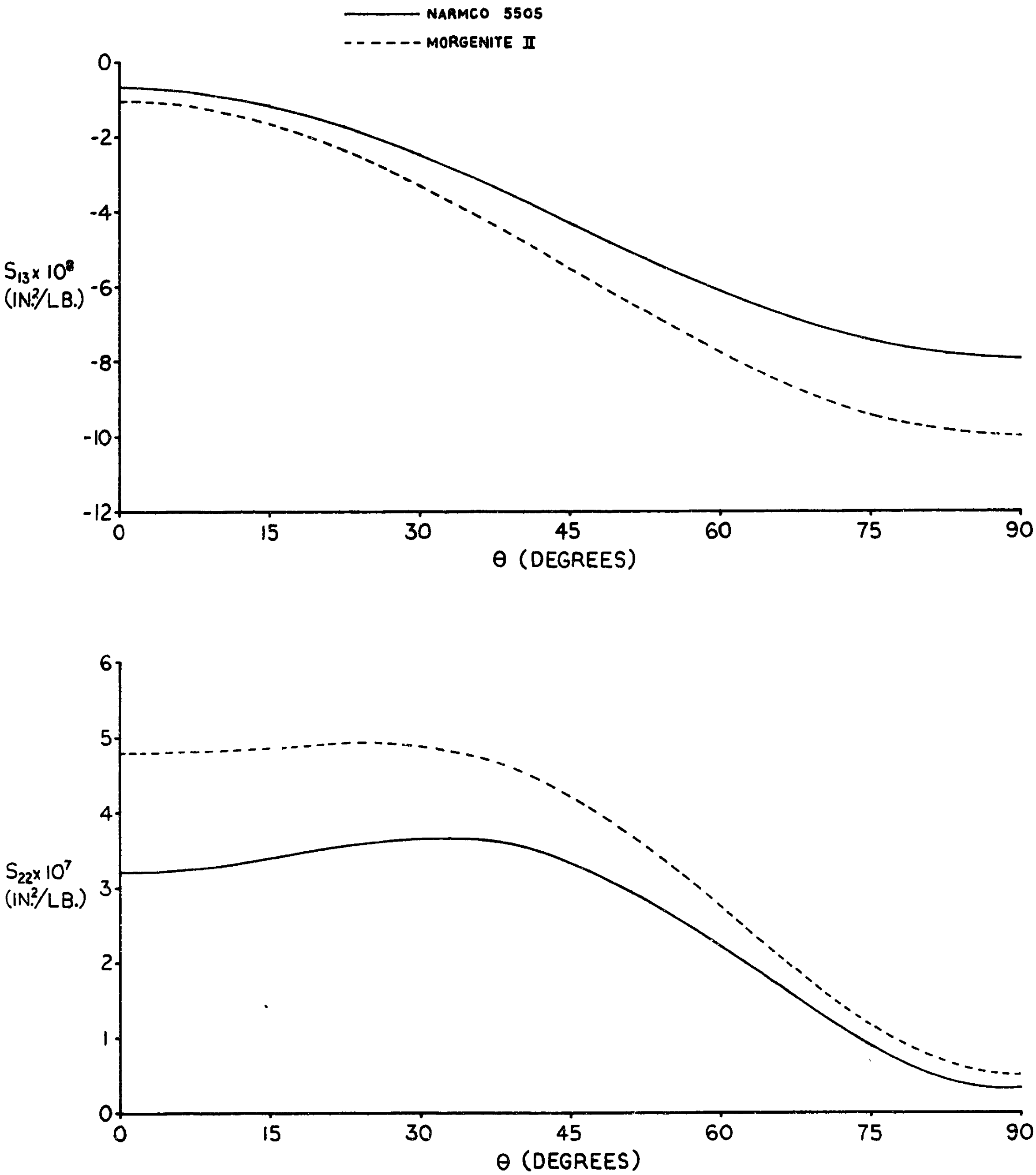


FIGURE 4

S_{23} AND S_{33} VS. θ

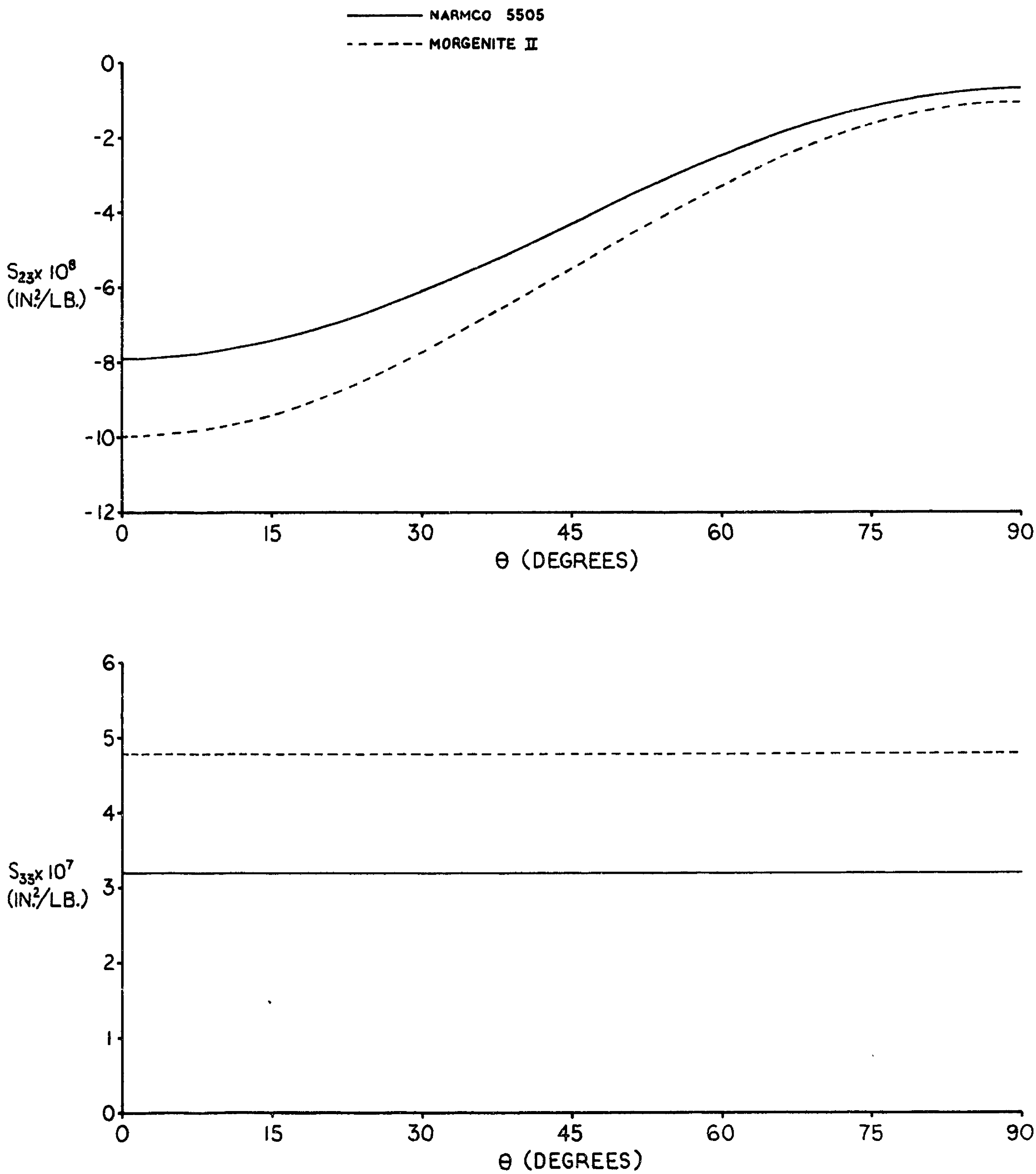


FIGURE 5

S_{16} AND S_{26} VS. θ

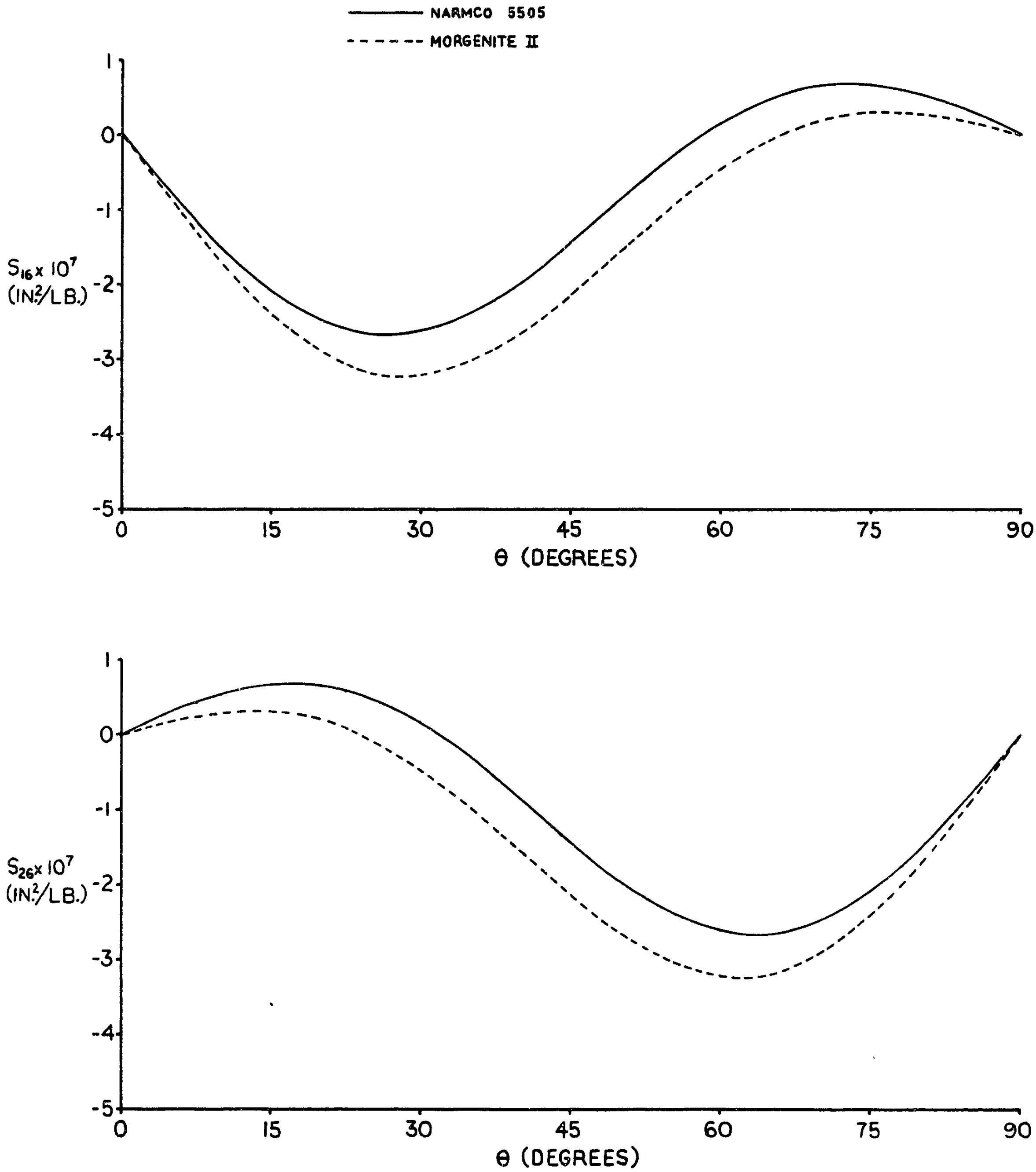


FIGURE 6

S_{36} AND S_{44} VS. θ

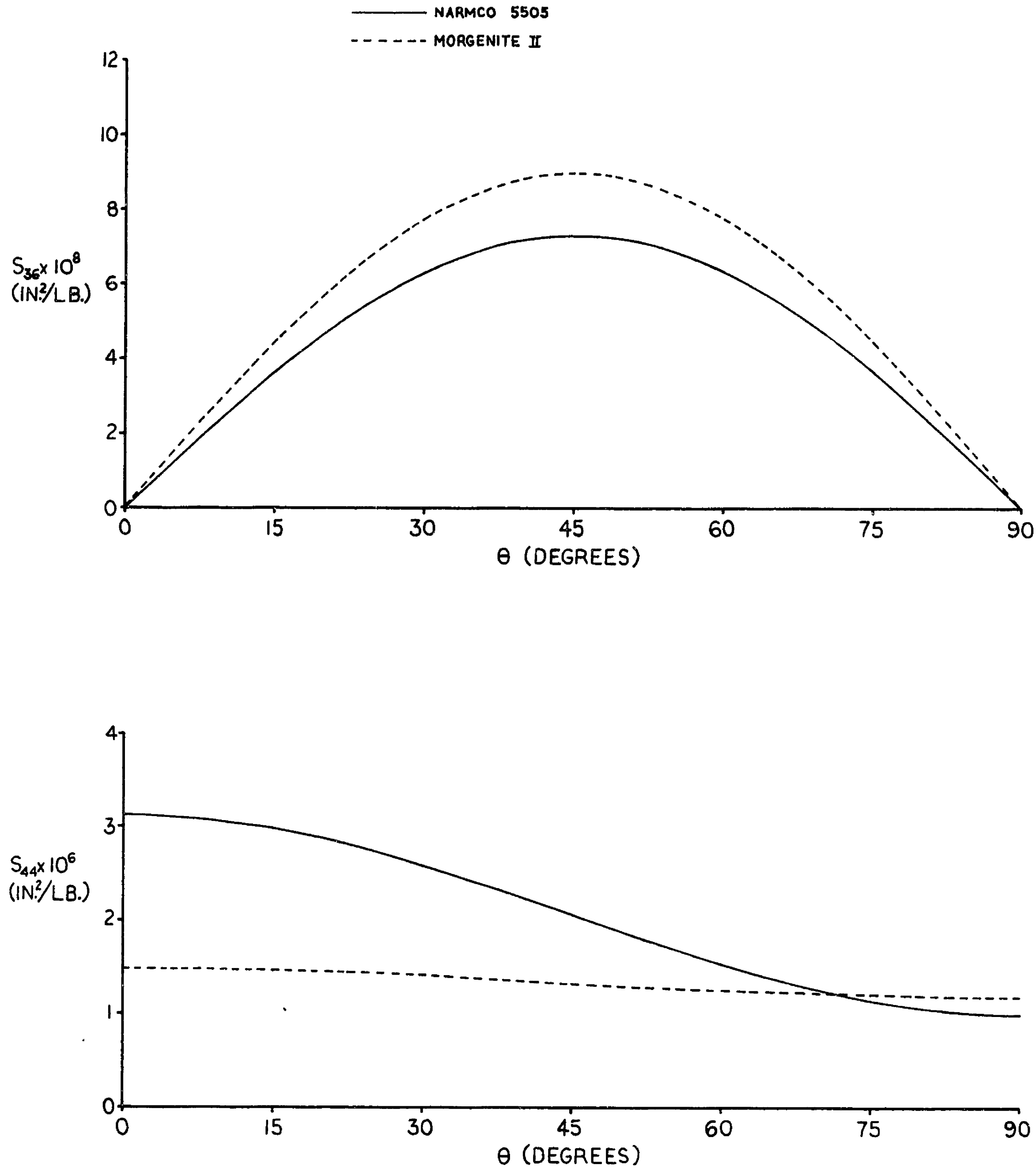


FIGURE 7

S_{45} AND S_{55} VS. θ

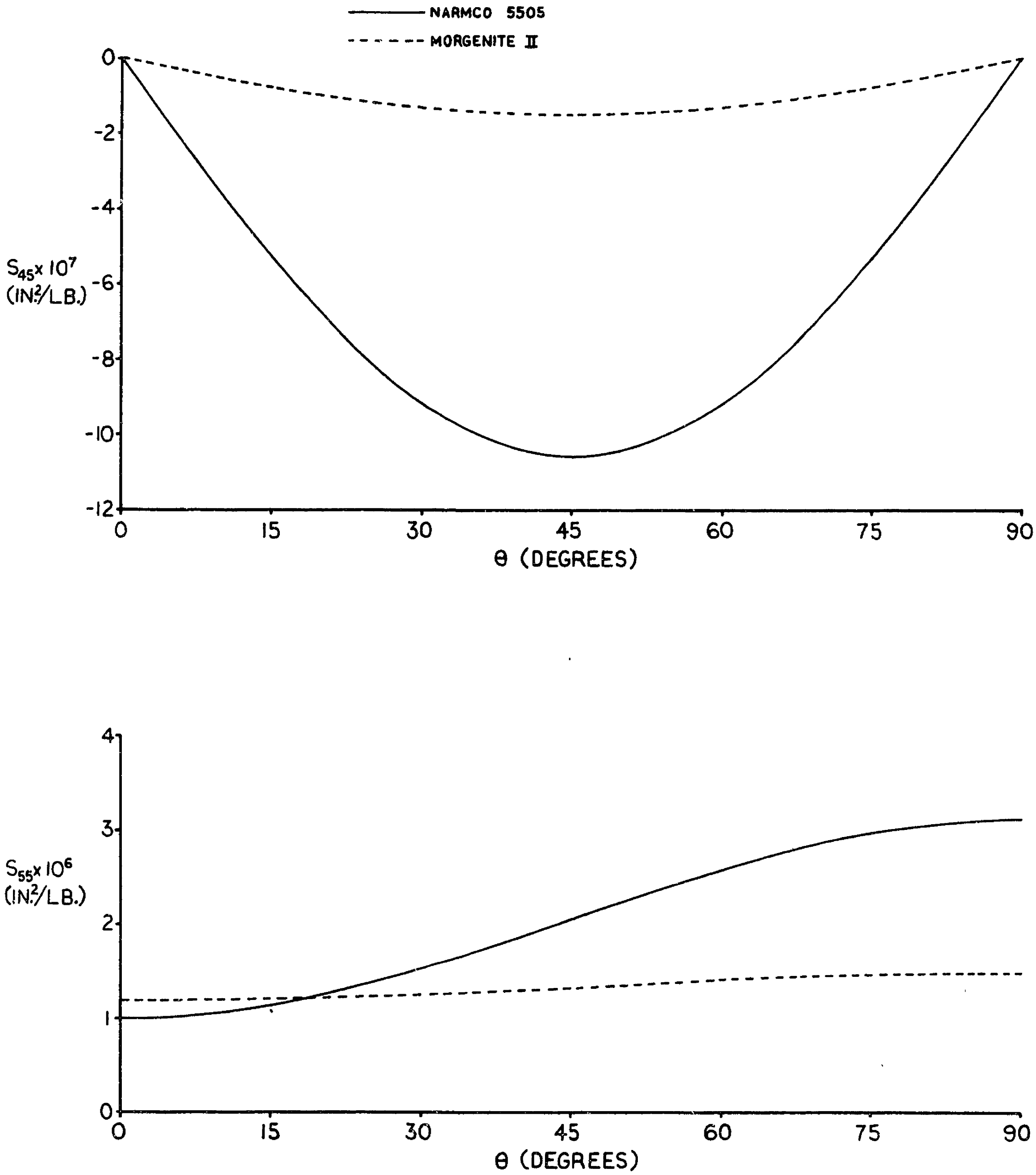
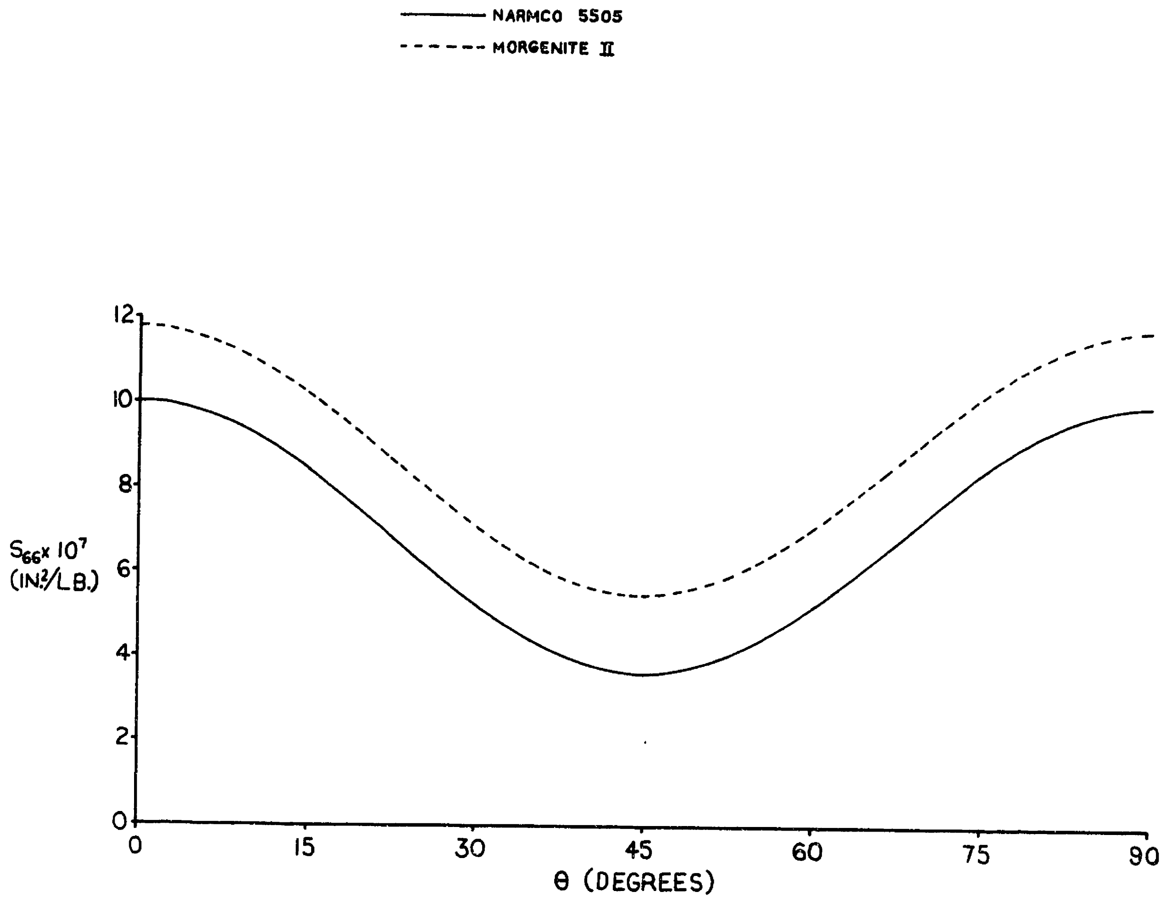


FIGURE 8

S_{66} VS. θ



strains ϵ_1, ϵ_2, and ϵ_6 remain constant over the entire cross-section of the laminate. For this particular problem, this is what is meant by the terminology "deformationally homogeneous over the cross-section" used in Section 1.1. Such a strain state requires that the stresses σ_1, σ_2, and σ_6 be constant over the cross-section of each lamina, but different in the $-\theta$ than in the $+\theta$ laminae. All other stresses are considered to be zero everywhere. This, in turn, is what is meant by "taking the elastic non-homogeneity of the laminate into account" as described in Section 1.1. Let the superscript (+) be used to identify the stresses in the $+\theta$ laminae and (-) to identify the stresses in the $-\theta$ lamina. Then by the stress-strain relations,

$$\epsilon_1 = S_{11}\sigma_1^{(+)} + S_{12}\sigma_2^{(+)} + S_{16}\sigma_6^{(+)} \tag{3.24}$$

$$\epsilon_1 = S_{11}\sigma_1^{(-)} + S_{12}\sigma_2^{(-)} - S_{16}\sigma_6^{(-)} \tag{3.25}$$

$$\epsilon_6 = S_{16}\sigma_1^{(+)} + S_{26}\sigma_2^{(+)} + S_{66}\sigma_6^{(+)} \tag{3.26}$$

$$\epsilon_6 = -S_{16}\sigma_1^{(-)} - S_{26}\sigma_2^{(-)} + S_{66}\sigma_6^{(-)} \tag{3.27}$$

$$\epsilon_2 = S_{12}\sigma_1^{(+)} + S_{22}\sigma_2^{(+)} + S_{26}\sigma_6^{(+)} \tag{3.28}$$

$$\epsilon_2 = S_{12}\sigma_1^{(-)} + S_{22}\sigma_2^{(-)} - S_{26}\sigma_6^{(-)} \tag{3.29}$$

where it should be noted that the values of S_{16} and S_{26} for a $-\theta$ lamina are the negatives of those for a $+\theta$ lamina.

To continue the development for general values of b from this point would result in the derivation of a stress state in which there are two nonzero stresses in the interior which must go to zero at the free edges: σ_2 and σ_6. This results in two edge effects which couple through the stress-strain relations to greatly complicate the second-order approximate solution. The behavior of edge effects due to the free-edge requirement on σ_2, moreover, will be investigated in the cross-ply problem described in Chapters 5 and 6, so the considerable increase in complexity associated with considering general values of b for the angle-ply problem would result in little if any additional physical insight. Finally, for values of b other than unity, it will be found that axial loading produces a lateral deformation of the laminate due to the existence of a uniform, nonzero value of ϵ_6 as predicted by lamination theory -- the laminate is not balanced about the y_1y_3 plane. As there are relatively few applications where coupling of lateral and axial deformations is desirable, one would not expect such an unbalanced configuration to be used very often in practice. For these reasons the subsequent treatment of the angle-ply problem contained herein will be restricted to cases where b = 1.0.

For a unity thickness ratio, the condition that the net shear force on the edges $y_2 = \pm a$ be zero requires

$$2t\,\sigma_6^{(+)} + 2t\,\sigma_6^{(-)} = 0 \tag{3.30}$$

while the condition that the net force normal to the cross-section equal the longitudinal loading yields the equation

$$2t\sigma_1^{(+)} + 2t\sigma_1^{(-)} = 4t\bar{\sigma}_1 \qquad (3.31)$$

Solution of (3.30) and (3.31) yields directly

$$\sigma_6^{(-)} = -\sigma_6^{(+)} \qquad (3.32)$$

$$\sigma_1^{(-)} = \sigma_1^{(+)} = \bar{\sigma}_1 \qquad (3.33)$$

Then (3.28) and (3.29) can only be satisfied if σ_2 is zero, which in turn causes (3.24) to become identical to (3.25) and the right side of (3.27) to become the additive inverse of the right side of (3.26). Thus, ϵ_6 must be zero -- which can also be reasoned physically from the fact that the laminate is balanced about the y_1y_3 plane. After these considerable simplifications, one rapidly obtains the value of $\sigma_6^{(+)}$ as

$$\sigma_6^{(+)} = -\frac{S_{16}}{S_{66}}\bar{\sigma}_1 \qquad (3.34)$$

3.4 Derivation of Assumed-Stress Solution for b = 1

The lamination-theory solution of Section 3.3 has an essential shortcoming in that it fails to satisfy the free-edge condition; i.e., that σ_6 is not zero at $y_2 = \pm a$. A second-order theory is proposed to remedy this defect and provide information about the stress distributions near the free edge. The major premises of the theory are as follows:

(a) All stress components become asymptotic to those predicted by lamination theory sufficiently far from the free lateral edges $y_2 = \pm a$.

(b) Those stresses which must go to zero at the free lateral edges do so exponentially with y_2, or at most as the product of a linear and an exponential function of y_2.

(c) All other stress components in the regions near the free lateral edges are such that the differential equations of equilibrium and all other exterior boundary conditions are identically satisfied.

To solve the angle-ply problem according to this theory, the assumed form of the stress σ_6 may be chosen as a starting point. Using the lamina numbers as superscripts to identify the stresses in the various laminae, one may write the lamination-theory asymptote condition on $\sigma_6^{(1)}$ as

$$\lim_{a \to \infty} \left[\sigma_6^{(1)} \Big|_{y_2 = 0} \right] = \sigma_6^{(+)} \tag{3.35}$$

The free-edge condition here is

$$\sigma_6^{(1)} \Big|_{y_2 = a} = 0 \tag{3.36}$$

A simple exponential form for $\sigma_6^{(1)}$ which satisfies both requirements for $y_2 \geq 0$ is

$$\sigma_6^{(1)} = \sigma_6^{(+)}\left[1 - \exp\left(\frac{y_2 - a}{kt}\right)\right] \tag{3.37}$$

where the value of k is, at this stage, unknown. There is only one nontrivial equilibrium equation for this problem, namely

$$\frac{\partial \sigma_6}{\partial y_2} + \frac{\partial \sigma_5}{\partial y_3} = 0 \tag{3.38}$$

This indicates that the dropping of the in-plane shear stress σ_6 to zero at the free lateral edges will give rise to an interlaminar shearing stress σ_5 in the vicinity of the free edges. Since this stress acts upon a y_3 plane, however, it must become zero at the free surfaces $y_3 = \pm 2t$. In the case of lamina (1) this requires

$$\sigma_5^{(1)}\Big|_{y_3 = 2t} = 0 \tag{3.39}$$

Solving (3.38) for $\frac{\partial \sigma_5^{(1)}}{\partial y_3}$, integrating, and applying (3.39) results in

$$\sigma_5^{(1)} = -\frac{1}{k}\sigma_6^{(+)}\left[2 - \frac{y_3}{t}\right]\exp\left(\frac{y_2 - a}{kt}\right) \tag{3.40}$$

Now by (3.32),

$$\sigma_6^{(2)} = -\sigma_6^{(1)} \tag{3.41}$$

and from the condition that σ_5 be continuous across the interface between laminae:

$$\sigma_5^{(1)}\Big|_{y_3=t} = \sigma_5^{(2)}\Big|_{y_3=t} \tag{3.42}$$

it follows that

$$\sigma_5^{(2)} = -\frac{1}{k}\sigma_6^{(+)}\left[\frac{y_2}{t}\right]\exp\left(\frac{y_2-a}{kt}\right) \tag{3.43}$$

To determine the longitudinal tension in each lamina it is then only necessary to apply (3.24) and (3.33), from which

$$\sigma_1^{(1)} = \frac{\epsilon_1}{S_{11}} - \frac{S_{16}}{S_{11}}\sigma_6^{(1)} \tag{3.44}$$

$$\sigma_1^{(2)} = \sigma_1^{(1)} \tag{3.45}$$

where ϵ_1 is obtained by substituting (3.33) and (3.34) into (3.24). The condition that σ_5 approach zero far from the lateral edges need not be formally applied, as it is automatically satisfied by (3.40) and (3.43). Since σ_5 is an odd function of y_2 and y_3, and σ_6 an even function of y_2 and invariant with y_3 within each lamina, a complete description of the stress field throughout the laminate has effectively been obtained by the derivation of equations (3.37), (3.40), (3.41), (3.43), (3.44), and (3.45), subject to a determination of k.

The solution as thus far obtained satisfies equilibrium and the exterior boundary conditions identically and is asymptotic to lamination theory in the interior. It is not an exact solution, however, since it cannot be made to satisfy the differential equations of strain compatibility.

The best one can do with a solution of this type is to guarantee satisfaction of compatibility in the mean by minimizing the complementary energy of the laminate [2] with respect to k:

$$\frac{d\Pi_c}{dk} = 0 \qquad (3.46)$$

where

$$\Pi_c = \iiint \tfrac{1}{2}\sigma_{mn}\epsilon_{mn}\,dv - \iint [\vec{\sigma}_m \cdot \vec{u}]\,n_m\,dA \qquad (3.47)$$

The condition of minimum complementary energy is thus used to determine the value of k. It has already been indicated that the stresses in the second, third, and fourth quadrants of the cross-section are mirror images of one sort or another of those in the first. By symmetry, therefore, the complementary energies of all four quadrants are equal and it is necessary to treat only the first quadrant in the analysis. And since none of the stresses or strains vary with y_1, it is necessary only to compute the complementary energy per unit length of laminate. When this is done, expressing σ_1 in terms of ϵ_1 and σ_6, one obtains

$$\Pi_c = \int_0^t\int_0^a \Big[-\frac{1}{2}\frac{\epsilon_1^2}{S_{11}} - S_{16}\frac{\epsilon_1}{S_{11}}\sigma_6^{(2)} + \frac{1}{2}\Big(S_{66} - \frac{S_{16}^2}{S_{11}}\Big)\sigma_6^{(2)2} + \frac{1}{2}S_{55}\sigma_5^{(2)2}\Big]\,dy_2\,dy_3$$

$$+ \int_t^{2t}\int_0^a \Big[-\frac{1}{2}\frac{\epsilon_1^2}{S_{11}} + S_{16}\frac{\epsilon_1}{S_{11}}\sigma_6^{(1)} + \frac{1}{2}\Big(S_{66} - \frac{S_{16}^2}{S_{11}}\Big)\sigma_6^{(1)2} + \frac{1}{2}S_{55}\sigma_5^{(1)2}\Big]\,dy_2\,dy_3 \qquad (3.48)$$

Since the constant terms involving ϵ_1^2 are independent of k, they may be ignored with no effect on the result of the subsequent differentiation. The values of the other integrals required are

$$\int_0^t \int_0^a \sigma_6^{(2)} dy_2\, dy_3 = -\sigma_6^{(+)} \left[at - kt^2\right] \quad (3.49)$$

$$\int_0^t \int_0^a \sigma_6^{(2)^2} dy_2\, dy_3 = \sigma_6^{(+)^2} \left[at - 1.5kt^2\right] \quad (3.50)$$

$$\int_0^t \int_0^a \sigma_5^{(2)^2} dy_2\, dy_3 = \frac{.1667}{k} \sigma_6^{(+)^2} t^2 \quad (3.51)$$

$$\int_t^{2t} \int_0^a \sigma_6^{(1)} dy_2\, dy_3 = \sigma_6^{(+)} \left[at - kt^2\right] \quad (3.52)$$

$$\int_t^{2t} \int_0^a \sigma_6^{(1)^2} dy_2\, dy_3 = \sigma_6^{(+)^2} \left[at - 1.5kt^2\right] \quad (3.53)$$

$$\int_t^{2t} \int_0^a \sigma_5^{(1)^2} dy_2\, dy_3 = \frac{.1667}{k} \sigma_6^{(+)^2} t^2 \quad (3.54)$$

It is important to note that the expressions presented in (3.49) through (3.54) are <u>not</u> the exact analytical forms of the definite integrals. They are based on the approximation that all terms involving $\exp(-\frac{a}{kt})$ or higher powers thereof are very nearly zero and may be neglected in comparison to all other terms in any given integral. This evidently places a restriction on the geometry of laminates for which the assumed form of solution is valid, but the precise severity of the restriction cannot be known until a determination of k is made. Substituting (3.49) through (3.54) into (3.48), differentiating with respect to k, and setting the result equal to zero requires that the value of k be given by

$$k = \sqrt{\frac{.1667\ S_{55}\,\sigma_6^{(+)}}{1.5\left(\frac{S_{16}^2}{S_{11}} - S_{66}\right)\sigma_6^{(+)} - 2S_{16}\frac{\epsilon_1}{S_{11}}}} \qquad (3.55)$$

CHAPTER 4

RESULTS OF ANGLE-PLY CALCULATIONS

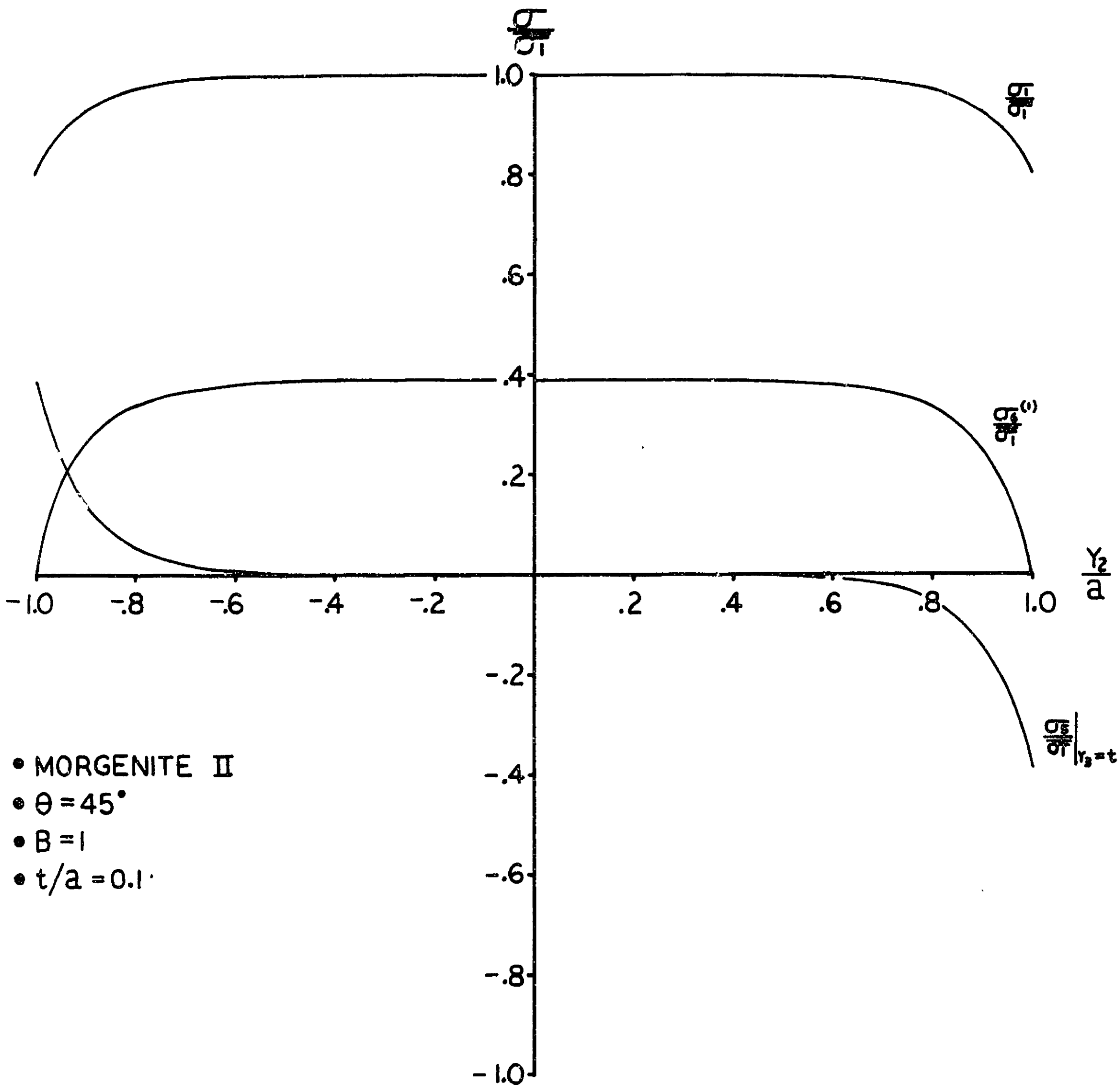
4.1 Sample Case

The approximate second-order solution obtained in Section 3.4 was coded into FORTRAN and assembled into a small computer program, ANGLE, in order to generate a matrix of solutions displaying variations in the stress field with the lamina angle θ. This was done merely for convenience, since the solution of Section 3.4 is simple enough to be worked out by hand. As the coding and logic of ANGLE are rather straightforward, we omit further discussion of the program here and proceed directly to a presentation of the results obtained. A listing of ANGLE and a glossary of its FORTRAN variables are presented for reference in Appendices A and B.

A typical angle-ply result is shown in Figure 9. For this particular case the value of k was found to be 1.07; as a consequence the edge effects are seen to be confined to a region about 4t in width directly adjacent to the free edges. Farther from the edges the first-order lamination theory is accurate to within 2% or better. This rather

FIGURE 9

ANGLE-PLY CASE: STRESS VS. Y_2



rapid subsidence of the edge effects, which one would intuitively expect *a priori* from Saint Venant's principle, permits the edge effects to be thought of as a *boundary layer* type of phenomenon with a boundary-layer thickness defined in terms of the thickness of the central lamina. One might appropriately define 4kt as the boundary-layer thickness for a problem of this type, as the value of the exponential terms in the assumed-stress distribution has decreased to 0.0182 at this distance into the laminate interior.

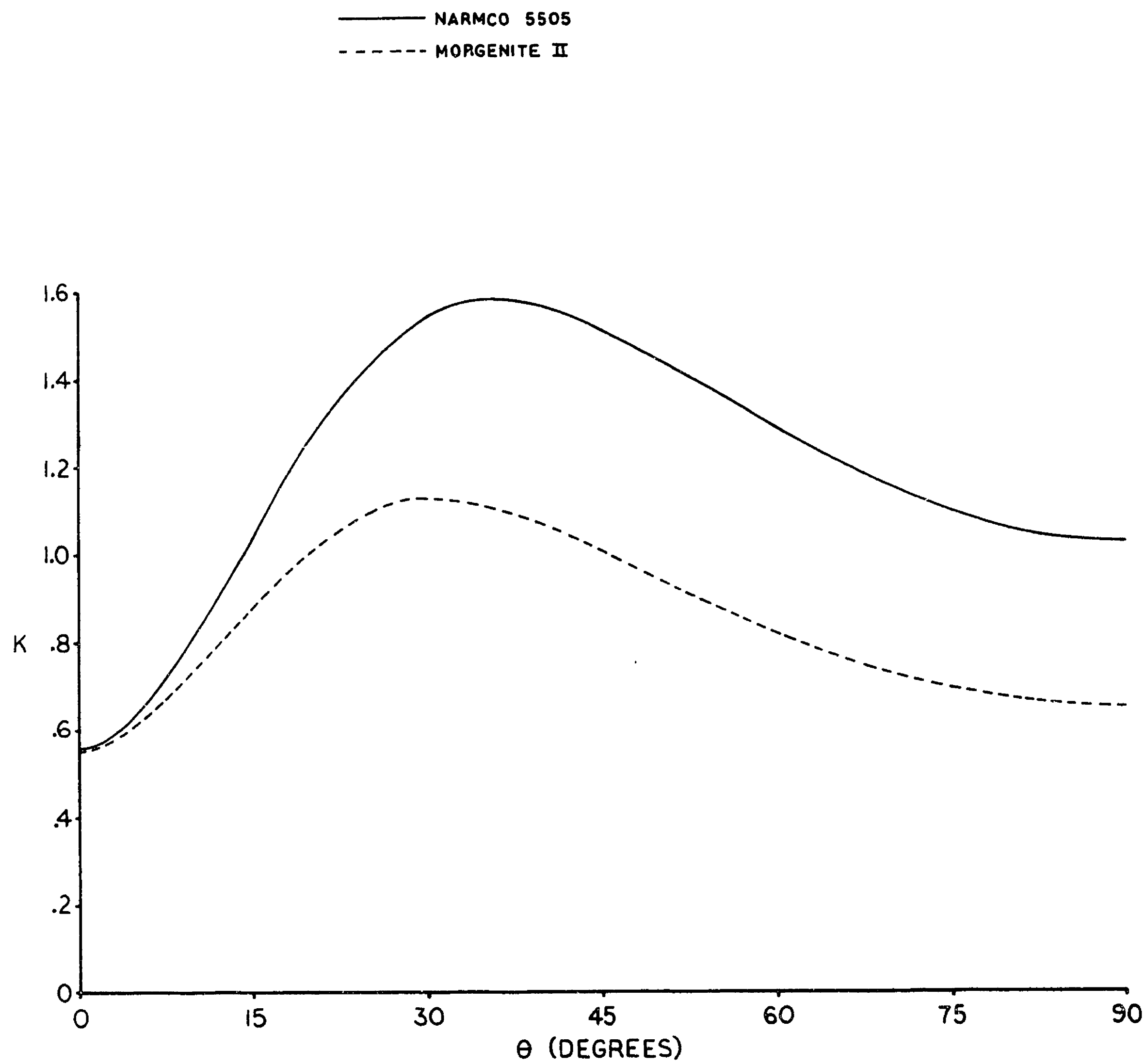
The maximum intensity of the interlaminar shearing stress σ_5 occurs at the intersection of the interlaminar interfaces, $y_3 = \pm t$, and the free edges, $y_2 = \pm a$. At these points its magnitude is 0.387 of the average applied composite longitudinal tension $\bar{\sigma}_1$ -- quite a considerable fraction. The longitudinal tensions in each lamina, however, are *reduced* to 0.81 of their maximum interior values by the action of σ_6 through the stress-strain relations. Despite the persistence of a constant ϵ_1 all the way out to the free edges, the shearing strains produced by the edge effects produce a marked "scissoring" of the $+\theta$ and $-\theta$ laminae with respect to each other. This effect has in fact been observed in tension tests of angle-ply laminates.

4.2 Variation of k and σ_5 with θ

The boundary-layer thickness parameter k as a function of θ for unity thickness ratio is shown in Figure 10. The

FIGURE 10

K VS. θ FOR B=1



value of k is on the order of unity for all lamina angles, but varies by a factor of two to three, depending on the material, as θ varies from 0° to 90°. The variation is more severe for NARMCO 5505, which has a maximum k of 1.58 at 35.5°, than for Morgenite II, which has a maximum k of 1.13 occurring at 29°. Since k is basically a function of the material properties, it does not go to zero at 0° or 90° even though the in-plane and interlaminar shearing stresses vanish at these orientations.

Figure 11 illustrates the variation in the maximum intensity of the interlaminar shear σ_5 with θ for unity thickness ratio. The free-edge effect is more severe in Morgenite II, where σ_5 reaches a magnitude of .443 of the applied tension at 37.5°, than in NARMCO 5505, where a maximum intensity ratio of .345 is reached at 35°. For each material there is a lamina orientation (other than 0° or 90°) for which there is no edge effect. In the case of NARMCO 5505 this angle is 58°, while for Morgenite II it is 66.5°. For angles less than the zero edge effect angle the ratio of σ_5 to $\bar{\sigma}_1$ is negative, while for greater angles it is positive, where σ_5 is taken at the point $y_2 = a$, $y_3 = t$. The existence of such an angle can be readily explained by recourse to Figure 14, which shows that σ_6 goes to zero for each material at the same angle as does σ_5. This, in turn, as may be seen from Figure 5, is because the tensor compliance S_{16} passes through zero at this point.

FIGURE 11

$\frac{\sigma_5}{\sigma_1}\Big|_{\substack{Y_2=a\\Y_3=t}}$ VS. θ FOR B=1

——— NARMCO 5505

- - - - - - MORGENITE II

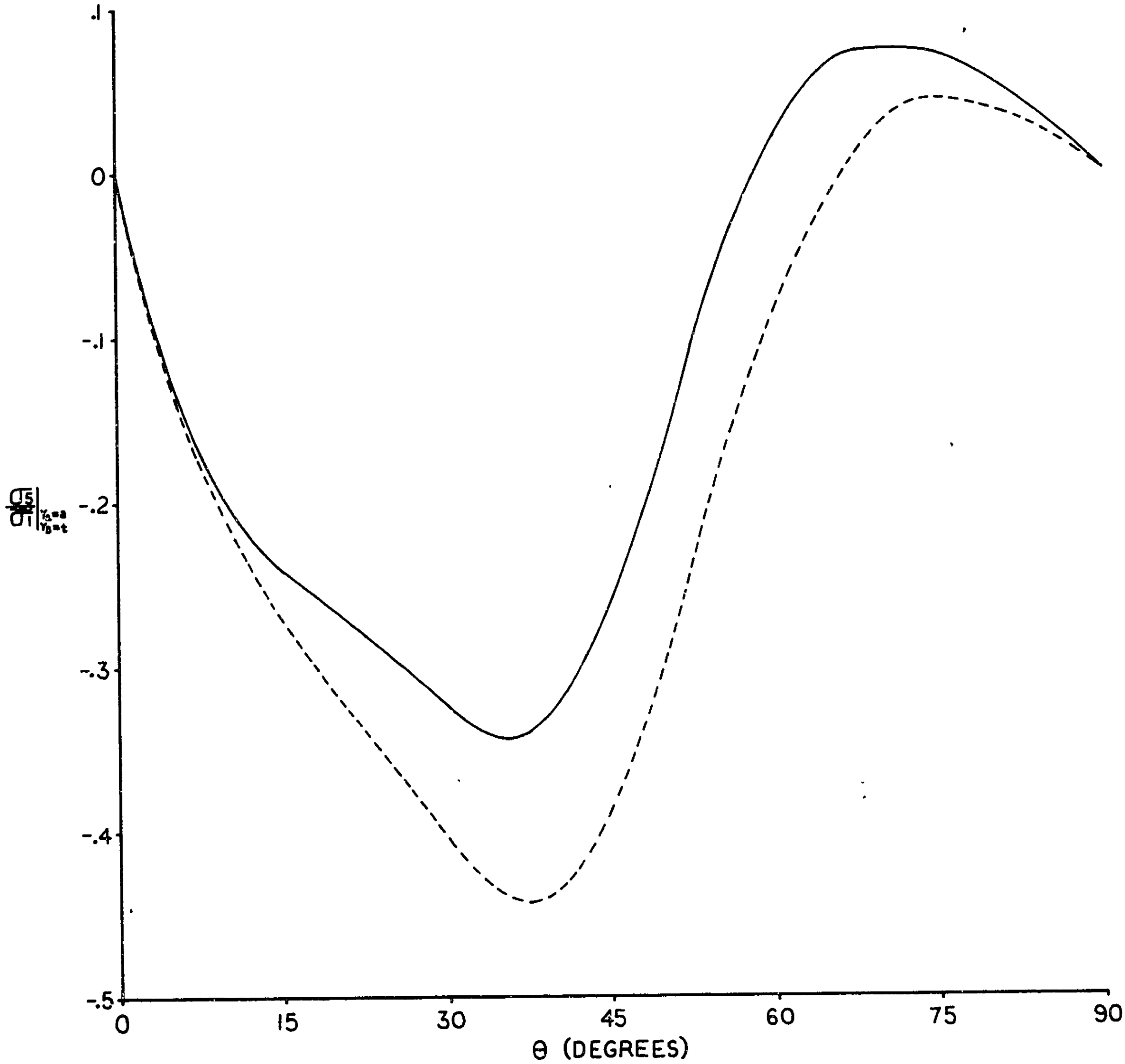
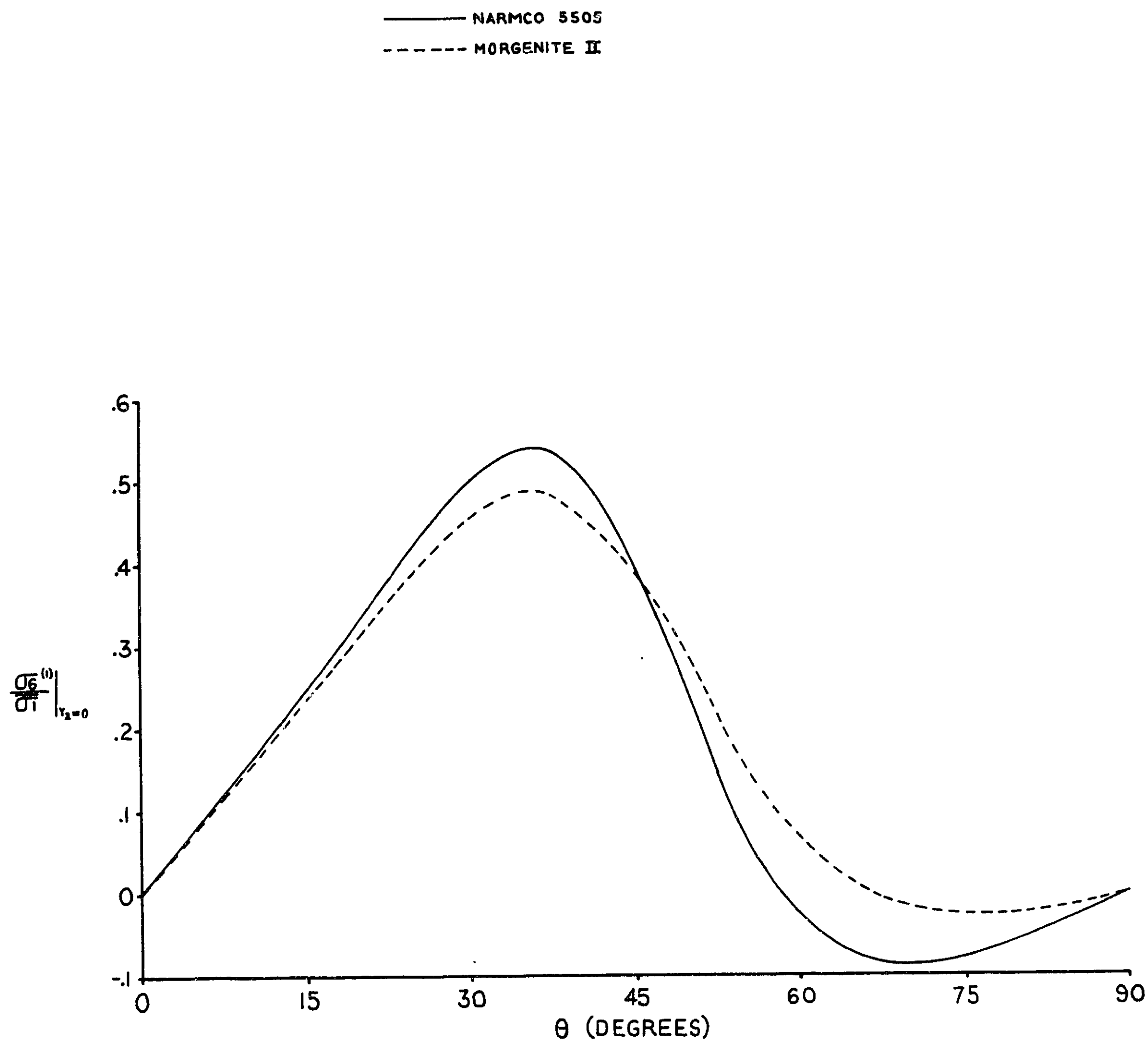


FIGURE 14

$\left.\frac{\sigma_6^{(1)}}{\sigma_1}\right|_{\gamma_2=0}$ VS. θ FOR B=1



Physically, this behavior results from the fact that angle plies oriented at angles less than the zero edge effect angle attempt to align themselves with the 0° direction, while those oriented at angles greater than the zero edge effect angle attempt to align themselves with the 90° direction, when the composite is placed in longitudinal tension or compression.

Much of the difference in interlaminar shear behavior between NARMCO 5505 and Morgenite II can be explained by reference to the ratio of the compliance constants S_{44} and S_{11} for the uniaxial material. Referring to equations (2.44) through (2.49) and (2.55) through (2.60), one finds that the value of S_{11} for Morgenite II is about 1.5 times that for NARMCO 5505, while S_{44} in Morgenite II is less than half that of NARMCO 5505. The values of S_{55} and S_{66} for the two uniaxial materials, moreover, are nearly equal. The graphite/epoxy material is thus considerably stiffer in shear in relation to its stiffness in tension than is the boron/epoxy. Thus, for all angles, Morgenite II exhibits smaller boundary layers than NARMCO 5505, and for most angles below the zero edge effect angle the interlaminar shear stresses are more severe in Morgenite II.

4.3 Importance of Results to Design

In designing composite laminates the free-edge effects are generally considered undesirable, since they may contribute to delamination failure or reduce the fatigue life of the finished structural element. Consequently, it is desirable to reduce the magnitude of such effects to a minimum and to confine them to as small a region near the edges as possible.

Since the orientation of an angle-ply laminate is usually chosen to satisfy overall stiffness and strength requirements, a reduction in the magnitude of the free-edge effect by adjusting θ will not always be possible. In particular, use of the zero edge effect angle will often be prohibited by the fact that the material is quite weak and compliant in longitudinal tension at this angle.

For any given orientation, however, the boundary-layer thickness 4kt (which is a measure of the edge effect's region of influence) is directly proportional to the lamina thickness. It follows that thin laminae, preferably consisting of no more than a single ply, are to be preferred over thick ones. The practice of using a great many plies of identical orientation to form a lamina in order to obtain desired stiffness properties while keeping the layup simple should be avoided.

The region of influence of the edge effects as measured by the value of k is uniformly greater in NARMCO 5505 than in Morgenite II. For most ply angles, however, the intensity of the edge effect stress σ_5 is greater in the graphite/epoxy than in the boron/epoxy. Except in applications where the intensity or the depth of penetration of the edge-effect stress is a critical consideration, therefore, considerations of overall strength-to-weight and stiffness-to-weight ratios will be considerably more important to materials choice than edge effects. Where edge-effect penetration depth is critical, it should be remembered that not only is k smaller for Morgenite II than for NARMCO 5505, but the graphite/epoxy tape is only about one-tenth as thick as the boron/epoxy, making Morgenite II clearly the preferable material for such applications.

4.4 Accuracy and Limitations

The solutions obtained for $b = 1.0$ were compared with solutions for a similar laminate arrangement in Morgenite II obtained by Pipes [3] of General Dynamics using a finite-difference approach. The agreement between the two solutions was found to be quite good, there being no detectable difference for $\theta = 45^o$ in the maximum predicted intensity of the interlaminar shear stress and only slight differences at other points which could as easily be due to the limits of graphing accuracy as to actual differences in the results. In this connection it may be mentioned that the angle-ply calculations of this study were performed at 7.5-degree increments of θ between $\theta = 0^o$ and $\theta = 90^o$, while those of [3] were done for somewhat fewer points. It should also be noted that θ has been redefined as $-\theta$ in the plot of normalized σ_ξ given in [3], so that this plot appears as the negative of Figure 11.

There is one essential way in which the approximate second-order solution differs from that of [3], and this difference points out a basic limitation on the permissible geometry of laminates which can be accurately solved by the approximate technique. The approximate solution predicts no dependence of the interlaminar shear intensity on laminate thickness. Reference [3], however, presents data indicating that there is a mild dependence on thickness. This is because [3] includes data for laminates that range all the way up to nearly half their width in thickness -- into regions for which the approximate solution is invalid. In laminates this thick lamination-theory values are not approached in the interior. In particular the longitudinal tension σ_1 is lowered throughout the laminate, so that the interlaminar shear normalized by this tension, as in [3], increases with increasing thickness. This behavior indicates that the second-order approximate solution should not be used unless the laminate width is at least 8kt, which is to say unless the dimension a of Figure 1 is at least 4kt -- one boundary-layer thickness. This amounts to stipulating a maximum permissible error of 1.82% in σ_6 at $y_2 = 0$.

CHAPTER 5

FORMULATION OF CROSS-PLY PROBLEM

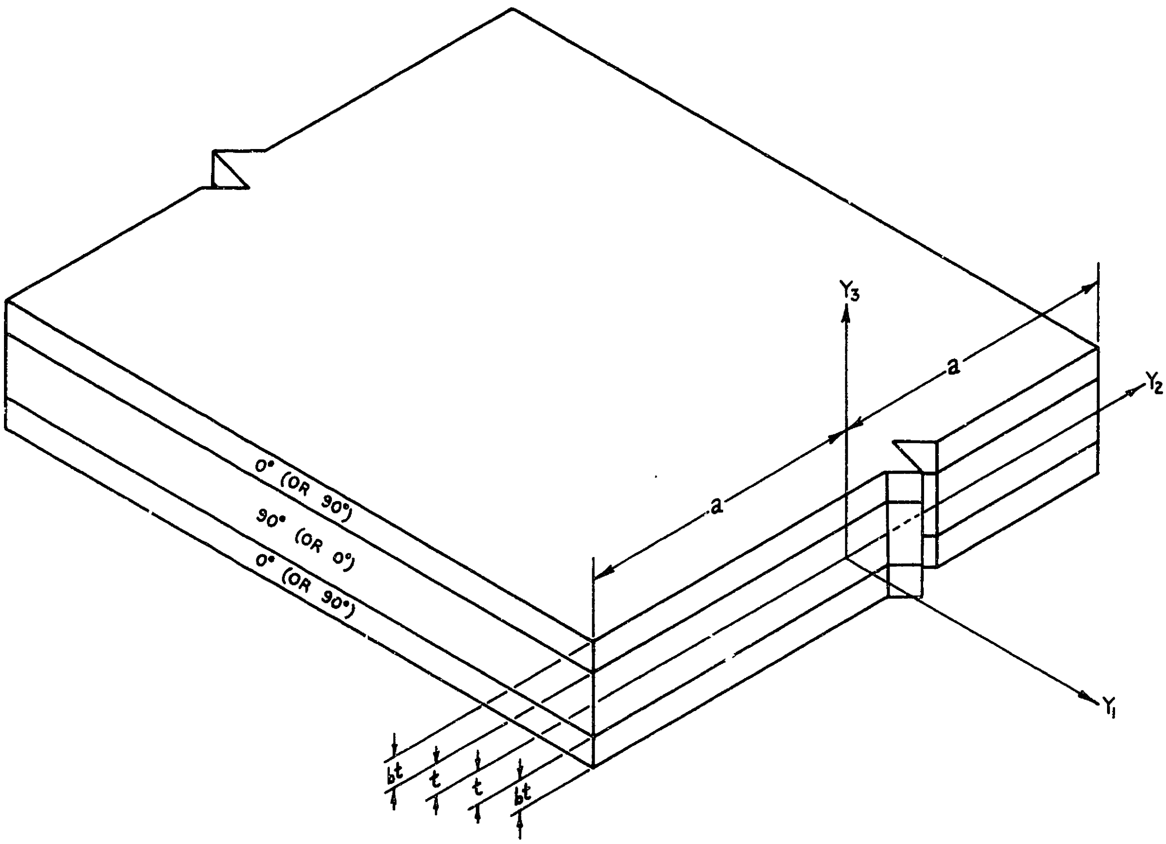
5.1 Laminate Configuration

In order to investigate edge effects caused by the free-edge condition on the transverse extensional stress σ_2, a cross-ply problem was formulated using the laminate configuration illustrated in Figure 15. A cross-ply laminate is defined as one composed entirely of laminae oriented at either 0^o or 90^o to the y_1 axis. The laminate of Figure 15 may be considered as either one of two possible configurations: 0/90/0 (laminae (1) and (3) at 0^o; lamina (2) at 90^o) or 90/0/90 (laminae (1) and (3) at 90^o; lamina (2) at 0^o), where the same lamina numbering scheme as used in Section 3.1 has been adopted. Both these arrangements produce a laminate that is balanced about both the y_1y_2 plane and the y_1y_3 plane, regardless of the value of b.

The same geometrical and loading considerations apply to the cross-ply laminate as to the angle-ply configuration: namely that the laminate length is assumed large compared to the dimension a and the specimen is loaded at its ends in longitudinal tension or compression, so that end effects

FIGURE 15

CROSS-PLY LAMINATE CONFIGURATION



have disappeared over most of the specimen's length and the loading state may be completely specified by the uniform longitudinal strain ϵ_1 and the average longitudinal stress $\bar{\sigma}_1$. Again, this insures that all stress and strain derivatives with respect to y_1 are zero. The cross-sections of the cross-ply laminate, unlike those of the angle-ply configuration, remain plane in the deformed state; the cross-ply specimen is thus placed in a state of plane strain (in all particulars except that ϵ_1 is constant rather than zero), as opposed to the "modified" plane strain of the angle-ply laminate, by the end loading condition.

5.2 Lamination-Theory Solution

Lamination theory predicts that a balanced cross-ply laminate of the type shown in Figure 15 will deform under longitudinal loading such that the strains ϵ_1 and ϵ_2 remain constant over the cross-section (the condition of compatibility, or "deformational homogeneity"). The stresses σ_1 and σ_2 will then be constant over the cross-section of each lamina, but different in the 90° than in the 0° laminae, and all other stresses will be uniformly zero. Suppose we stipulate, for the present, that the solution to a 0/90/0 configuration is desired and let the stresses in the 0° laminae be identified with a superscript (0), those in the 90° lamina by (90). Then the stress-strain relations become

$$\epsilon_1 = S_{11}\sigma_1^{(0)} + S_{12}\sigma_2^{(0)} \tag{5.1}$$

$$\epsilon_1 = S_{22}\sigma_1^{(90)} + S_{12}\sigma_2^{(90)} \tag{5.2}$$

$$\epsilon_2 = S_{12}\sigma_1^{(0)} + S_{22}\sigma_2^{(0)} \tag{5.3}$$

$$\epsilon_2 = S_{12}\sigma_1^{(90)} + S_{11}\sigma_2^{(90)} \tag{5.4}$$

The condition that there be no net normal force on the free edges requires

$$2bt\,\sigma_2^{(0)} + 2t\,\sigma_2^{(90)} = 0 \tag{5.5}$$

while the requirement that the net force normal to the cross-section equal the longitudinal applied loading yields the equation

$$2bt\,\sigma_1^{(0)} + 2t\,\sigma_1^{(90)} = 2(1+b)t\,\overline{\sigma_1} \tag{5.6}$$

Equations (5.5) and (5.6) are readily solved for the stresses in the 90° lamina in terms of $\overline{\sigma_1}$ and the stresses in the 0° laminae:

$$\sigma_2^{(90)} = -b\,\sigma_2^{(0)} \tag{5.7}$$

$$\sigma_1^{(90)} = (1+b)\overline{\sigma_1} - b\,\sigma_1^{(0)} \tag{5.8}$$

Substituting (5.7) and (5.8) into (5.1) through (5.4) and solving the resulting set of equations simultaneously determines $\sigma_1^{(0)}$ and $\sigma_2^{(0)}$ as follows:

$$\sigma_2^{(0)} = \frac{\bar{\sigma}_1 (S_{11} - S_{22})}{\left[\frac{(S_{22} + b S_{11})(S_{11} + b S_{22})}{(1+b) S_{12}} - (1+b) S_{12} \right]} \tag{5.9}$$

$$\sigma_1^{(0)} = \bar{\sigma}_1 - \frac{\sigma_2^{(0)} (S_{22} + b S_{11})}{(1+b) S_{12}} \tag{5.10}$$

5.3 Derivation of Assumed-Stress Solution

The lamination-theory prediction of a uniform σ_2 throughout each lamina of the cross-ply coupon is incorrect, just as the prediction of a constant σ_6 was in the angle-ply case, due to the free-edge requirement that all stresses acting on a y_2 plane go to zero at $y_2 = \pm a$. The second-order theory proposed in Section 3.4 is again applied in order to obtain a more accurate picture of the stress field near the free edges.

Since σ_2 must be asymptotic to the lamination-theory prediction in the interior but zero at the free edges, the equilibrium equation

$$\frac{\partial \sigma_2}{\partial y_2} + \frac{\partial \sigma_4}{\partial y_3} = 0 \tag{5.11}$$

requires that an interlaminar shearing stress σ_4 must exist in the regions adjacent to the edges. The forms of σ_2 and σ_4 which satisfy (5.11) cannot, however, be analogous to the forms of σ_6 and σ_5 which satisfy (3.38) for the angle-ply case since the behavior of σ_4 is complicated by the requirement that it, as well as σ_2, must go to zero at $y_2 = \pm a$. Application of the lamination-theory asymptote conditions

$$\lim_{a \to \infty} \left[\sigma_2^{(1)} \Big|_{y_2=0} \right] = \sigma_2^{(0)} \qquad (5.12)$$

$$\lim_{a \to \infty} \left[\sigma_4^{(1)} \Big|_{y_2=0} \right] = 0 \qquad (5.13)$$

the free-edge conditions

$$\sigma_2^{(1)} \Big|_{y_2=a} = 0 \qquad (5.14)$$

$$\sigma_4^{(1)} \Big|_{y_2=a} = 0 \qquad (5.15)$$

the free-surface condition

$$\sigma_4^{(1)} \Big|_{y_3=(1+b)t} = 0 \qquad (5.16)$$

and premise (b) of the second-order theory as stated in Section 3.4 permit $\sigma_2^{(1)}$ and $\sigma_4^{(1)}$ to be solved by the method of undetermined coefficients, resulting in

$$\sigma_2^{(1)} = \sigma_2^{(0)} \left[1 - \left(1 + \frac{a-y_2}{kt}\right) \exp\left(\frac{y_2-a}{kt}\right) \right] \qquad (5.17)$$

$$\sigma_4^{(1)} = -\frac{\sigma_2^{(0)}}{k} \left[(1+b) - \frac{y_3}{t} \right] \left[\frac{a-y_2}{kt} \right] \exp\left(\frac{y_2-a}{kt} \right) \qquad (5.18)$$

for $y_2 \geq 0$. The fact that there is a y_2-variation in σ_4, however, requires that there also be a nonzero σ_3 in the regions near the free edge in order that the equilibrium equation

$$\frac{\partial \sigma_4}{\partial y_2} + \frac{\partial \sigma_3}{\partial y_3} = 0 \qquad (5.19)$$

may be satisfied. Substituting the y_2 derivative of (5.18) into (5.19) to obtain $\frac{\partial \sigma_3^{(1)}}{\partial y_3}$, integrating, and applying the free-surface condition

$$\sigma_3^{(1)}\Big|_{y_3=(1+b)t} = 0 \tag{5.20}$$

determines $\sigma_3^{(1)}$ as

$$\sigma_3^{(1)} = \frac{\sigma_2^{(0)}}{k^2}\left[-\frac{y_3^2}{2t^2} + (1+b)\frac{y_3}{t} - \frac{(1+b)^2}{2}\right]\left[\frac{a-y_2}{kt} - 1\right]\exp\left(\frac{y_2-a}{kt}\right) \tag{5.21}$$

Then applying (5.7), $\sigma_2^{(2)}$ is found to be

$$\sigma_2^{(2)} = -b\,\sigma_2^{(1)} \tag{5.22}$$

and since σ_3 and σ_4 are required to be continuous across the interlaminar interfaces:

$$\sigma_3^{(2)}\Big|_{y_3=t} = \sigma_3^{(1)}\Big|_{y_3=t} \tag{5.23}$$

$$\sigma_4^{(2)}\Big|_{y_3=t} = \sigma_4^{(1)}\Big|_{y_3=t} \tag{5.24}$$

these stresses are determined as

$$\sigma_3^{(2)} = \frac{\sigma_2^{(0)}}{k^2}\left[\frac{b\,y_3^2}{2t^2} - \frac{b(1+b)}{2}\right]\left[\frac{a-y_2}{kt} - 1\right]\exp\left(\frac{y_2-a}{kt}\right) \tag{5.25}$$

$$\sigma_4^{(2)} = -\frac{\sigma_2^{(0)}}{k}\left[\frac{b\,y_3}{t}\right]\left[\frac{a-y_2}{kt}\right]\exp\left(\frac{y_2-a}{kt}\right) \tag{5.26}$$

The longitudinal tension in each lamina can now be determined from the stress-strain relations:

$$\sigma_1^{(1)} = \frac{\epsilon_1}{S_{11}} - \frac{S_{12}}{S_{11}}\sigma_2^{(1)} - \frac{S_{13}}{S_{11}}\sigma_3^{(1)} \tag{5.27}$$

$$\sigma_1^{(2)} = \frac{\epsilon_1}{S_{22}} - \frac{S_{12}}{S_{22}}\,\sigma_2^{(2)} - \frac{S_{23}}{S_{22}}\,\sigma_3^{(2)} \tag{5.28}$$

where ϵ_1 is known from substituting (5.9) and (5.10) into (5.1). The stress σ_2 is an even function of y_2 and is invariant with y_3 within each lamina. σ_3 is an even function of both y_2 and y_3, while σ_4 is an odd function of both y_2 and y_3. Equations (5.17), (5.18), (5.21), (5.22), and (5.25) through (5.28) therefore provide sufficient information to determine the stress field throughout the laminate by symmetry, subject to a determination of the value of k.

As in the angle-ply case, the solution thus defined satisfies stress equilibrium and the exterior boundary conditions identically and satisfies the lamination-theory interior asymptotes, but cannot be made to satisfy differential strain compatibility. Once again, k is determined by the principle of minimum complementary energy according to equations (3.46) and (3.47), which insure satisfaction of compatibility in the mean. It is again necessary to compute only the complementary energy of a unit length of the first quadrant of the laminate. With σ_1 expressed in terms of ϵ_1, σ_2, and σ_3 the energy integral becomes

$$\Pi_c = \int_0^t\int_0^a \Big[-\frac{1}{2}\frac{\epsilon_1^2}{S_{22}} + \frac{S_{12}}{S_{22}}\epsilon_1\sigma_2^{(2)} + \frac{S_{23}}{S_{22}}\epsilon_1\sigma_3^{(2)} + \frac{1}{2}\Big(S_{11} - \frac{S_{12}^2}{S_{22}}\Big)\sigma_2^{(2)2} + \frac{1}{2}\Big(S_{33} - \frac{S_{23}^2}{S_{22}}\Big)\sigma_3^{(2)2} + \Big(S_{13} - \frac{S_{12}S_{23}}{S_{22}}\Big)\sigma_2^{(2)}\sigma_3^{(2)} + \frac{1}{2}S_{55}\sigma_4^{(2)2}\Big]dy_2dy_3$$

$$+\int_t^{(1+b)t}\int_0^a\Big[-\frac{1}{2}\frac{\epsilon_1^2}{S_{11}} + \frac{S_{12}}{S_{11}}\epsilon_1\sigma_2^{(1)} + \frac{S_{13}}{S_{11}}\epsilon_1\sigma_3^{(1)} + \frac{1}{2}\Big(S_{22} - \frac{S_{12}^2}{S_{11}}\Big)\sigma_2^{(1)2} + \frac{1}{2}\Big(S_{33} - \frac{S_{13}^2}{S_{11}}\Big)\sigma_3^{(1)2} + \Big(S_{23} - \frac{S_{12}S_{13}}{S_{11}}\Big)\sigma_2^{(1)}\sigma_3^{(1)} + \frac{1}{2}S_{44}\sigma_4^{(1)2}\Big]dy_2\,dy_3 \qquad (5.29)$$

where the elastic properties of lamina (2) have been expressed in terms of uniaxial material properties. Again, the constant terms in ϵ_1^2 may be neglected, as they cannot affect the value of k. Making the same approximation used in the angle-ply problem, one obtains the following expressions for the required integrals:

$$\int_0^t\int_0^a \sigma_2^{(2)}dy_2\,dy_3 = -b\sigma_2^{(0)}\left[at - 2kt^2\right] \qquad (5.30)$$

$$\int_0^t\int_0^a \sigma_3^{(2)}dy_2dy_3 = 0 \qquad (5.31)$$

$$\int_0^t\int_0^a \sigma_2^{(2)2}dy_2dy_3 = b^2\sigma_2^{(0)2}\left[at - 2.75\,kt^2\right] \qquad (5.32)$$

$$\int_0^t\int_0^a \sigma_3^{(2)2}dy_2dy_3 = .1795\,\mathcal{F}_1\frac{\sigma_2^{(0)2}}{k^3}t^2 \qquad (5.33)$$

$$\int_0^t\int_0^a \sigma_2^{(2)}\sigma_3^{(2)}dy_2\,dy_3 = .2083\,\mathcal{F}_2\frac{\sigma_2^{(0)2}}{k}t^2 \qquad (5.34)$$

$$\int_0^t\int_0^a \sigma_4^{(2)2}dy_2\,dy_3 = .0833\,b^2\frac{\sigma_2^{(0)2}}{k}t^2 \qquad (5.35)$$

$$\int_t^{(1+b)t}\int_0^a \sigma_2^{(1)}dy_2\,dy_3 = b\sigma_2^{(0)}\left[at - 2kt^2\right] \qquad (5.36)$$

$$\int_t^{(1+b)t}\int_0^a \sigma_3^{(1)}dy_2dy_3 = 0 \qquad (5.37)$$

$$\int_t^{(1+b)t}\int_0^a \sigma_2^{(1)2}\,dy_2\,dy_3 = b\,\sigma_2^{(0)2}\left[at-2.75\,kt^2\right] \tag{5.38}$$

$$\int_t^{(1+b)t}\int_0^a \sigma_3^{(1)2}\,dy_2\,dy_3 = .0125\,\mathcal{F}_3\,\frac{\sigma_2^{(0)2}}{k^3}\,t^2 \tag{5.39}$$

$$\int_t^{(1+b)t}\int_0^a \sigma_2^{(1)}\sigma_3^{(1)}\,dy_2\,dy_3 = -.0417\,b\,\mathcal{F}_4\,\frac{\sigma_2^{(0)2}}{k}\,t^2 \tag{5.40}$$

$$\int_t^{(1+b)t}\int_0^a \sigma_4^{(1)2}\,dy_2\,dy_3 = .0833\,\mathcal{F}_4\,\frac{\sigma_2^{(0)2}}{k}\,t^2 \tag{5.41}$$

where

$$\mathcal{F}_1 = .349\,b^4 + .465\,b^3 + .186\,b^2 \tag{5.42}$$

$$\mathcal{F}_2 = .60\,b^3 + .40\,b^2 \tag{5.43}$$

$$\mathcal{F}_3 = (1+b)^5 - 5(1+b)^4 + 10(1+b)^3 - 10(1+b)^2 + 5(1+b) - 1 \tag{5.44}$$

$$\mathcal{F}_4 = (1+b)^3 - 3(1+b)^2 + 3(1+b) - 1 \tag{5.45}$$

Substituting (5.30) through (5.41) into (5.29), differentiating, and solving the resulting biquadratic equation for k yields the result

$$k = \sqrt{\frac{C_2+\sqrt{C_2^2+4C_1C_3}}{2C_1}} \tag{5.46}$$

where a considerably abbreviated notation has been adopted in which

$$C_1 = 2b\epsilon_1\mathcal{S}_1 - 1.375\,\sigma_2^{(0)}\left(b^2\mathcal{S}_2 + b\mathcal{S}_3\right) \tag{5.47}$$

$$C_2 = \sigma_2^{(0)}\left[.2083\,\mathcal{F}_2\mathcal{S}_4 + .0417\left(b^2S_{55} + \mathcal{F}_4S_{44} + b\mathcal{F}_4\mathcal{S}_5\right)\right] \tag{5.48}$$

$$C_3 = \sigma_2^{(0)}\left(.2688\,\mathcal{F}_1\mathcal{S}_6 + .0187\,\mathcal{F}_3\mathcal{S}_7\right) \tag{5.49}$$

and

$$\mathcal{S}_1 = \frac{S_{12}}{S_{22}} - \frac{S_{12}}{S_{11}} \tag{5.50}$$

$$\mathcal{S}_2 = S_{11} - \frac{S_{12}^2}{S_{22}} \tag{5.51}$$

$$\mathcal{S}_3 = S_{22} - \frac{S_{12}^2}{S_{11}} \tag{5.52}$$

$$\mathcal{S}_4 = S_{13} - \frac{S_{12}S_{23}}{S_{22}} \tag{5.53}$$

$$\mathcal{S}_5 = \frac{S_{12}S_{13}}{S_{11}} - S_{23} \tag{5.54}$$

$$\mathcal{S}_6 = S_{33} - \frac{S_{23}^2}{S_{22}} \tag{5.55}$$

$$\mathcal{S}_7 = S_{33} - \frac{S_{13}^2}{S_{11}} \tag{5.56}$$

Inherent in the above solution is the assumption, as stated in Section 5.2, that the laminate is of the 0/90/0 configuration. In order to work a 90/0/90 problem one must perform the following interchanges of compliance constants: S_{11} with S_{22}, S_{13} with S_{23}, and S_{44} with S_{55}. This formal procedure amounts to a temporary interchange of the definitions of the 0^o and 90^o directions.

CHAPTER 6

RESULTS OF CROSS-PLY CALCULATIONS

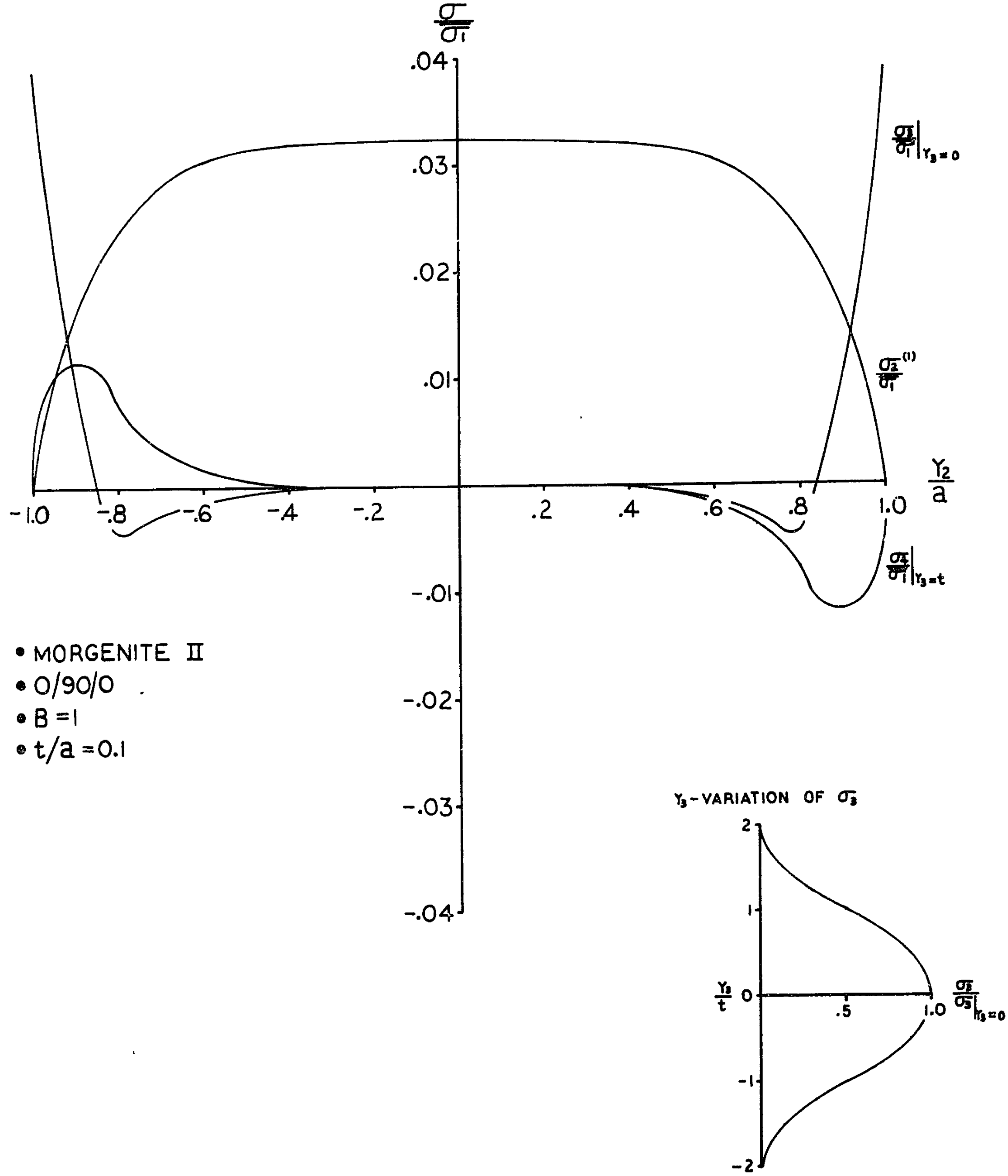
6.1 Sample Case

The solution of Section 5.3 was programmed in a manner similar to that of the angle-ply problem in order to generate a set of results which would yield information on the variation of the stress field with the thickness ratio b, and also on the behavioral differences distinguishing 90/0/90 from 0/90/0 configurations. Again, this was done principally for reasons of convenience, for although the solution of Section 5.3 is considerably more complicated than that of Section 3.4 it can still be worked by hand. As in the angle-ply case, the programming involved offers no special difficulty and requires no detailed discussion here. A listing of the program used, CROSS, and a glossary of its FORTRAN variables, appear in Appendices C and D for reference.

Figure 16 illustrates a result typical of those obtained for the cross-ply cases. Again, k was found to be of order one -- in this case, 0.91 -- so that the lamination-theory asymptotes are very closely approached for

FIGURE 16

CROSS-PLY CASE: STRESS VS. Y_2



regions 4t or deeper into the interior. A boundary-layer behavior is evident and one may, as for angle-ply laminates, define the boundary-layer thickness as 4kt.

Unlike the results of the angle-ply cases, the maximum intensity of the interlaminar shearing stress σ_4 does not occur at the free edges. Although it necessarily does occur on the interlaminar interfaces, it does so at a distance kt in from the free edges $y_2 = \pm a$. The intensity of σ_4 at these points is 0.0118 of the average composite longitudinal stress. The most severe free-edge effect, however, is not the interlaminar shear; it is the interlaminar tensile (or compressive) stress σ_3. For the case shown in Figure 16, the normalized σ_3 is seen to reach a maximum value of 0.0390 at the free edges. As illustrated in the insert drawing, the intensity of σ_3 is greatest at the laminate mid-plane and varies quadratically with y_3 within each laminate, having inflections at the interlaminar interfaces. As required by equations (5.31) and (5.37), σ_3 is compressive over part of the laminate width and tensile over the rest, such that its y_2 integral vanishes. One can see physically that this must necessarily be so, as the laminate is not subjected to any loading in the y_3 direction and hence the net normal force over any y_3 plane within the laminate must vanish. A similar argument, based on the absence of any y_2 loading, will be recalled from equation (5.5) as requiring that the net normal force on any y_2 plane be zero. Because the longitudinal tensions

$\sigma_1^{(1)}$ and $\sigma_1^{(2)}$ are too large to fit on the scale of Figure 16, and in any case vary hardly at all from the values predicted by lamination theory throughout the laminate, they are not shown.

6.2 Variation of k, σ_4, and σ_5 with b

The boundary-layer thickness parameter k is shown in Figure 17 as a function of the lamina thickness ratio b for both 0/90/0 and 90/0/90 configurations. Not surprisingly, since the boundary-layer thickness is in some sense a weighted average of the thicknesses of the 0° and 90° laminae, k starts out at zero for b = 0 and increases monotonically with b for all configurations. The 90/0/90 laminates display generally thinner boundary layers than 0/90/0 configurations, indicating that varying the thickness of a 90° lamina has less effect on the boundary-layer thickness than does varying the thickness of a 0° lamina, except in Morgenite II at small thickness ratios. The spread in behavior between 0/90/0 and 90/0/90 cases is less for Morgenite II than for NARMCO 5505 -- another effect of the lesser disparity between extensional and shear compliances exhibited by the graphite/epoxy material.

Figure 18 illustrates the variation of the maximum normalized interlaminar shear stress, $\frac{\sigma_4}{\overline{\sigma}_1}$, with b for laminates of 0/90/0 and 90/0/90 configurations. The stress intensity is seen to increase from zero fairly rapidly as b increases from zero to between 0.4 and 0.6, thereafter

FIGURE 17

CROSS-PLY CASES: K VS. B

——— NARMCO 5505

- - - - - - MORGENITE II

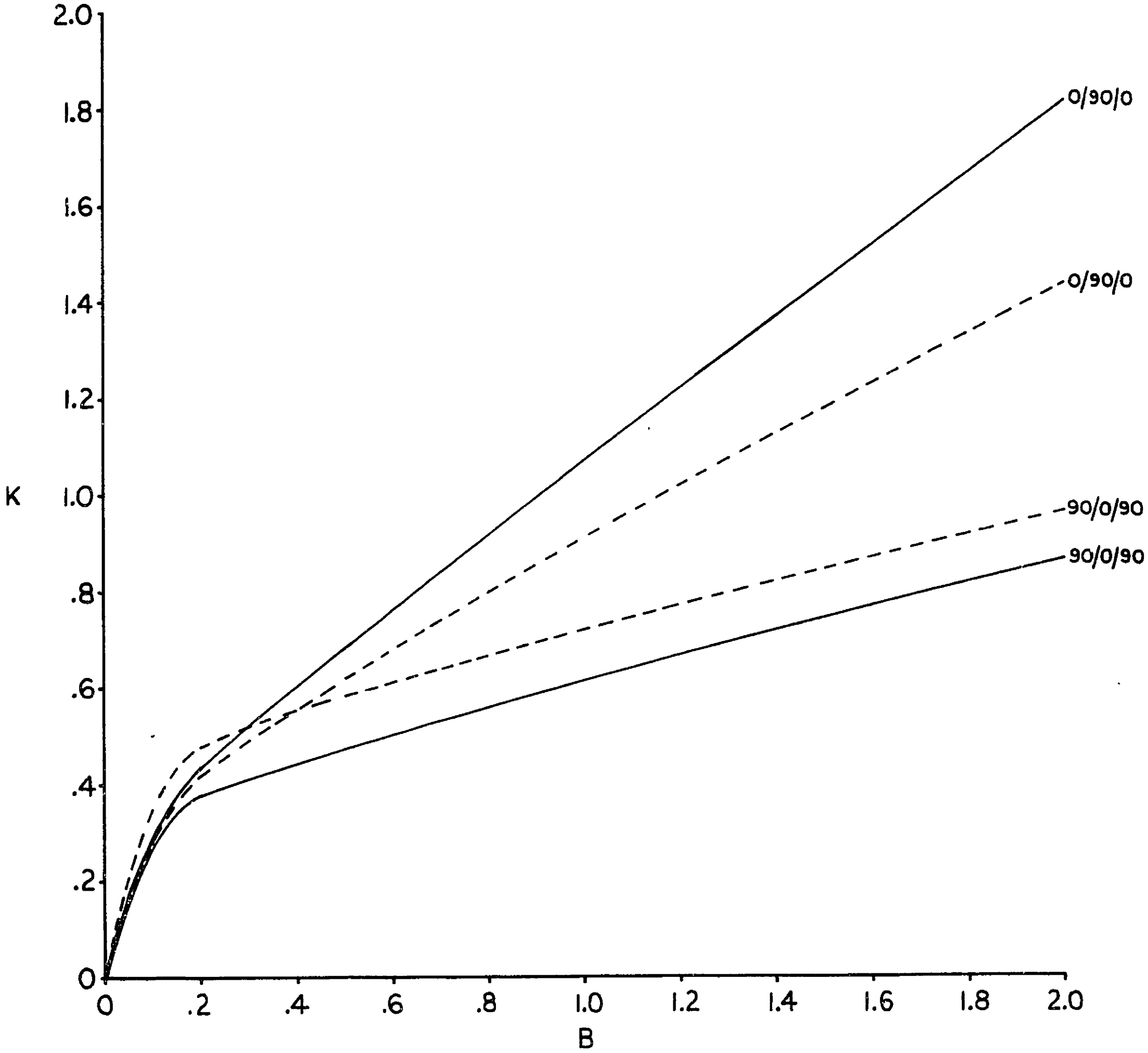
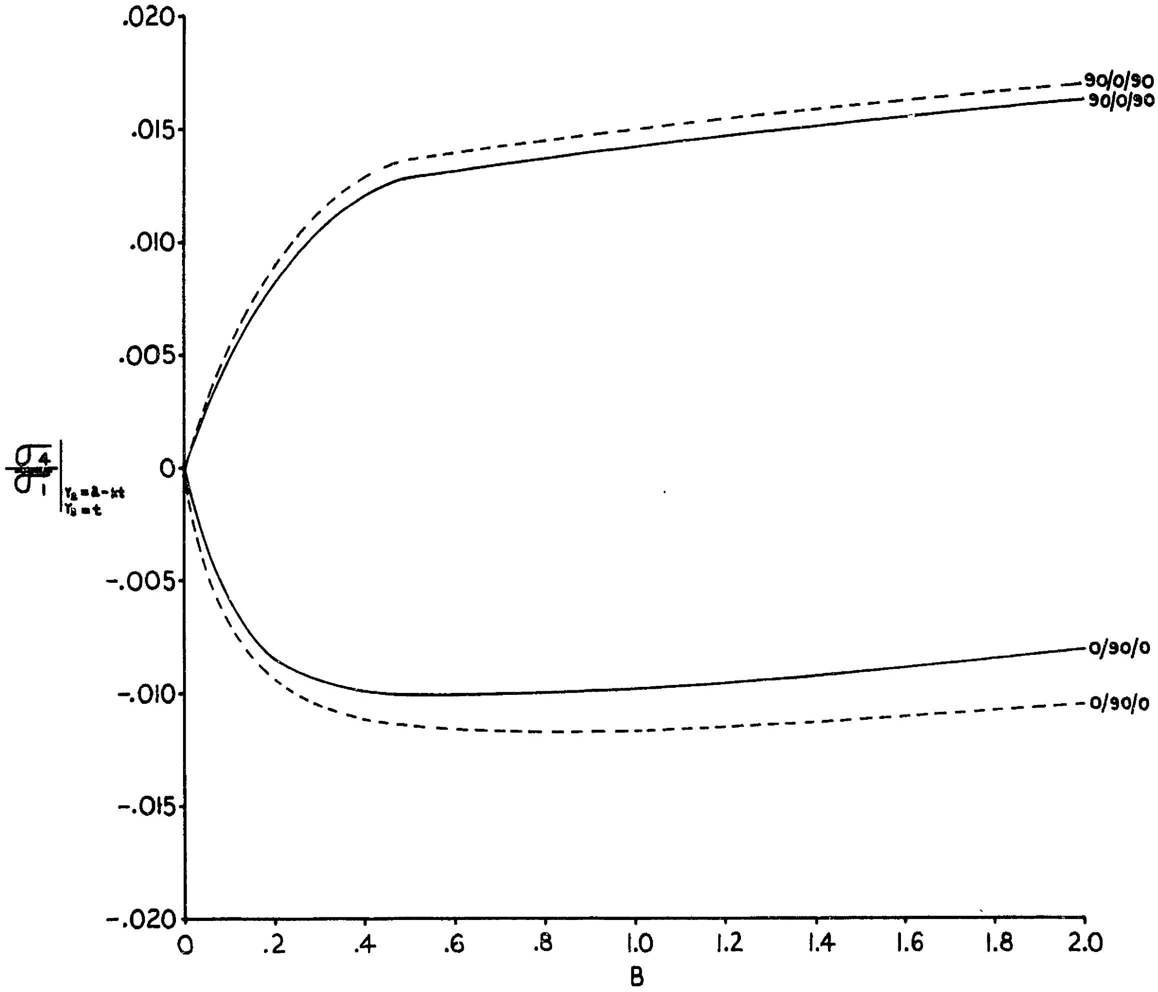


FIGURE 18

CROSS-PLY CASES: $\left.\frac{\sigma_4}{\sigma_1}\right|_{\substack{Y_2=a-kt \\ Y_3=t}}$ VS. B

——— NARMCO 5505
- - - - - MORGENITE II



leveling out to a virtually constant value of several percent of the average applied tension. This type of behavior is expected, both physically and from the nature of the assumed stress functions of the second-order solution. Physically, it is evident that a lamina of zero thickness cannot produce any effect on a lamina of finite thickness, and hence all free-edge effects (as well as the transverse tension σ_2 which produces them) must vanish for $b = 0$. Mathematically, the exponential function $\exp\left(\frac{y_2 - a}{kt}\right)$ is driven to zero more strongly than any negative power of k is driven to infinity as k itself goes to zero with b, and hence both σ_4 and σ_3 become zero as the thickness of the outer laminae becomes infinitesimal (the point $y_2 = a$ is singular and must be approached from the left). As b increases from zero so does k, and hence the y_2 derivative of $\sigma_2^{(1)}$ decreases even more than it would otherwise due to the decrease in the intensity of the lamination-theory value of $\sigma_2^{(1)}$ itself. Thus the y_3 derivative of $\sigma_4^{(1)}$ decreases in magnitude, compensating for the increase in b and eventually producing a relatively constant peak value of σ_4 at the interlaminar interface for large enough values of b. 0/90/0 laminates produce negative values of σ_4 for $y_2 > 0$, since laminae (1) and (3) are in transverse tension ($\sigma_2^{(1)}$ positive), while 90/0/90 laminates have positive shear stresses because the upper and lower laminae are in transverse compression. This statement applies to the shearing stress on the interface between laminae (1) and (2); the sign of the interlaminar shear is, of course, reversed on the interface between laminae (2) and (3). Morgenite II exhibits slightly higher interlaminar shear intensities than NARMCO 5505, again due to its greater shear stiffness in proportion to its longitudinal modulus, but neither material experiences interlaminar shear in excess of 2% of the average longitudinal tension for any value of b between 0 and 2.0.

Figure 19 illustrates the variation of the maximum normalized interlaminar tension $\frac{\sigma_3}{\sigma_1}$ with b. As does σ_4, σ_3 starts at zero and reaches a relatively constant intensity for values of b greater than about 0.6. This behavior is explained by physical and mathematical arguments which are analogous to those used in reference to Figure 18. 0/90/0 laminates are seen to experience an interlaminar tension which is more severe for Morgenite II than for NARMCO 5505, while 90/0/90 laminates exhibit a compression which is greater for NARMCO 5505 than for Morgenite II. This, too, is expected from the results of Figure 18 and the relationship between the y_2 derivative of σ_4 and the y_3 derivative of σ_3 defined by the equilibrium equation (5.19).

6.3 Importance of Results to Design

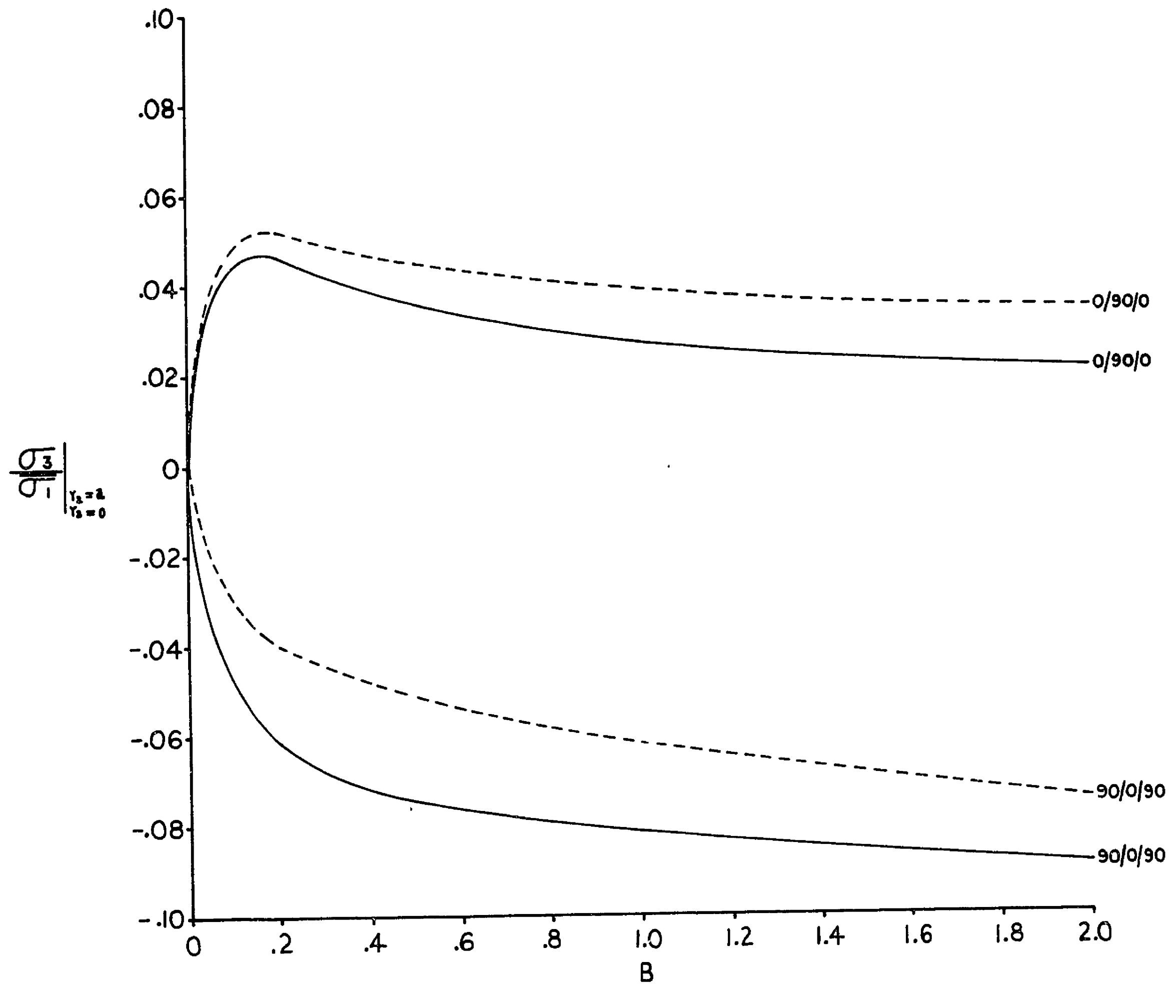
In cross-ply laminates, as for angle-ply laminates, it is desirable to keep both the magnitude of the edge effects and the distance over which they act to a minimum. Although the cross-ply edge effect stresses are generally far less intense than those for angle plies, the interlaminar tension

FIGURE 19

CROSS-PLY CASES: $\left.\frac{\sigma_3}{\sigma_1}\right|_{Y_2=a,\ Y_3=0}$ VS. B

——— NARMCO 5505

- - - - - MORGENITE II



in particular may nevertheless present a problem and could, in some applications, contribute to failure of the composite by delamination starting at the edges.

The value of b chosen for any given laminate is not too important as regards the edge-effect intensities, since these stresses are in any case relatively slight. It would still be unwise to use too great a value of b, however, since this does increase the edge-effect stresses somewhat in some configurations and (more importantly) increases the thickness of the boundary layer. Either a very large or a very small value of b, moreover, will produce a significant intensification of the lamination-theory stresses in the thinner laminae. Designers can use this effect to obtain a simultaneous-failure condition in both the 0^{o} and 90^{o} laminae, but more extreme variations in b than required by this condition will usually be found to be undesirable unless their use is forced by considerations of bending stiffness, as in some sandwich-construction applications.

Wherever bending stiffness in the y_1y_3 plane is a consideration, the 0/90/0 laminate will of course be preferable to the 90/0/90 arrangement. The same is true of applications where good resistance to buckling under axial compression is required -- and in fact a good general rule of thumb would be that the filaments in the outer laminae of any cross-ply laminate should run in the direction of the primary loading axis.

As in angle-ply laminates, overall stiffness-to-weight and strength-to-weight considerations may be expected to influence material choice more than differences in edge effects between the two materials studied. NARMCO 5505 has the larger value of k for 0/90/0 arrangements but Morgenite II has a slightly larger k value for most of the 90/0/90 cases. Shear intensities for the graphite/epoxy material are uniformly greater than those of the boron/epoxy, but only very slightly so. Interlaminar tension is greater for Morgenite II in 0/90/0 laminates, but more severe in NARMCO 5505 for 90/0/90 configurations -- and relatively slight in any case. If anything, edge-effect differences in cross-ply laminates may be expected to influence choices between the two materials even less than in the angle-ply cases. Again, however, it should be remembered that the smaller tape thickness of Morgenite II makes it the superior material in applications where edge-effect penetration depth is critical.

6.4 Accuracy and Limitations

The solution obtained in Section 5.3 was compared to results obtained from the assumed-stress hybrid finite-element method described in reference [4]. In this method the stresses within each element are assumed to be given by polynomial functions of the coordinates which identically satisfy stress equilibrium. The coefficients of the polynomial terms, initially unknown, constitute a column matrix $\{\beta\}$:

$$\{\sigma\} = [P]\{\beta\} \tag{6.1}$$

where [P] contains the various powers of the coordinates to be used. Interelement compatibility is maintained by assuming the edge displacements, {u}, of each polygonal (or polyhedral) element to vary linearly between the displacements at the corners, {q}:

$$\{u\} = [L]\{q\} \tag{6.2}$$

where [L] is the matrix of linear functions of the surface coordinates required. The resolved normal and shear stresses on the element surface, $\{\vec{\sigma}\}$, are required in terms of the stress coefficients $\{\beta\}$, as the hybrid method uses the principle of minimum complementary energy to determine the element stiffness matrix [k]. $\{\vec{\sigma}\}$ is related to $\{\beta\}$ by

$$\{\vec{\sigma}\} = [R]\{\beta\} \tag{6.3}$$

where the [R] matrix contains the required functions of element surface coordinates. With this information one can compute [k] as described in [4] according to

$$[k] = [T^T][H^{-1}][T] \tag{6.4}$$

where

$$[T] = \iint [R^T][L]\,dA \tag{6.5}$$

$$[H] = \iiint [P^T][S][P]\,dv \tag{6.6}$$

and [S] is the matrix of material compliances used in writing the compliance form of the stress-strain relations. The integration of (6.5) is carried out over the surface of the element; that of (6.6) over the element interior volume. A large digital computer program which has provisions for assembling the stiffness matrix of the complete body from those of the elements and accepting stress and displacement boundary conditions is then used to solve for $\{q\}$ and hence for $\{\beta\}$ according to

$$\{\beta\} = [H^{-1}][T]\{q\} \tag{6.7}$$

For the cross-ply problem the elements used were rectangular solids of unit length (y_1 dimension), width (y_2 dimension) a, and height (y_3 dimension) b, where a and b may be varied at will but must not differ from each other by too great a ratio or numerical accuracy problems will be encountered. Some investigators have used ratios greater than 5:1 successfully for some purposes, but 3:1 is more generally considered the safe limit and 2:1 was adopted as the upper limit in this study.

In solving the cross-ply problem by this method the following intraelement stress distribution was used:

$$\sigma_2 = \beta_1 + \beta_2 y_2 + \beta_3 y_3 \tag{6.8}$$

$$\sigma_3 = \beta_4 + \beta_5 y_2 + \beta_6 y_3 \tag{6.9}$$

$$\sigma_4 = \beta_7 - \beta_2 y_3 - \beta_6 y_2 \tag{6.10}$$

from which, by the stress-strain relations,

$$\sigma_1 = \frac{\epsilon_1}{S_{11}} - \frac{S_{12}}{S_{11}}\beta_1 - \frac{S_{12}}{S_{11}}\beta_4 - y_2\left(\frac{S_{12}}{S_{11}}\beta_2 + \frac{S_{12}}{S_{11}}\beta_5\right) - y_3\left(\frac{S_{13}}{S_{11}}\beta_3 + \frac{S_{13}}{S_{11}}\beta_6\right) \tag{6.11}$$

Since $\frac{\epsilon_1}{S_{11}}$ is a known constant it must be deleted from the $\{\beta\}$ matrix. If it is not, the [H] matrix will contain an extra diagonal element, and the [T] matrix an extra row, to account for the existence of a quantity -- the constant $\frac{\epsilon_1}{S_{11}}$ -- which cannot be adjusted by the principle of minimum complementary energy. The [H] matrix will then be singular and thus cannot be inverted. Upon performing the indicated deletion, one obtains a [T] matrix in the form

$$[T] = \begin{bmatrix} -\frac{b}{2} & \frac{b}{2} & \frac{b}{2} & -\frac{b}{2} & 0 & 0 & 0 & 0 & -ab\frac{S_{12}}{S_{11}} \\ 0 & \frac{ab}{2} & 0 & -\frac{ab}{2} & \frac{b^2}{6} & -\frac{b^2}{6} & -\frac{b^2}{3} & \frac{b^2}{3} & -\frac{a^2 b S_{12}}{2S_{11}} \\ -\frac{b^2}{6} & \frac{b^2}{6} & \frac{b^2}{3} & -\frac{b^2}{3} & 0 & 0 & 0 & 0 & -\frac{ab^2 S_{12}}{2S_{11}} \\ 0 & 0 & 0 & 0 & -\frac{a}{2} & -\frac{a}{2} & \frac{a}{2} & \frac{a}{2} & -ab\frac{S_{13}}{S_{11}} \\ 0 & 0 & 0 & 0 & -\frac{a^2}{6} & -\frac{a^2}{3} & \frac{a^2}{3} & \frac{a^2}{6} & -\frac{a^2 b S_{13}}{2S_{11}} \\ \frac{a^2}{6} & \frac{a^2}{3} & -\frac{a^2}{3} & -\frac{a^2}{6} & 0 & -\frac{ab}{2} & 0 & \frac{ab}{2} & -\frac{ab^2 S_{13}}{2S_{11}} \\ -\frac{a}{2} & -\frac{a}{2} & \frac{a}{2} & \frac{a}{2} & -\frac{b}{2} & \frac{b}{2} & \frac{b}{2} & -\frac{b}{2} & 0 \end{bmatrix} \tag{6.12}$$

and the [H] matrix becomes

$$[H] = \begin{bmatrix} ab\,\ell_3 & \frac{a^2b}{2}\ell_3 & \frac{ab^2}{2}\ell_3 & -ab\,\ell_5 & -\frac{a^2b}{2}\ell_5 & -\frac{ab^2}{2}\ell_5 & 0 \\ & \frac{a^3b}{3}\ell_3 + \frac{ab^3}{3}S_{44} & \frac{a^2b^2}{4}\ell_3 & -\frac{a^2b}{2}\ell_5 & -\frac{a^3b}{3}\ell_5 & \frac{a^2b^2}{4}[S_{44} - \ell_5] & -\frac{ab^2}{2}S_{44} \\ & & \frac{ab^3}{3}\ell_3 & -\frac{ab^2}{2}\ell_5 & -\frac{a^2b^2}{4}\ell_5 & -\frac{ab^3}{3}\ell_5 & 0 \\ & & & ab\,\ell_7 & \frac{a^2b}{2}\ell_7 & \frac{ab^2}{2}\ell_7 & 0 \\ & \text{SYMMETRIC} & & & \frac{a^3b}{3}\ell_7 & \frac{a^2b^2}{4}\ell_7 & 0 \\ & & & & & \frac{ab^3}{3}\ell_7 + \frac{a^3b}{3}S_{44} & -\frac{a^2b}{2}S_{44} \\ & & & & & & ab\,S_{44} \end{bmatrix} \quad (6.13)$$

where

$$\ell_3 = S_{22} - \frac{S_{12}^2}{S_{11}} \quad (6.14)$$

$$\ell_5 = \frac{S_{12}S_{13}}{S_{11}} - S_{23} \quad (6.15)$$

$$\ell_7 = S_{33} - \frac{S_{13}^2}{S_{11}} \quad (6.16)$$

With the help of Mr. James Kotanchik of the MIT Aeroelastic and Structures Research Laboratory, who performed all the programming necessary to incorporate this treatment into the STACUSS-I structural analysis program, solutions were generated for several test cases and compared with the results obtained from the solution of Section 5.2. In comparing results for a 0/90/0 laminate in NARMCO 5505 with a lamina thickness ratio of 2.0 agreement between the approximate second-order and finite-element solutions was found to be quite good in all respects. There was almost no detectable difference in σ_1, and the only difference in σ_2 was a y_3 dependence, causing an excursion about the lamination-theory value across the lamina thickness, which was indicated by the finite-element solution. The second-order theory, it will be recalled, does not predict any y_3 dependence of σ_2 within a given lamina. The finite-element solution indicated a slightly more rapid subsidence of the edge-effect stresses than predicted by the approximate theory, but the magnitudes of the edge-effect stresses were substantially the same for both solutions. There was no appreciable difference in the maximum intensity of σ_3, but the approximate theory proved to be somewhat conservative in predicting the maximum intensity of σ_4 -- .0072 of the average applied longitudinal tension as opposed to .0058 according to the finite-element solution. Elements of $a = t$ and $b = 0.5t$ were used in generating the finite-element results, making it likely that the behavior of the stresses in the regions directly adjacent to the edges was not determined as precisely as the behavior of the stresses deeper in the laminate interior, so that the slight disagreement between the two solutions encountered in the boundary-layer regions is not surprising. The shapes of the σ_2 subsidence and edge-effect concentration curves were well matched, and the approximate theory's assumption of a quadratic y_3 variation in σ_3 proved to be surprisingly accurate. The y_3 variation of σ_4, like that of σ_5 in the solution of [3] for the angle-ply problem, proved to have some curvature as opposed to the linear behavior assumed, but the disagreement in this respect was not serious.

The same geometrical limitations apply to laminates for which the cross-ply approximate solution is accurate as were found to apply for the angle-ply configuration. The laminate dimension a must be equal to or greater than 4kt to insure a boundary-layer behavior of the edge effects and a good approximation to lamination theory in the laminate interior.

CHAPTER 7

GENERAL APPLICABILITY OF THEORY

7.1 Problems of Potential Application

The uniaxial loading of three-lamina laminates in tension or compression, as investigated in this study, is the simplest type of problem to which the proposed second-order theory of laminated structures is applicable. It is by no means the only type of problem capable of being treated in this manner, however; there is a large class of more general composite laminate problems involving free-edge effects in which methods entirely analogous to those of Chapters 3 through 6 may be used. First, of course, it is relatively easy to extend the uniaxial loading analysis to laminates composed of any number of laminae, either cross-ply or angle-ply or both. Bending problems, which by lamination theory are found to involve linear distributions of the extensional strains ϵ_1 and/or ϵ_2 throughout the laminate cross-section, and torsion problems in which the concept of shear flow in thin-walled sections can be combined with lamination theory to provide first-order estimates of the intralaminar stresses and strains can also be treated.

The analysis of free-edge effects associated with holes or cutouts in wide, flat laminates, while certainly involving a considerable increase in mathematical complexity, is still another example of the types of problems amenable to solution by the second-order assumed-stress theory.

In short, the proposed method is applicable to free-edge effect problems of any kind which involve introduction of loading at the ends of the component under consideration, as long as laminates of suitable geometry are involved. By "suitable geometry" is meant that the width of laminate directly adjacent to the free edge in question must be great enough for the assumed boundary-layer behavior near the edge to become closely asymptotic to the lamination-theory solution in the interior, and that the laminate length must be great enough that the effects of load introduction at the ends need not be considered. Since a lamination-theory solution is the only initial information needed to apply the second-order theory, it follows that laminate balance about either the y_1y_2 or the y_1y_3 plane is not required. Certainly the mechanics of the solution are greatly complicated by laminate imbalance, but since lamination-theory solutions are obtainable for such cases it follows that second-order solutions are also obtainable.

7.2 Principles of General Application

In order to extend the second-order theory to more general edge-effect problems it is basically necessary only to keep in mind the fundamental premises of the theory as enumerated in Section 3.4. Suppose, by way of example, that one has obtained the lamination-theory solution for a balanced laminate of width 2a, containing a large number of both cross- and angle-ply laminae, subjected to uniaxial tension or compression. Such a solution will involve non-zero values of σ_1, σ_2, and σ_6 which are constant within each lamina. Within any given lamina, let the lamination-theory stresses and strains be identified by asterisks, as σ_1^*, σ_2^*, and so on. Let the stresses determined by the second-order theory be written without asterisks. Then the equilibrium equations may be written

$$\frac{\partial \sigma_6}{\partial y_2} + \frac{\partial \sigma_5}{\partial y_3} = 0 \tag{7.1}$$

$$\frac{\partial \sigma_2}{\partial y_2} + \frac{\partial \sigma_4}{\partial y_3} = 0 \tag{7.2}$$

$$\frac{\partial \sigma_4}{\partial y_2} + \frac{\partial \sigma_3}{\partial y_3} = 0 \tag{7.3}$$

Application of the second-order theory then yields the results, for $y_2 \geq 0$,

$$\sigma_6 = \sigma_6^* \left[1 - \exp\left(\frac{y_2 - a}{kt}\right)\right] \tag{7.4}$$

$$\sigma_5 = -\frac{1}{k}\sigma_6^* \left[C_1 + C_2 y_3\right] \exp\left(\frac{y_2 - a}{kt}\right) \tag{7.5}$$

$$\sigma_2 = \sigma_2^* \left[1 - \left(1 + \frac{a - y_2}{kt}\right) \exp\left(\frac{y_2 - a}{kt}\right)\right] \tag{7.6}$$

$$\sigma_4 = -\frac{\sigma_2^*}{k} \left[C_3 + C_4 y_3\right] \left[\frac{a - y_2}{kt}\right] \exp\left(\frac{y_2 - a}{kt}\right) \tag{7.7}$$

$$\sigma_3 = \frac{\sigma_2^*}{k^2}\left[C_5 y_3^2 + C_6 y_3 + C_7\right]\left[\frac{a-y_2}{kt} - 1\right]\exp\left(\frac{y_2-a}{kt}\right) \qquad (7.8)$$

$$\sigma_1 = \frac{\epsilon_1^*}{S_{11}} - \frac{S_{12}}{S_{11}}\sigma_2 - \frac{S_{13}}{S_{11}}\sigma_3 - \frac{S_{16}}{S_{11}}\sigma_6 \qquad (7.9)$$

The constants C_1 through C_7 are to be determined subject to the equilibrium equations (7.1) through (7.3), the conditions that σ_3, σ_4, and σ_5 be continuous across the interlaminar interfaces, and (if the lamina in question is the uppermost or lowermost of the laminate) the free-surface conditions at the upper and lower faces of the laminate. In this case the free-edge effects due to the vanishing of σ_6 at the edges are not coupled through the equilibrium equations to the effects due to the vanishing of σ_2. The determination of k by the principle of minimum complementary energy, however, will involve coupling of the stresses through the stress-strain relations.

Even in this rather simple extension of the theory, it may be seen that the number of constants to be determined from (7.1) through (7.7) becomes increasingly great as the number of laminae increases. Because of this, and because the energy integrations become very much more complicated, automatic computer techniques for solving systems of linear algebraic equations and performing numerical integrations will have to be brought into use. For more general problems which may involve additional stress coupling through the equilibrium equations the formulation will be more complicated still, and ultimately one can foresee that a level of complexity and generality will be reached beyond which the use of a finite-element or finite-difference technique will be simpler and more convenient than attempting to apply the second-order assumed-stress theory. One is reminded of an analogous situation encountered with finite-element methods, in which there is a compromise to be made between the complexity of the assumed intraelement stress or displacement functions and the size of the elements: there is a "point of diminishing returns" beyond which the quest for a reduction in the number of elements by increasing the number of terms in the intraelement functions is not worthwhile. Indeed, that such an analogy exists is not surprising, for the second-order theory is essentially an assumed-stress finite-element method in which each element encompasses an entire lamina.

In this connection the possibility of generalizing the functional forms of the second-order theory to include more than one exponential term may be mentioned. Thus far it has been tacitly assumed that all free-edge effects are boundary-layer phenomena involving a single characteristic length of subsidence: kt, where t is the thickness of one of the laminae in the laminate under consideration. Any problem amenable to solution by the second-order theory can be done with a single such exponential function, but the question arises whether or not greater accuracy might be achieved by the use of several exponential functions involving different boundary-layer thickness parameters

k_1, k_2, k_3, etc., or multiples of the single characteristic parameter k: 2k, 3k, and so on. The former choice of multiple-exponential treatment will require the solution of a matrix equation to determine the values of the k_i which satisfy the condition of minimum complementary energy, while the latter treatment essentially amounts to a form of eigenfunction expansion. It is certainly possible that such procedures could improve the accuracy of the second-order solution; judging from the results obtained for the elementary three-lamina cases solved in this study, however, the single-parameter solutions are already quite accurate and there is not likely to be a great deal of improvement in accuracy required in most cases. Moreover, the inclusion of more than one exponential term greatly complicates the analysis -- so much so, in fact, that anyone experiencing accuracy problems with the single-parameter second-order solution may be better advised to go to a finite-element solution than to attempt the introduction of additional exponentials into the second-order theory. Definite conclusions regarding such matters must, of course, await the results of experience in application.

CHAPTER 8

CONCLUSIONS

8.1 Theory

A workable second-order theory of laminated structures which accounts for free-edge effects has been developed and successfully applied to simple tension problems involving balanced laminates with good accuracy. The theory can, in principle, be extended to apply to free-edge problems involving tension, bending, and torsion of balanced and unbalanced laminates composed of any number of laminae, and to problems involving holes and cutouts in composite laminates, as long as the laminate under consideration is wide compared to its thickness and long compared to its width, with the loading condition being introduced at its ends.

As presently formulated, the theory assumes the free-edge effects to be of a boundary-layer nature with a subsidence behavior characterized by a single exponential function. The possibility of adding more exponential terms to improve accuracy in more difficult problems exists, but the use of finite-element solutions may prove simpler and more convenient than such a course, particularly in computing facilities where finite-element software already exists. The single-parameter second-order theory may also prove less convenient than a finite-element solution for certain of the more complicated free-edge problems. The merits of both techniques must be evaluated in each particular case before either one is chosen.

For the simple laminates investigated in this study, the second-order approximate technique did prove superior to the finite-element method in convenience of formulation and in computing time required. The results presented in Chapters 4 and 6 were generated on an IBM 1130 computing system which compiled both programs, listed them completely, executed 50 cases and printed detailed results all in about 50 minutes.

8.2 Results

Balanced angle-ply laminates of NARMCO 5505 boron/epoxy and Morgenite II graphite/epoxy were found to exhibit interlaminar shear concentrations of considerable magnitude near their free edges. Balanced cross-ply laminates of the same materials were found to be characterized by small interlaminar shear concentrations and slightly greater interlaminar tensions. In each case, the magnitudes of the edge-effect stresses were found to be of the same order as the magnitudes of the lamination-theory stresses that must vanish at the free edges to produce them. In the angle-ply cases the edge-effect stress σ_5 was found to reach a maximum intensity of the same order as the lamination-theory stress σ_6, while in the cross-ply cases the maximum intensities of σ_3 and σ_4 were of the same order as σ_2. The edge effects were in all cases found to act over a distance into the laminate interior of the same order as the thickness of a single lamina, being characterized by an exponential subsidence of the form $\exp\left(\frac{y_2-a}{kt}\right)$, where k is of order one.

8.3 Design Recommendations

In the fabrication of composite laminates thick laminae should be avoided in order to minimize the region of influence of free-edge effects. Where possible, single plies should form laminae. Wherever overall balance in a completed laminate is required and angle-plies are to be used, they should be used in balanced stacks of the configuration $+\theta/-\theta/-\theta/+\theta$, each ply of the four thus forming a lamina of equal thickness to each of the others. Where cross-plies are used, the difference in thickness between 0^o and 90^o laminae should be no greater than that required by simultaneous lamina failure criteria. In choosing between NARMCO 5505 and Morgenite II for a given application, edge-effect considerations will not normally be a significant factor. Where edge-effect penetration depth is critical, however, Morgenite II should be used.

APPENDIX A

DIGITAL COMPUTER PROGRAM ANGLE

Corrections

The listing of ANGLE presented herein is based on an erroneous analysis that attempts to account for the effect of varying thickness ratio without allowing for the effects of the transverse tension σ_2 produced by values of b other than 1.0. The discovery of this error led to the deletion of Figures 12 and 13 from the text. ANGLE as given here will produce correct results only if the value of b input is 1.0. To guard against the possibility of erroneous results due to the use of b values other than 1.0, the following changes in ANGLE can be made:

Change statements 1 and 2 to read

```
 1 READ (2,2) KASE, A, T, THETA
 2 FORMAT (I4, 3F5.2)
```

Change statement 10 and the subsequent three statements to read

```
10 S6P = -S16/S66
   S1P = 1.0
   S6M = -S6P
   S1M = 1.0
```

Remove statement 15 and the one following it, and change the next two statements to read

```
15 SF = S16**2/S11 - S66
   RK = SQRT(.1667*S55*S6P/(1.5*SF*S6P - 2.*S16*EPS1/S11))
```

```
C
C ANGLE6  PROGRAM FOR COMPUTING APPROXIMATE STRESS FIELDS IN ANGLE-PLY
C COMPOSITE LAMINATES OF THREE PLIES.  ALSO CALCULATES COMPLIANCES.
C
    1 READ (2,2) KASE,A,T,B,THETA
    2 FORMAT (I4,4F5.2)
      IF (KASE) 100,100,5
    5 READ (2,6) S110,S120,S130,S220,S230,S330,S440,S550,S660
    6 FORMAT (5E10.2/4E10.2)
      THETA = THETA/57.3
      STH = SIN(THETA)
      CTH = COS(THETA)
      S11 = S110*CTH**4+(2.*S120+S660)*STH**2*CTH**2+S220*STH**4
      S22 = S110*STH**4+(2.*S120+S660)*STH**2*CTH**2+S220*CTH**4
      S12 = (S110+S220-S660)*STH**2*CTH**2+S120*(STH**4+CTH**4)
      S66 = 4.*(S110+S220-2.*S120)*STH**2*CTH**2+S660*(CTH**4+STH**4-2.*
     1STH**2*CTH**2)
      S16 = 2.*S110*CTH**3*STH-2.*S220*STH**3*CTH+(2.*S120+S660)*(STH**3
     1*CTH-CTH**3*STH)
      S26 = 2.*S110*STH**3*CTH-2.*S220*CTH**3*STH+(2.*S120+S660)*(CTH**3
     1*STH-STH**3*CTH)
      S33 = S330
      S13 = S130*CTH**2+S230*STH**2
      S36 = 2.*(S130-S230)*STH*CTH
      S23 = S130*STH**2+S230*CTH**2
      S44 = S550*STH**2+S440*CTH**2
      S55 = S550*CTH**2+S440*STH**2
      S45 = (S550-S440)*STH*CTH
C
C COMPLIANCES HAVE NOW BEEN DETERMINED IN TERMS OF UNIAXIAL PROPERTIES.
C PROCEED TO CALCULATE LAMINATION THEORY RESULTS.
C
   10 S6P = 1./((S16/S11)*((B-1.)/2.+(1.-B)/(1.+B))-((1.+B)/2.*S66/S16))
      S1P = -S6P*((1.+B)/2.*S66/S16+(1.-B)/2.*S16/S11)
      S6M = -B*S6P
      S1M = -B*S1P+1.+B
      EPS1 = S11*S1P+S16*S6P
      EPS6 = S16*S1P+S66*S6P
      EPS2 = S12*S1P+S26*S6P
C
C NOW CALCULATE BOUNDARY LAYER THICKNESS FROM MINIMUM COMPLEMENTARY
C ENERGY.
C
   15 F1 = B**2+(1.+B)**3-3.*(1.+B)**2+3.*(1.+B)-1.
      F2 = B**2+B
      SF = S16**2/S11-S66
      RK = SQRT(.0833*S55*F1*S6P/(.75*F2*SF*S6P-2.*S16*B*EPS1/S11))
C
C PRINT RESULTS SO FAR AND PREPARE FOR STRESS CALCULATIONS.
C
      THETA = THETA*57.3
   20 WRITE (3,21) KASE,A,T,B,THETA
   21 FORMAT (1H1,5HCASE ,I4/1H ,4HA = ,F5.2,5X,4HT = ,F5.2,5X,4HB = ,F5
     1.2,5X,8HTHETA = ,F5.2,8H DEGREES//)
      WRITE (3,25) S11,S12,S13,S22,S23,S33,S16,S26,S36,S44,S45,S55,S66
   25 FORMAT (1H ,36HCOMPLIANCE CONSTANTS OF UPPER LAMINA/1H ,3X,3HS11,8
     1X,3HS12,8X,3HS13,8X,3HS22,8X,3HS23,8X,3HS33,8X,3HS16/1H ,7(E10.3,1
     2X)/1H ,3X,3HS26,8X,3HS36,8X,3HS44,8X,3HS45,8X,3HS55,8X,3HS66/1H ,6
     3(E10.3,1X)//)
      WRITE (3,27) S1P,S6P,S1M,S6M,EPS1,EPS2,EPS6
   27 FORMAT (1H ,25HLAMINATION THEORY RESULTS/1H ,3X,3HS1P,8X,3HS6P,8X,
     13HS1M,8X,3HS6M,8X,4HEPS1,7X,4HEPS2,7X,4HEPS6/1H ,1X,3(F6.3,5X),F6.
     23,4X,3(E10.3,1X)//)
      WRITE (3,30) RK
   30 FORMAT (1H ,31HASSUMED-STRESS SOLUTION6  RK = ,F5.3,2H T/1H ,3X,2H
     1Y2,9X,3HS1U,8X,3HS6U,8X,3HS1C,8X,3HS6C,8X,3HS5I/)
C
C STRESS CALCULATIONS BEGIN.
C
      DELT = T/5.
      Y2 = 0.
   35 EF = EXP((Y2-A)/(RK*T))
      S6U = S6P*(1.-EF)
      S1U = EPS1/S11-S6U*S16/S11
      S6C = -B*S6U
      S1C = EPS1/S11+S6C*S16/S11
      S5I = -B*S6P*EF/RK
      WRITE (3,40) Y2,S1U,S6U,S1C,S6C,S5I
   40 FORMAT (1H ,1X,F5.2,6X,5(F6.3,5X))
      Y2 = Y2+DELT
      IF (Y2-A) 35,35,1
  100 CALL EXIT
      END
```

APPENDIX B

FORTRAN VARIABLE GLOSSARY FOR ANGLE

FORTRAN Variable	Meaning
KASE	case number (input)
A	a (input)
T	t (input)
B	b (input, must be 1.0)
THETA	θ (input)
S110	$S_{11}^{(0)}$ (input)
S120	$S_{12}^{(0)}$ (input)
S130	$S_{13}^{(0)}$ (input)
S220	$S_{22}^{(0)}$ (input)
S230	$S_{23}^{(0)}$ (input)
S330	$S_{33}^{(0)}$ (input)
S440	$S_{44}^{(0)}$ (input)
S550	$S_{55}^{(0)}$ (input)
S660	$S_{66}^{(0)}$ (input)
STH	$\sin \theta$
CTH	$\cos \theta$
S11	S_{11}
S22	S_{22}

FORTRAN Variable	Meaning
S12	S_{12}
S66	S_{66}
S16	S_{16}
S26	S_{26}
S33	S_{33}
S13	S_{13}
S36	S_{36}
S23	S_{23}
S44	S_{44}
S55	S_{55}
S45	S_{45}
S6P	$\sigma_6^{(+)}/\bar{\sigma}_1$
S1P	$\sigma_1^{(+)}/\bar{\sigma}_1$
S6M	$\sigma_6^{(-)}/\bar{\sigma}_1$
S1M	$\sigma_1^{(-)}/\bar{\sigma}_1$
EPS1	$\epsilon_1/\bar{\sigma}_1$
EPS6	$\epsilon_6/\bar{\sigma}_1$
EPS2	$\epsilon_2/\bar{\sigma}_1$
F1	assembly function
F2	assembly function
SF	compliance function
RK	k
DELT	y_2 increment
Y2	y_2
EF	exponential function
S6U	$\sigma_6^{(1)}/\bar{\sigma}_1$

FORTRAN Variable	Meaning
S1U	$\sigma_1^{(1)}/\bar{\sigma}_1$
S6C	$\sigma_6^{(2)}/\bar{\sigma}_1$
S1C	$\sigma_1^{(2)}/\bar{\sigma}_1$
S5I	$\left[\sigma_5/\bar{\sigma}_1\right]_{y_3=t}$

APPENDIX C

DIGITAL COMPUTER PROGRAM CROSS

```
C
C CROSS& PROGRAM FOR COMPUTING APPROXIMATE STRESS FIELDS IN CROSS-PLY
C COMPOSITE LAMINATES OF THREE PLIES.
C
    1 READ (2,2) KASE,A,T,B
    2 FORMAT (I4,3F5.2)
      IF (KASE) 100,100,5
    5 READ (2,6) S11,S12,S13,S22,S23,S33,S44,S55
    6 FORMAT (8E10.2)
C
C FIRST CALCULATE LAMINATION THEORY RESULTS.
C
   10 S20 = (S11-S22)/((S22+B*S11)*(S11+B*S22)/((1.+B)*S12)-(1.+B)*S12)
      S10 = 1.-S20*(S22+B*S11)/((1.+B)*S12)
      S290 = -B*S20
      S190 = 1.+B-B*S10
      EPS1 = S10*S11+S20*S12
      EPS2 = S10*S12+S20*S22
C
C NOW CALCULATE BOUNDARY LAYER THICKNESS FROM MINIMUM COMPLEMENTARY
C ENERGY.
C
   15 F1 = .349*B**4+.465*B**3+.186*B**2
      F2 = .60*B**3+.40*B**2
      F3 = (1.+B)**5-5.*(1.+B)**4+10.*(1.+B)**3-10.*(1.+B)**2+5.*(1.+B)-
     11.
      F4 = (1.+B)**3-3.*(1.+B)**2+3.*(1.+B)-1.
      SF1 = S12/S22-S12/S11
      SF2 = S11-S12**2/S22
      SF3 = S22-S12**2/S11
      SF4 = S13-S12*S23/S22
      SF5 = S12*S13/S11-S23
      SF6 = S33-S23**2/S22
      SF7 = S33-S13**2/S11
      C1 = 2.*B*EPS1*SF1-1.375*S20*(B**2*SF2+B*SF3)
      C2 = S20*(.2083*F2*SF4+.0417*(B**2*S55+F4*S44+B*F4*SF5))
      C3 = S20*(.2688*F1*SF6+.0187*F3*SF7)
      RK = SQRT((C2+SQRT(C2**2+4.*C1*C3))/(2.*C1))
C
C PRINT RESULTS SO FAR AND PREPARE FOR STRESS CALCULATIONS.
C
   20 WRITE (3,21) KASE,A,T,B
   21 FORMAT (1H1,5HCASE ,I4/1H ,4HA = ,F5.2,5X,4HT = ,F5.2,5X,4HB = ,F5
     1.2//)
      WRITE (3,25) S10,S20,S190,S290,EPS1,EPS2
   25 FORMAT (1H ,25HLAMINATION THEORY RESULTS/1H ,3X,3HS10,8X,3HS20,8X,
     14HS190,7X,4HS290,7X,4HEPS1,7X,4HEPS2/1H ,3(F7.4,4X),F7.4,3X,2(E10.
     23,1X)//)
      WRITE (3,30) RK
   30 FORMAT (1H ,31HASSUMED-STRESS SOLUTION& RK = ,F5.3,2H T/1H ,3X,2H
     1Y2,9X,3HS1U,8X,3HS2U,8X,3HS1C,8X,3HS2C,8X,3HS4I,8X,3HS3C/)
C
C STRESS CALCULATIONS BEGIN.
C
      DELT = T/5.
      Y2 = 0.
   35 EF = EXP((Y2-A)/(RK*T))
      LF = (A-Y2)/(RK*T)
      S2U = S20*(1.-(1.+LF)*EF)
      S3U = S20*(-(1.+.5*B)**2/2.+(1.+B)*(1.+.5*B)-(1.+B)**2/2.)*(LF-1.)
     1*EF/RK**2
      S1U = EPS1/S11-S12*S2U/S11-S13*S3U/S11
      S2C = -B*S2U
      S3C = -S20*B*(1.+B)/2.*(LF-1.)*EF/RK**2
      S1C = EPS1/S22-S12*S2C/S22-S23*S3C/S22
      S4I = -S20*B*LF*EF/RK
      WRITE (3,40) Y2,S1U,S2U,S1C,S2C,S4I,S3C
   40 FORMAT (1H ,1X,F5.2,5X,6(F7.4,4X))
      Y2 = Y2+DELT
      IF (Y2-A) 35,35,50
   50 WRITE (3,55)
   55 FORMAT (1H /1H ,18HY3-VARIATION OF S3/1H ,3X,2HY3,5X,3HY3V/)
      DELT = T/10.
      Y3 = 0.
   56 Y3V = B*(Y3**2/(2.*T**2)-(1.+B)/2.)
   57 WRITE (3,58) Y3,Y3V
   58 FORMAT (1H ,2X,F4.2,2X,F7.4)
      Y3 = Y3+DELT
      IF (Y3-T) 56,56,60
   60 Y3V = -Y3**2/(2.*T**2)+(1.+B)*Y3/T-(1.+B)**2/2.
      IF (Y3-(1.+B)*T) 57,57,1
  100 CALL EXIT
      END
```

APPENDIX D

FORTRAN VARIABLE GLOSSARY FOR CROSS

FORTRAN Variable	Meaning
KASE	case number (input)
A	a (input)
T	t (input)
B	b (input)
S11	S_{11} (input)
S12	S_{12} (input)
S13	S_{13} (input)
S22	S_{22} (input)
S23	S_{23} (input)
S33	S_{33} (input)
S44	S_{44} (input)
S55	S_{55} (input)
S20	$\sigma_2^{(0)}/\bar{\sigma}_1$
S10	$\sigma_1^{(0)}/\bar{\sigma}_1$
S290	$\sigma_2^{(90)}/\bar{\sigma}_1$
S190	$\sigma_1^{(90)}/\bar{\sigma}_1$
EPS1	$\epsilon_1/\bar{\sigma}_1$
EPS2	$\epsilon_2/\bar{\sigma}_1$

FORTRAN Variable	Meaning
F1	$\mathcal{F}_1$
F2	$\mathcal{F}_2$
F3	$\mathcal{F}_3$
F4	$\mathcal{F}_4$
SF1	$\mathcal{S}_1$
SF2	$\mathcal{S}_2$
SF3	$\mathcal{S}_3$
SF4	$\mathcal{S}_4$
SF5	$\mathcal{S}_5$
SF6	$\mathcal{S}_6$
SF7	$\mathcal{S}_7$
C1	C_1
C2	C_2
C3	C_3
RK	k
DELT (first use)	y_2 increment
y_2	y_2
EF	exponential function
LF	linear function
S2U	$\sigma_2^{(1)}/\bar{\sigma}_1$
S3U	$[\sigma_3^{(1)}/\bar{\sigma}_1]_{y_3=(1+b/2)t}$
S1U	$[\sigma_1^{(1)}/\bar{\sigma}_1]_{y_3=(1+b/2)t}$
S2C	$\sigma_2^{(2)}/\bar{\sigma}_1$
S3C	$[\sigma_3^{(2)}/\bar{\sigma}_1]_{y_3=0}$
S1C	$[\sigma_1^{(2)}/\bar{\sigma}_1]_{y_3=0}$
S4I	$[\sigma_4^{(2)}/\bar{\sigma}_1]_{y_3=t}$

FORTRAN Variable	Meaning
DELT (second use)	y_3 increment
Y3	y_3
Y3V	y_3 variation of σ_3

REFERENCES

1. Structural Design Guide for Advanced Composite Applications, Advanced Composites Division, Air Force Materials Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, August 1969.

2. Bisplinghoff, R.L., Mar, J.W., and Pian, T.H.H., Statics of Deformable Solids, Addison-Wesley, Reading, Massachusetts, 1965.

3. Pipes, R. Byron, "Effects of Interlaminar Shear Stress Upon Laminate Membrane Performance", Air Force Materials Laboratory/Industry Sponsored IRAD Status Report on Advanced Composite Materials, General Dynamics Corporation, Fort Worth, Texas, April 8-9, 1970.

4. Pian, T.H.H., "Derivation of Element Stiffness Matrices by Assumed Stress Distributions", AIAA Journal, Vol. 2, No. 7, July 1964, pp. 1333-1336.

ADDENDUM

The following two figures may be consulted in conjunction with the discussion on pages 72 and 73 of the text in order to gain a more complete picture of the results of comparing the second-order approximate solution to the cross-ply problem with a solution generated by the finite-element assumed-stress hybrid method.

It may be noted that the approximate solution is somewhat more conservative than the finite-element solution, both as regards edge-effect magnitudes and with respect to edge-effect penetration depth. This behavior on the part of the approximate solution is favorable from the standpoint of the designer, since the interlaminar stresses (though slight) can contribute to delamination failure of the composite, especially in environments where there exists periodic loading that reverses the direction of the shear and alternates interlaminar tension with compression in an oscillatory fashion.

FIGURE A1

CROSS-PLY CASE: STRESS VS. Y_2

COMPARISON OF APPROXIMATE AND FINITE-ELEMENT SOLUTIONS

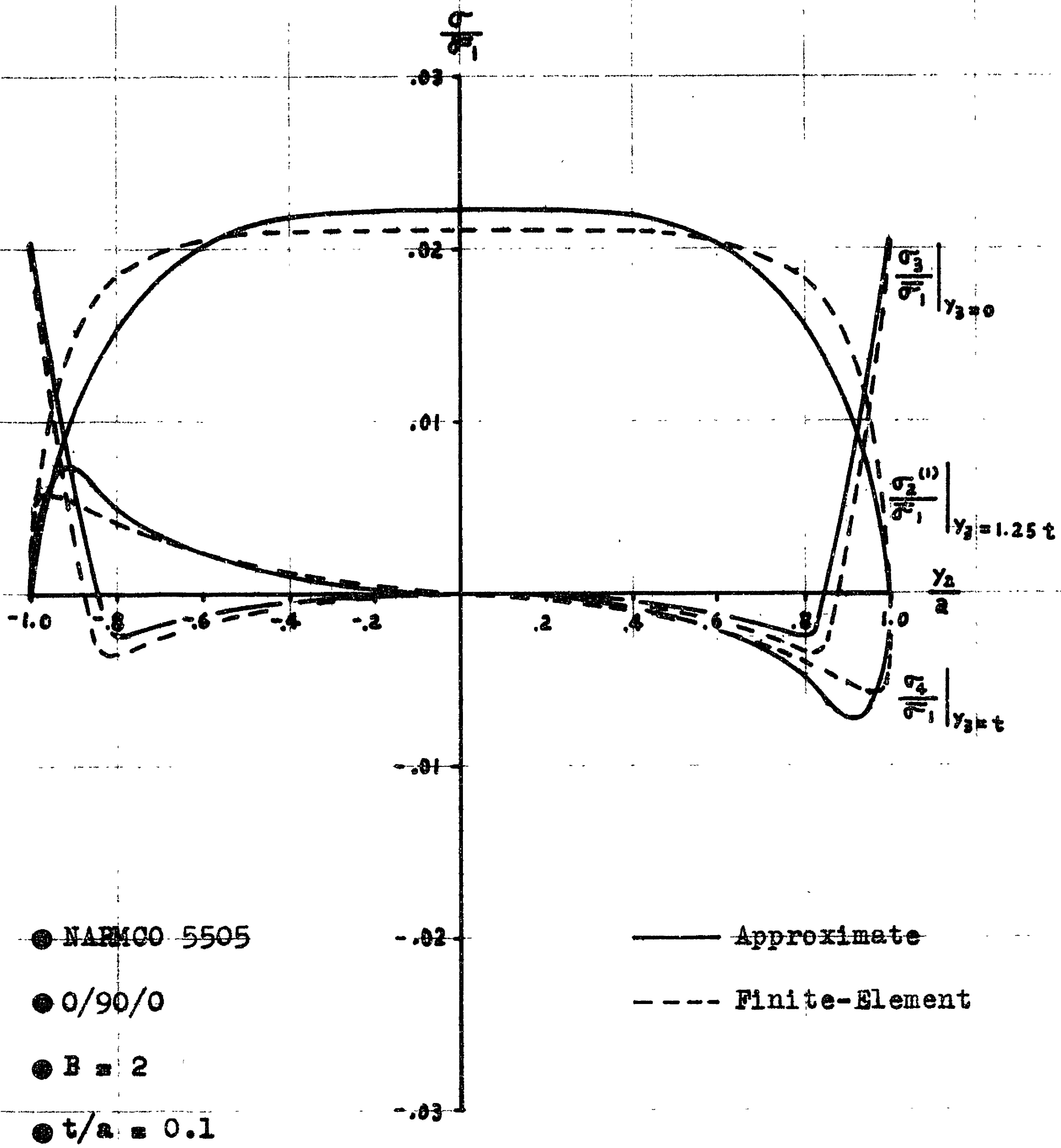
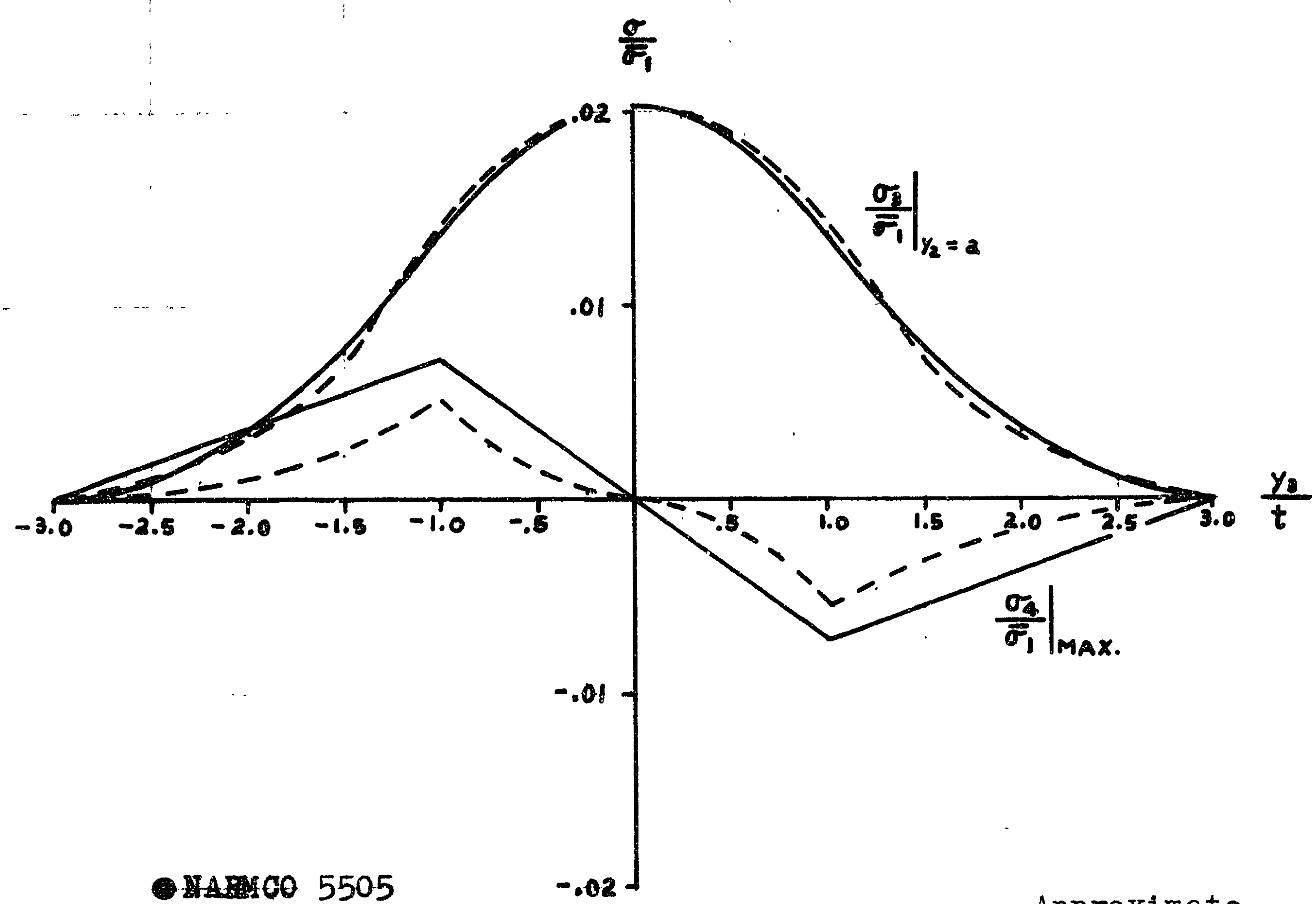


FIGURE A2

CROSS-PLY CASE: INTERLAMINAR STRESS VS. Y_3

COMPARISON OF APPROXIMATE AND FINITE-ELEMENT SOLUTIONS



- NARMCO 5505
- 0/90/0
- B = 2
- t/a = 0.1

——— Approximate

- - - - Finite-Element