CONDENSATION AND SQUARE IN A HIGHER CORE MODEL

by

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Abstract

A proof of global square in a core model with measures of order zero is given. The proof parallels Jensen's proof of global square in L : a hierarchy is chosen which allows a strong enough condensation property to establish global square. It is hoped that this method of proving square can be applied to core models with more general extender sequences.

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Introduction

Global square refers to the following principle discovered by Jensen:

There exists $\langle C_\nu : \nu$ is a singular limit ordinal$\rangle$ such that

a) C_ν is club in ν

b) otp $C_\nu < \nu$

c) $\bar{\nu} \in \lim C_\nu \longrightarrow C_{\bar{\nu}} = \bar{\nu} \cap C_\nu$

Global square is a deep combinatorial principle. It is used to prove the existence of κ^+-Souslin trees, and to establish many two-cardinal theorems. Jensen's elegant proof of square in L in [B], is based on a condensation property of the fine structure hierarchy for L. This thesis adapts a simplified version of Jensen's argument, due to Sy Friedman, to the core model context.

Mitchell has described a higher core model K which allows arbitrary sequences of measures. My thesis closely studies a special case of Mitchell's K in which all measures have order zero. In this thesis, first a special hierarchy for K called the "Delayed K-Hierarchy" is defined. Second, a condensation property for the Delayed K-Hierarchy is shown. Finally, global square is established in the Delayed K-Hierarchy using the condensation property in a proof that parallels Jensen's proof of global square in L.

With L in mind as a paradigm, I believe that there is a general approach to establishing combinatorial principles in core models, based on an appropriate condensation property. I believe that this condensation property appears when the right hierarchy for the core model is chosen. This thesis establishes global square for the special case of Mitchell's core model where all measures have order zero in a way that should generalize to core models with more general extender sequences.

The Delayed K-Hierarchy

In Jensen's definition of a premouse N, a measure E is added at $(\kappa^+)^N$. In this thesis, the definition of premouse is changed so that the measure is added later, at $(\kappa^{++})^{\mathrm{Ult}(N,E)}$. This delay leaves intact the related definitions of mice, core mice, strong mice, weasels and universal weasels; the only difference being that the new definition of premouse is substituted for the old. However, the new definition does change the hierarchy of universal weasels to make possible a condensation property from which a proof of square follows much as in L.

Here is the new definition of premouse:

Let $M = \langle J^F_\alpha, F_\alpha\rangle$. M is a premouse iff

A) $F = \{\langle\langle ab\rangle, \nu\rangle : \langle ab\rangle \in F_\nu\}$ and if $M|\nu =_{df} \langle J^F_\nu, F_\nu\rangle$ then for all $\nu \leq \alpha$, $M|\nu$ is acceptable, and for all $\nu < \alpha$, $M|\nu$ is sound. Let $M\|\nu =_{df} J^F_\nu$

B) If $\nu \leq \alpha$ and $F_\nu \neq \emptyset$, then there exist β and κ in $M|\nu$ such that

1) $M|\nu \models (\beta = \kappa^+$ and $\beta =$ greatest cardinal). (From now on, let g.c. stand for greatest cardinal.)

2) $F_\nu = \cup\{g :$ for some $z < \beta, g$ is a function with domain z such that for all $\alpha < z, g(\alpha) = \langle f_\alpha y_\alpha\rangle$ where f_α is the α^{th} function from κ to κ in $M|\nu$ such that for $\delta < \kappa, f_\alpha(\delta) < \delta^{++}.\}$. ($y_\alpha$ is defined below.)

3) Let $E_\nu = \{X :$ for some $\alpha < \beta, f_\alpha$ is the characteristic function of X and $y_\alpha = 1\}$. Then E_ν is a normal measure on κ in $M|\nu$.

4) Let $\pi : M|\nu \underset{E^*_\nu}{\longrightarrow} N$. (Here we are using Jensen's notation where $\pi : M \underset{E^*}{\longrightarrow} N$ means that the ultrapower is fine-structure preserving, or Σ^*). Then for all $\alpha < \beta, y_\alpha = \pi(f_\alpha)(\kappa)$

5) Let $\pi : M|\nu \xrightarrow{E^*_\nu} N$. Then

$$F^N \restriction \nu = F^{M|\nu} \restriction \nu, F^N_\nu = \emptyset, \text{ and } N \models (\nu = \kappa^{++}).$$

C) The order zero hypothesis:

Suppose that $F_\nu \neq \emptyset$ and $M|\nu \models (\beta = \kappa^+ \text{ and } \beta = gc)$.

Suppose that $\tau > \nu$ but that $J^F_\tau \models (\beta = \kappa^+)$. Then $F_\tau = \emptyset$.

D) The Initial Segment Condition:

Suppose that $F_\nu \neq \emptyset$ and $M|\nu \models (\beta = \kappa^+, \beta = gc)$

Let $W_\nu = \{f_\alpha\}_{\alpha<\beta}, U_\nu = \{y_\alpha\}_{\alpha<\beta}$.

Let $\kappa < \tau < \beta$ and suppose there exists $\beta' < \tau$ such that $J^F_\tau \models (\beta' = \kappa^+, \beta' = gc)$.

Let $\bar{W}_\nu = \{f_\alpha \in W_\nu : f_\alpha \in J^F_\tau\} = \{f_a\}_{\alpha<\beta'}$

Let $\bar{E}_\nu = \{X \in J^F_\tau : X \in E_\nu\} = \{X \in J^F_\tau : \exists f_\alpha \in \bar{W}_\nu, f_\alpha$ is the characteristic function of X and $y_\alpha = 1\}$.

Suppose that $\bar{E}_\nu$ is a normal measure on κ in J^F_τ. Let $\pi : M|\beta' \underset{\bar{E}^*_\nu}{\longrightarrow} N$ and suppose that $N \models (\tau = \kappa^{++})$

Let $\bar{U}_\nu = \{\pi(f_\alpha)(\kappa)\}_{\alpha<\beta'} =_{df} \{y'_\alpha\}\alpha < \beta'$

Let $\bar{F}_\tau = \{g \in J^F_\tau :$ for some $z < \beta'$, $\operatorname{dom} g = z$ and for all $\alpha < z, g(\alpha) = \langle f_\alpha y'_\alpha \rangle\}$

Suppose that $\langle J^F_\tau \bar{F}_\tau \rangle$ satisfies A and B. Then $\bar{F}_\tau = F_\tau$. ⊣

The definitions of mice, weasels, and their properties follow from the definition of premouse just as in the nondelayed hierarchy:

Let M be a premouse and suppose that $F^M_\nu = \emptyset$ codes a measure on κ where $M|\nu \models (\beta = \kappa^+)$. If $M \models (\neg\text{card } \beta)$ then we say that F^M_ν and E^M_ν die in M. If $M \models (\text{card } \beta)$ then F^M_ν and E^M_ν live in M. Because a premouse can code measures that die as well as live, there are two ways of iterating a premouse: simply and nonsimply.

Suppose $F_\nu = \emptyset$. Let $M|\alpha$ be the largest initial segment of M in which E_ν lives. The ultrapower of M by E_ν is the ultrapower of $M|\alpha$ by $E_\nu, \pi : M|\alpha \underset{E_\nu^*}{\longrightarrow} N$. If $M = M|\alpha$, the iteration is simple. If $M|\alpha \neq M$, the iteration is nonsimple and we say M must be "cutoff" before it can be iterated by E_ν.

Let $\bar{M} = \langle M_i \rangle_{i<\theta}$ be an iteration of $M = M_0$ of length θ. Let E_{ν_i} be a measure on κ_i. Let $\pi_{ii+1} : M_i|\alpha \underset{E_\nu^*}{\longrightarrow} M_{i+1}$ where $M_i|\alpha_i$ is the largest initial segment of M_i for which E_{ν_i} lives. Then $\langle \kappa_i \nu_i \rangle_{i<\theta}$ are the indices of the iteration. $\bar{M}$ is simple iff $\alpha_i = ht\ M_i$ for all i. $\bar{M}$ is nonsimple otherwise.

A premouse M is a mouse iff all iterations $\langle M_i \rangle_{i<\theta}$ of M can be continued, i.e. iff all iterates of M are wellfounded. If M is a mouse and $\langle M_i \rangle_{i\leq\theta}$ is a nonsimple iteration of M, then there are only a finite number of $i < \theta$ for which the iteration $\pi_{ii+1} : M_i \longrightarrow N_i$ is nonsimple. $\langle M_i \rangle_{i<\theta}$ is an iteration above τ iff $\kappa_i \geq \tau$ for all i. If the iteration $\pi : M \longrightarrow N$ is simple, it is Σ^* (fine structure preserving). If $\sigma : M \longrightarrow N$ is Σ^* and N is a mouse, M is a mouse. Given a mouse M, the core of M, or core(M), is the unique sound mouse that iterates simply above the ω^{th} projection of core(M) to M. If $\bar{M} =$ core(M), then $\omega\rho^\omega_{\bar{M}} = \omega\rho^\omega_M$ and $\omega\rho^n_{\bar{M}} \leq \omega\rho^n_M$ for all n. In fact, if $\tau : \bar{M} \longrightarrow M$ is the iteration, then $\cup\tau''\omega\rho^n_{\bar{M}} \subseteq \omega\rho^n_M$ for all n and τ sends the standard parameter of $\bar{M}$ to the standard parameter of M. If $\varphi : M \longrightarrow N$ is simple and above $\omega\rho^\omega_M$, then N is not sound. If $\langle M_i \rangle_{i<\theta}$ is a nonsimple iteration of M, let M_j be the place of last cutoff, $M'_j = M_j|\alpha_j$. Then $\pi_{j\theta} : M'_j \longrightarrow M_\theta$ is simple and above $\omega\rho^\omega_{M'_j}$, M_θ is not sound, and $M'_j =$ core(M_θ). To compare structures $M^1 = \langle J^A_z\ A_z \rangle$ and $M^2 = \langle J^B_r\ B_r \rangle$ (not necessarily premice), we will define a comparison iteration $\langle M^1_j \rangle_{j<\theta}\ \langle M^2_j \rangle_{j<\theta}$: Let $M^i_0 = M^i$ for $i = 1, 2$.

Suppose $\langle M_i^1 \rangle_{i \le z}$ and $\langle M_i^2 \rangle_{i \le z}$ have been defined. Let v_z be the place of first difference between M_z^1 and M_z^2. If either $A_{v_z}^z$ or $B_{v_z}^z$ is nonzero and does not code a measure, the iteration terminates. Otherwise, if $F_{v_z}^1 = \emptyset$, let $\pi_{zz+1}^1 : M_z^1 \longrightarrow M_z^2$ be the trivial iteration so that $M_z^1 = M_{z+1}^1$. If $F_{v_z}^1 \neq \emptyset$, let $M_z^1|\alpha_z$ be the largest initial segment of M_z^1 such that $E_{v_z}^1$ is live. Let $\pi_{zz+1}^1 : M_z^1|_{\alpha_z} \underset{E_{v_z}^{1*}}{\longrightarrow} M_{z+1}^1$. Similarly for M_z^2. If λ is a limit ordinal, let M_λ^i be the direct limit of $\langle M_{j\kappa}^i \pi_{j\kappa}^i \rangle_{j,\kappa<\lambda}$ for $i = 1, 2$. The comparison iteration will have indices $\langle \kappa_i v_i \rangle$ where $F_{v_i}^1$ and/or $F_{v_i}^2$ codes a measure on κ_i. A comparison iteration is always normal. If either M^1 or M^2 is a premouse and not a mouse, the comparison iteration may terminate at stage j because M_j^1 or M_j^2 may not be well founded. M^1 and M^2 are said to be coiterable iff the comparison iteration does not terminate because of a failure of wellfoundedness or because some predicate does not code a measure. Two mice are always coiterable, and their coiteration always terminates at a stage $< \infty$. Let $\langle M^1 \rangle_{i \le \theta} \langle M_i^2 \rangle_{i \le \theta}$ be a comparison iteration that terminates at stage $\theta \le \infty$. If one side of the iteration is nonsimple, the other side must be simple. Also, $M_\theta^1 \underset{+}{\subset} M_\theta^2, M_\theta^2 \underset{+}{\subset} M_\theta^1$, or $M_\theta^2 = M_\theta^1$. If the M^1-side of the iteration is nonsimple, then $M_\theta^2 \subseteq M_\theta^1$. (Here, $Q \subseteq R$ ($Q \underset{+}{\subset} R$) is shorthand for Q is an initial (proper initial) segment of R.) Recall that if the M^1-side of the iteration is nonsimple, then M_θ^1 is not sound and M_θ^1 cannot be a proper initial segment of M_θ^2. W is a weasel iff $W = J_\infty^F$ and for all α, $W|\alpha$ is a mouse. Any two weasels are coiterable, and so are a mouse and a weasel. If Q is a structure and N a mouse or weasel, say that in the comparison of N and Q, Q is ***passive below*** α iff $ht\ Q > \alpha$ and there is a normal iteration $N \longrightarrow N_i$ such that $N_i|\alpha = Q|\alpha$, and if z is the place of first difference between N_i and Q, then $F_z^Q = \emptyset$ or codes a measure on $\kappa \ge \alpha$. W is a universal weasel iff the comparison of

W and any coiterable premouse terminates. A mouse N is strong iff whenever M is a premouse such that $M|\alpha = N$ and M is iterable above α, then M is iterable and $N = \text{core}(M)|\alpha$. It is also true that N is strong iff N can be extended to a universal weasel, that is, there exists a universal weasel W such that $N = W|\alpha$ for some α. The core model K is the minimal universal weasel in the sense that any other universal weasel is an iterate of it by a simple iteration of length $\leq \infty$.

K is defined by:

$$K|\emptyset = \langle \emptyset\emptyset \rangle$$

$$K|\nu + 1 = \langle J^F_{\nu+1}, \emptyset \rangle$$

If λ is a limit ordinal, then

$$K|\lambda = \begin{cases} \langle J^F_\lambda G \rangle & \text{if } G \text{ codes a measure such that} \langle J^F_\lambda G \rangle \text{ is strong} \\ \langle J^F_\lambda \emptyset \rangle & \text{if no such } G \text{ exists .} \end{cases}$$

Then K is uniquely defined.

In this thesis, a condensation property for any universal weasel is shown that will be used to prove global square in any universal weasel including K.

Friedman Witnesses

If Q is a structure of the form $\langle J^F_\alpha, F_\alpha, A \rangle$ and $\sigma \leq \alpha$, then if A is empty, $Q|\sigma =_{df} \langle J^F_\sigma F_\sigma \rangle$. Otherwise, $Q||\sigma =_{df} J^F_\sigma$ and $Q \restriction \sigma =_{df} \langle J^F_\sigma, F_\alpha \restriction \sigma, A \restriction \sigma \rangle$, the *truncation* at σ.

If M is a premouse and $F^M_\nu \neq \emptyset$, then ν is active. Otherwise ν is inactive.

Let $\bar{h}^N_1(S)$ be the transitive collapse of $h^N_1(S)$, the Σ_1-hull of S in N. If $x \in h^N_1(S)$, let x collapse to $\bar{x}$ in $\bar{h}^N_1(S)$. If $x \in \bar{h}^N_1(S)$, let $\hat{x}$ in $h^N_1(S)$ collapse to x. If $M = \bar{h}^N_1(S)$, let $\hat{M} = h^N_1(S)$.

In general, if M is a mouse, $\bar{M}$ will stand for core(M).

If N is a structure, let γ_N be $\omega\rho_N^\omega$, γ_N^n be $\omega\rho_N^n$, and let p_N be the standard parameter of N.

Let $H = \bar{h}_1^N(\alpha \cup \langle p \rangle)$. p is *witnessed* in $h_1^N(\alpha \cup \langle p \rangle)$ iff p collapses to $\bar{p}$ in H, where $\bar{p}$ is the standard parameter of H above α. Let $N = (N^*)^{nq} \restriction \lambda$ and $H = h_1^N(\alpha \cup \langle p \rangle)$. Say that p is fully witnessed in $h_1^N(\alpha \cup \langle p \rangle)$ with respect to q iff

1) p is witnessed in $h_1^N(\alpha \cup \langle p \rangle)$

2) When $n > 0$, if $\sigma : H \xrightarrow{\Sigma_0} (N^*)^{nq}$ is the inverse of the collapse with $\sigma^* : H^* \longrightarrow N^*$ its canonical expansion, and $H = (H^*)^{n\bar{q}}$ and $\sigma^*(\bar{q}) = q$, then $\langle \bar{p}\bar{q} \rangle$ is the standard parameter of H^* above α

(So if $n = 0$ we can assume $q = \emptyset$. In this case, "p fully witnessed in $h_1^N(\alpha \cup \langle p \rangle)$ with respect to q" will mean p is witnessed in $h_1^N(\alpha \cup \langle p \rangle)$.)

Let M be such that $\gamma_M = \gamma_M^n$. Let $p_M = \langle p_1 \ldots p_n \rangle$, where p_{i+1} is the standard parameter of $(M)^{i,p\restriction i}$. Let $N = \bar{h}_1^{(M)^{n-1,p\restriction n-1}}(\gamma_M \cup \langle p_n \rangle)$. Let $\bar{\sigma} : N \longrightarrow (M)^{n-1,p\restriction n-1}$ be the inverse of the collapse with $\sigma : \bar{M} \longrightarrow M$ its expansion. Then $\bar{M} = \text{core}(M)$. (This defines core(M) even when M is not a mouse)

Let FW stand for Friedman witness. Let $p_N = \langle p_1 \ldots p_n \rangle$ be the standard parameter for N. X is the i^{th} Friedman witness for p in N iff $X = \bar{h}_1^N(p_i \cup \langle p_1 \ldots p_{i-1} \rangle)$. If N is a mouse, then all the FW's for p_N are members of N.

Let $X \in h_1^N(\alpha \cup \langle p \rangle)$. Say that X *witnesses* p in $h_1^N(\alpha \cup \langle p \rangle)$ iff $\bar{X}$ in $\bar{h}_1^N(\alpha \cup \langle p \rangle)$ is the set of Friedman witnesses for $\bar{p}$.

If Q is a structure, say that Q is the zeroth or trivial reduct of Q.

Supose $\sigma : H \longrightarrow M$ is Σ_1 where M is a premouse (mouse), or there exists a premouse (mouse) N such that $ht\ N = \lambda$, $F_\lambda^N \neq \emptyset$, F_λ^N codes a measure on some κ with $N \models (\beta = \kappa^+), M \models \beta = gc$ and $M = N \restriction \tau$. In the latter case also suppose that $F_\lambda^N \cap \tau$ measures all the subsets of κ in $\operatorname{ran}\sigma$ and for all $\alpha \in \operatorname{ran}\sigma$ there exists an f such that $\langle f\alpha\rangle \in F_\lambda^N \cap \operatorname{ran}\sigma$. If all these conditions exist, say that N makes M a *σ-premouse (mouse).*

If $\sigma : H \longrightarrow M$ and $\sigma \restriction \delta = id \restriction \delta$, and $\sigma(\delta) \not= \delta$, then say that δ is the critical point of σ, or $\delta = \operatorname{crit}(\sigma)$.

Condensation in a Universal Weasel

Let W be a universal weasel. It would be nice if whenever N is an initial segment of W, $\bar{h}_1^N(\alpha \cup \langle p \rangle)$ is also an initial segment of W. Unfortunately, this is not the case, but by putting certain conditions on p, we can get very close to W. With these conditions on p, $\bar{h}_1^N(\alpha \cup \langle p \rangle)$ will be at most one iterate (ultrapower) away from an initial segment of W. If $N = M^{nq}$ is a reduct of an initial segment of W, again by putting conditions on p, we can get $\bar{h}_1^N(\alpha \cup \langle p \rangle)$ to be the reduct of H where H is either an initial segment of W or one iterate away from an initial segment of W. Actually, it turns out that N does not have to be a reduct of an initial segment of W (recall that any M is the trivial reduct of itself). We can get the same result as long as N is "strong enough". If in the comparison of N and W N is passive below α, then the above is still true: $\bar{h}_1^N(\alpha \cup \langle p \rangle)$ will be at most one iterate away from being the reduct of an initial segment of W. This means, for example, that if M is an iterate or a truncation of N, where N is the reduct of an initial segment of W, then if we are careful, $\bar{h}_1^N(\alpha \cup \langle p \rangle)$ will still collapse to a reduct of a mouse at most one iterate away from W.

We will now prove some useful facts that will be needed for condensation and square. Keep in mind that when looking at Σ_1-maps $\sigma : H \longrightarrow M$ or Σ_1-hulls $\bar{h}_1^N(\alpha \cup \langle p \rangle)$, we need to be flexible enough to allow for M to be an iterate, reduct, truncation, or some combination. Since a truncation of a premouse is not a premouse, M may not even be a premouse! (Of course we need to be careful how we truncate.)

I. Let $\sigma : H \longrightarrow M$ be Σ_1 with $\text{crit}(\sigma) = \delta$. Let M be a σ-premouse. Then

$H|\delta = M|\delta$ and in a comparison of H and M, all iterations on the H-side take place above δ.

Proof: If M is not a premouse, let N make M a σ-premouse with $ht\ N = \lambda, F^N_\lambda$ coding a measure on $\kappa_0, N \models (\beta_0 = \kappa_0^+ \wedge \beta_0 = gc)$, and $M = N \restriction \tau$ where $\tau > \beta_0$.

First we will prove that in a comparison of H and M, all iterations on the H-side take place above δ. Suppose not. Let z be the place of first difference between H and M. Because $H|\delta = M|\delta$, $z \geq \delta$. So the only step of the iteration that can take place above δ is the first step. So assume $F^H_z \neq \emptyset$ and codes a measure on $\kappa < \delta$. Let $H|z \models (\beta = \kappa^+ \wedge \beta = gc)$. Because $H\|z = M\|z$, $M|z \models (\beta = \kappa^+ \wedge \beta = gc)$ also. Since $\text{crit}(\sigma) = \delta$, δ is a cardinal in H, so either $\kappa < \beta < \delta = z$, or $\kappa < \beta < z$ and $\beta = \delta$. Suppose first that M is a premouse. If $z < ht\ H$, let $\nu = \sigma(z)$. Otherwise let $\nu = htM$. Then $E^M_\nu \cap z = E^H_z$: Let $x \in E^H_z$. Then for some $\alpha < \beta$, $H \models$ (f_α is the characteristic function of x and $y_\alpha = 1$). But then $M \models$ (f_α is the characteristic function of x and $y_\alpha = 1$). So $x \in E^M_\nu$. Similarly, $E^M_\nu \cap z \subseteq E^H_z$, basically because both $H|z$ and $M|z$ have the same subsets of κ. So $E^M_\nu \cap z = E^H_z$. Similarly, $W^M_\nu \cap z = W^H_z$. Now because $H\|z = M\|z$, E^H_z is a normal measure on κ in $M|z$. So if $\pi : M\|z \underset{E^*_\nu}{\longrightarrow} N'$, then $N' \models (z = \kappa^{++})$. Clearly $\langle J^{F^M}_z, F^H_z \rangle$ satisfies all the requirements of a premouse and so $F^H_z = F^M_z$. Contradiction. Suppose now that M is not a premouse. If $z < htH$ or if $\text{ran}\,\sigma \subseteq M|\beta_0$, then the above proof shows that $F^H_z = F^M_z$. No.

So suppose $z = htH$ and $\sup(\text{ran}\,\sigma) > \beta_0$. $\kappa < \beta \leq \delta$, so $\sigma(\kappa) = \kappa$ and $\sigma(\beta) = (\kappa^+)^M$. If $\beta < \delta$ then $\sigma(\beta) = \beta = (\kappa^+)^M$ and $\delta = z$. But since $\delta = htH$ and $\sup(\text{ran}\,\sigma) > \beta_0$, κ must be κ_0, β must be β_0 and $P(\kappa_0)^M$ must be a subset of $\text{ran}\,\sigma$. But then $F^N_\lambda \cap \tau = F^N_\lambda$ since $F^N_\lambda \cap \tau$ must measure all subsets of κ in $\text{ran}\,\sigma$

and $P(\kappa_0)^M = P(\kappa_0)^N$. But if $F_\lambda^N \cap . = F_\lambda^N$ then $M = N$ and M is a premouse. No.

So assume $\beta = \delta$. Then since $z = ht\ H$ and $\sup(\operatorname{ran}\sigma) > \beta_0$, $\kappa = \kappa_0$ and $\sigma(\beta) = \beta_0$. But now, κ_0, β, and z are less than β_0. Regard σ as a Σ_0-map from H into N. Then the proof in the case that M is a premouse shows that $E_\lambda^N \cap z = E_z^H, W_\lambda^N \cap z = W_z^H$, and $E_z^N = F_z^H$. Since $z < \beta_0 < \tau, F_z^N = F_z^M$. Contradiction. So in a comparison of H and M, all iterations on the H-side take place above δ.

Now suppose that $M|\delta \neq H|\delta$. This means that $F_\delta^M \neq \emptyset$ and codes a measure on $\kappa < \beta < \delta$ where $M|\delta \models (\kappa = \beta^+)$. Clearly $H \models (\kappa = \beta^+)$ also. Then for all $x < \delta$, $M \models \exists\delta' < \sigma(\delta)(\delta' > x, F_{\delta'} \neq \emptyset, F_{\delta'}$ codes a measure on κ). So $H \models \exists\delta' < \delta(\delta' > x, F_{\delta'} \neq \emptyset, F_{\delta'}$ codes a measure on κ). It follows that in H there are at least two different places, say δ_1 and δ_2, both less than δ and greater than β where $F_{\delta_1}^H$ and $F_{\delta_2}^H$ both code up a measure on κ. But $(\beta = \kappa^+)^{H|\delta}$, so no new subsets of κ are added to $H|\delta$ after β. Therefore this is impossible by the initial segment condition on premice. So $H|\delta = M|\delta$.

II. Let $\sigma : H \longrightarrow M$ be Σ_1, $\sigma \restriction \delta = id \restriction \delta, \rho_H^1 \leq \delta$. Let M be a σ-mouse. Let W be a universal weasel and let $M|\rho_H^1 = W|\rho_H^1$. Assume that in a comparison of M and W, M is passive below ρ_H^1. Then H is a mouse.

Proof: If σ is Σ_1 and M a σ-premouse, σ captures enough "mouseness" of M for H to be a premouse. So we must show H is iterable. Suppose H is iterable above ρ_H^1. Consider the comparison iteration of W and H. Let it terminate at stage j. By I, the iteration on the H-side is above ρ_H^1. By the universality of W, the iteration $\tau_H : H \longrightarrow H_j$ is simple and therefore Σ^*. If the iteration $\tau_W : W \longrightarrow W_j$ is simple,

$H_j \subsetneq W_j$. If τ_W is nonsimple, $H_j \subseteq W_j$. Either way, H_j is a mouse. Since τ_H is Σ^*, this means H is a mouse.

It remains to show H is iterable above ρ^1_H. If M is a mouse, H is iterable above ρ^1_H because σ is Σ^0_1. So we must show H is iterable above ρ^1_H when M is not a mouse. Let N make M a σ-premouse: let $\lambda = ht\ N, F^N_\lambda$ code a measure on κ, $N \models (\beta = \kappa^+)$ and $M = N \restriction \tau$. Let $\langle H_i \rangle_{i<\theta}$ be an iteration of H above ρ^1_H. We will show that it can be continued. First assume $\langle H_i \rangle_{i<\theta}$ is simple. Let $\bar{\pi}_{ij}$ be the iteration maps, $\langle \bar{\kappa}_i \bar{v}_i \rangle$ the indices. We will define an iteration $\langle N_i \rangle_{i<\theta}$ of N with maps π_{ij} and indices $\langle \kappa_i v_i \rangle$, together with maps $\sigma_i : H_i \longrightarrow N_i$, in this way: Regard σ as a Σ_0-map from H to N and let σ_0 be σ. If σ_i has been defined, let $\kappa_i = \sigma_i(\bar{\kappa}_i)$. If $v_i \in H_i$, let $v_i = \sigma_i(\bar{v}_i)$. Otherwise, let $v_i = ht\ N_i$. Let $\pi_{ii+1} :\ N_i \underset{E^*_\nu}{\longrightarrow} N_{i+1}$. Define σ_{i+1} by $\sigma_{i+1}(\bar{\pi}_{ii+1}(f)\ (\bar{\kappa}_0)) = \pi_{ii+1}(\sigma_i(f)(\kappa_i))$ for $f \in H_i \cap {}^{\kappa}H_i$. Note that because $\tau < \beta, \pi_{ij}$ is simple iff $\bar{\pi}_{ij}$ is simple. For η a limit ordinal, let N_η be the direct limit of $\langle N_i, \pi_{ij} \rangle_{i,j<\eta}$ and let $\sigma_\eta(\bar{\pi}_{i\eta}(z)) = \pi_{i\eta}(\sigma_i(z))$.

If $\theta = \eta$, a limit ordinal, let $\bar{H}$ be the direct limit of $\langle H_i \bar{\pi}_{ij} \rangle_{i,j<\theta}$. Then σ_θ : $\bar{H} \longrightarrow N_\theta$ defined by $\sigma_\theta(\bar{\pi}_{1\theta}(z)) = \pi_{1\theta}(\sigma_i(z))$, is order preserving. It follows that $\bar{H}$ must be well founded and the iteration can be continued. Suppose $\theta = z + 1$. Let $F^{H_z}_{\bar{v}_z}$ code a measure on $\bar{\kappa}_z$ and suppose $E_{\bar{v}_z}$ lives in H_z. Then $F^{N_z}_{v_z}$ codes a measure on κ_z and E_{v_z} lives in N_z. Let $\bar{j} : H_z \longrightarrow \mathrm{Ult}(H_z, E_{\bar{v}_z})$ and $j : N_z \longrightarrow \mathrm{Ult}(N_z, E_{vz})$. Let $\bar{\sigma}(\bar{j}(f)(\bar{\kappa}_z)) = j(\sigma_z(f)(\kappa_z))$, where $f \in H_z \cap {}^{\bar{\kappa}_z}H_z$. Then $\bar{\sigma}$ is order preserving and since $\mathrm{Ult}(N_z, E_{vz})$ is well founded, so is $\mathrm{Ult}(H_z, E_{\bar{v}z})$. It follows that $\langle H_i \rangle_{i\leq z}$ can be continued simply above ρ^1_H. Now let $F^{H_z}_{\bar{v}_z}$ code a measure on $\bar{\kappa}_z$ in H_z where $H_{\bar{v}_z} \models (\bar{\beta}_z = \bar{\kappa}^+_z)$. Suppose that $E_{\bar{v}z}$ dies at $\bar{\alpha}_z < htH_z$. That is, $H_z|\bar{\alpha}_z$ is the largest initial segment of H_z that believes $\bar{\beta}_z$ is a cardinal. Then $\bar{\alpha}_z$ must be less than $\bar{\pi}_{0z}(\bar{\kappa})$ (where $\sigma(\bar{\kappa}) = \kappa$). It follows that $F^{N_z}_{v_z}$ codes a measure on κ_z that dies at $\alpha_z = \sigma(\bar{\alpha}_z)$. But then $\sigma_z \restriction H_z|\bar{\alpha}_z : H_z|\bar{\alpha}_z \longrightarrow N_z|\alpha_z$ is Σ^*. Since $N_z|\alpha_z$ is a mouse, $H_z|\bar{\alpha}_z$ is a mouse. So $\langle H_i \rangle_{i \leq z}$ can be continued nonsimply above ρ^1_H. But then it follows that if $\langle H_i \rangle_{i<\theta}$ is nonsimple it also can be continued.

So H *is* iterable above ρ^1_H and so H is a mouse.

III. Let $H|\gamma_H = W|\gamma_H$ where W is a universal weasel. Compare H and W. Let the comparison terminate at stage j. Assume that H is not an initial segment of W, that H is sound, and that $H = H_j$ or that $H \longrightarrow H_j$ is simple and above γ_H. Then $W_j \neq H_j$.

Proof. We need this fact: let M and N be coiterable and let the iteration have length θ. Let $\langle M_i \rangle_{i \leq \theta}$ $\langle N_i \rangle_{i \leq \theta}$ be the comparison iteration. Then $M_j = N_k$ iff $j = \kappa = \theta$, or ($j < \kappa$, $M_j \longrightarrow M_\kappa$ is the identity, and $\kappa = \theta$), or ($\kappa < j$, $N_\kappa \longrightarrow N_j$ is the identity, and $j = \theta$).

So suppose $W_j = H_j$. Clearly the iteration $W \longrightarrow W_j$, is nonsimple.

Case 1: The iteration on the W-side is always below γ_H. But then because $H|\gamma_H = W|\gamma_H$, we can assume $j = 1$ and $W_0|\beta_0 \longrightarrow W_1$ via a measure on $\kappa < \gamma_H$ coded at some $z > \gamma_H$. (Here stage zero is the last and only cutoff stage.) Then $\gamma_{W_1} \leq \kappa < \gamma_H$ since $W_0|\beta_0 = \text{core}(W_1)$. But $H = \text{core}(H_j)$ and $\gamma_H = \gamma_{H_j}$. Since $\gamma_{H_1} \neq \gamma_{W_1}$, $H_1 \neq W_1$. No.

Case 2: We can assume that after some stage s, the iteration on the W-side is above γ_H. Let t be the place of last cutoff on the W-side. Then $W_t|\beta_t \longrightarrow W_j$ is simple and above γ_H. Note that $t < j$. Because $W_j = H_j$, $\gamma_{W_j} = \gamma_{H_j} = \gamma_H$. It follows that $W_t|\beta_t = \text{core}(W_j) =$

$\mathrm{core}(H_j) = H$. But this is impossible by the preliminary fact.

IV Let $\sigma : H \longrightarrow M$ be Σ_1. Let $\sigma \restriction \delta = id \restriction \delta$ and let $\gamma_H \leq \delta$. Assume that H is the reduct of H^*, a mouse, and that $M|\gamma_H = W|\gamma_H$ where W is a universal weasel. Assume further that in a comparison of M and W, M is passive below γ_H. Then $\bar{H}^* = \mathrm{core}(H^*) \underset{+}{\subset} W$.

Proof. Suppose not. Compare $\bar{H}^*$ and W. Assume the comparison ends at stage j.

Case 1: The iteration $\bar{H}^* \longrightarrow \bar{H}^*_j$ is nontrivial. By I it is clear that because in a comparison of M and W, M is passive below γ_H, the $\bar{H}^*$ side of the iteration takes place above γ_H. By the universality of W, $\bar{H}^* \longrightarrow \bar{H}^*_j$ is simple. Because $\gamma_H \leq \delta$, $\bar{H}^*|\gamma_H = H^*|\gamma_H = M|\gamma_H = W|\gamma_H$. It follows by III that $W_j \neq \bar{H}^*_j$. Because $\bar{H}^* \longrightarrow \bar{H}^*_j$ is simple and above γ_H, $\bar{H}^*_j$ cannot be sound. Therefore it cannot be the case that $W_j \underset{+}{\supset} \bar{H}^*$. Therefore $W_j \underset{+}{\subset} \bar{H}^*$. But then $W \longrightarrow W_j$ is nonsimple and $\bar{H}^*_j \subseteq W_j$. Contradiction.

Case 2: $\bar{H}^* = \bar{H}^*_j$. So we have $W \longrightarrow W_j \subseteq \bar{H}^*$. By III we know that $\bar{H}^* \underset{+}{\subset} W_j$. As before, the first possible difference between W and $\bar{H}^*$ occurs at some $z > \gamma_H$. So $W_j \models (z = \beta^+)$ for some $\beta \geq \gamma_H$ and $ht\ \bar{H}^* \geq z$. But $\bar{H}^* \underset{+}{\subset} W_j$ and $\bar{H}^*$ collapses to γ_H. Contradiction.

So $\bar{H}^* \underset{+}{\subset} W$.

Let W be a universal weasel

Condensation in W

Let $H = \bar{h}_1^M(\delta \cup \langle p \rangle)$ and let $\sigma : H \longrightarrow M$ be the inverse of the collapse with $\text{crit}(\sigma) = \delta$. Let $W|\delta = M|\delta$ and assume that in a comparison of M and W, M is passive below γ_H. Assume that $M = (M^*)^{nq} \restriction \lambda$ for some n, q, λ and mouse M^* and that p is fully witnessed in $h_1^M(\delta \cup \langle p \rangle)$ with respect to q. Assume that if $\sigma^* : H^* \longrightarrow M^*$ is the canonical expansion of σ, then H^* is a mouse. Then H^* is either an initial segment of W or a one-step iterate of an initial segment of W.

Proof: By IV, $\bar{H}^* \underset{+}{\subset} W$. Since p is fully witnessed in $h_1^M(\delta \cup \langle p \rangle)$ with respect to q, $\bar{H}$, (the core of H) is the reduct of $\bar{H}^*$, the core of H^*. It follows that $H = \bar{H} \longleftrightarrow H^* = \bar{H}^*$, and that ($H$ is a one-step iterate of $\bar{H}$) $\longleftrightarrow$ (H^* is a one-step iterate of $\bar{H}^*$). Suppose $H \neq \bar{H}$. If $\delta = \gamma_H$, $H = \bar{H}$ (again by p being witnessed). So assume $\gamma_H < \delta$. Let $\tau : \bar{H} \longrightarrow H$ be the iteration map (whose canonical expansion is the iteration of $\bar{H}^*$ to H^*). Since p is fully witnessed in $h_1^M(\delta \cup \langle p \rangle)$ with respect to q, $\bar{p}$ is equal to p_H above δ and $\langle d\bar{\bar{p}} \rangle$ is equal to $p_{\bar{H}}$ where $\tau(d) < \delta$, $\tau(\bar{\bar{p}}) = \bar{p}$ and $\sigma(\bar{p}) = p$. Let z be the place of first difference between $\bar{H}$ and H. Let $F_z^{\bar{H}}$ code a measure on κ. By I, $H|\delta = W|\delta$, by hypothesis $W|\delta = M|\delta$, by IV $\bar{H}^*|\delta = W|\delta$, and we know $\bar{H}^*|\delta = \bar{H}|\delta$. So $\bar{H}|\delta = H|\delta$ and $z > \delta$.

Case 1: Suppose $\kappa < \delta$. Then $\kappa^+ = \delta$ in both $\bar{H}$ and H. (Recall that $\delta = \text{crit}(\sigma)$ is a cardinal in H.)

$$\bar{H} = \bar{h}_1^H(\kappa \cup \langle \tau(d), \bar{p}, \rangle)$$

and

$$\bar{H}_1 = \bar{h}_1^H(\kappa + 1 \cup \langle \tau(d), \bar{p} \rangle)$$

where $\tau_{0_1} : \bar{H} \longrightarrow \bar{H}_1$ is the first step of the iteration. But then since $\kappa^+ = \delta$ in $\bar{H}_1$ also, $\bar{H}_1 = \bar{h}_1^H(\delta \cup \langle p \rangle) = H$. So H here is exactly a one-step iterate of $\bar{H}$. So H^* in this case is exactly one iterate away from $\bar{H}^* \underset{+}{\subset} W$.

Case 2: Suppose $\kappa \geq \delta$.

Then $\bar{H} = \bar{h}_1^H(\kappa \cup \langle p \rangle) = H$. It follows that $H^* = \bar{H}^*$.

It's worth noting that the above proof shows that the only time H^* can fail to be an initial segment of W is when $M \models (|\delta| = \kappa$ and $\neg(\kappa$ measurable$))$.

Here are a few more definitions: A structure Q singularizes ν or is a singularizing structure for ν iff either $\nu = ht\ Q$ or $Q \models (\nu \text{ is regular})$ and for some n there is a partial $\Sigma_n(Q)$ function from $\gamma < \nu$ cofinally into ν. In this case, say that Q singularizes ν in a Σ_n-way. Let H be the n^{th} $*$-reduct of Q iff $H = Q^{n,p\restriction n}$ where p is the standard parameter of Q.

Global Square refers to the following principle:

There exists $\langle C_\nu : \nu \text{ is a singular limit ordinal}\rangle$ such that

a) C_ν is closed and unbounded in ν

b) $\operatorname{otp} C_\nu < \nu$

c) $\bar{\nu} \in \lim C_\nu \Longrightarrow C_{\bar{\nu}} = C_\nu \cap \bar{\nu}$

Global Square in K will be proved by choosing an appropriate singularizing structure $\mathcal{A}(\nu)$ for each singular limit ordinal. Once the $\mathcal{A}(\nu)$ are chosen, the proof will parallel the proof of square in L.

Before embarking on the proof of square, let me state a property of Skolem hulls which will be used over and over again:

Let $\alpha \in h_1^Q(\lambda \cup \langle x\rangle)$. Let $\sigma = \sup h_1^Q(\lambda \cup \langle x\rangle)$ and $\nu = \sup h_1^Q(\lambda \cup \langle x\rangle) \cap \alpha$.

Then $\nu = \sup h_1^{Q\restriction\sigma}(\nu \cup \langle x\rangle) \cap \alpha$.

Proof: Let $y \in h_1^Q(\nu \cup \langle x\rangle \cap \alpha$. Then for some ν' and σ' members of $h_1^Q(\lambda \cup \langle x\rangle), y \in h_1^{Q\restriction\sigma'}(\nu' \cup \langle x\rangle)$. But $h_1^{Q\restriction\sigma'}(\nu' \cup \langle x\rangle)$ is itself a member of $h_1^Q(\lambda \cup \langle x\rangle)$. So $\sup h_1^{Q\restriction\sigma'}(\nu' \cup \langle x\rangle) \cap \alpha < \sup h_1^Q(\lambda \cup \langle x\rangle) \cap \alpha = \nu$, and $y < \nu$.

Let me also prove this fact about Friedman witnesses before going on to square: Let p be the standard parameter of M. Let X be the FW for p in M. Let $M_1 = \bar{h}_1^{M\restriction\delta}(\alpha \cup \langle p \rangle)$ and suppose that $X \in h_1^{M\restriction\delta}(\alpha \cup \langle p \rangle)$. Then p is witnessed in $h_1^{M\restriction\delta}(\alpha \cup \langle p \rangle)$.

Proof: Let $\sigma : M_1 \longrightarrow M \restriction \delta$ be the inverse of the collapse. If $x \in h_1^{M\restriction\delta}(\alpha \cup \langle p \rangle)$, let $\bar{x} = \sigma^{-1}(x)$. Assume wlog that $p = \langle p_0 p_1 \rangle$. It will be enough to show that if $X = \bar{h}_1^M(p_1 \cup \langle p_0 \rangle)$, then there exists a Y in $h_1^{M\restriction\delta}(\alpha \cup \langle p \rangle)$ such that Y collapses to $\bar{h}_1^{M_1}(\bar{p}_1 \cup \langle \bar{p}_0 \rangle)$ in $\bar{h}_1^{M\restriction\delta}(\alpha \cup \langle p \rangle)$. Let $\tau : X \longrightarrow M$ be the inverse of the collapse and let $\tau(x_0) = p_0$.

Let $\bar{z}_0 = \min\{\bar{z} \in \operatorname{ord} \bar{X} :\ M_1 \models \forall y \neg\phi(y\beta\bar{p}_0)$ and
$\bar{X} \models \phi(\bar{z}\beta\bar{x}_0)$ where
ϕ is some Σ_0-formula and β is a finite
sequence of parameters less than $p_1\}$

Fix ϕ and β be such that $M_1 \models\ \forall y \neg\phi(y\beta\bar{p}_0)$ and $\bar{X} \models \phi(\bar{z}_0\beta\bar{X}_0)$. Let $\hat{z}_0 = \tau(z_0)$.

Claim: $\hat{z}_0 \notin M \restriction \delta$

Proof of Claim: $\bar{X} \models \phi(\bar{z}_0\beta\bar{x}_0) \iff X \models \phi(z_0\sigma(\beta)X_0 \iff M \models \phi(\hat{z}_0\sigma(\beta)p_0)$. But by definition of $\bar{z}_0, M_1 \models \forall y \neg\phi(y\beta\bar{p}_0)$. So $M \restriction \delta \models \forall y \neg\phi(y\sigma(\beta)p_0)$. So z_0 is not a member of $M \restriction \delta$.

But now it is true that if γ is a string of parameters less than $\bar{p}_1$ and ψ is a Σ_0-formula, then $M_1 \models \exists y\psi(y\gamma\bar{p}_0) \implies \bar{X} \restriction \bar{z}_0 \models \exists y\psi(y\gamma\bar{x}_0) : M_1 \models \exists y\psi(y\gamma\bar{p}_0) \implies M \restriction \delta \models \exists y\psi(y\sigma(\gamma)p_0) \implies M \models \exists y < \delta\psi(y\sigma(\gamma)p_0) \implies M \models \exists y < \hat{z}_0\psi(y\sigma(\gamma)p_0) \implies X \models \exists y < z_0\psi(y\sigma(\gamma)x_0) \implies \bar{X} \models \exists y < \bar{z}_0\psi(y\gamma\bar{x}_0) \implies \bar{X} \restriction \bar{z}_0 \models \exists y\psi(y\gamma\bar{x}_0)$. By the minimality of $\bar{z}_0, M_1 \models\ \forall y\neg\psi(y\gamma\bar{p}_0) \implies \bar{X} \restriction \bar{z}_0 \models\ \forall y \neg\psi(y\gamma\bar{X}_0)$. It follows that if $Q = \bar{h}_1^{\bar{X}\restriction \bar{z}_0}(\bar{p}_1 \cup \langle \bar{X}_0 \rangle), Q \in M_1$ and letting $\bar{Y} = Q$ and $\sigma(Q) = Y$ finishes the proof.

Now for the proof of square:

Assume that for each singular ordinal ν, the singularizing structure $\mathcal{A}(\nu)$ has been defined. Assume that $\mathcal{A}(\nu)$ singularizes ν in a Σ_1-way. Let p be the standard parameter for $\mathcal{A}(\nu)$ above ν. If $\nu < ht\ \mathcal{A}(\nu)$, let x be the least parameter string less than ν such that $\nu \in h_1^{\mathcal{A}(\nu)}(\langle px \rangle)$. If $\nu = ht\ \mathcal{A}(\nu)$, let $x = \phi$. Let $\alpha(\nu) = \sup\{\bar{\nu} < \nu : \bar{\nu} = \sup h_1^{\mathcal{A}(\nu)}(\bar{\nu} \cup \langle px \rangle) \cap \nu\}$. If $\nu = \sup h_1^{\mathcal{A}(\nu)}(\alpha(\nu) + 1 \cup \langle px \rangle) \cap \nu$, say that ν is of type I and define $\alpha_0 = \nu_0 = \alpha(\nu)$. Define:

$$C_\nu^1 = \{\bar{v} < \nu : \bar{v} = \sup h_1^{\mathcal{A}(\nu)}(\alpha \cup \langle \alpha_0 px \rangle) \cap \nu \text{ for some } \alpha\}$$

and leave C_ν^0 undefined.

If $\nu \neq \sup h_1^{\mathcal{A}(\nu)}(\alpha(\nu) + 1 \cup \langle px \rangle) \cap v$, say that ν is of type II and define

$$C_\nu^0 = \{\bar{v} < v : \text{ there exists } \alpha > \alpha(\nu) \text{ such that}$$

$$\bar{\nu} = \sup h_1^{\mathcal{A}(\nu)}(\alpha \cup \langle px \rangle) \cap v\}$$

If C_ν^n is defined and bounded in ν, define ν_n, α_n and $C_\nu^{n+1} : \nu_n = \sup(C_\nu^n) \cap \nu$. α_n is such that $\nu_n = \sup h_1^{\mathcal{A}(\nu)}(\alpha_n \cup \langle px\alpha_0 \ldots \alpha_{n-1} \rangle) \cap \nu$ and $\nu = \sup h_1^{\mathcal{A}(\nu)}(\alpha_n + 1 \cup \langle px\alpha_0 \ldots \alpha_{n-1} \rangle) \cap \nu$. $C_\nu^{n+1} = \{\bar{v} < \nu : \text{there exists } \alpha \text{ such that } \bar{\nu} = \sup h_1^{\mathcal{A}(\nu)}(\alpha \cup \langle px\alpha_0 \ldots \alpha_n \rangle) \cap \nu\}$. If C_ν^n is defined and unbounded in ν, let α_n be the least ordinal α such that $\nu = \sup h_1^{\mathcal{A}(\nu)}(\alpha \cup \langle px\alpha_0 \ldots \alpha_{n-1} \rangle) \cap \nu$. If $\bar{\nu} \in C_\nu^n$, let $\beta(\bar{\nu})$ the the least ordinal β such that $\bar{\nu} = \sup h_1^{\mathcal{A}(\nu)}(\beta \cup \langle px\alpha_0 \ldots \alpha_{n-1} \rangle) \cap \nu$ and let $\sigma(\bar{\nu})$ be such that $\sigma(\bar{\nu}) = \sup h_1^{\mathcal{A}(\nu)}(\beta \cup \langle px\alpha_0 \ldots \alpha_{n-1} \rangle)$.

Clearly, if C_ν^{n+1} is bounded in ν then $\alpha_{n+1} < \alpha_n$. It follows that there exists some least n for which either C_ν^n is unbounded in ν or for which $C_\nu^n = \emptyset$. If $C_\nu^n = \emptyset$ then $\operatorname{cof} \nu = \omega$.

As in L, for all n it is true that otp $C_\nu^n < \nu$ and C_ν^n is closed. If n is the least number such that either $C_\nu^n = \emptyset$ or C_ν^n is unbounded in ν, and $C_\nu^n = \emptyset$, let C_ν be the least ω-sequence in K cofinal in ν.

If n is as above, but C_ν^n is unbounded in ν, let $D_\nu = \{\bar{\nu} < \nu : \bar{\nu} > \sup\{x, \nu_0, \ldots \nu_{n-1}\}$, $\bar{\nu} \in C_\nu^n$, and if $\mathcal{A}(\nu)$ is the n^{th} reduct of a mouse M (so that $\mathcal{A}(\nu) = M^{mq}$ for some m, q, and mouse M), then p is fully witnessed in $h_1^{\mathcal{A}(\nu)\restriction\sigma(\bar{\nu})}(\bar{\nu} \cup \langle p \rangle)$ with respect to $q\}$

For all $\bar{\nu}$ sufficiently close to ν, p will be fully witnessed in $h_1^{\mathcal{A}(\nu)\restriction\sigma(\bar{\nu})}(\bar{\nu} \cup \langle p \rangle)$ with respect to q if $\mathcal{A}(\nu) = M^{mq}$ for some mouse M. So D_ν is a final segment of C_ν^n and is club in ν . If $\mathcal{A}(\nu) \models (\nu = \gamma^+)$ for some γ, let $D'_\nu = D_\nu \setminus \gamma$. Let $D'_\nu = D_\nu$ otherwise. If $\mathcal{A}(\nu) \models (\nu$, is inactive, $\nu = \kappa^{++}$, lim card $\kappa)$ and ν is not bad, let $C_\nu = D'_\nu \setminus \{\bar{\nu} < \nu : \bar{\nu}$ is bad$\}$. Otherwise, let $C_\nu = D'_\nu$.

Badness will be defined later and it will be shown that in the case that $\mathcal{A}(\nu) \models$ (ν is inactive, $\nu = \kappa^{++}$, lim card κ) and ν is not bad, C_ν is a final segment of D'_ν and so is club in ν. Assume for now that in all cases, C_ν is a final segment of C_ν^n.

Suppose that $\bar{\nu} \in \lim C_\nu$. Then define $B(\bar{\nu}) = \bar{h}_1^{\mathcal{A}(\nu)\restriction\sigma(\bar{\nu})}(\bar{\nu} \cup \langle p \rangle)$ and let $\tau :\ B(\bar{\nu}) \longrightarrow \mathcal{A}(\nu) \restriction \sigma(\bar{\nu})$ be the inverse of the collapse. Then $B(\bar{\nu})$ singularizes $\bar{\nu}$, and if $\nu \in \mathcal{A}(\nu), \bar{\nu} \in \operatorname{dom} \tau$ and $\tau(\bar{\nu}) = \nu$. If $\nu = ht\ \mathcal{A}(\nu)$ then $\bar{\nu} = ht\ B(\bar{\nu})$. Also, $\beta(\bar{\nu}) \models (\bar{\nu} = \gamma^+) \longleftrightarrow \mathcal{A}(\nu) \models (\nu = \gamma^+)$.

Claim: $\mathcal{A}(\bar{\nu}) = \beta(\bar{\nu})$

Assuming the claim, it follows that for $\bar{\nu} \in \lim C_\nu, \bar{\nu} \cap C_\nu = C_{\bar{\nu}}$:

Proof: Let C_ν be a final segment of C_ν^n. Let $\bar{\nu} \in \lim C_\nu$. For ease of notation, let $\beta = \beta(\bar{\nu})$ and $\sigma = \sigma(\bar{\nu})$. $\mathcal{A}(\bar{\nu}) = h_1^{\mathcal{A}(\nu)\restriction\sigma}(\bar{\nu} \cup \langle p \rangle)$. (By claim.) Let $\tau : \mathcal{A}(\bar{\nu}) \longrightarrow$

$\mathcal{A}(\nu) \upharpoonright \sigma$ be the inverse of the collapse. Note that by the witnessing condition on p in $h_1^{\mathcal{A}(\nu)\upharpoonright\sigma}(\bar{\nu} \cup \langle p \rangle)$, p collapses to $\bar{p}$, the standard parameter for $\mathcal{A}(\bar{\nu})$ above $\bar{\nu}$. Recall that if $\bar{\nu} \in \operatorname{dom} \tau$ then $\tau(\bar{\nu}) = \nu$. Also, $\tau(x) = x$ and x is the smallest parameter string less than $\bar{\nu}$ in $\mathcal{A}(\bar{v})$ such that $\bar{\nu} \in h_1^{\mathcal{A}(\bar{\nu})}(\langle x\bar{p} \rangle)$.

First, $\alpha(\nu) = \alpha(\bar{\nu})$:

Since $\alpha(\nu) < \bar{\nu}, \alpha(\nu) = \sup h_1^{\mathcal{A}(\nu)\upharpoonright\sigma}(\alpha(\nu) \cup \langle px \rangle) \cap \nu$, so $\alpha(\nu) \leq \alpha(\bar{\nu})$. If $\alpha(\nu) < y, \mathcal{A}(\nu) \models \exists\sigma'\exists\delta(\delta \in h_1^{\mathcal{A}(\nu)\upharpoonright\sigma'}(y \cup \langle px \rangle) \cap \nu$ and $\delta > y)$. So if $y \in h_1^{\mathcal{A}(\nu)}(\beta \cup \langle px\alpha_0 \ldots \alpha_{n-1} \rangle) \cap \bar{\nu}$ and $y > \alpha(\nu)$, then $\sup h_1^{\mathcal{A}(\nu)\upharpoonright\sigma}(y \cup \langle px \rangle) \cap \bar{\nu} > y$. Let $\alpha(\nu) < y < \bar{\nu}$. Find ν' minimal such that $\nu' \in h^{\mathcal{A}(\nu)}(\beta \cup \langle px\alpha_0 \ldots \alpha_{n-1} \rangle) \cap \nu$ and $y \leq \nu'$. Then by the above, for some $\delta < \nu'$, $\delta \in h_1^{\mathcal{A}(\nu)}(\beta \cup \langle px\alpha_0 \ldots \alpha_{n-1} \rangle)$, $\sup h_1^{\mathcal{A}(\nu)\upharpoonright\sigma}(\delta \cup \langle px \rangle) \cap \bar{\nu} > y$, and $\alpha(\nu) = \alpha(\bar{\nu})$.

If ν is of type II, and $n = 0$, then if $\nu' \in \bar{\nu} \cap C_\nu, \nu' = \sup h_1^{\mathcal{A}(\nu)}(\beta(\nu') \cup \langle px \rangle) \cap \bar{\nu}$. Since $\nu' < \bar{v}, \beta(v') < \beta$ and $\sigma(v') < \sigma$. So $v' = \sup h_1^{\mathcal{A}(v)\upharpoonright\sigma}(\beta(v') \cup \langle px \rangle) \cap \bar{v}$ and $v' \in C^0_{\bar{\nu}}$. If $v' \in C^0_{\bar{\nu}}$ let $v' = \sup h_1^{\mathcal{A}(\bar{v})}(\beta(v') \cup \langle \bar{p}x \rangle) \cap \bar{v} = \sup h_1^{\mathcal{A}(v)\upharpoonright\sigma}(\beta(v') \cup \langle px \rangle) \cup \bar{v}$. Again, since $\beta(v') < \beta$, $\sigma' = \sup h_1^{\mathcal{A}(v)}(\beta(v') \cup \langle px \rangle)$, is less than σ and $\nu' = \sup h_1^{\mathcal{A}(v)\upharpoonright\sigma}(\beta(v') \cup \langle px \rangle) \cap \nu$ and $v' \in C^0_\nu \cap \bar{v}$. So here $C_{\bar{\nu}} = \bar{\nu} \cap C_\nu$ if $\bar{\nu} \in \lim C^0_\nu$.

If ν is of type I then since $\alpha(v) = \alpha_0, \sup\{\beta, \alpha_0, \ldots \alpha_{n-1}\}$ is less than or equal to $\alpha(v)$. It follows that $\bar{v} = \sup h_1^{\mathcal{A}(v)\upharpoonright\sigma}(\alpha(v) + 1 \cup \langle px \rangle) \cap \bar{\nu}$. So $\bar{\nu}$ is also of type I. The same proof as in the case $n = 0$ shows that if $n = 1$ and ν is of type I then $C_{\bar{\nu}} = \bar{\nu} \cap C_\nu$ when $\bar{\nu} \in \lim C_\nu$.

To prove that $C_{\bar{\nu}} = \bar{\nu} \cap C_\nu$ when $n > 0$ and ν is of type II or when $n > 1$ and ν is of type I, we need to establish that $\alpha_i^\nu = \alpha_i^{\bar{\nu}}$ and $\nu_i = \bar{\nu}_i$ for $i < n$: Suppose ν is of type I, so that $\alpha_0 > \alpha(\nu)$. $\nu_0 = \sup h_1^{\mathcal{A}(\nu)}(\alpha_0 \cup \langle px \rangle) \cap \nu$. Let $\sigma_0 = \sup h_1^{\mathcal{A}(v)}(\alpha_0 \cup \langle px \rangle)$. So $\nu_0 = \sup h_1^{\mathcal{A}(v)\upharpoonright\sigma_0}(\nu_0 \cup \langle px \rangle) \cap \nu$ and $\nu_0 > \alpha_0$. Recall that $\bar{\nu} > \nu_0$. Suppose $\sigma_0 \le \sigma$. Then $\nu_0 = \sup h_1^{\mathcal{A}(v)\restriction\sigma}(\nu_0 \cup \langle px\rangle) \cap \nu$. But $\nu_0 > \alpha_0 \ge \sup\{\beta, \alpha_0, \ldots, \alpha_{n-1}\}$. So $\bar{\nu} = \sup h_1^{\mathcal{A}(v)\restriction\sigma}(\nu_0 \cup \langle px\rangle) \cap \nu$. Contradiction. So $\sigma_0 < \sigma$. But then $\nu_0 = \sup h_1^{\mathcal{A}(\nu)\restriction\sigma}(\alpha_0 \cup \langle px\rangle) \cap \bar{\nu}$. Because $\alpha_0 + 1 \ge \sup\{\beta, \alpha_0, \ldots \alpha_{n-1}\}$, $\bar{\nu} = \sup h_1^{\mathcal{A}(\nu)\restriction\sigma}(\alpha_0 + 1 \cup \langle px\rangle) \cap \bar{\nu}$. So $\alpha_0^{\nu} = \alpha_0^{\bar{\nu}}$ and $\nu_0 = \bar{\nu}_0$. A similar proof shows that $\alpha_i^{\nu} = \alpha_i^{\bar{\nu}}$ and $\nu_i = \bar{\nu}_i$ for $0 < i < n$ for ν of either type I or type II.

$\bar{\nu} \cap C_{\nu}^n = \{v' < \bar{v} : v' = \sup h_1^{\mathcal{A}(v)}(\beta' \cup \langle px\alpha_o \ldots \alpha_{n-1}\rangle \cap \bar{\nu}\}$. But now, because $\alpha_i^{\nu} = \alpha_i^{\bar{\nu}}$ and $\nu_0 = \bar{\nu}_i$ for $i < n$, just as before, $\bar{\nu} \cap C_{\nu}^n = C_{\bar{\nu}}^n$ and so for $\bar{v} \in \lim C_{\nu}$, $C_{\bar{\nu}}^n$ must be unbounded in $\bar{\nu}$ and $C_{\bar{\nu}} = \bar{\nu} \cap C_{\nu}$. So square will be established for K once $\mathcal{A}(v)$ is defined and the claim that $B(v) = \mathcal{A}(v)$ is proven.

Let X_{ν} be the shortest initial segment of K that singularizes ν, and suppose it singularizes ν in a Σ_n-way. Let N_{ν} be the $*$-$(n-1)^{st}$ reduct of X_{ν}. If there is a γ such that $N_{\nu} \models (\nu = \gamma^+$ and γ is measurable), let M_{ν} be the result of iterating N_{ν} once by its measure on γ. (Note that both N_{ν} and M_{ν} singularize ν in a Σ_1-way).

In most cases, $\mathcal{A}(\nu)$ will be N_{ν}. However, look at what can happen if $N_{\nu} \models (\nu = \gamma^+$ and $\neg$ (γ measurable) and lim card γ) : Suppose C_{ν} is the club sequence derived from N_{ν} and $\bar{\nu} = \sup h_1^{N_{\nu}}(\beta(\bar{\nu}) \cup \langle\alpha_0 \ldots \alpha_n px\rangle) \cap \nu, \sigma(\bar{\nu}) = \sup h_1^{N_{\nu}}(\beta(\bar{\nu}) \cup \langle\alpha_0 \ldots \alpha_n px\rangle)$ as usual. Then $B(\bar{\nu}) = \bar{h}_1^{N_{\nu}\restriction\sigma(\bar{\nu})}(\bar{\nu} \cup \langle p\rangle)$. Since $N_{\bar{\nu}} \models (\neg\gamma$ measurable), γ cannot be measurable in $B(\bar{\nu})$. But it's possible that "locally" in K, γ is measurable, i.e. $N_{\bar{\nu}} \models$ (γ measurable). It turns out in this case that $B(\bar{\nu})$ equals $M_{\bar{\nu}}$. The solution here is to let $\mathcal{A}(\nu) = M_{\nu}$ when $N_{\nu} \models (\nu = \gamma^+$ and γ measurable). Another problem arises when ν is active: Let $N_{\nu} \models \nu = \kappa^{++}$, and let C_{ν} be the club sequence derived from N_{ν}, so $\bar{\nu} = \sup h_1^{N_{\nu}}(\beta(\bar{\nu}) \cup (\alpha_0 \ldots \alpha_n))$ and $B(\bar{\nu}) = \bar{h}_1^{N_{\nu}\restriction\bar{\nu}}(\bar{\nu})$. Here $B(\bar{\nu})$ is no longer even a reduct of a mouse. Furthermore, since $\bar{\nu} > \kappa^+$ cannot be active in K, $N_{\bar{\nu}}$ would seem at first to be too horribly different from N_ν for $C_{\bar{\nu}}$ to equal $\bar{\nu} \cap C_\nu$. In this case, the singularizing structure for ν remains N_ν, but for appropriate $\bar{\nu}$ between κ and κ^+, $\mathcal{A}(\bar{\nu})$ will be $N_\nu \upharpoonright \bar{\nu} = \langle K\|\nu, F_\nu \upharpoonright \bar{\nu}\rangle$.

The Definition of $\mathcal{A}(\nu)$ and the proof that $\mathcal{A}(\nu) = \beta(\nu)$:

Suppose that $N_\nu \models (\nu = \gamma^+$ and lim card $\gamma)$. If $N_\nu \models (\gamma$ measurable), let $\mathcal{A}(\nu) = M_\nu$. If $N_\nu \models \neg(\gamma$ measurable), let $\mathcal{A}(\nu) = N_\nu$. Then it follows that for $\bar{\nu} \in \{\nu : N_\nu \models (\nu = \gamma^+$ and lim card $\gamma)\}$, $B(\bar{\nu}) = \mathcal{A}(\bar{\nu})$.

Proof: First, suppose that $N_\nu \models (\nu = \gamma^+$ and γ measurable). Then $\mathcal{A}(\nu) = M_\nu$. Let C_ν be the club sequence derived from M_ν. Let $\bar{\nu} \in \lim C_\nu$, $\bar{\nu} = \sup h_1^{M_\nu}(\beta(\bar{\nu}) \cup \langle \alpha_0 \ldots \alpha_{n-1} px\rangle) \cap \nu$. Then $B(\bar{\nu}) = \bar{h}_1^{M_\nu \upharpoonright \sigma(\bar{\nu})}(\bar{\nu} \cup \langle p\rangle)$ where $\sigma(\bar{\nu}) = \sup h_1^{M_\nu}(\beta(\bar{\nu}) \cup \langle \alpha_0 \ldots \alpha_{n-1} px\rangle)$. Let $\tau : B(\bar{\nu}) \longrightarrow M_\nu \upharpoonright \sigma(\bar{\nu})$ be the inverse of the collapse. Clearly $\nu < ht\ M_\nu$, so if M_ν is a mouse, $M_\nu \upharpoonright \sigma(\bar{\nu})$ is a τ-mouse and $B(\bar{\nu})$ is a mouse. If $M_\nu = (N)^{nq}$ for some $n > 0$, then regard τ as a Σ_0 map from $B(\bar{\nu})$ to M_ν. Then the canonical expansion of τ from B' to N is at least Σ_1. Since N is a mouse, B' is a mouse and $B(\bar{\nu})$ is the reduct of a mouse. $M_\nu\|\nu = K\|\nu$ and in a comparison of M_ν and κ, M_ν is obviously passive below $\nu > \gamma_{B(\bar{\nu})}$. Since $\bar{\nu} \in C_\nu, p$ is witnessed in $h_1^{M_\nu \upharpoonright \sigma(\bar{\nu})}(\bar{\nu} \cup \langle p\rangle)$ and if $M_\nu = (N)^{nq}$ for some $n > 0$ then p is fully witnessed in $h_1^{M_\nu \upharpoonright \sigma(\bar{\nu})}(\bar{\nu} \cup \langle p\rangle)$ with respect to q. It follows by condensation that $B(\bar{\nu}) = N_{\bar{\nu}}$ or $B(\bar{\nu}) = M_{\bar{\nu}}$. If $B(\bar{\nu}) = N_{\bar{\nu}}$, then $N_{\bar{\nu}} \models (\bar{\nu} = \gamma^+$, lim card γ, and $\neg$ (γ measurable). If $B(\bar{\nu}) = M_{\bar{\nu}}$, then $N_{\bar{\nu}} \models (\bar{\nu} = \gamma^+$, lim card γ, and γ measurable). Either way, $B(\bar{\nu}) = \mathcal{A}(\bar{\nu})$. Now, suppose that $N_\nu \models (\nu = \gamma^+$ and $\neg\gamma$ measurable). Then $\mathcal{A}(\nu) = N_\nu$. If $\bar{\nu} \in$ lim C_ν where C_ν is the club sequence derived from N_ν, exactly as above, it follows that $B(\bar{\nu}) = \mathcal{A}(\bar{\nu})$.

Now consider R, the union of $\{\nu : \nu < ht\ N_\nu$ and $\neg(N_\nu \models (\nu = \gamma^+$ and lim card $\gamma))\}$ and $\{\nu : \nu = ht\ N_\nu$ and $\neg(N_\nu \models \nu$ is active$)\}$. Let $\nu \in R$. Let C_ν be the club sequence derived from N_ν and let $\bar{\nu} \in \lim C_\nu, \bar{\nu} = \sup h_1^{N_\nu}(\beta(\bar{\nu}) \cup \langle \alpha_0 \ldots \alpha_{n-1} px \rangle) \cap \nu$, $\sigma(\bar{\nu}) = \sup h_1^{M_\nu}(\beta(\bar{\nu}) \cup \langle \alpha_0 \ldots \alpha_{n-1} px \rangle)$. Then by condensation, $N_{\bar{\nu}} = h_1^{N_\nu \restriction \sigma(\bar{\nu})}(\bar{\nu} \cup \langle p \rangle)$. If $\tau : N_{\bar{\nu}} \longrightarrow N_\nu \restriction \sigma(\bar{\nu})$ is the inverse of the collapse, then since $\tau(\bar{\nu}) = \nu$, $(N_{\bar{\nu}} \models$ lim card $\bar{\nu} \iff N_\nu \models$ lim card $\nu)$, $(N_{\bar{\nu}} \models \bar{\nu}$ is an n^{th} successor cardinal $\iff N_\nu \models \nu$ is an n^{th} successor cardinal), and $(N_{\bar{\nu}} \models \bar{\nu}$ is active $\iff N_\nu \models \nu$ is active).

It follows that if $\mathcal{A}(\nu)$ is defined to be N_ν for ν such that for no κ $N_\nu \models (\nu = \kappa^{++}$ and lim card $\kappa)$, then $\mathcal{A}(\nu) = \mathcal{B}(\nu)$. It remains to define $\mathcal{A}(\nu)$ for those ν such that for some κ, $N_\nu \models (\nu = \kappa^{++}$ and lim card $\kappa)$. First, badness must be defined:

ν is bad iff

1) $N_\nu \models (\nu = \kappa^{++}, \nu < ht\ N_\nu)$

2) Let $B = \bar{h}_1^{N_\nu}(\kappa + 1 \cup \langle p_\nu \rangle)$ where p_ν is the standard for N_ν above ν. Let $\pi : B \longrightarrow N_\nu$ be the inverse of the collapse. Then there exists ν^* such that $\pi(\nu^*) = \nu$ and ν^* is active in K.

Condition 2 requires B is not 1-sound above κ: Let $\bar{B} = \text{core}(B)$, then $\bar{B}$ is the reduct of an initial segment of K. Since ν^* is active in K, ν^* is active in $\bar{B}$ and since ν^* is inactive in $B, \bar{B} \neq B$. As $\bar{B}$ is the reduct of an initial segment of $K, ht\ \bar{B} > \alpha = \sup h_1^{N_\nu}(\kappa + 1 \cup \langle p_\nu \rangle) \cap \kappa^+$. So the first difference between B and $\bar{B}$ must occur at or above κ. It follows that $\nu^* = \kappa^{++}$ in both B and $\bar{B}$. But since the iteration of $\bar{B}$ to B is simple, the first difference between $\bar{B}$ and B must occur at ν^*. It must also be that $\bar{B} = N_\alpha$, since $\bar{B}$ collapses α. Note also that $\kappa = \sup h_1^{N_\nu}(\kappa \cup \langle p_\nu \rangle) \cap \nu$, since $\bar{B}$ iterates to B with critical point κ.

Define F'_ν to be $\pi[F_{\nu^*}]$. (Here $F_{\nu^*} = F^K_{\nu^*}$). Then because $K|\nu^* = \bar{h}_1^{K|\nu^*}(\alpha), \nu =$ $\sup\{y : \langle xy\rangle \in F'_\nu$ and $x \in K|\alpha\}$. Define $P_\nu = \langle K\|\nu, F'_\nu\rangle$. Then $K|\nu^* = \bar{h}_1^{P_\nu}(\alpha)$ $h_1^{P_\nu}(\alpha) = \{y : \langle xy\rangle \in F'_\nu$ and $x \in K|\alpha\}$, $\nu = \sup h_1^{P_\nu}(\alpha)$ and $\alpha = \sup h_1^{P_\nu}(\alpha) \cap \kappa^+$. If ν is such that for some $\kappa, N_\nu \models (\nu = \kappa^{++}$ and lim card $\kappa)$ *and* ν is bad, define $\mathcal{A}(\nu) = P_\nu$. Otherwise set $\mathcal{A}(\nu)$ to be N_ν.

Now that $\mathcal{A}(\nu)$ has been defined, it remains to show that $\mathcal{A}(\nu) = B(\nu)$ for ν such that for some κ, $N_\nu \models (\nu = \kappa^{++}$ and lim card $\kappa)$.

Claim 1: If ν is bad and $\bar{\nu} \in \lim C_\nu$, then $\bar{\nu}$ is bad and $P_\nu \restriction \bar{\nu} = P_{\bar{\nu}} = B(\bar{\nu})$.

Proof: Let $\nu, \nu^*, \kappa, \alpha, B$ and $\bar{B}$ be as above.

Let $\bar{\nu} = \sup h_1^{P_\nu}(\lambda \cup \langle\alpha_0, \ldots \alpha_n\rangle)$ where $\beta(\bar{\nu}) = \lambda$ is less than α and for all $i, \alpha_i < \alpha$.

Let $\beta = \sup h_1^{P_\nu}(\lambda \cup \langle\alpha_0 \ldots \alpha_n\rangle) \cap \kappa^+$

For the sake of notation, let $P \restriction \bar{\nu}$ stand for $P_\nu \restriction \bar{\nu} = \langle\kappa\|\bar{\nu}, F'_\nu \cap \bar{\nu}\rangle$

Then, $h_1^{P\restriction\bar{\nu}}(\beta) = \{y : (xy) \in F'_\nu\ x \in K|\beta\}$

$$\bar{\nu} = \sup h_1^{P\restriction\bar{\nu}}(\beta)$$

$$\beta = \sup h_1^{P\restriction\bar{\nu}}(\beta) \cap \kappa^+$$

Let $j : \bar{B} \longrightarrow B$ be the iteration map and choose τ such that j sends N_β cofinally into $j(N_\beta) \restriction \tau$. Let p_β be the standard parameter for N_β. Then:

$\nu^* \cap h_1^{j(N_\beta)\restriction\tau}(\kappa + 1 \cup (j\langle\rho_\beta\rangle)) = \{j(f)(\kappa) : f \in K|\beta \cap {}^\kappa\kappa$ and for all $\delta < K, f(\delta) <$ $\delta^{++}\}$

Proof: $\subseteq$: Suppose that x is the unique least member of $j(N_\beta) \restriction \tau$ such that $j(N_\beta) \restriction \tau \models \exists z\phi(z, \delta, \kappa, j(\rho_\beta))$ where ϕ is Σ_0 and $\delta < \kappa$. Then there exists some t in N_β such that $j(N_\beta)|j(t) \models \exists z\phi(z\delta\kappa j(\rho_\beta))$. Define $f(\alpha) = x_\alpha \longleftrightarrow x_\alpha$ is the unique member of $N_\beta|t$ such that $N_\beta|t \models \exists z\phi(z\delta\alpha\rho_\beta)$, for all $\alpha < \kappa$. Then $f \in N_\beta$ and $j(f)(\kappa) = x$. Since $x < \nu^*$ we can assume $f \in {}^\kappa\kappa$ and for all $\delta < \kappa, f(\delta) < \delta^{++}$.

$\supseteq$: Let $f \in K|\beta$. Then $f \in N_\beta$ and so $f \in h_1^{N_\beta}(\kappa \cup \langle \rho_\beta \rangle)$. But then if $f(\delta) < \delta^{++}$ for all $\delta < \kappa$ and $f \in {}^\kappa\kappa$, $j(f)(\kappa) \in \nu^* \cap h_1^{j(N_\beta)\restriction\tau}(\kappa + 1 \cup \langle j(\rho_\beta) \rangle)$. $\dashv$

But $\{j(f)(\kappa) : f \in K|\beta, f \in {}^\kappa\kappa, \text{ and } f(\delta) < \delta^{++} \text{ for } \delta < \kappa\} = \{y : (xy) \in F_{\nu^*}, x \in K|\beta\}$. But then, $\pi[\{y : (xy) \in F_{\nu^*}, x \in K|\beta\}] = \{y : (xy) \in F'_\nu\ x \in K|\beta\} = h_1^{P\restriction\bar\nu}(\beta)$. So $\sup h_1^{\pi(j(N_\beta))\restriction\mu}(\kappa + 1 \cup \langle \pi j(p_\beta) \rangle) \cap \nu = \bar\nu$ (Here μ is chosen so that πj sends N_β cofinally into $\pi j(N_\beta) \restriction \mu$). As usual, $\bar\nu = \sup h_1^{\pi j(N_\beta)\restriction\mu}(\bar\nu \cup \langle \pi j(p_\beta) \rangle) \cap v$. Note that because $N_\beta = h_1^{N_\beta}(\kappa \cup (p_\beta))$, all the FW's for $\pi j(p_\beta)$ in $\pi j(N_\beta)$ are contained in $\dot h_1^{\pi j(N_\beta)\restriction\mu}(\kappa \cup \langle \pi j(p_\beta) \rangle)$. So $\pi j(p_\beta)$ is fully witnessed in $h_1^{\pi j(N_\beta)\restriction\mu}(\bar\nu \cup \langle \pi j(p_\beta) \rangle)$. N_β is a reduct of an initial segment of N_α. It follows that $\pi j(N_\beta)$ is a reduct of an initial segment of N_ν, a reduct of an initial segment of K. It follows by condensation that $N_{\bar\nu} = \bar h_1^{\pi j(N_\beta)\restriction\mu}(\bar\nu u \langle \pi j(p_\beta) \rangle)$ and $\pi j(p_\beta)$ collapses to the standard parameter of $N_{\bar\nu}$ above $\bar\nu$.

Let $M = \bar h_1^{N_\beta}(\kappa + 1 \cup \langle p_\beta \rangle) = \bar h_1^{\pi j(N_\beta)\restriction\mu}(\kappa + 1 \cup \langle \pi j(p_\beta) \rangle)$. We showed that $\nu \cap h_1^{\pi j(N_\beta)\restriction\mu}(\kappa + 1 \cup \langle \pi j(p_\beta) \rangle) = h_1^{P\restriction\bar\nu}(\beta)$. Let $z : M \longrightarrow \pi j(N_\beta)$ be the inverse of the collapse. Then $z(\omega^*) = \nu$ where $K|\omega^* = \bar h_1^{P\restriction\bar\nu}(\beta)$. If $\bar z : M \longrightarrow N_{\bar\nu}$ is the inverse of that collapse, then $\bar z(\omega^*) = \bar\nu$ and ω^* is active in K.

So $\bar\nu$ *is* bad.

Let $z_0 : M \longrightarrow \bar h_1^{\pi j(N_\beta)\restriction\mu}(\bar\nu \cup \langle \pi j(p_\beta) \rangle) = N_{\bar\nu}$, and let $z_1 : N_{\bar\nu} \longrightarrow h_1^{\pi j(N_\beta)\restriction\mu}(\bar\nu \cup (\pi j \langle p_\beta \rangle))$ be inverses of the collapses. $z_0[F_{\omega^*}] = \nu \cap h_1^{\pi j(N_\beta)\restriction\mu}(\kappa + 1 \cup \langle \pi j(p_\beta) \rangle) = F_{\nu'} \cap \bar\nu$. Since $z_1 \restriction \bar\nu = id \restriction \bar\nu$, $z_1 z_0[F_{\omega^*}] = F_{\nu'} \cap \bar\nu$ and $P_{\bar\nu} = \langle K\|\bar\nu, F_{\nu'} \cap \bar\nu \rangle$. Finally, $B(\bar\nu) = \bar h_1^{P_\nu\restriction\bar\nu}(\bar\nu) = P_\nu \restriction \bar\nu = P_\nu$ $\dashv$

Claim 2: If ν is active and $\bar{\nu} \in \lim C_\nu$, then $\bar{\nu}$ is bad, and $P_{\bar{\nu}} = \langle K \| \bar{\nu}, F_\nu \rangle = B(\bar{\nu})$.

Proof: Exactly as in Claim 1 with some substitutions: Let E_ν be the measure on κ in N_ν where $N_\nu \models (\nu = \kappa^{++})$. Let $j : N_\nu \underset{E_\nu^\bullet}{\longrightarrow} (N_\nu)'$ be the iteration via E_ν. Then, substitute N_ν for N_α, N_ν for P_ν, F_ν for F_ν' $(N_\nu)'$ for $B, j : N_\nu \underset{E_\nu^\bullet}{\longrightarrow} (N_\nu)'$ for $j' : \bar{B} \longrightarrow B$, κ^+ for α, ν for ν^*, and id for π.

Note: at the point in the proof where one contemplates $M = \bar{h}_1^{\pi j(N_\beta)\restriction\mu}(\kappa + 1 \cup (\pi j \langle p_\beta \rangle))$, (in this case $M = \bar{h}_1^{j(N_\beta)\restriction\mu}(\kappa + 1 \cup \langle (j(p_\beta)) \rangle)$, M still turns out to be the reduct of an initial segment of K because even though $j(N_\beta)$ is now the reduct of an initial segment of $(N_\nu)'$, instead of N_ν, M is very small. M is of size κ, and below $\nu, (N_\nu)'$ is equal to N_ν. So the proof is not affected by this difference.

Claim 3: Let $\nu < ht\ N_\nu$ and suppose ν is not bad. Then C_ν is a final segment of D_ν.

Proof: For ease of notation, let $N = N_\nu$ and $p = p_\nu$ where p_ν is the standard parameter of N_ν above ν. We can assume that $N \models$ (ν is inactive, $\nu = \kappa^{++}$, lim card κ, $\nu < ht\ N$). Then for all ν_i in D_ν, $N_{\nu_i} \models$ (ν_i is inactive, $\nu_i = \kappa^{++}$, lim card κ, $\nu_i < ht\ N_{\nu_i}$). Again, for ease of notation, let $N_{\nu_i} = N_i$ and $p_{\nu_i} = p_i$ where p_{ν_i} is the standard parameter of N_{ν_i} above ν_i. For each $i, N_i = \bar{h}_1^{N\restriction\sigma_i}(\nu_i \cup \langle p \rangle)$ for some appropriate σ_i with $\sup_i(\sigma_i) = ht\ N$. Let $B = \bar{h}_1^N(\kappa + 1 \cup \langle p \rangle)$ and for each i let $B_i = \bar{h}_1^{N\restriction\sigma_i}(\kappa + 1 \cup \langle p \rangle)$. Then for all i, $B_i = \bar{h}_i^{B\restriction\alpha_i}(\kappa + 1 \cup \langle \bar{p} \rangle)$ for some appropriate α_i with $\sup_i(\alpha_i) = ht\ B$ and p collapsing to $\bar{p}$ in B. Suppose B is 1-sound above κ. Then $B = h_1^B(\kappa \cup \langle z \rangle)$ where z is the standard parameter for B above κ. But then for i large enough, $z \in h_1^{B\restriction\alpha_i}(\kappa + 1 \cup \langle \bar{p} \rangle)$, z is witnessed in $h_1^{B\restriction\alpha_i}(\kappa + 1 \cup \langle \bar{p} \rangle)$, and $B_i = \bar{h}_1^{B\restriction\alpha_i}(\kappa \cup \langle z \rangle)$ is 1-sound above κ. Therefore, for i large enough, ν_i is not bad.

So in this case, C_ν is a final segment of D_ν.

Suppose B is not 1-sound above κ. Let $\pi : B \longrightarrow N$ be the inverse of the collapse. Let $\pi(\nu^*) = \nu$. Then ν^* must be inactive in κ and if $\bar{B} = \operatorname{core} B$, then $\bar{B}$ iterates to B above ν^*. This can only happen if $\kappa \in h_1^B(\kappa \cup \langle z \rangle)$ where z is the standard parameter of B above κ. But then for i large enough, κ and z are members of $h_1^{B \restriction \alpha_i}(\kappa \cup \langle z \rangle)$ and z is witnessed in $h_1^{B \restriction \alpha_i}(\kappa \cup \langle z \rangle)$. This means that core$(B_i)$ iterates to B_i above $(\kappa^{++})^{B_i}$. Therefore ν_i cannot be bad. Again, a final segment of D_ν is not bad and C_ν is club in ν.

Claim 3 Completes the proof of global square in κ. The proof that global square holds in κ is really a proof that global square holds in any universal weasel. (And universality was used only in the proof of condensation.)

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