The Laplacian for Spaces with Cone-Like Singularities

by

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Abstract: Let M be a compact manifold and S an embedded orientable hypersurface of M. We consider a one parameter family of metrics on M of the form $\mathcal{G}_t = dy^2 + f(t,y)g_M$ where $f(t,y) = (t^2+y^2)^{\frac{1}{2}}$ and y is a defining function for the hypersurface S. We remark that in the limit $t \longrightarrow 0$, $\mathcal{G}_t$ is a conic metric on $[M \setminus S]$. Let $\Delta_{\mathcal{G}_t} = d\delta_{\mathcal{G}_t} + \delta_{\mathcal{G}_t} d$ be the associated family of Laplace-Beltrami operators and $\mathcal{R}(\Delta_{\mathcal{G}_t} f^2, \lambda)$ the resolvent family. Our main result is the construction of an algebra of pseudodifferential operators which contains both $\Delta_{\mathcal{G}_t}$ and $\mathcal{R}(\Delta_{\mathcal{G}_t} f^2, \lambda)$ uniformly in t.

The construction follows the technique developed by Melrose to analyze degenerate elliptic boundary value problems. We define a Lie algebra of vector fields (on an appropriately "blown-up" parameter space X_s) whose enveloping algebra, $\mathrm{Diff}^*_s(X;\Omega^{\frac{1}{2}})$, contains the operator $\Delta_{\mathcal{G}_t}$ as an elliptic element. We then introduce an algebra of pseudodifferential operators which "microlocalizes" the ring $\mathrm{Diff}^*_s(X;\Omega^{\frac{1}{2}})$. Operators in the algebra are defined by reference to a class of distribution kernels on a product space constructed to model the geometry of the family $\Delta_{\mathcal{G}_t}$. We examine the basic properties of the algebra and introduce the notion of "uniformly finite rank" operators on the modified parameter space X_s. To establish the above result we produce a good paramatrix with error term of uniformly finite rank. The work concludes with an investigation of this error term in which we prove among other things that the error can be taken to be conormal.

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To Barbara.

Contents

1 Introduction

Let M be a compact n-manifold equipped with a Riemannian metric g_M and suppose S is an embedded orientable submanifold. The tubular neighborhood theorem establishes an isomorphism between an open neighborhood of S in M and a neighborhood of the zero section of the normal bundle NS of S. The metric g_M induces a metric g_F on the fibre of the normal bundle near S. In the sequel we will be interested in deformations of g_M depending on a parameter t. More specifically suppose ρ is a distance function for the fibres of NS induced by the introduction of normal coordinates. We will consider parametrized families of Riemannian structures which, near S, are of the form

$$\mathcal{G}_t = f^2(t,\rho)g_M + g_F \tag{1}$$

where $f(t,\rho) : [0,1) \times F \longrightarrow \mathbf{R}^+$ is chosen to model the passage from one metric structure (e.g. smooth) to a more singular metric structure (e.g. conic).

Of particular interest is the case of S a hypersurface. In this case M admits a defining function for S which we denote by y. We then consider deformations of the metric structure near S similar to (1). We write

$$G_t = f^2(t,y)g_M + dy^2 \tag{2}$$

where $f(t,y)$ is given by $f(t,y) = (t^2 + y^2)^{\frac{1}{2}}$. We proceed to analyze the resulting one-parameter family of Laplace-Beltrami operators using a pseudodifferential calculus constructed on the parameter space $X = [0,1) \times M$. The methods employed borrow heavily from Melrose's development of analysis on manifolds with corners.

For $t > 0$ fixed, G_t is a Riemannian metric on M. We denote the adjoint of the exterior differential d by δ_t and the associated Laplace-Beltrami opertor $d\delta_t + \delta_t d$ by Δ_{G_t}. As t varies we obtain a one parameter family of elliptic operators on M. We denote by $\mathcal{H}_t^k$ the harmonic k-forms on M with respect to the metric G_t and attempt to study in detail the singular limit $\lim_{t\to 0}\mathcal{H}_t^k$. To do so we construct a good approximation to the resolvent family for Δ_{G_t} where by good we mean with "uniformly finite rank" in the parameter t. Here "uniformly finite rank" refers to the range of the error term and is defined precisely in chapter 6. This notion replaces the classical notion of compactness and is central to both this thesis and future applications. We proceed by constructing an algebra of pseudodifferential operators containing Δ_{G_t} and a paramatrix for Δ_{G_t} in the above sense, i.e., an inverse modulo error of uniformly finite rank.

As a starting point we introduce the manifold with boundary $X = [0,1)_t \times M$ and consider the one parameter family of operators Δ_{G_t} as an operator along the fibres of X which we denote by Δ_G. As such it is elliptic in the fibre direction for $t > 0$ and at $t = 0$ is a degenerate elliptic operator in the sense of Melrose [7].

The first step in the construction of the desired family of pseudodifferential operators is the isolation of a family of vector fields, $\mathcal{V}_s$, on X tangent to the fibres of X and "dual" to the metric (1). This family is singular along the submanifold S at $t = 0$ and does not form a Lie algebra, a basic requirement of the Melrose program. To correct for this we introduce a space X_s which is obtained by "blowing-up" the submanifold S at $t = 0$. The space X_s is the natural domain for polar coordinates around the submanifold S. In particular, the function $f(t, y)$ is smooth in the natural C^∞-structure on X_s. Moreover, the C^∞-span of $\mathcal{V}_s$ becomes a Lie algebra and a $C^\infty(X)$ module with respect to the new structure. We denote by $\mathrm{Diff}_s^*(X)$ the enveloping algebra at $\mathcal{V}_s$ and note there is a bundle ${}^sTX$ satisfying $\mathcal{V}_s = C^\infty(X_s, {}^sTX)$. The dual bundle ${}^sT^*X$ and its exterior powers ${}^s\Lambda^k$ are the natural spaces upon which Δ_G acts.

To proceed with the microlocalization we construct an appropriate "s-stretched product space" as a domain for kernels which are to be associated to operators in the calculus. We note that operators in $\mathrm{Diff}_s^*(X)$ correspond to all δ-sections along the diagonal of the stretched product and obtain a calculus by replacing these δ-sections with arbitrary conormal sections. In addition we impose vanishing conditions at boundary faces other than those intersecting the diagonal. We then begin the construction of a paramatrix for Δ_G. We observe that the residual terms in the calculus $\Psi_s^{-\infty}(X)$ are not necessarily of uniform finite rank. To obtain errors which are of uniform finite rank we are forced to solve various intermediate model problems at the boundary hypersurfaces of X_s^2. The solution of these problems necessarily introduces kernels with nontrivial behavior at boundary faces other than those intersecting the diagonal submanifold. To account for this we modify the definition of the calculus to include kernels with conormal behavior along off-diagonal boundary faces. This behavior is completely desribed by an "exponent family". Exponent families are discrete subsets of $\mathbb{C} \times \mathbb{N}$ whose elements satisfy a growth requirement at infinity:

$$(z_l, m_l) \in E_i, \quad |(z_l, m_l)| \longrightarrow \infty \; \Rightarrow \Re(z_l) \longrightarrow \infty.$$

We associate one of these discrete sets to each boundary hypersurface of the s-stretched product space. For a fixed boundary hypersurface the elements of the exponent family corresponding to the fixed hypersurface represent the allowable exponents for a class of functions admitting expansions at the boundary hypersurface [see appendix A]. We then establish a composition theorem for this general calculus and investigate its basic properties. Finally, in Chapter 7 we prove our main result:

Theorem Let $\mathcal{R}(\Delta_G f^2; \lambda)$ be the resolvent family for $\Delta_G f^2$. Then $\mathcal{R}(\Delta_G f^2; \lambda)$ extends to be meromorphic as a family of operators

$$\mathbb{C} \setminus \mathbb{N}i \ni \lambda \;\longrightarrow\; \mathcal{R}(\Delta_G f^2; \lambda) \in \Psi_s^{-2}(X, \Omega^{\frac{1}{2}}) + \Psi_s^{-\infty;E}(X, \Omega^{\frac{1}{2}}) \tag{3}$$

with poles of finite rank.

The work concludes with an appendix whose contents are central to many of the proofs and/or motivating ideas. Apppendix A is a study of differential analysis on manifolds with corners and includes a discussion of the properties of conormal distributions which is of central importance. The material can be found in [3] and [8] and is included here to make the work essentially self-contained. In addition, the paper of Mazzeo and Melrose [5] should be referred to as a less complicated model for the central problem of this thesis: how to treat the degenerating parameter.

We remark upon our assumptions on S and $f(t,y)$:
S a hypersurface is in some sense the topologically most interesting case. Depending on the global geometry of M and S we have one of two possibilities: Either S disconnects M (e.g. $S^1 \subset S^2$) or S does not disconnect M (e.g. $S^1 \subset T = S^1 \times S^1$). That we choose $f(t,y)$ of the form $f(t,y) = (t^2+y^2)^{\frac{1}{2}}$ reflects the fact that we were initially interested in studying the passage from the smooth to the conic case. Replacing f by some higher power of the boundary defining function, say,

$$g = f^{\sigma}(t,y), \quad \sigma \geq 1$$

we can study cuspidal degenerations with a minimum of complications in the analysis. It is hoped that in the near future both these restrictions will be lifted and the general case dealt with.

Before we proceed, we mention a few possible future applications. The first application of the above results concerns the relationship of the behavior of the harmonic forms $\mathcal{H}_t^k$ as the parameter $t \longrightarrow 0$ and the "Mayer-Vietoris" sequence for the manifold M as the hypersurface S is collapsed to a point. Before elaborating further we recall several topological facts.

Let Y be a manifold with boundary, B. There is a long exact sequences in deRham Cohomolgy given by

$$\cdots \longrightarrow H^{k-1}_{dR}(B) \xrightarrow{\beta} H^k_{dR,R}(Y) \longrightarrow H^k_{dR,A}(Y) \longrightarrow H^k_{dR}(B) \longrightarrow \cdots \tag{4}$$

where the relative and absolute deRham spaces are given by

$$H^k_{dR,A}(Y) = \left[\mathrm{null}(d) \cap \mathcal{A}(Y,\Lambda^k)\right] / d\mathcal{A}(Y,\Lambda^{k-1}) \tag{5}$$

$$H^k_{dR,R}(Y) = \left[\mathrm{null}(d) \cap \dot{\mathcal{A}}(Y,\Lambda^k)\right] / d\dot{\mathcal{A}}(Y,\Lambda^{k-1}) \tag{6}$$

[Here $\mathcal{A}(Y)$ and $\dot{\mathcal{A}}(Y)$ denote the space of absolute and relative conormal distributions on the space Y.] The boundary map β is obtained using the wedge product :

$$\beta \;:\; H^{k-1}_{dR}(B) \ni \alpha \longrightarrow [u], \quad u = \delta(y)dy \wedge \alpha \in \dot{\mathcal{A}}(Y,\Lambda^k). \tag{7}$$

This sequence is often given by the more familiar sequence in singular cohomology for Y relative to B:

$$\cdots \longrightarrow H^{k-1}(B) \longrightarrow H^k(Y,B) \longrightarrow H^k(Y) \longrightarrow H^k(B) \longrightarrow \cdots \tag{8}$$

however our real interest is in the deRham complex.

Suppose $p \in B$ and fix local coordinates $y,\ x_1,\ \cdots\ x_{n-1}$ in a neighborhood of p where y is a boundary defining function. Consider the class of conic metrics on Y, i.e. those metrics locally of the form

$$g_\sigma \;=\; y^{2\sigma}\left(\frac{dy^2}{y^2}+dx^2\right)$$

and denote by Δ_σ the g_σ Laplacian: $\Delta_\sigma = d\delta_\sigma + \delta_\sigma d$. Associated to Y there is a natural Lie algebra of smooth vector fields, $\mathcal{V}_b$, consisting of those vector fields tangent to the boundary B. Associated to $\mathcal{V}_b$ there is a smooth vector bundle, ${}^bTY$, with dual bundle ${}^bT^*Y$. The kth-exterior product of the bundle ${}^bT^*Y$ is called the bundle of compressed k-forms and is denoted by ${}^b\Lambda^k$. The operator $y^{2\sigma}\Delta_\sigma$ acts on the bundle of compressed k-forms [see [8] for a detailed discussion of this topic]. We denote the null space of Δ_σ acting on compressed k-forms (the Kodaira-Hodge cohomology) by H^k_{KH}. Motivated in part by the Hodge theorem for compact manifolds, much has been done to establish a relationship between the Kodaira-Hodge groups and the deRham groups [1], [2], [7], [8]. In particular, we have [see [7]]:

Theorem If $\sigma > 0$ then

$$H^k_{KH} \longleftrightarrow H^k_{dR,R}, \quad \forall k \geq \frac{n+1}{2} \tag{9}$$

$$H^k_{KH} \longleftrightarrow \iota^*[H^k_{dR,R}] \subset H^k_{dR,A}, \quad k = \frac{n}{2} \tag{10}$$

$$H^k_{KH} \longleftrightarrow H^k_{dR,A}, \quad \forall k \leq \frac{n-1}{2}. \tag{11}$$

We propose to use the analysis of this thesis to examine the behavior of harmonic forms as one passes from the smooth case to the above conic case. To do so we let M be a manifold with an embedded oriented hypersurface S. Supose y is a defining function for S. As above, the submanifold S may or may not disconnect M. We assume for the moment that it does and form the subspaces M_+ and M_- as follows:

$$M_+ \;=\; \{m \in M;\ y(m) \geq 0\} \tag{12}$$

$$M_- \;=\; \{m \in M;\ y(m) \leq 0\} \tag{13}$$

We note that M_+ and M_- are manifolds with boundary given by S. By the above observation we have

$$\cdots \longrightarrow H^{k-1}(S) \longrightarrow H^k(M_+, S) \longrightarrow H^k(M_+) \longrightarrow H^k(S) \longrightarrow \cdots. \tag{14}$$

This gives a commutative diagram

$$\begin{array}{ccccccc}
H^{k-1}(M_+) & \longrightarrow & H^k(M, M_+) & \longrightarrow & H^k(M) & \longrightarrow & H^k(M_+) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{k-1}(S) & \longrightarrow & H^k(M_+, S) & \longrightarrow & H^k(M_+) & \longrightarrow & H^k(S)
\end{array}$$

By excision, the second vertical arrow in the diagram is an isomorphism. By a straightforward diagram chase we obtain the following long exact sequence

$$\cdots \longrightarrow H^{k-1}(S) \longrightarrow H^k(M_+) \oplus H^k(M_-) \longrightarrow H^k(M) \longrightarrow H^k(S) \longrightarrow \cdots \tag{15}$$

where the two maps are given by

$$H^k(M_+) \oplus H^k(M_-) \ni (\phi, \psi) \mapsto [\phi - \psi] \in H^k(M) \tag{16}$$
$$H^k(M) \ni \alpha \mapsto \alpha_{|S} \in H^k(S). \tag{17}$$

Now let $\mathcal{G}_t$ be a one parameter family of metrics on M as in (2). As the parameter t tends to zero we obtain conic metrics on the spaces $[M_+ \setminus S]$, $[M_- \setminus S]$. The above observation concerning the Kodaira-Hodge cohomology gives an explicit limit for the cohomology of the spaces $[M_+ \setminus S]$, $[M_- \setminus S]$. Using a paramatrix with uniform behavior behavior in the parameter we can actually describe how this limit takes place. Details will appear elsewhere.

This work began as an approach to studying intersection cohomology for stratified spaces. Borrowing from Hodge theory, the initial idea was to describe the intersection cohomology groups as the null space of an elliptic operator obtained as a "natural" limit of operators on a smooth version of the stratified space. This smooth version was simply given by an embedding of the space in some Euclidean space together with a family of tubular neighborhoods and corresponding projection and distance functions - a version of Thom-Mather data. Boundary conditions for the limiting operator were to reflect Goresky-MacPherson perversity conditions. Although the technical problems are considerable this program may yet be carried out using the analysis of this thesis as a foundation.

2 Tangent Algebra

2.1 Introduction

As in chapter one, we begin by fixing a compact manifold M equipped with a Riemannian metric g_M. Suppose S is an embedded orientable hypersurface . Let y is a defining function for S.

We form the one parameter family of metrics on M given by

$$\mathcal{G}_t \quad = \quad f^2(t,y)g_M + dy^2, \quad t \in [0,1) \tag{18}$$

where $f = (t^2+y^2)^{\frac{1}{2}}$. Let $\Delta_{\mathcal{G}_t}$ be the corresponding one parameter family of Laplace-Beltrami operators:

$$\Delta_{\mathcal{G}_t} = d\delta_{\mathcal{G}_t} + \delta_{\mathcal{G}_t} d \tag{19}$$

where $\delta_{\mathcal{G}_t}$ is the L^2 adjoint of d with respect to the metric $\mathcal{G}_t$.

We form the manifold with boundary $X = [0,1) \times M$. We can view the family of operators $\Delta_{\mathcal{G}_t}$ as an operator $\Delta_{\mathcal{G}}$ on the fibres of X depending on t as a parameter. In the future we will reserve the notation "$\Delta_{\mathcal{G}_t}$" for the operator obtained by fixing the parameter at the value t. Our strategy for extending $\Delta_{\mathcal{G}}$ to the boundary is to invert the operators $\Delta_{\mathcal{G}_t}$ uniformly in the parameter t. Our technique hinges on being able to define a differential structure on X in which the operator $\Delta_{\mathcal{G}}$ is in some sense elliptic. The approach we use is that of Melrose in his investigation of degenerate elliptic boundary value problems. We begin by defining a family of vector fields on X:

$$\mathcal{V} = \text{span}\{f\partial_y, \partial_{\theta_i}\}$$

where ∂_{θ_i} span TS locally, f is as before and the span is taken over all C^∞ functions on X. Unfortunately, $\mathcal{V}$ does not form a Lie algebra, a basic requirement for the construction of a microlocalizing algebra. To adjust for this shortcoming we introduce a manifold with corners for which polar coordinates in (y,t) are admissable local coordinates (in particular, f will be smooth). The construction is an example of "blowing up" a manifold along a submanifold. We define a submanifold $Q \subset \partial X$ by setting $Q = \{t = y = 0\}$. We define a new space X_s by

$$X_s \quad = \quad (X)_Q = [X \setminus Q] \bigsqcup S^+NQ \tag{20}$$

where S^+NQ is the inward pointing spherical normal bundle to Q:

$$^+N_pQ = T_p^+X/T_pQ$$

Quotienting out by the $\mathbf{R}^+$ action we obtain the spherical normal fibre

$$S^+N_pQ = \left(T_p^+X/T_pQ\backslash\{0\}\right)\backslash\mathbf{R}^+$$

and finally

$$S^+NQ = \bigsqcup S^+N_pQ.$$

The fact that Q is an embedded submanifold of the boundary insures that the space X_s is a manifold with corners equipped with a C^∞ structure such that the blow-down map $\beta : X_s \longrightarrow X$ is smooth. In fact, β is a diffeomorphism everywhere with the exception of the new boundary component $ff(X_s)$ where it has rank $n-1$. Depending on the global geometry of S there will be either one or two other boundary components. We assume that there are two which we label $fr(X_s)$ and $fl(X_s)$. We write

$$\mathcal{V}_s \;=\; \text{span}\{f\partial_y,\ \partial_{\theta_i}\} \tag{21}$$

where the span is now taken over $C^\infty(X_s)$. Then $\mathcal{V}_s$ is a Lie algebra of C^∞-vector fields on X_s. We have

Proposition 2.1 *There is a C^∞ bundle, ${}^sTX$, over X_s such that the vector fields $\mathcal{V}_s$ are exactly the smooth sections of ${}^sTX$:*

$$\mathcal{V}_s \;=\; C^\infty(X_s, {}^sTX). \tag{22}$$

PROOF Recalling the definition of $\mathcal{V}_s$ we see that a local basis is given by $f\partial_y,\ \partial_{\theta_i}$. This fixes the vector bundle structure on ${}^sTX$. To show that ${}^sTX$ is C^∞ it is then a simple exercise to to see that the coefficients in the local coordinate representation transform properly under a smooth change of coordinates. Recall, $\mathcal{V}_s$ is given by vector fields locally of the form

$$V = a(r,\omega,\theta)r\partial_y + \sum_{i=1}^{n-1} b_i(r,\omega,\theta)\partial_{\theta_i} \tag{23}$$

where the variables $r,\ \omega,\ \theta$ on X_s are given by

$$r = \left(y^2+t^2\right)^{\frac{1}{2}} \tag{24}$$

$$\omega = \arctan(\frac{y}{t}) \tag{25}$$

$$\theta = (\theta_1, \theta_2, \ldots, \theta_n) \tag{26}$$

and the coefficeints $a,\ b_i$ are smooth. Let

$$r' = \phi_1(r,\omega,\theta)$$
$$\omega' = \phi_2(r,\omega,\theta)$$
$$\theta' = \phi_3(r,\omega,\theta)$$

be a smooth change of coordinates. Then we can arrange to have

$$r' = r\alpha_1(r, \omega, \theta)$$

with $\alpha_1 > 0$ and the claim follows.

The Lie algebra structure of $\mathcal{V}_s$ allows us to consider the universal enveloping algebra of $\mathcal{V}_s$ which we denote by $\mathrm{Diff}_s^*(X)$, the s-differential operators. Our goal is to microlocalize the ring $\mathrm{Diff}_s^*(X)$.

2.2 Laplace-Beltrami Operators on s-Bundles

The dual bundle of ${}^sTX$, ${}^sT^*X$, will play a central role in the discussion that follows. Locally, ${}^sT^*X$ is spanned by

$$\frac{dy}{f},\ d\theta_1,\ \dots,\ d\theta_{n-1}.$$

Taking the the exterior powers of ${}^sT^*X$ we obtain the s-k-forms ${}^s\Lambda^k$ and note that the exterior differential lifts from M to X_s where we have

$$\begin{aligned} d\left(a(r,\omega,\theta)\frac{dy}{f}^I \wedge d\theta^J\right) &= [(\partial_r a)y - (\partial_\omega a)sin\omega]\,\frac{dy}{f} \wedge \left(\frac{dy}{f}\right)^I \wedge d\theta^J + \\ &\quad +(-1)^{|\alpha|}(\partial_\theta a)\left(\frac{dy}{f}\right)^I \wedge d\theta \wedge d\theta^J. \end{aligned} \tag{27}$$

In particular we note

Lemma 2.1 *For $k \in \mathbb{N}$,*

$$d \in Diff_s^1(X,\ {}^s\Lambda^k,\ {}^s\Lambda^{k+1}).$$

□

We wish to describe the (induced) action of $\Delta_{\mathcal{G}_t}$ on the various s-form bundles. We begin by replacing the metric in (18) with a model metric:

$$\tilde{g}_t = f^2\left(\left(\frac{dy}{f}\right)^2 + g_S\right) \tag{28}$$

where g_S is a metric on S. This is a metric of the form

$$G_\sigma = f^{2\sigma} g_t$$

where $g_t = \left(\frac{dy}{f}\right)^2 + g_S$. For metrics of the form G_σ we have

$$<\,,\,>_{G_\sigma} = f^{-2\sigma k} <\,,\,>_{g_t}$$

on k-forms. The metric densities are related by $dG_\sigma = f^{\sigma n} dg_t$. To compute the adjoint of d for the metric G_σ we use duality. For k-forms $d\alpha,\ \beta$ we have:

$$\begin{aligned}
\int_M < d\alpha, \beta >_{G_\sigma} d_{G_\sigma} &= \int_M < d\alpha, \beta >_{g_t} f^{-2\sigma k} f^{\sigma n} dg_t \\
&= \int_M < d\alpha, f^{\sigma(n-2k)} \beta >_{g_t} \\
&= \int_M < \alpha, \delta_{g_t} f^{\sigma(n-2k)} \beta >_{g_t} \\
&= \int_M < \alpha, \delta_{g_t} f^{\sigma(n-2k)} \beta >_{G_\sigma} f^{2\sigma(k-1)} f^{-\sigma n} d_{G_\sigma} \\
&= \int_M < \alpha, f^{-\sigma(n-2(k-1))} \delta_{g_t} f^{\sigma(n-2k)} \beta >_{G_\sigma} d_{G_\sigma}.
\end{aligned}$$

Hence, we see that

$$\delta_{G_\sigma} = f^{-\sigma(n-2k+2)} \delta_{g_t} f^{\sigma(n-2k)}. \tag{29}$$

We note that

$$\begin{aligned}
\delta_{g_t} f^\alpha &= f^\alpha \delta_{g_t} + \alpha f^{\alpha-1} \iota_{\nabla f} \\
d f^\alpha &= f^\alpha d + \alpha f^{\alpha-1} y \frac{dy}{f} \wedge
\end{aligned}$$

where the gradient is taken with respect to the metric g_t. Explicitly, $\nabla f = y f \partial_y$. The G_σ Laplacian is therefore given by

$$\begin{aligned}
\Delta_{G_\sigma} &= d\delta_{G_\sigma} + \delta_{G_\sigma} d \\
&= d\left(f^{-\sigma(n-2k+2)} \delta_{g_t} f^{\sigma(n-2k)}\right) + \left(f^{-\sigma(n-2k+2)} \delta_{g_t} f^{\sigma(n-2k)}\right) d \\
&= d\left(f^{-\sigma(n-2k+2)} (f^{\sigma(n-2k)} \delta_{g_t} + \sigma(n-2k) f^{\sigma(n-2k)-1} \iota_{\nabla f})\right) + \\
&\quad f^{-\sigma(n-2k+2)} \left(f^{\sigma(n-2k)} \delta_{g_t} + \sigma(n-2k) f^{\sigma(n-2k)-1} \iota_{\nabla f})\right) d \\
&= d\left(f^{-2\sigma} \delta_{g_t} + \sigma(n-2k) f^{-2\sigma-1} \iota_{\nabla f}\right) + \\
&\quad f^{-2\sigma} \delta_{g_t} d + \sigma(n-2k) f^{\sigma(n-2k)-1} \iota_{\nabla f} d \\
&= f^{-2\sigma}\left(\Delta_{g_t} + \sigma(n-2k) f^{-1} \mathcal{L}_{\nabla f} - 2\sigma y f^{-1} \frac{dy}{f} \wedge \delta_{g_t} - 3(n-2k) y f^{-2} \frac{dy}{f} \wedge \iota_{\nabla f}\right).
\end{aligned}$$

where $\mathcal{L}_{\nabla f}$ denotes the Lie derivative with respect to the vector field ∇f and and ι is interior multiplication with respect to the metric g_t. We are particularly interested in the case $\sigma = 1$:

$$\Delta_G = f^{-2}\left(\Delta_{g_t} + (n-2k)f^{-1}\mathcal{L}_{\nabla f} - 2yf^{-1}\frac{dy}{f}\wedge\delta_{g_t} - 3(n-2k)yf^{-2}\frac{dy}{f}\wedge\iota_{\nabla f}\right). \quad (30)$$

To compute the action of $f^2\Delta_G$ we first note that we have a decomposition of the various s-form bundles near the boundary:

$$^s\Lambda^k = {}^s\Lambda^{0,k} + {}^s\Lambda^{1,k-1} \quad (31)$$

where $^s\Lambda^{0,k}$ and $^s\Lambda^{1,k-1}$ are given by reference to local coordinates:

$$\begin{aligned} a(r,\omega,\theta)d\theta^J &\in {}^s\Lambda^{0,k} \quad |J| = k \\ a(r,\omega,\theta)\frac{dy}{f}\wedge d\theta^J &\in {}^s\Lambda^{1,k-1} \quad |J| = k-1. \end{aligned}$$

The exterior derivative has a decomposition relative to the above decomposition of s-forms:

$$d = d_{nor} + \tau d_{tan} \quad (32)$$

where τ is multiplication by $(-1)^i$ on $^s\Lambda^{i,j}$ and the action of d on each component may be read off from (27). The adjoint of d with respect to the metric g_t is then given by

$$\delta_{g_t} = *^{-1}d*\tau \quad (33)$$

where the Hodge star operator decomposes as

$$\begin{aligned} * &= *_{nor}*_{tan}\tau = *_{tan}*_{nor}\tau \\ *_{nor} &: \Lambda^i \longrightarrow \Lambda^{1-i} \\ *_{tan} &: \Lambda^i \longrightarrow \Lambda^{n-1-i}. \end{aligned}$$

The action of each piece of the Hodge star operator is given by

$$*_{nor}\left[a(r,\omega,\theta)\left(\frac{dy}{f}\right)^I\wedge d\theta^J\right] = *_{nor}\left[a(r,\omega,\theta)\left(\frac{dy}{f}\right)^I\right]\wedge d\theta^J \quad (34)$$

$$*_{tan}\left[a(r,\omega,\theta)\left(\frac{dy}{f}\right)^I\wedge d\theta^J\right] = \left(\frac{dy}{f}\right)^I\wedge *_{tan}\left[a(r,\omega,\theta)d\theta^J\right] \quad (35)$$

where

$$*_{nor} : 1 \mapsto \frac{dy}{f} \quad (36)$$

$$*_{nor} : \frac{dy}{f} \mapsto -1. \quad (37)$$

Note that d_{nor} and $*_{nor}$ commute with d_{tan} and $*_{tan}$. Hence,

$$\begin{aligned}
\delta_{g_t} &= *_{nor}^{-1} d_{nor} *_{nor} + *_{tan}^{-1} d_{tan} *_{tan} \\
&= *_{nor}^{-1} d_{nor} *_{nor} + \delta_S.
\end{aligned} \tag{38}$$

The Laplacian on k-forms is then given by

$$\begin{aligned}
\Delta_{g_t} &= d\delta_{g_t} + \delta_{g_t} d \\
&= [d_{nor} + \tau d_{tan}][*_{nor}^{-1} d_{nor} *_{nor} + \delta_S] + [*_{nor}^{-1} d_{nor} *_{nor} + \delta_S][d_{nor} + \tau d_{tan}] \\
&= \Delta_S + (d_{nor}\delta_{nor} + \delta_{nor} d_{nor})
\end{aligned}$$

where the cross terms cancel or vanish identically. On forms of type ${}^s\Lambda^{0,k}$, $\delta_{nor} \equiv 0$ and we have

$$\begin{aligned}
\Delta_{g_t} &= \delta_{nor} d_{nor} + \Delta_S \\
&= f\partial_y (f\partial_y) + \Delta_S \\
&= -(f\partial_y)^2 + \Delta_S.
\end{aligned} \tag{39}$$

On forms of type ${}^s\Lambda^{1,k-1}$, $d_{nor} \equiv 0$ and we have

$$\begin{aligned}
\Delta_{g_t} &= d_{nor}\delta_{nor} + \Delta_S \\
&= -(f\partial_y)^2 + \Delta_S
\end{aligned} \tag{40}$$

where the operator $(f\partial_y)^2$ is applied to the coefficient of the form. In addition, we are now able to compute the action of $f^2\Delta_G$ on the s-form bundles. We write

$$\begin{aligned}
f^2\Delta_G &= \Delta_{g_t} + \sigma(n-2k)f^{-1}\mathcal{L}_{\nabla f} - 2\sigma y f^{-1}\frac{dy}{f} \wedge \delta_{g_t} - 3(n-2k)yf^{-2}\frac{dy}{f} \wedge \iota_{\nabla f} \\
&= T_1 + T_2 + T_3 + T_4.
\end{aligned} \tag{41}$$

On forms of type ${}^s\Lambda^{0,k}$ with $\omega = a\,d\theta^J$ we have

$$\begin{aligned}
T_2\omega &= (n-2k)(y\partial_y a)d\theta^J \\
T_3\omega &= 0 \\
T_4\omega &= 0.
\end{aligned}$$

On forms of type ${}^s\Lambda^{1,k-1}$ with $\omega = a\left(\frac{dy}{f}\right) \wedge d\theta^J$ we have

$$\begin{aligned}
T_2\omega &= (n-2k)(y\partial_y a)\left(\frac{dy}{f}\right) \wedge d\theta^J + f^{-1}y\sum(\partial_{\theta_i} a)\, d\theta_i \wedge d\theta^J + \omega \\
T_3\omega &= -2(y\partial_y a)\left(\frac{dy}{f}\right) \wedge d\theta^J \\
T_4\omega &= -3(n-2k)f^{-2}y^2\omega.
\end{aligned}$$

Summarizing, on forms of type ${}^s\Lambda^{0,k}$ with $\omega = ad\theta^J$ we have

$$f^2\Delta_G\omega = \Delta_{g_t}\omega + (n-2k)(y\partial_y a)d\theta^J. \tag{42}$$

On forms of type ${}^s\Lambda^{1,k-1}$ with $\omega = a\left(\frac{dy}{f}\right)\wedge d\theta^J$ we have

$$\begin{aligned} f^2\Delta_G\omega &= \Delta_{g_t}\omega + \omega + (n-2k)(y\partial_y a)\left(\frac{dy}{f}\right)\wedge d\theta^J + f^{-1}y\sum(\partial_{\theta_i}a)\,d\theta_i\wedge d\theta^J \\ &\quad -2(y\partial_y a)\left(\frac{dy}{f}\right)\wedge d\theta^J - 3(n-2k)f^{-2}y^2\omega. \end{aligned} \tag{43}$$

From (42) and (43) it follows easily that:

Lemma 2.2 *For every* $k\in\mathbb{N}$,

$$f^2\Delta_G \in Diff_s^2(X,\ {}^s\Lambda^k).$$

□

2.3 s-Algebra, Model Problems and Boundary Spectrum

Next we study in detail the structure of the bundle ${}^sTX$ near the front face of X_s. We begin by noting that the front face of X_s is an integral submanifold for the Lie algebra $\mathcal{V}_s$. We set

$$\mathcal{I}_{ff} = \{h\in C^\infty(X_s);\ h\equiv 0 \text{ on } ff(X_s)\}. \tag{44}$$

Note that the elements of $\mathcal{V}_s$ are tangent to $ff(X_s)$ and hence $\mathcal{I}_{ff}\cdot\mathcal{V}_s$ is an ideal in $\mathcal{V}_s$. The quotient

$$\mathcal{W}_{ff(X_s)} = \mathcal{V}_s/\mathcal{I}_{ff}\cdot\mathcal{V}_s$$

is therefore a Lie algebra. The exact sequence of Lie algebra homomorphisms

$$0\longrightarrow \mathcal{I}_{ff}\cdot\mathcal{V}_s\longrightarrow\ \mathcal{V}_s\ \xrightarrow{N_1}\mathcal{W}_{ff(X_s)}\longrightarrow 0. \tag{45}$$

lifts to enveloping algebras to give, for every m, an exact sequence

$$0\longrightarrow \mathcal{I}_{ff}\cdot \mathrm{Diff}_s^m(X)\longrightarrow\ \mathrm{Diff}_s^m(X)\ \xrightarrow{N_1}\mathcal{D}^m(\mathcal{W}_{ff(X_s)})\longrightarrow 0. \tag{46}$$

Denote by $\mathcal{V}_{ff(X_s)}$ the restriction of $\mathcal{V}_s$ to the front face of X_s. We then have a map

$$\mathcal{W}_{ff(X_s)}\longrightarrow\mathcal{V}_{ff(X_s)}$$

given by $[V] \longrightarrow V_{|ff(X_s)}$. This is well defined since if $[V] = [V']$, we have $V - V' \in \mathcal{I}_{ff} \cdot \mathcal{V}_s$ and hence $(V - V')_{|ff(X_s)} = 0$. In fact, this map is an isomorphism.

We define the normal operator as the map

$$N : \mathrm{Diff}_s^m(X) \longrightarrow \mathrm{Diff}^m(ff(X_s)) \tag{47}$$

obtained by lifting the map N_1 to enveloping algebras.

As we will see, the normal map defines the model problem whose solution is the key to finding the appropriate extension of $\Delta_{\mathcal{G}_t}$.

We now develop a local model which realizes the normal operator. Let M_δ denote the coordinate dilation map

$$M_\delta \quad : \quad (t, y, \theta) \longrightarrow (\delta t, \delta y, \theta).$$

For $V \in \mathcal{V}_s$, define

$$(M_\delta^{-1})_* V \quad = \quad V_\delta.$$

V_δ is a vector field defined on a neighborhood of the interior of $ff(X_s)$. As $\delta \longrightarrow 0$ this neighborhood increases and we obtain a smooth vector field on $T_q^+ X$, the inward pointing tangent space to X_s at q:

$$\lim_{\delta \to 0} V_\delta \quad = \quad V'.$$

Fixing $q \in ff(X_s)$, this procedure defines a map

$$V \longrightarrow V'.$$

If the image of V under the above map is zero, then given the local coordinate representation of V in (23) and the formula for M_δ, we see that V must vanish in a boundary neighborhood of q. Hence, we have a local realization of the map N_1.

We are particularly interested in the normal operator of $f^2 \Delta_G$. Note that since f gives a defining function for $ff(X_s)$ [see (24)], the metric (28) restricts to zero at $ff(X_s)$. We can however induce a metric on $ff(X_s)$ by examining the second order behvior of the metric as we approach points belonging to the interior of $ff(X_s)$. We have

Proposition 2.2 *The metric g_t induces a complete conic metric, $\tilde{g}$, on the front face of X_s. The normal operator for the induced Laplace-Beltrami operator is the Laplace-Beltrami operator in the induced metric modulo lower order terms:*

$$N(\Delta_{g_t}) = \Delta_{\tilde{g}}. \tag{48}$$

PROOF In the interior of $ff(X_s)$ we have projective coordinates

$$t,\ \phi = \frac{y}{t},\ \theta_i\ 1 \leq i \leq n-1. \tag{49}$$

We note that $dy = td\phi + \phi dt$ and that the vector ∂_ϕ is tangent to $ff(X_s)$. The length of ∂_ϕ in the induced metric is then given by

$$|\partial_\phi|^2 = \frac{1}{1+\phi^2}$$

and the first claim follows. Using the formulas (39) and (40) for the action of the g_t Laplacian on k-forms we see that

$$N(f\partial_y) = \left(1+\phi^2\right)^{\frac{1}{2}} \partial_\phi \tag{50}$$

and hence

$$N(\Delta_{g_t}) = \left[\left(1+\phi^2\right)^{\frac{1}{2}} \partial_\phi\right]^2 + N(\Delta_S). \tag{51}$$

This is the highest order term of the Laplacian on $ff(X_s)$ for the metric $\tilde{g} = \frac{d\phi^2}{1+\phi^2} + g_S$ and the proposition follows.

□

The following proposition establishes an equivalence between the model problems for metrics of the form (18) and metrics of the form (28).

Proposition 2.3 *Let* $\mathcal{G} = f^2\left(\frac{dy^2}{f^2} + g_M\right)$ *and denote by* $\Delta_{\mathcal{G}}$ *the corresponding family of Laplace-Beltrami operators. Then there is a family metrics,* $G = f^2\left(\frac{dy^2}{f^2} + g_S\right)$, *(where* g_S *is a metric on* S*) and corresponding family of Laplace-Beltrami operators* Δ_G *satisfying*

$$N(f^2\Delta_{\mathcal{G}}) = N(f^2\Delta_G).$$

PROOF The metric g_M induces a metric on S by restriction. Denote this metric by $\tilde{g}_S$. Consider the family

$$G = f^2\left(\frac{dy^2}{f^2} + \tilde{g}_S\right)$$

where f is as before. The metric densities, $d\mathcal{G}$, dG are then related by

$$d\mathcal{G} = \left(1 + fh + O(f^2)\right) dG$$

where h is smooth on M. Hence for k-forms $d\alpha$, β

$$\int < d\alpha, \beta >_{\mathcal{G}} d\mathcal{G} \;\; = \;\; \int < \alpha, \delta_G \beta >_{\mathcal{G}} d\mathcal{G} + \int (< d\alpha, \beta >_{fH} - < \alpha, \delta_G \beta >_{fH}) \, d\mathcal{G} \quad (52)$$

where δ_G is the G-adjoint of d. Hence, we write $\delta_{\mathcal{G}} = \delta_G + P$ where P corresponds to the second term in the expression (52). We note that the inner product with respect to fH vanishes at $ff(X_s)$ and hence $N(P) = 0$. Hence $N(\delta_{\mathcal{G}}) = N(\delta_G)$. Since the exterior differential does not depend on the metric, we obtain

$$N(f^2 \Delta_{\mathcal{G}}) = N(f^2 \Delta_G).$$

□

Following the above procedure with the obvious substitutions we can restrict operators in $\mathrm{Diff}^m_s(X; \Omega^{\frac{1}{2}})$ and define a normal operator at the side face $fr(X_s)$. Here the results are much simpler.

Proposition 2.4 *The metric G induces an incomplete conic metric on $fr(X_s)$. The normal operator of the Laplace-Beltrami operator, denoted by $I_{fr}(\Delta_G)$, is the Laplace-Beltrami operator for the induced metric. We write*

$$I_{fr}(\Delta_G) = \Delta_{g_{ic}}.$$

PROOF For $q \in fr(X_s)$, q in the interior, we have coordinates near q given by

$$f, \;\; T = \frac{t}{f}, \;\; \theta \qquad (53)$$

where $fr(X_s)$ is defined by $T \equiv 0$ and the boundary $\partial fr(X_s) = fr(X_s) \cap ff(X_s)$ is given by $f = T = 0$. The vector ∂_y is tangent to $fr(X_s)$ and has length $|\partial_y|^2 = 1$. Similarly, ∂_θ is tangent to $fr(X_s)$ and has length $|\partial_\theta|^2 = y^2$ proving the first claim. To prove the second claim we examine the normal operator of d, the exterior differential acting on the s-forms. From the formula (27) we see immediately that the normal operator of the exterior differential is the exterior differential on the b-compressed bundles for $fr(X_s)$. Similarly, the normal operator of the adjoint is seen to be the adjoint for d on the b-compressed bundles with respect to the induced metric. Hence, the second claim follows:

$$I_{fr}(\Delta_G) = \Delta_{g_{ic}}$$

where the subscript ic refers to the incomplete conic metric.

□

This proposition comes as no surprise as we set out to build the conic problem into the algebra $\mathcal{V}_s$ as part of the limit as $t \longrightarrow 0$. Clearly, the same argument applies to the boundary face $fl(X_s)$ and hence the normal operator at the face $fl(X_s)$ is also an incomplete conic Laplacian. Of immediate interest is the connection between the two incomplete operators as one approaches the boundaries $fl(X_s) \cap ff(X_s)$ and $fr(X_s) \cap ff(X_s)$. To further investigate this relationship we associate to $ff(X_s)$ a family of operators, the indicial family at $ff(X_s)$ which we denote by $I_{ff(X_s)}(f^2\Delta_G)(z)$. To define $I_{ff(X_s)}(f^2\Delta_G)(z)$ we fix a defining function ρ_{ff} for $ff(X_s)$ and set

$$I_{ff(X_s)}(f^2\Delta_G)(z) = \left[\rho_{ff}^{-iz}(f^2\Delta_G)\rho_{ff}^{iz}\right]_{|ff(X_s)}. \tag{54}$$

Recall, near the interior of $ff(X_s)$ we have projective coordinates (49) with f a defining function for $ff(X_s)$. Hence,

$$I_{ff(X_s)}(f^2\Delta_G)(z) = \left[f^{-iz}(f^2\Delta_G)f^{iz}\right]_{|ff(X_s)}$$

From the decomposition (41) to compute the indicial family for $f^2\Delta_G$ we need only compute the conjugates of the operators T_i. We begin by noting that $[T_3, f^{iz}] = [T_4, f^{iz}] = 0$. Recall, $T_1 = \Delta_{g_t} = -(f\partial_y)^2 + \Delta_S$. Hence,

$$\begin{aligned} T_1 f^{iz} &= -(f\partial_y)^2 f^{iz} + f^{iz}\Delta_S \\ &= -f^{iz}\left(f\partial_y + iz\frac{y}{f}\right)^2 + f^{iz}\Delta_S \\ &= f^{iz}\left[T_1 + z^2\left(\frac{y^2}{f^2}\right) - 2izy\partial_y - iz(1 - \left(\frac{y^2}{f^2}\right))\right]. \end{aligned}$$

Finally, $T_2 = (n-2k)f^{-1}\mathcal{L}_{\nabla f}$ and

$$\begin{aligned} T_2 f^{iz} &= f^{iz}T_2 + (n-2k)f^{-1}(\mathcal{L}_{\nabla f} f^{iz}) \\ &= f^{iz}\left(T_2 + (n-2k)iz\frac{y^2}{f^2}\right). \end{aligned}$$

Summarizi

$$\Delta_G f^{iz} = f^{iz}\Delta_G + f^{iz-2}\left[z^2\left(\frac{y^2}{f^2}\right) - 2izy\partial_y - iz(1 - \left(\frac{y^2}{f^2}\right)) + (n-2k)iz\frac{y^2}{f^2}\right]. \tag{55}$$

In particular,

$$\Delta_G f^2 = f^2\Delta_G + -4y\partial_y - 2((1 + \left(\frac{y^2}{f^2}\right)) + (n-2k)\frac{y^2}{f^2}. \tag{56}$$

We can also define the indicial family at $fr(X_s)$. Near $q \in fr(X_s)$, q in the interior of $fr(X_s)$, we have coordinates given by (53) with $fr(X_s)$ defined by $T = 0$ and the boundary $\partial fr(X_s) = fr(X_s) \cap ff(X_s)$ given by $T = f = 0$. The indicial family at the face $fr(X_s)$ is then given by

$$\begin{aligned} I_{fr(X_s)}(f^2\Delta_G)(s) &= \left[\left(\frac{t}{f}\right)^{-is} f^2\Delta_G \left(\frac{t}{f}\right)^{is}\right]_{|fr(X_s)} \\ &= \left[f^{is}(t^{-is}f^2\Delta_G t^{is})f^{-is}\right]_{|fr(X_s)}. \end{aligned}$$

The two indicial operators admit a simple relation. We define the indicial family along the intersection of the boundary faces $ff(X_s) \cap fr(X_s)$ by

$$I_{ff(X_s)\cap fr(X_s)}(f^2\Delta_G)(\theta, \partial_\theta, s, z) = \left[f^{-iz}\left(\frac{t}{f}\right)^{-is}(f^2\Delta_G)\left(\frac{t}{f}\right)^{is} f^{iz}\right]_{|ff(X_s)\cap fr(X_s)}. \quad (57)$$

The dependence of $I_{ff(X_s)\cap fr(X_s)}(f^2\Delta_G)(\theta, \partial_\theta, s, z)$ on the exponent for t is trivial and we see that $I_{ff(X_s)\cap fr(X_s)}$ depends only on the difference $s - z$:

$$I_{ff(X_s)\cap fr(X_s)}(\theta, s, z) = I_{ff(X_s)\cap fr(X_s)}(\theta, s - z).$$

We note that by setting $z = 0$ we obtain the indicial family $I_{fr(X_s)}(s)$ and similarly, setting $s = 0$ we recover the indicial family $I_{ff(X_s)}(-z)$. In particular this indicates that the indicial roots for the model problem at $ff(X_s)$ are given as the indicial roots for the model problem at $fr(X_s)$ with a sign change. The indicial roots for the model problem at $fr(X_s)$ are well known [7]. Here we simply record the result: For Δ acting on ${}^b\Lambda^k$, $spec_b(y^2\Delta)$ consists of the eight sequences

$$z = i\left[\left(k - \tfrac{n}{2} + 1\right) \pm \sqrt{\left(k - \tfrac{n}{2}\right)^2 + \mu_{k,d}^2} \pm 1\right] \quad (58)$$

$$z = i\left[\left(k - \tfrac{n}{2} + 1\right) \pm \sqrt{\left(k - \tfrac{n}{2} + 1\right)^2 + \mu_{k,\delta}^2}\right] \quad (59)$$

$$z = i\left[\left(k - \tfrac{n}{2} + 1\right) \pm \sqrt{\left(k - \tfrac{n}{2} + 1\right)^2 + \mu_{k-1,d}^2}\right] \quad (60)$$

where $\mu_{k,d}^2$ and $\mu_{k,\delta}^2$ are respectively the eigenvalues of the tangential Laplacian on k-forms which are closed and exact. These sequences describe the asymptotic behavior of formally harmonic forms near the boundary $ff(X_s) \cap fr(X_s)$. As indicated above, the indicial roots for the model problem at $ff(X_s)$ can be obtained from the indicial roots at the side face $fr(X_s)$ (i.e., from the above sequences (58)-(59)) by a change of sign. This reflects the obsevation made in Proposition 2.2 and Proposition 2.4 that the metric for the model problem at $fr(X_s)$ changes from an incomplete conic metric to a complete conic metric as one passes to the model problem on $ff(X_s)$. In fact the induced metric on $ff(X_s)$ is obtained by examining the second order behavior of the incomplete metric near the boundary. This establishes a stronger relationship between the model problems which sharpens the conclusion of Proposition 2.2. We have

Proposition 2.5 *The operator induced at the front face of X_s is the Laplacian in the induced metric.*

$$N(f^2\Delta_G) = \Delta_{\tilde{g}}.$$

□

In particular, the normal operator at $ff(X_s)$ is a complete conic Laplacian with solutions whose boundary behavior is completely specified by (58)-(60).

3 Stretched Product

3.1 Introduction

Recall that for Y a compact manifold the Schwartz kernel theorem associates to every continuous linear operator $A : C^\infty(Y) \longrightarrow C^{-\infty}(Y)$ a distribution, $K(A)$, on the product manifold $Y^2 = Y \times Y$. Pseudodifferential operators on Y are then identified with operators whose kernels are smooth off the diagonal $\Delta \subset Y^2$ with conormal singularity along Δ.

In the following we modify this procedure to obtain an algebra of operators on X which restricts to an algebra on ∂X and includes the Laplace-Beltrami operator $f^2\Delta_G$. The first step in this modification is to replace the Cartesian product X^2 with a space whose boundary structure models the geometry of $f^2\Delta_G$.

We begin by observing that the variable t is to be treated as a parameter. Our first step then is to introduce the product

$$\mathcal{P}_2 = [0,1)_t \times M \times M = X \times M$$

We single out the submanifold $Q \subset \partial\mathcal{P}_2$ defined by the set of equations

$$Q \;=\; \{t = y^1 = y^2 = 0\} \subset \partial\mathcal{P}_2. \tag{61}$$

Next we pass from the space $\mathcal{P}_2$ to an appropriately blown up space. Our choice of blow up is governed by the following considerations: First, The blown up space should allow us to treat singularities ocurring at the boundary of $\mathcal{P}_2$ and singularities occuring along the diagonal

$$\Delta = \{(t,p,p) \in \mathcal{P}_2\}$$

seperately. In addition, we want the blow down map to factor smoothly through the space $X_s \times M$ and there to be a lifting of the family of vector fields $\mathcal{V}_s$ such that the lifted family of vector fields is transverse to the lifted diagonal Δ_ν. As we shall see such a space does exist.

3.2 Construction of the Stretched Product X_s^2

We begin by defining several embedded submanifolds of $\partial\mathcal{P}_2$. Let $Q \subset \partial\mathcal{P}_2$ be defined as in (61) above. Define

$$S_1 = \{t = y^1 = 0\}$$
$$S_2 = \{t = y^2 = 0\}.$$

Our next step in the construction is to form the blown-up space

$$Z = (\mathcal{P}_2)_Q = [\mathcal{P}_2 \setminus Q] \bigsqcup SN^+Q$$

where SN^+Q is the inward pointing spherical normal bundle of Q. The fibres of SN^+Q are obtained as before: For $q \in Q$, the normal fibre at q is

$$N_q^+Q = T_q^+\mathcal{P}_2/T_qQ.$$

Quotienting out the $\mathbf{R}^+$ action, we obtain

$$SN_q^+Q = \left(T_q^+\mathcal{P}_2/T_qQ\backslash\{0\}\right)/\mathbf{R}^+.$$

The new boundary face, SN^+Q, has the structure of a half sphere bundle over the submanifold Q :

$$\begin{array}{ccc} S_+^2 & \longrightarrow & SN^+Q \\ & & \downarrow \\ & & Q \end{array}$$

The space Z is a manifold with corners equipped with a smooth blow-down map

$$\beta_1 : Z \longrightarrow \mathcal{P}_2.$$

The lifted diagonal,

$$\Delta_\nu = \text{clos}(\beta_1^{-1}(\Delta \setminus Q)),$$

is an embedded submanifold of Z. The lifts

$$\tilde{S}_i = \text{clos}(\beta_1^{-1}(S_i \setminus Q)) \quad i = 1, 2$$

are embedded submanifolds in the boundary of Z which do not intersect. Hence we may blow up the space Z first along $\tilde{S}_1$ and then along the lift of $\tilde{S}_2$ or vice versa and obtain diffeomorphic results [For a discussion of the commutativity of the blow up procedure see Appendix A]. Setting

$$\tilde{S} = \tilde{S}_1 \cup \tilde{S}_2$$

we define the $\mathcal{V}_s$-strecthed product by

$$X_s^2 = (Z)_{\tilde{S}} = ((Z)_{\tilde{S}_1})_{\tilde{S}_2}.$$

Again, the space X_s^2 is a manifold with corners equipped with a C^∞ structure such that the blow-down map

$${}^s\beta_2 : X_s^2 \longrightarrow \mathcal{P}_2$$

is smooth. Note that $\Delta_\nu \cap \tilde{S} = \emptyset$ and hence the diagonal lifted to Z and the diagonal lifted to X_s^2 are diffeomorphic under blow down. In particular, near the boundary of Δ_ν the front face of X_s^2 has the structure of an half sphere bundle over the submanifold Q. Figure 1 represents the boundary hypersurfaces of X_s^2. As the figure indicates, X_s^2 has 9 different boundary hypersurfaces which we number as shown. We denote a specific boundary hypersurface as "fn" where n refers to the numbering in the diagram. We often use the notation $ff(X_s^2)$ for the front face of X_s^2, i.e, $f1$.

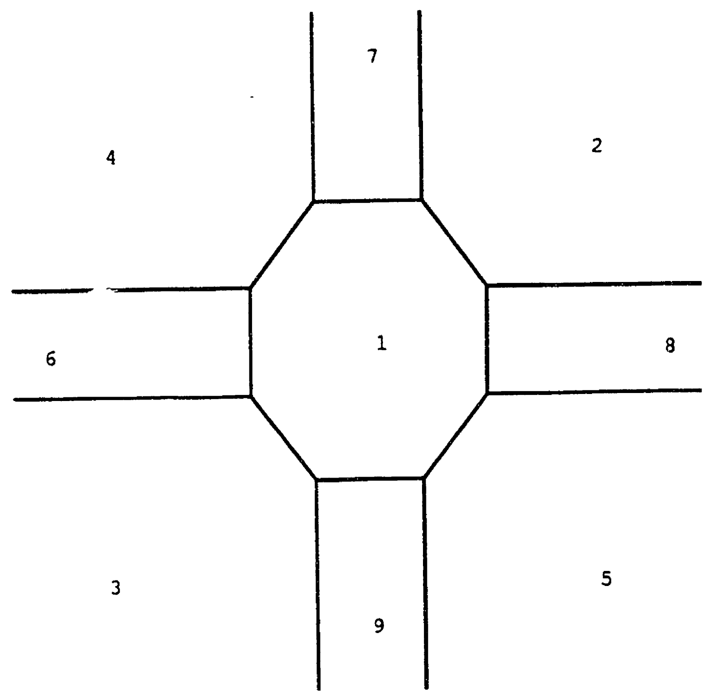

Figure 1: Boundary hypersurfaces of X_s^2.

3.3 Basic Properties of X_s^2

Recall, the space X_s was defined in Chapter 2 as

$$X_s = (X)_B$$

where $B = \{t = y = 0\}$. We denote the boundary faces of X_s by $fl(X_s)$, $ff(X_s)$, $fr(X_s)$ as in the previous chapter.

Proposition 3.1 *The blow down map ${}^s\beta_2 : X_s^2 \longrightarrow \mathcal{P}_2$ factors smoothly through the spaces $X_s \times M$ and $M \times X_s$. We have smooth blow down maps ${}^s\beta_{2L}$, ${}^s\beta_{2R}$ and projections π_{2L}, π_{2R} mapping as indicated in figure 3.3.*

PROOF We restrict our attention to the right half of the diagram, the results for the left half follow by an interchange of indices. We begin by forming the space

$$W_1 = (\mathcal{P}_2)_{S_1}$$

where as before,

$$S_1 = \{t = y^1 = 0\}.$$

The spaces W_1 and $X_s \times M$ are diffeomorphic. To pass from $X_s \times M$ to X_s^2 set

$$B_1 \subset X_s \times M, \quad B_1 = \{R = y^2 = 0\}$$

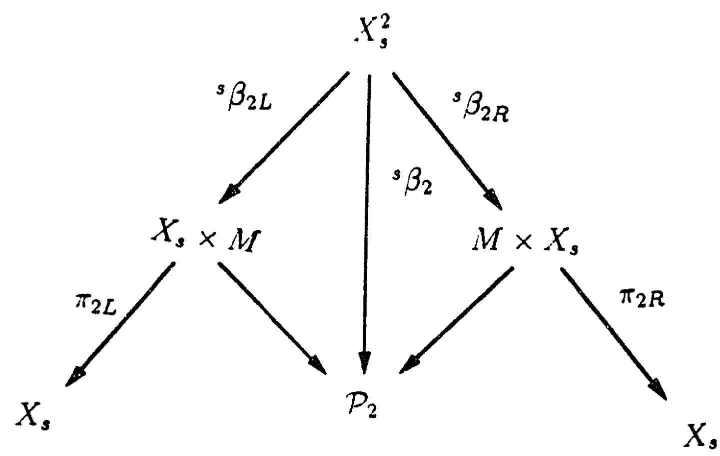

Figure 2: Factoring ${}^s\beta_2$ through $X_s \times M$ and $M \times X_s$.

$$B_2 \subset X_s \times M, \quad B_2 = \{y^2 = 0\} \cap \partial(X_s \times M).$$

Note that B_1 is the lift of the submanifold $Q \subset \mathcal{P}_2$ to the space W_1. Likewise, B_2 is the lift of the submanifold $S_2 \subset \mathcal{P}_2$ to W_1. Hence, to prove the proposition, it suffices to show that we can commute the blow-up of $\mathcal{P}_2$ along Q and S_1. As we have seen, Q and S_1 are both embedded and hence the proposition follows [see appendix A1].

□

The maps $\tilde{\pi}_{2L} = \pi_{2L} \circ^s \beta_{2L}$, $\tilde{\pi}_{2R} = \pi_{2R} \circ^s \beta_{2R}$ provide $ff(X_s^2)$ with additional structure. For $q \in ff(X_s)$, $\tilde{\pi}_{2R}^{-1}(q) \simeq S \times \mathbf{R}$ and we have a fibration

$$\begin{array}{ccc} S \times \mathbf{R} & \longrightarrow & ff(X_s^2) \\ & & \downarrow \\ & & S \times S^1_+ \end{array}$$

where we have used the identification $ff(X_s) \simeq SN^+Q \simeq S \times S^1_+$. Figure 3.3 displays the geometry of this fibration and will be useful later in the solution of the model problem for the front face of X_s^2.

Next, we check that the vector fields

$$\mathcal{V}_s = C^\infty(X_s, {}^sTX)$$

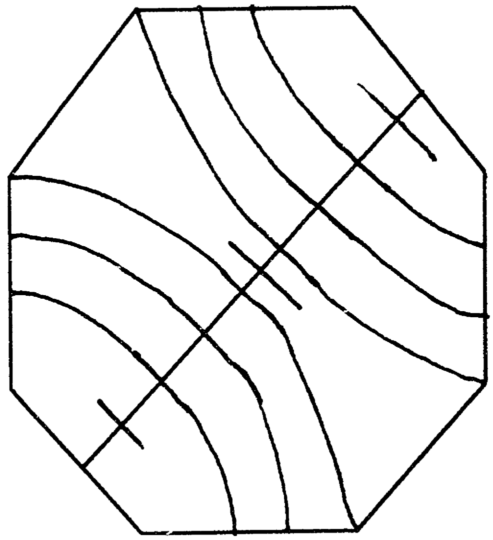

Figure 3: Fibration of $ff(X_s^2)$.

lift as required under the maps $\tilde{\pi}_{2L}, \tilde{\pi}_{2R}$.

Proposition 3.2 *The Lie algebra of vector fields $\mathcal{V}_s$ lifts under $\tilde{\pi}_{2L}, \tilde{\pi}_{2R}$ to be transversal to the lifted diagonal Δ_ν.*

PROOF The claim is local. Away from the set $\Delta_\nu \cap ff(X_s^2)$ the proposition is clear. Near $\Delta_\nu \cap ff(X_s^2)$ we have two sets of local (projective) coordinates. In the region where $t > |y|$ we have coordinates

$$t,\ \frac{y^1}{t} = Y^1,\ \frac{y^2}{t} = Y^2,\ \theta_i^1,\ \theta_i^2 \quad 1 \le i \le n-1 \tag{62}$$

valid when $t > 0$. In these coordinates we compute the lifts of a spanning set for $\mathcal{V}_s$:

$$\partial_{\theta_i} \mapsto \partial_{\theta_i},\ 1 \le i \le n-1 \tag{63}$$

$$r\partial_y \mapsto t\left(1 + (Y^1)^2\right)^{\frac{1}{2}} \frac{1}{t}\partial_{Y^1} \tag{64}$$

$$= F(Y)\partial_{Y^1} \quad F(Y) > 0 \tag{65}$$

In region 2 where $y^1 > t$, we have coordinates

$$\frac{t}{y^1} = T,\ y^1,\ \frac{y^2}{y^1} = Y^2,\quad \theta_i^1,\ \theta_i^2 \quad 1 \le i \le n-1. \tag{66}$$

Computing the lifts in these coordinates, we obtain

$$\begin{aligned} \partial_{\theta_i} &\mapsto \partial_{\theta_i},\ 1 \le i \le n-1 && (67)\\ r\partial_y &\mapsto y^1\left(1+T^2+(Y^2)^2\right)^{\frac{1}{2}}[\partial_{y^1} - \frac{T}{y^1}\partial_T - \frac{Y^2}{y^1}\partial_{Y^2}] \\ &= G(Y,T)[y^1\partial_{y^1} - T\partial_T - Y^2\partial_{Y^2}]\ \ G(Y,T) \neq 0. && (68) \end{aligned}$$

The boundary of Δ_ν in region 1 is given by $\{Y^1 = Y^2,\ \theta_i^1 = \theta_i^2\ \ 1 \le i \le n-1\}$ while in region 2 it is given by $\{Y^2 = 1,\ \ \theta_i^1 = \theta_i^2\ \ 1 \le i \le n-1\}$. Hence the proposition follows.

□

Proposition 3.2 allows us to construct a calculus of pseudodifferential operators microlocalizing the boundary fibration structure $\mathcal{V}_s$. We have

Lemma 3.1 *The lifted diagonal $\Delta_\nu \subset X_s^2$ is isomorphic to X_s under blow-down. The map*

$$\begin{aligned} {}^sT_pX \ &\ni\ v \longrightarrow V_{2R}(p) \in\ {}^sT_p(X_s^2)/{}^sT_p(\Delta_\nu) && (69)\\ V\ &\in\ \mathcal{V}_s,\ \ V(p) = v \end{aligned}$$

where V_{2R} is the lift of $V \in \mathcal{V}_s$ from the right factor gives a bundle isomorphism

$$ {}^sTX \ \longleftrightarrow\ N(\Delta_\nu). \tag{70}$$

Dually, we have an isomorphism

$$ {}^sT^*X \longleftrightarrow N^*(\Delta_\nu).$$

PROOF This is essentially contained in Proposition 3.2. To prove the first claim it suffices to check that the map $\tilde{\pi}_{2R}$ restricts to a diffeomorphism over Δ_ν. This is clear everywhere except perhaps at $\Delta_\nu \bigcap ff(X_s^2)$. Over this region we note that β_{2R} is a b-submersion onto the clean submanifold Q. By transversality of π_{2R}, and a dimension count, we see that $\tilde{\pi}_{2R}$ must also be a diffeomorphism over the region in question.

The transversality of $\mathcal{V}_s$ to Δ_ν under lifting insures that the map (69) is in fact an isomorphism. The smoothness of the above map with respect to the base variable is clear and hence we have a bundle map .

□

The following result will ultimately allow us to keep track of singularities occuring along different boundary faces.

Proposition 3.3 *The maps $\tilde{\pi}_{2L}$, $\tilde{\pi}_{2R} : X_s^2 \longrightarrow X_s$ are b-fibrations.*

PROOF We restrict our attention to $\tilde{\pi}_{2R}$, the proof for $\tilde{\pi}_{2L}$ follows by a change of indices. We first compute:

$$(\tilde{\pi}_{2R})^{-1}\rho_{fl} = \rho_1\rho_6\rho_8 \tag{71}$$
$$(\tilde{\pi}_{2R})^{-1}\rho_{ff} = \rho_2\rho_7\rho_4 \tag{72}$$
$$(\tilde{\pi}_{2R})^{-1}\rho_{fr} = \rho_3\rho_9\rho_5. \tag{73}$$

Hence, $\tilde{\pi}_{2R}$ is a b-map. To establish the submersion condition we begin by noting that π_{2R} is trivially a fibration of $\mathcal{P}_{2R}$. The space X_s^2 is obtained by blowing up submanifolds whose lifts are b-transversal to the fibres of π_{2R}. To finish the proof we need a lemma:

Lemma 3.2 *Suppose $\phi : X \longrightarrow Y$ is a b-map between two manifolds with corners which is a b-fibration of X. Suppose W is a p-submanifold contained in ∂X which is b-transversal to the fibres of the map ϕ. Suppose $Z = (X)_W$ and $\tilde{W} = \mathrm{clos}\beta^{-1}(W)$. Then the map $\tilde{\phi} = \beta \circ \phi : Z \longrightarrow Y$ is a b-submersion.*

PROOF This statement contains little more than the definition of b-transversality. We must show that ${}^b\tilde{\phi}_*$ is surjective onto the compressed bundle of each boundary component. Fix $p \in \partial_k Z$ and suppose $p \in \tilde{W}$ since otherwise ${}^b\tilde{\phi}_*$ is clearly surjective and we are done. Then

$${}^b\beta_* : {}^bT_p\tilde{W} \longrightarrow {}^bT_qW$$

is surjective where $q = \beta(p)$. By b-transversality of W to the fibres of ϕ we have

$${}^bT_xW + {}^bT_xF_y = {}^bT_xX.$$

where $F_y = \phi^{-1}(x)$ is the fibre of ϕ containing x. The map ϕ has surjective b-differential and the result follows. □

The lemma and Proposition A.1 conclude the proof of the previous proposition. □

3.4 Half Densities

We now turn to the question of half densities. As an operator on half-densities the kernel of the identity is given by

$$\delta(y-y')\delta(\theta-\theta')|dydy'd\theta d\theta'|^{\frac{1}{2}}.$$

To express the kernel as a half-density on the space $\mathcal{P}_2$ we write

$$\delta(y-y')\delta(\theta-\theta')|dydy'd\theta d\theta' dt|^{\frac{1}{2}} \otimes |dt|^{-\frac{1}{2}}$$

and consider the bundle $\Omega^{\frac{1}{2}}(\mathcal{P}_2) \otimes |dt|^{-\frac{1}{2}}$. To simplify notation we denote this bundle by $\Omega^{\frac{1}{2}}_{\mathrm{F}}(\mathcal{P}_2)$. As outlined above, the space X^2_s is obtained from the space $\mathcal{P}_2$ by successively blowing-up submanifolds in the boundary of the corresponding spaces. Hence we have a map

$$^s\beta_2^* \ : \ C^{-\infty}(\mathcal{P}_2) \longleftrightarrow C^{-\infty}(X^2_s)$$

and a map

$$^s\beta_2^* \ : \ C^{-\infty}(\mathcal{P}_2, \Omega^{\frac{1}{2}}(\mathcal{P}_2)) \longleftrightarrow C^{-\infty}(X^2_s, \ {}^s\beta_2^*\Omega^{\frac{1}{2}}).$$

We note that

$$^s\beta_2^*\Omega^{\frac{1}{2}}(\mathcal{P}_2) \equiv \rho\Omega^{\frac{1}{2}}(X^2_s)$$

where

$$\rho_{\text{lift}} = \rho_1 \prod_{i=6}^{9} \rho_i^{\frac{1}{2}}. \tag{74}$$

Finally we set

$$\Omega^{\frac{1}{2}}_{\mathrm{F}}(X^2_s) \equiv \Omega^{\frac{1}{2}}(X^2_s) \otimes |dt|^{-\frac{1}{2}} \tag{75}$$

and note that we have an identification

$$^s\beta_2^* \ : \ C^{-\infty}(\mathcal{P}_2, \Omega^{\frac{1}{2}}_{\mathrm{F}}(\mathcal{P}_2)) \longleftrightarrow C^{-\infty}(X^2_s, \rho_{\text{lift}} \ {}^s\beta_2^*\Omega^{\frac{1}{2}}_{\mathrm{F}}(X^2_s)). \tag{76}$$

As in Proposition 3.1 we can construct X^2_s using the partially blown-up spaces $\mathcal{P}_{2i}$ as a starting point. Hence we have identifications

$$^s\beta_{2L}^* : C^{-\infty}(X_s \times M) \longleftrightarrow C^{-\infty}(X^2_s)$$
$$^s\beta_{2R}^* : C^{-\infty}(M \times X_s) \longleftrightarrow C^{-\infty}(X^2_s).$$

Pulling back the half density bundles we have

$$ {}^s\beta_{2L}^*\Omega^{\frac{1}{2}}(X_s \times M) \equiv (\rho_1\rho_6\rho_8)^{\frac{1}{2}}\,\Omega^{\frac{1}{2}}(X_s^2) \tag{77} $$

$$ {}^s\beta_{2R}^*\Omega^{\frac{1}{2}}(M \times X_s) \equiv (\rho_1\rho_7\rho_9)^{\frac{1}{2}}\,\Omega^{\frac{1}{2}}(X_s^2). \tag{78} $$

Hence we get a lifting

$$ {}^s\beta_{2L}^* : C^{-\infty}(X_s \times M; \Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s \times M)) \longleftrightarrow C^{-\infty}(X_s^2; (\rho_1\rho_6\rho_8)^{\frac{1}{2}}\,\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2)) \tag{79} $$

$$ {}^s\beta_{2R}^* : C^{-\infty}(M \times X_s; \Omega_{\mathrm{F}}^{\frac{1}{2}}(M \times X_s)) \longleftrightarrow C^{-\infty}(X_s^2; (\rho_1\rho_7\rho_9)^{\frac{1}{2}}\,\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2)). \tag{80} $$

We will use these liftings to express the action of $A \in \mathrm{Diff}_s^*(X;\Omega^{\frac{1}{2}})$ on polyhomogeneous conormal functions taking values in half-densities on X_s.

If A is a linear operator depending on t as a parameter, $A : \dot{C}^\infty(X;\Omega^{\frac{1}{2}}) \longrightarrow C^{-\infty}(X;\Omega^{\frac{1}{2}})$ acting via a kernel $K(A) \in C^{-\infty}(\mathcal{P}_2;\Omega_{\mathrm{F}}^{\frac{1}{2}})$ we can use the map ${}^s\beta_2^*$ to identify $K(A)$ with a distribution kernel $K_A \in C^{-\infty}(X_s^2;\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2))$. As we shall see this lifting often results in an easier to handle expression for the action of A. We will be particularly interested in distributions conormal with respect to the lifted diagonal :

Definition 3.1 For $m \in \mathbf{R}$ the space of s-pseudodifferential operators of order m, $\Psi_s^m(X;\Omega^{\frac{1}{2}})$, is the space of linear operators $A : \dot{C}^\infty(X;\Omega^{\frac{1}{2}}) \longrightarrow C^{-\infty}(X;\Omega^{\frac{1}{2}})$ with kernel $K_A \in C^{-\infty}(X_s^2;\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2))$ satisfing

$$ K_A \in I^m(X_s^2,\Delta_\nu;\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2)); \quad K_A \equiv 0 \text{ on } \partial(X_s^2)\setminus\{f1\cup f2\cup f3\}. $$

Briefly,

$$ \begin{aligned} \Psi_s^m(X;\Omega^{\frac{1}{2}}) \;=\; & \{A;\ K_A \in I^m(X_s^2,\Delta_\nu;\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2)) \text{ and } K_A \text{ vanishes to} \\ & \text{infinite order on } \partial(X_s^2)\setminus\{f1\cup f2\cup f3\}\ \}. \end{aligned} \tag{81} $$

In the next chapter we show that $\Psi_s^*(X;\Omega^{\frac{1}{2}})$ is a graded algebra and investigate the various mapping properties. For now, we settle for

Proposition 3.4 *The algebra* $\Psi_s^*(X;\Omega^{\frac{1}{2}})$ *contains the ring* $\mathrm{Diff}_s^*(X;\Omega^{\frac{1}{2}})$.

Proof We must check that the kernel of an operator $P \in \mathrm{Diff}_s^m(X;\Omega^{\frac{1}{2}})$ lifted to X_s^2 is conormal of the proper order. The kernel of the identity acting on $\dot{C}^\infty(X;\Omega^{\frac{1}{2}})$ is

$$ K(Id) \;=\; \delta(y^1-y^2)\delta(\theta^1-\theta^2)|dy^1dy^2d\theta^1d\theta^2dt|^{\frac{1}{2}} \otimes |dt|^{-\frac{1}{2}}. \tag{82} $$

Lifting to X_s^2 we have two sets of coordinates. Using coordinates

$$t,\ \frac{y^1}{t} = Y^1,\ \frac{y^2}{t} = Y^2,\ \theta_i^1,\ \theta_i^2 \quad 1 \le i \le n-1.$$

and lifting $K(Id)$ to X_s^2 we see that the lifted kernel K_{Id} is given by

$$K_{Id} = \delta(Y^1 - Y^2)\delta(\theta^1 - \theta^2) \mid dY^1 dY^2 d\theta^1 d\theta^2 dt \mid^{\frac{1}{2}} \otimes |dt|^{-\frac{1}{2}} \tag{83}$$

where we have used the homogeneity of the delta function. Similarly, in coordinates

$$\frac{t}{y^1} = T,\ y^1,\ \frac{y^2}{y^1} = Y^2,\ \ \theta_i^1,\ \theta_i^2 \quad 1 \le i \le n-1$$

we have

$$K_{Id} = \delta(1 - Y^2)\delta(\theta^1 - \theta^2) \mid dy^1 dY^2 d\theta^1 d\theta^2 dT \mid^{\frac{1}{2}} \otimes |dt|^{-\frac{1}{2}}. \tag{84}$$

Suppose $P \in \mathrm{Diff}_s^m(X; \Omega^{\frac{1}{2}})$, P of the form

$$P = \sum_{j \le m} V_{i_1} \cdots V_{i_j} \tag{85}$$

with $V_{i_k} \in \mathcal{V}_s$. Each V_{i_k} lifts from the left factor to be tangent to all boundary faces of X_s^2. The kernel of P is then obtained by repeated application of elements of $\mathcal{V}_s$ to the kernel of the identity. Hence, since delta sections are conormal of order zero, the proposition follows.

□

This result justifies our claim that $\Psi_s^*(X; \Omega^{\frac{1}{2}})$ microlocalizes the boundary structure $\mathcal{V}_s$. We conclude this chapter with

Proposition 3.5 *The family $\Psi_s^*(X; \Omega^{\frac{1}{2}})$ acts on the space $\dot{C}^\infty(X; \Omega^{\frac{1}{2}})$.*

PROOF. Let $A \in \Psi_s^m(X, \Omega^{\frac{1}{2}})$ and $\phi \in \dot{C}^\infty(X; \Omega^{\frac{1}{2}})$. Then the action of A on ϕ is given by

$$A\phi = (\tilde{\pi}_{2L})_*[K_A \cdot (\tilde{\pi}_{2R})^* \phi] \in C^{-\infty}(X; \Omega^{\frac{1}{2}}). \tag{86}$$

By Proposition 3.3 $\tilde{\pi}_{2L}$ and $\tilde{\pi}_{2R}$ are both b-submersions. Hence we may use the mapping properties discussed in Appendix A (in particular proposition A.7) to obtain

$$(\tilde{\pi}_{2R})^* \phi \in \mathcal{A}_{phg}^{\emptyset}(X_s^2, \Omega^{\frac{1}{2}}) = \dot{C}^\infty(X_s^2, \Omega^{\frac{1}{2}})$$

Hence, we obtain

$$K_A \cdot (\tilde{\pi}_{2R})^* \phi \in \mathcal{A}_{phg}^{\emptyset}(X_s^2, \Omega^{\frac{1}{2}}) = \dot{C}^\infty(X_s^2, \Omega^{\frac{1}{2}}) \tag{87}$$

and using Proposition A.7 we see that all conormal singularity is integrated out and

$$A\phi = (\tilde{\pi}_{2L})_*[K_A \cdot \tilde{\pi}_{2R}^* \phi] \in \mathcal{A}_{phg}^{\emptyset}(X; \Omega^{\frac{1}{2}}) \tag{88}$$

i.e, $A\phi \in \dot{C}^\infty(X; \Omega^{\frac{1}{2}})$.

4 Small Calculus

4.1 Triple Stretched Product

To prove that the family of operators $\Psi_s^*(X, \Omega^{\frac{1}{2}})$ form an algebra filtered by a symbol mapping we must establish a composition formula. This section is devoted to the key (geometric) lemma needed to prove that such a formula exists. We begin by constructing the space

$$\mathcal{P}_3 = X \times M \times M = [0,1)_t \times M_1 \times M_2 \times M_3.$$

We note that by dropping any one of the M factors of $\mathcal{P}_3$ we obtain the space $\mathcal{P}_2$. On this copy of $\mathcal{P}_2$ we can construct the space X_s^2. Hence, we can construct the partially blown-up spaces

$$\mathcal{P}_{3L} = X_s^2 \times M$$
$$\mathcal{P}_{3R} = M \times X_s^2$$

and $\mathcal{P}_{3C}$ which has X_s^2 constructed from the first and third factors of M.

Next we form the blown-up space

$$X_s^3 = (((\mathcal{P}_3)_{\tilde{\mathcal{Q}}})_{\tilde{\mathcal{S}}})_{\tilde{\mathcal{W}}}.$$

where $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{W}$ are subsets of the boundary $\partial\mathcal{P}_3$ and the superscript tilde indicates the appropriate lift. The submanifolds $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{W}$ are chosen such that X_s^3 satisfies the following conditions: First, the blow-up procedure should be symmetric under an exchange of the factors M_i. Next, the blow-down map should factor smoothly through each of the spaces $\mathcal{P}_{3i}$. This results in a commutative diagram:

$$\begin{array}{ccc} X_s^3 & \longrightarrow & \mathcal{P}_3 \\ \downarrow & & \downarrow \\ X_s^2 & \longrightarrow & \mathcal{P}_2 \end{array}$$

where the vertical arrows on the right are given by the appropriate projection and those on the left by a composition of blow-down and projection. Finally, we check that the left column maps in the above diagram are b-fibrations.

We begin the actual construction of X_s^3 by defining (embedded) submanifolds contained in the boundary of $\mathcal{P}_3$. We set

$$\mathcal{Q} = \{t = y^1 = y^2 = y^3 = 0\} \tag{89}$$

where the superscript refers to coordinates on the corresponding factor of M. We also set

$$\begin{aligned} \mathcal{S}_1 &= \{t = y^2 = y^3 = 0\} \\ \mathcal{S}_2 &= \{t = y^1 = y^3 = 0\} \\ \mathcal{S}_3 &= \{t = y^1 = y^2 = 0\}. \end{aligned}$$

Finally, set

$$\begin{aligned} \mathcal{W}_{12} &= \{t = y^3 = 0\} \\ \mathcal{W}_{13} &= \{t = y^2 = 0\} \\ \mathcal{W}_{23} &= \{t = y^1 = 0\}. \end{aligned}$$

Let

$$\mathcal{B}_1 = (\mathcal{P}_3)_{\mathcal{Q}}. \tag{90}$$

Then $\mathcal{B}_1$ is a manifold with corners which carries a C^∞ structure such that the blow down map $\beta_1 : \mathcal{B}_1 \longrightarrow \mathcal{P}_3$ is smooth.

The lifts

$$\tilde{\mathcal{S}}_i = \operatorname{clos}(\beta_1^{-1}(\mathcal{S}_i - \mathcal{Q}))$$

intersect $ff(\mathcal{B}_1)$ at six locations in the boundary $(\partial ff(\mathcal{B}_1))$ and are mutually disjoint. Hence, we can blow them up in any order we choose and obtain diffeomorphic results. Setting $\mathcal{S} = \bigcup \tilde{\mathcal{S}}_i$ we define

$$\mathcal{B}_2 = (\mathcal{B}_1)_{\mathcal{S}} = (((\mathcal{B}_1)_{\tilde{\mathcal{S}}_1})_{\tilde{\mathcal{S}}_2})_{\tilde{\mathcal{S}}_3}.$$

The lifts $\widetilde{\mathcal{W}_i} = \operatorname{clos}\beta_1^{-1}(\mathcal{W}_i - \mathcal{S})$ are mutually disjoint embedded submanifolds in the boundary of $\mathcal{B}_2$ intersecting the $ff(\mathcal{B}_2)$ along its boundary in twelve disjoint segments. Setting $\mathcal{W} = \bigcup \widetilde{\mathcal{W}_i}$ we define

$$X^3_s = (\mathcal{B}_2)_{\mathcal{W}} = (((\mathcal{B}_2)_{\widetilde{\mathcal{W}_{12}}})_{\widetilde{\mathcal{W}_{13}}})_{\widetilde{\mathcal{W}_{23}}}. \tag{91}$$

Not surprisingly, X^3_s is a manifold with corners equipped with a smooth blow down map

$${}^s\beta_3 : X^3_s \longrightarrow \mathcal{P}_3.$$

Figure 4.1 displays the geometry of the front face and of the boundary of X^3_s; it has 27 distinct boundary hypersurfaces. Each boundary hypersurface intersects $\partial ff(X^3_s)$. The intersection of the lifted spaces, $\tilde{\mathcal{S}}_i$, with the front face can be pictured as vertices of an octahedron each of which is itself an octagon. The intersection of the blown-up planes, $\tilde{\mathcal{W}}_i$, with the front face can be thought of as the edges of the octahedron each of which is a rectangle. The faces

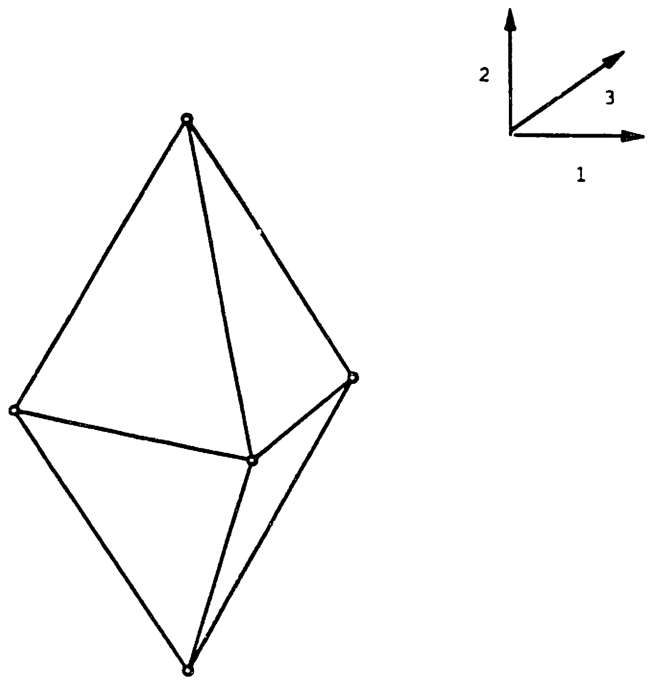

Figure 4: Geometry of the boundary of X^3_s

of the octahedron correspond to lifts of the boundary components not associated with any of the blow-ups. Finally, the front face of X^3_s is represented by an octahedron truncated by first removing the vertices and then removing the edges. Note that we can number the boundary hypersurfaces of X^3_s by fixing an ordering of the six vertices in the picture and numbering the resulting vertex-edge-face-body system lexicographically. For notational convenience we begin with $ff(X^3_s)$. We choose an ordering consistent with the orientation of the three M axes as indicated in the figure.

Proposition 4.1 *The map ${}^s\beta_3$ factors smoothly through each of the spaces $\mathcal{P}_{3i}$. We have blow-down maps ${}^s\beta_{3j}$ and projections π_{3i} mapping as indicated in figure 4.1:*

PROOF We restrict to the case $\mathcal{P}_{3R}$; the other cases follow by a permutation of indices. The space $\mathcal{P}_{3R}$ is diffeomorphic to $(((\mathcal{P}_3)_{\mathcal{S}_1})_{\tilde{\mathcal{W}}_{12}})_{\tilde{\mathcal{W}}_{13}}$. Let ${}^s\beta_{3R} : \mathcal{P}_{3R} \longrightarrow \mathcal{P}_3$. Let $\bar{\mathcal{Q}}$ denote the lift of $\mathcal{Q}$ to $\mathcal{P}_{3R}$. To prove the proposition, we first show that we can commute the blow up of $\mathcal{Q}$ with that of $\mathcal{S}_1$, $\tilde{\mathcal{W}}_{12}$, and $\tilde{\mathcal{W}}_{13}$. This follows by the cleanness observed previously. We repeat this argument for the remaining submanifolds; $\bar{\mathcal{S}}_2$, $\bar{\mathcal{S}}_3$ and $\bar{\mathcal{W}}_{23}$ and the proposition follows. □

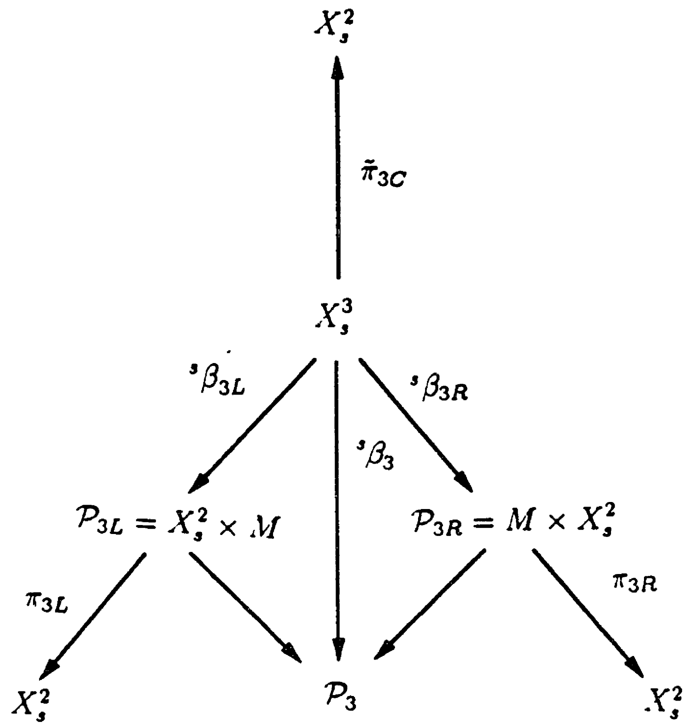

Figure 5: Factoring ${}^s\beta_3$ through $\mathcal{P}_{3i}$.

Proposition 4.2 *The maps $\tilde{\pi}_{3i} : X_s^3 \longrightarrow X_s^2$ are b-fibrations.*

PROOF We first show that each $\tilde{\pi}_{3i}$ is a b-map. Table 1 is an explicit computation of the lifting matrices of the $\tilde{\pi}_{3i}$ which indicates that the result holds. To check the submersion condition we restrict our attention to $\tilde{\pi}_{3R}$; the other two cases are exactly analogous. The argument is very similar to the analogous statement for maps from the double stretched product to the left and right factors. The map $\pi_{3R} : \mathcal{P}_{3R} \longrightarrow X_s^2$ is a fibration of $\mathcal{P}_{3R}$. The boundary submanifolds $\bar{Q}$, $\bar{S}_2$, $\bar{S}_3$, and $\bar{\mathcal{W}}_{23}$ are b-transversal to the fibres of the map π_{3R}. Hence, by Lemma 3.2 the maps are b-submersions. To conclude the proof we note that Table 1 insures that the hypotheses of Proposition A.1 are met and hence the maps are actually b-fibrations.

□

$\tilde{\pi}_{3L}$	$\tilde{\pi}_{3C}$	$\tilde{\pi}_{3R}$	1	2	3	4	5	6	7	8	9
ff	ff	ff	1	0	0	0	0	0	0	0	0
1	1	5	0	0	0	0	0	1	0	0	0
2	2	6	0	0	0	0	0	0	0	1	0
3	5	3	0	0	0	0	0	0	0	0	1
4	6	4	0	0	0	0	0	0	1	0	0
5	3	1	1	0	0	0	0	0	0	0	0
6	4	2	1	0	0	0	0	0	0	0	0
13	15	35	0	0	1	0	0	0	0	0	0
14	16	45	0	0	0	1	0	0	0	0	0
15	13	15	0	0	0	0	0	1	0	0	0
16	14	25	0	0	0	0	0	1	0	0	0
23	25	36	0	0	0	0	1	0	0	0	0
24	26	46	0	1	0	0	0	0	0	0	0
25	23	16	0	0	0	0	0	0	0	1	0
26	24	26	0	0	0	0	0	0	0	1	0
35	35	13	0	0	0	0	0	0	0	0	1
36	45	23	0	0	0	0	0	0	0	0	1
45	36	14	0	0	0	0	0	0	1	0	0
46	46	24	0	0	0	0	0	0	1	0	0
135	135	135	0	0	1	0	0	0	0	0	0
136	145	236	0	0	1	0	0	0	0	0	0
145	136	145	0	0	0	1	0	0	0	0	0
146	146	245	0	0	0	1	0	0	0	0	0
235	235	136	0	0	0	0	1	0	0	0	0
236	245	236	0	0	0	0	1	0	0	0	0
245	236	146	0	1	0	0	0	0	0	0	0
246	246	246	0	1	0	0	0	0	0	0	0

Table 1. Lifting matrices for the maps $\tilde{\pi}_{3i}$, $i = L, C, R$.

We denote the partial diagonals of $\mathcal{P}_3$ by

$$\delta_{3R} = \{(t, q, p, p) \in \mathcal{P}_3; q, p \in M\}$$

$$\delta_{3C} = \{(t, p, q, p) \in \mathcal{P}_3; q, p \in M\}$$

$$\delta_{3L} = \{(t, p, p, q) \in \mathcal{P}_3; q, p \in M\}$$

We denote the triple diagonal in $\mathcal{P}_3$ by

$$\begin{aligned}\Delta_T &= \{(t,p,p,p) \in \mathcal{P}_3;\ p \in M\} \\ &= \delta_{3i} \cap \delta_{3j},\quad i \neq j.\end{aligned} \tag{92}$$

Note that the boundary of Δ_T meets $\mathcal{Q} = \bigcap \mathcal{S}_i = \bigcap \mathcal{W}_i$. We denote the lift of Δ_T to $\mathcal{B}_1$ [see (90)] by

$$\tilde{\Delta}_T = \mathrm{clos}\beta^{-1}(\Delta_T \setminus \mathcal{Q}).$$

We remark that $\tilde{\Delta}_T$ is disjoint from $\tilde{\mathcal{S}}_i,\ \tilde{\mathcal{W}}_i$ for $i = 1,2,3$. The lift of Δ_T to X^3_s,

$$\Delta_{T,s} = \mathrm{clos}\beta_3^{-1}(\tilde{\Delta}_T) = \cap_i \delta_{3i,s} \tag{93}$$

(where $\delta_{3i,s}$ denotes the lift of the *ith* partial diagonal) is therefore diffeomorphic to $\tilde{\Delta}_T$.

Proposition 4.3 *The maps $\tilde{\pi}_{3i} : X^3_s \longrightarrow X^2_s$ embedd $\tilde{\Delta}_{T,s}$ in X^2_s as a submanifold diffeomorphic to Δ_ν.*

PROOF Away from $ff(X^3_s) \bigcap \Delta_{T,s}$ there is nothing to prove. Near the remaining region it suffices to show that $\tilde{\pi}_{3i}$ is a b-submersion onto Δ_ν when restricted to $\Delta_{T,s}$. We restrict our attention to $\tilde{\pi}_{3R}$. We observe as above that the map ${}^s\beta_{3R}$ is a product of blow-down maps the last of which is with respect to the submanifold $\tilde{\mathcal{Q}}$. The submanifold $\Delta_{T,s}$ intersects only one new boundary component, $ff(X^3_s)$. Consider the space $(\mathcal{P}_{3R})_{\mathcal{Q}}$. We have local coordinates near $p \in \mathcal{Q}$ given by

$$t,\ y^1,\ Y^2 = \frac{y^2}{t},\ Y^3 = \frac{y^3}{t},\ \theta^j_i,\ 1 \leq i \leq n-1,\ 1 \leq j \leq 3. \tag{94}$$

$\mathcal{Q}$ is then given locally by $t = y_1 = 0$. Hence, coordinates for $x \in ff\left((\mathcal{P}_{3R})_{\mathcal{Q}}\right)$ are given by

$$t,\ Y^1 = \frac{y^1}{t},\ Y^2 = \frac{y^2}{t},\ Y^3 = \frac{y^3}{t},\ \theta^j_i,\ 1 \leq i \leq n-1,\ 1 \leq j \leq 3. \tag{95}$$

The intersection of the diagonal with the front face is then given by $Y^1 = Y^2 = Y^3,\ \theta^j_i = \theta^k_i$. The b-differential of $\tilde{\pi}_{3i}$ at $x \in ff(\mathcal{P}_{3R})_{\mathcal{Q}}$ is then seen to map $\partial_{Y_1} + \partial_{Y_2} + \partial_{Y_3}$ to $\partial_{Y_2} + \partial_{Y_3}$ with the diagonal in X^2_s given by $Y^2 = Y^3$ and hence the proposition follows.

□

Proposition 4.4 *The maps $\tilde{\pi}_{3i} : X^3_s \longrightarrow X^2_s$ are transversal to the lifted partial diagonal $\delta_{3j,s},\ i \neq j$.*

PROOF. We restrict our attention to the case of $\tilde{\pi}_{3R}$ and $\delta_{3L,s}$. The other cases are in essence the same due to the symmetry in the construction of X^3_s. Away from the region where $\delta_{3L,s}$

meets the new boundary hypersurfaces there is nothing to prove. Fix $x \in \delta_{3L,s} \cap ff(X_s^3)$. Using the coordinates given in the proceeding proposition, near x the submanifold $\delta_{3L,s}$ is given by $Y^1 = Y^2$. Hence, we see immediately that near x $\delta_{3L,s}$ is transverse to the fibres of $\tilde{\pi}_{3R}$. The same is true for the other boundary hypersurfaces meeting $\delta_{3L,s}$ ($\delta_{3L,s}$ transverse to the fibres of $\tilde{\pi}_{3R}$) and the treatment is similar.

□

Before investigating the composition formula and various mapping properties, we record the natural liftings of densities on X_s^2 to X_s^3.

Beginning with the space $\mathcal{P}_3$ we constructed the space X_s^3 by blowing up a sequence of clean submanifolds each contained in the boundary of the respective space. The composite blow-down map

$$^s\beta_3 : X_s^3 \longrightarrow \mathcal{P}_3$$

induces an identification

$$^s\beta_3^* : C^{-\infty}(\mathcal{P}_3) \longleftrightarrow C^{-\infty}(X_s^3)$$

and a map

$$^s\beta_3^* : C^{-\infty}(\mathcal{P}_3; \Omega^{\frac{1}{2}}(\mathcal{P}_3)) \longleftrightarrow C^{-\infty}(X_s^3;\ ^s\beta_3^*\Omega^{\frac{1}{2}}(\mathcal{P}_3)).$$

We note that

$$^s\beta_3^*\Omega^{\frac{1}{2}}(\mathcal{P}_3) \equiv \rho^{D_T}\Omega^{\frac{1}{2}}(X_s^3) \tag{96}$$

where

$$\rho^{D_T} = \rho_{ff}^{\frac{3}{2}} \left(\prod_{i=1}^{6} \rho_i\right) \left(\prod_{i=1}^{2} \rho_{i3}\rho_{i4}\right)^{\frac{1}{2}} \left(\prod_{i=5}^{6}\prod_{j=1}^{4} \rho_{ji}\right)^{\frac{1}{2}}.$$

As with half densities on the parametrized double product space $\mathcal{P}_2$, we consider the bundle

$$\Omega_{\mathrm{F}}^{\frac{1}{2}}(\mathcal{P}_3) \equiv \Omega^{\frac{1}{2}}(\mathcal{P}_3) \otimes |dt|^{-\frac{1}{2}}.$$

We then have an identification

$$^s\beta_3^* : C^{-\infty}(\mathcal{P}_3; \Omega_{\mathrm{F}}^{\frac{1}{2}}(\mathcal{P}_3)) \longleftrightarrow C^{-\infty}(X_s^3; \rho^{D_T}\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^3)). \tag{97}$$

Recall that X_s^3 is also otained from the space $X_s^2 \times M$ by blowing up a sequence of embedded submanifolds contained in $\partial(X_s^2 \times M)$. Hence we have an identification

$$^s\beta_{3L}^* \ : \ C^{-\infty}(X_s^2 \times M) \longleftrightarrow C^{-\infty}(X_s^3)$$

and a map

$$^s\beta_{3L}^* \ : \ C^{-\infty}(X_s^2 \times M, \Omega^{\frac{1}{2}}(X_s^2 \times M)) \longrightarrow C^{-\infty}(X_s^3, \ ^s\beta_{3L}^*\Omega^{\frac{1}{2}}(X_s^2 \times M)).$$

We note that

$$^s\beta_{3L}^*\Omega^{\frac{1}{2}}(X_s^2 \times M) \equiv \rho^{\alpha_L}\Omega^{\frac{1}{2}}(X_s^3)$$

where

$$\rho^{\alpha_L} = \rho_{ff}^{\frac{1}{2}} \left(\prod_{i=3}^{6} \rho_i^{\frac{1}{2}}\right) \left(\prod_{i=3}^{4} \rho_{i5}\rho_{i6}\right)^{\frac{1}{2}}. \tag{98}$$

Likewise,

$$^s\beta_{3R}^*\Omega^{\frac{1}{2}}(M \times X_s^2) \equiv \rho^{\alpha_R}\Omega^{\frac{1}{2}}(X_s^3) \tag{99}$$

where

$$\rho^{\alpha_R} = \rho_{ff}^{\frac{1}{2}} \left(\prod_{i=1}^{4} \rho_i^{\frac{1}{2}}\right) \left(\prod_{i=1}^{2} \rho_{i3}\rho_{i4}\right)^{\frac{1}{2}} \tag{100}$$

and

$$^s\beta_{3C}^*\Omega^{\frac{1}{2}}(P_{3C}) \equiv \rho^{\alpha_C}\Omega^{\frac{1}{2}}(X_s^3) \tag{101}$$

where

$$\rho^{\alpha_C} = \rho_{ff}^{\frac{1}{2}} \left(\prod_{i=1}^{2} \rho_i^{\frac{1}{2}}\right) \left(\prod_{i=5}^{6} \rho_i^{\frac{1}{2}}\right) \left(\prod_{i=1}^{2} \rho_{i5}\rho_{i6}\right)^{\frac{1}{2}}. \tag{102}$$

As with the double stretched product we have a natural lifting of half densities under the maps $\tilde{\pi}_{3i}$. Fix a smooth nonvanishing half-density μ on M. Given $\nu \in C^\infty(X_s^2, \Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2))$ we have liftings

$$\nu \boxtimes \mu \in C^\infty(X_s^2 \times M; \Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2 \times M))$$
$$\mu \boxtimes \nu \in C^\infty(M \times X_s^2; \Omega_{\mathrm{F}}^{\frac{1}{2}}(M \times X_s^2)).$$

Using the above liftings from the partially stretched spaces $\mathcal{P}_{3i}$ to X_s^3 we obtain

$$\mu \mapsto \tilde{\pi}_{3i}^*\mu \in C^\infty(X_s^3; \rho^{\alpha_i}\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^3)) \tag{103}$$

where ρ^{α_i} is as above.

4.2 Symbol Mapping and Composition Formula

The symbol mapping for s-pseudodifferential operators arises through the symbol mapping for conormal distibutions. Recall, the symbol mapping for conormal distributions is given by sending $K_A \in I^m(X_s^2, \Delta_\nu; \Omega^{\frac{1}{2}})$ to its symbol in some local coordinate representation (more accurately, to an equivalence class of symbols in some local coordinate representation).

$$K_A = \int e^{i<x',\eta>} a(x'',\eta) d\eta \tag{104}$$

$$K_A \mapsto [a(x'',\eta)|d\eta|] \ni S^m(N^*(\Delta_\nu)) \otimes \Omega_{fib}(N^*(\Delta_\nu)) \otimes \Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2) \tag{105}$$

Our first order of business is to simplify the density factor.

Proposition 4.5 *There are bundle maps covering the diffeomorphism* $\Delta_\nu \longleftrightarrow X_s$ *giving the identifications*

$$\Omega_{fib}(N^*(\Delta_\nu)) \longleftrightarrow \Omega_{fib}({}^sT^*X) \quad \textit{and} \tag{106}$$

$$\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2) \longleftrightarrow \Omega_{fib}({}^sTX) \tag{107}$$

PROOF The first map is an immediate corollary of Proposition 3.1. For the second we note that over any submanifold the density bundles split:

$$\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2) \simeq \Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s) \otimes \Omega_{fib}^{\frac{1}{2}}(N(\Delta_\nu)).$$

Hence, using the above identification we have

$$\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2) \simeq \Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s) \otimes \Omega_{fib}^{\frac{1}{2}}({}^sTX).$$

We note that $T_q({}^sT_pX) \simeq {}^sT_pX$ Hence, we have the identifications

$$\Omega_{fib}^{\frac{1}{2}}({}^sTX) \simeq \Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s).$$

This concludes the proof of the proposition.

□

Returning to our discusion of the symbol map for conormal distributions we conclude that the density factor is canonically trivial and we have a map

$$I^m(X_s^2, \Delta_\nu; \Omega^{\frac{1}{2}}) \xrightarrow{\sigma_m} S^m({}^sT^*X)/S^{m-1}({}^sT^*X).$$

The kernel of the map σ_m is $I^{m-1}(X_s^2, \Delta_\nu; \Omega^{\frac{1}{2}})$ and hence we have an exact sequence

$$0 \longrightarrow I^{m-1}(X_s^2, \Delta_\nu; \Omega^{\frac{1}{2}}) \hookrightarrow I^m(X_s^2, \Delta_\nu; \Omega^{\frac{1}{2}}) \xrightarrow{\sigma_m} S^m({}^sT^*X)/S^{m-1}({}^sT^*X) \longrightarrow 0. \quad (108)$$

We record this as part of the following

Theorem 4.1 *$\Psi_s^*(X; \Omega^{\frac{1}{2}})$ is an algebra filtered by the symbol map σ_m. For every $m \in \mathbf{R}$, there is an exact sequence*

$$0 \longrightarrow \Psi_s^{m-1}(X; \Omega^{\frac{1}{2}}) \hookrightarrow \Psi_s^m(X; \Omega^{\frac{1}{2}}) \xrightarrow{\sigma_m} S^m({}^sT^*X)/S^{m-1}({}^sT^*X) \longrightarrow 0 \quad (109)$$

and composition gives

$$\Psi_s^m(X; \Omega^{\frac{1}{2}}) \cdot \Psi_s^{m'}(X; \Omega^{\frac{1}{2}}) \subset \Psi_s^{m+m'}(X; \Omega^{\frac{1}{2}}) \quad (110)$$
$$\sigma_{m+m'}(A \cdot B) = \sigma_m(A) \cdot \sigma_{m'}(B) \ \mathrm{mod} S^{m+m'-1}. \quad (111)$$

PROOF. The proof of the composition formula is similar in spirit to the proof that $\Psi_s^*(X; \Omega^{\frac{1}{2}})$ acts on $\dot{C}^\infty(X; \Omega^{\frac{1}{2}})$. Fix nonvanishing smooth half-densities $\nu \in C^\infty(X_s^2; \Omega^{\frac{1}{2}})$ and $\mu \in C^\infty(X_s^2; \Omega_{\mathrm{F}}^{\frac{1}{2}})$. Choose $A \in \Psi_s^m(X; \Omega^{\frac{1}{2}})$ and $B \in \Psi_s^{m'}(X; \Omega^{\frac{1}{2}})$. Define

$$K_C \cdot \nu \ = \ (\tilde{\pi}_{3C})_*[(\tilde{\pi}_{3L})^* K_A \cdot (\tilde{\pi}_{3R})^* K_B \cdot (\tilde{\pi}_{3C})^* \nu]. \quad (112)$$

By Proposition 4.2 we know that the maps $\tilde{\pi}_{3i}$ are b-fibrations. Hence we can use the results of Appendix A to compute pull-backs and push-forwards by the maps $\tilde{\pi}_{3i}$. In particular, since K_A and K_B have nontrivial exponent sets only at the faces $f1(X_s^2)$, $f2(X_s^2)$, and $f3(X_s^2)$ and we have

$$\tilde{\pi}_{3L}^*(K_A/\mu) \in \mathcal{A}_{phg}^E I^m(X_s^3, \delta_{3L,s})$$
$$\tilde{\pi}_{3R}^*(K_B/\mu) \in \mathcal{A}_{phg}^F I^{m'}(X_s^3, \delta_{3R,s})$$

were the exponents $E = \{E_i\}_{i=1}^{27}$ and $F = \{F_i\}_{i=1}^{27}$ satisfy

$$E_i \ = \ \emptyset \text{ if } i \neq \ 1,\ 6,\ 7,\ 8,\ 13,\ 20,\ 21,\ 26,\ 27$$
$$F_j \ = \ \emptyset \text{ if } j \neq \ 1,\ 2,\ 3,\ 16,\ 19,\ 20,\ 23,\ 24,\ 27.$$

Proposition 4.4 and Proposition 4.3 show that the hypotheses of Proposition A.8 are satisfied. Therefore we have

$$\mathcal{A}_{phg}^E I^m(X_s^3, \delta_{3R,s}) \cdot \mathcal{A}_{phg}^F I^{m'}(X_s^3, \delta_{3L,s}) \subset \mathcal{A}_{phg}^G I^{m+m'}(X_s^3, \delta_{3R,s} \cup \delta_{3L,s} \cup \Delta_{T,s}).$$

where $\Delta_{T,s}$ is the lifted triple diagonal (93). The product

$$\tilde{\pi}_{3L}^*(K_A/\mu) \cdot \tilde{\pi}_{3R}^*(K_B/\mu) \in \mathcal{A}_{phg}^G I^{m+m'}(X_s^3, \delta_{3R,s} \cup \delta_{3L,s} \cup \Delta_{T,s}) \quad (113)$$

were in particular, the exponent set G satisfies

$$G_i \;=\; \emptyset \text{ if } i \neq 1,\ 20,\ 27. \tag{114}$$

Under the b-submersion $\tilde{\pi}_{3C}$ we have a map

$$(\tilde{\pi}_{3C})_* : \mathcal{A}^G_{phg} I^{m+m'}(X^3_s, \delta_{3R,s} \cup \delta_{3L,s} \cup \Delta_{T,s};\ {}^b\Omega) \longrightarrow A^{\tilde{\pi}_{3L\flat} G} I^{m+m'}(X^2_s, \Delta_\nu;\ {}^b\Omega) \tag{115}$$

To complete the proof of the theorem we note that the faces $f1,\ f20,\ f27$ push-forward under central projection to the faces $f1,\ f2,\ f3$. Hence the kernel K_C has nontrivial expansion only at the faces $f1,\ f2,\ f3$. For completeness we note the density factor

$$\tilde{\pi}^*_{3L}\mu \cdot \tilde{\pi}^*_{3C}\nu \cdot \tilde{\pi}^*_{3R}\mu \in \rho^{\alpha_L+\alpha_C+\alpha_R-D_T}\Omega_F(X^3_s) \otimes |dt|^{\frac{1}{2}} \tag{116}$$

where ρ, α_i and D_T are as in (96). A quick check of the lifting formulas for half densities indicates that in fact, $\alpha_L+\alpha_C+\alpha_R-D_T=0$. Although we do not need this fact to establish the composition formula for the small calculus, it will prove useful later on. Piecing together the above information we see that the formula for K_C define an element of $\Psi^{m+m'}_s(X;\Omega^{\frac{1}{2}})$ and the composition rule is established. Finally, the formula for the symbol map of the composition is an immediate corollary of the definition of the symbol map and the composition formula.

□

4.3 Normal Operator and Approximate Paramatrices

Proposition 6.3 and the definition of weighted Sobolev spaces suggest that we consider a second filtration on the algebra $\Psi^m_s(X;\Omega^{\frac{1}{2}})$. Let $\varrho = \rho_1\rho_2\rho_3$. We filter $\Psi^m_s(X;\Omega^{\frac{1}{2}})$ by the order of vanishing along faces intersecting the lifted diagonal Δ_ν:

$$\varrho^t\Psi^m_s(X;\Omega^{\frac{1}{2}}) \cdot \varrho^{t'}\Psi^{m'}_s(X;\Omega^{\frac{1}{2}}) \subset \varrho^{t+t'}\Psi^{m+m'}_s(X;\Omega^{\frac{1}{2}}). \tag{117}$$

Associated to this filtration there is a mapping

$$N \;:\; \Psi^m_s(X;\Omega^{\frac{1}{2}}) \longrightarrow \Psi^m_s(ff(X_s);\Omega^{\frac{1}{2}}) \tag{118}$$

given by

$$N(A) \;=\; [K_A]|_{f1}. \tag{119}$$

$N(A)$ represents the leading term in Taylor series expansion of the operator A at the front face of X^2_s. We note that

$$A \in \Psi^m_s(X;\Omega^{\frac{1}{2}}),\ \ N(A) = \sigma_{f2}(A) = \sigma_{f3}(A) = 0 \iff \exists B \in \Psi^m_s(X;\Omega^{\frac{1}{2}}),\ \ A = \rho B$$

where ρ is a total boundary defining function for X_s^2.

Theorem 4.1 and Proposition 3.1 allow us to construct a first approximation to a paramatrix for Δ_G: Proposition 3.1 implies that Δ_G is transversally elliptic with respect to Δ_ν, i.e., Δ_G is elliptic in the algebra $\Psi_s^m(X;\Omega^{\frac{1}{2}})$. In view of Theorem 4.1 a standard symbolic argument will produce an operator $O_1 \in \Psi_s^{-2}$ satisfying

$$\Delta_G \cdot O_1 \;=\; Id - R_1, \quad R_1 \in \Psi_s^{-\infty}(X;\Omega^{\frac{1}{2}}). \tag{120}$$

Although residual terms in the small calculus have smooth kernels, they do not in general produce decay at the boundary necessary to conclude compactness in the (strong) sense of operators on $L^2(X_s;\Omega^{\frac{1}{2}})$ [see Proposition 6.3]. In part this reflects a shortcoming of the calculus $\Psi_s^*(X;\Omega^{\frac{1}{2}})$ but the real problem is the notion of compactness that is being employed. In chapter six we formulate the correct notion of "compactness". Here we restrict ourselves to a brief discussion of the problem we are trying to circumvent.

Recall, the goal is to define paramatrices for the operators Δ_{G_t} which are uniform in the parameter t as $t \longrightarrow 0$. If this is possible, we should be able to "guess" at a first approximation for the paramatrix. The first step in the procedure should be given by freezing the kernel of Δ_G at the front face, extending trivially in a neighborhood of this boundary component and inverting the resulting operator. In general there will be an obstruction to invertibility given by null space for the induced operator. This obstruction will be represented in the small calculus as error terms which have nontrivial expansion at the boundary of the front face. They will not be removeable.

The case of one fibre dimension is sufficient to understand the problems that arise in the higher dimensional case. In one fibre dimension Δ_G is given by

$$f^2\partial_y^2 + (f\partial_y f)\,\partial_y + \partial_\theta^2 \tag{121}$$

where $f = (y^2+t^2)^{\frac{1}{2}}$. Lifting to X_s^2 we see that the kernel of A near the front face is given by

$$\left[\left(\left[(Y^1)^2+1\right]^{\frac{1}{2}}\partial_{Y^1}\right)^2 + \partial_{\theta^1}^2\right]\delta(Y^1-Y^2)\delta(\theta^1-\theta^2)\Gamma \tag{122}$$

where

$$t,\; Y^1 = \frac{y^1}{t},\; Y^2 = \frac{y^2}{t}\;\; \theta^1,\; \theta^2$$

are (projective) coordinates near the front face of X_s^2 and Γ is a section of the bundle $\Omega_F^{\frac{1}{2}}(X_s^2)$. Hence, the normal operator is given by

$$N(A) \;=\; \left(\left[Y^2+1\right]^{\frac{1}{2}}\partial_Y\right)^2 + \partial_\theta^2. \tag{123}$$

This checks with our previous definition of the normal operator given in Proposition 2.2. Making the coordinate change $r = \int \frac{1}{(Y^2+1)^{\frac{1}{2}}} dY$ we obtain

$$N(A) \quad = \quad \partial_r^2 + \partial_\theta^2. \tag{124}$$

We proceed to produce a fundamental solution of $N(A - \lambda^2 I)$. Using Fourier series in the compact direction we obtain a one parameter family of ODE's indexed by $\mathbf{Z}$:

$$\left(\frac{d^2}{dr^2} - (k^2 + \lambda^2) \right) u = 0. \tag{125}$$

These ODE's admit solutions of the form $u = e^{\alpha_{\pm(k,\lambda)} r}$ where

$$\alpha_{\pm(k,\lambda)} \quad = \quad \pm\sqrt{k^2 + \lambda^2}. \tag{126}$$

For $k \neq 0$ we see that we have solutions decaying by a power law for large r. For $k = 0$ solutions behave asymptotically as a linear function in Y and we therefore must make a nontrivial choice for boundary conditions. Setting

$$G_{k,\lambda}(r) \quad = \quad \frac{e^{\alpha_{\pm(k,\lambda)}|r|}}{2\alpha_{\pm(k,\lambda)}} \tag{127}$$

and

$$G \quad = \quad \sum_{k \in \mathbf{Z}} G_{k,\lambda}(r) e^{ik\theta} \tag{128}$$

we obtain a fundamental solution for the operator $\partial_r^2 + \partial_\theta^2 - \lambda^2$ and hence a fundamental solution for $N(A - \lambda^2 I)$. The kernel $K(r, r', \theta, \theta')$ for the fundamental solution is given by

$$K(r, r', \theta, \theta') = \sum_{k \in \mathbf{Z}} G_{k,\lambda}(r - r') e^{ik(\theta - \theta')}.$$

Note that the kernel is nontrivial along the boundary of the front face corresponding to the intersection with faces other than $f2(X_s)$ and $f3(X_s)$. Extension of our inverse will therefore involve the introduction of nontrivial expansions along boundary hypersurfaces other than $f1$, $f2$, $f3$. These boundary terms are not represented in $\Psi_s^*(X; \Omega^{\frac{1}{2}})$. Rather than abandoning our approach thus far we are led to generalize $\Psi_s^*(X; \Omega^{\frac{1}{2}})$ to account for such boundary behavior. This is the content of the next chapter.

5 General Calculus

5.1 Definition of $\Psi_s^{*;E}(X;\Omega^{\frac{1}{2}})$

As we noted at the close of the last chapter, to invert $N(\Delta_G f^2)$ we must suitably generalize the calculus $\Psi_s^*(X;\Omega^{\frac{1}{2}})$. Specifically, we must allow operators whose kernels have classical expansions along the boundary faces other than $f1(X_s^2)$, $f2(X_s^2)$, and $f3(X_s^2)$. With this in mind, we establish the following

Definition 5.1 For any exponent family E for X_s^2, the space $\Psi_s^{-\infty;E}(X;\Omega^{\frac{1}{2}})$ is the set of linear operators $A : C^\infty(X;\Omega^{\frac{1}{2}}) \longrightarrow C^{-\infty}(X;\Omega^{\frac{1}{2}})$ with kernel K_A satisfying

$$K_A \in \mathcal{A}^E_{phg}(X_s^2;\Omega_{\mathrm{F}}^{\frac{1}{2}}) \cup \{K \in I^m(X_s^2,\Delta_\nu;\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2)) \text{ and } K \text{ vanishes to infinite order on } \partial(X_s^2)\setminus\{f1\cup f2\cup f3\}\ \}.$$

Briefly, we write

$$\Psi_s^{-\infty;E}(X;\Omega^{\frac{1}{2}}) = \Psi_s^{-\infty}(X;\Omega^{\frac{1}{2}}) + \mathcal{A}^E_{phg}(X_s^2;\Omega_{\mathrm{F}}^{\frac{1}{2}}). \tag{129}$$

We define the mth order general pseudodifferential operators by setting

$$\Psi_s^{m;E}(X;\Omega^{\frac{1}{2}}) = \Psi_s^{m}(X;\Omega^{\frac{1}{2}}) + \Psi_s^{-\infty;E}(X;\Omega^{\frac{1}{2}}). \tag{130}$$

We have several possible choices for extending the symbol mapping to $\Psi_s^{*;E}(X;\Omega^{\frac{1}{2}})$. If $A \in \Psi_s^{m;E}(X;\Omega^{\frac{1}{2}})$, then A decomposes as

$$A = A' + A'',\ \text{where } A' \in \Psi_s^m(X;\Omega^{\frac{1}{2}}),\quad A'' \in \Psi_s^{-\infty;E}(X;\Omega^{\frac{1}{2}}).$$

We define the first or diagonal symbol map by

$$\sigma_m(A) = \sigma_m(A'). \tag{131}$$

This gives an exact sequence

$$0 \longrightarrow \Psi_s^{m-1;E}(X;\Omega^{\frac{1}{2}}) \hookrightarrow \Psi_s^{m;E}(X;\Omega^{\frac{1}{2}}) \xrightarrow{\sigma_m} S^m({}^sT^*X)/S^{m-1}({}^sT^*X) \longrightarrow 0. \tag{132}$$

As before, we record this in the following propositions:

Theorem 5.1 *Given two operators $A \in \Psi_s^{m;E}(X;\Omega^{\frac{1}{2}})$ and $B \in \Psi_s^{m';F}(X;\Omega^{\frac{1}{2}})$ there is a well defined composite operator $C \in \Psi_s^{m+m';G}(X;\Omega^{\frac{1}{2}})$ where the exponent set G is given by*

$G = \left(\tilde{\pi}_{3C^b}[\tilde{\pi}^b_{3L}E \bar{+} \tilde{\pi}^b_{3R}F \bar{+} -^b 1]\right) \bar{+} {}^b 1$. *Moreover, the symbol mapping along the diagonal is preserved and we have an exact sequence*

$$0 \longrightarrow \Psi_s^{m-1;E} \hookrightarrow \Psi_s^{m;E} \xrightarrow{\sigma_m} S^m({}^sT^*X)/S^{m-1}({}^sT^*X) \longrightarrow 0. \tag{133}$$

Composition gives

$$\Psi_s^{m;E}(X;\Omega^{\frac{1}{2}}) \cdot \Psi_s^{m';F}(X;\Omega^{\frac{1}{2}}) \subset \Psi_s^{m+m';G}(X;\Omega^{\frac{1}{2}}) \tag{134}$$

$$\sigma_{m+m'}(A \cdot B) = \sigma_m(A) \cdot \sigma_{m'}(B) \; mod S^{m+m'-1}. \tag{135}$$

PROOF Table 5.2 and the remarks concerning the lifts of densities to X_s^3 record all the necessary data. The proof is modeled on the proof of the composition formula for the small calculus. Let μ and ν be novanishing half densities $\nu \in C^\infty(X_s^2;\Omega^{\frac{1}{2}})$, $\mu \in C^\infty(X_s^2;\Omega_{\mathrm{F}}^{\frac{1}{2}})$. Given $K_A \in \mathcal{A}^E_{phg}I^m(X_s^2,\Delta_\nu;\Omega_{\mathrm{F}}^{\frac{1}{2}})$ and $K_B \in \mathcal{A}^F_{phg}I^{m'}(X_s^2,\Delta_\nu;\Omega_{\mathrm{F}}^{\frac{1}{2}})$ we form

$$K_C \cdot \nu = (\tilde{\pi}_{3C})_*[\tilde{\pi}^*_{3L}K_A \cdot \tilde{\pi}^*_{3R}K_B \cdot \tilde{\pi}^*_{3C}\nu]. \tag{136}$$

By Proposition 4.2 the maps $\tilde{\pi}_{3L}$, $\tilde{\pi}_{3R}$, $\tilde{\pi}_{3C}$ are b-fibrations. Hence, by Proposition A.6

$$\tilde{\pi}^*_{3L}(K_A/\mu) \in \mathcal{A}^{\tilde{E}}_{phg}I^m(X_s^3,\delta_{3L,s})$$

$$\tilde{\pi}^*_{3R}(K_B/\mu) \in \mathcal{A}^{\tilde{F}}_{phg}I^{m'}(X_s^3,\delta_{3R,s})$$

Where $\tilde{E} = \tilde{\pi}^b_{3L}E$, and $\tilde{F} = \tilde{\pi}^b_{3R}F$. We note the hypotheses of Proposition A.3 are satisfied and hence

$$\tilde{\pi}^*_{3L}(K_A/\mu) \cdot \tilde{\pi}^*_{3R}(K_B/\mu) \in \mathcal{A}^{\tilde{E}\bar{+}\tilde{F}}_{phg}I^{m+m'}(X_s^3,\delta_{3L,s}\cup\delta_{3R,s}\cup\Delta_{T,s}) \tag{137}$$

The density factors combine to give

$$\tilde{\pi}^*_{3C}\nu \cdot \tilde{\pi}^*_{3L}\mu \cdot \tilde{\pi}^*_{3R}\mu \in \rho^{\alpha_C+\alpha_L+\alpha_R-D_T}\Omega_F(X_s^3) \otimes |dt|^{\frac{1}{2}} \tag{138}$$

where D_T is as in the pull-back formula for triple product densities. As we saw in Theorem 4.1, $\alpha_C + \alpha_L + \alpha_R - D_T = 0$ and we see that the product satisfies

$$\tilde{\pi}^*_{3L}K_A \cdot \tilde{\pi}^*_{3R}K_B \cdot \tilde{\pi}^*_{3C}\mu \;\in\; \mathcal{A}^{\tilde{E}\bar{+}\tilde{F}}_{phg}I^{m+m'}(X_s^3,\delta_{3L,s}\cup\delta_{3R,s}\cup\Delta_{T,s};\Omega_F\otimes|dt|^{\frac{1}{2}}).$$

Shifting the exponent family down one so as to work on b-compressed densities we obtain a new index family K. By Proposition A.8 the product pushes forward and we obtain

$$(\tilde{\pi}_{3C})_*[\tilde{\pi}^*_{3L}K_A \cdot \tilde{\pi}^*_{3R}K_B \cdot \tilde{\pi}^*_{3C}\mu] \in \mathcal{A}^{\tilde{\pi}_{3C^b}K}_{phg}I^{m+m'}(X_s^2,\Delta_\nu,\; {}^b\Omega_F\otimes|dt|^{\frac{1}{2}}) \tag{139}$$

and the composition law follows after shifting the exponent up one to work over half-densities. The symbolic property is an easy consequence of composition formula.

□

fv	fn	$\tilde{\pi}^b_{3L}E$	$\tilde{\pi}^b_{3C}E$	$\tilde{\pi}^b_{3R}E$	$K_A \cdot K_B$	$\tilde{\pi}_{3Cb} \mapsto$
ff	f1	E_1	E_1	E_1	$E_1 + F_1$	1
1	f2	E_6	E_6	E_1	$E_6 + F_1$	6
2	f3	E_8	E_8	E_1	$E_8 + F_1$	8
3	f4	E_9	E_1	E_9	$E_9 + F_9$	1
4	f5	E_7	E_1	E_7	$E_7 + F_7$	1
5	f6	E_1	E_9	E_6	$E_1 + F_6$	9
6	f7	E_1	E_7	E_8	$E_1 + F_8$	7
13	f8	E_3	E_6	E_9	$E_3 + F_9$	6
14	f9	E_4	E_6	E_7	$E_4 + F_7$	6
15	f10	E_6	E_3	E_6	$E_6 + F_6$	3
16	f11	E_6	E_4	E_8	$E_6 + F_8$	4
23	f12	E_5	E_8	E_9	$E_5 + F_9$	8
24	f13	E_2	E_8	E_7	$E_2 + F_7$	8
25	f14	E_8	E_5	E_6	$E_8 + F_6$	5
26	f15	E_8	E_2	E_8	$E_8 + F_8$	2
35	f16	E_9	E_9	E_3	$E_9 + F_3$	9
36	f17	E_9	E_7	E_5	$E_9 + F_5$	7
45	f18	E_7	E_9	E_4	$E_7 + F_4$	9
46	f19	E_7	E_7	E_2	$E_7 + F_2$	7
135	f20	E_3	E_3	E_3	$E_3 + F_3$	3
136	f21	E_3	E_4	E_5	$E_3 + F_5$	4
145	f22	E_4	E_3	E_4	$E_4 + F_4$	3
146	f23	E_4	E_4	E_2	$E_4 + F_2$	4
235	f24	E_5	E_5	E_3	$E_5 + F_3$	5
236	f25	E_5	E_2	E_5	$E_5 + F_5$	2
245	f26	E_2	E_5	E_4	$E_2 + F_4$	5
246	f27	E_2	E_2	E_2	$E_2 + F_2$	2

Table 5.2. Lifting data for general composition.

As an immediate corollary, we have

Corollary 5.2 *Suppose* $A \in \Psi_s^{m;E}(X;\Omega^{\frac{1}{2}})$ *and* $B \in \Psi_s^{-\infty;F}(X;\Omega^{\frac{1}{2}})$ *and the exponent families* E *and* F *satisfy* $E_i > \epsilon_1$, $F_i > \epsilon_2$. *Then* $A \cdot B \in \Psi_s^{-\infty;G}(X;\Omega^{\frac{1}{2}})$ *where* G *satisfies* $G_i > \epsilon_1 + \epsilon_2$. *In particular,* $B^k \in \Psi_s^{-\infty;G'}(X;\Omega^{\frac{1}{2}})$ *with* $G'_i > k\epsilon_2$.

□

Composing elements in the small calculus with elements in the general calculus considerably simplifies the table of relevent lifting data. Given $A \in \Psi_s^{m;E}(X;\Omega^{\frac{1}{2}})$ where $E = \{E_i\}$ satisfies $E_i = \emptyset$ $i > 3$, and an operator $B \in \Psi_s^{m';F}$, the exponent set for the product $A \cdot B$ can be obtained from the following table:

fn	$\tilde{\pi}_{3Cb} \mapsto$	fn	$\tilde{\pi}_{3Cb} \mapsto$	fn	$\tilde{\pi}_{3Cb} \mapsto$
f1	$E_1 + F_1$	f4	$E_3 + F_5$	f7	$E_1 + F_8$
f2	$E_2 + F_2$	f5	$E_2 + F_4$	f8	$E_2 + F_7$
f3	$E_3 + F_3$	f6	$E_3 + F_9$	f9	$E_1 + F_6$

Table 5.3. Table for composition with small calculus

In particular, the small calculus $\Psi_s^*(X;\Omega^{\frac{1}{2}})$ is a module for the general calculus and we have another proof of Theorem 4.1.

5.2 Secondary Symbols and Elliptic Operators

In addition to the diagonal symbol map, we can also define a symbol mapping at boundary hypersurfaces. To do so we begin by recalling the results recorded in Appendix A. We obtain symbol mappings at various boundary hypersurfaces for a manifold with corners X and a clean submanifold Y, [see lines (205), (208)]. Setting $X_s^2 = X$, $\Delta_\nu = Y$ and $H = fi(X_s^2)$, $1 \le i \le 9$ we obtain symbol mappings and exact sequences:

$$0 \longrightarrow \mathcal{A}_{phg}^{E+1_{fi}} I^m(X_s^2, \Delta_\nu) \hookrightarrow \mathcal{A}_{phg}^{E} I^m(X_s^2, \Delta_\nu) \xrightarrow{\sigma_H} \xrightarrow{\sigma_H} \mathcal{A}^{\{E\}_{fi}} I^m(N^+(fi), N^+_{fi\cap\Delta_\nu}(fi)) \longrightarrow 0 \tag{140}$$

$$0 \longrightarrow \mathcal{A}_{phg}^{E+1_{fj}} I^m(X_s^2, \Delta_\nu) \hookrightarrow \mathcal{A}_{phg}^{E} I^m(X_s^2, \Delta_\nu) \xrightarrow{\sigma_H} \xrightarrow{\sigma_H} \mathcal{A}^{\{E\}_{fj}}(N^+(fj)) \longrightarrow 0. \tag{141}$$

In chapter two we defined the normal map

$$N : \mathrm{Diff}_s^*(X;\Omega^{\frac{1}{2}}) \longrightarrow \mathrm{Diff}_s^*(ff(X_s);\Omega^{\frac{1}{2}})$$

where $\mathrm{Diff}_s^*(ff(X_s))$ was the enveloping algebra of $\mathcal{V}_{ff(X_s)}$. We extend this to a map on $\Psi_s^{*;E}(X;\Omega^{\frac{1}{2}})$ by setting

$$N(A) = [K_A]_{|f1}, \quad A \in \Psi_s^{*;E}(X;\Omega^{\frac{1}{2}}).$$

We remark that this gives an exact sequence

$$0 \longrightarrow \Psi_s^{m;E\oplus 1_{f1}}(X;\Omega^{\frac{1}{2}}) \hookrightarrow \Psi_s^{m;E}(X;\Omega^{\frac{1}{2}}) \longrightarrow A_{phg}I^m(ff(X_s^2), N_{f1}^{+}\Delta_\nu;\Omega^{\frac{1}{2}}) \longrightarrow 0$$

which is (140). We are particularly interested in the operator $\Delta_G f^2$ of chapter 2. We recall that $ff(X_s^2)$ fibres over $ff(X_s)$ with fibres given by $S \times \mathbf{R}$. The operator $\Delta_G f^2$ lifts to X_s^2 from the left. Restricting to $ff(X_s^2)$ we obtain an operator $(\Delta_G f^2)_1 = N(\Delta_G f^2)$ on $ff(X_s^2)$.

Proposition 5.1 *The operator $(\Delta_G f^2)_1$ is invertible modulo finite rank error on $L^2(ff(X_s^2);\Omega^{\frac{1}{2}})$.*

PROOF Near q in the interior of $ff(X_s^2)$ we have coordinates

$$t,\ Y = \frac{y}{t},\ Y' = \frac{y'}{t},\ \theta,\ \theta'. \tag{142}$$

Lifting $\Delta_G f^2$ from the left we obtain a second order operator on the fibres of $ff(X_s^2)$ depending smoothly on the θ' variables in the base. The operator on the fibres corresponds to the model operator at $ff(X_s)$ and hence by Proposition 2.5 of chapter 2, is inverible modulo finite rank error. To conclude the proof it would suffice to demonstrate smoothness in the remaining variable. To see this is the case we recall that the lift from $\mathcal{P}_{2R}$ to X_s^2 factors as the composition of two blow-down maps. The blow-down map is transverse to the fibres of $ff(\mathcal{P}_{2R}) \simeq M \times ff(X_s)$ and hence the lift of $\Delta_G f^2$ to the intermediate space depends smoothly on the base parameter. Finally we observe that the final blow-down is with respect to a submanifold intersecting the boundary of the front face of the intermediate space and hence does not affect the smoothness of the lifted operator in the base parameter.

□

Similarly, we can lift $\Delta_G f^2$ from the right and restrict to the side face $f2(X_s^2)$ to obtain an operator $(\Delta_G f^2)_2$ along the fibres of $f2(X_s^2)$. Here the work is considerably simpler.

Proposition 5.2 *The operator $(\Delta_G f^2)_2$ is invertible modulo finite rank error on $L^2(f2(X_s^2);\Omega^{\frac{1}{2}})$.*

PROOF Near q in the interior of $f2(X_s^2)$ we have coordinates

$$y,\ Y' = \frac{y'}{y},\ T = \frac{t}{y},\ \theta,\ \theta' \tag{143}$$

with $f2(X_s^2)$ defined by $T = 0$ and the boundary $\partial f2(X_s^2) = f2(X_s^2) \cap ff(X_s^2)$ given by $T = y = 0$. The metric G lifts to a 2-cotensor on X_s^2 and induces an incomplete conic metric along the fibres of $f2(X_s^2)$. The operator $(\Delta_G f^2)_2$ restricts to the fibre to be the incomplete conic Laplacian of the model problem at $fr(X_s)$ [see Proposition 2.4]. As we noted above, the blow-down map ${}^s\beta_2$ factors smoothly through the space $\mathcal{P}_{2R}$. Here however the fibration is immediately seen to correspond to the fibration of the $\mathcal{V}_b$-stretched product of a manifold with boundary [see [7], [8]] as the above coordinates near the intersection $f2(X_s^2) \cap ff(X_s^2)$ indicate. Hence, the operator $(\Delta_G f^2)_2$ depends smoothly on the base variables and the proposition follows.

□

Proposition 5.3 *Let* $P \in \mathrm{Diff}_s^m(X;\Omega^{\frac{1}{2}})$, $A \in \Psi_s^{m';E}(X;\Omega^{\frac{1}{2}})$. *Then* $P \cdot A \in \Psi_s^{m+m';E}(X;\Omega^{\frac{1}{2}})$ *satisfies*

$$\begin{aligned} N(P\cdot A) &= N(P)\cdot N(A) && (144)\\ \sigma_{m+m'}(P\cdot A) &= \sigma_m(P)\cdot\sigma_{m'}(A) && (145)\\ \sigma_{fj}(P\cdot A) &= I_{fj}(P)\cdot\sigma_{fj}(A) && (146)\end{aligned}$$

Proof That $P \cdot A \in \Psi_s^{m+m';E}(X;\Omega^{\frac{1}{2}})$ follows from the theorem as does formula (145). To prove formula (144) it suffices to show that it holds for P given by the Lie action of a vector field $V \in \mathcal{V}_s$. In this case,

$$K_{\mathcal{L}_V\cdot A} = \tilde{\mathcal{L}}_V K_A$$

i.e., the kernel of the product is the kernel of A acted on by V lifted from the left factor where the action is the Lie action. Hence, restricting to the front face and using the tangency of the lifted vector field, we obtain (144). Similarly, the tangency of the lifted vector fields to the other boundary hypersurfaces gives (146).

□

As an immediate corollary we have

Corollary 5.3 *Suppose* $P \in \mathrm{Diff}_s^m(X;\Omega^{\frac{1}{2}})$ *is elliptic, that* E *is an exponent family for* X_s^2 *and that* $R \in \Psi_s^{m';E}(X;\Omega^{\frac{1}{2}})$. *Then there exists* $O \in \Psi_s^{m'-m;E}(X;\Omega^{\frac{1}{2}})$ *such that*

$$PO - R \in \Psi_s^{-\infty;E}(X;\Omega^{\frac{1}{2}}). \tag{147}$$

Proof By proposition 5.3 we can invert P along the diagonal via a direct symbolic construction.

□

For future reference we record

Corollary 5.4 *There exists an operator* $O \in \Psi_s^{-2}(X;\Omega^{\frac{1}{2}})$ *satisfying*

$$\Delta_G f^2 \cdot O = Id - R, \qquad R \in \Psi_s^{-\infty}(X;\Omega^{\frac{1}{2}}).$$

□

6 Mapping Properties

6.1 Conormal Spaces

Recall that the kernels of operators in the small calculus take values in the half-density bundle $\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2)$. In order to establish the various mapping properties of the general calculus we will lift half-densities on X_s from the left and from the right to X_s^2 where the kernel of the operator is defined as a conormal distribution. We then multiply the lift from the right with the operator kernel and push the product forward to the left factor to define an action of the operator on the space in question.

Before proceeding we recall several formulae for lifting densities. We showed (80)

$$^s\beta_{2L}^*\Omega^{\frac{1}{2}}(X_s \times M) \equiv (\rho_1\rho_6\rho_8)^{\frac{1}{2}}\,\Omega^{\frac{1}{2}}(X_s^2) \tag{148}$$

$$^s\beta_{2R}^*\Omega^{\frac{1}{2}}(M \times X_s) \equiv (\rho_1\rho_7\rho_9)^{\frac{1}{2}}\,\Omega^{\frac{1}{2}}(X_s^2). \tag{149}$$

We note that for any index set E,

$$\mathcal{A}_{phg}^{E}(X_s^2;\rho\Omega^{\frac{1}{2}}(X_s^2)) \equiv \mathcal{A}_{phg}^{E\tilde{+}D}(X_s^2;\Omega^{\frac{1}{2}}(X_s^2)) \tag{150}$$

$$\mathcal{A}_{phg}^{E}(X_s^2;{}^b\Omega(X_s^2)) \equiv \mathcal{A}_{phg}^{E\tilde{+}-^b1}(X_s^2;\Omega(X_s^2)) \tag{151}$$

where D is the index set corresponding to the exponents in $\rho = \rho_1\left(\prod_{i=6}^{9}\rho_i^{\frac{1}{2}}\right)\prod_{j=2}^{5}\rho_j^0$ and $-^b1$ is the index set with the single entry -1 for every hypersurface.

Proposition 6.1 *Suppose $A \in \Psi_s^{m;E}(X;\Omega^{\frac{1}{2}})$. Then A defines a linear map*

$$A : \mathcal{A}_{phg}^{F}(X_s;\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{A}_{phg}^{G}(X_s;\Omega^{\frac{1}{2}})$$

where G is given by $\left(\tilde{\pi}_{2Lb}\left[\left(E\tilde{+}\tilde{\pi}_{2R}^{b}F\right)\tilde{+}-^b1\right]\right)\tilde{+}\,{}^b1$.

Proof Choose $A \in \Psi_s^{m;E}(X;\Omega^{\frac{1}{2}})$ and $\phi \in \mathcal{A}_{phg}^{F}(X_s;\Omega^{\frac{1}{2}})$. Fix a nonvanishing smooth half-densities $\nu \in C^\infty(X_s;\Omega^{\frac{1}{2}}(X_s))$ and $\mu \in C^\infty(X_s^2;\Omega_{\mathrm{F}}^{\frac{1}{2}}(X_s^2))$. For $\phi \in \mathcal{A}_{phg}^{F}(X_s;\Omega^{\frac{1}{2}})$ the action of A on ϕ is as in Proposition 3.5:

$$A\phi\cdot\nu \;=\; (\tilde{\pi}_{2L})_*[K_A\cdot\tilde{\pi}_{2R}^*\phi\cdot\tilde{\pi}_{2L}^*\nu]. \tag{152}$$

Let μ be a smooth nonvanishing half-density on X_s^2. Proposition 3.3 and Proposition A.6 allow us to lift conormal distributions on X_s to conormal distributions on X_s^2. We have

$$K_A/\mu \in \mathcal{A}_{phg}^{E}I^m(X_s^2,\Delta_\nu)$$

$$\tilde{\pi}_{2R}^*(\phi/\nu) \in \mathcal{A}_{phg}^{\tilde{\pi}_{2R}^{b}F}(X_s^2)$$

The density factors combine to give a full density on X_s^2:

$$\mu \cdot \tilde{\pi}_{2R}^* \nu \cdot \tilde{\pi}_{2L}^* \nu \in C^\infty(X_s^2; \Omega(X_s^2)). \tag{153}$$

Hence, the product

$$\begin{aligned} K_A \cdot \tilde{\pi}_{2R}^* \phi \cdot \tilde{\pi}_{2L}^* \nu &\in \mathcal{A}_{phg}^{E \bar{+} \tilde{\pi}_{2R}^\flat F} I^m(X_s^2, \Delta_\nu; \Omega) \\ &\equiv \mathcal{A}_{phg}^{(E \bar{+} \tilde{\pi}_{2R}^\flat F) \bar{+} -{}^b 1} I^m(X_s^2, \Delta_\nu; {}^b\Omega). \end{aligned} \tag{154}$$

By Proposition A.7 the product pushes forward and all conormal singularity is integrated out:

$$\begin{aligned} (\tilde{\pi}_{2L})_* [K_A \cdot \tilde{\pi}_{2R}^* \phi \cdot \tilde{\pi}_{2L}^* \nu] &\in \mathcal{A}_{phg}^{\tilde{\pi}_{2L\flat}[(E \bar{+} \tilde{\pi}_{2R}^\flat F) \bar{+} -{}^b 1]}(X_s; {}^b\Omega) \\ &\equiv \mathcal{A}_{phg}^{(\tilde{\pi}_{2L\flat}[(E \bar{+} \tilde{\pi}_{2R}^\flat F) \bar{+} -{}^b 1]) \bar{+} {}^b 1}(X_s; \Omega) \end{aligned} \tag{155}$$

as required.

□

	1	2	3	1	2	3	$(\tilde{\pi}_{2R}^\flat F)_j$	$K_A \cdot \tilde{\pi}_{2R}^* \phi \cdot \tilde{\pi}_{2L}^* \nu$	$\tilde{\pi}_{2L\flat} \mapsto$
f1	0	1	0	0	1	0	F_c	$E_1 + F_c$	fc
f2	0	0	1	1	0	0	F_r	$E_2 + F_r$	fl
f3	1	0	0	0	0	1	F_l	$E_3 + F_l$	fr
f4	1	0	0	1	0	0	F_l	$E_4 + F_l$	fl
f5	0	0	1	0	0	1	F_r	$E_5 + F_r$	fr
f6	1	0	0	0	1	0	F_l	$E_6 + F_l$	fc
f7	0	1	0	1	0	0	F_c	$E_7 + F_c$	fl
f8	0	0	1	0	1	0	F_r	$E_8 + F_r$	fc
f9	0	1	0	0	0	1	F_c	$E_9 + F_c$	fr

Table 1. Lifting data for Proposition 6.1

We note the following special case of the proposition: Suppose $\phi \in L^2(X_s, \Omega^{\frac{1}{2}}) \cap \mathcal{A}_{phg}^F(X_s, \Omega^{\frac{1}{2}})$. Then necessarily $F_i > -\frac{1}{2}$, $i = l, f, r$. Lifting ϕ, we obtain an exponent family $\tilde{\pi}_{2r}^\flat F = \{F_j\}_{j=1}^9$ with

$$\begin{aligned} F_j &= F_l \text{ if } j = 3,\ 4,\ 7 \\ &= F_r \text{ if } j = 2,\ 5,\ 9 \\ &= F_c \text{ if } j = 1,\ 6,\ 8. \end{aligned}$$

The exponent family $E\tilde{+}\tilde{\pi}_{2R}^{b}F$ takes the form

$$\begin{array}{ccc} F_l+E_4 & F_l+E_7 & F_l+E_2 \\ F_c+E_6 & F_c+E_1 & F_c+E_8 \\ F_r+E_3 & F_r+E_9 & F_r+E_5 \end{array}$$

where the position in the array corresponds to the face given in figure 1. Pushing forward we obtain an exponent family $G=\{G_i\}_{i=1}^{3}$ with

$$\begin{aligned} G_i &= \tilde{\pi}_{2Lb}(E\tilde{+}\tilde{\pi}_{2R}^{b}F) \\ &= \{(Z,M);\exists\ (z_j,m_j)\in\tilde{G}_j \ with\ e(i,j)Z=z_j\ for\ one\ j\ and\ M+1=\sum(m_j+1)\} \end{aligned}$$

Here the $e(i,j)$ are either zero or one and G_i is given by the union over the column corresponding to i under $\tilde{\pi}_{2L}$. Minimal elements of G therefore correspond to minimal elements of $\tilde{G}$ over the face (in X_s) in question. Of particular interest is the case $min\{E_i+\tilde{F}_i\}>-\frac{1}{2}$. Given the assumption on F we can insure the above bound on minimum by requiring

$$E_i\ >\ 0\quad 1\leq i\leq 9.$$

6.2 Boundedness for Sobolev Spaces

We begin with the fundamental mapping property of $\Psi_s^{*;\mathcal{E}}(X;\Omega^{\frac{1}{2}})$.

Proposition 6.2 *If $m\leq 0$ and E satisfies $E_i\geq 0\quad 1\leq i\leq 9$, then each $A\in\Psi_s^{m;E}(X;\Omega^{\frac{1}{2}})$ defines a bounded operator on $L^2(X_s;\Omega^{\frac{1}{2}})$.*

PROOF We follow the argument given by Hormander which reduces to the case $m=-\infty$ and then use an argument of Melrose to finish the proof. We begin by fixing $A\in\Psi_s^{m;E}(X,\Omega^{\frac{1}{2}})$ as above. We note that the adjoint of A (acting on $C^{-\infty}(X,\Omega^{\frac{1}{2}})$) is an operator $A^*\in\Psi_s^{m;E'}(X,\Omega^{\frac{1}{2}})$ where the index family E' is given by

$$\begin{array}{ccc} E_5 & E_8 & E_2 \\ E_9 & E_1 & E_7 \\ E_3 & E_6 & E_4. \end{array}$$

In particular, the adjoint of A satisfies the above hypotheses if A does. Suppose that $m<0$. Then the L^2 continuity of A follows from the L^2 continuity of B where $B=A^*A$:

$$<A\phi,A\phi>=<A^*A\phi,\phi>\leq||B||\cdot||\phi||.$$

We note that B is of order $2m<m$. Moreover by the composition formula and the above observations concerning the exponent family of the adjoint the exponent family of B satisfies

the hypothesis of the theorem if the exponent family of A does. If $m = 0$ we can choose a large positive constant M satisfying $M > 2sup|\sigma_A(x,\eta)|$. We then construct an approximate square root for A. We begin by setting

$$c(x,\eta) = (M - \sigma_A(x,\eta))^{\frac{1}{2}}.$$

By choice of M, $c(x,\eta)$ is a symbol and we may choose an operator C with $\sigma_C(x,\eta) = c(x,\eta)$. Then,

$$C^2 = M - A^*A - R, \quad R \in \Psi_s^{-1;E'}(X;\Omega^{\frac{1}{2}})$$

where the exponent family E' is the exponent for A^*A and hence satisfies the hypotheses. Hence the continuity of A is implied by the continuity of R and we may therefore assume that $m = -\infty$.

So suppose $A \in \Psi_s^{-\infty;E}(X;\Omega^{\frac{1}{2}})$. Fix a nonvanishing smooth half-density ν on X_s. For $f,\ g\ \in \dot{C}^\infty(X;\Omega^{\frac{1}{2}})$ we have

$$\begin{aligned}\int_{X_s} f\nu A(g\nu) &= \int_{X_s} (\tilde{\pi}_{2L})_*[\tilde{\pi}_{2L}^*(f\nu)\cdot K_A\cdot\tilde{\pi}_{2R}^*(g\nu)] \\ &= \int_{X_s^2} \tilde{\pi}_{2L}^*(f\nu)\cdot K_A\cdot\tilde{\pi}_{2R}^*(g\nu)\end{aligned}$$

Combining half densities,

$$= \int_{X_s^2} \tilde{\pi}_{2L}^*(f)\cdot\kappa_A\cdot\tilde{\pi}_{2R}^*(g)\rho\gamma, \quad \gamma\in C^\infty(X_s^2,\Omega), \quad \rho = \rho_1\left(\prod_{i=6}^{9}\rho_i\right)^{\frac{1}{2}} \tag{156}$$

where $\kappa_A \in \mathcal{A}_{phg}^E(X_s^2)$. Set

$$\varrho_L = \rho_1\rho_6\rho_8, \qquad \varrho_R = \rho_1\rho_7\rho_9. \tag{157}$$

Then we have

$$\int_{X_s} f\nu\cdot A(g\nu) \leq \int_{X_s^2} |\kappa_A|\cdot|\tilde{\pi}_{2L}^*(f)|\cdot|\tilde{\pi}_{2R}^*(g)|\rho\gamma \tag{158}$$

$$\leq \left(\int_{X_s^2}|\kappa_A|\cdot|\tilde{\pi}_{2L}^*(f)|^2\varrho_L\gamma\right)^{\frac{1}{2}}\cdot\left(\int_{X_s^2}|\kappa_A|\cdot|\tilde{\pi}_{2R}^*(g)|^2\varrho_R\gamma\right)^{\frac{1}{2}} \tag{159}$$

Choose a volume form ω on X_s and denote by ω_R its lift to X_s^2. Choose an n form α, $\alpha = d\varrho_R\wedge\ldots\wedge\beta_n$ such that

$$\omega_R\wedge\alpha = \varrho_R\gamma. \tag{160}$$

Our assumption on the exponent family gives

$$|\kappa_A| \leq C \tag{161}$$

Hence,

$$\left(\int_{X_s^2} |\kappa_A|| \tilde{\pi}_{2R}^*(g)|^2 \varrho_R\gamma\right)^{\frac{1}{2}} \leq C\left(\int_{X_s^2} |\tilde{\pi}_{2R}^*(g)|^2 \omega_R \wedge \alpha\right)^{\frac{1}{2}}$$
$$\leq C'\left(\int_{X_s} |g|^2 \omega\right)^{\frac{1}{2}}. \qquad (162)$$

Repeating the argument concludes the proof of the proposition.

□

We define the L^2-based Sobolev spaces by requiring regularity under the vector fields $\mathcal{V}_b$ where $\mathcal{V}_b$ is as in Appendix A:

$$\mathcal{H}_b^k = \{u \in L^2(X_s, \Omega^{\frac{1}{2}});\ \mathcal{V}_b^j u \subset L^2 \quad \forall j \leq k\}.$$

The dual space to $\mathcal{H}_b^k$ denoted by $\mathcal{H}_b^{-k}$ is defined by the condition

$$u \in \mathcal{H}_b^{-k} \iff u = \sum P_i u_i \;\; P_i \in Diff_b^k(X, \Omega^{\frac{1}{2}}),\;\; u_i \in L^2(X_s, \Omega^{\frac{1}{2}}).$$

For general $m \in \mathbf{R}$, we define

$$\mathcal{H}_b^m(X, \Omega^{\frac{1}{2}}) = \{u \in L^2(X_s, \Omega^{\frac{1}{2}});\ \Psi_s^m u \subset L^2(X_s, \Omega^{\frac{1}{2}})\}\ m \geq 0 \qquad (163)$$
$$\mathcal{H}_b^m(X, \Omega^{\frac{1}{2}}) = \{u;\ u = \sum P_i u_i \;\; P_i \in \Psi_s^m(X, \Omega^{\frac{1}{2}}),\;\; u_i \in L^2(X_s, \Omega^{\frac{1}{2}})\}\ m \leq 0. \qquad (164)$$

In addition, we will need to consider Sobolev spaces weighted by real powers of boundary defining functions:

$$\rho^t \mathcal{H}_b^m(X; \Omega^{\frac{1}{2}}) = \{u \in C^{-\infty}(X, \Omega^{\frac{1}{2}});\ \rho^{-t} u \in \mathcal{H}_b^m(X; \Omega^{\frac{1}{2}})\}. \qquad (165)$$

By a generalization of Rellich's lemma we have

$$\rho^t \mathcal{H}_b^m(X; \Omega^{\frac{1}{2}}) \stackrel{\iota}{\hookrightarrow} \rho^{t'} \mathcal{H}_b^{m'}(X; \Omega^{\frac{1}{2}}) \qquad (166)$$

is compact if and only if $t > t'$, $m > m'$. In view of proposition 6.2 we have

Proposition 6.3 *If r, m, and t are real numbers then each $A \in \Psi_s^m(X; \Omega^{\frac{1}{2}})$ defines a continuous linear map*

$$A : \rho^t \mathcal{H}_b^r \longrightarrow \rho^t \mathcal{H}_b^{r-m}(X; \Omega^{\frac{1}{2}}). \qquad (167)$$

□

6.3 Uniform Maps

Recall, our goal is to produce a paramatrix for the operator $\Delta_G f^2$. Using the above notion of compactness would require producing a paramatrix with error term vanishing at every boundary face of the product X_s^2. This condition is manifestly problematic [see chapter 4.4] since we have no hope of removing the cohomology which survives in the limit as $t \longrightarrow 0$. We circumvent this problem by returning to our initial description of $\Delta_G f^2$ as an opertor along the fibres of the parameter space X. For fixed $t_0 > 0$, we can produce a paramatrix with compact, in fact finite rank, error. Moreover, as t varies $t_0 > t > 0$, we can choose generalized inverses whose error term has range of fixed finite dimension. With the benefit of hindsight we establish this notion as the appropriate requirement for the behavior of our error term.

Definition 6.1 Suppose $P : C^\infty(X_s;\Omega^{\frac{1}{2}}) \longrightarrow C^{-\infty}(X_s;\Omega^{\frac{1}{2}})$. For $\phi \in C^\infty(M;\Omega^{\frac{1}{2}})$, let $\tilde{\phi}$ denote the natural lift of ϕ to X_s. Denote the lift of $C^\infty(M;\Omega^{\frac{1}{2}})$ by $\tilde{C}^\infty(X_s;\Omega^{\frac{1}{2}})$ and consider the action of P on $\tilde{C}^\infty(X_s;\Omega^{\frac{1}{2}})$. We say that the operator P is of uniformly finite rank if the range of P is of finite rank when the domain of P is restricted to $\tilde{C}^\infty(X_s;\Omega^{\frac{1}{2}})$.

A simple yet central example of such an operator is given by fixing smooth functions e', $e \in C^\infty(X_s;\Omega_F^{\frac{1}{2}})$ and considering the operator E corresponding to the kernel

$$K_E = e'(\xi) \cdot e(\eta).$$

For ϕ as above we have

$$\begin{aligned} E\tilde{\phi}(\xi) &= \int_{X_s} K_E(\xi,\eta)\tilde{\phi}(\eta) \\ &= < e,\tilde{\phi} > e'(\xi). \end{aligned}$$

In the sequel we will always use the terms "L^2-bounded" and "finite rank" to refer to maps which are uniformly L^2-bounded and uniformly of finite rank.

7 Paramatrix Construction

7.1 Introduction

In this chapter we prove the main result of this thesis: The resolvent for Δ_G is in the space of s-pseudodifferential operators. We will work with half densities throughout. By definition, the resolvent is a bounded family of operators on $L^2(X_s, \Omega^{\frac{1}{2}})$:

$$\mathcal{R}(\Delta_G f^2; \lambda) = (\Delta_G f^2 - \lambda^2)^{-1}; \quad \lambda \in \mathbf{C}. \tag{168}$$

Clearly, this family may fail to be analytic at integer points along the imaginary axis. We prove

Theorem 7.1 *Let $\mathcal{R}(\Delta_G f^2; \lambda)$ be the resolvent family for $\Delta_G f^2$. Then $\mathcal{R}(\Delta_G f^2; \lambda)$ extends to be meromorphic as a family of operators*

$$\mathbf{C} \setminus \mathbf{N}i \ni \lambda \longrightarrow \mathcal{R}(\Delta_G f^2; \lambda) \in \Psi_s^{-2}(X, \Omega^{\frac{1}{2}}) + \Psi_s^{-\infty;E}(X, \Omega^{\frac{1}{2}}) \tag{169}$$

with poles of finite rank.

PROOF We constuct a paramatrix which approximates $\mathcal{R}(\Delta_G f^2; \lambda)$. For notational convenience we set $A(\lambda) = \Delta_G f^2 - \lambda^2$. By Proposition 5.3, there is an operator $\hat{O}_1(\lambda) \in \Psi_s^{-2}(X, \Omega^{\frac{1}{2}})$ satisfying

$$A(\lambda) \cdot \hat{O}_1(\lambda) - Id = \hat{R}_1(\lambda), \quad \hat{R}_1(\lambda) \in \Psi_s^{-\infty}(X, \Omega^{\frac{1}{2}}). \tag{170}$$

As indicated in Proposition 6.2 the residual term $\hat{R}_1(\lambda)$ is not necessarily compact let alone finite rank. To obtain a compact error we modify the choice of $\hat{O}_1(\lambda)$ to remove as much of the error term along the boundary of X_s^2 as possible. We begin by investigating the behavior of the kernel of $A(\lambda)$ near $ff(X_s^2)$.

Proposition 2.5 and Proposition 5.1 allow us to invert $N(A(\lambda))$ modulo a finite dimensional error. We write

$$N(A(\lambda)) \cdot O_2(\lambda) = N(\hat{R}_1(\lambda)) + P_1 \tag{171}$$

where P_1 is of finite rank. We note that the kernel of $O_2(\lambda)$ can be chosen to decay at the boundary components of $ff(X_s^2)$ which do not intersect the lifted diagonal and to be bounded below by 0 at the remaining faces. Extending $O_2(\lambda)$ and using the surjectivity of the normal operator, we obtain an operator $\hat{O}_2(\lambda)$ satisfying

$$\begin{aligned} A(\lambda) \cdot [\hat{O}_2(\lambda)] &= \tilde{R}_2(\lambda) \text{ where} \\ N(\tilde{R}_2(\lambda)) &= -N(\hat{R}_1(\lambda)) + P_1. \end{aligned}$$

Hence, we obtain

$$A(\lambda)\cdot[\hat{O}_1(\lambda)+\hat{O}_2(\lambda)] \;=\; Id-\hat{R}_2(\lambda)+\hat{P}_1;\;\; \hat{R}_2(\lambda)\in\Psi_s^{-\infty;\mathcal{E}} \tag{172}$$

where $\hat{P}_1$ is an extension of P_1 to X_s^2 and the exponent family $\mathcal{E}$ satisfies $\mathcal{E}_i \geq \epsilon_i > 0$ for $i = 1,\ 4 \leq i \leq 9$. In particular we note that the error term $\hat{R}_2(\lambda)$ for the "rough paramatrix" $\hat{O}_1(\lambda)+\hat{O}_2(\lambda)$ vanishes at the boundary faces $ff(X_s^2),\ \ fi(X_s^2),\ \ 4\leq i\leq 9$.

To remove the error at the remaining boundary faces we produce generalized inverses along the side faces intersecting the lifted diagonal. We note that our choice of solution along the front face gives a relation for the solutions along the side faces $f2(X_s^2)$ and $f3(X_s^2)$. In particular, in order to preserve the vanishing of the error term on the front face of X_s^2 we must produce a kernel on $f2(X_s^2)$ which agrees with the kernel of $\hat{O}_1(\lambda)+\hat{O}_2(\lambda)$ along the common boundary $ff(X_s^2)\cap f2(X_s^2)$.

By Proposition 5.2 the model problem at face $f2(X_s^2)$ admits a solution with finite rank error. Hence, there exists an operator $O_3(\lambda)$ on $f2(X_s^2)$ satisfying

$$I_{f2(X_s^2)}(A(\lambda))\cdot O_3(\lambda) \;=\; -I_{f2(X_s^2)}(\hat{R}_2(\lambda))+P_3$$

where P_3 is of finite rank. In fact, the kernel of P_3 can be expressed as

$$K_{P_3} \;=\; \sum_{j=1}^{l}(y)^{s_j}(y')^{-s_j}F_j(\theta)F_j(\theta')$$

where s_j are the indicial roots for the incomplete conic problem associated to the face $fr(X_s)$. Extending $O_3(\lambda)$ and using the surjectivity of the symbol map at $f2(X_s^2)$ we obtain an operator $\hat{O}_3(\lambda)$ satisfying

$$A(\lambda)\cdot[\hat{O}_1(\lambda)+\hat{O}_2(\lambda)+\hat{O}_3(\lambda)] \;=\; Id+\hat{R}_3(\lambda)+\hat{P}_2+\hat{P}_3 \text{ where } \hat{R}_3(\lambda)\in\Psi_s^{-\infty;\mathcal{E}'}$$

where $\hat{P}_2$ is an extension of P_2 to X_s^2 and the exponent family $\mathcal{E}'$ satisfies

$$\mathcal{E}'_i \geq \epsilon_i > 0 \text{ where } i\neq 3.$$

Similarly, we can use Proposition 2.4 and the obvious symmetry of X_s^2 to construct an operator $\hat{O}_4(\lambda)$ satisfying

$$A(\lambda)\cdot[\hat{O}_1(\lambda)+\hat{O}_2(\lambda)+\hat{O}_3(\lambda)+\hat{O}_4(\lambda)] \;=\; Id-\hat{R}_4(\lambda)+\hat{P}_2+\hat{P}_3+\hat{P}_4$$

We write

$$A(\lambda)\cdot\left[\sum_{i=1}^{4}\hat{O}_i(\lambda)\right] \;=\; Id-\hat{R}_4(\lambda)+\hat{P},\quad \hat{R}_4(\lambda)\in\Psi_s^{-\infty;\mathcal{E}''} \tag{173}$$

where the exponent family $\mathcal{E}''$ satisfies $\mathcal{E}''_i \geq \epsilon_i > 0$. In addition, the exponent family for $\sum_{i=1}^4 \hat{O}_i(\lambda)$ remains bounded below by 0. We now use Proposition 5.2 to remove the error term $\hat{R}_4(\lambda)$ completely. We note that repeated composition results in an operator with exponent family bounded below by an arbitrarily large constant. Hence, near the boundary of X_s the operator $\hat{R}_4(\lambda)$ is small and decreases upon composition. We sum the Neumann series and obtain

$$\hat{O}_5(\lambda) = \sum_{k=1}^{\infty} \hat{R}_4^k(\lambda)$$

and note

$$(Id - \hat{R}_4(\lambda)) \cdot (Id - \hat{O}_5(\lambda)) = Id.$$

Hence, setting

$$\hat{O}_6(\lambda) = [\hat{O}_1(\lambda) + \hat{O}_2(\lambda) + \hat{O}_3(\lambda) + \hat{O}_4(\lambda)] \cdot (Id - \hat{O}_5(\lambda))$$

we have

$$A(\lambda) \cdot \hat{O}_6(\lambda) = Id + \hat{P} \cdot (Id - \hat{O}_5(\lambda)) \tag{174}$$

$$= Id + \hat{Q} \tag{175}$$

where $\hat{Q}$ is finite rank. We denote the transpose of $\hat{O}_6(\lambda)$ by $\hat{O}_6^t(\lambda)$. This operator is a right inverse for $A(\lambda)$ modulo finite rank. The operators $\hat{O}_6(\lambda)$ and $\hat{O}_6^t(\lambda)$ differ by a finite rank operator. Since $A(\lambda)$ is self adjoint, we can construct a two sided self adjoint paramatrix modulo finite rank by symmetrzing. We write

$$A(\lambda) \cdot O(\lambda) = Id - E(\lambda) \tag{176}$$

where $O(\lambda) \in \Psi_s^{-2;E}(X; \Omega^{\frac{1}{2}})$ is self adjoint with exponent family E bounded below by 0. The error term E is finite rank and is given by a kernel

$$K_E = \sum_{i \leq N} e_i' \cdot e_i. \tag{177}$$

In fact, we have somewhat more:

Proposition 7.1 *Suppose e_i is given as in (177). Then e_i can be chosen to be conormal to the boundary of X_s:*

$$e_i \in \mathcal{A}(X_s; \Omega^{\frac{1}{2}}).$$

PROOF By the composition formula for the general calculus we know that the error term $E(\lambda)$ is an element of the calculus, $E(\lambda) \in \Psi_s^{-\infty;F}(X; \Omega^{\frac{1}{2}})$ where the exponent family F is easily computed by keeping track of the exponents at each step of the above construction. But $\Psi_s^{-\infty;F}(X; \Omega^{\frac{1}{2}}) = \Psi_s^{-\infty}(X; \Omega^{\frac{1}{2}}) + \mathcal{A}_{phg}^F(X_s^2; \Omega^{\frac{1}{2}})$.

□

A Appendix A

A.1 Introduction and Disclaimer

This appendix contains the basic results of differential analysis on manifolds with corners. Central to theory are the notions of boundary fibration structure, b-maps, and the role played by conormal distributions.

The material here can be found in [8] to which we refer the reader for any proofs not presented.

A.2 Basic Results For Analysis on Manifolds With Corners

We begin by fixing a manifold with corners Y. Let $\{H_i\}_{i=1}^{n}$ be an enumeration of the boundary hypersurfaces of Y. Let $\mathcal{V}_b$ be the family of smooth vector fields tangent to all boundary faces of Y. Near a boundary face of codimension k we have cooridinates x_1, x_k, y_1, ... y_{n-k} for which elements of $\mathcal{V}_b$ are locally of the form

$$\mathcal{V}_b \ni V = \sum_{i=1}^{k} a_i(x,y) x_i \partial_{x_i} + \sum_{i=1}^{n-k} b_i(x,y) \partial_{y_i} \tag{178}$$

Where the coefficients $a_i(x,y)$, $b_i(x,y)$ are smooth. This immediately implies that there is a smooth vector bundle, ${}^bTX$ satisfying

$$\mathcal{V}_b = C^\infty(Y, {}^bTY). \tag{179}$$

We call ${}^bTY$ the compressed tangent bundle. We have a natural C^∞ vector bundle map $e_b : {}^bTX \longrightarrow TX$ which is of rank k over $\partial_k Y$. For $y \in Y$ we denote the null space of $e_b : {}^bT_yY \longrightarrow T_yY$ by ${}^bN_yY$. These spaces form natural subbundles of ${}^bTX$. We have

Lemma A.1 *Over the interior of each $B \in M_kY$, the b-normal spaces ${}^bN_yY$ form a bundle which is the sum of k naturally trivial 1-dimensional subbundles. This subbundle extends smoothly to a trivial bundle ${}^bNB \subset {}^bT_BY$ over the submanifold B.*

There is a natural category of maps to consider when working over manifolds with corners

Definition A.1 suppose $F : Y_1 \longrightarrow Y_2$ is a smooth map of manifolds with corners. Suppose $\{\rho_j\}_{j=1}^{m}$ are defining functions for the boundary hypersurfaces of Y_1 and that $\{\rho_i\}_{i=1}^{n}$ are defining functions for the boundary hypersurfaces of Y_2. We say that F is a b-map if

$$F^*(\rho_i) = h \prod_{j=1}^{m} \rho_j^{e(i,j)} \quad h \in C^\infty(Y_1), \quad h \neq 0.$$

The matrix $(e(i,j))$ is called the lifting matrix for the map F. Note that the composition of two b-maps is a b-map.

Lemma A.2 *For any b-map $F : Y_1 \longrightarrow Y_2$ the differential $F_* : T_yY_1 \longrightarrow T_{F(y)}Y_2$ and its transpose $F^* : T^*_{F(y)}Y_2 \longrightarrow T^*_yY_1$ extend by continuity from $y \in Y_1^\circ$ to define the b-differentials ${}^bF^* : {}^bT^*_{F(y)}Y_2 \longrightarrow {}^bT^*Y_1$ ${}^bF_* : {}^bT_yY_1 \longrightarrow {}^bT_{F(y)}Y_2$.*

We note that for any b-map F, ${}^bF_* : {}^bN_yY_1 \longrightarrow {}^bN_{F(y)}Y_2$.

Definition A.2 A b-submersion is a b-map $F : Y_1 \longrightarrow Y_2$ between manifolds with corners with ${}^bF_*$ surjective $\forall y \in Y_1$. Similarily, a b-fibration $F : Y_1 \longrightarrow Y_2$ is a b-submersion which is in addition surjective when restricted to a map between b-normal bundles.

We have the following further relation:

Proposition A.1 *A b-submersion $F : Y_1 \longrightarrow Y_2$ is a b-fibration iff the exponent set $e(i,j)$ satisfies for each i $e(i,j) \neq 0$ for at most one j.*

Maps which are b-submersions preserve local geometric models and will play a fundamental role in all that follows.

If $X_2 \subset Y_2$ is a clean submanifold we have an injection

$$ {}^bT_xX_2 \hookrightarrow {}^bT_xY_2 \quad x \in X_2. $$

We say that a b-map F is b-transversal to a submanifold X_2 if

$$ {}^bT_{F(x)}X_2 \ + \ F_*({}^bT_xY_1) \ = \ {}^bT_{F(x)}Y_2. $$

An immediate consequence of b-transversality is that ${}^bF^*$ defines a linear isomorphism

$$ {}^bF^* : \ {}^bN^*_{F(y)}Y_2 \longleftrightarrow {}^bN^*_yY_1 \quad \forall y \in Y_1. $$

Hence, the fibre-b-density bundles over $\mathbf{P}N^*Y_2$ lift under ${}^bF^*$ to the fibre-b-density bundles over $\mathbf{P}N^*Y_1$. Hence, we have a b-map

$$ \tilde{F} : \mathbf{P}N^*Y_1 \longrightarrow \mathbf{P}N^*Y_2 $$

which is an isomorphism on fibres.

A.3 Polyhomogeneity

Suppose Y is a manifold with corners with boundary hypersurfaces $\{H_i\}_{i=1}^m$ and corresponding defining functions $\{\rho_i\}_{i=1}^m$. We begin with

Definition A.3 An exponent set E_i for the hypersurface H_i is a discrete subset $E_i \subset \mathbb{C} \times \mathbb{N}$ satisfying

$$(z_l, m_l) \in E_i, \quad |(z_l, m_l)| \longrightarrow \infty \Rightarrow \Re(z_l) \longrightarrow \infty.$$

An exponent set for Y is a collection of exponent sets $E = \{E_i\}_{i=1}^m$ one for each boundary hypersursurface H_i. We define the space $\mathcal{A}^E_{phg}(Y)$ of polyhomogeneous conormal functions associated to an exponent set E by induction on the maximum codimension of a boundary face. If Y is a compact manifold without boundary there are no exponent sets and we set

$$\mathcal{A}_{phg}(Y) \;=\; C^\infty(Y). \tag{180}$$

Assume that $\mathcal{A}^E_{phg}(Y)$ is defined as a topological vector space whenever Y has boundary codimension at most $k-1$. In particular, the spaces

$$C^\infty_c([0,1); \mathcal{A}^E_{phg}(Y)) \quad \text{and} \quad \dot{C}^j_c([0,1); \mathcal{A}^E_{phg}(Y)), j \in \mathbb{N} \cup \{\infty\} \tag{181}$$

are well defined as topological vector spaces. The second space consists of j-times continuously differentiable functions on [0,1) with values in $\mathcal{A}^E_{phg}(Y)$ and which vanish to order j at 0 and in some neighborhood of 1. Fix a boundary hypersurface H_j. Let $E(j)$ denote the collection of exponent sets for boundary hypersurfaces of Y which meet H_j other than H_j itself. Consider a function on $[0,1) \times H_j$ which has an asymptotic expansion

$$u \;\sim\; \sum_{(z,m)\in E_j(r)} \sum_{k \le m} \rho_j^z (log \rho_j)^k f_{z,k}, \;\; f_{z,k} \in C^\infty_c([0,1); \mathcal{A}^{E(j)}_{phg}(H_j)) \tag{182}$$

where the meaning of the asymptotic expansion is given as follows: For each $r \in \mathbb{R}$ set $E_j(r) = \{(z,m) \in E_j; \Re(z) < r\}$. Note that this is a finite set. Then the meaning of (182) is that for each $j \in \mathbb{N}$ there exists $R = R(j)$ such that

$$u \;-\; \sum_{(z,m)\in E_j(r)} \sum_{k \le m} \rho_j^z (log \rho_j)^k f_{z,k}, \;\; f_{z,k} \in \dot{C}^j_c([0,1); \mathcal{A}^{E(j)}_{phg}(H_j)) \;\; \forall r > R(p) \tag{183}$$

We remark that the asymptotic completeness of these spaces is a consequence of E. Borel's lemma. Finally, we introduce

Definition A.4 *By induction over the maximun codimension of a boundary face, we define*

$$\mathcal{A}^E_{phg}(Y) \;=\; \{u = \tilde{u} + \sum_{j=1}^q u_j; \tag{184}$$

$$\tilde{u} \in \dot{C}^\infty(Y) \textit{ and } u_j \textit{ satisfies (182) for each } j.\} \tag{185}$$

The topology of $\mathcal{A}^E_{phg}(Y)$ is given by the seminorms on the coefficients $f_{z,k}$ and on the remainders.

We note that the definition of $\mathcal{A}^{E}_{phg}(Y)$ is independent of the boundary defining functions chosen and that $\mathcal{A}^{E}_{phg}(Y)$ is a $C^{\infty}(Y)$-module. In particular, we can define the space of polyhomogeneous sections of any C^{∞} vector bundle over Y:

$$\mathcal{A}^{E}_{phg}(Y;G) \;=\; \mathcal{A}^{E}_{phg}(Y)\otimes_{C^{\infty}(Y)} C^{\infty}(Y;G). \tag{186}$$

Elements of $\mathcal{A}^{E}_{phg}(Y)$ should be considered functions which admit series expansions in powers specified by E at the boundary hypersurfaces of Y. With this "working definition" we explore several elementary operations on exponent sets. Given two exponent sets E, F we can define a third exponent set G by setting

$$G_i = E_i \bigcup F_i.$$

We write $G = E\tilde{\cup}F$. Clearly, if $f_1 \in \mathcal{A}^{E}_{phg}(Y)$, $f_2 \in \mathcal{A}^{F}_{phg}(Y)$ then $f_1 + f_2 \in \mathcal{A}^{G}_{phg}(Y)$ where G is given by the above. The union of exponent sets corresponds to addition of polyhomogeneous distributions and we have the following

Proposition A.2 *Addition gives a seperately continuous bilinear map*

$$\mathcal{A}^{E}_{phg}(Y)\times\mathcal{A}^{F}_{phg}(Y) \;\longrightarrow\; \mathcal{A}^{G}_{phg}(Y) \tag{187}$$

where $G = E\tilde{\cup}F$.

Similarly, given E, F we can define an exponent set G for which a multiplication formula holds. We set

$$G_i \;=\; \{(Z,M); Z = z_j + z_k,\; M = m_j + m_k \text{ where } (z_j,m_j)\in E_i,\;\; (z_k,m_k)\in F_i\} \tag{188}$$

We write $G = E\tilde{+}F$ and note, as expected,

Proposition A.3 *Multiplication gives a seperately continuous bilinear map*

$$\mathcal{A}^{E}_{phg}(Y)\times\mathcal{A}^{F}_{phg}(Y) \;\longrightarrow\; \mathcal{A}^{G}_{phg}(Y) \tag{189}$$

where $G = E\tilde{+}F$.

In addition, we can examine the behavior of polyhomogeneous functions under pull-back and push-forward. Given a b-map between manifolds with corners, $F : Y_1 \longrightarrow Y_2$ with E an exponent set for Y_2 we define a mapping of exponent sets

$$F^{b}(E) \;=\; \{G_1,\ldots,G_q\} \tag{190}$$

$$G_j \;=\; \{(Z,M);\; Z_j = \sum e(i,j)z_i,\;\; M_j = \sum m_i,\;\; (z_i,m_i)\in E_i\} \tag{191}$$

We then have

Proposition A.4 *Let $F : Y_1 \longrightarrow Y_2$ be a b-map between manifolds with corners. Then for any exponent set E of Y_2, the pullback of C^∞ functions in the interior extends to a continuous linear map*

$$F^* \;:\; \mathcal{A}^E_{phg}(Y_2) \longrightarrow \mathcal{A}^G_{phg}(Y_1) \tag{192}$$

where $G = F^\flat(E)$.

To obtain results for the push-forward along a b-map we add the b-fibration condition:

$${}^bF^*_p : {}^bT^*_{F(p)}Y_2 \longrightarrow {}^bT^*_pY_1 \text{ injective } \forall p \in Y_1. \tag{193}$$

To define the push-forward of an exponent set we first decompose the exponent set E for Y_1 as

$$E \;=\; E' \oplus E'' \tag{194}$$

where E' corresponds to those exponent sets for hypersurfaces H_j with $e(i,j) \neq 0$. We define the push-forward $G = F_\flat(E)$ by setting

$$\begin{aligned} G_i \;=\; & \{(Z, M);\; \exists (z_j, m_j) \in E'_j,\; E'_j \text{ a component of } E', \text{ such that} \\ & e(i,j)Z = z_j \text{ for one } j \text{ and } M + 1 = \sum_j (m_j + 1)\} \end{aligned} \tag{195}$$

Recalling that the b-density bundle for Y_1 is given by

$${}^b\Omega \cong \left[\prod_{1 \le i \le m} \rho_i\right]^{-1} \Omega$$

we have

Proposition A.5 *Let $F : Y_1 \longrightarrow Y_2$ be a b-fibration. Then for any exponent set E for Y_1 with decompositon as in (194) such that*

$$\Re(z) > 0 \;\; \forall (z, m) \in E''_j, \tag{196}$$

push-forward gives a continuous map

$$F_* \;:\; \mathcal{A}^E_{phg}(Y_1;\, {}^b\Omega) \longrightarrow \mathcal{A}^{F_\flat(E)}_{phg}(Y_2;\, {}^b\Omega). \tag{197}$$

We can use this framework to define a symbol mapping at boundary hypersurfaces. To do so we begin by recalling several facts about conormal distributions on $\mathbf{R}^n$ and their symbols. Let SP denote stereographic compactification on $\mathbf{R}^n$. Then

$$SP^* : \mathcal{A}^m_{phg}(\mathbf{P}^n) \longleftrightarrow S^{-m}(\mathbf{R}^n).$$

and we may write the symbol mapping for distributions conormal to the point 0 as

$$I^m(\mathbf{R}^n, \{0\}) \xrightarrow{\sigma_m} \mathcal{A}^{\{SP(m)\}}_{phg}(\mathbf{P}^n)$$

where the exponent set $SP(m)$ is the exponent set consisting of the single entry $(m, 0)$ and the brackets indicate that the space is actually a quotient by polyhomogeneous functions vanishing to one higher order. If X is a manifold and $Y \subset X$ a clean submanifold then the local coordinate symbol immediately generalizes the above to

$$I^m(X,Y) \xrightarrow{\sigma_m} \mathcal{A}^{\{SP(m)\}}_{phg}(\mathbf{P}N^*Y, \Omega_{b,fib})$$

where $\{SP(m)\}$ is as before and $\mathbf{P}N^*Y$ denotes the projective normal bundle compactified fibre-wise.

The notion of a polyhomogeneous conormal function immediately generalizes to that of a polyhomogeneous distribution conormal with respect to a clean submanifold. As before we fix X a manifold with corners and Y a clean submanifold of X. We then modify Definition A.4 by working locally and replacing $C^\infty(Y)$ with $I^m(X,Y)$. We denote the resulting space by $\mathcal{A}^E_{phg}I^m(X,Y)$. The properties of polyhomogeneous conormal functions then admit analogs for polyhomogeneous conormal distributions.

Proposition A.6 *If $F = X_1 \longrightarrow X_2$ is a b-map of compact manifolds with corners and $Y_2 \subset X_2$ is a clean submanifold b-transversal to F, then for any exponent family E for X_2 and any $m \in \mathbf{R}$ there is a naturally defined continuous pull-back map*

$$F^* : \mathcal{A}^E_{phg}I^m(X_2, Y_2) \longrightarrow I^m(X_1, Y_2)$$

where

$$M = m + \frac{1}{4}(dim X_1 - dim X_2).$$

The symbol at Y_1 is given by

$$\sigma_M(F^*u) = F^\# \sigma_m(u) \quad \forall u \in \mathcal{A}^E_{phg}I^m(X_2, Y_2).$$

To obtain interesting results for push-forward we must (as in the case of conormal functions) add several hypotheses:

Proposition A.7 *Suppose $F : X_1 \longrightarrow X_2$ is a b-submersion b-transversal to a clean submanifold $Y_1 \subset X_1$. Then for any exponent family E for X_1 with decomposition as in (194), pushforward gives a continuous linear map*

$$F_* : \mathcal{A}^E_{phg} I^m(X_1, Y_1;\ {}^b\Omega) \longrightarrow A^{F_b E}_{phg}(X_2;\ {}^b\Omega).$$

In particular, push-forward integrates out all conormal singularity. The final result of this section is fundamental to all our work on manifolds with corners. In essence it is the composition formula for pseudodifferential operators.

Suppose $Y_1,\ Y_1' \subset X_1$ are clean submanifold which intersect b-transversally along a submanifold $W_1 = Y_1 \cap Y_1'$;

$${}^bT_yY_1 +\ {}^bT_yY_1' =\ {}^bT_yX_1 \quad \forall y \in W_1 = Y_1 \cap Y_1'.$$

In particular, W_1 is clean in X_1. We suppose that $F : W_1 \longrightarrow X_2$ embedds W_1 as a clean submanifold $W_2 \subset X_2$. In particular, we then have an embedding

$$F^* \ :\ N^*W_2 \longrightarrow N^*W_1. \tag{198}$$

Since Y_1 and Y_1' intersect b-transversally we have

$$N^*W_1 = N^*_{W_1}Y_1 \oplus N^*_{W_2}Y_1'. \tag{199}$$

Since F is transversal to both Y_1 and Y_1', the image of the embedding (198) meets with neither component of (199). Hence, there is a well-defined product

$$A^{SP(m_1);E(y_1)}_{phg}(\mathbf{P}N^*Y_1) \cdot A^{SP(m_2);E_1'}_{phg}(\mathbf{P}N^*Y_1') \longrightarrow A^{SP(m_1+m_2);(E\bar{+}F)w_1}_{phg}(\mathbf{P}N^*W).$$

In addition we have a natural identification of compressed density bundles

$$[\ {}^b\Omega(S_1) \otimes\ {}^b\Omega_{fibre}(\mathbf{P}N^*Y_1) \otimes\ {}^b\Omega_{fibre}(\mathbf{P}N^*Y_1')]_{|W} =\ {}^b\Omega(X_2)_{|W}.$$

Combining these maps we have a bilinear map

$$F_\# : A^{\{SP(m_1),E_1(Y_1)\}Y_1}_{phg}(\mathbf{P}N^*Y_1;\ {}^b\Omega(X_1) \otimes \Omega_{b,fibre}) \cdot A^{\{SP(m_2);F(Y_2)\}Y_2}_{phg}(\mathbf{P}N^*Y_2;\ {}^b\Omega(X_1) \otimes \Omega_{b,fibre})$$

$$\longrightarrow A^{\{SP(m_1+m_2),E\bar{+}F\}W_1}_{phg}(\mathbf{P}N^*W_1;\ {}^b\Omega(X_2) \otimes \Omega_{b,fibre}).$$

Finally, we state

Proposition A.8 *Let $F : X_1 \longrightarrow X_2$ be a b-submersion between manifolds with corners satisfying (B.1) and suppose that $Y_1, Y_2 \subset X_1$ are clean submanifolds meeting transversally in $W_1 = Y_1 \cap Y_2$ such that F is transversal to Y_1 and Y_2 and embeds W_1 as a submanifold W_2 of X_2, then for any exponent families E_1 and E_2 for the boundary hypersurfaces of X_1 and any $m_1,\ m_2 \in \mathbb{R}$ multiplication on X_1 followed by push-forward to X_2 gives a separately continuous bilinear form*

$$u \times v \longrightarrow F_*(u \cdot v),$$

$$A^{E_1}_{phg} I^{m_1}(X_1, Y_1;\ {}^b\Omega) \times A^{E_2}_{phg} I^{m_2}(X_1, Y_2;\ {}^b\Omega) \longrightarrow A^{F_b(E_1 \bar{+} E_2)}_{phg} I^{m_1+m_2}(X_2, W_2;\ {}^b\Omega) \qquad (200)$$

such that the symbol map at W_2 satisfies

$$\sigma_{m_1+m_2}[F_*(u \cdot v)] = F_\#[\sigma_{m_2} v].$$

Similarly, if X is a manifold with corners and Y is a clean submanifold then the local product decomposition allows us to define a map

$$\mathcal{A}^E_{phg} I^m(X, Y) \xrightarrow{\sigma_m} \mathcal{A}^{\{SP(m);E(Y)\}_Y}(\mathbb{P}^n N^* Y) \qquad (201)$$

where $E(Y)$ consists of those exponents sets which correspond to boundary hypersurfaces that intersect with Y. We have a long exact sequence

$$0 \longrightarrow \mathcal{A}^E_{phg} I^{m-1}(X, Y) \hookrightarrow \ \mathcal{A}^E_{phg} I^m(X, Y) \ \xrightarrow{\sigma_m} \\ \xrightarrow{\sigma_m} \qquad \mathcal{A}^{\{SP(m);E(Y)\}_Y}(\mathbb{P}^n N^* Y) \longrightarrow 0. \qquad (202)$$

We note that if $X = Z \times Z$ is the product of manifolds Z and H is the diagonal in X then (201) gives the standard symbol mapping. This framework also allows us to define "symbol" mappings at various hypersurface of X which give additional information concerning distributional behavior along the specified hypersurface. If H is a boundary hypersurface of X not meeting Y and $u \in \mathcal{A}^m_{phg} I^m(X, Y)$ we have a decomposition

$$u = u' + u'',\ \ u' \in \mathcal{A}^{(E)_H}(X),\ \ u'' \in \mathcal{A}^{(E)_{H^c}} I^m(X, Y) \qquad (203)$$

where $(E)_{H^c}$ is the exponent set E with the component corresponding to H dropped. We have a map

$$\mathcal{A}^E_{phg} I^m(X, Y) \xrightarrow{\sigma_H} \mathcal{A}^{\{E\}_H}(N^+ H) \qquad (204)$$

given by

$$u \longrightarrow [u']. \qquad (205)$$

where the brackets again indicate a quotient. This gives an exact sequence

$$0 \longrightarrow \mathcal{A}_{phg}^{E+1_H} \hookrightarrow \mathcal{A}_{phg}^{E} I^m(X,Y) \xrightarrow{\sigma_H} \\ \xrightarrow{\sigma_H} \mathcal{A}^{\{E\}_H}(N^+H) \longrightarrow 0. \tag{206}$$

Similarly, if $H \cap Y \neq \emptyset$ we have a map

$$\mathcal{A}_{phg}^{E} I^m(X,Y) \xrightarrow{\sigma_H} \mathcal{A}^{\{E\}_H}(N^+H, N_{H\cap Y}^+ H) \tag{207}$$

given by

$$u \longrightarrow [u']$$

where

$$u = u' + u'', \quad u' \in \mathcal{A}^{\{E\}_H} I^m(N^+H, N_{H\cap Y}^+ H), \quad u'' \in \mathcal{A}^{\{E\}_{H^c}} I^m(X,Y). \tag{208}$$

This results in a third exact sequence

$$0 \longrightarrow \mathcal{A}_{phg}^{E+1_H} \hookrightarrow \mathcal{A}_{phg}^{E} I^m(X,Y) \xrightarrow{\sigma_H} \\ \xrightarrow{\sigma_H} \mathcal{A}^{\{E\}_H}(N^+H, N_{H\cap Y}^+ H) \longrightarrow 0 \tag{209}$$

A.4 Blowing Up Submanifolds

In order to simplify analysis near a submanifold it is often useful to introduce polar coordinates around the submanifold. In this section we give an explanation of the process and investigate several of its basic properties.

We begin by blowing up the origin of a vector space V. To do so we simply replace the origin of V with its link in V, the projective sphere:

$$V_{\{0\}} = (V \setminus \{0\})/\mathbb{R}^+ \sqcup [V \setminus \{0\}].$$

The associated blow-down map

$$\beta_{\{0\}} : V_{\{0\}} \longrightarrow V$$

is a diffeomorphism away from the new boundary face in $V_{\{0\}}$ where it has rank 1. To prove that $V_{\{0\}}$ has a well defined C^∞-structure we prove

Lemma A.3 *The action of $GL(n)$ on $\mathbb{R}^n$ lifts to a C^∞-action of $GL(n)$ on $\mathbb{R}^n_{\{0\}}$.*

Note that the above construction and Lemma (with the obvious modifications) apply equally well to the case of $\mathbb{R}^n_k$:

$$\begin{aligned} (\mathbb{R}^n_k)_{\{0\}} &= (\mathbb{R}^n_k \setminus \{0\})/\mathbb{R}^+ \sqcup [\mathbb{R}^n_k \setminus \{0\}] \\ &= S^{n-1} \cap \mathbb{R}^n_k \sqcup [\mathbb{R}^n_k \setminus \{0\}]. \end{aligned}$$

If V is a vector bundle over a base manifold with corners Y (which we identify with the zero section of V) we can use the above to blow-up V along the zero section Y by simply blowing up the origin of each fibre:

$$V_Y = \bigsqcup_{y \in Y} (V_y)_{\{0\}} \; ; \qquad \beta_Y : V_Y \longrightarrow V.$$

The space V_Y has a natural C^∞-structure as a manifold with corners which is given by reference to local coordinates and the use of Lemma A.3.

In general, given a manifold with corners X and a submanifold $Y \subset X$, the above construction should apply whenever X admits a neighborhood of Y which is diffeomorphic to a neighborhood of the zero section of the normal bundle of Y. A sufficient hypothesis for this condition to hold is that Y is a p-submanifold:

Definition A.5 A submanifold $Y \subset X$ of a manifold with corners is a d-submanifold if for each $y \in Y$ there are local coordinates ϕ at y, with coordinate neighborhood $\Omega \subset X$, such that

$$\phi(\Omega \cap Y) = L \cap \phi(\Omega)$$

where L is of the form

$$\begin{aligned} L &= \{x \in \mathbb{R}^n_k; x_{l+1} = \cdots = x_k = 0, \ x_{k+1} \geq 0, \cdots, x_{k+j} \geq 0, \\ &\quad x_{n+j+1} = \cdots x_{n+j+r} = 0\} \end{aligned}$$

with $l, \ r, \ j \geq 0, \quad l \leq k$ and $r + j + k \leq n$. The submanifold Y is a p submanifold if L can be taken of the above form with $j = 0$.

If Y is a p-submanifold of X we define X blown-up along Y to be

$$X_Y = S^+NY \bigsqcup [X \setminus Y]. \tag{210}$$

Here S^+NY is the inward pointing spherical normal bundle to Y in X. This bundle is obtained by first defining the inward pointing spherical normal fibres. Recall, the inward pointing normal fibres are given by

$$N^+_y Y = T^+_y X / T_y Y.$$

Quotienting out the $\mathbb{R}^+$ action we obtain the inward pointing spherical normal fibres:

$$SN^+_y Y = \left(T^+_y X / T_y Y \setminus \{0\}\right) / \mathbb{R}^+.$$

The bundle is then given by

$$SN^+Y = \bigsqcup_{y \in Y} SN^+_y Y.$$

We denote the new boundary component by $ff(X_Y)$. We then have

Theorem A.1 *The space X_Y has a unique C^∞-structure as a manifold with corners such that the blow-down map β_Y is C^∞, is a diffeomorphism of $X_Y \setminus ff(X_Y)$ onto $X \setminus Y$ and has rank $\dim Y + 1$ at each point of $ff(X_Y)$.*

We will also need to know how vector fields lift under blow-down. We have

Proposition A.9 *If $Y \hookrightarrow X$ is a closed p-submanifold then the blow-down map $\beta_Y : X_Y \longrightarrow X$ is a b-map, the space of smooth vector fields on X tangent to Y lifts to span, over $C^\infty(X_Y)$, the space of smooth vector fields on X_Y tangent to $ff(X_Y)$ and the subspace $\mathcal{V}_b(X)$ consisting of vector fields tangent to Y lifts to span, over $C^\infty(X_Y)$, the whole of $\mathcal{V}_b(X_Y)$.*

Finally, we consider the commutativity of the blow-up procedure. Suppose $Y \subset \partial X$ is a p-submanifold, $Z \subset X$ is a p-submanifold and that Z and Y intersect cleanly. If $Z \subset Y$ we define the lift of Z to X_Y by

$$\beta_Y^*(Z) = \beta_Y^{-1}(Z) \subset ff(X_Y).$$

If $Z = cl(Z \setminus Y)$ we define the lift of Z to X_Y by

$$\beta_Y^*(Z) = cl[\beta_Y^{-1}(Z)].$$

We have

Proposition A.10 *Let $Z \hookrightarrow Y \hookrightarrow X$ be clean submanifolds, Y in X, and Z in Y. Define*

$$Z' = \beta_Y^*(Z) = \beta_Y^{-1}(Z) \subset X_Y$$
$$Y' = \beta_Z^*(Y) = cl[\beta_Z^{-1}(Y \setminus Z)] \subset X_Z.$$

Then these are clean submanifolds and there is a natural diffeomorphism

$$(X_Y)_{Z'} \longleftrightarrow (X_Z)_{Y'}. \tag{211}$$

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