NONLINEAR FEEDBACK CONTROL OF CONTINUUM SYSTEMS

by

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ABSTRACT

This thesis develops methods for the analysis and design of nonlinear feedback control for a class of continuum systems with a finite number of unstable modes. Emphasis is upon switched or "bang-bang" feedback. Energy functions and Lyapunov stability criteria are the primary analytical tools. Practical design criteria are presented based on a feedback system with a finite number of memoryless control elements, or with simple lead or lag dynamics added.

Null stability criteria are derived, which emphasize the matching of spatial weighting in the sensor and enforcer elements of the feedback system. A region of stability is calculated, and maximized by requiring strong feedback coupling to unstable modes and weak coupling to stable ones. The minimum number of feedback stations required for stability is shown to be equal to the number of unstable modes. The effects of feedback mechanisms such as hysteresis, time delay, and time sampling are estimated, and the estimates verified by digital computer simulation. Scanning control and spatially continuous feedback are also analyzed. Continuum systems with bang-bang forces due to field discontinuities, such as fluid orientation problems, are treated using the same analysis. Prototype systems are described and designed as examples.

Feedback stabilization of kink modes in TOKAMAK-type plasma confinement devices, such as the M.I.T. ALCATOR, is investigated as an example of such a control system. Preliminary design of that control system is carried out, and its feasibility shown to be limited by the high-speed, high-current switching components needed in the feedback loops. Such feedback may allow higher heating currents in devices with lower magnetic fields.

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Finally, I wish to thank the faculty of the M.I.T. Electrical Engineering Department for providing the education which allowed me to complete this work successfully.

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TABLE OF CONTENTS

1. Introduction

The purpose of this thesis is to provide a guide for the system designer who may want to use bang-bang feedback to control or stabilize some kind of continuous system. It will be assumed that he is familiar with the standard analysis techniques of lumped systems and of linear systems, and that a linearized model of his system can be developed which applies over the operating range of interest. This work will concentrate on the properties of nonlinear controls for distributed systems, with special emphasis on the use of "bang-bang" feedback control, which switches its output very suddenly compared to other scales of interest in the system.

As an example of such a control system, the problem of feedback stabilization of kink modes in the M.I.T. ALCATOR will be investigated in detail. Such instabilities in toroidal plasma-confinement devices pose a serious problem in the design of future controlled fusion plants. Practical considerations make bang-bang feedback particularly interesting for such devices.

1.1 Problems of Feedback System Design

The designer of a feedback control system will face certain problems if he is working with nonlinear controls or with continuous systems, which he may not have encountered before.

Nonlinear systems have the perplexing property that their behavior depends in what may be a drastic way upon the initial conditions of the system, or upon the scale of perturbation introduced into the system. This makes it even more critical than might otherwise be the case for the control system to be planned with enough amplitude in its feedback loops to cope with any expected level of disturbance or control requirement. To make sure of this, it is necessary to be able to describe such a subset of the possible controls and disturbances of the system in a precise manner, and relate it to the feedback requirements.

Continuous systems introduce other complications. Lumped systems may be modelled by a finite number of coupled ordinary differential equations, and so the state of the system at any time can be described exactly by a finite number of parameters. This is not true of distributed or continuous systems, because they typically store energy at every point rather than at discrete locations. Thus a feedback controller, which probably has only a few scalar inputs and outputs, has a real problem in observing the state of the system and so determining the appropriate response. For the designer, this has a great deal to do with the spacial distribution of his sensing elements. The same problem occurs again in the ability to control the system, involving the forcing elements of the controller. Indeed, the total numbers of sensing and forcing elements must be determined by the designer.

In addition to these problems, there are always questions of deviation of a real control system from its "ideal" mathematical representation. How fast must the controller change output levels, and how can such lags be represented? How sudden must the transition be, and what are the effects of dead bands or linear bands of operation near the transition points? How can the designer account for linear or higher order terms in the feedback which might accompany the bang-bang force, intentionally or not? And, of course, if control as well as stabilization is desired, what dynamic limitations will there be on the input?

Each of the above problems will be considered in this thesis, and examples worked to show how the analytical tools may be applied.

1.2 A Guide for Readers

This thesis is intended for several audiences. It is organized so that certain parts may be read independently for different purposes. The physicist or system designer with little background in the field of nonlinear systems will find Chapter 3 a valuable development of the concept, notation, and reasoning in the simplified context of lumped parameter systems. This section is of little interest to those already familiar with nonlinear control systems. The analysis of the continuum control problem begins in Chapter 4 with a specialized model, and the most general form of the problem is posed in Section 4.2.

The example of stabilization of MHD modes in Tokamak-type plasma devices is worked out in detail, with preliminary design for a control system and a feasibility study. This is all in Chapter 5, and is meant primarily for those readers concerned with the problem.

While most of the work in this thesis will apply to continuous physical systems, the results can also be applied in many cases to "large-scale" system design; the structuring of economic and political policies, control of social systems, and the design of management systems at all levels. Such systems may often be modelled quite well by nonlinear coupled systems of differential equations. The considerations governing their control are quite similar in many instances to those governing the control of distributed physical system; both have a very large number of parameters describing the state, both have the problems of choosing the parameters to be sensed and the resulting distribution of force or feedback action over the system. Both are often interested in stable dynamics about a particular choice of equilibrium. Therefore, the designer of these large scale systems might well benefit from a knowledge of the principles of design which emerge from the control of small physical systems. He may also gain from an understanding of the properties of solutions and the relation between system structure and resulting behavior.

1.3 Example: Plasma Confinement

Typically, a continuous system is characterized by several parameters, and often it is desirable to operate the system in a range of parameters which makes certain modes of the system unstable. This is the case in the problem of plasma confinement. One proposed geometry for future fusion power plants is a toroidal hydrogen plasma, surrounded by vacuum and enclosed in a conducting metal shell. The plasma is primarily heated to fusion temperatures by a current which flows around the torus, induced by a changing magnetic flux through the center of the torus. However,as the heating current is raised, certain kinking motions of the plasma are observed to become unstable. The plasma touches the walls of the chamber, hydrogen and energy are lost, and fusion conditions are not reached. Therefore, it is desirable to stabilize such perturbations of the plasma while raising the heating current, using magnetic fields to force the plasma back toward its equilibrium position when it approaches the walls. Because the necessary fields and currents are very large, and the plasma motions rapid, it is probably more practical to switch the feedback fields on and off rather than keep them on and try to linearly vary their strength.

The resulting bang-bang feedback control of a continuum is precisely the kind of problem under discussion. The designer must decide how large he expects the pertrubations of the system to be. He must choose the spacial distribution of his sensing elements around the surface of the torus, and the number required. He must then decide how to construct his feedback current windings to control the expected form of unstable perturbations and determine the number needed. He must design the feedback controller so that fields are applied at the proper times by correct use of the input (sensor) signals.

His analysis must include calculation of the required amplitude of feedback currents and fields. Also, he has to be sure his switching elements can act quickly enough to keep up with the plasma motions. He therefore needs to evaluate the effects of switching lags, as well as other non-ideal effects associated with his equipment and design.

This designer might make a number of rough estimates of the above, based on a simple lumped-parameter model of the unstable modes involved. However, he could not trust his results unless he had some kind of firm theoretical basis for believing that his approximations were good. He would need some information on the behavior of the other modes of the system, and the effect of the feedback on them.

Once these questions were answered, the designer could use his approximate model to examine the feasibility of the feedback scheme and begin to design with hardware. This thesis will answer such questions, and show how the answers may be obtained for other systems.

2. Literature Survey

The literature pertaining to the design of the class of systems of interest is quite large. To organize its content, it will be divided into several broad categories.

First, the literature of nonlinear lumped-parameter systems will be reviewed, with emphasis on the stability of systems and then on their control. Then the background of distributed parameter systems will be reviewed, broken into the same two groups. The stability section will contain more linear analysis in the latter case, since most work has been limited to that approach. Also, the important results for analysis of MHD stability of plasma columns is included. Finally, applications to large-scale systems will be outlined as they have appeared in the literature. All papers are referred to by author, with additional information in the Bibliography.

2.1 Nonlinear Lumped-Parameter Systems

2.1.1 Stability

There are a large number of texts covering the subject of dynamics of nonlinear finite-dimensional systems. Several good ones are those by Lefschetz, Kuo, and Brockett.* Approaches to the subject are generally one of two types. The describing function technique, developed simultaneously by Kochenburger and Goldfarb, approximates the nonlinear elements in the system by linear elements, with a transfer function which depends upon the input amplitude and frequency. This allows various linear stability criteria to be applied, with modifications such as the graphical technique described by McAllister.

Unfortunately, it is so far impossible to generalize these techniques to continuous systems, because the infinite sets of coupled equations do not lend themselves to approximation by a single sinusoidal signal. Therefore, this technique will be restricted in this thesis to qualitative descriptions of effects which can not be easily handled otherwise, and mainly those which are properties of the feedback elements rather than including the distributed system. The literature of this area does provide useful background on the effects of hysteresis,

*References to the Bibliography are cited by author. If more than one work by the same author is listed, the publication date is also cited.

time delay, phase lag, dead and linear bands, and other non-ideal aspects of bang-bang control. Mahalanbis points out the various types of delays and lags in discontinuous feedback systems, and notes the analogy to negative damping which will be developed further. Recent numerical work by Thomassen compares the tolerance of time delay in linear and bang-bang systems for long delays.

The second major approach to stability of nonlinear systems is Lyapunov's direct, or second, method. This is described in detail by Hahn and by Bellman and Kalaba (1964). The mathematics of the method are well established, although a certain amount of variation in exact definitions can be seen in the literature. The identification of Lyapunov functions with energy has been made in several places, including work by Willems and by Wall. A similar approach used in electrical networks can be derived from Tellegen's Theorem and is thoroughly surveyed by Penfield, Spence, and Duinker. Most surveys of Lyapunov's direct method, such as that by Antosiewicz, require that time-derivatives of the state variables be smooth in order that the solution be uniquely determined by, and depend continuously on, the initial conditions. This would rule out use of such a method for bang-bang problems. However, Rosenbrock allows the derivatives of the Lyapunov function to be discontinuous by showing that only the right-hand derivative is required for unambiguous results. This method will therefore be used as the main analytical tool in investigating the stability of nonlinear distributed parameter systems. The theory of Lyapunov's direct method will be described in Chapter 3.

2.1.2 Control

The nonlinear control of finite-dimensional systems has a large literature, full of rigorous mathematics but often deficient in physical insight. The most pertinent work to bang-bang control is concerned with minimum-time optimal control problems, which assume the system to be completely observable. Observability of linear systems is discussed thoroughly by Brockett and will be referred to later. The earliest influential work appears to be that of Solncev, which emphasized stability. This problem was developed into a time-optimization problem and a solution described by Bellman, Glicksberg and Gross. Their work, with modifications and extensions by Gamkrelidze and by LaSalle, showed that the general equation of motion of an N-dimensional linear system with forcing terms of constrained magnitude could be driven to zero in minimum time, by forcing terms with maximum magnitude and variable sign. Also, if the system had real distinct negative eigenvalues, the sign changed at most N-1 times. Such work does not generalize to infinite-dimensional systems.

The problems of discontinuous forcing terms were thoroughly classified by Andre and Siebert in a rigorous work on the subject. The first really complete solution of the time-optimal problem was produced by Bushaw for second-order nonlinear systems. Again, however, his reasoning implies complete knowledge of the system and of all possible options, so that his bang-bang result has little direct bearing on distributed systems. Further works by Athanassiades and Smith on higher order systems and by Doeser on variational techniques lead to the conclusion, however, that minimum-time controllers will generally use bang-bang feedback. A contribution by Chandaket and Leondes generalizes these results to systems with complex, stable roots and time-varying magnitude limits. While this body of literature indicates new reason for interest in bang-bang control, it does not suggest that generalization to unstable or continuous systems will follow easily. Also, observability is assumed with disturbing ease in order to obtain a well-posed problem.

2.2 Distributed Parameter Systems

2.2.1 Stability

Much of the more important early work in the area of stability of distributed systems is described by H. Bateman. Just as Parker analyzes linear lumped systems in terms of normal modes, linear continuous systems are most commonly analyzed that way. A good introduction to such analysis is that of Woodson and Melcher. Modes are defined by the equations of motion of the bulk materials and fields, plus boundary conditions. Any allowed perturbation of the equilibrium condition may be expressed as a sum of such modes. For systems without feedback, or with feedback included in the definition of the modes, each mode evolves in time in a simple exponential fashion. If all the modes are stable, then the equilibrium is stable. Relations between mode spatial structure and temporal behavior are summarized in the dispersion relation. Instabilities can often be classified as absolute or convective by use of the Bers-Briggs stability criterion. Such modal analysis will be important background for this thesis for two reasons. First, the modes of the linear system will be used as a series representation of perturbations with nonlinear feedback, resulting in more complicated time-dependence. Second, bang-bang feedback can be viewed as a saturating linear feedback system in the limit of infinite gain. Thus, for small amplitudes of perturbation, a bang-bang feedback system will behave like a linear system with very large feedback gain. This limit is a useful check on predicted behavior.

The types of feedback which have been investigated for distributed systems can be divided into two types. The first is continuous feedback, in which the feeding back occurs continuously at all points along the distributed system; and discrete feedback, with lumped elements interacting with the distributed system.

In the case of continuous feedback, analytical work has been done with linearized systems and linearized feedback, and experimental work including nonlinear feedback, in a series of publications by Melcher and by Melcher and Warren[(1966)] studying the case of unstable fluid interfaces. A particularly pertinent example involving stabilization of MHD kink modes in plasmas was investigated theoretically by Canales for linear feedback. Another prototype system was investigated by Crowley, again with linear feedback. In each case above, the analysis of the effects of feedback is linearized.

A different approach to the problem, also linearized, involves describing the feedback as a coupling between modes of two distinct systems. Conservation theorems are generated for these systems by an algorithm of Bers and Penfield, and coupling of modes is discussed by Haus. A nonlinear analysis of feedback, taking one mode at a time, is attempted by Melcher, Guttman, and Hurwitz, but does not account for the coupling of modes due to the feedback. It does, however, use an energy picture which is very similar to the approach to be developed later. Essentially, this makes a lumped approximation of the continuous system. A review by Melcher(1970) points out the current problems in this field, as well as the problems of discrete feedback. Use of wave train representations for long systems is outlined, and the problems of mode coupling and feedback nonlinearity as well as the limitations of linear stabilization of plasmas in the presence of interchange modes are discussed but not solved. These problems will be dealt with in later chapters.

In the area of discrete feedback to continuous systems, Melcher(1965) again analyzes linear feedback for a prototype system pointing out the minimum necessary number of such stations and the overstability problems at high gain. This was also considered in Crowley's work and certainly suggests problems for bang-bang systems, as well as a limiting check of results. By taking such linear arguments to their logical conclusion, design criteria for bang-bang systems will be derived. Problems of time-sampled systems and spatially discrete, periodic feedback structures are analyzed by Dressler for linear feedback, using a Z-transform technique. Separation of modes, and linear stabilization of these one at a time, is discussed by Gould and Murray-Lasso. One large problem with such separation is the practical difficulty of implementation of such separation, discussed by Murray-Lasso. This is again essentially a lumped approximation to the extent that separation is imperfect, and will be discussed later. Further work in this direction has been done using dynamic (Luenberger) observers for partial state estimation and pole-moving algorithms to achieve desired modal dynamics. Applications of pole-moving techniques are given by Berkman, and a firm conceptual basis laid by Prado. Prado's work in particular provides a more rigorous examination of some of the control problems not treated in this thesis, such as observability of hyperbolic systems. Luenberger observers of finite dimension, however, are unable to give full state estimates for a continuum.

A linear system with discrete feedback from a scanning linear element is analyzed by Heller. Scanning is converted into a time-sampled, continuous feedback system by a change of time coordinates, and then linear modal analysis used to examine stability. This same coordinate change will be used later in this thesis to examine scanning by a nonlinear discrete feedback station. Heller's linear analysis points out, as does Dressler elsewhere, that time-sampled systems will always destabilize the higher modes of the system unless some damping effect dominates the behavior of such modes.

The work on linear systems does not apply in detail to nonlinear controls, however, because the simple exponential time dependence no longer applies. This complicates the analysis, and requires that this thesis look to other methods to describe the effects of the feedback.

The energy approach to stability analysis is more general. An outline of Lyapunov's direct method for distributed systems was given by Baker in very general terms. It avoids use of the function-space approach which is popular among control theorists, mentioned below. In a more physical context, a variational principle was produced by Bernstein and others which is in fact a form of Lyapunov stability criterion for linear or nonlinear hydromagnetic systems. It is derived in detail and in greater generality by Chandrasekhar. The principle applies to perfectly conducting fluid systems with arbitrary internal and external magnetic fields, either isolated in space or enclosed in a conducting shell, with possible compressibility, viscosity, and gravitational fields. If the minimum possible potential energy due to perturbations is positive, then the system is stable. Eigenvalues may be calculated directly from the perturbations if the form of the modes is known, such as in Rayleigh's earlier work. A tutorial development of the subject is given by Kadomtsev. Feedback is not included in any of this work in explicit terms, although it is not impossible to add it. This thesis shall do so, and review the principle in greater detail.

Since the example of plasma confinement will be considered in detail later, the primary sources of modal analysis for such systems are of interest. The first pertinent work was the analysis of the MHD modes of a cylindrical conducting fluid or plasma with equilibrium currents restricted to the surface, by Kruskal and Schwartzchild. This work was extended by R.J. Tayler to a broader class of currents, and the best simple cylindrical model for fusion devices analyzed by Shafranov (1957). This included both uniform distributed currents and surface currents, and passive feedback in the form of a conducting shell. Extension to toroidal systems was made by Lust and by Mercier using the energy principle. That method has been used by Shafranov (1969) for extended discussion of stability of plasma columns with fixed and free surfaces and arbitrary radial current distributions.

Linear feedback effects have been evaluated for helical field cases by Ribe and Rosenblith, and numerical studies done by Friedberg and by many others in the field. Experimental feedback work has been mainly linear, such as that of Parker and Thomassen. Nonlinear experiments are still in the planning stages, such as a possible bang-bang feedback experiment on the M.I.T. Alcator device which is described in this thesis.

A major problem of feedback to large scale plasma systems is the huge power-bandwidth product required. It is hoped that switched feedback will be easier to implement than linear feedback in these extreme conditions. Another problem with linear feedback is the vanishing of feedback coupling near the interchange condition (field lines parallel to flutes of a mode), discussed by Melcher (1970). These problems will be evaluated in this thesis, and possible solutions explored.

2.2.2 Control

The control of distributed parameter systems shows strong analogy to that of lumped systems. Key concepts such as observability and controllability are developed by Butkovsky as well as an introduction to the problems of optimal control. Unfortunately, as Robinson points out in his review of the topic, most work on optimal control has avoided the questions of stability and observability. Several modifications of the concept of observability are described by Prado.

Work by Lions rigorously treats the existence of optimal controls for many common systems and cost functions. Examples are worked out in detail, including bang-bang results for time-optimal problems. Observability is not discussed, and controllability is assumed. Bellman and Kalaba (1962) use extensions of dynamic programming to approach a very general class of optimal control problems, with the same limitations. Wang (Advances,1964) and others have followed this approach in more specific cases, to obtain feedback forms for the control. Axelband discusses the case of approximate optimal control by use of limited controllers. Stability and observability are again omitted. Stability of a system with optimal control is studied by Wang (IEEE Transactions, 1964) in the case of a system with time delays, using a Lyapunov stability criterion. He assumes observability, and in general assumes the uncontrolled system to be stable in order for the controlled system to be stable. This thesis will specifically consider unstable systems and apply controls without assuming the usual observability criteria. In his examples above, Wang assumes spatially band-limited perturbations, so that a Fourier series representation may be truncated. Such approximations are discussed by Gould; they are essentially a lumped-parameter model, and truncation error is difficult to estimate.

Many case studies of optimal control can be found such as that of Wang (1970) in the area of plasma confinement, where a bang-bang control results. His model does not consider the limitations imposed by real physical measurements.

2.3 Scope of the Thesis

Like the literature discussed in this survey, the thesis results will be divided into lumped and distributed systems.

The section on lumped parameter systems will be essentially a review of known results, in the context of nonlinear system design as it will be described for distributed systems. Observability and Lyapunov stability theory will be explained and defined as found in the literature. They will be applied to a lumped-parameter example, and results derived for bang-bang feedback. The expressions derived in this section will then be directly applicable to distributed parameter systems, and thus facilitate handling any effects which may be described by lumped characteristics, such as lag and related non-ideal feedback effects.

The sections on distributed parameter systems will then concentrate on the practical problems of controlling and stabilizing a continuum. Lyapunov stability criteria will be developed as is done in the literature. However, the state estimation of a continuous system will be modified to account for the practical difficulties of working with lumped sensor elements. Then the bang-bang control problem will be worked out in detail under these limitations. Optimal control will not be attempted. The problems of accounting for all the modes, lumped approximation error, and evaluation of non-ideal effects will be specifically taken into account. Limitations on such control will be discussed, and design criteria developed.

The limit of continuous feedback will be explored. Situations where such an analysis might apply will be modelled, and analytical difficulties outlined. The advantages of such a description will be noted, and the problem of a scanning feedback system converted into a sampled continuous feedback in the nonlinear case.

Several alternative pictures of bang-bang stabilized distributed systems will be compared. Direct modal analysis, wave train analysis, energy transport and velocity pictures, and the method of characteristics will be presented as they might be implemented on a computer. Actual computer simulation will be limited to the modal approach.

The application of bang-bang feedback to plasma confinement will be approached as a specific experimental proposal. The problems and probable success of such an experiment will be discussed in terms of a simple MHD model.

Possible areas for further work include the solution of the time-optimal control problem in the case of limited observability, and the application of design criteria to various specific systems. These range from large-scale systems to plasma confinement experiments.

3. Lumped Parameter Systems

This section is used as a vehicle for developing the concepts, notation, and approximations to be used later in the distributed parameter case. Section 1 reviews the control theory applicable to our problem. Section 2 develops a simple model of an oscillator with a bang-bang feedback force added. This oscillator will correspond to a single mode of the distributed problem in Chapter 4. Section 3 introduces variations on the basic type of feedback, with approximations which are useful in calculating the effects of various corrections for gross prediction of system behavior.

3.1 Review of Concepts

3.1.1 System

We define a system to be a specified set of variables. This may seem unphysical, but in order to isolate one physical system from all others we must specify what variables belong to it and what belong to the external world. Thus, the set may consist of quantities which may be measured inside a certain volume in space, or quantities having some assumed casual relationship to each other.

We then define the state of a system to be the minimum information needed to determine the value of every variable in the system at a given time. If some variables are functions of other variables at an instant of time, then they need not be explicitly specified in order to determine the state. Any subset of variables, then, which can be specified independently, is an acceptable set of state variables. They each represent a coordinate in state space, and so at any instant the state is given by the state vector $\bar{s}(t)$. This allows us to write a general equation of motion for the system

$$\frac{d\bar{s}}{dt} = \bar{K}(\bar{s}, u, t)$$

$$\bar{s}(t=t_o) = \bar{s}_o \qquad (3.1.1.1)$$

where $\bar{K}$ is a function of the state, and of time, and of a vector of input variables $\bar{u}(t)$ not dependent on the state of the system. We will also denote as $\bar{y}(t)$ a vector of variables representing measurements made on the system, with

$$\bar{y} = C(\bar{s}, \bar{u}, t) \qquad (3.1.1.2)$$

and $\bar{C}$ some specified function of its arguments. Equations (1) and (2) will then define a representation of the system, its input, and its output. If the functions $\bar{K}$ and $\bar{C}$ are sums of linear terms in $\bar{s}$ and $\bar{u}$, then the system is said to be linear; if not, then it is nonlinear.

A lumped system is one which has a state vector of finite dimension, as opposed to a distributed or continuous system, which does not.

We refer to the equilibrium of such a system as any state $\bar{s}$ such that when $\bar{u} = o$, $\frac{d\bar{s}}{dt} = o$ for all t. We will in general choose the origin of our coordinates in state space so that $\bar{s} = \bar{o}$ is such an equilibrium.

This origin condition will also be referred to as the null solution, or simply the null.

When feedback is applied to a system in the form $\bar{u}(t) = \bar{F}(\bar{s}, t)$ it is expedient to define a new system including the feedback elements. Thus, the behavior of the original system with $\bar{u} = \bar{o}$ may be quite different from that of the total system with a given feedback law.

In general, it is necessary that $\bar{F}$ satisfy a global Lipschitz condition on $\bar{s}$ to insure a unique solution to the differential equation with initial condition. In particular, the bang-bang force terms which will be considered in this thesis will not satisfy this condition. However, these terms may be treated as a limit of a series of functions which do satisfy the Lifschitz condition, where the limit may be taken at any point in the development. Therefore, no further comment need be made as to the existence or uniqueness of solutions.

3.1.2 Stability

3.1.2.1 Definitions

There are a number of types of stability conditions discussed in the literature. Since we are primarily interested in nonlinear feedback systems, the following definitions are chosen from the sources dealing with Lyapunov's direct method. The systems with which we intend to work are traditionally described by energy methods. We introduce Lyapunov theory here to show the basis for such arguments in a stability analysis.

Combining the equation (3.1.1.1) with a feedback law relating $\bar{u}(t)$ to $\bar{y}(t)$, we may write the equation of motion of the system as

$$\frac{d\bar{s}}{dt} = \bar{f}(t,\bar{s}(t)\ ) \qquad (3.1.2.1)$$

Here we allow $\bar{s}$ and $\bar{f}$ to make on values in Euclidean N-space R^N, defined and with $\bar{f}$ bounded and $\bar{s}$ continuous on the set $\{t,\bar{s}|t>o,||\bar{s}||<\infty\}$ referred to as I x S. It will be assumed that $\bar{f}(t,\bar{o}) = \bar{o}$ on I, and that $\bar{f}$ is sufficiently smooth that there exists a unique solution $\bar{x}(t,t_o,\bar{x}_o)$ in S which depends continuously upon $(t_o,\bar{x}_o)$ and equals $\bar{x}_o$ at $t = t_o$.

We introduce the <u>norm</u> $||\bar{x}||$ as a measure of the length of $\bar{x}$, with properties $||\bar{o}||=o$, $||\bar{x}||>o$ if $\bar{x} \neq o$, and $||\bar{x}||\to\infty$ if any component $x_i\to\infty$, and $||\bar{x}_1 + \bar{x}_2||\leq||\bar{x}_1||+||\bar{x}_2||$.

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For purposes of this thesis, the norm will be

$$||\bar{x}|| = [\sum_{i=1}^{M} x_i^2]^{1/2} \qquad (3.1.2.1.2)$$

where M is the dimension of S, (finite or not). Let N be a subset of S containing the neighborhood of $\bar{x}=\bar{o}$, say $N = \{\bar{x} \mid ||\bar{x}||<r, r>o\}$. Let $I_o=\{t \mid t\geq t_o \geq o\}$.

We may now define the following types of stability:

1) The null solution $\bar{x}=\bar{o}$ is <u>stable</u> if given any $\varepsilon>o$ and $t_o \varepsilon I$, there exists a $\delta(\varepsilon,t_o) > o$ such that $||\bar{x}_o||<\delta$ implies that $||\bar{x}(t)||<\varepsilon$ for all $t > t_o$.

2) The null solution is <u>uniformly stable</u> if the above $\delta(\varepsilon,t_o) = \delta(\varepsilon)$ independent of t_o can be found for any $\varepsilon > o$.

3) The null solution is <u>quasi-asymptotically stable</u> if, given any $t_o \varepsilon$, there exists a $\delta(t_o)>o$ such that $||\bar{x}_o|| < \delta$ implies $||\bar{x}(t)||\to o$ as $t \to \infty$; which means that for any (t_o), there exists a $T(\varepsilon,t_o)$ such that $||\bar{x}(t)|| < \varepsilon(t_o)$ for all $t > t_o + T$.

4) The null solution is <u>asymptotically stable</u> if it is stable and quasi-asymptotically stable.

5) The null solution is <u>uniformly asymptotically stable</u> if it is uniformly stable and if 3) can be satisfied for $\varepsilon > o$ with δ and T not dependent on t_o.

6) The system is globally stable if all solutions are bounded; that is, if given any $t_o \in I$ and $r_o > o$, there exists a finite $r(t_o, r_o) > o$ such that $||\bar{x}_o|| < r_o$ implies that $||\bar{x}(t, t_o, \bar{x}_o)|| < r$ for all $t > t_o$.

7) A system is uniformly globally stable if all solutions are uniformly bounded; which means that the above $r(t_o, r_o) = r(r_o)$ independent of t_o.

In addition, we introduce the following definition of our own, because it is often useful to know what subset of possible initial conditions leads to bounded solutions.

8) A system is said to be stable in a region $R(t_o)$, if for all $\bar{x}_o$ in the region R_o, there exists a finite $r(t_o) > o$ such that $||\bar{x}(t, t_o, \bar{x}_o)|| < r$ for all $t \geq t_o$. (Clearly, it is necessary for $R(t)$ to be bounded for all $t \geq t_o$, in the sense that all vectors $\bar{x}(t)$ in $R(t)$ have bounded norm.)

9) A system is said to be uniformly stable in a region $R(t_o)$ if the above r can be chosen independent of t_o.

3.1.2.2 Lyapunov's Direct Method

Now that we have defined stability, we will develop criteria for determining whether a system is stable according to one of the preceding definitions.

Let $V(t,\bar{x})$ be a real scalar function, defined and locally Lipschitzian on $I_o \times S_o$, $S_o=\{\bar{x} \mid ||\bar{x}|| \leq \rho\}$ (the subset of $I \times S$ of interest), such that given $\bar{x}_o$ in S_o, $\bar{V}(t,\bar{x})$ is continuous on I_o, and require that $V(t,\bar{o}) = o$ on I_o.

Define on $I_o \times S_o$ the functions

$$V' = \frac{dV}{dt} = \lim_{h \to o} \frac{V(t+h,\ \bar{x} + h\ \bar{f}(t,\bar{x})) - V(t,\bar{x})}{h} \tag{3.1.2.2.1}$$

This is the right-hand total derivative of $V(t,\bar{x})$ when $x(t)$ satisfies the equation of motion.

Such a function is <u>positive definite</u> on $I_o \times S_o$ if, given any ε, $o < \varepsilon < \rho$ there exists a $\mu(\varepsilon) > o$ such that $V(t,\bar{x}) \geq \mu$ for all $t \in I_o$, $\varepsilon \leq ||\bar{x}|| \leq \rho$. The function is <u>negative definite</u> if $-V$ is positive definite.

It is decrescent if $V(t,\bar{x}) \to o$ with $\bar{x}$ uniformly on I; that is, for any $\mu > o$ there exists a $\delta(\mu) > o$ such that $||\bar{x}|| < \delta$ implies that $V(t,\bar{x}) < \mu$ for all $t \in I$.

It is <u>radially unbounded</u> if $V(t,\bar{x})$ goes to ∞ with $\bar{x}$ uniformly on I; that is, for any μ there exists an $r(\mu)$ such that $||\bar{x}|| > r$ implies that $V(t,\bar{x}) > \mu$ for all $t \varepsilon I$.

A function $V(t,\bar{x})$ is said to be a Lyapunov function for the system (3.1.3) on $I_o \times S_o$ if:

1) It is defined and locally Lipschitzian on $I_o \times S_o$.

2) It is continuous on I_o and $V(t,\bar{o}) = o$.

3) $V' = \frac{dV}{dt} \leq o$ on $I_o \times S_o$.

We now invoke the following theorems from the literature (the survey by Antosiewicz) to relate properties of $V(t,\bar{x})$ to stability. Proofs are included here to point out the key steps in the reasoning.

Theorem 1) If there exists on I X N a positive definite Lyapunov function $V(t,\bar{x})$, then $\bar{x} = \bar{o}$ is stable. This can be established by a power series in $||\bar{x}||$.

Proof: Given any ε, $o < \varepsilon < \rho$, there is a $\mu(\varepsilon)$ such that $V(t,\bar{x}) \geq \mu$ for all $t \varepsilon I$, $\varepsilon \leq ||\bar{x}|| \leq \rho$ because V is positive definite.

Given any $t_o \varepsilon I$, there is a $\delta(t_o,\varepsilon)$ such that $V(t_o, \bar{x}_o) < \mu$ for all $||\bar{x}_o|| < \delta$. But because $V' \leq o$,

$V(t,\bar{x}) \leq V(t_o \bar{x}_o)$. Thus it is not possible that $||\bar{x}(t)|| \geq \varepsilon$; so $x = o$ is stable.

Theorem 2) If there exists on I x N a decrescent, positive definite Lyapunov function $V(t,\bar{x})$, then $\bar{x} = o$ is uniformly stable.

Proof: The proof is exactly like theorem 1, but because of the added condition it is possible to choose S independent of t_o.

Theorem 3) If there exists on I x N a positive definite Lyapunov function $V(t, \bar{x})$ such that V is negative definite on I x N, then $\bar{x} = \bar{o}$ is stable. Also, for any $t_o \varepsilon I$, any $\rho_o < \rho$, and any ε, $o < \varepsilon < \rho$, there exists a $\delta(t_o, \rho_o)$ and a $\tau_o(\rho_o, t_o, \varepsilon)$ such that if $||\bar{x}_o|| < \delta$, then for some $t_1 \leq t_o + \tau_o$, $||\bar{x}(t_1,t_o, \bar{x}_o)|| < \varepsilon$.

Proof: Given ε, we can find δ by theorem 1. If $||\bar{x}_o|| < \varepsilon$, then let $\tau_o = o$. If $||\bar{x}_o|| > \varepsilon$, then let λ = the largest value of $V(t,\bar{x})$ such that $||\bar{x}|| < \delta$. Then let $\nu > o$ be the minimum value of V such that $||\bar{x}|| > \varepsilon$. Let $\tau_o = \lambda/\nu$.

Then if $||\bar{x}|| > \varepsilon$ for all $t \varepsilon [t_o, t_o \;\; \tau_o]$ it is required that $\mu(\varepsilon) < V(t_o + \tau_o, t_o, \bar{x}_o)$ but

$V(t,\bar{x}) \leq V(t_o \bar{x}_o)$. Thus it is not possible that $||\bar{x}(t)|| \geq \varepsilon$; so $x = o$ is stable.

Theorem 2) If there exists on I x N a decrescent, positive definite Lyapunov function $V(t,\bar{x})$, then $\bar{x} = o$ is uniformly stable.

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Theorem 3) If there exists on I x N a positive definite Lyapunov function $V(t, \bar{x})$ such that V is negative definite on I x N, then $\bar{x} = \bar{o}$ is stable. Also, for any $t_o \in I$, any $\rho_o < \rho$, and any ε, $o < \varepsilon < \rho$, there exists a $\delta(t_o, \rho_o)$ and a $\tau_o(\rho_o, t_o, \varepsilon)$ such that if $||\bar{x}_o|| < \delta$, then for some $t_1 \leq t_o + \tau_o$, $||\bar{x}(t_1,t_o, \bar{x}_o)|| < \varepsilon$.

Proof: Given ε, we can find δ by theorem 1. If $||\bar{x}_o|| < \varepsilon$, then let $\tau_o = o$. If $||\bar{x}_o|| > \varepsilon$, then let λ = the largest value of $V(t,\bar{x})$ such that $||\bar{x}|| < \delta$. Then let $\nu > o$ be the minimum value of V such that $||\bar{x}|| > \varepsilon$. Let $\tau_o = \lambda/\nu$.

Then if $||\bar{x}|| > \varepsilon$ for all $t \in [t_o, t_o \ \tau_o]$ it is required that $\mu(\varepsilon) < V(t_o + \tau_o , t_o, \bar{x}_o)$ but

$V(t_o + \tau_o, t_o, \bar{x}_o) \leq V(t_o, \bar{x}_o) - \nu \tau_o < o$ which is not possible. Therefore, at some t_1 in $[t_o, t_o + \tau_o]$ there must be an $||\bar{x}(t_1)|| < \varepsilon$.

Theorem 4) If there exists on I x N a decrescent positive definite Lyapunov function $V(t,\bar{x})$ and $V'(t,\bar{x})$ is negative definite on I x N, then $\bar{x} = \bar{o}$ is uniformly asymptotically stable.

Proof: By theorem 3, there exists a τ_o and ρ_o and t_1 with $t_1 \varepsilon[t_o, t_o + \tau_o]$ such that if $||\bar{x}_o|| < \rho_o$, then $||\bar{x}(t_1, t_o, \bar{x}_o|| < \delta$ for any δ. By theorem 2, there exists a $\delta(\varepsilon)$ such that if $||\bar{x}(t_1, t_o, \bar{x}_o)|| < \delta(\varepsilon)$, then $||\bar{x}|| < \varepsilon$ for all $t > t_1$. Therefore, $\bar{x} = o$ is uniformly asymptotically stable.

Theorem 6) If there exists a real scalar radially unbounded function $V(t, \bar{x})$ on I x N_c, where $N_c = \{\bar{x} | \ ||x|| > \rho\}$ which is locally Lipschitzian and positive definite there, and if $V' < o$, then the system is globally stable.

Proof: Given any $t_o \varepsilon$ I and $r_o > \rho$, let $V_o(t_o, r_o)$ represent the largest value of $V(t,\bar{r})$ such that $t \geq t_o$, $\rho \leq ||\bar{x}|| \leq r_o$. Let $r(t_o, r_o)$ be chosen so that $V(t, \bar{x}) > V_o$ for all $t \geq t_o$, $||\bar{x}|| > r$.

Then if $||\bar{x}|| < r_o$, it follows that $V(t_o, \bar{x}_o) < V_o$ and for $t > t_o, V(t,\bar{x}) \leq V(t_o,\bar{x}_o)$. Thus for any $t > t_o$, $||\bar{x}|| < r$.

There are other theorems, involving asymptotic stability for all $\bar{x}_o$ and uniform boundedness, which are quite similar to those given here. Thus we will omit these proofs. Primarily, our concern will be with construction of Lyapunov functions to show stability; first of the null solution, and second for some larger region including the null. Since we will not generally be dealing with systems which are globally stable, it is useful to develop theorems for stability in a region. These are original with this work and are given below.

Theorem 7) If there exists a connected, bounded, open region $R(t)$ in S such that $V(t, \bar{x})$ is a Lyapunov function for the system on $I \times R(t)$ and $V(t, \bar{x})$ takes on only its maximum value V_o everywhere on the boundary of $R(t)$, then the system is stable in the region $R(t)$.

Proof: Since the boundary of $R(t)$ is given by $V(t, \bar{x}) = V_o$, and since for all $\bar{x}_o$ inside $R(t_o)$ for all $t_o \in I$ we have $V(t_o, \bar{x}_o) < V_o$, then for all $t \geq t_o$ it is required that $V(t, \bar{x}) < V(t_o, \bar{x}_o)$. Thus at no time $t \geq t_o$ can the state $\bar{x}$ lie on the boundary of $R(t)$, and so $\bar{x}$ remains inside $R(t)$. Since $R(t)$ is bounded, $||\bar{x}||$ remains bounded for all $t \geq t_o$.

Theorem 8) If $V(t, \bar{x})$ is a Lyapunov function on $I \times R(t)$, where $R(t)$ is a connected, open region of S containing the null and bounded by $V(t, \bar{x}) = V_o < \infty$, and if for all $t \in I$ and $\bar{x}(t) \in R(t)$ the quantity $|| \nabla_s V || > \varepsilon > o$ for $\bar{s} \neq \bar{o}$, and all components of $\nabla_s V$ are continuous for $\bar{s} \neq \bar{o}$, then the region $R(t)$ is bounded.

Proof: Let $\bar{s}(t)$ represent any point in $R(t)$. Since $V(t, \bar{o}) = o$, then

$$\infty > V_o > V(t, \bar{s}_1) = \int_{\bar{o}}^{\bar{s}} \nabla_s V \cdot d\,\bar{s} > \varepsilon\, \ell \geq \varepsilon\, ||\bar{s}_1||$$

$$\text{PATH } \ell \text{ in } R(t)$$

$$\text{where} \quad d\bar{\ell} = \frac{\nabla_s V \cdot |ds|}{|\nabla_s V|}$$

Therefore, $||\bar{s}_1||$ is finite, and so $R(t)$ is bounded.

These theorems allow us to establish the stability of a system by constructing a Lyapunov function for it and examining the properties of that function. In general, we will assume that the positive definiteness of the function near the null can be established by the first two terms of a power series expansion in $||\bar{x}||$ of $\nabla_s V \cdot \bar{s}$. (This means that we ignore cases of zero linear force terms.) We do this simply because such cases are physically unlikely in the distributed system situation, and we would need information about the detailed nonlinear behavior of the system to investigate such cases. Once null stability is established, we look for the limits of the region of stability R(t) by looking for a point at which $\nabla_s V = \bar{o}$. This point represents a local maximum for V in the direction normal to $V(t, \bar{x}) = V_o$, and a neighborhood of constant $V(t,\bar{x})$ in all directions along the surface. The point $\bar{s}_o$, given by $\nabla_s V = \bar{o}$ which has smallest value $V(t, \bar{s}_o)$ for all t, sets the limit of R(t), and V_o is the value of $V(t,\bar{s}_o)$ which is minimum.

Note that these theorems give sufficient, but not necessary, conditions for stability. Therefore, the estimates obtained here may be very conservative in some cases.

3.1.3 Observability and Controllability

The concept of observability is quite straightforward: given the output $\bar{y}$ of the system for some time $t_o \leq t \leq t_1$, if it is possible to uniquely reconstruct the state of the system $\bar{s}(t)$ at $t = t_o$, then the system is observable at that time. For a linear system, we may write

$$\frac{d\bar{s}}{dt} = \bar{\bar{A}}(t) \cdot \bar{s} + \bar{\bar{B}}(t) \cdot u$$

$$\bar{y} = \bar{\bar{C}}(t) \cdot \bar{s} + \bar{\bar{D}}(t) \cdot \bar{u} \tag{3.1.3.1}$$

The behavior of the system may be written as

$$\bar{s}(t) = \bar{\bar{\Phi}}(t,t_o) \cdot \bar{s}(t_o) + \int_{t_o}^{t} \bar{\bar{\Phi}}(x, t_o) \cdot \bar{\bar{B}}(x) \cdot \bar{u}(x)dx \tag{3.1.3.2}$$

where $\bar{\bar{\Phi}}(t,t_o)$ is the transition matrix and satisfies the equation and initial condition

$$\frac{d\,\bar{\bar{\Phi}}\,(t,t_o)}{d\,t} = \bar{\bar{A}}(t) \cdot \bar{\bar{\Phi}}(t,t_o)$$

$$\bar{\bar{\Phi}}(t_o,t_o) = \bar{\bar{I}} \tag{3.1.3.3}$$

Since the input and output are supposedly known, we may simplify the problem to one of

$$\bar{y}(t) = \bar{\bar{C}}(t) \cdot \bar{\bar{\Phi}}(t,t_o) \cdot \bar{s}(t_o) \text{ where } \bar{u} = \bar{o}. \tag{3.1.3.4}$$

Given $\bar{y}(t)$ over some interval $t_o \leq t \leq t_1$, can we reconstruct any $\bar{s}(t)$? If so, the system is completely observeable.

This mathematical concept is closely related to the practical problem of state estimation. Given the available output information on a system, how can we estimate the state? This is clearly important for feedback control, since our desired input will depend on certain aspects of the state.

Controllability is the dual of the observeability concept. A system is completely controllable if there exists an input which will drive the state between any two points in the state space in a given time. Theoretically a system which is both observeable and controllable can be handled almost at will, and one begins to look for optimal control schemes.

The relationship between observeability and state estimation becomes less obvious as the number of state variables increases. Although a complex system may be observeable, any algorithm for state estimation may require too many steps to be practical. In the limit of an infinite number of degrees of freedom, as in a distributed system, the lumped concept of observeability loses its value. Much of the literature of optimal control, as was pointed out in Chapter 2, simply ignores the state estimation problem and assumes the state to be known.

The most common method of state estimation is the use of a Luenberger observer. This consists of an external dynamical system with a finite number K of state variables. If it is designed properly, the effect of the outputs $\bar{y}(t)$ on the observer will be to make these K state variables take on the value of the desired information about the original system state $\bar{s}(t)$.

Because of the restricted class of problems we will deal with in this thesis, it will be possible to achieve the desired stability or control with a very simple observer; either one with no memory (K=o), or one including only single time integrations or time-differentiations of the state.

3.2 Model: The Dielectric Slab

3.2.1 The Linear System

In order to clarify the important concepts of bang-bang feedback, a lumped-parameter system is a useful model. In this simplified context, we can develop the notation to be used later with distributed systems, and examine the analytical tools which will be needed then.

Our model will be that of a slab of solid dielectric material, free to move in one direction ξ, and constrained by the effects of an idealized spring and dashpot. This represents the original second-order system, which is linear and time-invariant, before feedback is applied. The purpose of any feedback, unless stated otherwise, is assumed to be to keep the block near $\xi = o$, and minimize the effect of any perturbations.

We can take the mass of the slab to be $M > o$, the spring constant K, the damping b, and the coordinate measured from equilibrium ξ. The equation of the motion is

$$M \frac{d^2 \xi}{d t^2} = -K \xi - b \frac{d \xi}{d t} \tag{3.2.1.1}$$

where K and b may be negative.

Since the system is linear, we may assume solutions of the form

$$\xi(t) = \xi(o) \text{ Cos } \omega t + \frac{\dot{\xi}(o)}{\omega} \text{ Sin } \omega t \tag{3.2.1.2}$$

where ω is in general complex. In fact,

$$j\omega = \frac{-b \pm \sqrt{b^2 - 4 M K}}{2 M} \tag{3.1.2.3}$$

Thus, to assure that the solution be stable, we require

a) $K > o$ and

b) $b \geq o$ (3.2.1.4)

We will refer to violation of the first conditon as a static instability, and the second as a dynamic instability, when the effects are separable. When the condition b) is an inequality, the solution is asymptotically stable.

We may cast the system in state variable form by assuming an external force $F_e(t)$ as the input and the position to be the output. Thus we have

$$\bar{s}' = [\ \xi,\ \dot{\xi}]\quad \frac{d\bar{s}}{dt} = \begin{bmatrix} o & 1 \\ -K & -b \end{bmatrix} \bar{s} + \begin{bmatrix} o \\ 1 \end{bmatrix} F_e(t)$$

$$y = [1 \quad o] \cdot \bar{s}$$

(3.2.1.5)

3.2.2 Feedback

Now we apply a feedback force to the slab. This might be done by creating electric fields near the ends and letting polarization forces give the feedback, as in figure 3.2.2.1. The dielectric material $\varepsilon > \varepsilon_o$ is of dimensions L x W x d, mass M. It exactly fills the gap between two conducting plates, but is held against the polarization forces by the spring and dash-pot. The effect of the voltage is to pull the slab into the plates.

To calculate the feedback force, we note that (neglecting fringing fields) the capacitance of the plates is

$$\frac{Q}{V} = C(\xi) = \frac{W\,L\,\varepsilon_o}{d} + \frac{W\,(\frac{L}{2} + \xi)\,(\varepsilon - \varepsilon_o)}{d} \qquad (3.2.2.1)$$

where Q is the total charge on the top plate. Energy conservation may be written for the electrical sub-system as

$$d\,W = V\,d\,Q - F_e\,d\xi \qquad (3.2.2.2)$$

where $W(Q,\varepsilon)$ is the energy in the capacitor and F_e the force exerted on the slab, measured positive in the

Figure 3.2.2.1 The Dielectric Slab

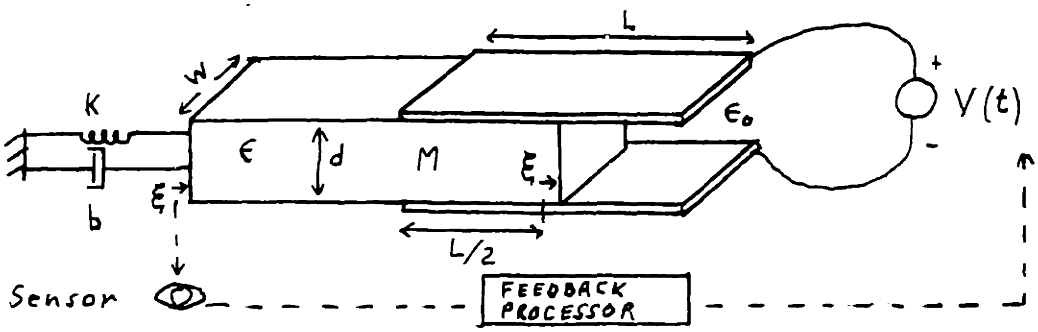


Figure 3.2.3.1 Bang-Bang Feedback

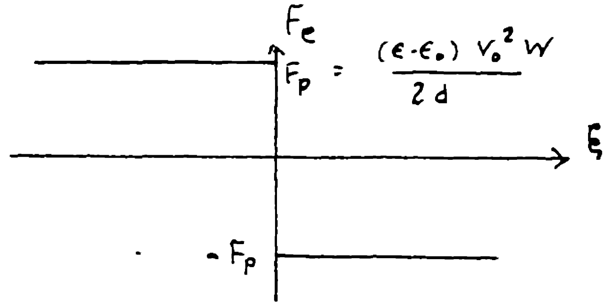
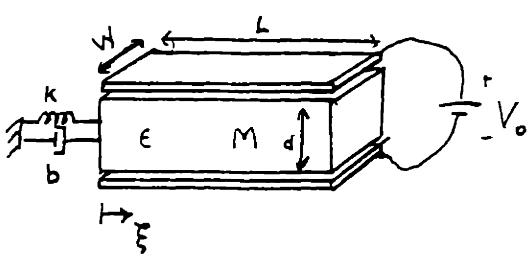


Figure 3.2.3.2 Internal Bang-Bang Force



direction of ξ. Choosing W = o when Q = ξ = o, we may combine 5) and 6) to integrate and find

$$W(Q,\varepsilon) = \frac{1}{2} \frac{Q^2}{C(\xi)} \tag{3.2.2.3}$$

This gives

$$F_e = -\frac{\partial W}{\partial \xi}\Big|_Q = \frac{1}{2} \left(\frac{Q}{C} \right)^2 \frac{\partial C}{\partial \xi} \tag{3.2.2.4}$$

Setting Q = C V and differentiating, we get

$$F(V,\varepsilon) = \frac{V^2 \; W(\varepsilon-\varepsilon_o)}{2d} \tag{3.2.2.5}$$

Thus the force may be made to take on any desired positive value by adjustment of V(t). A negative force could be achieved by placing a similar capacitor-plate arrangement on the other end of the block. The equation of motion of the block may then be written

$$M \frac{d^2\varepsilon}{d\,t^2} = -K\,\varepsilon - b\,\frac{d\varepsilon}{dt} + F_e(t) \qquad (3.2.2.6$$

Our choice of sensors and feedback processing will then determine the dependence of F_e on the position or velocity of the block, on time, or on any other parameters we choose.

3.2.3 Bang-Bang Feedback

One such dependence will be of particular interest. Suppose that the second pair of plates is used, and the feedback arranged so that the second capacitor has voltage V_o applied when $\xi > o$, and the first has the same voltage applied when $\xi < o$. The resulting force depends only on ξ and is graphed in figure 3.2.3.1. It is piecewise constant, both as a function of ξ and as a function of time. Because of its two-state nature we refer to it as bang-bang feedback.

For analytical purposes, we may write this feedback force as

$$F_e = -F_p \left[\frac{\xi}{|\xi|} \right] \qquad \text{where } F_p = \frac{V_o^2 W (\varepsilon - \varepsilon_o)}{2d} \tag{3.2.3.1}$$

where we interpret the case of $\xi = o$ as a limiting form of

$$F_e = \lim_{a \to o^+} - F_p \left[\frac{\xi}{a + |\xi|} \right] \tag{3.2.3.2}$$

There is another way in which we might implement the above feedback law. This would be to arrange the system of Fig. 3.2.3.2 so that in equilibrium, the dielectric block is centered under the plates, exactly filling the space between them and with $V(t) = V_o$. We would then have a situation in which the feedback law defined by 2) exactly describes the electrical force on the slab, but the discontinuity arises from the sharply bounded region of electric fields in space, rather than an artificially imposed feedback law. Thus, an analysis of bang-bang feedback control will also be useful in describing certain classes of systems in which the feedback is an inherent part of the system, caused by spatial discontinuities.

Just as the above feedback law was a bang-bang positional feedback force, we could arrange our feedback to be of the form

$$F_e = - F_v\left[\frac{\dot{\xi}}{|\dot{\xi}|} \right] \qquad (3.2.3.3)$$

which is a bang-bang velocity feedback. Just as positional feedback depended only upon ξ and so represented a sort of nonlinear spring, the velocity feedback represents a nonlinear damping term. A superposition of the two forces is clearly possible as another special case.

Let's be a bit more precise about our definition of bang-bang feedback. We assume, at least for the moment, that an equilibrium exists with $\frac{\partial}{\partial t} = o$. For a linear system, any small deviation on the order of ε from equilibrium produces a force (or analogous expression) proportional to ε. In any practical problem, there will be some range of ε of interest, $\varepsilon_1 < \varepsilon < E_2$. If there is a "feedback" element in the system, with nonlinear characteristics, such that it produces a force on the order of 1, or much larger than the linear term for $\dot{\varepsilon} = \varepsilon_1$, then that nonlinear force appears to be discontinuous on the scale of interest. We therefore refer to it as "bang-bang".

In the next section, the effect of such feedback laws upon the stability and equilibrium of the system will be investigated.

3.2.4 Stability

We will use the total energy of the system, including a contribution due to feedback, as a Lyapunov function. Multiplying equation (3.2.2.6) by $\dot{\xi}$ and integrating by parts, we have

$$\frac{d}{dt}\left[\frac{1}{2} M \dot{\xi}^2 + \frac{1}{2} K \xi^2 \right] = - b \dot{\xi}^2 + F_e(t) \dot{\xi} \tag{3.2.4.1}$$

We may identify the kinetic energy $T = \frac{1}{2} M \dot{\xi}^2$, the potential energy of the original linear system $\psi = \frac{1}{2} K \xi^2$, and the damping "internal" to the original system $B_o = b \dot{\xi}^2$. With $F_e = o$, the energy function $E = T + \psi$ is a Lyapunov function for the system if $b \geq o$, since $d E/ dt = -B_o$. It is positive definite and radially unbounded if $K > o$, giving stability of the null and global stability.

Let us then assume that our feedback processing is such that

$$F_e(t) = - F_p (\xi) - F_v(\dot{\xi}) \tag{3.2.4.2}$$

The feedback force is a superposition of a positioning force and a damping force, henceforth referred to as position and velocity feedback. We may then write

$$\frac{d}{dt}[E] = -B \qquad (3.2.4.3)$$

$$E = T + \psi + U \qquad B = B_o + B_1$$

$$U = \int_o^{\xi} F_p(x)\, dx \qquad B_1 = F_v(\dot{\xi})\ \dot{\xi}$$

This expression will serve as a prototype for all future stability analysis of the class of feedback problems treated in this thesis. In order to dominate any negative contributions to E given by ψ, the form of V requires that $F_p(\xi)$ be restricted to the first and third quadrants. In order to dominate any negative contributions to B from B_o, the function F_v is likewise restricted.

Given this feedback structure, the key points in establishment of stability of the null follow easily. We must find out if $B \geq o$ for all $\bar{s}$ such that $|\bar{s}| = \varepsilon > o$, and for the same $\bar{s}$ establish whether $E(\bar{s}) > o$, or $\nabla_s E \cdot \bar{s} > o$. To do this, we consider consecutively higher-order terms in an expansion of B and $\nabla_s E \cdot \bar{s}$ in ascending powers of ε. The first terms encountered, since F_p and F_v are finite, are first order in ε. If there is a place where these vanish, the second order terms are considered, and so on. In general we need not consider higher than second order terms, because these include the effects of the linear original system and are unlikely to vanish precisely in a physical situation.

Now consider the case of bang-bang feedback:

$$F_p(\xi) = F_p \frac{\xi}{|\xi|} \quad , \quad F_v(\xi) = F_v \frac{\dot{\xi}}{|\dot{\xi}|} \quad . \tag{3.2.4.4}$$

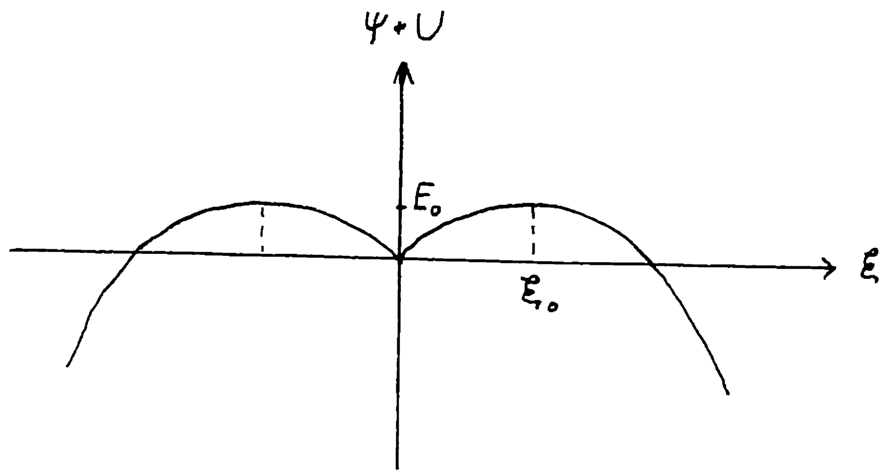
These lead to $U = F_p|\xi|$, $B_1 = F_v|\dot{\xi}|$. Thus to lowest order $B > o$ if $F_v > o$ regardless of b, and if $F_v = o$ is still positive for $b > o$.

$$\nabla_s E \cdot \bar{s} = \frac{\partial E}{\partial \xi} \xi + \frac{\partial E}{\partial \dot{\xi}} \dot{\xi} = K \xi^2 + F_p |\xi| + M \dot{\xi}^2 \tag{3.2.4.5}$$

To lowest order this expression is positive if $M > o$ and $F_p > o$. Thus, even if $K < o$, the system displays null stability if $F_p > o$ because the bang-bang term is dominant at small amplitudes, and a similar argument

Figure 3.2.4.1

Potential Energy Picture of Bang-Bang Stabilization



holds for damping terms. Figure 3.2.4.1 displays the effect graphically, showing potential energy $\psi + U$ as a function of ξ for the case $K < o$. Note that if $F_p < o$ or $F_v < o$, the null is unstable regardless of K and b. We shall also note that if the feedback is constrained by $|F_p(t)| \leq$ some constant, then bang-bang positional feedback results in the fastest energy increase as a function of ξ of all possible feedback functions. Therefore, it results in confining ξ to the smallest range of values for a given energy of perturbations. Bang-bang feedback also has the advantage of compatability with digital processing equipment and simplicity of implementation; only switches are necessary, as opposed to amplifiers.

However, the same reasoning leads to the conclusion that at some large amplitude the linear term will dominate, and we expect instability for $K < o$. Therefore, it is desirable to determine the limits of the region of stability R. Note that if null stability applies, the region R is known to exist in the neighborhood of the null, and so its existence need not be proved.

Since $F_p > o$ and $F_v \geq o$, the region of stability can not be limited by the feedback. We must look for a region with boundary determined by one of the following.

1) B becoming negative outside the boundary;

2) E decreasing outside the boundary.

The first condition requires B = o on the boundary, since B is continuous. The second could result from the derivative of E normal to the boundary going to zero or jumping discontinuously to a negative value.

At this point we notice that along switching surfaces D = ξ=o, it is always true that $\nabla_s U = o$. Thus, the value of $\nabla_s E$ along such surfaces is determined entirely by the linear force terms. The condition that these give nonzero $\nabla_s E$ is precisely the condition for null stability along such directions: $\nabla_s E \cdot \bar{s} > o$ for D = o. Therefore, we know that the limit of R due to E reaching a maximum may be found from $\nabla_s E = o$, (stability theorem 8) because the points of discontinuous $\nabla_s E$ have been ruled out. This gives a region of stability as $E(\bar{s}) < E_o$ where $E_o = E(\bar{s}_o)$ and $\bar{s}_o$ is the point of lowest E such that $\nabla_s E = \bar{o}$. If E(s) has been chosen wisely, the region so defined includes most initial conditions leading to bounded responses.

We therefore find two regions: R_1 bounded by the surface B = o, and R_2 bounded by the surface $E(\bar{s}) = E_o$. The region of stability R will then be the region $E(\bar{s}) < E_1$ where E_1 is the minimum value of E on either of the two boundaries. This must be true, because inside E_1 we know that $E < E_1$ and $B \geq o$. Therefore, the energy can not grow to the value E_1, and it must do so to cross the boundary. Thus if the system ever enters the region R, it can not leave.

Consider for example $F_v = o$, $b > o$, $F_p > o$, $K \neq o$. Then $B \geq o$ everywhere, but $\nabla_s E = M \dot{\xi} \dot{\xi} + (K \xi + F_p \frac{\xi}{|\xi|}) \xi$. This gives

$$\nabla_s E = o \text{ at } \dot{\xi}_o = o, \ \xi_o = \frac{-F_p}{K} \frac{\xi_o}{|\xi_o|} \tag{3.2.4.8}$$

Note that this is simply the condition that the feedback and spring forces are balanced. We can find a self-consistent solution for ξ_o only if $K < o$; hence, for $K > o$, the region R has no boundary and the system is globally stable. However, if $K < o$, $\xi_o = -\frac{F_p}{K} \frac{\xi_o}{|\xi_o|}$, which can be verified by taking cases for the sign of ξ_o. Thus $\bar{s}_o = \xi_o, \dot{\xi}_o$, giving $E_o = - \frac{F_p^2}{2K} > o$, from $E_o = E(\xi_o, \dot{\xi}_o)$. Thus, the region of stability is bounded by the surface

$$E(\bar{s}) = \frac{1}{2} M \dot{\xi}^2 + \frac{1}{2} K \xi^2 + F_p |\xi| = E_o \tag{3.2.4.7}$$

Inside this surface, $o \leq E(\bar{s}) < E_o$ and all solutions having such initial conditions are stable. In the case K = o, we find that ξ_o is infinite, as in E_o. This simply means that once again the system is globally stable. The case K = o is clearly no problem.

Any finite number of coupled second-order systems can be analyzed in an analogous manner. Clearly, however, if there are higher-order nonlinear terms in any part of the system, they must be included in determination of the boundary of R, and will lead to more difficult solution.

We may then say that one system is more stable than another, if (all other factors being equal) it has a larger region of stability; assuming that E is well chosen.

We can plot these stability results in several ways. First, we can directly draw the surfaces of constant $E(\bar{s},t)$ for any given t in the state space, and shade in the region of stability. This gives a measure of the amplitude of disturbances which have been stabilized by any given feedback scheme.

Second, we can assume some simple form of perturbation, and plot the limits of stability in terms of the feedback amplitude required versus some destabilizing parameter; say, F_p versus (-K) or F_v versus (-b) for fixed ξ_o and $\dot{\xi}_o$. Such a plot might be useful to a system designer. The coordinates could be put in

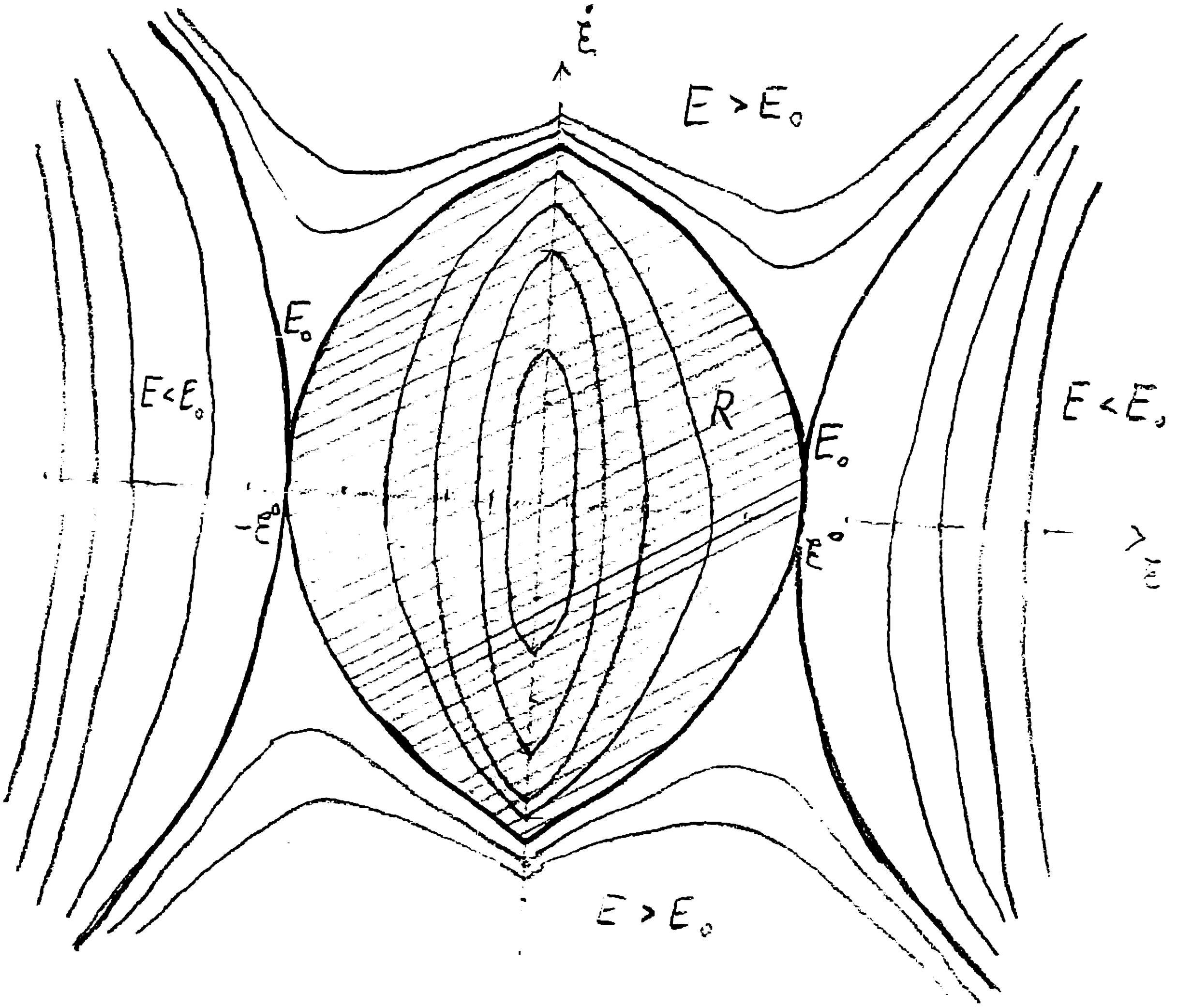


figure 3.2.4.2
Region of Stability

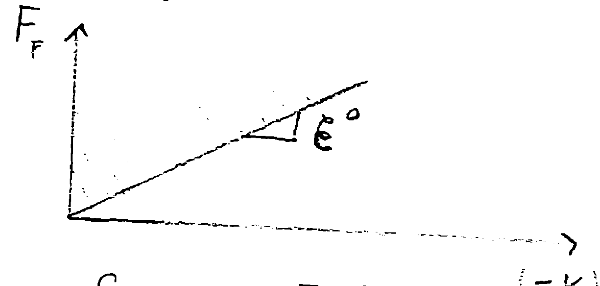


figure 3.2.4.3
Range of Stable Parameters

dimensionless form by use of the mass M and some appropriate lengths and times.

Consider the example where we scale our quantities so that $M = 1$, take $b = o$, and set $K < o$, $F_p > o$ constant. A sketch of $E(\bar{s})$ = constant is shown in figure 3.2.4.2. A plot of stability parameters showing F_p vs. $(-K)$ at constant ξ_o, is shown in fig. 3.2.4.3.

3.2.5 Comparison with Other Approaches

In order to examine the advantages and disadvantages of this type of analysis, we will look at the positional bang-bang feedback problem from two other points of view.

First, we can directly integrate the equation of motion to obtain, from

$$M \ddot{\xi} = -K \xi - F_p \frac{\xi}{|\xi|} \tag{3.2.5.1}$$

the exact solution, where $\omega = K/M$, as long as $\xi > o$

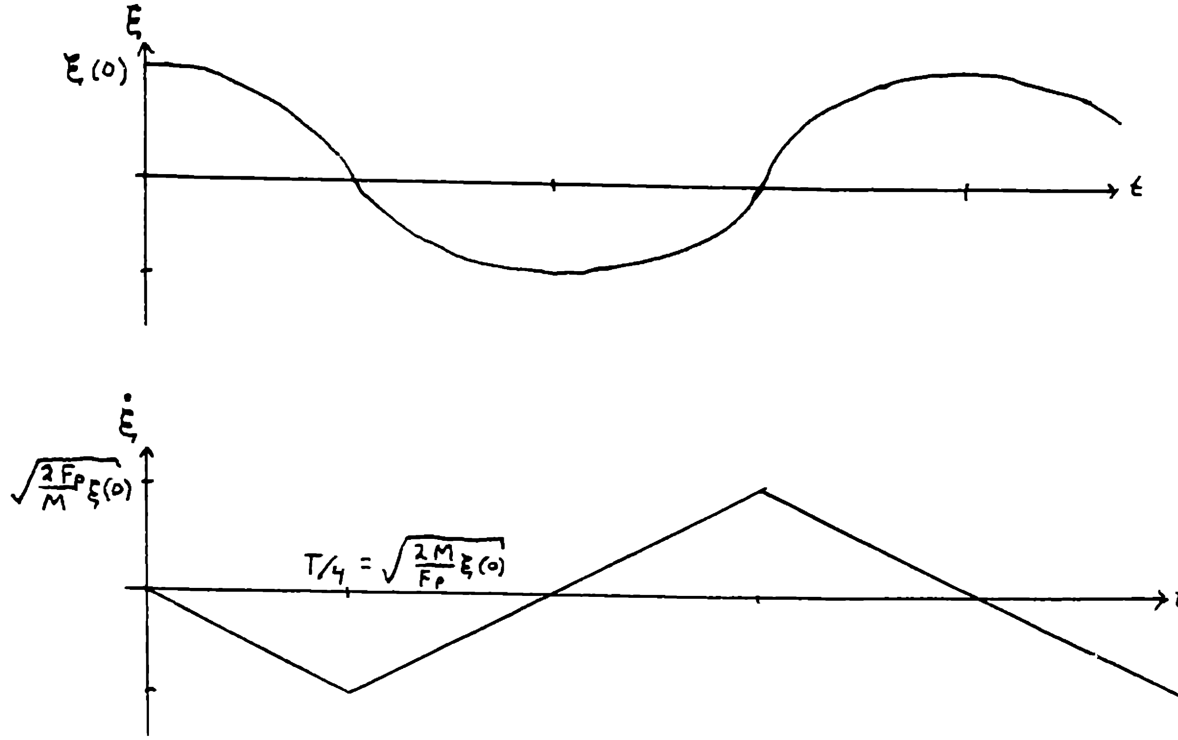
$$\xi(t) = \frac{-F_p}{K} + [\ \xi(t=o) + \frac{F_p}{K}] \cos \omega t + \dot{\xi}(t=o) \frac{\text{Sin } \omega t}{\omega}$$

$$\xi(t) = -[\xi(t=o) + \frac{F_p}{K}]\omega \text{ Sin } \omega t + \dot{\xi}(t=o) \cos \omega t \tag{3.2.5.2}$$

When ξ changes sign, the solution is re-started with new initial conditions. A special case with $K = o$, $\dot{\xi}(t=o) = o$ is graphed in figure 3.2.5.1, where ξ is a piecewise-parabolic function of time. Notice that the time between zero-crossings (half a period) is $T/2 = 2\sqrt{\frac{2M}{F_p} \xi(o)}$, which depends upon the initial conditions. The energy approach does not tell us the

Figure 3.2.5.1

Exact Solution to Bang-Bang Force Problem



exact form of solution, but loses that information in order to simplify the analysis.

The second method to be used for comparison is the describing function approach. If a sinusoidal signal $\xi = \xi(o) \cos \omega t$ enters the bang-bang feedback element, then the output has a fundamental component

$$F_p(t) = F_p \frac{4}{\pi} \cos \omega t \tag{3.2.5.3}$$

We may therefore model the feedback element as a linear filter

$$G(\omega,\xi) = \frac{4 F_p}{\pi \xi(o)} \tag{3.2.5.4}$$

to the extent that higher harmonics may be ignored. Oscillations will then occur at a frequency and amplitude $\xi(o)$ such that $\left(- \frac{1}{M^2} \right) \left(\frac{4 F_p}{\pi \xi(o)} \right) = -1 =$ open loop gain, or

$$\omega = 2\sqrt{\frac{F_p}{\pi M \xi(o)}} \text{ giving a half-period of } \frac{T}{2} = 2\pi \sqrt{\frac{\pi M \xi(o)}{F_p}} .$$

Comparing with the exact solution, we see that the error is a factor of $\pi \sqrt{\frac{\pi}{2}}$ in cycle time prediction.

Both these methods depend upon having an exact expression for the state of the system at all times, at least to a sinusoidal steady-state limit. Thus they will have limited utility in a distributed problem.

3.2.6 Feedback-coupled Oscillators

The purpose of looking at lumped systems in this way is to develop a prototype for feedback control of distributed systems. Thus the mass, spring, and dash-pot of this example will correspond to a single mode of a distributed system. Since in most cases there will be more modes than feedback elements, we now consider the case of two dielectric slabs coupled together by a single feedback element. This example involves only positional feedback.

Let the equation of motion for each slab be as follows:

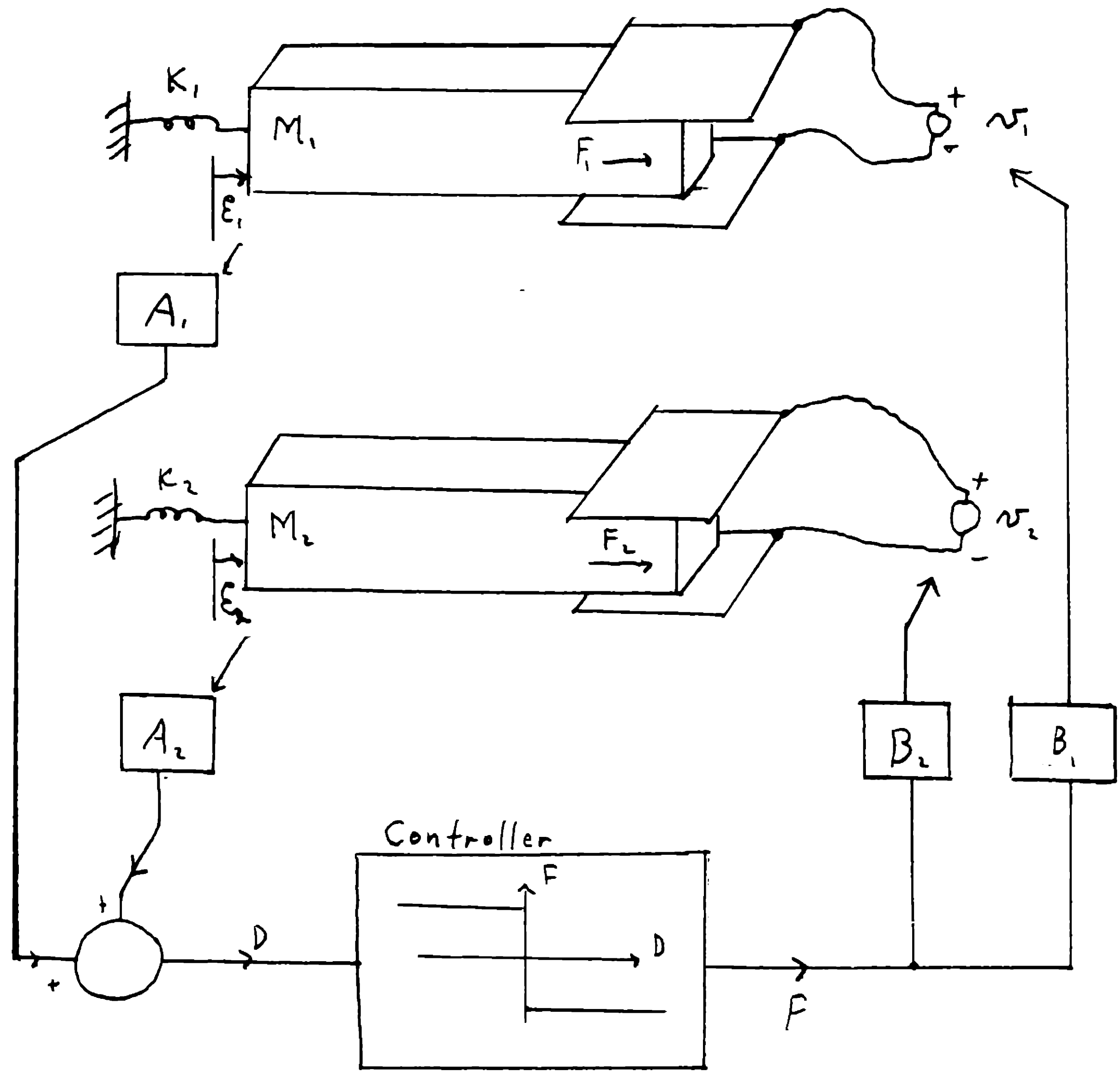
$$M_1 \frac{d^2 \xi_1}{dt^2} = -K_1 \xi_1 - b_1 \frac{d\xi_1}{dt} + F_1(t)$$

$$M_2 \frac{d^2 \xi_2}{dt^2} = -K_2 \xi_2 - b_2 \frac{d\xi_2}{dt} + F_2(t) \qquad (3.2.6.1)$$

The sensor of the single feedback element will be taken to sense a linear sum of the displacements. The resulting sensor signal will be referred to as the discriminant $D(t) = A_1 \xi_1(t) + A_2 \xi_2(t)$.

The force due to the feedback element will also be taken to be linearly distributed to the two slabs

Figure 3.2.6.1 Feedback-Coupled Oscillators



The controller input is a linear combination of slab positions, and its output results in a feedback force of different weight for each slab.

$F_1(t) = -B_1\ F_p(t)$, $F_2(t) = -B_2\ F_p(t)$, where the force $F_p(t) = F_p(D(t))$ depends only on the discriminant.

The state of the composite system is then described by the two displacements and the two velocities, giving a four-dimensional state space. We form the energy (Lyapunov) function as before by multiplying each equation of motion by its velocity and integrating by parts:

$$\frac{d}{dt}\left[\ \frac{1}{2} M_1\ \dot{\xi}_1^{2} + \frac{1}{2} K_1\ \xi_1^{2}\ \right] = -\ b_1\ \dot{\xi}_1^{2} - b_1\ \dot{\xi}_1\ F_p(D) \tag{3.2.6.2}$$

$$\frac{d}{dt}\left[\ \frac{1}{2} M_2\ \dot{\xi}_2^{2} + \frac{1}{2} K_2 \xi_2^{2}\ \right] = -b_2\ \dot{\xi}_2^{2} - b_2\ \dot{\xi}_2\ F_p(D)$$

The equations are then added together to obtain a single energy equation. We then make use of the fact that

$$\frac{d}{dt}\left[\ \int_o^{C(t)} F_p(x)\ dx\right] = F_p(C)\ \frac{dC}{dt} = F_p(D)\ \frac{dC}{dt} + \frac{dC}{dt}\left[F_p(C)-F(D)\right] \tag{3.2.6.3}$$

to rewrite the equation as

$$\frac{dE}{dt} = -B,\ E = T + \psi + U,\ B = B_1(t) + B_2(t)$$

$$T = \frac{1}{2} M_1 \dot{\xi}_1^{\,2} + \frac{1}{2} M_2 \xi_2^{\,2} \qquad B_1(t) = b_1 \dot{\xi}_1^{\,2} + b_2 \dot{\xi}_2^{\,2}$$

$$\psi = \frac{1}{2} K_1 \xi_1^{\,2} + \frac{1}{2} K_2 \xi_2^{\,2}$$

$$U = \int_0^{C(t)} F_p(x)\, dx \quad \text{where } C(t) = B_1 \xi_1(t) + B_2 \xi_2(t)$$

$$B_2(t) = \frac{d}{dt} \left[F_p(D(t)) - F_p(C\,(T)) \right] \tag{3.2.6.4}$$

Thus, for any nonlinear feedback function $F_p(D)$, in order to stablize the equilibrium we want $F_p(D)$ to be in the first on third quadrant, and we want as closely as possible for D=C to eliminate the second damping term. Otherwise, the energy or damping could become negative leading to static or dynamic instability.

This condition may be contrasted with that of a linear feedback system under the same circumstances. If $D \neq C$, then it must be because the weighting of the two displacements $\ddot{\xi}_1$ and ξ_2 is different in the sensing signal D(t) and the forcing signal C(t). This can produce a situation in which the force is applied in phase with the displacement for a given oscillator, and so drive it unstable. Such a mismatching of weights in a linear feedback system would put an upper bound on the allowable gain in the feedback loop consistent with stability. In the bang-bang case, the "gain" for small signals is infinite, so that any mismatch causes instability of the null. This is discussed further in Section 3.3.8.

If bang-bang feedback is used, $F_p(D) = F_p \dfrac{D(t)}{|D(t)|}$ is the feedback law, and so

$$U = F_p |C(t)| \text{ and } B_2(t) = \frac{dC}{dt} F_p \left[\frac{D(t)}{|D(t)|} \quad \frac{C(t)}{|C(t)|} \right] \tag{3.2.6.5}$$

This last term clearly shows that $B_2 < o$ when $DC < o$ and $\dot{C}$ is of the proper sign. This occurs when the single sensor is telling the feedback element to drive the system the wrong way. An extreme case might be if $A_1 = B_2 = o$, so that the sensing of ξ_2 resulted in a force applied to M_1 which could drive it unstable.

Assuming that $A_1 = B_1$ and $A_2 = B_2$ gives $B_2(t) = o$ and $U = F_p \; |D(t)|$. We then can examine stability of the null.

If $F_p > o$, then we are assured that to first-order terms $\nabla_s E(\bar{s}) \cdot \bar{s} > o$ for $|\bar{s}| = \varepsilon > o$. However, second-order terms dominate E for all directions in state space near $\bar{s} = \bar{o}$ such that $D = o = A_1 \xi_1 + A_2 \xi_2$. Given that D = o, we must then require

$$\nabla_s \bar{E} \cdot \bar{s} = M_1 \ddot{\xi}_1^{\,2} + M_2 \dot{\xi}_2^{\,2} + K_1 \xi_1^{\,2} + K_2 \xi_2^{\,2} > o.$$

Let's assume that we have numbered our slab systems so that $K_2 > K_1$. Thus, if only one of them was originally unstable, it was system one. We then eliminate ξ, using $D_K = o$. Also, since $M_1 > o$, $M_2 > o$, we may be assured that the kinetic terms above are positive or zero. We then write for the most critical direction

$$\nabla_s \bar{E} \cdot \bar{s} = K_1 \left(\frac{A_2 \xi_2}{A_1} \right)^2 + K_2 \xi_2^{\,2} + [K_1 (\frac{A_2}{A_1}) + K_2] \xi_2^{\,2}. \tag{3.2.6.6}$$

For stability of the null, we require that the coefficient $K_1 (\frac{A_2}{A_1})^2 + K_2$ be positive. This is a restriction on the effective spring constant of the system in a direction which does not excite the feedback.

Certain conclusions become immediately apparent. First, such feedback can not stabilize more modes than there are stations; if $K_2 < o$ and $K_1 \leq K_2$, the system is unstable regardless of F_p. Second, there must be a certain margin of stability to the stable modes, in this case $K_2 > K_1 (\frac{A_2}{A_1})^2$.

The effect might be thought of as balancing an unstable mode against a stable one when the feedback has no effect. This margin of stability increases as the feedback coupling to higher, stable modes decreases and as the coupling to lower modes increases. Thus for most efficient use of feedback, coupling to stable modes (A_2) should be minimized, and coupling to unstable modes (A_1) maximized. At best, $A_2 = o$ and the unstable modes are controlled without affecting the higher modes.

The region of stability, assuming the null to be stable, is given by $\nabla_s E = \bar{o}$, or

$$\dot{\xi}^o{}_1 = o, \quad \xi^o{}_2 = o$$

$$K_1 \xi_1{}^o + F_p A_1 \frac{D^o}{|D^o|} - o \rightarrow \xi_1{}^o = \frac{-F_p}{K_1} A_1 \frac{D^o}{|D^o|}$$

$$K_2 \xi_2{}^o + F_p A_2 \frac{D^o}{|D^o|} = o \rightarrow \xi_2{}^o = \frac{-F_p}{K_2} A_2 \left|\frac{D^o}{D^o}\right|$$

(3.2.6.7)

Clearly, if $K_1 > o$ and $K_2 \geq K_1$ no self-consistent solution exists and the system is globally stable. Plugging into the energy form, we find

$$E_o = -\frac{1}{2}\left(\frac{\left[F_p A_1 \frac{D^o}{|D^o|}\right]^2}{K_1}\right) + \left(\frac{\left[F_p A_2 \frac{D^o}{|D^o|}\right]^2}{K_2}\right)$$

(3.2.6.8)

so that the region of stability is $E(\xi_1, \dot{\xi}_1, \xi_2, \dot{\xi}_2) < E_o$. The condition that E_o be positive, assuming $K_1 < o$, $K_2 > o$, is simply that $K_1 \left(\frac{A_2}{A_1}\right)^2 + K_2 > o$, the condition for null stability, which checks with intuition.

3.3 Complications

There are many deviations from the idealized systems discussed so far which will enter into the analysis of any real system. Many such complications are characteristics of the feedback elements alone, and so can be expressed in terms of the input and output of the system without reference to the internal state. In such a case, it is expedient to investigate the complications in the context of the lumped-parameter model, and let the results carry over into the domain of distributed systems. In the following sections we continue to let $D_n(t)$ represent the sensory outputs. The purpose of these sections is to develop algorithms for estimating the effects of non-ideal aspects of the problem. Therefore several expressions concerning hysteresis, lag, delay, etc. are approximations for use in the problems of implementation, rather than general expressions for use in the unrestricted analysis of these effects.

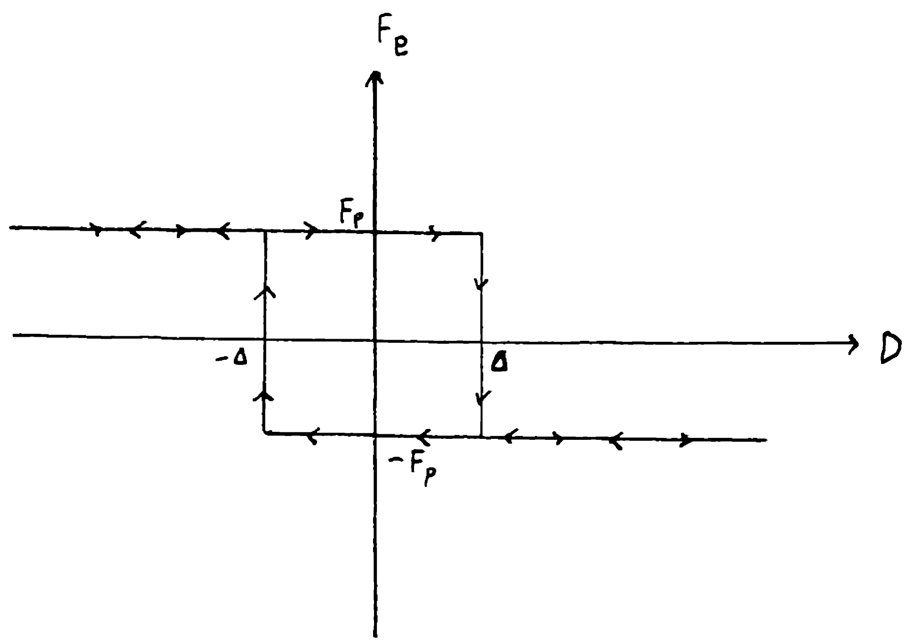
3.3.1 Hysteresis as Negative Damping on Energy

When the state of the system changes so that the discriminant D(t) passes through zero, a real controller will not immediately react and change the sign of the feedback force. This delay might take the form of hysteresis, so that the feedback changed state when the discriminant had passed some value $\Delta > o$ beyond the nominal switching value of zero. Such a feedback function is diagrammed in figure 3.3.1.1.

Hysteresis may occur naturally due to the mechanisms used in the feedback loop, or it may be introduced deliberately to avoid "chatter". In either case we may view the phenomenon as a deviation from ideal feedback, representing a difference between the feedback force as it should be and the force actually applied. We may, therefore, write it as an additional force superimposed on the ideal feedback, so that the sum of the two equals the actual force imposed. We write the hysteretic force contribution as a term due to misapplication of positional feedback,

$$F_e = -F_p \frac{D}{|D|} + H_p \qquad \text{where}$$

Figure 3.3.1.1
Hysteresis



$$H_p = F_p \left[\frac{1}{2} \left(1 + \frac{\Delta - |D|}{|\Delta - |D||} \right) \right] \left[\frac{1}{2} \left(1 + \frac{D\dot{D}}{|D\dot{D}|} \right) \right] \frac{D}{|D|}$$

(3.3.6.1)

The first term in brackets assures us that this force appears only for $|D| < \Delta$. The second term makes it nonzero only when $\dot{D}$ and D are in the same direction. The last gives it the sign of **D**, so that it cancels the ideal restoring force.

Insertion into the equations of motion and formation of an energy equation result in this term becoming a negative energy contribution so that $E = T + \psi + U + U_p$, where

$$U_p = - \frac{F_p}{4} \left[1 + \frac{\Delta - |D|}{|\Delta - |D||} \right] \left[1 + \frac{D\dot{D}}{|D\dot{D}|} \right] |D|$$

(3.3.1.2)

This shows that the negative energy term appears only in the immediate viscinity of the null, where $|D| < \Delta$. It allows us to examine stability with $U_p = o$, and then add this effect to consider its perturbing influence. Being negative, it may result in a loss of null stability, but if the region of stability extended beyond $|D|=\Delta$ it should not eliminate the entire region of stability.

By rearranging the last factors in the expression for H_p, we may express the term as part of the damping instead of an energy contribution. This leads to a damping term B_{Hp} added to the other terms in B of the form

$$B_{Hp} = \frac{-F_p}{4}\left[1 + \frac{\Delta - |D|}{|\Delta - |D||}\right]\left[1 + \frac{D\dot{D}}{|D\dot{D}|}\right] |\dot{D}| \tag{3.3.1.3}$$

Such a negative damping term results in energy being pumped into the system each time $|D| < \Delta$. Thus if $b = o$ and $F_v = o$, this hysteresis effect would result in a slow growth of all oscillations of the system, until finally the energy grew so large that the state exceeded the bounds of the region of stability. At that point a second stage of growth would occur characterized by the fastest-growing unstable modes of the system.

One would expect that since hysteresis creates a negative damping, its effects might be minimized by the presence of positive damping. This could be supplied by internal damping mechanisms of the system ($b > o$) or by velocity feedback ($F_v > o$). Internal damping, being linear, would grow with the norm more quickly than B_p, and so if this effect were to limit the growth of E it would do so at some sufficiently large amplitude of perturbations which could be estimated from the expressions given. If F_v were introduced, then it would probably be subject to the same hysteretic problems as F_p, where the new force term would be

$$H = \frac{F_v}{4}\left[1 + \frac{\Delta' - |\dot{D}|}{|\Delta' - |\dot{D}||}\right]\left[1 + \frac{\dot{D}\dot{D}}{|\dot{D}\dot{D}|}\right]\frac{\dot{D}}{|\dot{D}|} \tag{3.3.1.4}$$

This could again be expressed as a negative damping term in the energy equation, given by $B = B + B_H$, where $B = F_v|\dot{D}|$ and

$$B_H = -\frac{F_v}{4}\left[1 + \frac{\Delta' - |\dot{D}|}{|\Delta' - |\dot{D}||}\right]\left[1 + \frac{\dot{D}\dot{D}}{|\dot{D}\dot{D}|}\right]|\dot{D}| \tag{3.3.1.5}$$

This shows that in general the system will grow until its average $\dot{D}$ is sufficiently larger than Δ' so that the two terms give approximately equal contributions. Clearly there will be a range of $F_p >> F_v, b = o$

where energy will be primarily pumped into the system by B_{Hp} and removed by $F_v|\dot{D}|$.

The introduction of a dead band into the bang-bang force term, wider than the hysteresis loop, would of course eliminate the hysteresis effect, unless a new hysteretic effect accompanied the edges of the dead band.

3.3.2 Time Delay as Hysteresis

Perhaps a more realistic model of certain types of switch lag or processing delay in a feedback loop would be a pure time delay τ. This model says that τ seconds after D crosses the switching input value, the output changes state. If the time delay is short enough compared to motions of the system (say, two successive zero crossings of D(t))then the time lag might be described as a hysteresis with $\Delta = \tau \dot{D}$ where $\dot{D}$ is evaluated at the instant that D = o. (This form is investigated in a different context in Chapter 4 as a way of representing combined position-plus-rate feedback.)

To ascertain just what conditions would allow this approximation, we may employ the describing function analysis of hysteresis and time lag and compare results. Time lag τ gives a describing function, comparing with (3.2.3.4),

$$G(\omega) = \frac{4 F_p}{\pi \, \xi(o)} \; e^{-j\omega\tau} \qquad (3.3.2.1)$$

Hysteresis Δ, on the other hand, has a phase lag ϕ such that $\xi_o \sin\phi = \Delta$ so that the describing function is

$$G(\omega) = \frac{4\ F_p}{\pi\xi(o)}\ e^{-j\ \sin^{-1}[\frac{\Delta}{\xi(o)}]} \quad . \tag{3.3.2.2}$$

To compare the two, we replace τ by $\frac{\Delta}{\dot{D}(t_1)}$ where $\dot{D}(t_1) = \omega\ \xi(o)$, giving describing function

$$G(\omega) = \frac{4\ F_p}{\pi\xi(o)}\ e^{-j\frac{\Delta}{\xi(o)})} \tag{3.3.2.3}$$

Thus we see that the approximation is good if $\frac{\Delta}{\xi(o)} << 1$ or substituting for Δ in terms of τ, if $\omega\ \tau << 1$. The role of ω must be carefully considered in a nonlinear feedback system. Therefore, in the limit of small time delays compared to successive zero-crossing times of the sensor signal, the effect of time delay may be approximated by the hysteresis formula with $\Delta = \xi\ \tau$. Note that no amount of deadband could ever eliminate a time delay, since a new delay results at the edge of the band.

3.3.3 Time Sampling as Time Delay or Lag

If the control system under discussion is to be run by a digital machine, or if it is simulated on a digital computer, then the input D(t) will be time-sampled at regular intervals $t = n\,T$. Thus, when a zero-crossing occurs, the controller will not learn of it until the next sample arrives. Unless by some coincidence the sampling rate is equal to the period of oscillation, in general the result will be a time delay which is of varyingduration, uniformly distributed between zero and T, with an average value of $T/2 = \tau$. Under what circumstances can this approximation be made?

Suppose that the input signal D(t) to the feedback is band-limited, with Fourier transform D(f). The sampling process can be considered multiplication by $\sum_{n=-\infty}^{\infty} U_o(t-n\,T)$. To avoid loss of information, we assume D(f) is limited to the Nyquist frequency, so $D(f) = o$ for $|f| > \frac{1}{2T}$.

To find the output of the sampler, we convolve D(f) with $\sum_{n=-\infty}^{\infty} \frac{1}{T} U_o(f - \frac{n}{T})$ to get a sampler output

$$Z(f) = \sum_{n=-\infty}^{\infty} \frac{1}{T} D(f - \frac{n}{T}) \qquad (3.3.3.1)$$

If we now take the feedback to be of the sample and hold type, the signal z(t) must then be fed into a T-second holding circuit, which we model as a linear element with impulse response

$$h(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \tag{3.3.3.2}$$

This has Fourier transform

$$\begin{aligned} H(f) &= \frac{1}{j2\pi f}\left[1 - e^{-j2\pi fT}\right] \\ &= e^{-j\pi fT} \frac{\sin \pi f T}{\pi f} \end{aligned} \tag{3.3.3.3}$$

Thus the output of the sample and hold would have the form

$$y(f) = \sum_{n=-\infty}^{\infty} D(f - \frac{n}{T}) \frac{\sin \pi f T}{\pi f T} e^{-j2\pi f(\frac{T}{2})} \tag{3.3.3.4}$$

If we then low-pass filter this output for use by eliminating all frequencies $|f| > \frac{1}{2T}$, the resulting signal has frequency-domain form

$$D^*(f) = D(f) \frac{\sin \pi f T}{\pi f T} e^{-j2\pi f(\frac{T}{2})} \tag{3.3.3.5}$$

Thus the signal $D^*(t)$ is simply the input signal $D(t)$ with time delay $\tau = \frac{T}{2}$ provided that

$$\pi f T = \omega \tau << 1.$$

Hence, once again if the effective time-lag is small compared to successive zero-crossings of the sensing signal, the formula for hysteresis may be applied, when this time

$$\Delta = \xi \frac{T}{2} \tag{3.3.3.6}$$

We also note that the time-lag factor $e^{-j2\pi f\tau}$ in (3.3.9) may be approximated, for $\omega \tau << 1$, by a simple first-order lag so that

$$D^*(t) \simeq \frac{D(t)}{1 + j\omega\tau} \tag{3.3.3.7}$$

Hence this development also shows that for the above approximation, a first-order lag in the feedback loop may also be treated as time delay or hysteresis.

Numerical work by Thomassen shows that, unless $\omega\tau <$ about .5, time delay results in instability regardless of the velocity feedback used. Thus, in the range of parameters giving stable operation, the above approximations should be reasonably good. Also, since they were made with minimal reference to the internal description of the system, they apply to continuous systems as well as lumped ones.

3.3.4 Control with Bang-Bang Feedback

The situations considered thus far have been restricted to stabilization of lumped systems for sufficiently small perturbations about a pre-existing equilibrium. However, in general one may want to drive the system along some predetermined trajectory in state space, or at least specify the output variables as functions of time. When this case is analyzed, we will also acquire the ability to handle bang-bang feedback which is asymmetric: the force applied in the two possible states of each feedback element may be different. Also, these forces may be functions of time. In this section, then, we will describe such systems and relate their analysis to the simple cases already considered.

In the most general case, the desired outputs of the system will be denoted S(t). We will then assume that, for positional feedback, the feedback force is applied so that when

$$D(t) > S(t),\ F_e = -F_p^{+}(t) - F_V$$

$$D(t) < S(t),\ F_e = F_p^{-}(t) - F_V$$

This may be written formally as

$$D^*(t) = D(t) - S(t)$$

$$F_e(t) = -\frac{F_p^+(t)}{2}\left[1 + \frac{D^*(t)}{|D^*(t)|}\right] + \frac{F_p^-(t)}{2}\left[1 - \frac{D^*(t)}{|D^*(t)|}\right] - F_v\left(t, \frac{dD^*}{dt}\right)$$

(3.3.4.1)

Note that the value of F_e when $D^* = o$, designated $F_S(t)$, is left undefined at this point. Although we restrict it to finite values, it is unnecessary to define it precisely, since the value of F_e at a point in time if bounded can not affect the state of the system, which responds only to the integral of F_e. Thus the equation of motion

$$M \frac{d^2\xi}{dt^2} = -K\xi - b\frac{d\xi}{dt} + F_e(t) \qquad (3.3.4.2)$$

may be separated into two parts; one representing a quasistatic equilibrium condition at $D^* = o$ and the other representing deviations from that condition. We therefore define ξ_S such that $\xi_S(t) = S(t)$ and $\xi^*(t) = \xi(t) - \xi_S(t)$. Static equilibrium then gives $-K\,\xi_S(t) + F_S(t) = o$, defining $F_S(t)$.

We then define the remaining part of the feedback positioning force as

$$F_R(t) = F_e(t) - F_S(t)$$

$$= \frac{-[F_S(t) + F_p^+(t)]}{2} \left[1 + \frac{D^*}{|D^*|} \right]$$

$$+ \frac{[-F_S(t) + F_p^-(t)]}{2} \left[1 - \frac{D^*}{|D^*|}\right] - F_V\left(t, \frac{dD^*}{dt}\right)$$

(3.3.4.3)

$$F_R(t) = \frac{-F_R^+(t)}{2} \left[1 + \frac{D^*}{|D^*|}\right] + \frac{F_R^-(t)}{2} \left[1 - \frac{D^*}{|D^*|}\right] - F_V\left(t, \frac{dD^*}{dt}\right)$$

(3.3.4.4)

At this point we know that when $D^* = o$, $F_R = o$.

The remaining terms in the equation of motion can all be written in terms of ξ^*, $\frac{d\xi^*}{dt}$ and $\xi_S(t)$,

$$M \frac{d^2\xi^*}{dt^2} + M \frac{d^2\xi_S}{dt^2} = - K \xi^* - b \frac{d\xi_S}{dt}$$

$$- b \frac{d\xi^*}{dt} + F_R(t)$$

(3.3.4.5)

We then form our energy equation, representing a Lyapunov function in the state space $\bar{s}^* = (\xi^*, \frac{d\xi^*}{dt})$, by multiplying the above by $\frac{d\xi^*}{dt}$ and integrating by parts. The additional relation

$$\frac{d}{dt}(F_p \quad |D^*|) = \frac{dD^*}{dt} F_p \frac{D^*}{|D^*|} + \frac{dF_p}{dt} |D^*|$$

gives the result

$$\frac{dE}{dt} = -B$$

$$E = T + \psi + U \qquad B = B_S + B_o^* + B_T$$

$$T = \frac{1}{2} M \left(\frac{d\xi^*}{dt}\right)^2 \qquad B_S = b \frac{d\xi^*}{dt} \frac{d\xi_S}{dt} + M \frac{d\xi^*}{dt} \frac{d^2\xi_S}{dt^2}$$

$$\psi = \frac{1}{2} K \xi^{*2} \qquad B_o^* = b \left(\frac{d\xi^*}{dt}\right)^2 + F_V\left(t, \frac{dD^*}{dt}\right) \frac{dD^*}{dt}$$

$$B_T = \frac{-d F_p^*}{dt} |D^*|$$

$$U = \frac{F_R^+(t)}{2}\left[1 + \frac{D^*}{|D^*|}\right] |D^*| + \frac{F_R(\bar{t})}{2}\left[1 - \frac{D^*}{|D^*|}\right] |D^*| = F_p^* |D^*|$$

$$D^* = \xi^*$$

(3.3.4.6)

From these expressions, we are in a position to analyze the stability of the new equilibrium using the same analytical tools as before. For stability of the null, we will require $F_p^+(t) \geq \varepsilon > o$, $F_p^-(t) \geq \varepsilon > o$; and $B \geq o$ for small perturbations, or $F_V(t, \frac{dD*}{dt}) \frac{D*}{dt} \geq -B - B_T$. The region of stability becomes time-dependent, determined once again by $E(\bar{s})<E_o(t)$ where $E_o(t) = E(\bar{s}*(t))$ where $\bar{s}_o*(t)$ is a saddle point found from $\nabla_{\bar{s}*} E(t) = \bar{o}$.

Special cases are clearly called for here. One such case could be simply one of time-varying feedback amplitude, with $S(t) = o$. In this case, B_T gives the effect of the time-variation of F_p on the system. This mechanism corresponds to the phenomenon of parametric instability in a linear case. One could think of this method of pumping energy into a system as follows: if the system were to remain at $\bar{s} = \bar{s}_1$ while the value of F_p were suddenly raised, the energy due to U would be increased without any other effects taking place. This could eventually result in driving the system out of the region of stability.

Probably the most interesting limit is that in which the desired displacement is time-independent, or $S(t) = S$. This makes B vanish, and reduces the above to the creation of a new static equilibrium. The new equilibrium is stabilized by bang-bang feedback provided that the original feedback force was larger than the force needed to hold the system against its linear restoring terms. Thus in this case a system stable at $\xi = o$ might be unstable at $\xi = S$.

Similarly, if $\frac{dS}{dt}$, $\frac{d^2S}{dt}$, and $\frac{dF_p^*}{dt}$ are sufficiently small, any internal or externally applied damping in the system will dominate B and so give a quasistatic region of operation. The magnitudes of the rates involved could be calculated for any given case from the expressions given above.

In terms of our dielectric slab model, for instance, it might be desired to make the slab move such that $\xi(t) = A \cos \omega t = S(t)$. To do this, the sensor output $D = \xi(t)$ is compared with the desired motion, resulting in an error function $D^*(t) = \xi(t) - A \cos \omega t = \xi^*(t)$. We will assume that b is negligible. The applied feedback force will be

$$F_e = - F_p \frac{D^*}{|D^*|} - F_V \frac{\dot{D}^*}{|\dot{D}^*|} = F_S(t) + F_R(t) \tag{3.3.4.7}$$

which is to be accomplished by adjusting the voltages on capacitor plates at either end of the slab, as described earlier. The force required for equilibrium is $F = + K A \cos \omega t$, so that the feedback force remaining for stabilization is

$$F_R = \frac{[F_p + K A \cos \quad t]}{2} \left[1 + \frac{D^*}{|D^*|}\right] + \frac{[F_p - K A \cos \omega t]}{2} \left[1 + \frac{D^*}{|D^*|}\right]$$

(3.3.4.8)

Thus, for adequate positioning force, it is required that for all t, $F_p > K A \cos \omega t$.

For null stability, one further requires that, for $B > o$,

$$F \frac{\dot{D}^*}{|\dot{D}^*|} > M A \omega^2 \frac{\dot{D}^*}{|\dot{D}^*|} \cos \omega t \pm K A \omega \sin \omega t$$

The minimum value of F_p insures that the feedback can hold the slab against spring tension. The second constraint insures that there is enough feedback damping force to dominate the inertia of the slab and the effects of variation of F_R. Clearly, if $F = o$ and the frequency

ω were very high, the mass M would be left behind its desired position.

With null stability assured, the region of stability will be bounded in this case by $\nabla_{\xi} E = o$ giving $\xi^*_o = o$, and either

$$\xi_o^* = -\frac{[F_p + K\,A \cos \omega t]}{K} \quad \text{for } D_o^* > o$$

$$\xi_o^* + \frac{[F_p - K\,A \cos \omega t]}{K} \quad \text{for } D_o^* < o.$$

(3.3.4.9)

Assuming that null stability occurred, these equations have self-consistent solutions only if $K < o$. Otherwise the system is globally stable. If $K < o$, $A > o$, then the value of ξ_o^* to use is the one giving the minimum value of E_o. Thus at $t = o$ we have

$$\xi_o^* = -\frac{F_p}{K} - A \quad \text{and so}$$

$$E_o(t-o) = -\frac{1}{2} \frac{(F_p + K\,A)^2}{K}$$

(3.3.4.10)

Thus the system is stable if at $t = o$ the initial conditions are such that $\bar{s}^*$ lies inside $E(\bar{s}^*) = E_o$. Note that the statement $\xi_o^* = -\frac{F_P}{K} - A$ corresponds to $\xi = \xi_o + A = \frac{-F_P}{K}$, the simple result that the feedback force balances the spring force.

In the more complicated case where there are N feedback stations, $1 \leq K \leq N$, and M oscillators, $1 \leq m \leq M$, the same analysis may be done, where $D_n = \sum_{m=1}^{M} A_{nm} \xi_m$ are linearly independent outputs of the sensor elements. The only difficult step is the determination of equilibrium forces F_S and displacements ξ_m, given the desired sensor outputs $D_n = S_n(t)$. These forces and displacements total M + N unknowns , and therefore we need M + N equations. N of these are given by $D_n = S_n(t)$ where D is written in terms of the displacements ξ_m. The other M equations come from the force equilibrium condition. If the feedback force on the m^{th} slab is $F_{e_m} = \sum_{n=1}^{N} B_{nm} F_e(t)$, then for each slab we have

$$-K_m \xi_{S_m} + \sum_{n=1}^{N} B_{nm} F_{S_n} = o \quad \text{for all } m.$$

Thus we may simultaneously solve these M + N equations for the ξ_{S_m} and F_{S_n}, and rewrite our equations in terms of $\xi_m^* = \xi_m - \xi_{Sm}$, $D_n^* = D_n - S_n = \sum_{m=1}^{M} A_{nm} \xi_m^*$, and $F_{R_n} = F_{e_n} - F_{S_n}$. Problems involving $N > 1$ will be discussed further in the next section.

3.3.5 Multiple Feedback Systems

All of this work is developed in order to describe a distributed system as a set of modes, each modelled by an oscillator. Hence, we now look at the cost of multiple oscillators.

All the previous work that has been described can be generalized to the case of M oscillating slabs with N feedback stations, $M \geq N$. Each feedback station is characterized by its sensor, its feedback force function, and its force coupling to each oscillator. Let the n^{th} sensor output be a linear sum of oscillator displacements. (Velocity feedback behaves similarly.)

$$D_n = \sum_{m=1}^{M} A_{nm} \xi_m , 1 \leq n \leq N \tag{3.3.5.1}$$

Any convenient normalization of A_{nm} may be employed. If desired, an intermediate cross-coupling network might be inserted between the sensors and the feedback elements, so that the input to the n^{th} feedback element is of the form

$$Y_n = \sum_{i=1}^{N} H_{ni} D_i \tag{3.3.5.2}$$

In this case, it is simple to combine the linear operators $\bar{\bar{H}}$ and $\bar{\bar{A}}$ into a single effective $\bar{\bar{A}}$-matrix. Thus this case need not be treated separately.

Let the feedback force $F_e = -F_n(D_n(t))$ due to the n^{th} feedback element have coupling coefficient B_{nm}

to the m^{th} oscillator, so that the force term in the m^{th} equation of motion due to the n^{th} feedback station is $B_{nm} F_n$. Then each such equation of motion, by superposition of forces, is

$$M_m \frac{d^2\xi_m}{dt^2} = - K_m \xi_m - b_m \frac{d\xi_m}{dt} + \sum_{n=1}^{N} B_{nm} F_n (D_n) \tag{3.3.5.3}$$

For convenience we define the dual of the discriminant D_n to be

$$C_n(t) = \sum_{m=1}^{M} B_{nm} \xi_m(t) \tag{3.3.5.4}$$

We then note that

$$\frac{d}{dt} \int_o^{C_n(t)} F_n(x)\, dx = \dot{C}_n F_n (C_n)$$

Thus, by multiplying the above equation of motion by $\frac{d\xi_m}{dt}$ and summing over all m, the energy equation becomes

$$\frac{dE}{dt} = - B, \qquad B = B_o + B_p$$

$$E = T + \psi + U \qquad T = \sum_{m=1}^{M} \frac{1}{2} M_m \left(\frac{d\xi_m}{dt}\right)^2 \tag{3.3.5.5}$$

$$\psi = \sum_{m=1}^{M} \frac{1}{2} K_m \xi_m^{\ 2} \qquad B_p = \sum_{n=1}^{N} \dot{C} \, [F_n(C\ ) - F_n(D\ )]$$

$$U = \sum_{n=1}^{N} \int_o^{C_n(t)} F_n(x)\ dx \qquad B_o = \sum_{m=1}^{M} b_m \left(\frac{d\xi_m}{dt}\right)^2$$

These can easily be recognized simple generalizations of the corresponding formula for N = 1, M = 1 derived in the previous sections. They are more complicated to use only because the scalar operations carried out to determine stability are replaced by matrix operations. The analysis of velocity feedback is precisely analogous and so will be omitted here.

We will specialize to bang-bang feedback at this point: $F_n(D) = - F_{p_n} \dfrac{D_n}{|D_n|}$ so that

$$U = \sum_{n=1}^{N} F_{p_n} |C_n| \text{ and } B_p = \sum_{n=1}^{N} \dot{C}_n F_{p_n} \left[\frac{D_n}{|D_n|} - \frac{C_n}{|C_n|}\right] \tag{3.3.5.6}$$

Once we have ascertained that the above conditions hold, we can test for stability of the null: $\nabla_s E \cdot \bar{s} > o$ and $B \geq o$. For $B \geq o$ we require $C_n = D_n$, which will happen for all possible ξ_m only if $A_{nm} = B_{nm}$ for all n and M. For $\nabla_s E \cdot \bar{s}$ o, we require $F_{p_n} > o$ for all n to lowest order.

Let $R \leq N$ be the number of linearly independent discriminants D_n. Then, the next higher order terms will appear in $\nabla_s E \cdot \bar{s} > o$ only if $M > R$, for only then can there be a nonzero solution of

$$D_n = \sum_{m=1}^{M} A_{nm} \xi_m = o, \; n = 1 \text{ to } N. \tag{3.3.5.7}$$

To consider the quadratic terms in $\nabla_s E \cdot \bar{s}$, we will first make sure that the oscillators are ordered so that two criteria are satisfied:

1) The first R oscillators, $1 \leq m \leq R$, are chosen so that there is no nontrivial solution of $\sum_{m=1}^{R} A_{nm} \xi_m = o$, $n = 1$ to N. This is always possible because there are R linearly independent discriminants.

2) Subject to the limitations of 1), the ξ_m are ordered so that $K_i \geq K_j$ if $i > j$.

We then rewrite the equations $D_n = o$, as a single vector equation

$$\bar{\bar{A}} \cdot \bar{\bar{\xi}} = \bar{o}, \text{ so that } [\bar{\bar{A}}]_{nm} = A_{nm}, \; [\bar{\xi}]_m = \bar{\xi}_m.$$

This can be split into two parts,

$$\bar{\bar{A}}_R \cdot \bar{\xi}_R + \bar{\bar{A}}_{M-R} \cdot \bar{\xi}_{M-R} = \bar{o} .$$

where $\bar{\xi}_R$ contains the displacements $1 \leq m \leq R$. Because of the ordering of m, we know that $\bar{\bar{A}}_R$ is an invertible R x R matrix. Thus we may solve for the displacements $\bar{\xi}_R$ in terms of the displacements $\bar{\bar{\xi}}_{M-R}$ as follows:

$$\bar{\xi}_R = - \bar{\bar{A}}_R^{-1} \cdot \bar{\bar{A}}_{M-R} \cdot \overline{\xi_{M-R}} \tag{3.3.5.8}$$

We then use the fact that, if the linear (D_n) terms vanish, we may write

$$\nabla_s E \cdot \bar{s} = \sum_{m-1}^{M} M_m \dot{\xi}_m^2 + K_m \xi_m^2 \tag{3.3.5.9}$$

Since $M_m > o$, the kinetic term is clearly positive definite. We must then show, for stability of the null, that $2\psi = \sum_{m=1} K_m \xi_m^2 > o$ for all $D_n = o$.

Let $\bar{\bar{K}}$ be the diagonal matrix with $[\bar{\bar{K}}]_m = K_m$ and split it into two smaller diagonal matrices $\bar{\bar{K}}_R$ and $\bar{\bar{K}}_{M-R}$ as was done with $\bar{\xi}$. We may then write

$$2\psi = \bar{\xi}'_R \cdot \bar{\bar{K}}_R \cdot \bar{\xi}_R + \bar{\xi}_{M-R}' \cdot \bar{\bar{K}}_{M-R} \cdot \dot{\xi}_{M-R}$$

(when prime denotes adjoint) or

$$2\psi = \bar{\xi}'_{M-R} \cdot [\bar{\bar{A}}'_{M-R} \cdot \bar{\bar{A}}^{-1'}_{R} \cdot \bar{\bar{K}}_R \cdot \bar{\bar{A}}^{-1}_{R} \cdot \bar{\bar{A}}_{M-R} + \bar{\bar{K}}_{M-R}] \cdot \bar{\xi}_{M-R}$$

(3.3.5.10)

We must then show that the matrix

$$\bar{\bar{Q}} = \bar{\bar{A}}_{M-R}' \cdot \bar{\bar{A}}_R^{-1'} \cdot \bar{\bar{K}}_R \cdot \bar{\bar{A}}_R^{-1} \cdot \bar{\bar{A}}_{M-R}' + \bar{\bar{K}}_{M-R}$$

(3.3.5.11)

is positive definite, in order to have $2\psi > o$ for any choice of $\bar{\xi}_{M-R}$. This can be done using Sylvester's test, which consists of requiring det $\bar{\bar{Q}}_I > o$, $1 \leq I \leq M-R$, when $\bar{\bar{Q}}_I$ represents the matrix formed of the first I rows and columns of $\bar{\bar{Q}}$.

Because of the ordering of K_m, if $K_{R+1} < o$ and $K_{R+1} > K_{m<R}$ then the first term of Sylvester's test will clearly be negative, and the null is not stable. Thus, the minimum number of linearly independent feedback stations needed in order to achieve stability of the null is equal to the number of oscillators with $K_m \leq o$.

Once we know that the minimum number of feedback stations will be used, we might want to maximize their effectiveness. If our Lyapunov function is well chosen, this means making $\bar{\bar{Q}}$ as positive as possible. To do so, we want to vary the coupling constants A_n so as to minimize the effect of $\bar{\bar{K}}_R$ on $\bar{\bar{Q}}$, and make sure that all the unstable oscillators ($K_m \leq o$) are contained in $\bar{\bar{K}}_R$. Therefore, our feedback should:

1) Be sure to have each sensor observe a linearly independent combination of the unstable displacements.

2) Maximize the coupling to those oscillators by maximizing the determinant of $\bar{\bar{A}}_R$. This makes the terms of $\bar{\bar{A}}^{-1}_R$ small and so minimizes the effect of $\bar{\bar{K}}_R$ on $\bar{\bar{Q}}$.

3) Minimize coupling to the stable oscillators. This makes all elements of $\bar{\bar{A}}_{M-R}$ small, and so minimizes effects of $\bar{\bar{K}}_R$ upon $\bar{\bar{Q}}$.

In the limit where each feedback station couples to only one unstable oscillator, the minimum number is clearly sufficient to establish stability of the null.

Once the problem of stability of the null is analyzed, the region of stability problem emerges. Once again, the velocity feedback and damping considerations are exactly like those involving positional feedback, so we take $b = F_V = o$ and look for the limit on the region R of stability, using the saddle point condition;

$$\nabla_s E = \bar{o} \quad \text{at } \bar{s}_o, \quad E(\bar{s}_o) = E_o.$$

This leads by direct substitution of the formula for E to

$$\dot{\xi}^o_m = o$$

$$K_m \xi^o_m + \sum_{n=1}^{N} A_{n_m} F_{p_n} \frac{D_n^o}{|D_n^o|} = o$$

where $D_n^o = \sum_{m=1}^{M} A_{n_m} \xi_m^o$

(3.3.5.12)

Solution of the above equations for ξ^o_m is not difficult; it merely requires trying all possible combinations of $\frac{D_n^o}{|D_n^o|} = \pm 1$, which gives 2^N possibilities. Often physical insight will motivate the choice.

In any case, if no self-consistent solution exists, then the system is globally stable. This occurs, for instance, if $K_m > o$ for all m. If more than one such solution exists, the region of stability is bounded by $E(\bar{s}) = E_o$ such that E_o is the minimum of those found. This value will be

$$E_o = - \sum_{m=1}^{M} \frac{\left[\sum_{n=1}^{N} F_{p_n} A_{nm} \frac{D_n^o}{|D_n^o|} \right]}{K_m}$$

(3.3.5.13)

The existence of such a solution implies that E_o will be positive.

This concludes the analysis of the stability of M oscillators coupled together by $N < M$ bang-bang feedback elements. All the other complicating effects discussed in this chapter may be included in this analysis in a straightforward manner. The results of this section will be discussed further in Chapter 4, when a clear interpretation of A_{nm} and B_{nm} is available.

3.3.6 Higher Order Terms

We have considered in detail the case of bang-bang feedback, where the positional or velocity feedback is a piecewise-constant function of the input variable

$$F_p(D) = -F_p \frac{D}{|D|}$$

A more general form of feedback designed to keep $D = o$ might in many cases be a force function with a discontinuity at $D = o$ and some continuous variation for $D \neq o$. On either side of the discontinuity, then, we could expand the force in a power series in D, so that (for $D > o$ for instance)

$$F_p(D) = \frac{-F_{p_o}}{2} \left(1 + \frac{D}{|D|}\right) - F_{p_1} D - F_{p_2} D^2 \quad \ldots \tag{3.3.6.1}$$

How can our analysis be generalized to include these higher order terms?

First, let's consider the stability of the null. Clearly the zero-order force terms will dominate, since they lead to first-order terms in $\nabla_s E \cdot \bar{s}$. Thus the higher-order terms will not affect these calculations at all. If there is a non-zero perturbation of the null possible which does not excite the feedback zero-order forces (D=o) then it will not excite the higher-order forces either. Thus, the quadratic stability test is unaffected, also. If there were a feedback system with no zero-order force but a first-order term, such as a saturating linear feedback force, then it would have to be added into the first-order terms due to the linear system in the quadratic-term stability test $\nabla_s E \cdot \bar{s} > o$. This would be simple enough, and need not be carried further, for it is unrealistic to expect all the first-order terms to vanish precisely for any perturbation of the system.

When the region of stability question arises, however, the situation is quite different. The condition $\nabla_s E = \bar{o}$ would still represent the saddle point $\bar{s}_o$ such that $E(\bar{s}) = E_o$ gives the boundary of a region of stability. Calculation of that $\bar{s}_o$, however, would have to include all the terms in the feedback function. Thus, instead of solving a linear equation for ξ^o, the equation would be of whatever degree the feedback function turned out to be. Thus, if there were N feedback stations and M oscillators, one would have to solve M simultaneous nonlinear equations, a task not to be envied. Hence this problem will not be pursued further.

3.3.7 Dead Band and Linear Band

The bang-bang feedback force can only be approximated in any real situation. One possible deviation from the ideal, discontinuous force term is a linear band of feedback which smoothly allows the force to change from its negative to positive extrema. Another possible alteration in the force law is a region of zero force near the discontinuity at D = o. Thus for a linear band, the force is

$$F_p(D) = \begin{cases} -F_p & D > \Delta \\ -F_p \dfrac{D}{\Delta} & |D| < \Delta \\ +F_p & D < -\Delta \end{cases} \qquad (3.3.7.1)$$

For a dead band, the force would be identically zero for $|D| < \Delta$, and the same as bang-bang feedback otherwise.Dead-band may be deliberately introduced to reduce the duty cycle of feedback elements, or to eliminate chatter.

Since both these force functions can be fitted into the general form of energy equation given in section 3.2, the resulting form of feedback energy U will be given below:

Linear band: $U = \frac{1}{2} F_p D^2 \left[\frac{1}{2} \left(1 + \frac{\Delta - |D|}{|\Delta - |D||}\right) \right]$

(3.3.7.2)

$$+ \left[\frac{1}{2} F_p \Delta^2 + F_p (|D| - \Delta) \right] \left[\frac{1}{2}\left(1 + \frac{|D| - \Delta}{||D| - \Delta|}\right)\right]$$

Dead band: $U = [F_p(|D| - \Delta)] \left[\frac{1}{2} \left(1 + \frac{|D| - \Delta}{||D| - \Delta|}\right) \right]$

(3.3.7.3)

In the case of linear band, stability of the null must be investigated by expanding D in terms of the displacements and seeing if $\Delta_s E \cdot \bar{s} > o$ for all possible displacements. In the case of dead band, the null will be stable only if it was stable without feedback, since the feedback will not turn on until the state of the system is removed from the neighborhood of $\bar{s} = \bar{o}$. However, there may still be a region of bounded solutions (a stable oscillator).

For a linear band case, if the null is not stable then there will be no region of stability. If the null is stable, then a region of stability may be found by setting $\nabla_s E = \bar{o}$ and solving for the saddle point $\bar{s}_o$, assuming it to lie outside the linear band. When found, this assumption can be checked. If it lies inside the linear band, then the largest value of E_o such that $E(\bar{s}) = E_o$ lies entirely within the linear band will be the limit of the region of stability.

For the dead band case, even if the null is unstable there may be a region of stability, but it will not include the origin unless $E_o > o$. If $E_o < o$, the region of stability lies between two surfaces each defined by $\dot{E}(\bar{s}) = E_o$. If there is a region of stability but no null stability, then the system will always be subject to disturbances on the scale of $|D| = \Delta$ regardless of damping.

A dead band might well be designed into a bang-bang feedback system deliberately to decrease the switching frequency or to cut down the duty cycle. Such effects can be clearly visualized in the case of a single oscillator. Since $D(t)$ is a linear combination of state variables, it changes continuously with time, so that the time lag between turn-off and turn-on of the feedback force represents an interval of rest for the feedback elements during which the system "drifts" according to the original linear equations of motion.

3.3.8 Non Self-Adjoint Feedback

So far in this discussion, we have said little about the case of feedback with different weighting functions for A_{nm} and B_{nm}, the sensor and enforcer element coupling to different modes. This is because our energy function is not a Lyapunov function in such a case; the feedback part of the system is not self-adjoint: If $\bar{\eta}$ and $\bar{\xi}$ are perturbations, $\bar{\eta}' \cdot \bar{B}_n \; F_n(\bar{A}_n' \cdot \bar{\xi}) \neq \bar{\xi}' \; \bar{B}_n \; F_n \; (\bar{A}_n' \cdot \bar{\eta})$.

To examine this case, let's consider a linear feedback scheme with one station and two slabs (modes). The equations of motion, with masses normalized to unity, are

$$\frac{d^2 \xi_1}{dt^2} = - \omega_1^2 \; \xi_1 - B_1 \; F(A_1 \; \xi_1 + A_2 \; \xi_2)$$

$$\frac{d^2 \xi_2}{dt^2} \; - -\omega_2^2 \; \xi_2 - B_2 \; F(A_1 \; \xi_1 + A_2 \; \xi_2)$$

(3.3.8.1)

where F is the feedback gain. For very small perturbations, the limiting case $F \to \infty$ should approximate bang-bang behavior.

The exact eigenvalues of this problem can be found from the matrix equation

$$\det \begin{vmatrix} \omega^2 - \omega_1^2 - B_1 F A_1 & -B_1 F A_2 \\ -B F A_1 & \omega^2 - \omega_2^2 - B_2 F A_2 \end{vmatrix} = 0$$

(3.3.8.2)

This gives the quadratic equation in ω^2:

$$\omega^4 - \omega^2 (\omega_1^2 + \omega_2^2 + F(A_1 B_1 + A_2 B_2))$$

$$+ (\omega_1^2 \omega_2^2 + F(\omega_1^2 A_2 B_2 + \omega_2^2 A_1 B_1)) = 0$$

(3.3.8.3)

Letting

$$b = \omega_1^2 + \omega_2^2 + F(A_1 B_1 + A_2 B_2)$$

$$c = \omega_1^2 \omega_2^2 + F(\omega_1^2 A_2 B_2 + \omega_2^2 A_1 B_1)$$

the solutions are given by

$$\omega^2 = \frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - c}$$

(3.3.8.4)

We note several things at this point. First, the roots depend only on the products $A_m B_m$, not on the separate factors. Second, if all $A_m B_m > o$, then $\omega^2 > o$ for sufficiently large F. Third, if ω^2 is complex, then at least one unstable root will always exist with ω complex. This will occur if $(\frac{b}{2})^2 - c < o$. This criterion can be restated by making the following substitutions:

$$\Omega = \frac{1}{2}(\omega_2^{\,2} - \omega_1^{\,2})$$

$$S = \frac{1}{2}(A_1 B_1 + A_2 B_2)$$

$$D = \frac{1}{2}(A_1 B_1 - A_2 B_2)$$

Then, overstability (instability at nonzero frequency) is between the limits

$$F_o = \frac{\Omega}{S^2}\left[D \pm \sqrt{D^2 - S^2} \right] \qquad (3.3.8.5)$$

Hence there is a range of overstability provided that $|D| > |S|$, or (assuming $A_1 B_1 > o$) provided that $A_2 B_2 < o$.

To learn the restrictions on F placed by stability of systems with $A_2 B_2 < o$, consider various ranges of gain.

If the gain is low, so that

$$|\omega_1^2 \quad \omega_2^2| >> F^2 \, (A_1 \, B_1 + A_2 \, B_2)^2$$

(or alternatively, if $\omega_m^2 >> F \, A_m \, B_m$ for all m) then the roots of (3.3.7.3) are given approximately by

$$\omega^2 = \omega_1^2 + F \, A_1 \, B_1$$

$$\omega^2 = \omega_2^2 + F \, A_2 \, B_2 \qquad (3.3.8.6)$$

This is the regime investigated by Crowley (1971) where he concludes that an estimate of the allowable gain F can be found by requiring that all modes m of a continuous system satisfy $\omega_m^2 + F \, A_m \, B_m > o$.

This limit does not apply in the high-gain regime where we may assume $|F \; A_m \, B_m| >> \omega_m^2$ for at least a finite number of modes M.

In this high-gain limit, the roots of (3.3.7.3) reduce to one given by

$$\omega^2 = F \, (A_1 \, B_1 + A_2 \, B_2) \qquad (3.3.8.7)$$

and a second, which to lowest nonzero terms in the inverse feedback gain, are

$$\omega^2 = \frac{\omega_1^{\ 2} A_2 B_2 + \omega_2^{\ 2} A_1 B_1}{A_1 B_1 + A_2 B_2}$$

(3.3.8.8)

The first root tells the system designer that he must be surs that $\sum_m A_m B_m > o$ for his feedback system to be stable in the high-gain limit. The second root is independent of F, and in fact corresponds to perturbations which do not excite the feedback. Stability then corresponds to null stability in the self-adjoint bang-bang case $(A_m = B_m)$ with stability requirement

$$\omega_1^{\ 2} A_2 B_2 + \omega_2^{\ 2} A_1 B_1 > o.$$

Now consider the case of M modes, and N feedback stations. Let's suppose that for modes $m \leq M_1$ the coupling to the feedback is strong;

$$\left| \sum_{n=1}^{N} F A_{nm} B_{nm} \right| >> \left| {}_m^{\ 2} \right|$$

and for $m > M_1$ the reverse is true. The eigenvalues ω^2 are then shifted as the gain is turned up, so that for $m > M_1$ we have roots

$$\omega^2 \cong \omega_m^{\ 2} + F A_m B_m$$

from the low-gain limit, and for $m < M_1$ there are N roots given approximately by the equation

$$\det \left| \omega^2 \bar{\bar{I}}_N - \sum_{n=1}^{N} F_n \bar{B}_n \cdot \bar{A}'_n \right| = o$$

(3.3.8.9)

(where $\bar{B}_n$ and $\bar{A}_n$ are truncated at M_1 entries A_{nm}, B_{nm}) plus $(M_1 - N)$ roots (the zero eigenvalues above) independent of F_n and representing combinations of modes which do not excite the feedback. The N modes given by (9) above will all be stable provided that the eigenvalues w_n^2 of the matrix $\sum_{n=1}^{N} F_n \bar{B}_n \cdot \bar{A}_n'$ are positive. We can conclude several things from this. First

$$\sum_{n=1}^{N} \omega_n^2 = \sum_{n=1}^{M_1} \sum_{k=1}^{N} F_k A_{kn} B_{kn} = \sum_{n=1}^{N} F_n \bar{B}_n' \cdot \bar{A}_n$$

Second, if each feedback station n is separately designed so that $\sum_{m=1}^{M_1} A_{nm} B_{nm} > o$, then all the eigenvalues will be positive and the corresponding modes will be stable. However, as long as any mode m of the original system had $\sum_{n=1}^{N} A_{nm} B_{nm} < o$, the resulting feedback-stabilized system will be overstable for some lower value of gain. Hence, some m $A_{nm} B_{nm} < o$, the best one can achieve is conditional stability.

Our conclusions for feedback system design are as follows. If possible, choose A_{nm} and B_{nm} so that they are of the same sign for all m. If this is not achieved, there will probably be at least some mode m which is overstable at finite gain. If that design is not possible, then at least let $A_m B_m$ be small when it is negative to make sure that $\sum_{m=1}^{M_1} A_{nm} B_{nm} > o$ for each feedback station, so that at high gain the system won't be driven unstable. Then, the structure of the coefficients A_{nm} should be chosen so that combinations of modes which don't excite the feedback will all be stable, and that for weakly coupled modes $w_m^2 + F A_m B_m > o$.

In the case of continuum systems, the easiest way to make sure that $A_{nm} B_{nm} > o$ for all m will be to make the spatial weighting functions for sensor and enforcer elements $A_n(z)$ and $B_n(z)$ the same for all n. This will be clarified in the following chapter.

3.4 Evaluation of Lumped Cases

The material in this section represents in itself a design guide for certain types of lumped linear systems, with bang-bang feedback. It presents analytical tools, estimation formulas, and design criteria for a class of bang-bang feedback systems with a variety of modifications.

However, the primary purpose of developing this material in energy-function form is to be able to apply it to the control and stabilization of continuous systems. These formulas will appear again, but each oscillator will be replaced by a mode of the linear distributed system. Coupling of the feedback forces and sensors will be accomplished by spatial shaping of weighting functions. Therefore, it is important to understand the methods and ideas put forth in this section in order to build up a picture of the continuum as a state space with an energy function.

4) Distributed Systems

4.1) Quasi-One Dimensional Systems

Chapter 4 contains most of the general results of this thesis. Most of these can be most easily explained in the context of a one-dimensional mathematical model. This model approximates the behavior of many systems after their multi-dimensional aspects have been suppressed by analysis.

The model will be developed in the first section, as a quasi-one dimensional model of a vibrating membrane which is subject to external electrical forces. Subsequent sections will expand the model to include bang-bang feedback in a number of situations, and present design criteria for such feedback control of continuous systems.

4.1.1.1) Linear System

Consider a membrane under tension with mass per unit length ρ, stretched over some interval $o \leq z \leq L$. We will assume that, for simplicity, the perturbations $\xi(z,t)$ from equilibrium are slowly varying in z, so that they look locally constant in z. At a distance d above and below the membrane, we position conducting plates and raise them to a voltage V with respect to the membrane, which is taken to be perfectly conducting. The electric fields above and below the membrane are then respectively, in the x-direction.

$$E_a \doteq - \frac{V}{d-\xi} \qquad\qquad E_b \doteq + \frac{V}{d+\xi} \tag{4.1.1.1.1}$$

Hence the electrical force on the membrane (per unit length in z,) is given by

$$F_e = \frac{1}{2} \varepsilon_o \left(E_a{}^2 - E_b{}^2\right) = \frac{\varepsilon_o V^2}{\partial} \left[\frac{1}{(d-\xi)^2} - \frac{1}{(d+\xi)^2} \right] \tag{4.1.1.1.2}$$

If we assume that for small perturbations $|\xi| < < d$, we may linearize to get

$$F_e \doteq \frac{\varepsilon_o V^2}{d^3} \quad \xi = \quad P \quad \xi \qquad (4.1.1.1.3)$$

Viscosity might give an additional force term $-b \frac{\partial \xi}{\partial \xi}$.
These give an equation of motion

$$\rho \frac{\partial^2 \xi}{\partial t^2} = \gamma \frac{\partial^2 \xi}{\partial z^2} + \quad P \xi - b \frac{\partial \xi}{\partial t}$$

(4.1.1.1.4)

with boundary conditions

$$\xi(o, t) = \xi(L, t) = o \qquad (4.1.1.1.5)$$

This will be our prototype of a continuous hyperbolic linear syster The purpose of the model is merely to develop the prototype, so that details of the modelling process are unimportant. See figure 4.1.1.1. for a diagram of the system.

At this point we introduce the following normalized variables:

$$z' = \frac{z}{L}$$

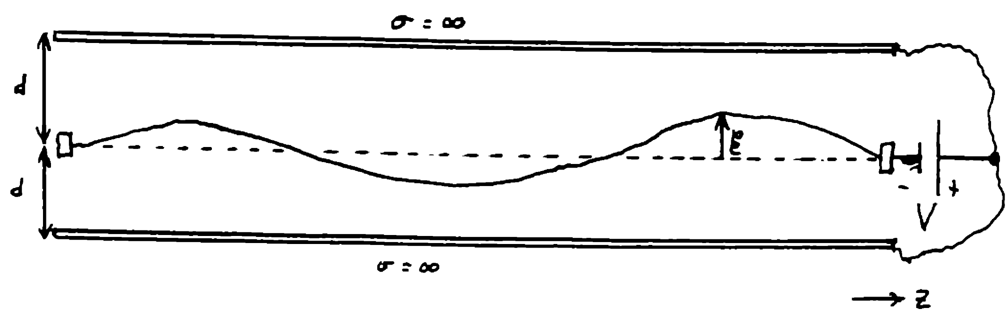
$$\xi = \frac{\xi}{d}$$

$$t' = \frac{t}{T} \quad \text{where } T^2 = \frac{\rho L^2}{\gamma}$$

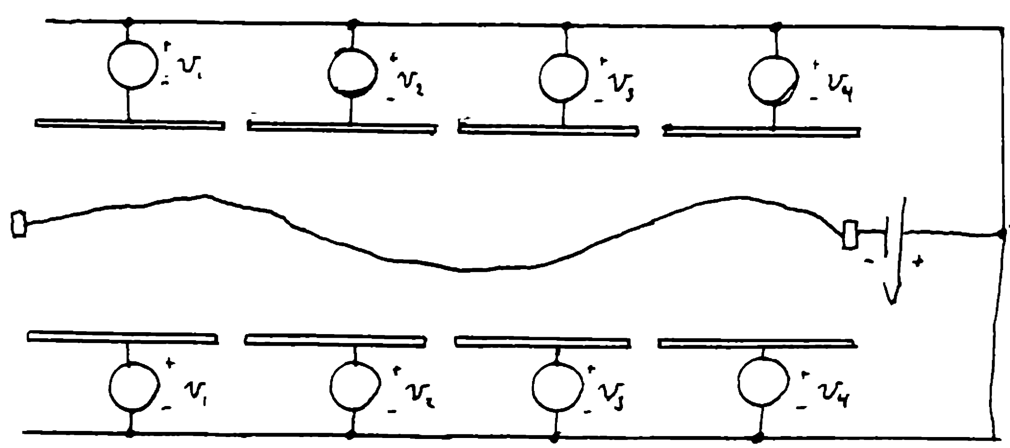
$$b' = \frac{bT}{\rho}$$

$$P' = \frac{PT^2}{\rho} \qquad (4.1.1.1.6)$$

Figure 4.1.1.1 Membrane System



a. Without Feedback



b. With Feedback (N=4)

All primes will be suppressed in the following development.

The resulting equation of motion is

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \xi}{\partial z^2} + P\,\xi - b\,\frac{\partial \xi}{\partial t}$$

$$\xi(o,t) = \xi(1,t) = o \qquad (4.1.1.1.7)$$

Such a system is traditionally analyzed in terms of modes. Let

$$\xi(z,t) = \sum_{m=1}^{\infty} a_m(t)\ \xi_m(z) \qquad (4.1.1.1.8)$$

where ξ_m is an eigenfunction for the spatial operator, or a mode:

$$\xi_m(z) = \sqrt{2}\ \sin m\pi z$$

normalized so that the modes form a complete orthonormal set of functions on the spatial interval of the system. Then our equation of motion reduces to a set of ordinary constant-coefficient differential equations for $a_m(t)$:

$$\frac{d^2 a_m}{dt^2} + b\,\frac{d\,a_m}{dt} + [\,m^2\,\pi^2 - P\,]\,a_m = o \qquad (4.1.1.1.9)$$

We define $m^2 \pi^2 - P = K_m$, and solve for a_m with solutions

$$a_m(t) = a_m(o) \cos \omega_m t + \dot{a}_m(o) \frac{\sin \omega_m t}{\omega_m} \quad \text{where}$$

$$-\omega_m^2 + j\, b\, \omega_m + K_m = o \quad \text{leading to}$$

$$j\, \omega_m = \frac{-b}{2\rho} + \sqrt{\left(\frac{b}{2}\right)^2 - K_m} \qquad (4.1.1.1.10)$$

Stability requires that $b \geq o$, $K_m > o$ for all m, or $P < \pi^2$. Thus P represents a destabilizing influence, and tension a stabilizing influence which dominates at least the higher modes of the system.

4.1.1.2 Feedback

We now wish to add feedback to the system to stabilize the equilibrium for higher values of P. By optical or other means, we will sense some spatially weighted averages $A_n(z)$ of the deflection ε, and feed these signals D_n into nonlinear force feedback elements. The resulting action on the membrane is an additional force distribution, which is the sum of the forces due to each of the feedback elements. The force due to the n^{th} feedback element will have spatial distribution $B_n(z)$ and amplitude $F_n(D_n(t))$. (This will be normalized by $F_n' = \frac{F_n T^2}{\rho}$ with prime suppressed.)'

We may establish the conventions

$$\int_o^1 A_n^2(z)\, d\, z = \int_o^1 B_n^2(z)\, d\, z = 1 \qquad (4.1.1.2.1)$$

to avoid confusion, and formally define the sensor signals

$$D_n(t) = \int_o^1 A_n(z)\, \xi(z,t)\, d\, z, \quad n = 1 \text{ to } N. \qquad (4.1.1.2.2)$$

The new equation of motion is then

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial\ \xi}{\partial z^2} + P\xi - b\frac{\partial \xi}{\partial t} + \sum_{n=1}^{n} B_n(z)\, F_n(t) \qquad (4.1.1.2.3)$$

Added feedback terms dependent on the velocity rather than position can be added in an obvious manner.

We may analyze the system with feedback by using a Galerkin series which expresses the new $\xi(z,t)$ in terms of the modes of the original linear system. This gives a sensor output signal

$$D_n(t) = \sum_{m=1}^{\infty} A_{nm}\, a_m(t)$$

$$A_{nm} = \int_0^1 A_n(z)\, \xi_m(z)\, d\,z \qquad (4.1.1.2.4)$$

The equation of motion can then also be multiplied by $\xi_m(z)$ and integrated over z to give

$$\left[\frac{d^2}{dt^2} + K_m\right] a_m(t) = -b\,\frac{d\,a_m(t)}{dt} + \sum_{n=1}^{N} B_{nm}\, F_n(t)$$

where

$$B_{nm} = \int_0^1 B_n(z)\, \xi_m(z)\, d\,z \qquad (4.1.1.2.5)$$

This equation represents the equation of motion of the m^{th} mode of the system, showing explicitly the coupling of the feedback terms. It is an exact parallel to the equation of motion of the m^{th} oscillator in the previous chapter.

In the case of bang-bang feedback, the force due to the n^{th} feedback station will be assumed to be of the form

$$F_n(t) = - F_{P_n} \frac{D_n}{|D_n|} - F_{V_n} \frac{\dot{D}_n}{|\dot{D}_n|} \tag{4.1.1.2.6}$$

This is a superposition of positional and velocity feedback forces, each dependent on only the n^{th} sensor output or its time derivative. Cross-coupling is considered later.

Applying these relations to the membrane model, the force from the n^{th} feedback station is due to the feedback voltage on the n^{th} segment of the electrodes above and below the membrane, as shown in Fig. 4.1.1(b). If there are N equally spaced segments, then in the quasi-one dimensional approximation

$$B_n(z) = \begin{cases} 1 & \left(\frac{n-1}{N}\right) < z < \frac{n}{N} \\ 0 & \text{elsewhere} \end{cases} \tag{4.1.1.2.7}$$

The electrical force including the voltage v_n on the n^{th} segment is given by (4.1.1.2) as

$$F_n = \frac{\varepsilon_o V^2}{2} \left[\frac{(1 - \frac{v_n}{V})^2}{(d-\xi)^2} - \frac{(1 + \frac{v_n}{V})^2}{(d+\xi)^2} \right]$$

The highest-order effect of v_n is to introduce an additional (feedback) force on the membrane

$$\bar{F}_n(t) = - \frac{2 \varepsilon_o V v_n(t)}{d^2} \qquad (4.1.1.2.8)$$

where v_n is determined by D_n and $\dot{D}_n$.. Letting be a sum of bang-bang terms,

$$v_n(t) = v_{n} \frac{D_n}{|D_n|} + v_{V_n} \frac{\dot{D}_n}{|\dot{D}_n|}$$

we may write

$$\bar{F}_n(t) = - F_{P_n} \frac{D_n}{|D_n|} - F_{V_n} \frac{\dot{D}_n}{|\dot{D}_n|}$$

where (before normalization)

where (before normalization)

$$F_{P_n} = \frac{2\,\varepsilon_o\, V\, v_{P_n}}{d^2} \qquad F_{V_n} = \frac{2\,\varepsilon_o\, V\, v_{V_n}}{d^2} \tag{4.1.1.2.9}$$

The next higher-order effects due to v_n are at least linear in both of the factors $\left(\frac{v_n}{V} \right) \left(\frac{\xi}{d} \right)$. If we assume that $|v_n| < < V$, then these terms are negligible. compared to the forces already calculated, in spite of the bang-bang nature of v_n.

4.1.2 Canonical Form

Looking at the equation of motion for the m^{th} mode (4.1.1.2.5) and comparing with the previous chapter, we see that the next step is to form an energy equation by multiplying (4.1.1.2.5) by $\dot{a}_m(t)$ and summing over all modes m. This gives

$$\frac{d}{dt}\left[\sum_{m=1}^{\infty}\left(\frac{1}{2}\dot{a}_m^{\,2}+\frac{1}{2}K_m\,a_m^{\,2}\right)\right]=-\sum_{m=1} b\,\dot{a}_m^{\,2}$$

$$-\sum_{n=1}^{N}\sum_{m=1}^{\infty} B_{nm}\,\dot{a}_m\left[F_{P_n}\frac{D_n}{|D_n|}+F_{V_n}\frac{\dot{D}_n}{|\dot{D}_n|}\right]$$

We now define the dual of the sensor signal

$$C_n(t)=\int_0^1 B_n(z)\,\xi(z,t)\,dz=\sum_{m=1}^{\infty}B_{nm}\,a_m(t)$$

The last term of (4.1.3.1) is then of the form

$$-\sum_{n=1}^{N}\frac{dC_n}{dt}\left[F_{P_n}\frac{D_n}{|D_n|}+F_{V_n}\frac{\dot{D}_n}{|\dot{D}_n|}\right] \qquad (4.1.2.2)$$

Since this expression involves only terminal variables (independent of m) it may be treated precisely as its equivalent in the lumped-parameter case.

It is clear that, as nearly as possible, the designer should try to arrange the sensor elements so that their spatial weighting is the same as the forcing elements which they control. This gives the desired result $B_{nm} = A_{nm}$, and since the modes form a complete set, requires that $B_n(z) = A_n(z)$. This means that the sensor signal $D_n(t)$ is simply an average of the deflection over the length of the n^{th} forcing segment. In this case the positional feedback terms form an exact time-derivative, so that

$$U = \sum_{n=1}^{N} F_{P_n} |C_n| \qquad (4.1.2.3)$$

which is the bang-bang form of the more general expression

$$U = \sum_{n=1}^{N} \int_{o}^{C_n} F_{P_n}(x)\, d\, x \qquad (4.1.2.4)$$

where $(-F_{P_n}(D_n))$ is the nonlinear feedback force law.

The complete Lyapunov function is then

$E = T + \psi + U$ where

$$T = \sum_{m=1}^{\infty} \frac{1}{2} \dot{a}_m^{\,2} = \int_o^1 \frac{1}{2}\left(\frac{\partial \xi}{\partial t}\right)^2 dz$$

$$\psi = \sum_{m=1}^{\infty} \frac{1}{2} K_m \, a_m^{\,2} = \int_o^1 \frac{1}{2}\left[\left(\frac{\partial \xi}{\partial t}\right)^2 - P \, \xi^2\right] dz$$

(4.1.2.5)

The right-hand derivative of E is

$$\frac{dE}{dt} = -B, \qquad B = B_o + B_1 + B_P + B_V$$

where

$$B_o = \sum_{m=1}^{\infty} b \, \dot{a}_m^{\,2} = \int_o^1 b \left(\frac{\partial \xi}{\partial t} \right)^2 dz$$

$$B_V = \sum_{n=1}^{N} F_{V_n} |\dot{C}_n| \qquad (4.1.2.6)$$

For $A_n(z) = B_n(z)$, the terms B_P and B_1 vanish. They represent deviations from the ideal spatial matching condition, and may be considered a means of evaluating the errors introduced. From equations (4.1.2.2) and (4.1.2.4), we require that for $D_n \neq C_n$,

$$B_P = \sum_{n=1}^{N} F_{P_n} \dot{C}_n \left[\frac{D_n}{|D_n|} - \frac{C_n}{|C_n|} \right]$$

$$B_1 = \sum_{n=1}^{N} F_{V_n} \dot{C}_n \left[\frac{\dot{D}_n}{|\dot{D}_n|} - \frac{\dot{C}_n}{|\dot{C}_n|} \right] \quad (4.1.2.7)$$

Note that the spatial matching condition does <u>not</u> require that the feedback couple only to one mode, as many older schemes in the literature require. Any spatial distribution is acceptable, as long as the sensing and forcing distributions are the same. Clearly some will work better than others, and the design criteria for choosing a good one come directly from the stability analysis.

4.1.2.1 Null Stability

Under what conditions is E a positive definite Lyapunov function in the neighborhood of the null $\dot{\xi}(z, t) = \xi(z,t) = o$? For if it is, then the null is stable.

In the particular case which we have chosen, $b > o$, so $\frac{dE}{dt} \leq o$ if $A_n(z) = B_n(z)$ and $F_{P_n} \geq o$. However, the requirement that $E > o$ for any combination of $(a_m, \dot{a}_m) \neq o$ in the neighborhood of the null may be written $\nabla_s E \cdot \bar{s} > o$ where $\bar{s}$ is the state vector in the space of mode amplitudes a_m and velocities $\dot{a}_m$. Since the space is infinite-dimensional, stability depends upon the choice of norm, as we define

$$||\bar{s}|| = [\sum_{m=1}^{\infty} a_m^2 + \dot{a}_m^2]^{1/2} .$$

We then write

$$\nabla_s E \cdot \bar{s} = \sum_{m=1}^{\infty} \dot{a}_m^2 + \sum_{m=1}^{\infty} K_m a_m^2 + \sum_{n=1}^{N} F_{P_n} |D_n|$$

$$D_n = \sum_{m=1}^{\infty} A_{nm} a_m \qquad (4.1.2.1.1)$$

Clearly the first term is positive definite. Clearly, also, the last term dominates near the null and is positive definite for $F_{P_n} > o$, except in directions where $D_n = o$ for n = 1 to N.

The critical case is whether the remaining term (equal to 24) is positive definite for all a_m such that $D_n = o$, n = 1 to N:

$$\sum^{\infty} K_m a_m^2 > o, \quad D_n = o \tag{4.1.2.1.2}$$

What makes this problem difficult is the infinite summation. Certainly if $K_m > o$ for all m, the system is stable, but it would be even without feedback so that isn't very enlightening. In general, there will be a finite number of unstable modes M, with $K_m \leq o$ for $m \leq M$. This number will be determined, in our membrane model, by γ and P.

We proceed as in the lumped-parameter case: define vectors, $\bar{a}$, $\bar{a}_N$ and $\bar{a}_\infty$ so that

$$\left[\frac{\bar{a}_N}{\bar{a}_\infty}\right] = [\,\bar{a}\,], \quad a_m = \text{the } m^{th} \text{ component of } \bar{a}$$

so that $\bar{a}_N$ is an N-vector containing the components of $\bar{a}$ for $m \leq N$, and $\bar{a}_\infty$ contains the rest. Then we may write

$$\bar{D} = \bar{\bar{A}} \cdot \bar{a} \tag{4.1.2.1.3}$$

where $[D]_n = D_n$, $[\bar{\bar{A}}]_{nm} = A_{nm}$.

Define also a diagonal matrix $\bar{\bar{K}}$ so that

$$[\bar{\bar{K}}]_{mm} = K_m$$

Then the above problem may be written

$$\bar{a}' \cdot \bar{\bar{K}} \cdot \bar{a} > o \text{ for all } \bar{a} \text{ such that}$$
$$\bar{D} = \bar{\bar{A}} \cdot \bar{a} = \bar{o}. \qquad (4.1.2.1.4)$$

where prime denotes transpose. Now divide $\bar{\bar{K}}$ into $\bar{\bar{K}}_N$ and $\bar{\bar{K}}_\infty$, and $\bar{\bar{A}}$ into $\bar{\bar{A}}_N$ and $\bar{\bar{A}}_\infty$ in a manner analogous to the division of $\bar{a}$. Thus $\bar{\bar{K}}_N$ and $\bar{\bar{A}}_N$ are N x N matrices containing the information for $m \leq N$, where N is the number of stations. $\bar{\bar{K}}_\infty$ is a diagonal matrix of infinite dimension containing K_m for $m > N$, and $\bar{\bar{A}}_\infty$ is $N \times \infty$ representing the rest of $\bar{\bar{A}}$. We then rewrite the above as

$$\bar{a}'_N \cdot \bar{\bar{K}}_N \cdot \bar{a}_N + \bar{a}'_\infty \cdot \bar{\bar{K}}_\infty \cdot \bar{a}_\infty > o$$

for all $\bar{a}_N$ and $\bar{a}_\infty$ such that

$$\bar{D} = \bar{A}_N \cdot \bar{a}_N + \bar{A}_\infty \cdot \bar{a}_\infty = \bar{o}. \qquad (4.1.2.1.5)$$

At this point we recognize that if $\bar{\bar{A}}_M$ is not invertible there is no hope of stabilizing the system. For if that is the case, then it is possible to find a collection of mode amplitudes a_m, $m \leq M$ such that $\bar{D} = \bar{o}$ for $a_m = o$, $m > M$. Let $\bar{a}_o$ be a vector composed of these mode amplitudes. Since we know that $K_m \leq o$ for $m < M$, then $\bar{a}'_o \cdot \bar{\bar{K}} \cdot \bar{a}'_o$ is negative, and so E is not positive definite throughout the neighborhood of the null.

We therefore establish the minimal design criterion that, for stability of the null, $A_n(z)$ must be chosen so that D_n represent the least M independent samples of the M lowest modes: the unstable modes. This can be tested independently of the higher modes of the system.

If $\bar{\bar{A}}_N$ is not invertible (of rank $M \leq R \leq N$) then reduce N by one until an invertible matrix remains. The previous reasoning requires that $R \geq M$ for guaranteed stability. Then, solve for $\bar{a}_N = -A_N{}^{-1} \cdot \bar{\bar{A}}_\infty \cdot \bar{a}_\infty$. Plugging into the expression for 2ψ, we require

$$\bar{a}'_\infty \cdot [\bar{\bar{K}}_\infty + \bar{\bar{A}}'_\infty \cdot \bar{\bar{A}}_N{}^{-1\prime} \cdot \bar{K}_N \cdot \bar{\bar{A}}_N{}^{-1} \cdot \bar{\bar{A}}_\infty] \cdot \bar{a}_\infty > o.$$

for all $\bar{a}_\infty$.

(4.1.2.1.6)

Let $\bar{\bar{Q}}_\infty = \bar{\bar{K}}_\infty + \bar{\bar{A}}_\infty' \cdot \bar{\bar{A}}_N^{-1\,'} \cdot \bar{\bar{K}}_N \cdot \bar{\bar{A}}_N^{-1} \cdot \bar{\bar{A}}_\infty$. Use Sylvester's test for positive definiteness of quadratic forms, which says let $\bar{\bar{Q}}_L$ be the matrix representing the first L rows and columns of $\bar{\bar{Q}}_\infty$. Thus, by analogous notation,

$$\bar{\bar{Q}}_L = \bar{\bar{K}}_L + \bar{\bar{A}}_L' \cdot \bar{\bar{A}}_N^{-1\,'} \cdot \bar{\bar{K}}_N \cdot \bar{\bar{A}}_N^{-1} \cdot \bar{\bar{A}}_L$$

If $\bar{\bar{Q}}_\infty$ is positive definite, then det $\bar{\bar{Q}}_L > o$ for all $1 \leq L \leq \infty$. This would not be very practical if we had to find det $\bar{\bar{Q}}_L$ for all L. However, det $\bar{\bar{Q}}_L$ should become large and positive as L becomes large, for the following reasons.

First, $\bar{\bar{K}}_\infty$ is diagonal and has entries which become large as $L \to \infty$. In our example, $K_m \sim m^2$ as $m \to \infty$.

Second, as long as our feedback sensors and forcing elements are of finite width, they must have small coupling to the higher modes. This says that $|A_{nm}| \to o$ as $m \to \infty$. Thus, the effect of $\bar{\bar{K}}_N$, representing the unstable modes, is minimized as m becomes large. Our later examples bear out this premise. Essentially, each $\bar{\bar{Q}}_L$ of Sylvester's test approximates the whole system by (N+L) modes.

The first few terms, then, will tell whether $\bar{\bar{Q}}_\infty$ is positive definite. In many cases the first term of the series of tests, det $\bar{\bar{Q}}_1$, will determine the entire test. This first term requires

$$K_{N+1} > - \bar{A}'_{N+1} \cdot \bar{\bar{A}}_N^{-1'} \cdot \bar{\bar{K}}_N \cdot \bar{\bar{A}}_N^{-1'} \cdot \bar{A}_{N+1}$$

(4.1.2.1.7)

where $\bar{A}_{N+1}$ is the vector with elements $A_{n(N+1)}$. This says that not only must the $(N+1)^{th}$ mode be stable, but it must be stable by a certain margin to avoid being destabilized by the feedback coupling to the unstable modes. A good feedback design should minimize this margin to maximize the stable range of P. This can be done by maximizing det $|\bar{\bar{A}}_N|$, so that $\bar{\bar{A}}_N^{-1}$ is small, and by minimizing all elements of $\bar{\bar{A}}_\infty$. This says that the coupling to unstable modes should be strong, and the coupling to stable modes weak, for best use of feedback stations.

Let's analyze our example case for stability of the null. Assume $A_n(z) = B_n(z)$ as given by the example. Then we find A_{nm} to be

$$A_{nm} = \int_{\frac{n-1}{N}L}^{\frac{nL}{N}} \sqrt{2} \quad \sin m\pi z \, dz$$

$$= \frac{\sqrt{2}}{m\pi} \left[\cos \frac{m(n-1)\pi}{N} - \cos \frac{mn\pi}{N} \right]$$

(4.1.2.1.8)

Consider the case of N = 2 feedback stations. How large can P become before the stability of the null can not be sustained by two feedback stations? An approximate answer can be gotten by requiring that the mode m = 3 be stable, giving $P < 9\pi^2$. However, the exact answer will be a stronger condition on P, due to the coupling to unstable modes through the feedback.

To find this limiting value, first it is necessary to calculate $\bar{\bar{K}}_N$ and $\bar{\bar{A}}_N$ for N = 2.

$$K_1 = \pi^2 - P$$

$$K_2 = 4\pi^2 - P$$

$$\bar{\bar{A}}_N = \begin{bmatrix} \frac{\sqrt{2}}{\pi} & \frac{\sqrt{2}}{\pi} \\ \frac{\sqrt{2}}{\pi} & \frac{-\sqrt{2}}{\pi} \end{bmatrix}$$

$\det \bar{\bar{A}}_N = \frac{-4}{\pi^2} \neq 0$ so $\bar{\bar{A}}_N$ is invertible:

$$\bar{\bar{A}}_N^{-1} = \begin{bmatrix} +\frac{\sqrt{2}\,\pi}{4} & +\frac{\sqrt{2}\,\pi}{4} \\ +\frac{\sqrt{2}\,\pi}{4} & -\frac{\sqrt{2}\,\pi}{4} \end{bmatrix}$$

We now calculate the matrix $\bar{\bar{Q}}_\infty$ by stages as $\bar{\bar{Q}}_L$.

$$\bar{\bar{A}}_N^{-1'} \cdot \bar{\bar{K}}_N \cdot \bar{\bar{A}}_N^{-1} = \begin{bmatrix} \frac{\pi^2}{8}(5\pi^2 - 2P) & \frac{-3\pi^4}{8} \\ \frac{-3\pi^4}{8} & \frac{\pi^2}{8}(5\pi^2 - 2P) \end{bmatrix}$$

We will examine only the first two matrices in the test for positive definiteness of $\bar{\bar{Q}}_\infty$; that is, we will require det $\bar{\bar{Q}}_2 > o$. Therefore, we need only

$$K_3 = (3\pi)^2 - P$$
$$K_4 = (4\pi)^2 - P$$

and the first two columns of $\bar{\bar{A}}_\infty$:

$$\begin{matrix} \frac{\sqrt{2}}{3\pi} & o \\ \frac{\sqrt{2}}{3\pi} & o \end{matrix}$$

We see that the mode m=4 does not couple at all to our feedback. This is not surprising, since m=4 goes through a complete cycle of the sine wave at each feedback station. Hence, if Q_1 is positive definite, Q_2 will also be.

$$\det \bar{\bar{Q}}_1 = Q_1 = K_3 + \left[\frac{\sqrt{2}}{3\pi} \quad \frac{\sqrt{2}}{3\pi} \right] \cdot \bar{\bar{A}}^{-1'} \cdot \bar{\bar{K}}_N \cdot \bar{\bar{A}}^{-1} \cdot \begin{bmatrix} \frac{\sqrt{2}}{3\pi} \\ \frac{\sqrt{2}}{3\pi} \end{bmatrix}$$

$$Q_1 = K_3 + \frac{1}{9} (\pi^2 - P) = \frac{82\pi^2 - 10P}{9}$$

The requirement $\det Q_1 > o$ gives the exact condition for null stability with $N = 2$:

$$P < 8.2\ \pi^2 \ . \qquad (4.1.2.1.10)$$

It may be verified that this condition is stronger than any higher-mode approximations $\det \bar{\bar{Q}}_L > o$ for $L > 1$, for the reasons discussed previously. The above condition is a limit on the destabilizing voltage V in terms of the tension γ and other geometrical factors. If a higher value of P is desired, a rearrangement of the feedback geometry with 2 stations is theoretically possible which could give up to $P < 9\pi^2$ as a stability limit. Higher values of P would require more feedback stations. Raising the feedback voltage F_p does not allow null stability at higher P.

4.1.2.2 Region of Stability

Once we have established the existence of a neighborhood of stability near the null, it is desirable to define the region of state space which represents possible initial conditions leading to stable solutions. The initial condition of the membrane, given by $\xi(z, t_o)$ and $\frac{\partial \xi}{\partial t}$ (z, t_o), can be analyzed in terms of modes as done before. Therefore, the initial condition of the system is given by a double infinity of mode amplitudes $a_m(t_o)$ and velocities $\dot{a}_m(t_o)$. The region of stability represents a closed, bounded region in state space $R(t_o)$ which has the property that if $\bar{s}(t_o) = \{a_m(t_o), \dot{a}_m(t_o)\}$ lies in $R(t_o)$ at $t = t_o$, then $||\bar{s}(t)||$ will be bounded for all $t > t_o$.

The boundary of R will be established by the condition $E(\bar{s}) = E_o$, with $B \geq o$ and $E(\bar{s}) < E_o$ for all $\bar{s} \in R$. Using the same reasoning as was introduced in the last chapter, $E_o = E(\bar{s}_o)$ where $\bar{s}_o$ is the point of minimum E (not including the null) where $\nabla_s E = \bar{o}$. This condition gives an equation for each component of $\nabla_s E$. The mode velocity components are simply

$$\dot{a}_m^o = o. \tag{4.1.2.2.1}$$

The mode amplitude components are more complicated and give

$$a^o_m = - \sum_{n=1}^{N} \frac{A_{nm} F_{Pn}}{K_m} \frac{D^o_n}{|D^o_n|} \tag{4.1.2.2.2}$$

where $D^o_n = \sum_{m=1}^{\infty} A_{nm} a^o_m$.

The solution can be found by assigning test signs to $\frac{D^o_n}{|D^o_n|}$ and solving for a^o_m. Since there are a finite number of discriminants, this procedure is not too difficult. A self-consistent solution is required, so that the a^o_m found must produce the assumed sign of D^o_n. This check can be made with a small number of a^o_m answers, because a well-designed system will have small coupling to the higher modes. If more than one set of results is self-consistent, the one with lowest $E(\bar{s}_o) = E_o$ should be chosen. If no self-consistent set exists, then there could be no boundary to R and so the system is globally stable. The final value of E_o is calculated directly as

$$E_o = \sum_{m=1}^{\infty} \frac{1}{2} K_m {a_m^o}^2 + \sum_{n=1}^{N} F_{P_n} |D_n^o|$$

$$= \sum_{m=1}^{\infty} \frac{1}{2} K_m \left[\sum_{n=1}^{N} \frac{A_{nm} F_{P_n} D_n^o}{K_m |D_n^o|} \right]^2$$

$$+ \sum_{n=1}^{N} F_{Dn} \frac{D_n^o}{|D_n^o|} \left[\sum_{m=1}^{\infty} A_{nm} \left(\sum_{j=1}^{N} - \frac{A_{jm} F_{pj} D_j^o}{K_m |D_j^o|} \right) \right]$$

$$E_o = - \sum_{m=1}^{\infty} \frac{1}{2 K_m} \left[\sum_{n=1}^{N} A_{nm} F_{Pn} \frac{D_n^o}{|D_n^o|} \right]^2 = - \sum_{n=1}^{N} \frac{1}{2} K_m {a_m^o}^2$$

(4.1.2.2.3)

This formula can be calculated to arbitrary accuracy very quickly, since $K_m \to \infty$ and $A_{nm} \to o$ by hypothesis as $m \to \infty$. We may use E_o as a measure of the stability of the system, since a large E_o means a large region of stability. The condition of null stability assures us that $E_o > o$.

In the case that all $K_m > o$, the system is globally stable. Hence, there is no self-consistent solution for $\bar{s}_o$, as can be inferred from (4.1.2.2.2): if $D_n{}^o > o$ for example for all n, then $A_{nm}\, a_m{}^o < o$, but that implies that $D_n{}^o < o$ which is inconsistent.

The example of the membrane with N = 2 may be used to illustrate the process. Suppose $P = 5\pi^2$. Then the previous section assured us that a region of stability exists, because the null is stable. How does the size of the region of stability vary with F_{Pn}? We will assume for simplicity that $F_{P1} = F_{P2} = F_P$, and make use of the previously calculated values of A_{nm}.

The point $\bar{s}_o$ occurs when $\dot{a}_m = o$ and, by (4.1.2.2.2)

$$a_m{}^o = \frac{-F_P}{K_m} \sum_{n=1}^{N} A_{nm} \frac{D_n{}^o}{|D_n{}^o|}$$. Assume $D_n{}^o > o$, for n=1,2.

Then $$a_1{}^o = \frac{2F_P\sqrt{2}}{\pi\,(P-\pi^2)}$$

$$a_2{}^o = o = a_4{}^o$$

$$a_3{}^o = \frac{-2F_P\sqrt{2}}{3\pi\,(9\pi^2-P)}$$

The terms $m \geq 5$ will have negligible effect, since we can see that K_m goes as m^2 getting large and positive while A_{nm} goes as $\frac{1}{m}$. This gives

$$D_1^{o} = D_2^{o} = \left(\frac{\sqrt{2}}{\pi} \right) \left(\frac{2\ F_P\ \sqrt{2}}{\pi(P-\ \pi^2)} \right) + \left(\frac{\sqrt{2}}{3\pi} \right)\left(\frac{-2F_P\sqrt{2}}{3\pi\ (9\pi^2-P)} \right) + \ldots$$

$$= \frac{8\ F_P}{9\pi^4}$$

When $P = 5\ \pi^2$ it is clear that $D_1^{o} = D_2^{o} > o$. Hence this is a self-consistent solution. It gives a value of approximately

$$E_o \cong -\frac{1}{2}(\pi^2-P)\left(\frac{2F_P\ \sqrt{2}}{\pi\ (P-\pi^2)} \right)^2 - \frac{1}{2}(9\pi^2-p)\left(\frac{2\ F_P\sqrt{2}}{3\pi\ (9\pi^2-P)} \right)^2$$

$$E_o \cong \frac{8\ F_P^{\ 2}}{9\ \pi^2} \qquad (4.1.2.2.4)$$

Note that the critical mode amplitudes a_m^{o} scale linearly with F_P , while E_o scales as the square.

What other sign possibilities exist? We could choose $D_1^o > o$, $D_2^o < o$. This gives

$$a_1^o = a_3^o = a_4^o = o, \quad a_2^o = \frac{2\sqrt{2}\, F_P}{\pi(P-4\pi^2)}$$

which is also self-consistent. However, this leads to a value

$$E_o = -\frac{1}{2}(4\pi^2-P)\left(\frac{2\sqrt{2}\, F_P}{\pi(P-4\pi^2)}\right)^2 = \frac{4\, F_P^2}{\pi^4}$$

which is larger than the first. Hence, the first value should be chosen. Other choices for the signs of D_n lead to merely the negative of the previous choices in terms of a_m^o, D_n^o, and so the same values of E_o. Thus, the region of stability consists of all $\bar{s}(t_o)$ such that $\bar{s}(t_o)$ falls inside the closed surface (which encloses the null) $E(\bar{s}) = E_o$ given by (4.1.2.2.4). Inside that surface, $E(\bar{s}) < E_o$ and $B \geq o$.

These results may be expressed graphically in several ways. One interesting graph for the system designer is the range of parameters P and F_P such that the null is stable. This plot corresponds closely to those drawn for linear systems. The example N = 2 is plotted in

Figure 4.1.2.2.1 Range of Stable Parameters

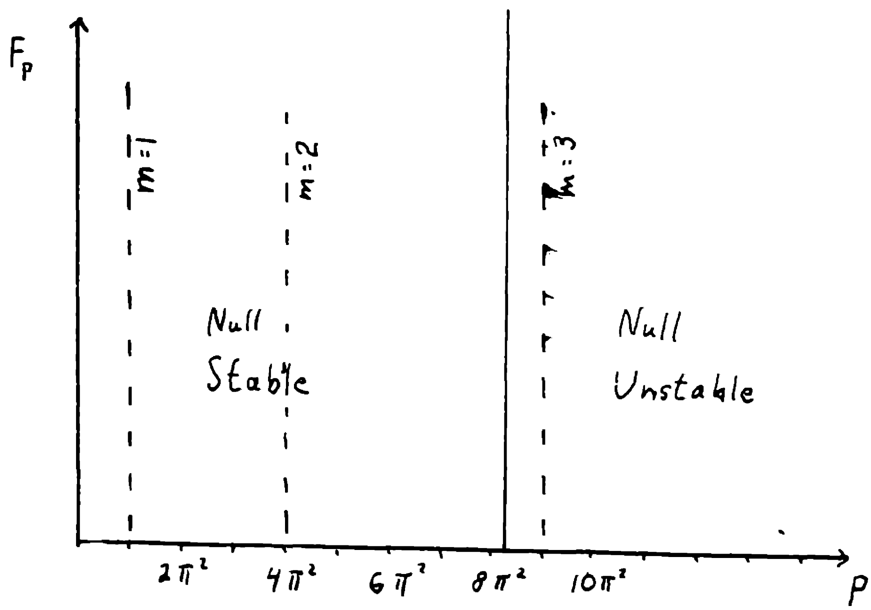


Figure 4.1.2.2.2 Region of Stability

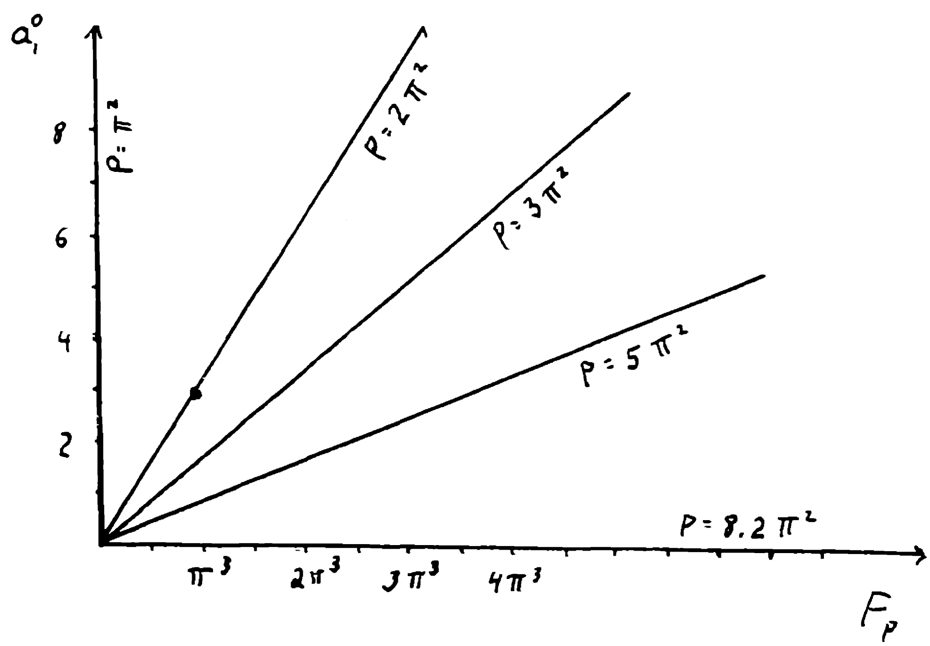


figure 4.1.2.2.1 with modal stability limits shown for reference to the original system. This plot is not as informative as the linear feedback equivalent because it is independent of F_P as would be the linear case if $A_n = B_n$ and $F \to \infty$. If desired, one could calculate the limits on P as a function of N, the number of feedback stations, and plot this dependence.

To give graphical information about the region of stability, figure 4.1.2.2.2 shows the largest stable amplitude of the lowest mode a_1^{o}, as a function of feedback amplitude F_P for various values of P. Many other plots are possible and made practical by the use of a computer to solve the stability problem for various parameter values.

Now that this example has been completely analyzed, a more careful consideration of the more general continuum control problem will be presented in the next chapter.

4.2 Modes and State Space

What are the minimum conditions on a system which allow us to treat it as was done in the previous section?

4.2.1 Form

Until now, we have assumed the following general characteristics:

1) The system must be governed by a set of partial differential equations over $\bar{r} \varepsilon V$ (any number of dimensions) which are at most second-order in time, and boundary conditions on all S_j bounding V, of the form given below

$$\bar{\bar{\rho}}(r) \cdot \frac{\partial^2 \bar{x}}{\partial t^2} + b\bar{\bar{\rho}}(\bar{r}) \cdot \frac{\partial \bar{x}}{\partial t} + \bar{\bar{K}}(\bar{r}) \cdot \bar{x} + \sum_{n=1}^{N} \bar{B}_n(\bar{r}) F_n(D_n) + \sum_{n=1}^{N} B_n'(\bar{r}) F_n'\left(\frac{\partial D_n'}{\partial t}\right) = o$$

(4.2.1.1)

At boundaries S_j, $\bar{\bar{L}}_j \cdot \bar{x} + \sum_{n_j=1}^{NJ} \bar{B}_{n_j} \cdot (\bar{r}) F_{n_j}(D_{n_j})$

$$+ \sum_{n_j=1}^{NJ} B_{n_j}'(\bar{r}) F_{n_j} \left(\frac{dD_{n_j}}{dt} \right) = o$$

(4.2.1.2)

where $\bar{\bar{K}}$ and $\bar{\bar{L}}_j$ are linear operators including spacial partial derivatives. With F_n and $F_{nj} = o$, the linear part will be referred to as the "original system". The scalar functions F_n or F_{nj} will be referred to as "feedback". The "sensor signals" D_n or D_{nj} are spacial averages of the dependent variables,

$$D_n = \int_V \bar{A}_n(\bar{r}) \cdot \bar{x} \, ds$$
$$D_{nj} = \int_{S_j} \bar{A}_{nj}(\bar{r}) \cdot \bar{x} \, ds \qquad (4.2.1.3)$$

We will assume that $\bar{\bar{\rho}}(\bar{r})$ is positive definite; that is, $\bar{x}' \cdot \bar{\bar{\rho}}(\bar{r}) \cdot \bar{x} > o$.

<u>Linear Part</u> - 2) The linear spatial operator $\bar{\bar{K}}(r) \cdot \bar{x}$ with boundary conditions $\bar{\bar{L}}_j \cdot \bar{x} = o$ at S_j has eigenfunctions $\bar{x}_m(\bar{r})$ and eigenvalues K_m;

$$\bar{\bar{K}}(\bar{r}) \cdot \bar{x}_m(\bar{r}) = K_m \, \bar{\bar{\rho}}(r) \, \bar{x}_m(\bar{r}) \qquad (4.2.1.1.1)$$

where the $\bar{x}_m$ are normalized so that

$$\int_V \bar{x}_m(\bar{r})' \cdot \overset{\circ}{\bar{\bar{\rho}}}(\bar{r}) \, \bar{x}_m(\bar{r}) \, d V = 1 \qquad (4.2.1.1.2)$$

and the $\bar{x}_m$ form a complete set over V:

$$\bar{x}(r,t) = \sum_m a_m(t) \, \bar{x}_m(\bar{r}) \qquad (4.2.1.1.3)$$

for any reasonable $\bar{x}(\bar{r},t)$ satisfying the boundary conditions. Note that this limits the system to one with a countable number of eigenmodes and eigenvalues. This avoids the problems of continuous spectra.

Thus the coefficients $\{ a_m, \dot{a}_m\} = \bar{s}$ form a vector space which contains all possible states of the system $\{ \bar{x}, \frac{d\bar{x}}{dt} \}$.

3) The $\bar{x}_m$ are orthogonal for $m \neq n$,

$$\int_V \bar{x}_n(\bar{r})' \cdot \bar{\bar{\rho}}(\bar{r}) \cdot \bar{x}_m(\bar{r}) \, dV = \delta_{mn} \tag{4.2.1.1.4}$$

This is equivalent to the statement that $\bar{\bar{K}}(\bar{r})$ is a self-adjoint operator:

$$\int_V \bar{x}_1(\bar{r})' \cdot \bar{\bar{K}}(\bar{r}) \cdot \bar{x}_2(\bar{r}) \, dV = \int_V \bar{x}_2(\bar{r})' \cdot \bar{\bar{K}}(\bar{r}) \cdot x_1(\bar{r}) \, dv \tag{4.2.1.1.5}$$

for any $\bar{x}_1$ and $\bar{x}_2$ satisfying the boundary conditions $L_j(\bar{x}_i) = o$ at S_j. Below we show the necessity and sufficiency of this statement to prove 3). Necessity: For if the $\bar{x}_m$ are orthogonal, then let

$$\bar{x}_1 = \sum_m a_m \bar{x}_m \qquad \bar{x}_2 = \sum_m a_n \bar{x}_n$$

$$\int_V \bar{x}_1 \cdot \bar{\bar{K}} \cdot \bar{x}_2 \, dV = \Sigma \, \Sigma \, a_m a_m \int_V \bar{x}_m \cdot \bar{\bar{K}} \cdot \bar{x}_n \, dV$$

$$= \sum_m \sum_n a_m a_n K_m \delta_{mn} = \sum_m a_m^2 K_m$$

reversing x_1 and x_2 leads to the same expression.

Sufficiency: If K is self-adjoint, then

$$\int_V \bar{x}_m' \cdot \bar{\bar{K}} \cdot \bar{x}_n \, dV = \int_V \bar{x}_n \cdot \bar{\bar{K}} \cdot \bar{x}_m \, dV$$

$$K_n \int_V \bar{x}_m' \cdot \bar{\bar{\rho}}(\bar{r}) \cdot \bar{x}_n \, dV = K_m \int_V \bar{x}_n' \cdot \bar{\bar{\rho}}(\bar{r}) \cdot \bar{x}_m \, dV$$

For $m \neq n$, this implies that either $\bar{x}_m$ and $\bar{x}_n$ are orthogonal, or $K_m = K_n$. The cases of non-distinct eigenvalues encountered in this work are those in which, for instance, both sines and cosines of the same argument make up the eigenfunctions $\bar{x}_m$. These are clearly orthogonal.

4) The linear original system then has solutions as given in 2) with $a_m(t) = C_m e^{j\omega_m t}$. It will be assumed that the modes are numbered so that $Im(\omega_m) \geq o$ for $m > M$. If an infinite number of modes are unstable, then feedback stabilization by a finite number of stations is not possible.

This restriction implies that if $m > M$, then $\dot{b} \geq o$ and $K_m > o$.

Nonlinear Part 5) The feedback functions $F_n(D_n)$, $F_n'\left(\frac{dD'_n}{dt}\right)$, $F_{nj}(D_{nj})$, and $F'_{nj}\left(\frac{dD'_{nj}}{dt}\right)$ are assumed to be piecewise continuous scalar functions, independent of time, with at most one discontinuity at $D = o$.

Under these conditions, it is possible to dot the equation with $\frac{\partial \bar{x}}{\partial t}$ and integrate over V, using boundary conditions, to obtain an energy equation $\frac{dE}{dt} = -B$. The function E may then be used as a Lyapunov function on the slab space defined by the mode amplitudes $a_m(t)$ and velocities $\frac{da_m}{dt}$ for stability analysis.

Results:

The general form of the energy equation for this system is

$$\frac{dE}{dt} = -B, \qquad E = T + \psi + U \qquad (4.2.1.3.1)$$

$$T = \frac{1}{2} \int_V \frac{d\bar{x}'}{dt} \cdot \bar{\bar{\rho}} \cdot \frac{d\bar{x}}{dt} d\tau = \sum_m \frac{1}{2} \dot{a}_m^{\,2}$$

$$\psi = \frac{1}{2} \int_V \bar{x} \cdot \bar{\bar{K}} \cdot \bar{x} \, d\tau = \sum_m \frac{1}{2} K_m a_m^{\,2}$$

$$U = \sum_{n=1}^{N} U_n \quad \text{where } U_n = \int_0^{C_n} F_n(x) \, dx$$

$$C_n = \int_V \bar{B}_n' \cdot \bar{x} \, d\tau = \sum_m B_{nm} a_m$$

$$B = B_o + B_P + B_F \qquad (4.2.1.3.2)$$

$$B_o = \int_V \frac{d\bar{x}}{dt} \cdot \bar{\bar{b}} \cdot \frac{d\bar{x}}{dt} = \sum_m b_m \dot{a}_m$$

$$B_P = \sum_n \frac{dC_n}{dt} [F_n(D_n) - F_n(C_n)]$$

$$B_V = \sum_{n'} \frac{dC'_n}{dt} \cdot F_n' \left(\frac{dD_n'}{dt} \right)$$

Null stability is determined by

$$\nabla_S E \cdot \bar{s} = o \qquad \text{for small } |\bar{s}|$$

$$B \geq o \qquad (4.2.1.3.3)$$

and the region of stability bounded by $E(\bar{s}) = E_o$ where E_o is theis smallest value of E such that either $\nabla_S E = \bar{o}$ at $\bar{s}_o$ or $B > o$ at $\bar{s}_o$.

Note that in the continuum problem, the spatial distribution of the n^{th} sensing and forcing elements determine $A_n(z)$ and $B_n(z)$. Clearly, we desire $A_n(z) = B_n(z)$ to make $B \geq o$. However, the deviation from this condition is measured by $F_n(D_n) - F_n(C_n)$. Therefore, if $A_n(z) - B_n(z)$ is small (perhaps only

$A_{nm} - B_{nm} \neq o$ for large m, and then very small) then the effects of $\dot{B}_p$ may be masked for motions on the scale of interest by B_o or B_V. Then we may ignore this deviation from ideal feedback design.

4.2.2 Comparison with Modal Control

The control of continuous systems one mode at a time has been discussed in the literature, in the context of linear feedback. Clearly, if we can make each feedback station interact with only one mode by spacial filtering, the control problem is enormously simplified. If $A_{nm} = \delta_{nm}$, then design of each feedback station can alter the dynamics of the unstable modes at will, one at a time. Pole-moving schemes for linear feedback are only one possibility, since nonlinear feedback based on position or velocity information could also be used. However, it is usually very difficult to achieve the required spacial distributions $A_n(z)$ and $B_n(z)$. Luenberger observers relieve this constraint somewhat, but only by removing the problem to higher modes which they do not estimate. The scheme proposed in this work is more general, in that $A_n(z)$ and $B_n(z)$ can be chosen arbitrarily, as long as they are almost identical. After this criterion is satisfied, one can then try to improve the design to the point where it begins to approximate modal control.

Looking at the null stability criterion, it is clear that the most efficient feedback scheme is modal control. However, if that can not be conveniently achieved because of physical limitations on $A_n(z)$ and $B_n(z)$, one can still achieve stability by keeping $A_n(z) = B_n(z)$. The deviation from modal control results in combining unstable modes with stable ones in the null stability criterion, and gives a measure of system degradation.

4.2.3 Generalization

What restrictions in the preceding class of problems can be relaxed? Let us consider them in order.

1) The system of partial differential equations may be of higher order I in time derivatives. In general, the derivatives of order I-2K, $o \leq K < {}^{I}/2$, will become part of the Lyapunov function E, and the odd derivatives become part of its total derivative B in the resulting energy equation $\frac{dE}{dt} = -B$. If $\bar{\bar{\rho}}(\bar{r}) = o$, then the equation becomes first order in time, and might, for example, be represented by the diffusion equation rather than the equation of a vibrating string. The same analysis can be used, but with the simplification that the state consists only of mode amplitudes. Details of such a system are described in Appendix C.

Also, $\bar{\bar{\rho}}$ and $\bar{\bar{b}}$ may be spatial derivative operators instead of constants. However, they must be self-adjoint as defined in 3), for otherwise non-orthogonality of modes would not allow drawing of conclusions about stability. Once again, the space of states having negative damping must be finite-dimensional if feedback is to be successfully applied. Note that if the $\bar{x}_m$ are not the eigenvectors of $\bar{\bar{\rho}}$ and $\bar{\bar{b}}$, then finding ω_m may be complicated, but the energy analysis can still be used. In such a case, analysis of $\frac{dE}{dt} = -B$ would require defining two different basis sets in the state space for proving $E > o$ and $B \geq o$.

Also, the boundary conditions $L_j(\overline{x}) = o$ may include time-derivatives. In this case, the set of eigenfunctions is not limited by such boundary conditions, and the boundary condition is used to determine additional terms in the energy equation. Clearly, boundary interactions are just a special case of interactions which are distributed through the volume V.

2) Clearly, this normalization is arbitrary. However, no other has any obvious advantages.

3) Note that the requirement that $\overline{\overline{K}}$ be self-adjoint is in some cases related to the property of exchange of stability. If some parameter of a system is varied continuously, and the system as a result becomes unstable with zero frequency, then the system is said to exhibit exchange of stability. If b remains non-negative as $\overline{\overline{K}}$ varies, this property clearly results. One common form of the $\overline{\overline{b}}$ operator is that of the viscous stress tensor. Appendix A shows that this is self-adjoint by showing that $\overline{V} \cdot \overline{F}$ has a symmetric form. We also note here that convective systems (those with non-zero equilibrium velocities $\overline{V}_o$) are in general not self-adjoint if $\overline{V}_o \cdot \overline{x} \neq o$.

4) Our division of the system into linear "original system" and nonlinear "feedback" need not be correct. Perhaps something in the system changes rapidly as a function of the state as compared with other processes. In that case, the "feedback" may be an internal mechanism with effectively an infinite number of stations. Such a case will be discussed later, in the example of a fluid with a sharply bounded field region above it. Then, any eigenvalues are allowed for the linear part, and the summation over N becomes an integral.

5) The feedback functions $F_n(D_n)$ etc. may be functions of time. In this case, their derivatives enter the energy equation as additional terms in B. In general, all effects which can not be handled as a part of E must be included as corrections to B. In many cases, as will be discussed, such correction terms can be bounded and their effect understood this way.

Also, it may be that a programmed value may be subtracted from the sensor signal,

$$D_n*(t) = D_n(t) - S_n(t),$$

and the feedback is a function of the error $D_n*(t)$. This might apply to the possibility of shaping surfaces of plastics or metals in an extrusion process, or adding time-variation

to control for stability. In this case, the prc
can be put in the form of a static or quasistati
equilibrium and a dynamic stability about that e
with time-varying F_n. These aspects will be dis
later.

In the work which follows, we will continue
with a one-dimensional model in which ρ and b ar
stants. $K(\bar{r})$ will be a linear sum of spatial de
all of even order. Thus, we can examine the prc
of the entire class of systems in terms of the s
model we have already developed.

4.2.4 Norm, State, and Stability

The system can now be formally described in terms of the state space $\{a_m, \dot{a}_m\}$. Since the feedback requires no energy storage, it does not add state elements. If the feedback is truly external to the physical system, then it makes sense to refer to the $D_n(t)$ as the outputs and the $F_n(t)$ as inputs. We know that the stability of an infinite-dimensional system depends upon the definition of norm. Our definition will be

$$|\bar{x}| = [\sum_m a_m^2 + \dot{a}_m^2]^{1/2} \qquad (4.2.4.1)$$

This will converge for most common perturbations $\bar{x}$, which have components falling off at least as $1/m$.

Lyapunov stability of such a system can then be defined as it was for finite-dimensional systems. The development is fully described by the model in section 4.1.

4.3 Complications

Most of the complications which are of interest for this class of control systems are complications in the form of the feedback law. As such, they can be treated independently from the continuous system itself, and so are handled just as in the lumped parameter problem. Therefore, we refer to Chapter III for the following topics.

Hysteresis as Negative Damping or Energy

Time Delay as Hysteresis

Time-Sampling as Time Delay or Lag

Control with Bang-Bang Feedback

Multiple Feedback Station Problems

Higher Order Terms

Dead Band and Linear Band

We simply treat the modes of the continuous system as oscillators, coupled together by the feedback. Our algorithms for treating stability questions have all been worked out so that the modal series can be truncated, knowing that the effects will be minimal. This is based on two assumptions:

1) The higher modes of the system, having rapid spacial variations in their structure, are very stable

(K_m and/or b_m large and positive.)

2) The higher modes do not couple strongly to the feedback, because the feedback sensing and forcing elements have finite size. (A_n and B_n do not have rapid spacial variations in their structure, and so A_{nm} and B_{nm} become small as m becomes large.) These assumptions become more true as the design of the feedback system improves, since good design implies:

1) b_m and $K_m >> o$ for $m > N$

2) $B_{nm} \stackrel{\circ}{=} A_{nm}$

3) det $\bar{\bar{A}}_N$ large

4) $|A_{nm}|$ small for $m > N$.

In other words, the sensors and forcing elements should be designed so as to give spacial weighting patterns with maximum coupling to the unstable modes and minimum coupling to the others.

With this in mind, the algorithms for testing for null stability and determining region of stability should be re-read at this point. The tests given in Chapters III and used in IV-1 are identical except that the number of modes (oscillators) has become infinite.

With this in mind, we will focus on several problem areas in the above group of topics which are peculiar to the continuous system.

4.3.1 Combined Position and Rate Feedback

To minimize system complexity, it is desirable to use a single feedback station with one bang-bang output for both position and rate feedback. Thus, some algorithm based on both D_n and $\dot{D}_N$ will be used to determine $G_n(D_n, \dot{D}_n)$, and $F = -F_n\, G_n/|G_n|$.

The simplest case, of course, is a linear weighting of the two. Let the discriminant of the feedback be

$$G_n = D_n + T\,\dot{D}_n = \sum_m A_{nm}\,(a_m + T\,\dot{a}_m) \tag{4.3.1.1}$$

We can form an energy function for this type of feedback by taking our modal equation of motion, multiplying by $\dot{a}_m$ as usual, sum over m, and then notice that the feedback terms look like the first term of

$$\frac{d}{dt}\,[\,F_n\,|G_n|\,] = F_n\,\frac{G_n}{|G_n|}\,\dot{G}_n = F_n\,\frac{G_n}{|G_n|}\,\sum_m A_{nm}(\dot{a}_m + T\,\ddot{a}_m) \tag{4.3.1.2}$$

The second term is of the form

$$F_n\,\frac{G_n}{|G_n|}\,\ddot{D}_n$$

which may be either positive or negative. However, let's assume for a moment that the amplitude a_m are small, so that the discontinuous feedback force dominates the equation of motion. Thus

$$\ddot{a}_m = - \sum_n A_{nm} F_n \frac{G_n}{|G_n|} \tag{4.3.1.3}$$

Therefore, combining the above with our equations of motion, we get

$$\frac{d}{dt} [T + \psi + U] = -B \tag{4.3.1.4}$$

where $U = \sum_n F_n |\dot{G}_n|$, T and ψ are as given previously, and the feedback part of B is approximately

$$B_F \cong T \sum_m [\sum_n A_{nm} F_n \frac{G_n}{|G_n|}]^2 \tag{4.3.1.5}$$

Thus we once again have an energy formulation. Now, however, the feedback energy involves the velocities $\dot{a}_m$ as well as amplitudes a_m. If $T \to o$, we recover the original energy function. As T becomes large, however, the energy equation should approach that of a pure velocity feedback system. It does not, because as $T \to \infty$ the force as written here takes on an indeterminate form.

Long before that happens, however, the null becomes unstable if $\omega_1^2 < o$. To see this rigorously, one has to solve the problem of proving $\nabla_s E \cdot \bar{s} > o$ for all $\{a_m, \dot{a}_m\}$ such that $G_n = o$; the same operation as done in positional feedback. However, $G_n = o$ implies (in vector notation from chapter III).

$$\bar{\bar{A}}_N \cdot \bar{a}_N + T\,\bar{\bar{A}}_N \cdot \dot{\bar{a}}_N + \bar{\bar{A}}_\infty \cdot \bar{a}_\infty + \bar{\bar{A}}_\infty \cdot \dot{\bar{a}}_\infty = o \tag{4.3.1.6}$$

Solving for $\bar{a}_N$ and substituting into the energy matrix to find $T + \psi$, one gets the result $\nabla_s E \cdot \bar{s} = 2(T + \psi)$ which is given in figure 4.3.1.1, as a matrix divided into nine parts. A computer could use Sylvester's test for positive definiteness by truncating W_∞ and A_∞ in the usual way. A more intuitive look, however, is more rewarding.

If the feedback is well designed, so that elements of $\bar{\bar{A}}_N^{-1}$ and $\bar{\bar{A}}_\infty$ are small, then two terms give most of the information in this test. The static considerations come from the terms in the center submatrix, and are identical to the null stability test for positional feedback. The velocity contribution is most critical in the upper-left submatrix, which is purely diagonal. Stability requires that

$$
\begin{array}{c|c|c|c}
 & \bar{\dot{a}}_N & \bar{a}_\infty & \bar{\dot{a}}_\infty \\
\hline
\bar{\dot{a}}_N & \bar{\bar{I}}_N + T^2 \; \bar{\bar{W}}_N & T \; \bar{\bar{W}}_n \cdot [\bar{\bar{A}}_N^{-1} \cdot \bar{\bar{A}}_\infty] & T^2 \; \bar{\bar{W}}_N \cdot [\bar{\bar{A}}_N^{-1} \cdot \bar{\bar{A}}_\infty] \\
\hline
\bar{a}_\infty & T[\bar{\bar{A}}'_\infty \cdot \bar{\bar{A}}_N^{-1'}] \cdot \bar{\bar{W}}_N & \bar{\bar{W}}_\infty + [\bar{\bar{A}}'_\infty \cdot \bar{\bar{A}}_N^{-1'} \cdot \bar{\bar{W}}_N \cdot \bar{\bar{A}}_N^{-1} \cdot \bar{\bar{A}}_\infty] & T[\bar{\bar{A}}'_\infty \cdot \bar{\bar{A}}_N^{-1} \cdot \bar{\bar{W}}_N \cdot \bar{\bar{A}}_N^{-1} \cdot \bar{\bar{A}}_\infty] \\
\hline
\bar{\dot{a}}_\infty & T^2[\bar{\bar{A}}_\infty \cdot \bar{\bar{A}}_N^{-1'}] \cdot \bar{\bar{W}}_N & T[\bar{\bar{A}}_\infty' \cdot \bar{\bar{A}}_N^{-1'} \cdot \bar{\bar{W}}_N \cdot \bar{\bar{A}}_N^{-1} \cdot \bar{\bar{A}}_\infty] & \bar{\bar{I}}_\infty + T^2[\bar{\bar{A}}_\infty' \cdot \bar{\bar{A}}_N^{-1'} \cdot \bar{\bar{W}}_N \cdot \bar{\bar{A}}_N^{-1} \cdot \bar{\bar{A}}_\infty]
\end{array}
$$

Energy Matrix for $\nabla_s E \cdot \bar{s}$ when $G_n = o$

Figure 4.3.1.1

$$1 + T^2 \omega_m^2 > o \text{ for all } m \leq N. \qquad (4.3.1.7)$$

The most dangerous mode is m = 1 since it has most negative ω^2. The upper limit on T consistent with stability, then, is predicted to be

$$T^2 < \frac{-1}{\omega_1^2} \qquad (4.3.1.8)$$

The existence of off-diagonal submatrices have the effect of requiring slightly lower values of T^2, because using the generalized Gauss theorem: if a matrix $\bar{\bar{K}} = \begin{vmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ \bar{\bar{B}}' & \bar{\bar{D}} \end{vmatrix}$ and let $\bar{\bar{D}} \neq o$, then $\det \bar{\bar{K}} = \det (\bar{\bar{A}} - \bar{\bar{B}}\bar{\bar{D}}^{-1} \cdot \bar{\bar{B}}^{1})$. Hence, for $\det K > o$, we require not only $\det \bar{\bar{A}} > o$, but $\det (\bar{\bar{A}} - \bar{\bar{B}} \cdot \bar{\bar{D}}^{-1} \cdot \bar{\bar{B}}') > o$ Of course, if $\omega_m^2 > o$ for all m, then any T is allowed, as we would expect. Also, if T = o, the original stability criterion results.

Region of stability considerations take the form from $\nabla_s E = o$ of

$$\dot{a}_m^o = - T \sum_n F_n A_{nm} G_n^o / |G_n^o|$$

$$a_m^o = \frac{-\sum_n F_n A_{nm} \dfrac{G_n^o}{|G_n^o|}}{\omega_m^2}$$

(4.3.1.9)

The resulting maximum energy of stable perturbations is given by

$$E_o = - \sum_m (T^2 + \frac{1}{\omega_m^2}) [\sum_n F_n A_{mn} \frac{G_n^o}{|G_n^o|}]^2 \qquad (4.3.1.10)$$

which can be shown to be positive definite from the preceding discussion. The effect of $T > o$ is to lower the value of E_1 slightly, making the region of stability slightly smaller.

We must remember, however, that the form for B written above is an approximation. Actual results of computer experiments show that in fact instability results for T less than some critical value, but the precise value is about ten times greater than $1/\omega_1^2$ and depends on initial conditions. Optimal values of T to give fast quenching of perturbations without overshoot generally depend on the initial conditions. For static initial conditions in which the lowest mode makes up of the initial condition, good choices for the value of T can be derived from the lumped approximation

$$\ddot{a}_1 + b_1 \dot{a}_1 + K_1 a_1 = - \sum_n F_n A_n \frac{G_n}{|G_n|}$$

where

$$G_n = (a_1 + T \dot{a}_1) A_{n1} \tag{4.3.1.11}$$

The optimization problem has explicit solutions for finite-dimensional problems. (See for instance, LaSalle's work.)

Here is another approach to combined feedback for small T. Let

$$F_n(t) = -F_n \frac{D_n + T \dot{D}_n}{|D_n + T \dot{D}_n|} \tag{4.3.1.12}$$

Thus, examining the feedback force in the $D_n - \dot{D}_n$ plane, we see that $F_n(t)$ changes sign across the line $D_n + T \dot{D}_n = o$.

This scheme can be related to the one given by simple position feedback, since our energy function can be formed from it. We note that the feedback force here is of proper sign for position feedback provided that $D_n(D_n + T \dot{D}_n) > o$. When this inequality is violated, then the force is in the proper direction for velocity feedback. Thus we form the total force

$$F_n(t) = - F_n \frac{D_n}{|D_n|} - F_n[1 - \frac{D_n(D_n + T\dot{D}_n)}{|D_n(D_n + T\dot{D}_n)|}] \frac{\dot{D}_n}{|\dot{D}_n|}$$

(4.3.1.13)

This gives

$$U_n = F_n|D_n| \geq o$$

$$B_n = F_n [1 - \frac{D_n(D_n + T\dot{D}_n)}{|D_n(D_n + T\dot{D}_n)|}] |\dot{D}_n| \geq o$$

(4.3.1.14)

This effect of the rate feedback, then, is to add a damping term without changing the energy function. The damping term is nonzero only part of the time. If τ is the time between zero-crossings and $\tau >> T$, then $B_n \neq o$ for approximately $\frac{T}{\tau}$ of the time. Thus averaged over many oscillations, the effective feedback is that of a term

$$F_n' = F_n \frac{D_n}{|D_n|} - F_n \frac{T}{\tau} \frac{\dot{D}_n}{|D_n|}$$

(4.3.1.15)

Since the average zero-crossing time τ approaches zero as the amplitude decreases to zero, the damping term given here will be dominated by other effects, such as time delays, for small amplitudes.

We may conclude that the effect of lead or lag compensation in a positional feedback loop, formed by placing a linear element with transfer function

$$\frac{1 + T_1 s}{1 + T_2 s} \quad \text{with } T_1, T_2 << \tau \tag{4.3.1.16}$$

between the sensor and the nonlinear element in the feedback loops, will be:

If $T_1 > T_2$, damping is introduced

If $T_2 > T_1$, negative damping is introduced.

The approximate value of the damping term is for $T_i << \tau$, given by

$$B_n = F_n \frac{(T_1 - T_2)}{\tau} \quad |\dot{D}_n| \tag{4.3.1.17}$$

These results match those of Kochenberger, who used a describing function approach to treat this class of lumped control problems.

4.3.2) Control of a Continuum with Discrete Bang-Bang Feedback

Suppose that, in addition to stabilizing a continous system, we actually wish to control its shape. Perhaps we wish to mold some material without touching its surface, using electrical or magnetic fields. The membrane model will be used, so that

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \xi}{\partial z^2} + P\xi - \sum_{n=1}^{N} B_n(z)\, F_n(t)$$

where

$$\xi(o,t) = \xi(1,t) = o \qquad (4.3.2.1)$$

Suppose we want to force the surface into some shape. This time, the discriminant will have an offset S_n so that

$$D_n(t) = \int_o A_n(z)'\xi(z,t)\, d\, z - S_n \qquad (4.3.2.2)$$

The feedback force $F_n(t)$ will be of the form

$$F_n(t) = \frac{F_{n+}}{2} \left[1 + \frac{D_n}{|D_n|} \right] + \frac{F_{n-}}{2} \left[1 - \frac{D_n}{|D_n|} \right] \qquad (4.3.2.3)$$

That is, $F_n(t)$ takes on F_{n+} for $D_n(t) > o$, and F_{n-} for $D_n(t) < o$. We are free to define $F_n(t)$ for $D_n = o$.

To analyze this system, we expand the displacements in terms of the normal modes of the system without the feedback, as before.

$$\xi(z,t) = \sum_m a_m(t)\ \xi_m(z),\ \xi_m = \sqrt{2}\ \sin m\pi z \tag{4.3.2.4}$$

where without feedback

$$\frac{\partial^2\ a_m(t)}{\partial t^2} = -\omega_m{}^2\ a_m(t) \tag{4.3.2.5}$$

$$\omega_m{}^2 = m^2\ \pi^2 - P \tag{4.3.2.6}$$

With the feedback, the equation of motion can be dotted with ξ_m and integrated over z to give

$$\frac{\partial^2\ a_m}{\partial\ t^2} + \omega_m{}^2\ a_m = \int_o^1 \xi_m\ (\sum_n -B_n(z)F_n(t)\ )\ d\ z$$

Let $B_n(z) = \sum_m\ B_{nm}\ \xi_m(z)$

$$B_{nm} = \int_o^1 B_n(z)\ \xi_m(z)\ d\ z \tag{4.3.2.7}$$

Then

$$\left[\frac{\partial^2}{\partial t^2} + \omega_m^2 \right] a_m = \sum_{n=1}^{N} F_n(t)\ B_{nm} \tag{4.3.2.8}$$

We now want to cast the problem in the form of a static equilibrium at D_n - o. For that reason, we write $F_n(t) = F_{n_1} + F_{n_2}\ (D_n(t))$ where F_{n_1} is constant and F_{n_2} = o at D_n = o. We then have

$$a_m^e = \sum_{n=1}^{N} \frac{B_{nm}\ F_n}{\omega_m^2} \tag{4.3.2.9}$$

at the point D_n = o. Let

$$A_{nm} = \int_0^L \xi_m(z)\ A_n(z)\ d\ z$$

so that

$$D_n = \sum_m A_{nm}\ a_m - S_n \tag{4.3.2.10}$$

Thus we have, at the equilibrium (denoted by primes) $D_n = o$, or using (9) and (10),

$$S_n = \sum_m A_{nm} \left(\sum_{J=1}^{N} \frac{B_{jm} F_{j1}}{\omega_m^2} \right)$$

$$= \sum_j^N \left(\sum_m \frac{A_{nm} B_{jm}}{\omega_m^2} \right) F_{j1} \qquad (4.3.2.11)$$

Now, we will have to assume the usual things about the system. One is that ω_m^2 increases with increasing m. The other is that A_{nm} and B_{nm} decrease with increasing m. Both of these are reasonable for any system with stability in the short-wavelength perturbation limit and feedback electrodes of finite size. Thus, the coefficients in (11) will converge quickly. We can write (11) as

$$\bar{S} = \bar{\bar{M}} \cdot \bar{F}_1 \qquad (4.3.2.12)$$

where $\bar{S}$ and $\bar{F}_1$ are n-vectors and $\bar{\bar{M}}$ is the matrix of coefficients

$$M_{nj} = \sum_m \frac{A_{nm} B_{jm}}{\omega_m^2} \qquad (4.3.2.13)$$

If one mode has $\omega_m^2 \to o$, we see that it dominates the series and $\bar{F}_1 \to \bar{o}$.

If each of the N feedback stations is spatially discrete, and if $A_n(z)$ and $B_n(z)$ are not spatially orthogonal, then we expect that $\bar{\bar{M}}$ will probably be invertible. Otherwise there would exist a set of nonzero forces that could exist in static equilibrium with zero output. Thus we may write

$$\bar{F}_1 = \bar{\bar{M}}^{-1} \ \bar{S} \qquad (4.3.2.14)$$

This defines the values F_{n1} in terms of the offsets S_n. We also have the complete expression for the shape of the equilibrium from (9)

$$\xi^e(z) = \sum_m a_m^e \ \xi_m(z) = \sum_m \left(\sum_{n=1}^{N} \frac{B_{nm} F_{n_1}}{\omega_m^2} \right) \xi_m(z)$$

(4.3.2.15)

This summation over m again converges quickly, as did (11).

We then have solved the equilibrium problem, and can examine the dynamics. What we've done is to fix the values of N scalar quantities, representing weighted average characteristics of the equilibrium. Any number of techniques can be used to decide what these should be in a given practical situation.

For example, let N = 1 feedback station, with

$$F_1(t) = \pi^2 \frac{D}{|D|}$$

and S = .1 with A(z) = B(z) = 1. Let $P = 2\pi^2$ so that the m = 1 mode is unstable

$$\omega_1^{\,2} = -\pi^2$$
$$\omega_2^{\,2} = 2\pi^2$$
$$\omega_3^{\,2} = 7\pi^2$$

Thus we're trying to keep the system at an average position of .1 by applying a force of $\pm\ \pi^2$ uniformly to it whenever its average deflection varies from this figure.

The parameters

$$A_m = B_m = \int_o^1 \sqrt{2}\ \sin m\pi z dz = \begin{cases} \frac{2\sqrt{2}}{m\pi}, & m \text{ odd} \\ o, & m \text{ even} \end{cases}$$

So

$$M_{11} = \sum_{\substack{m \\ odd}} \left(\frac{2\sqrt{2}}{m} \right)^2 \left(\frac{1}{\omega_m^2} \right) = \frac{8}{\pi^4} [-1 + \frac{1}{63} + \ldots] \doteq -.08$$

So

$$M^{-1} \doteq -12.5, \quad F_1 = M^{-1} S = -1.25$$

Hence the equilibrium is given by

$$\xi_e(z) = \sum_{\substack{m \\ odd}} (-1.25) \left(\frac{2\sqrt{2}}{\pi m} \right) \left(\frac{1}{\omega_m^2} \right) \sin m\pi z$$

$$\xi_e(z) \doteq .114 \sin \pi z - .0054 \sin 3\pi z \quad \ldots$$

Now for the dynamic problem.

Let $a_m^* = a_m - a_m^e$ represent the motion in state space $\left(\frac{\partial a_m^*}{\partial t}, a_m^* \right)$ away from equilibrium. We have from (8)

$$\left[\frac{\partial^2}{\partial t^2} + \omega_m^2 \right] a_m^* = \sum_n B_{nm} F_{n_2}(t) \tag{4.3.2.16}$$

where

$$F_{n2}(t) = F_n(t) - F_{n1} = \frac{-F_{n+}^*}{2}\left(1 + \frac{D_n}{|D_n|}\right) + \frac{F_{n-}^*}{2}\left(1 - \frac{D_n}{|D_n|}\right)$$

$$= \frac{(F_{n+} - F_{n1})}{2}\left(1 + \frac{D_n}{|D_n|}\right) + \frac{(F_{n-} - F_{n1})}{2}\left(1 - \frac{D_n}{|D_n|}\right)$$

$$D_n = \sum_m A_{nm} a_m^*$$

(4.3.2.17)

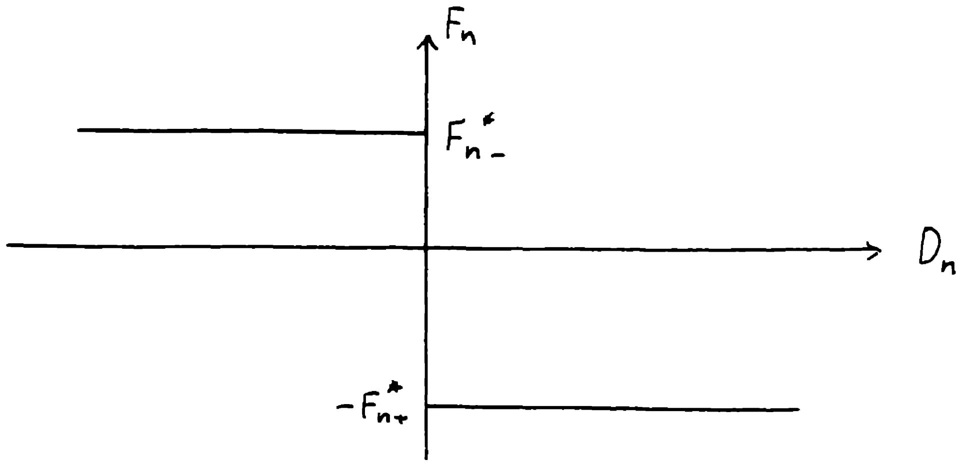
The dynamics are determined completely by (16) and (17). In particular, if we rewrite (17) as

$$F_{n_2}(t) = \left(\frac{F_{n+} + F_{n-} - 2F_{n1}}{2}\right) - \left(\frac{-F_{n+} + F_{n-}}{2}\right) \frac{D_n}{|D_n|}$$

That is, we have a new problem with no offset, and a different feedback force! If we wished, we could make this in turn into a problem with offset, and a symmetric feedback term. The two effects are equivalent and can be transformed linearly by (12) and (14).

We would now like to examine the region of stability of the system in the state space ($\frac{\partial a_m^*}{\partial t}$, a_m^*).

Figure 4.3.2.1 Feedback at n^{th} Station



We multiply (16) by $\frac{\partial a_m^*}{\partial t}$ and integrate by parts to get

$$\frac{dE_m}{dt} = \frac{d}{dt} \left[\frac{1}{2} \left(\frac{\partial a_m^*}{\partial t} \right)^2 + \frac{1}{2} \omega_m^2 a_m^{*2} - \sum_n B_{nm} a_m^* F_{n2} \right] = o$$

(4.3.2.18)

This tells us that we want B_{nm} to be large for modes such that $\omega_m^2 \leq o$ to ensure a large energy contribution for our limited force F_{n2}. Let $\sum_m B_{nm} a_m^* = C_n$.

Summing over all m, we then get

$$\frac{dE}{dt} = o, \quad E = T + \psi + U \qquad (4.3.2.19)$$

$$T = \sum_m \frac{1}{2} \left(\frac{\partial a_m^*}{\partial t} \right)^2 = \int_o^1 \frac{1}{2} \left(\frac{\partial \xi}{\partial t} \right)^2 dz \geq o$$

$$\psi \quad \sum_m \frac{1}{2} \omega_m^2 a_m^{*2} = \int_o^1 \frac{1}{2} \left[\frac{\partial}{\partial z} (\xi-\xi_e)^2 \right] - \frac{1}{2} P (\xi-\xi_e)^2 \, dz$$

$$U = \sum_n C_n F_{n2}$$

In order to ensure stability, we want the energy $E > o$ in a small region near the new origin $\bar{o}*: \frac{\partial a_m^*}{\partial t} = a_m^* = o.$

This requires that, for all n,

$$C_n = D_n \qquad (4.3.2.20)$$

$$F_n^* = \frac{-F_{n+} + F_{n1}}{2} > o \qquad (4.3.2.21)$$

$$F_n^* = \frac{F_{n-} - F_{n1}}{2} > o \qquad (4.3.2.22)$$

(20) ensures that the feedback is in the right direction for all modes. It implies that $B_{nm} = A_{nm}$ for all n and m, or $B_n(z) = A_n(z)$; that is, each feedback station has the same spatial distribution in its forcing and sensing functions. This can be relaxed only at the cost of driving some higher modes unstable for very small oscillations about $\bar{o}*$.

(21) and (22) ensure that the feedback drives the system toward $D_n = o$ rather than away from it, for both signs of D_n. If $|F_{n1}|$ gets too large, one of these conditions will be violated, so this is a limitation on the possible offsets S_n consistent with stability.

We then have a feedback function for the n^{th} station looking like Figure 4.3.2.1. The number of stations needed for bang-bang stabilization is independent of the amplitude of feedback as long as it is of the proper sign, so we will not discuss it here. Generally, N must at least equal the number of independent modes with $\omega_m^2 < o$.

To determine a region of stability, we look for the minimum energy E_o such that $\nabla E = o$ at $E(\bar{s}_o) = E_o$. The state vector $\bar{s}_o$ can be determined from the minimum of F_{n+} and F_{n-}, denoted $F_{no} > o$.

This requires a self-consistent solution to

$$a_m^o = -\sum_n \frac{A_{nm} F_{no}}{\omega_m^2} \frac{D_n^o}{|D_n^o|} \quad , \quad \frac{\partial a_m^o}{\partial t} = o \tag{4.3.1.23}$$

where D_n has the sign appropriate to F_{no} and results from a_m^o. A finite number of trials will always yield a solution if it exists. If it does not, then global stability applies, as is the case when $\omega_m^2 > o$ for all m. Otherwise, the region of guaranteed stability, including $\bar{o}^*$, is then bounded by the surface

$$E(\bar{s}) \leq E_o$$

where $E(\bar{s})$ is given by (16) and

$$E_o = \sum_m \frac{1}{2\ \rho_m\ \omega_m^2} \left[\sum_{n=1}^{N} A_{nm}\ F_{no} \frac{D_n^o}{|D_{n_o}|} \right]^2$$

(4.3.2.24)

$$+ \sum_n F_{no}\ |D_n^o|$$

where

$$D_{n_o} = -\sum_m \frac{A_{nm}}{\rho_m\ \omega_m^2} \left[\sum_{J=1}^{N} A_{jm}\ F_{jo} \frac{D_j^o}{|D_{jo}|} \right]$$

or

$$E_o = \sum_m - \frac{1}{2\ \rho_m\ \omega_m^2} \left[\sum_{n=1}^{N} A_{nm}\ F_{no} \frac{D_n^o}{|D_{n_o}|} \right]^2$$

(4.3.2.25)

(which is positive, and converges quickly)

$$= \sum_m - \frac{1}{2}\ \rho_m\ \omega_m\ a_m^{o2}$$

Continuing our specific example, we see that (20), (21), and (22) are all satisfied. We may assume from previous work that this feedback arrangement has an energy minimum at $\bar{o}^*$. We have

$$F^* = \pi^2 + 1.25^o = 11.1 > o$$

$$F^* = \pi^2 - 1.25^o = 8.6 > o$$

So the minimum is $F_{n_o} - F_{n-} = 8.6$

Thus we assume that $D_n^o < o$ and use (23) to find

$$a_m^o = + \left(\frac{2\sqrt{2}}{m\pi} \right) \frac{(8.6)}{\omega_m^2}$$

$$a_1^o = - \left(\frac{2\sqrt{2}}{\pi} \; \frac{8.5}{\pi^2} \right) = .77$$

$$a_2^o = o \qquad a_3^o = \left(\frac{2\sqrt{2}}{3\pi}\right)\left(\frac{8.6}{7\pi^2}\right) \cong .037$$

So $D^o = \sum_m A_m a_m^o < o$ as assumed, so the solution is self-consistent

$$E_o = \sum_m - \frac{1}{2} \omega_m^2 a_m^2$$

$$\cong \left(\frac{1}{2} \right) (.77)^2 (\pi^2) \cong 30$$

So any initial perturbation with initial values of

$a_m^*, \frac{\partial a_m^*}{\partial t}$ within the volume enclosed by the surface

$$E(\bar{s}) = \sum_m \frac{1}{2} \rho_m \left(\frac{\partial a_m^*}{\partial t}\right)^2 + \sum_m \frac{1}{2} \rho_m \omega_m^2 a_m^{*2}$$

$$+ D \left[\frac{F^*_+}{2}\left(1 + \frac{D}{|D|}\right) - \frac{F^*_-}{2}\left(1 - \frac{D}{|D|}\right)\right. < E_o$$

where

$$D = \sum_m A_m a_m^* \qquad (4.3.2.26)$$

will result in a bounded response for all times.

The problem is solved.

We may complicate the issue by making S_n or $F_{n\pm}$ functions of time. The same steps apply, with static equilibrium separated from the dynamics, but the time-derivatives must be lumped into the expression for B, as done previously and in the continuous feedback section.

4.3.3 Boundary Control

We will often encounter, in the design of control systems, the limitation that the system is only accessible to us at some boundary. In one dimension, this means that feedback can only be applied at the ends. In two dimensions, the boundary is the perimeter, and in three dimensions this is the enclosing surface. It is important to know what properties the system must have to allow us to stabilize or control its behavior with such restricted feedback.

We examine the one-dimensional case in the context of our membrane example. The equation of motion is simplified to:

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \xi}{\partial z^2} + P\xi, \quad o \leq z \leq 1 \tag{4.3.3.1}$$

P, T, P constants

$$\text{at } z=o, \quad \frac{\partial \xi}{\partial z} = + F_o(\xi)$$

$$\text{at } z=1, \quad \frac{\partial \xi}{\partial z} = -F_1(\dot{\xi}) \tag{4.3.3.2}$$

where F_o and F_1 will be our feedback functions. Thus, if $F_o = F_1 = o$, we have the "free end" string problem, with solutions

$$\xi = \sum_{m=o}^{\infty} a_m(t)\ (\sqrt{2}\cos m\pi z),\quad \omega_m^{\ 2} = m^2\pi^2 - P \tag{4.3.3.3}$$

We may then calculate the effect of the feedback by considering overall energy conservation; we multiply the equation by $\frac{\partial\xi}{\partial t}$ and integrate over z to get

$$\frac{dE}{dt} = o,\ E = T + \psi + U \tag{4.3.3.4}$$

$$T = \int_o^1 \tfrac{1}{2}\rho\left(\frac{\partial\xi}{\partial t}\right)^2 = \sum_{m=o}^{\infty} \frac{1}{2}\left(\frac{da_m}{dt}\right)^2 ,$$

$$\psi = \int_o^1 \frac{1}{2}\left(\frac{\partial\xi}{\partial z}\right)^2 - \frac{P}{2}\xi^2\,dz = \sum_{m=o}^{\infty}\frac{1}{2}\ \omega_m^{\ 2}\,a_m^{\ 2}$$

$$U = \int_o^{\xi(o)} F_o(D)\,dD + \int_o^{\xi(1)} F_1(D)\,dD$$

We may compare this with feedback from two distributed feedback stations by noting that the same result is derived from our general development when

$A_o(z) = U_o(z) \qquad A_1(z) = U_o(z-1)$

Giving

$D_o(t) = \xi(t, z=o) \qquad D_1(t) = \xi(t, z=1)$

Thus we can use the same general approach to evaluate the stability of the system. One difficulty arises, however. Since the feedback has a singular spacial distribution, the coupling coefficients A_{nm} do not drop off as they would for more distributed feedback. In fact, $A_{om} = 1$, $A_{1m} = (-1)^m$ is our example. In effect, this is saying that the expansion of the system's perturbations in terms of modes of the original system does not necessarily converge.

This can be seen in the case of bang-bang feedback;

$$F_o(D) = F_1(D) = F \frac{D}{|D|} \qquad (4.3.3.5)$$

When we test for stability of the null, we have from D=o

$\xi(o) = \xi(1) = o$ giving

$$a_o = - \sum_{i=1}^{\infty} a_{2i} \qquad (4.3.3.6)$$

$$a_1 = - \sum_{i=1}^{\infty} a_{2i+1}$$

Substituting into ψ and requiring $\nabla\psi \cdot \bar{s} > o$ for all $\bar{s}$ results in the Sylvester's test for the quadratic form

$$\begin{vmatrix} \omega_2^{\,2}+\omega_o^{\,2} & o & \omega_o^{\,2} & o & \\ o & \omega_3^{\,2}+\omega_1^{\,2} & o & \omega_1^{\,2} & \cdots \\ \omega_o^{\,2} & o & \omega_4^{\,2}+\omega_o^{\,2} & o & \\ o & \omega_1^{\,2} & o & \omega_5^{\,2}+\omega_1^{\,2} & \\ & \vdots & & & \end{vmatrix} \qquad (4.3.3.7)$$

The first few inequalities resulting here give

$$(2\pi)^2 - 2P^2 > o \qquad (4.3.3.8)$$

$$(3\pi)^2 - 2P^2 > o$$

$$(4\pi)^2 - 2P^2 \quad \frac{P^4}{(2\pi)^2-2P^2} > o$$

$$(5\pi)^2 - 2P^2 - \frac{P^4}{(3\pi)^2-2P^2} > o$$

and so on.

Instead of becoming more and more easily satisfied, these <u>may</u> become less and less so.

We can not be sure what that limit would be, without some new physical insight. That comes from noticing that the conditions $\xi(o) = \xi(1) = o$ impose a new set of boundary conditions on the system, and we can solve for the new modes, getting

$$\xi = \sum_{m=1}^{\infty} a_m \sqrt{2} \sin m\pi z, \quad \omega_m^2 = m^2 \pi^2 - P \tag{4.3.3.9}$$

Clearly, the stability requirement is

$$P < \pi^2 \tag{4.3.3.10}$$

Thus, we can get an exact expression for the limit of the series of tests above by considering the stability of the system with the argument of the feedback set to zero. We assume the bang-bang feedback has pinned down the boundary, and require that the resulting system be stable. If it is not, then no rearrangement of boundary feedback can be effective for this system.

For region-of-stability considerations, we can rely on the previous work with distributed feedback, since even if the coupling coefficients do not decrease for large m, the increasing frequencies will be enough to cause the series calculation of E_0 to converge. In our example, the series will approach $1/m^2$ for large m.

4.3.4 Continuous Feedback

Since we have considered in detail most of the interesting aspects of bang-bang feedback from an aribtrary number of feedback stations, it is logical to consider the limit of an infinite number of such stations. This would be the case of a force applied at every point in the system as a response to perturbations at that point. While it is hardly feasible to consider such large numbers of feedback loops that the system is well modelled by this approximation, there are physical systems where the description is quite good. These are systems in which some property of the continuum is sufficiently nonlinear that it is worthwhile to model its effects as a discontinuous feedback force imposed on a linear system.

Such a system might for instance be a fluid interface which has its equilibrium surface very near a sharp boundary in some field. Suppose, for instance, in order to orient a fluid with high dielectric constant, we impose an electric field over the region we would like it to occupy. If the field drops off suddenly, the surface of the fluid feels a bang-bang force whenever it enters the field region. Thus incompressible fluids

can be oriented in a zero-gravity environment
We can model such a situation as quasi-one dimensional with obvious generalizations to more dimensions.

We now take up a concrete example of a system which we propose to analyze for stability in terms of our energy function for bang-bang forces. We choose a fluid suspended against gravity, with $\varepsilon > \varepsilon_o$ and a sharply bounded region of electric field E_o inside its equilibrium surface.

Consider the problem of an insulating incompressible fluid, bounded by rigid walls at $x = o$, $z = o$, and $z = 1$; of density ρ, dielectric constant $\varepsilon > \varepsilon_o$, surface tension T, equilibrium surface at $x = b$, in a gravitational field $g\hat{x}$. Within this fluid, we establish a region of electric fields of constant magnitude E_o by means of closely spaced conducting plates of alternating potential, spaced less than one unstable wavelength apart. We then write conservation of momentum within the fluid as

$$\rho \frac{\partial V}{\partial t} = \rho g\hat{x} - \nabla P \qquad (4.3.4.1)$$

where

$$\nabla \cdot V = o \qquad (4.3.4.2)$$

Taking the divergence of (1), we have

$$\nabla^2 P = o \qquad (4.3.4.3)$$

with solutions of the form

$$P = P_o + \rho g x + \pi$$

where

$$\pi = \sum_{m=o}^{\infty} \pi_m \frac{\cosh K (x+\phi_1)}{\cosh K b} \cos K(z+\phi_3) \qquad (4.3.4.4)$$

$$K = m\pi$$

We then apply our boundary conditions on the fluid by setting the velocity normal to each rigid wall equal to zero. Using (1), this gives the results that $\phi_1 = \phi_3 = o$. Thus π_m is the Fourier amplitude of the pressure at the plane $x = b$. The equilibrium condition gives $P_o = -\rho g b$. Thus, to first order, the Fourier amplitude of the pressure at the fluid surface is given by

$$P_m = \rho g a_m + \pi_m \qquad (4.3.4.5)$$

where the displacement at the surface is given by

$$\xi(z,t)=\sum_{m=o}^{\infty} a_m \cos Kz \tag{4.3.4.6}$$

noting that $a_o = o$.

We now may write the surface tension force on the interface as

$$\tau_x = T \frac{\partial^2 \xi}{\partial z^2} \tag{4.3.4.7}$$

with components $\tau_{mn} = - TK^2 a_m$.

The force on the perturbed interface due to the electric field is zero for positive displacements and constant for negative ones. It is given by

$$F_e = \frac{(\varepsilon-\varepsilon_o)}{2} E_o{}^2 \frac{(1 - \frac{\xi}{|\xi|})}{2} \tag{4.3.4.8}$$

since it is either opposing the gravitational effect or zero. If the fluid is incompressible, the integral of the constant part of F_e over the surface gives a simple constant pressure, and the second term in F_e remains as the effective feedback force.

We consider the simplest cases first, in order to show parallels to the discrete feedback case. Later, we will add the possibility of offset C(z) in the input to the nonlinear effect, asymmetry in the response $F_e(\xi)$, and time variation of either of these quantities.

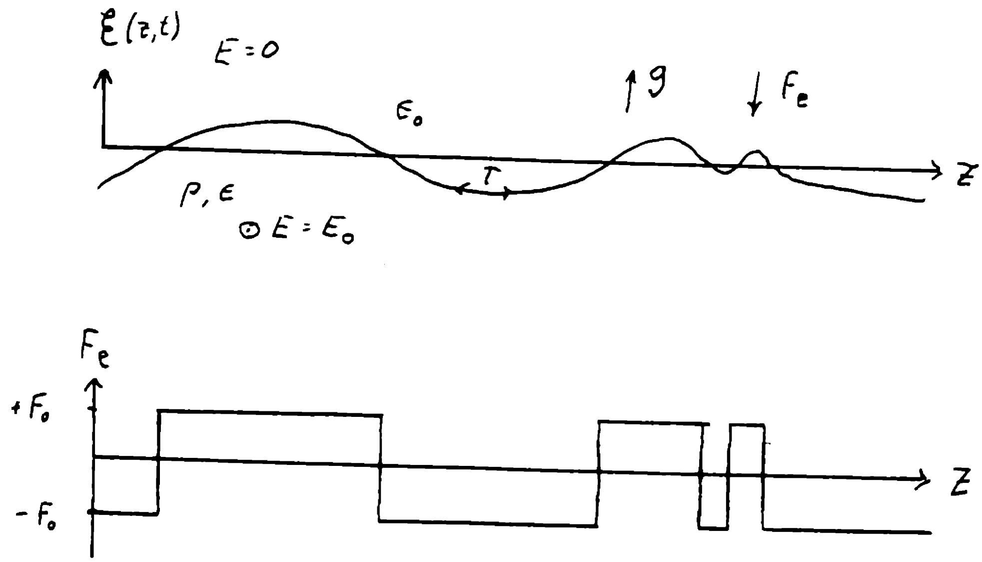
In a sense, the continuous feedback problem is easier than the discrete one, because we need not worry about whether we have enough feedback stations. The questions of region of stability, quasistatic approximation, etc. are still valid, however, and the answers are complicated by the fact that we can no longer find the feedback force at all points given the perturbation, without considering an infinite number of points. The feedback force is now a piecewise-constant function with an unknown number of zero-crossings at unknown locations, as for example in Figure 4.3.4.1. The force has Fourier components

$$F_m^{\ e} = \frac{(\varepsilon-\varepsilon_o)\ E_o^{\ 2}}{2} \int_o^1 (1 - \frac{\xi}{|\xi|}) \cos Kz \ dz \qquad (4.3.4.9)$$

We note that, to first order, the pressure at the interface is totally directed normal to the plane x = b. Thus, we may write equilibrium at the interface as

$$\pi_m + \rho g\ a_m - T\ K^2\ a_m + \mathbf{F}_m^{\ e} = o \qquad (4.3.4.10)$$

Figure 4.3.4.1 Continuous Feedback



$$F_o = \frac{1}{4}(\epsilon - \epsilon_o) E_o^2$$

Writing the x-component of (1) at the interface,

$$\rho \frac{\partial^2 a_m}{\partial t^2} = -K \text{ Tanh } Kb \ \pi_m \qquad (4.3.4.11)$$

Solving (10) for π_m and substituting, we have

$$\rho \frac{\partial^2}{\partial t^2} \varepsilon \ a_m = K \text{ Tank } Kb[[\rho g - TK^2] a_m + F_m^{\ e}]$$

(4.3.4.12)

We now rewrite $F^e_m = (-\partial U/\partial a_m$ for all $\xi \neq o$, where

$$U = \frac{-(\varepsilon-\varepsilon_o)E_o^{\ 2}}{2} \sum_m a_m \int_o^1 (1 - \frac{\xi}{|\xi|}) \cos Kz \, dz$$

(4.3.4.13)

$$= \frac{-(\varepsilon-\varepsilon_o)E_o^{\ 2}}{2} \int_o^1 [\sum_{m=o}^{\infty} a_m \cos Kz](1 - \frac{\xi}{|\xi|}) \, dz$$

(4.3.4.14)

We now recognize the quantity in brackets as ξ from (6) and note that

$$\int_o^1 \xi \, dz = o = a_o \qquad (4.3.4.15)$$

so that

$$U = \frac{(\varepsilon-\varepsilon_o)E_o^2}{2} \int_o^1 |\xi| \, dz \geq o \tag{4.3.4.16}$$

We multiply (12) by $\frac{d\,a_m}{dt} \frac{1}{K \text{ Tanh } Kb}$ and sum over m to get

$$\sum_m \left[\frac{\rho}{K \text{ Tanh } Kb} \frac{d^2\,a_m}{dt^2} \frac{d\,a_m}{dt} + (TK^2-\rho g]a_m \frac{d\,a_m}{dt} \right. \tag{4.3.4.17}$$

$$\left. + \frac{\partial U}{\partial\,a_m} \frac{d\,a_m}{dt} \right] = o$$

which we rewrite as dE/dt = o

$$E = T + \psi + U \tag{4.3.4.18}$$

$$T = \sum_m \frac{\rho}{2 \text{ K Tanh Kb}} \left(\frac{d\,a_m}{dt} \right)^2 \geq o$$

$$\Psi = \sum_m \frac{1}{2} (T\,K^2 - \rho g)\,(a_m)^2$$

$$U = \frac{(\varepsilon-\varepsilon_o)\,E_o^2}{2} \int_o^1 \left| \sum_m a_m \text{ Cos } Kz \right| dz \geq o$$

and U as given previously. Note that $\frac{\rho}{K \tanh Kb}$ plays the role of an effective mass density ρ_m of the surface. Clearly, adding viscosity will simply place a negative term on the R.H.S. of (18), by analogy to the Hamiltonian of the system (See Appendix A).

We then recognize that E is a positive definite, decrescent Lyapunov function in a region sufficiently near the origin in the state space defined by our mode amplitudes. The system therefore has a stable null solution. It is globally stable for $g < o$, as would be expected.

The region of stability for the system is given by the set of states $\bar{s}$ in a closed volume of state space including the origin for which $E(\bar{s}) < E_o$. We find E_o by solving $\nabla E = o$ at $\bar{s} = \bar{s}_o$ and, choosing the minimum value of $E(\bar{s}_o) = E_o$.

However, at this point the problem becomes difficult, if not impossible. Taking $\nabla_s E = o$ gives

$$\dot{a}_m^{\,o} = o$$

$$a_m^{\,o} = - \frac{(\varepsilon - \varepsilon_o)\, E_o^{\,2}}{2(TK^2 - \rho g)} \int_o^1 \frac{\xi^o}{|\xi^o|} \cos Kz \, dz$$

$$\xi^o = \sum_m a_m^{\,o} \cos Kz$$

We no longer have a finite number of choices for $\xi^o(z)$, but can pick its value at any point independently. Therefore, it is not feasible to try an exhaustive search of the possible solutions. If we postulate the form of $\frac{\xi^o}{|\xi^o|}$ we may get an answer, but it need not be the only set of self-consistent solutions. In this case, assuming $\frac{\xi^o}{|\xi^o|}$ changes sign only at $z = \frac{1}{2}$ gives solutions, if they exist, of

$$a_m^o = \left[- \frac{(\varepsilon-\varepsilon_o) \quad E_o^2}{m\pi(TK^2-\rho g)} \right] \cdot \left[\begin{matrix} o, & m = \text{even} \\ 1, & m = \text{odd} \end{matrix} \right]$$

If no solutions exist, then the system is globally stable. In any case,

$$E_o = -\sum_m \frac{1}{2\rho m(TK^2-\rho g \;)} \left[\int_o \frac{1(\varepsilon-\varepsilon_o)E_o^2}{4} \; \frac{\xi}{|\xi|} \cos KZ \, dz \right]^2 > o. \tag{4.3.4.20}$$

Now let's consider continuum control. Let the modal equation of motion be

$$\rho_m \frac{\partial^2 a_m}{\partial t^2} + K_m \, a_m + b_m \frac{\partial a_m}{\partial t} + \int_o^1 F_e(t,z) \cos Kz \, dz = o \tag{4.3.4.21}$$

including the possibility of internal damping. Suppose

$$F_e = \frac{+F_+(z)}{2}\left(1 + \frac{D(z,t)}{|D(z,t)|}\right) - \frac{F_-(z)}{2}\left(1 - \frac{D(z,t)}{|D(z,t)|}\right)$$

Let $D(z,t) = \xi(z,t) - C(z,t)$

Define

$$\xi(z,t) = \sum_m a_m(t)\,\xi_m(z) \qquad C(z,t) = \sum_m C_m(t)\,\xi_m(z)$$

$$\xi_m(z) = \sqrt{2}\cos m\pi z,$$

Let

$$\rho_m = \int_0^1 \rho\,\xi_m^2(z)\,dz$$

$$(4.3.4.22)$$

$$K_m = \rho_m W_m^2 = Tm^2\pi^2 - P$$

$$\rho_m W_m^2 = T m^2 \pi^2 - P + b_m j\omega$$

Let $F_1(z)$ represent the static equilibrium part of the feedback and $F_2(z,t)$ be the dynamic part.

$$a_m = a_m^* + C_m \qquad \rho_m W_m^2 C_m(t) = -\int_0^1 F_1(z,t)\,\xi_m(z)\,dz$$

$$= -F_{1m}(t)\int_0^1 \xi_m^2\,dz$$

$$(4.3.4.23)$$

So

$$F_1(z,t) = \sum_m F_{1m}(t)\ \xi_m(z)$$

can be calculated to arbitrary accuracy. We know

$$F_2(z,t) = F_e(z,t) - F_1(z,t)$$

$$F_{1m} = \frac{K_m\ C_m}{\int_0^1 \xi_m^{\ 2}\ dz} \quad \text{and } C_m = \frac{\int_0^1 C(z,t)\xi_m(z)\ dz}{\int_0^1 \xi_m^{\ 2}\ dz} \tag{4.3.4.24}$$

So

$$F_{1m} = K_m \int_0^1 C(z,t)\ \xi_m(z)\ dz$$

Then let

$$E(t) = T + \psi + U \tag{4.3.4.25}$$

$$T = \sum_m \frac{1}{2}\ \rho_m\ \dot{a}_m^{*2} \qquad \psi = \sum_m \frac{1}{2}\ K_m\ a_m^{\ *2}$$

$$U(t) = \int_0^1 \int_0^{D(z,t)} F_2(x,z,t)\ dx\ dz$$

The resulting energy equation is

$$\frac{dE}{dt} = -\ (B_o + B_1 + B_2)$$

$$B_o = \sum_m b_m \dot{a}_m^{*2} \qquad B_1 = \sum_m (b_m \dot{a}_m^* \dot{C}_m - \rho_m \dot{a}_m^* \ddot{C}_m)$$

$$B_2 = \int_o^1 [F_2(D(z,t),z,t)\ \dot{C}(z,t) - \int_o^{D(z,t)} \frac{\partial F_2}{\partial t}(x,z,t)dx]dz$$

Here we have

$$F_2(z) = F_e(z) - F_1(z) \qquad (4.3.4.27$$

$$= \frac{[-F_+(z) - F_1(z)]}{2}\left(1 + \frac{D(z)}{|D(z)|}\right)$$

$$+ \frac{[F_-(z) - F_1(z)]}{2}\left(1 - \frac{D(z)}{|D(z)|}\right)$$

$$+ \frac{-F_+^*(z)}{2}\left(1 + \frac{D(z)}{|D(z)|}\right) + \frac{+F_-^*}{2}\left(1 - \frac{D(z)}{|D(z)|}\right.$$

For stable equilibrium at $D(z) = 0$, we require

$$F_+^*(z) > o \qquad F_-^*(z) > o \qquad (4.3.4.28)$$

We can interpret each term in B just as we did in the discrete feedback case. In the quasi-static approximation, the terms B_1 and B_2 will be small because

1) Time-derivatives of C and F_e are assumed small since their characteristic times $\tau >> |\frac{1}{\omega_m}|$

2) Variables $\dot{a}_m^*$ and $\dot{a}_m^*$ oscillate many times and so products of these terms and $\dot{C}$ or $\dot{F}_e$ will average to zero over times of order τ.

Note that

$$U = \int_o^L \left[\frac{F_+(z)}{2}\left(1 + \frac{D(z)}{|D(z)|}\right) + \frac{F_-^*(z)}{2}\left(1 - \frac{D(z)}{|D(z)|}\right)\right]|D(z)|dz$$

$$D(z) = \sum_m a_m^* \ \xi_m(z)$$

So $\nabla\bar{E} \cdot \bar{S}^* > o$ for all $\bar{S}^* \neq \bar{o}^*$

if

$F_+^* > o,\ F_-^* > o$

then the equilibrium $\bar{o}^*$ is stable.

Now let's ignore B and concentrate on E for B > o. To examine the region of stability, let's look at the worst-case situation in which

$\nabla\bar{E} = \bar{o}$

$\dot{a}^*_m = o$

We must find the value of a_m^* at this point $\bar{S}_o^*$, and evaluate E_o there

Let $F_o(z) = \mathrm{Min}(F_+^*(z),\ F_-^*(z)) > o$. Then we are guaranteed stability inside a state-space surface defined as $E(\bar{S}^*) = E_o$ where $E_o = E(\bar{S}_o^*)$ is the minimum energy for which $\nabla E = o$.

$\bar{S}_o^*$ is given by $\dot{a}_m^{*o} = o$ (4.3.4.29)

$$a_m^{*o} = - \frac{\int_o^L F_o(z)\ \xi_m(z) \frac{D^o(z)}{|D^o(z)|}\ dz}{K_m}$$

If more than one solution can be found, choose the one with lowest $E(\bar{S}^{*o}) = E_o$. Assume that the solution a_m^{*o} exists. This leads to

$$E_o = \sum_m \frac{1}{2} \frac{[\int_o^L F_o(z)\ \xi_m(z) \frac{D^o(z)}{|D^o(z)|}\ dz\ ]^2}{K_m} \qquad (4.3.4.30)$$

$$- \int_o^L F_o(z)\ |\ \sum_m \xi_m(z) \frac{[\int_o^L F_o(x)\ \xi_m(x) \frac{D^o(x)}{|D^o(x)|}\ dx]}{K_m}|\ dz$$

$$= - \sum_n \frac{1}{2} K_m \ [\int_o^L F_o(z)\ \xi_m(z) \frac{D^o(z)}{|D^o(z)|}\ dz\ ]^2 > o$$

$E(\bar{S}) = E_o$ gives the surface of the stability region.

In summary, then, the stability of a continuum system with a discontinuous forcing term may be analyzed using the same Lyapunov function approach as was used for discrete bang-bang feedback. One example of such a problem is the orientation of dielectric fluids using electric fields. The discontinuity in the force, which is traceable to field discontinuities near the surface of the fluid, can be modelled as a continuum of feedback stations. The existence of a stable null position is easier to show than the discrete feedback case. The region of stability, however, is more difficult to find unless simplifying assumptions can be made.

4.3.5 Scanning Control

One ideal form of control for a continuous system would be one which scans the system with a single sensor, processes a single output variable, and forces the system with another scanner. This often minimizes the processing hardware.

A clever change of coordinates, suggested for linear feedback analysis in an unpublished paper by W. G. Heller allows use of the energy approach for nonlinear stability analysis.

Let the system be governed by the equation

$$\rho \frac{\partial^2 \xi}{\partial t^2} = -b \frac{\partial \xi}{\partial t} + \gamma \frac{\partial^2 \xi}{\partial z^2} + P\xi - F(t)\ V_o(z-z_m)$$

$$\xi(o) = \xi(L) = o \qquad (4.3.5.1)$$

where z_m is the location of a scanner as it sweeps the length of the system in a time T. Then, we can let $V_s = L/\tau$, $V_o = \sqrt{\gamma/\rho}$, and write the feedback force as a superposition of all the scans: for $o \leq z \leq 1$,

$$(4.3.5.2)$$

$$F(t)\ B(z-z_m) = -\sum_{m=-\infty}^{\infty} F(t)\ U_o(z-z_m)$$

$$z_m = V_s t - m L \qquad (4.3.5.3)$$

Now, we introduce a transformation of coordinates which makes the scanner stationary in time, but keeps the boundaries stationary in space. (Note that this is not a moving frame of reference!)

$$t' = t - z/V_s \qquad z' = z \tag{4.3.4.4}$$

So

$$t'_m = t - z_m/V_s = mT \qquad z'_m = V_s t' + z' - mL$$

And

$$\frac{\partial\xi}{\partial t} = \frac{\partial\xi}{\partial t'} \qquad \frac{\partial\xi}{\partial z} = \frac{\partial\xi}{\partial z'} - \frac{1}{V_s}\frac{\partial\xi}{\partial t'}$$

(4.3.5.5)

$$\frac{\partial^2\xi}{\partial t^2} = \frac{\partial^2\xi}{\partial t'^2} \qquad \frac{\partial^2\xi}{\partial z^2} = \frac{\partial^2\xi}{\partial z'^2} - \frac{2}{V_s}\frac{\partial^2\xi}{\partial t'\partial z'} + \frac{1}{V_s^2}\frac{\partial^2\xi}{\partial t'^2}$$

So the equation of motion becomes

$$\rho\frac{\partial^2\xi}{\partial t'^2}\left[1 - \frac{V_o^{\,2}}{V_s^{\,2}}\right] + 2\rho\frac{V_o^{\,2}}{V_s^{\,2}}\frac{\partial^2\xi}{\partial t'\partial z'} - \gamma\frac{\partial^2\xi}{\partial z'^2}$$

$$+ b\frac{\partial\xi}{\partial t'} - P\xi + \sum_{m=-\infty}^{\infty} F\left(t' + \frac{z'}{V_s}\right) U_o(mL - V_s t') = o$$

If we let $F(t) = F \dfrac{\xi(z_m,t)}{|\xi(z_m,t)|}$ for bang-bang feedback, then in the new coordinates, at $t' = mT$

$$F(mT + \frac{z'}{V_s}) = F \frac{\xi(mT,z')}{|\xi(mT,z')|} \qquad (4.3.5.7)$$

We can form an energy equation by multiplying by $\frac{\partial\xi}{\partial t'}$, and integrating over z', and use eigenfunctions $\xi(t',z') = \sum_m a_m(t') \sqrt{\frac{2}{\rho L}} \text{sim} \frac{m\pi z'}{L}$.

What we have done is to convert a scanning feedback system in unprimed coordinates into a spacially continuous, time-sampled system in primed coordinates.

Since $t \to o$ as $t' \to \infty$, our stability arguments apply to either coordinate system. Note that the mixed derivative term vanishes

$$\int_o^L \frac{\partial\xi}{\partial t'} \frac{\partial^2\xi}{\partial t'\partial z'} dz' = \left[\frac{1}{2} \left(\frac{\partial\xi}{\partial t} \right)^2 \right]_o^L = o \qquad (4.3.5.8)$$

Thus the resulting energy equation is

$$\frac{dE}{dt} = -B \qquad E = T + \psi + U \qquad (4.3.5.9)$$

$$T = \int_o^L \frac{\rho}{2} \left(1 - \frac{V_o}{V_s}\right)^2 \left(\frac{\partial \xi}{\partial t'}\right)^2 dz' = \sum_m \left(1 - \frac{V_o^2}{V_s^2}\right) \dot{a}_m^2$$

$$\psi = \int_o^L \frac{1}{2} \left[T \left(\frac{\partial \xi}{\partial z'}\right)^2 - P\,\xi^2\right] dz' = \sum_m \frac{K_m}{2} a_m^2 .$$

$$K_m = \left(\gamma \left(\frac{m\pi}{L}\right)^2 - P\right)$$

$$U = \frac{F}{L} \int_o^L |\xi|\, dz'$$

$$B = b \int_o^L \left(\frac{\partial \xi}{\partial t'}\right)^2 dz' - F \int_o^L \dot{\xi}\left[\frac{1}{L}\ \frac{\xi}{|\xi|} - \sum_m U_o(mL - V_s t')\ \frac{\xi(t',z')}{|\xi(t',z')|}\right] dz$$

The time-averaged effect of B would be zero if ξ were always equal to its sampled value. Since it is not,it has a slightly negative component during the intervals between samples. This negative damping could be tolerated without instability if internal damping had a greater effect. It is clear from the form of T above that stability can only be achieved by scanning feedback $V_o < V_s$; that is, if the scanning speed is greater than the wave speed for P = o.

Both these conclusions parallel the results of Heller for linear systems. Using the results of our section on time-sampling, we know that for scanning bang-bang control to be feasible, we require $(\frac{L}{T})^2 < \frac{Y}{P}$, and $T << \tau$, the characteristic zero-crossing time interval for disturbances of the appropriate amplitude. If this latter is not true, then the negative damping will dominate the entire region of state with $E < E_o$.

4.4 Computer Simulation

4.4.1 Options and Tradeoffs

Several options will be discussed in this chapter and the next for possible computer simulations of continuous systems with nonlinear feedback. The purpose of our simulation is to examine the aspects of system behavior which are not easily accessible to our analytical tools. Several effects have been discovered in some approximation, and a simulation checks the results of these approximations. Some types of feedback are not subject to our analysis at all, and simulation is the only available test of their feasibility. Finally, the simulation serves as a check on analytically predicted results.

The limitations of digital computer simulation are the discretization of spa ial information, the discretization of temporal information, and the computational expense of a given scheme.

If digital computer simulation is the only desired result, then the most basic technique involves converting the differential equations of motion into difference equations and integrating directly. This tends to be extremely time-consuming, and will often involve use of a very fine network of spatial points to get accurate results. It does not take advantage of the simple nature of the original linear system. This approach was not implemented.

In the systems which we are studying, solutions tend to be well represented by a truncated Fourier series in space. The schemes involving spatial discretization or methods of characteristics do not take advantage of that fact. Therefore, the modal approach, or Galerkin series, is the most efficient method of representing the system. The time-dependence of the system can then be handled in two ways. One is by direct discretization, numerically integrating the difference equations for a truncated set of mode amplitudes $\{a_m, \dot{a}_m\}$. The second and most efficient uses the extra information that the feedback forces are piecewise-constant to use an exact analytical solution for the mode amplitudes between increments. The first method was used as a check on the second, and showed better than 5% agreement in energy values (only at a cost of double the computing time).

The computing method used, then is approximate in two ways:

1) The modal series is truncated,

2) The feedback is allowed to change sign only at the beginning of sampling intervals.

If the system is assumed to be feedback controlled by a time-sampled controller, then this second approximation is simply a representation of the sampling effects. Hence, our analytical predictions of time sampling can be checked this way.

A program was developed to simulate a bang-bang feedback control system with the following parameters:

$$\rho = L = 1 \qquad \xi_m = \sin m\pi z \tag{4.4.1.1}$$

$$B_n(z) = A_n(z) = \begin{cases} 1 & \frac{n-1}{N} \le z \le \frac{n}{N} \\ o & \text{otherwise} \end{cases} \qquad 1 \le n \le N$$

$K_m = m^2 \pi^2 - P$

$b_m = o$

F_p = positional feedback force

F_V = velocity feedback force

Truncation at M = 16 modes

DT = time interval used (effective sampling)

4.4.2 Check of Analytical Results

Varying P and F_P with F_V - o, N = 4 feedback stations, the following predictions were verified:

1) Null stability was exactly predicted, within the limitations of the simulation.

2) Region of stability was also predicted to available accuracy.

3) When the null was unstable, growth occured in two stages; first, the modes combined to grow while oscillating near D_n = o; then at large amplitudes (a_1^o = 100), D_n departed rapidly from zero and the system took on the spatial characteristics and growth rate of the fastest-growing mode, m = 1.

4) Time sampling caused a slow increase in E, until $E > E_o$.

Then, taking $F_P \neq$ o, the following predictions were verified:

5) Velocity feedback alone can not stabilize an unstable system; it merely makes it oscillate about $\dot{D}_n$ = o while growing.

6) Velocity feedback dominated the sampling effects at sufficiently large amplitudes.

7) Optimum velocity feedback force was comparable to the net restoring force acting on the lowest mode excited. By optimum, we mean largest dE/dt.

The criterion for determining whether one feedback scheme was better than another was to take a standard initial condition ($a_1 = 1.$, $a_2 = .5$, $a_3 = .1$, $a_4 = .05$, $a_5 = .01$, $\dot{a}_m = a_{m>5} = o$) and standard system N=4, P=45, and determine which led to the quickest reduction in total energy E to below 1/e of its initial value.

Then, a single feedback force F was switched according to the sign of ($D_n + T \dot{D}_n$), for various values of T. This combined position and rate feedback showed that

8) Stability with damping was achieved as predicted for $T^2 < \left| \frac{1}{\omega_1{}^2} \right|$. However, simulation showed

9) That instability did not occur until T reached values five times greater than those calculated. It then grew while oscillating about ($D_n + T \dot{D}_n$) = o. This implies that there is probably a better Lyapunov function.

10) Optimum values of T were found to be well below this figure, however, in the range predicted by theory.

Observations of simulated systems also showed certain interesting characteristics:

1) Bang-bang control of multi-mode systems shows no noticeable temporal periodicity in its detailed behavior. The potential and kinetic energies show alternating maxima and minima, with period of oscillation depending mainly upon the amplitude of the lowest mode and the net force on that mode.

2) The spatial shape of the displacement, if initially made up of lower modes only, continues to be well approximated by the first N modes. The relative magnitude of these amplitudes changes in a major way between successive zero-crossing of D_n.

3) Except for the effects of time-sampling, the behavior of a system in all respects is well represented by (N+1) modes. Time-sampling effects are also well represented if damping is present. (More modes would be needed if the coupling coefficients did not drop off as quickly with m as they do for this geometry.)

4) The increases in E due to time-sampling effects tend to occur in sudden jumps of as much as E/2 when a zero-crossing occurs, rather than as a gradual effect. These jumps become less frequent as the amplitude of the disturbance increases.

4.4.3 Estimation of Sample and Hold Effects

If the formalism we have developed is to be of use, it should suggest aporoximate prediction formulas for the non-ideal effects we have discussed.

In particular, assuming a sample and hold feedback scheme which is simulated by our computer model, how can the energy growth rate be estimated? The desired form of the answer would be a long-term average growth formula.

The "exact" expression, assuming hysteresis Δ and no damping forces, is

$$\frac{dE}{dt} = \sum_{K=1}^{N} \frac{F_{P_K}}{4} \left[1 + \frac{\Delta - |D_K|}{|\Delta - |D_K||}\right]\left[1 + \frac{D_K \dot{D}_K}{|D_K \dot{D}_K|}\right] |\dot{D}_K|$$

We will make the following simplifying assumptions:

1) The sampling time DT is very small compared to the time between successive zero-corssings of $D_K(t)$.

2) The state of the system is **far** inside the region of stability.

3) The system can be approximately represented by the lowest mode.

The first assumption lets us use our approximation $\Delta \overset{o}{=} \frac{DT}{2} \left| \frac{dD_K}{dt} \right|$; that is, the time delay in the feedback response is well modelled by a hysteresis effect, represented by the average time delay (half the sampling interval) times the output rate at zero-crossing.

The second assumption lets us ignore, for a first approximation, the potential energy term ψ due to linear forces.

The third assumption allows us to approximate the kinetic energy as a single term.

$$T = \frac{1}{2} \dot{a}_1^{\,2}$$

While the first two assumptions can be justified by appropriate system parameters, the last is highly questionable.

However, we are only looking for an estimate, so we accept this limitation.

Then the system spends an equal amount of time between the extreme values of D_n, or the average value of

$$\frac{1}{2}\left[1 + \frac{\Delta - |D_n|}{|\Delta - |D_n||} \right] \cong \frac{\Delta}{A_{n_1} a_{max}}$$

Also, when $D_n = o$, $|\dot{D}_n|$ is a maximum, so

$$|\dot{D}_n| \cong |A_{n_1} \dot{a}_{max}| \text{ and } \frac{1}{2} \dot{a}_{max}^{\,2} = \sum_{n=1}^{N} F_{P_n} |A_{n_1}| a_{max}|$$

(4.4.3.2)

This relation uses the knowledge that the maximum values of kinetic and potential energies are equal.

We then assume that $D_n \dot{D}_n > o$ about half the time, so that on the average

$$\frac{1}{2} \left[1 + \frac{D_n \dot{D}_n}{|D_n \dot{D}_n|} \right] \overset{o}{=} \frac{1}{2} \qquad (4.4.3.3)$$

We then substitute into our equation to get

$$\frac{dE}{dt} = \sum_{n=1}^{N} F_{p_n} \left| \frac{DT \quad \dot{a}_{max}}{2 \quad a_{max}} \right| \left[\frac{1}{2}\right] \left| A_{n_1} \dot{a}_{max} \right| \qquad (4.4.3.4)$$

$$= \frac{DT}{4} \frac{\dot{a}_{max}^{\,2}}{a_{max}} \sum_{n=1}^{N} F_{p_n} A_{n_1}$$

$$= \frac{DT}{2} \, ' \left[\sum_{n=1}^{N} F_{p_n} A_{n_1} \frac{D_n}{|D_n|} \right]^2$$

This predicts a linear energy increase, proportional to DT. This same problem can be solved more precisely using (4.3.1.5) with $T = -DT/2$ to get the same result, but summed over all modes: E is modified slightly for $\omega_1^{\,2} T^2 << 1$ and the energy increase is

$$\frac{dE}{dt} = \frac{DT}{2} \sum_m \left[\sum_{n=1}^{N} F_n A_{nm} \frac{G_n}{|G_n|} \right]^2$$

(4.4.3.5)

For the given system we may estimate

$$\sum_n A_{nm} \overset{\circ}{=} \frac{2\sqrt{2}}{m\pi} \quad \text{for } m = \text{odd, o for } m = \text{even}$$

With $F_p = 100$ this becomes

$$\frac{dE}{dt} = 10^4 \; DT$$

Computer runs to check this result for DT = .004 and .008 are shown in figure 4.4.3.1 As an estimate, this is quite good, giving the average dE/dt within about 25% of actual values.

We would expect this estimate to break down for large DT (comparable to $1/|\omega_1|$) or for $E \geq E_o$.

The purpose of these estimation methods is to help in designing a system. Therefore, we should be able to predict such things as the amount of velocity weighting which is needed to balance the effects of sampling. With DT = .008 and F = 100, N = 4 we predict that the effects of sampling and velocity feedback should roughly cancel at T = DT/2 = .004. We also predict that the instability

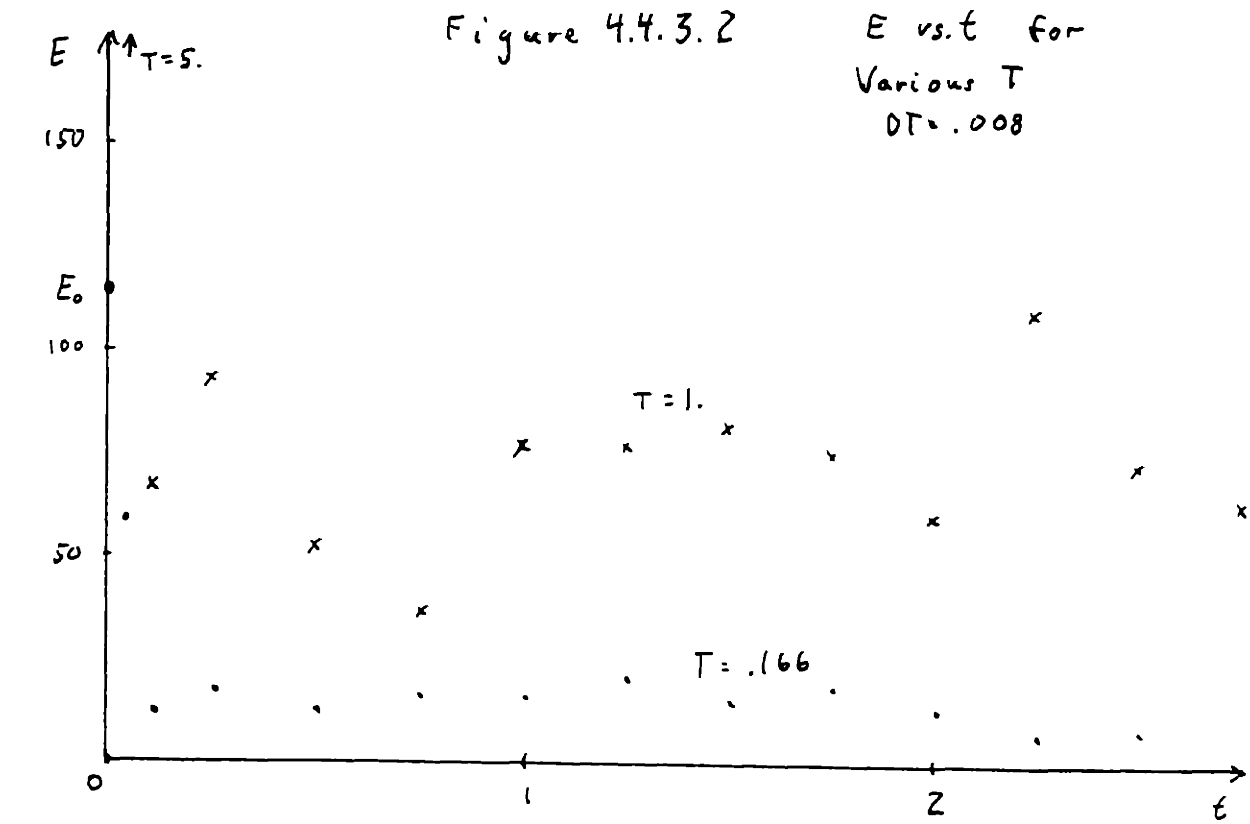


Figure 4.4.3.2 E vs. t for Various T DT= .008

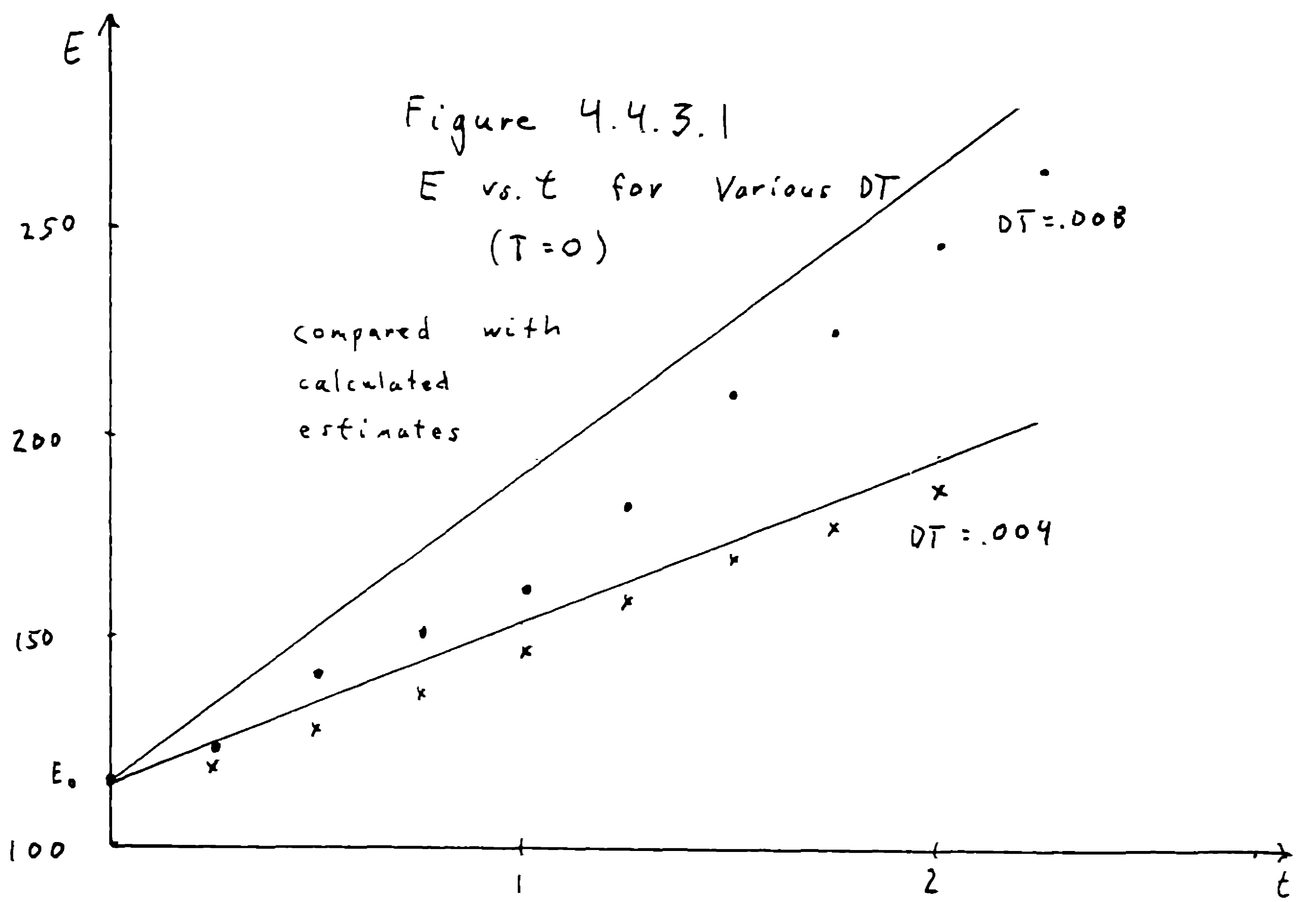
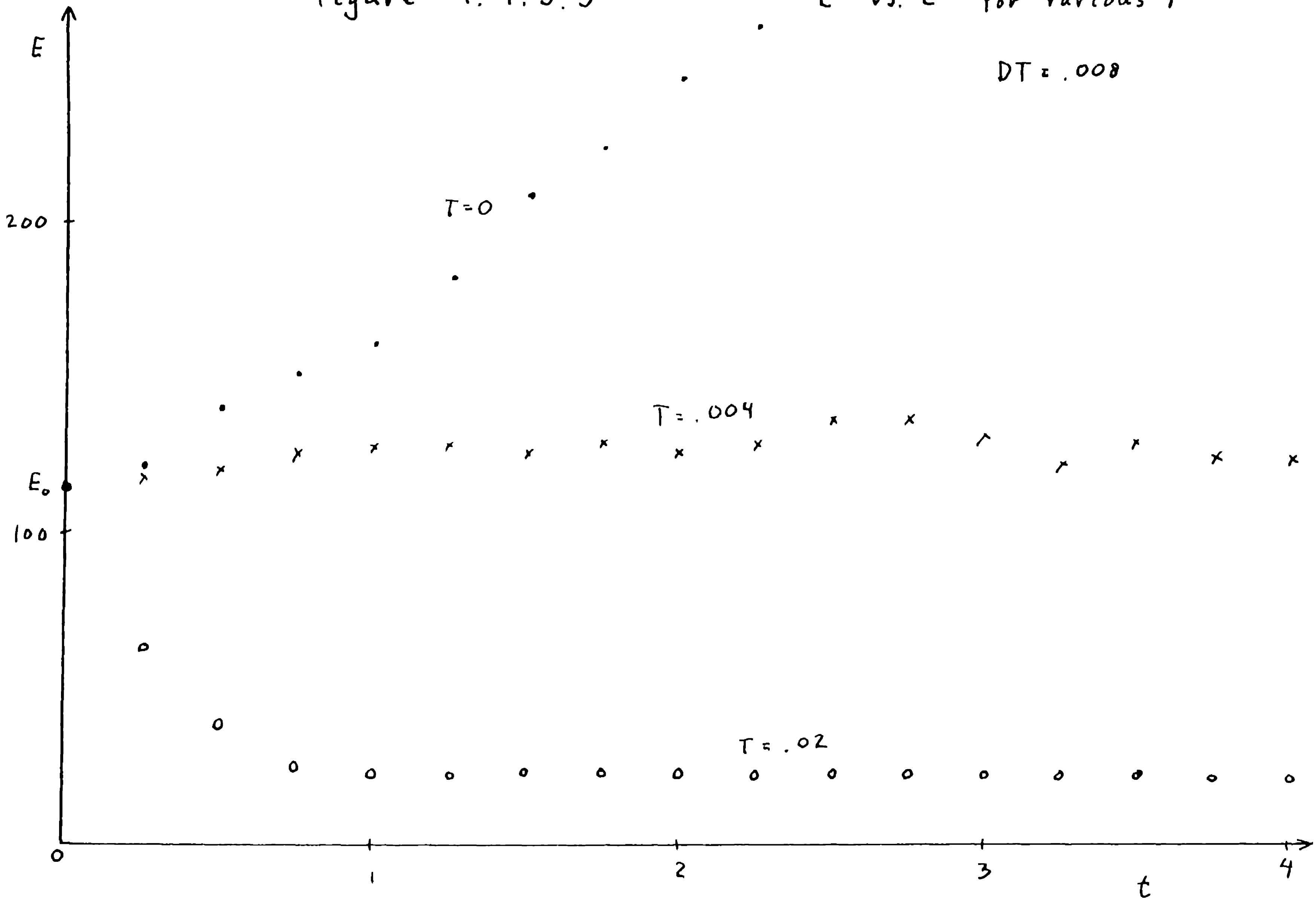


Figure 4.4.3.1 E vs. t for Various DT (T=0)

Figure 4.4.3.3 E vs. t for Various T



results for $T > \frac{1}{|\omega_1|} = \frac{1}{6}$. We check these predictions by simulation and plot E vs. t for the first few seconds in the next graphs in Figure 4.4.3.2 and 4.4.3.3. Note that the system is stable but shaky for T as large as 1, and that E would be roughly constant at the T value predicted. All in all our estimates are pretty good predictions of the magnitudes involved.

4.5 Alternatives

It is always valuable to have more than one approach to a subject. Even if the alternatives are not useful in the present context, they may later hold the answer to a different question.

Several alternative approaches may be applied to the present problem. One is the analysis by use of wave trains rather than by modes of the system. This is most efficient in long systems, or systems which close on themselves. Another approach to such a system involves the concept of energy velocity, useful in power transmission problems. A third possibility is the use of the method of characteristics to give information about a system, especially connective systems, even supersonic ones. These are presented in the following sections. The results show that wave train analysis can be treated just as modal analysis was to give stability results; energy velocity concepts grow naturally from energy stability arguments and may be useful in transport problems; and characteristics give an alternative picture of the state space.

4.5.1 Long Systems

When a system is long enough that disturbances with interesting dimensions can travel for several characteristic lengths before they contact a boundary, it may be enlightening to describe the system in terms of travelling waves rather than stationary modes. Linear systems are easy to understand both ways, because a stationary mode may be constructed as a sum of equal waves travelling in opposite directions. Thus, we would expect that a formulation of system dynamics in terms of energy would look very similar if it were based on either the modes of the linear system without feedback, or the wave amplitudes of the linear system without feedback. This is in fact the case.

Suppose the linear system supports wave trains of the form

$$\xi(z,t)=\Sigma_m\, X_{1_m} \cos(\omega_m t-K_m z) + X_{2_m} \sin(\omega_m t-K_m z) \qquad (4.5.1.1)$$

where

$$\omega_m = \omega_m(K_m) \text{ and } K_m = \frac{m\,\pi}{L},\; m = -\infty \text{ to } +\infty.$$

We may write this as

$$\xi(z,t) = \mathrm{Re} \sum_{m=-\infty}^{\infty} a_m(t) \quad \xi_m(z)$$

$$a_m = [\ X_{1_m} - j \quad X_{2_m}\ ]\ e^{j\omega_m t} \quad \xi_m = e^{-jK_m z} \tag{4.5.1.3}$$

We may then use the complex conjugate (*) to write

$$\xi(z,t) = \sum_{m=-\infty}^{\infty} \sum_{n'=o}^{1} a_{mn'}(t)\ \xi_{mn'}(z) \tag{4.5.1.4}$$

where

$$a_{m_o}(t) = \frac{1}{2}\, a_m, \quad a_{m_1}(t) = \frac{1}{2}\, a_m^*$$

$$\xi_{m_o}(t) = e^{+jK_m z}, \quad \xi_{m_1}(z) = \xi_m^*$$

$$\omega_m^{\ 2} = \omega_{m_o}^{\ 2} = \omega_{m_1}^{\ 2}$$

Thus we have an expression for the perturbation which is in the same form as a modal expansion. The steps leading to an energy formulation are exactly the same, where we treat each (m,n) as a separate mode, even though the n'=o and n'=1 modes are not independent. The resulting modal equation is gotten by dot multiplying by $\xi_m^*(z)$ and integrating:

$$\rho_m \left[\frac{\partial_2 a_{mn'}}{\partial t^2} + \omega_m^2 a_{mn'}\right] = - \sum_{n=1}^{N} F \frac{D_n}{|D_n|} A_{mn} \tag{4.5.1.5}$$

where

$$\rho_{mn} = \int_o^L \rho |\xi_m|^2 \, dz$$

$$A_{mn} = \int_o^L A_n(z)\, \xi_{mn'}(z)\, dz, \quad A_{m_o} = A_{m_1}{}^* = A_m$$

$$D_n^2 = \int_o^L A(z)\, \xi(z,t)\, dz = \sum_m \sum_n A_{mn}\, a_{mn'}$$

We then multiply by $\frac{d\, a_{mn'}^*}{dt}$ and sum over all m and n′ to get

$$\frac{dE}{dt} = o, \quad E = T + \psi + U \tag{4.5.1.6}$$

$$T = \sum_{m=-\infty}^{\infty} \sum_n \rho_m \frac{d\, a_{mn}^*}{dt} \frac{d^2 a_{mn}}{dt^2} = \sum_n \sum_{m=-\infty}^{\infty} \frac{1}{2} \rho_m \left|\frac{d\, a_{mn}}{dt}\right|^2$$

$$= \int_o^L \frac{1}{2} \rho \left(\frac{\partial \xi}{\partial t}\right)^2 \tag{4.5.1.7}$$

[Since the ξ_{mn} are mutually orthogonal and $a_{m_1}(t) = a_{m_o}{}^*(t)$]. Similarly,

$$\psi = \sum_{n'} \sum_{m=-\infty}^{\infty} \frac{1}{2} \rho_m \omega_m^2 |a_{mn'}|^2(t) = \int_0^L [\tfrac{1}{2} T (\frac{\partial \xi}{\partial z})^2 + \tfrac{1}{2} P \xi^2] dz \tag{4.5.1.8}$$

if

$$F_e = - \sum_{n=1}^{N} A_n(z) F \frac{D_n}{|D_n|} , \quad D_n(t) = \int_0^L A_n(z) \xi(z,t) dz \tag{4.5.1.9}$$

then

$$\frac{dE}{dt} = o, \quad E = T + \psi + U \tag{4.5.1.10}$$

$$U = \sum_{n=1}^{N} F |D_n| , \quad D_n = \sum_{m=-\infty}^{\infty} \sum_{n} A_{mn} a_{mn}$$

where

$$A_{mn} = \int_0^L A(z) \xi_{mn}(z) dz$$

We may then make our series more compact by noting that $|a_{mn}|^2$ is independent of n, $A_{nm_1} a_{m_1} = (A_{nm_o} a_{m_o})^*$ so that the m < o terms are determined by the m > o ones. Thus we really have double the sum over m = $-\infty$ to ∞ with n=o, and $D_n = 2 \text{ Re} \sum_{m=-\infty}^{\infty} A_{nm_o} a_{m_o}$. Thus we can suppress the subscript n′ altogether. The factor of two drops out of the equation.

$$T = \sum_m \frac{1}{2} \rho_m \left|\frac{da_m}{dt}\right|^2 = \int_o^L \rho \left(\frac{d\xi}{dt} \right)^2 dz$$

$$\psi = \sum_m \frac{1}{2} \rho_m \omega_m^2 |a_m|^2 \qquad (4.5.1.11)$$

$$U = \sum_{n=1}^{N} F_n |D_n| , \qquad D_n = \text{Re} \sum_m A_{nm} a_m(t)$$

There is not enough new information in this formulation to merit a lengthy discussion. We simply have a different series to sum. If the boundary conditions on the system are not periodic, then the reflection condition will place restrictions on a_m, $m < o$ in terms of a_m, $m > o$. These must be considered in order to determine the number of feedback stations N needed for stabilization, since the number of independent unstable modes is the critical number. In general, if there are two modes $n = o$ and $n = 1$ for each m, then double the number of feedback stations will be needed.

One special case is particularly interesting. A periodic system (such as a circle or torus) with N feedback stations evenly spaced around it, such as in the example, can be shown to <u>always</u> have det $|\bar{\bar{A}}_N| \neq o$ if the complex notation is used. See Appendix F for details.

Can we formulate a local energy conservation relationship for our system? In the limit of point-for-point feedback (an infinite number of stations) it is clear that we can. Taking the equation of motion with bang-bang feedback at every point

$$\rho \frac{\partial^2 \xi}{\partial t^2} = T \frac{\partial^2 \xi}{\partial z^2} + P\,\xi - F \frac{\xi}{|\xi|} - b \frac{\partial \xi}{\partial t} \qquad (4.5.1.12)$$

multiplying by $\frac{\partial \xi}{\partial t}$, and rearranging, we get

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho\left(\frac{\partial \xi}{\partial t}\right)^2 + \frac{1}{2} T \left(\frac{\partial \xi}{\partial z}\right)^2 - \frac{\rho}{2}\xi^2 + F|\xi|\right] + \frac{\partial}{\partial z}\left[-T \frac{\partial \xi}{\partial z}\frac{\partial \xi}{\partial t}\right] + b\left(\frac{\partial \xi}{\partial t}\right)^2 = 0 \qquad (4.5.1.13)$$

where we may interpret the first term as the time-variation of local energy storage W_1 the second as the divergence of energy flow S, and the third as dissipation P_d. A logical definition of local energy velocity is $\overline{V}_e = \overline{S}/W$, and $\partial W/\partial t + \nabla \cdot \overline{S} + Pd = 0$.

Suppose, however, that we wish to define a non-local energy velocity, which corresponds to the time rate of change of the "location" of a pulse. We define

$$V_e(t) = \frac{d}{dt} \frac{\int_0^L z\ W(z,t)\ dz}{\int_0^L W(z,t)\ dz} = \frac{d}{dt}\left[\frac{L(t)}{W(t)}\right]$$ or the rate

(4.5.1.14)

of change of the 1st moment of energy as the energy velocity. Clearly, this depends upon the form of perturbation of the system.

We note that Jeffries and Jeffries define a different energy velocity for linear systems, which is a local property and may be described as a local group velocity. It has the property that the energy contained between two points, each moving at their energy velocity, is constant. The concept is different from that presented here, since this velocity is not local. The two ideas complement each other.

We may wish to show that energy flows at a spatial average velocity equal to the energy velocity. This may be done by multiplying our energy conservation relation by z and integrating over the system to get

$$\frac{\partial}{\partial t} \int_0^L [z\ W(z,t)] = + \int_0^L S\ dz - \int_0^L \frac{\partial}{\partial z}\ (zS)\ dz - \int_0^L Pd\ dz$$

(4.5.1.15)

The second term on the right vanishes unless reflections are occuring. The third is not interesting since it represents deviation from energy conservation in any case. Thus we get

$$V_e(t) = \frac{\int_0^L S\,dz}{\int_0^L W\,dz} \quad \text{as desired.} \qquad (4.5.1.16)$$

Appendix B considers evaluation of V_e for linear systems in more detail, and shows that V_e reduces to a weighted average of the group velocity as we would expect. Also, several generalizations of V_e are discussed in the context of linear systems. End effects are ignored, since they are not usually of interest for velocity concepts. Hence, this development applies mainly to long systems, or periodic ones.

This definition of energy velocity,since it is not local, can be extended to any system with nonlinear feedback of either discrete or continuous form.

Is it possible to extend local energy-conservation arguments to systems with discrete feedback stations? We are faced with a problem, since the feedback contributes to the energy. How can we assign this energy to some point in the system? Let's avoid the problem by integrating over the entire feedback element. We will assume that the feedback spatial distributions A (z) are bounded on $z_{no} < z < z_{n1}$, and do not overlap. Thus, from the local equation of motion

$$\rho \frac{\partial^2 \xi}{\partial t^2} = T \frac{\partial^2 \xi}{\partial z^2} + P\, \xi - F \frac{D_n}{|D_n|} A(z) - b \frac{\partial \xi}{\partial t}$$

$$D_n(t) = \int_{z_{no}}^{z_{n1}} A_n(z)\, \xi(z,t)\, dz \qquad (4.5.1.17)$$

Multiplying by $\frac{\partial \xi}{\partial t}$ and integrating from z_{no} to z_{n1} gives

$$\frac{dE_n}{dt} + \Delta(S_n) + B_n = 0, \quad E_n = T_n + \psi_n + U_n, \quad S_n = S(z_{n1}) - S(z_{no}) \qquad (4.5.1.18)$$

$$B_n = \int_{z_{no}}^{z_{n1}} b \left(\frac{\partial \xi}{\partial t} \right)^2 dz \quad S(z) = -T \frac{\partial \xi}{\partial z} \frac{\partial \xi}{\partial t}$$

$$T_n = \int_{z_{no}}^{z_{n1}} \frac{1}{2} \rho \left(\frac{\partial \xi}{\partial t}\right)^2 dz \quad \psi_n = \int_{z_{no}}^{z_{n1}} \frac{1}{2} T \left(\frac{\partial \xi}{\partial z}\right)^2 - \frac{P}{2} \xi^2 dz$$

$$U_n = F_n \, |D_n|$$

Thus we can write local energy conservation near each feedback station. Letting these regions fall side by side ($z_{n1} = z_{(n+1)o}$ etc.) so that they fill the system will allow examination of the transfer of energy from one to another. Summing over all feedback stations gives our total energy results.

The logical extension of our energy velocity, then, is to let

$$V_e(t) = \frac{d}{dt}\left[\frac{\sum_{n=1}^{N} n\, E_n(t)}{E}\right] \qquad E = \sum_{n=1}^{N} E_n$$

(4.5.1.19)

This gives the number of elements traversed by an energy pulse per unit of time. Setting b = o, we multiply by n and sum over n, giving

$$V_e(t) = \frac{1}{E}\frac{d}{dt}\left[\sum_{n=1}^{N} n\, E_n\right] = \frac{1}{E}\sum_{n=1}^{N} n\,\frac{dE_n}{dt} =$$

$$-\frac{1}{E}\sum_{n=1}^{N} n\,[S(z_{n1})-S(z_{no})]$$

(4.5.1.20)

Since $S(z_{n1}) = S(z_{(n+1)o})$ we may write

$$V_e(t) = -\frac{1}{E}\sum_{n=1}^{N} [(n+1)\, S(z_{(n+1)o}) - n\, S(z_{no}) - S(z_{(n+1)o})]$$

(4.5.1.21)

Ignoring reflections or end effects, this is just

$$V_e = \frac{1}{E}\sum_{n=1}^{N} S(z_{no}) \qquad (4.5.1.22)$$

or the normalized inter-element energy transfer rate.
If each segment is of uniform length $z_{n1} - z_{no} = \Delta L = \frac{L}{N}$,
we may multiply by this factor to get

$$V_e(t) = \frac{1}{E}\sum_{n=1}^{N} S(z_{no})\ \Delta L \qquad (4.5.1.23)$$

which approaches the continuum limit as $N \to \infty$.

4.5.2 Numerical Methods and Method of Characteristics

So far, we have two ways of integrating the equations of motion of the nonlinear system. Let's assume that we intend to use a digital computer for the job. Then we may wish to make the partial differential equation into a difference equation by sampling at regular intervals. Thus, our approach looks like this:

We are given $\xi(z_i, t_i)$ and $\frac{\partial\xi}{\partial t}(z_i, t_i)$, $z_i = i\Delta z$ with the equation

$$\rho \frac{\partial^2 \xi}{\partial t^2} = T \frac{\partial^2 \xi}{\partial z^2} + P\xi - \sum_{n=1}^{N} F_n \frac{D_n}{|D_n|} A_n(z) \tag{4.5.2.1}$$

1) We may approximate

$$\frac{\partial \xi}{\partial z}(z_i) = \frac{\xi(z_{i+1}) - \xi(z_{i-1})}{2\Delta z}$$

$$\frac{\partial^2 \xi}{\partial z^2}(z_i) = \frac{\xi(z_{i+1}) - 2\xi(z_i) + \xi(z_{i-1})}{(\Delta z)^2}$$

$$D_n(t_i) = -\int_0^L A(z)\xi(z,t_i)dz \cong \sum_{i=0}^{L/\Delta z} A(z_i)\,\xi(z_i,t_i)\Delta z \tag{4.5.2.2}$$

Thus we can evaluate the left side [L.H.S.] of the equation at z_i, t_i. We then can write

2)

$$\frac{\partial \xi}{\partial t}(z_i, t_{i+1}) = \frac{\partial \xi}{\partial t}(a_i, t_i) + \frac{1}{\rho}[\text{L.H.S.}]\,\Delta t \tag{4.5.2.3}$$

3)

$$\xi(z_i, t_{i+1}) = \xi(z_i, t_i) + \left[\frac{\partial \xi}{\partial t}(z_i, t_i) + \frac{\partial \xi}{\partial t}(z_i, t_{i+1})\right]\frac{\Delta t}{2} \tag{4.5.2.4}$$

Other more complicated approximations may save convergence time (or step size) or computation time, but the philosophy is the same.

A second approach is to use the modal equations to approximate the system. This is a fundamentally different approach, and its spatial convergence properties are governed by different considerations. Given the same information, we can write

1)

$$a_m(t_i) = \int_o^L \xi(z, t_i)\,\xi_m(z)\,dz \tag{4.5.2.5}$$

for any desired number of modes M. Also,

$$\frac{da_m}{dt}(t_i) = \int_o^L \frac{\partial \xi}{\partial t}(z, t_i)\,\xi_m(z)\,dz \tag{4.5.2.6}$$

We then have the equation from previous work,

$$\rho_m \frac{d^2 a_m}{dt^2} = -\rho_m \omega_m^2 a_m - \sum_{n=1}^{N} F_n A_{nm} \frac{D_n}{|D_n|}$$

(4.5.2.7)

$$D_n = \sum_{m=1}^{\infty} A_{nm} a_m$$

2) Thus we write

$$D_n(t_i) \cong \sum_{m=1}^{M} A_{nm} a_m(t_i)$$

This allows evaluation of the [L.H.S.] of the equation at t_i.

3) Thus

$$\frac{da_m}{dt}(t_{i+1}) = \frac{da_m}{dt}(t_i) + [\text{L.H.S.}]\, \Delta t/\rho_m$$

$$a_m(t_{i+1}) = a_m(t_i) + \left[\frac{d\,a_m}{dt}(t_i) + \frac{d\,a_m}{dt}(t_{i+1}) \right] \frac{\Delta t}{2}$$

(4.5.2.8)

Again, fancier approximations can be used. One variation on the above method is especially suited to the bang-bang problem. It involves recognizing that the above equation is a linear inhomogeneous differential equation with constant coefficients, as long as the D_n's do not change sign. Thus we can write the exact solution to the piecewise uncoupled equations:

$$a_m(t + \Delta t) = a_m(t) \cos \omega_m \Delta t + \frac{d\, a_m}{dt}(t) \frac{\sin \omega_m \Delta t}{\omega_m}$$

$$- \frac{1}{\rho_m \omega_m^2} \sum_{n=1}^{N} F_n \frac{D_n(t)}{|D_n(t)|} [1 - \cos \omega_m \Delta t]$$

(4.5.2.9)

$$\frac{d\, a_m}{dt}(t+\Delta t) = \frac{d\, a_m}{dt}(t) \cos \omega_m \Delta t - a_m(t) \omega_m \sin \omega_m \Delta t$$

$$- \frac{1}{\rho_m} \sum_{n=1}^{N} F_n \frac{D_n(t)}{|D_n(t)|} \frac{\sin \omega_m \Delta t}{\omega_m}$$

$$D_n(t+\Delta t) \overset{o}{=} \sum_{m=1}^{M} A_{nm}\, a_m(t+\Delta t)$$

(4.5.2.10)

These are correct until $D_n(t+\Delta t)$ changes sign. At this value of t, the value of D_n should be updated and the process continued. It is more convenient to keep Δt fixed and accept the error of the time-sampling. (A hunting procedure could be easily envisioned, if desired.) This has the effect of a sample-and-hold sensing device on the feedback. The modal truncation is accurate to the extent that A_{nm} is small for $m > M$. Thus, Δt may be made considerably larger without loss of accuracy. This method has proved to be the most economical and easiest implemented.

There is a third method, however, which could be used as a check on the preceding ones. This involves use of the method of characteristics. Given the system

$$\rho \frac{\partial^2 \xi}{\partial t^2} = \gamma \frac{\partial^2 \xi}{\partial z^2} + P\,\xi + F_e(\xi) \qquad (4.5.2.11)$$

where ρ, T, P are constant and $F_e(\xi)$ may be either discrete or continuous given feedback force, we may write

$$\phi_1 = \frac{\partial \xi}{\partial t} - \sqrt{\frac{\gamma}{\rho}}\,\frac{\partial \xi}{\partial z} \qquad C_1 = \sqrt{\frac{T}{\rho}}$$

$$\phi_2 = \frac{\partial \xi}{\partial t} + \sqrt{\frac{\gamma}{\rho}}\,\frac{\partial \xi}{\partial z} \qquad C_2 = -\sqrt{\frac{T}{\rho}}$$

$$\frac{\partial \phi_1}{\partial t} + C_1 \frac{\partial \phi_1}{\partial z} = \frac{P}{\rho}\,\xi + \frac{F_e}{\rho}(\xi)$$

$$\frac{\partial \phi_2}{\partial t} + C_2 \frac{\partial \phi_2}{\partial z} = \frac{P}{\rho} \xi + \frac{F_e}{\rho}(\xi) \tag{4.5.2.12}$$

Thus we may define characteristic lines

$$C_1: z_1 = z_o + \int_o^t C_1 \, dt$$

$$C_2: z_2 = z_o + \int_o^t C_2 \, dt \tag{4.5.2.13}$$

Along C_1, we have

$$\frac{d\phi_1}{dt} = \frac{P}{\rho} \xi + \frac{F_e}{\rho}(\xi)$$

Along C_2, we have

$$\frac{d\phi_2}{dt} = \frac{P}{\rho} \xi + \frac{F_e}{\rho}(\xi)$$

Now, suppose we want to find $\xi(z_1, t_1)$ and $\frac{\partial \xi}{\partial z}(z_1, t_1)$. We may let

$$z_a = z_1 - \sqrt{\frac{\gamma}{\rho}}\, t_1, \quad z_b = z_1 + \sqrt{\frac{\gamma}{\rho}}\, t_1, \tag{4.5.2.14}$$

Let's assume we are doing a computer simulation. Our initial data will be $\xi(z_i,t_o)$ and $\frac{\partial\xi}{\partial t}$ (z_i,t_o) in the segment $z_a \le z \le z_b$. We then let

$$\Delta t = \frac{t_1 - t_o}{I}, \quad t_j = t_o + j\Delta t, \qquad \Delta z = 2C_1\,\Delta t, \quad z_i = z_a + i\Delta z$$

We find new initial data at $t = t_o + \Delta t$ for all $z_a + \sqrt{\frac{T}{\rho}}\,\Delta t \le z_i \le z_b - \sqrt{\frac{T}{\rho}}\,\Delta t$ by the following algorithm:

1) $$\frac{\partial\xi}{\partial z}(z_i,t_o) = \frac{\xi(z_{i+1},t_o) - \xi(z_{i-1},t_o)}{2\Delta z} \quad \text{for all } i$$

(4.5.2.15)

2) $$\phi_1(z_i,t_o) = \frac{\partial\xi}{\partial t}(z_i,t_o) - \sqrt{\frac{Y}{\rho}}\,\frac{\partial\xi}{\partial z}(z_i,t_o)$$

$$\phi_2(z_i,t_o) = \frac{\partial\xi}{\partial t}(z_i,t_o) + \sqrt{\frac{Y}{\rho}}\,\frac{\partial\xi}{\partial z}(z_i,t_o) \qquad \text{for all } i_o$$

$$F_e(z_i,t_o) = F_e(\xi_i(z_i,t_o))$$

(4.8.2.16)

3) $$\phi_1(z_i + C_1\,\Delta t,\ t_o+\Delta t) = \phi_1(z_i,t_o) + \left[\frac{P}{\rho}\,\xi(z_i,t_o) + F_e(z_i,t_o)\right]\Delta t$$

$$\phi_2(z_i + C_2\,\Delta t,\ t_o+\Delta t) = \phi_2(z_i,t_o) + \left[\frac{P}{\rho}\xi(z_i,t_o) + F_e(z_i,t_o)\right]\Delta t$$

for all i (4.5.2.17)

We then renumber the new points z_i' where ϕ_1 and ϕ_2 are known at the new time t_i': $z_i' = z_i + C_1 \Delta t$, $t_i = t_i + \Delta t$

4)

$$\frac{\partial \xi}{\partial t}(z_i',t_i') = \frac{(\phi_1 + \phi_2)}{2}(z_i',t_i') \quad \frac{\partial \xi}{\partial z}(z'_i,t'_i) = \frac{(\phi_2 - \phi_1)}{2\sqrt{\frac{T}{\rho}}}(z'_i,t'_i)$$

(4.5.2.18)

5)

$$\xi(z'_i,t'_i) = \frac{\xi(z_i,t_i)+\xi(z_i+\Delta z,t_i)}{2} + \frac{\Delta t}{4}\Big[\frac{\partial \xi}{\partial t}(z_i,t_i)$$

$$+ \frac{2\partial \xi}{\partial t}(z'_i,t'_i) + \frac{\partial \xi}{\partial t}(z_i-\Delta z,t_i)\Big]$$

(4.3.2.19)

We are then ready to begin a new cycle. Although this method is rather complex, it will give results which converge in a manner quite different from the previous ones, and so is a valuable check.

The conceptual point to be made is that there is another "state space" description available to us:

$$\frac{d\phi_1}{dt} = \frac{P}{\rho} \xi + \frac{F_e}{\rho} (\xi)$$

$$\frac{d\phi_2}{dt} = \frac{P}{\rho} \xi + \frac{F_e}{\rho} (\xi) \qquad (4.5.2.20)$$

$$\frac{dz_1}{dt} = C_1$$

$$\frac{dz_2}{dt} = C_2$$

where $\frac{\partial \xi}{\partial z}(z,t)$ is a function of $\phi_1(z,t)$ and $\phi_2(z,t)$.

A higher order system with N characteristics could be described the same way, by letting $\bar{\phi}$ represent the vector of state variables, $\bar{z}$ represent the corresponding locations along characteristic lines, and

$$\frac{d\bar{\phi}}{dt} = \bar{F}(\bar{\phi})$$

$$\frac{d\bar{z}}{dt} = \bar{C}(\bar{\phi})$$

govern the state of the system. Stability might perhaps be analyzed using a Lyapunov function approach.

5.0 Plasmas and Controlled Fusion

5.1 The Problem

5.1.1 Strategy

The successful operation of plasma devices for fusion power will depend upon the production of high-temperature plasmas in a confined volume. One type of device which looks promising for this function is the TOKAMAK. This consists of a toroidal chamber, usually with highly conducting walls, containing the plasma and encircling the flux of a transformer. When the transformer primary is excited, the plasma forms the secondary and a heating current flows in the plasma. A very large static toroidal magnetic field is also present to help stabilize the current-carrying plasma. A major limitation on the heating currents in Tokamak-type devices is the problem of MHD kink modes of the system, which become unstable at certain threshold current values, with a resulting loss of plasma and energy.

One possible solution to the problem is feedback stabilization of the system. Because such a system would require very large feedback currents and bandwidths, it is most practical to use nonlinear feedback in the form of switches rather than linear amplifiers.

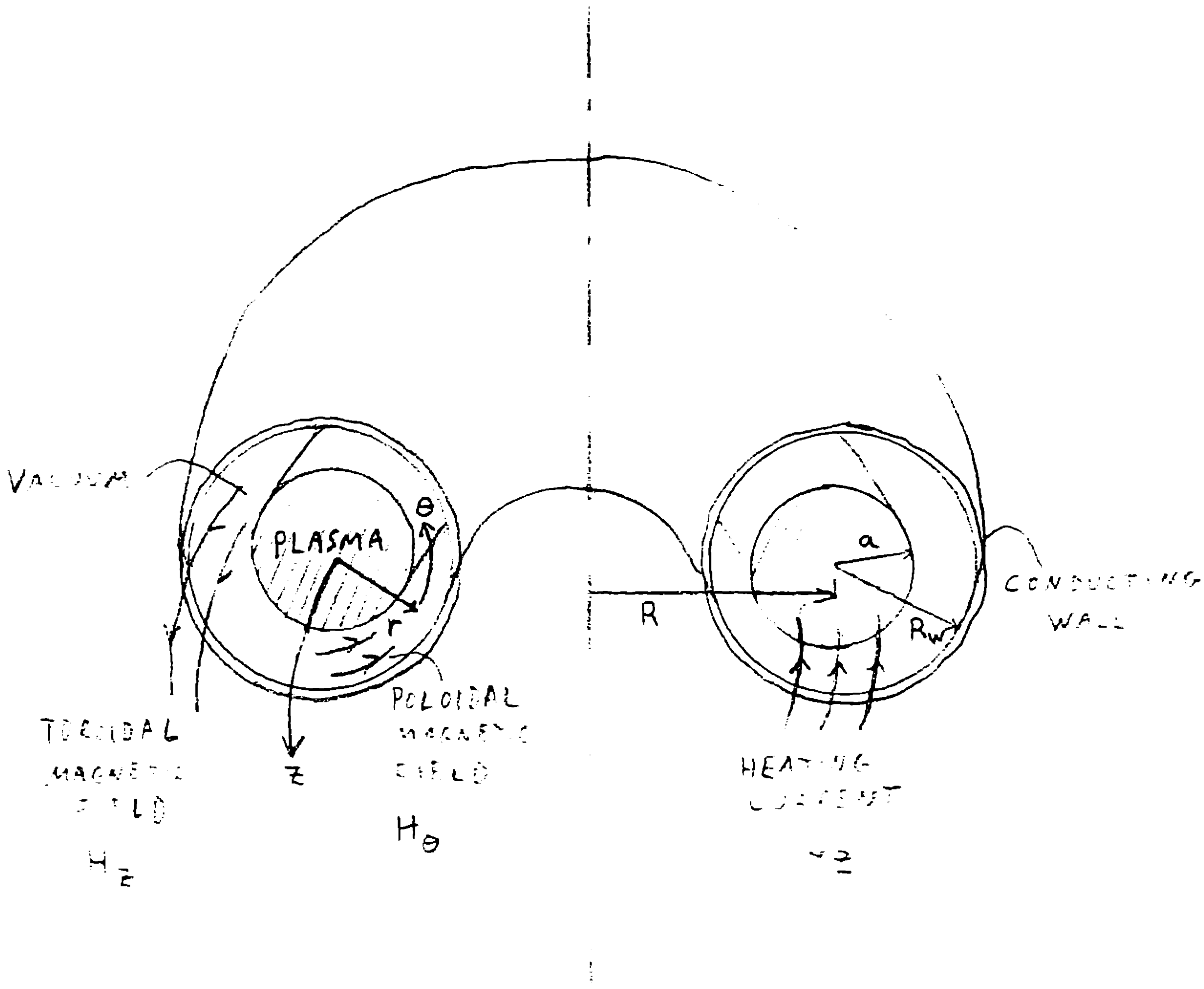


Figure 5.1.1.1
TOKAMAK SCHEMATIC

We therefore envision a Tokamak device, enclosed in a conducting shell, with feedback current straps projecting into the vacuum region between the plasma and the shell. A sensing signal is generated for each strap which is a weighted average of the local displacement of the plasma surface. This signal is operated on in some nonlinear fashion, feedback currents flow in the strap, and the result is an additional force on the plasma surface. This force is distributed locally according to a second weighting function. Clearly, with a finite number of sensor outputs, the state of the entire system is not known.

We must now answer certain questions before designing our feedback system. How will feedback affect the stability of the equilibrium? How much current will be needed? How fast must these currents be switched? How many straps are needed, and what is the best geometry for sensing and forcing?

Once these questions have been answered, more pointed conclusions about feasibility, hardware, and experimental questions can be drawn. To approach these questions, we shall first extend the energy principle for hydromagnetic stability to systems with nonlinear feedback. We shall then develop a general description of switched, or "bang-bang" feedback, with stability criteria and design considerations. Then we shall apply our results to a proposal for feedback stabilization of the M.I.T. Alcator device, a high-field Tokamak now under construction.

5.1.2 Notation

Perturbations of the plasma are denoted $\xi(\bar{r},t)$ where $\bar{r}$ is the position in terms of a periodic cylindrical model, with coordinates $o < r < a$, $o < \theta < 2\pi$, and $o < Z < 2\pi R$. Here a and R are the minor and major radii of the torus. Equilibrium mass density ρ, current $J_Z(r)$, toroidal field H_Z, and poloidal field $H_\theta(r)$ are assumed, with $\bar{H}_o(\bar{r}) = H_\theta(r)\theta + H_Z Z$.

The sensor distribution for the n^{th} feedback strap is denoted $A_n(\bar{r})$, and the resulting signal referred to as the discriminant $D_n(t)$. We will refer to the equilibrium surface S of the plasma, which encloses the equilibrium volume V_i. The vacuum region is denoted V_o. A feedback field $\bar{H}_F$ from the n^{th} feedback strap produces a normal force distribution $B_n(\bar{r})$ on S, and the nonlinear signal processing is denoted by the function $F_n(D_n)$. This conforms with the general model of the previous chapter.

The proof of self-adjointness of perfectly conducting hydromagnetic systems, which is the basis of the energy principle described by Bernstein et al and explained in detail by Chandrasekhar, implies the existence of orthonormal modes of such systems. Therefore we write

$$\bar{\xi}(\bar{r},t) = \sum_m a_m(t)\, \bar{\xi}_m(\bar{r}), \qquad (5.1.2.1)$$

modes that are orthogonal in the sense that

$$\int_{V_i} \rho \xi_m \cdot \xi_n \, d\tau = \delta_{mn} \, \rho_m \, . \qquad (5.1.2.2)$$

The self-adjointness property of the system will be discussed later. Any convenient normalization may be assumed.

Here we denote by ρ_m the effective mass of the m^{th} mode. Without feedback, the modes behave as $a_m(t) = a_m(o) \cos \omega_m t + \dot{a}_m(o) \sin \omega_m t/\omega_m$. Therefore, we have a positional feedback force normal to S produced by each of N feedback stations, producing a total force

$$\bar{F}_p(\bar{r},t) = \sum_{n=1}^{N} -B_n(\bar{r})\, F_n(D_n(t)) \qquad (5.1.2.3)$$

with discriminant

$$D_n(t) = \int_S A_n(\bar{r})\bar{\xi}(\bar{r},t) \cdot n \, dS = \sum_m A_{nm}\, a_m(t)$$

$$(5.1.2.4)$$

where we have defined

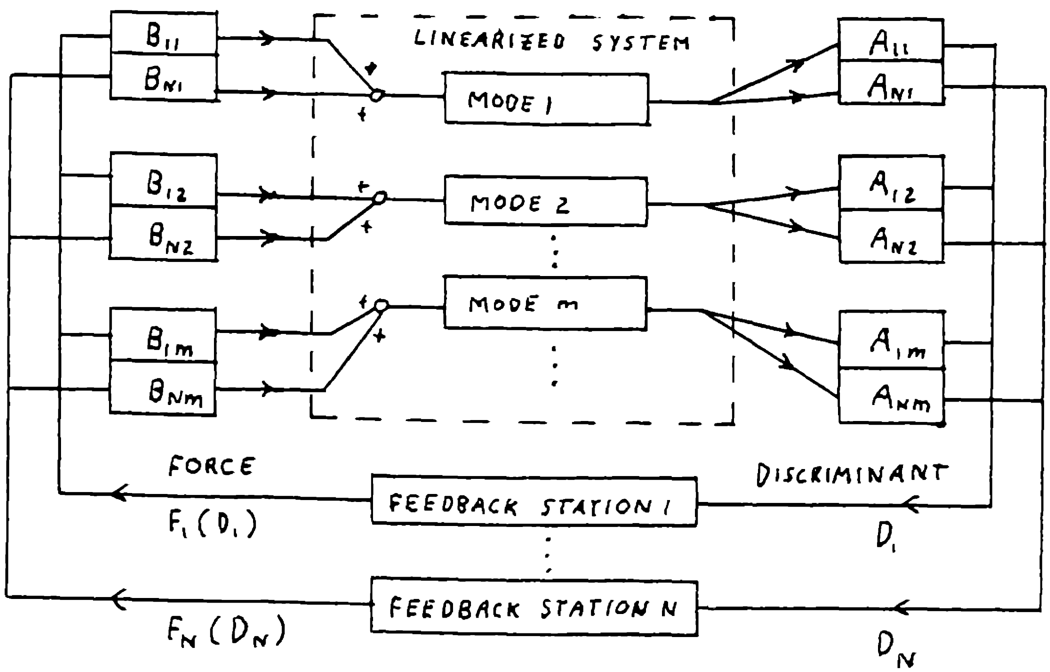
$$A_{nm} = \int_S A_n(\bar{r})\, \bar{\xi}_m(\bar{r}) \cdot n \, dS.$$

$$(5.1.2.5)$$

The system is diagrammed in Fig. 5.1.2.1.

Figure 5.1.2.1

Nonlinear Feedback to a Linearized Distributed System



If velocity feedback is also desired on S_1 it is assumed to be an additional force of the form

$$F_V(r,t) = \sum_{n=1}^{N} -B'_n(\bar{r}) \quad F'_n\left(\frac{dD'_n(t)}{dt} \right) , \tag{5.1.2.6}$$

with D'_n defined analogously in terms of $A'_n(r)$ and $\xi(r,t)$.

5.2 The Energy Principle

5.2.1 Description

A very general approach to the stability analysis of plasmas is referred to in the literature as the energy principle. It takes the linearized equations of a perfectly conducting fluid and develops from them a small-signal energy conservation result. For static equilibrium, the resulting kinetic and potential energy terms are quadratic in the local fluid displacement $\bar{\xi}(\bar{r})$, and both are purely real. Thus, exchange of stabilities applies independent of the geometry of the system. Also, stability can be determined without solving for the exact modes or associated growth rates of the system.

The principle was first formalized in connection with the stability of interstellar plasmas and is recently of great interest in the stabilization of plasmas for controlled fusion. It applies in general form to any nonrelativistic, isotropic, perfectly conducting fluid, either isolated in space or enclosed in a perfectly conducting shell. The fluid may be compressible, and it may reside in an external potential field such as gravity. Equilibrium currents and magnetic fields are allowed both inside ($\bar{H}_i$) and outside ($\bar{H}_o$) the fluid volume.

To avoid unnecessary algebraic complexity, the energy principle will be derived for a special case in the following section. Then the more general form will be stated.

5.2.2 Example

This section will derive the energy principle for a special case as an example. For simplicity, we will consider an incompressible, perfectly conducting fluid with no equilibrium volume currents in some arbitrary closed volume V_i. It is surrounded by some vacuum region V_o, which is enclosed by a perfectly conducting shell. The equilibrium magnetic fields are $\bar{H}_i$ and $\bar{H}_o(\bar{r})$ inside and outside the fluid. We will prove, via the energy principle, that exchange of stabilities applies (ω^2 real) and that stability of the static equilibrium can be reduced to a condition on the zero-order magnetic fields near the fluid-vacuum interface.

In fact, the incompressibility assumption can be relaxed without serious consequences, because the potentially unstable modes of a plasma column in fact have no divergence in the case of marginal stability.

5.2.2.1 Equations

We begin with the linearized equations of the fluid. Perturbation quantities are lower case, equilibrium values capitals, and $\overline{\xi}$ is the fluid displacement.

$$\rho \frac{\partial \overline{v}}{\partial t} = -\nabla p + (\nabla \times \overline{h}_i) \times \mu_i \overline{H}_i \qquad (5.2.2.1.1)$$

$$\nabla \cdot v = o \qquad (5.2.2.1.2)$$

$$\overline{e}_i + \overline{v} \times \mu_o \overline{H}_i = o \qquad (5.2.2.1.3)$$

$$\nabla \times \overline{e}_i = -\mu_i \frac{\partial \overline{h}_i}{\partial t} \qquad (5.2.2.1.4)$$

$$\mu_i \nabla \cdot \overline{h}_i = o \qquad (5.2.2.1.5)$$

where $\overline{H} = \overline{H}_i + \overline{h}_i$, $P = P_i + p$, $\overline{v} = \frac{\partial \xi}{\partial t}$.

In the vacuum region,

$$\nabla \times \overline{h}_o = o \qquad (5.2.2.1.6)$$

$$\nabla \cdot \overline{h}_o = o \qquad (5.2.2.1.7)$$

Boundary conditions come from force equilibrium at the interface S:

$$P_i + \frac{1}{2}\mu_i H_i^2 = \frac{1}{2}\mu_o H_o^2 \qquad (5.2.2.1.8)$$

$$p + \mu_i \bar{h}_i \cdot \bar{H}_i = \mu_o \bar{h}_o \cdot \bar{H}_o + \frac{\mu_0}{2} \bar{\xi} \cdot \bar{n}_s \left[\frac{\partial H_o^2}{\partial n_s} \right] \qquad (5.2.2.1.9)$$

and non-penetration of flux at the perfectly conducting surfaces

$$(\bar{e}_i + \bar{v} \times \mu_i \bar{H}_i) \times \bar{n} = o \quad \text{on } S \qquad (5.2.2.1.10)$$

$$(\bar{e}_o + \bar{v} \times \mu_o \bar{H}_o) \times \bar{n} = o \quad \text{on } S \qquad (5.2.2.1.11)$$

$$(\bar{H}_o + \bar{h}_o) \cdot \bar{n}_o = o \quad \text{on } S_o$$

where, at the fluid surface S,

$$\bar{n} = \bar{n}_s + \bar{n}_1 \qquad (5.2.2.1.12)$$

and $\bar{n}_1$ is the first-order perturbation at a point on S, the unperturbed fluid surface, while at the conducting shell S_o,

$$\bar{n} = \bar{n}_o \qquad (5.2.2.1.13)$$

For simplicity, we take $\overline{H}_i$ and P_i constant over V_i and $\overline{H}_i \cdot \overline{n}_s = o$, $\overline{H}_o \cdot \overline{n}_s = \overline{H}_o \cdot \overline{n}_o = o$.

We can now combine (3) and (4) to find

$$\overline{h}_i = \nabla \times (\overline{\xi} \times \overline{H}_i) \qquad (5.2.2.1.14)$$

5.2.2.2 Energy Form

To obtain our energy equation, we dot (1) with the velocity, and integrate over the equilibrium volume of the fluid V_i:

$$\int_{V_i} \rho \, \overline{v} \cdot \frac{\partial \overline{v}}{\partial t} \, d\tau = \int_{V_i} \frac{\partial \overline{\xi}}{\partial t} \cdot \overline{F}(\overline{\xi}) \, d\tau \qquad (5.2.2.2.1)$$

where

$$F(\overline{\xi}) = - \nabla p + \nabla x \, (\nabla \, x \, \overline{\xi} \, x \, \overline{H}_i) \, x \, (\mu_o \, \overline{H}_i) \qquad (5.2.2.2.2)$$

It is easy to rearrange the left-hand side of (15) into a perfect time derivative $\frac{dT}{dt}$ where

$$T = \int_{V_i} \frac{1}{2} \rho \, \overline{v} \cdot \overline{v} \, d\tau \geq o \qquad (5.2.2.2.3)$$

represents the kinetic energy of the perturbation. We now would like to represent the right hand side of (15) as the time derivative of a potential energy

$$\psi = \int_{V_i} - \frac{1}{2}\overline{\xi} \cdot \overline{F} \, (\overline{\xi}) \, d\tau \qquad (5.2.2.2.4)$$

To do this, we must show that

$$\int_{V_i} \frac{\partial \overline{\xi}}{\partial t} \cdot \overline{F}(\overline{\xi})\, d\tau = \int_{V_i} \overline{\xi} \cdot \frac{\partial}{\partial t} \overline{F}(\overline{\xi})\, d\tau$$

$$= \int_{V_i} \overline{\xi} \cdot \overline{F}\left(\frac{\partial \overline{\xi}}{\partial t} \right) d\tau$$

(5.2.2.2.5)

We note that $\overline{v} = \frac{\partial \overline{\xi}}{\partial t}$ satisfies the same equations and boundary conditions as $\overline{\xi}$, so we can reduce the problem to one of showing that $\overline{F}(\overline{\xi})$ is "self adjoint": that is

$$\int_{V_i} [\overline{\eta} \cdot \overline{F}(\overline{\xi}) - \overline{\xi} \cdot F(\overline{\eta})]\, d\tau = o \qquad (5.2.2.2.6)$$

where $\overline{\xi}$ and $\overline{\eta}$ are any two possible solutions of the fluid displacement which satisfy the system's equations and boundary conditions.

Before proving this, let's note a few things about self-adjoint systems. If (6) is true, then we can show that distinct modes of the system are orthogonal, since letting

$$\overline{\eta} = \overline{\xi}_1 (\overline{r})\, e^{j\omega_1 t}$$

$$\overline{\xi} = \overline{\xi}_2 (\overline{r})\, e^{j\omega_2 t}$$

(5.2.2.2.7)

and using (1) and (6) gives

$$(\omega_1^2 - \omega_2^2) \int_{V_i} \rho \; \bar{\xi}_1 \cdot \bar{\xi}_2 \; d\tau = 0 \qquad (5.2.2.2.8)$$

Hence the energies of different modes of the system are simply additive. This means that (6) is a very powerful result, and a great deal can be said about the systems which are self-adjoint.

So, let's prove (6) for our system.

5.2.2.3 Proof of Self-Adjoint Property

We will do it by calculating $I = \int_{V_i} \bar{\eta} \cdot F(\bar{\xi})\, d\tau$ and showing that it is symmetric in $\bar{\eta}$ and $\bar{\xi}$. This will also give us an expression for ψ, the potential energy, which we need later anyway.

$$I = \int_{V_i} \bar{\eta} \cdot [-\nabla p + \nabla \times (\nabla \times \bar{\xi} \times \bar{H}_i) \times \mu_i \bar{H}_i]\, d\tau \tag{5.2.2.3.1}$$

The first term becomes $-\nabla \cdot (\bar{\eta}\, p) + p\, \nabla \cdot \bar{\eta}$ and since η is a solution of the system, $\nabla \cdot \bar{\eta} = o$.

For the second term, we rewrite as

$$-(\bar{\eta} \times \mu_i \bar{H}_i) \cdot \nabla \times (\nabla \times \bar{\xi} \times \bar{H}_i)$$

$$= + \nabla \cdot [(\bar{\eta} \times \mu_i \bar{H}_i) \times (\nabla \times \bar{\xi} \times \bar{H}_i)] - (\nabla \times \bar{\xi} \times \bar{H}_i) \cdot (\nabla \times \bar{\eta} \times \bar{H}_i) \tag{5.2.2.3.2}$$

Hence

$$I = \int_S [\bar{\eta}\, p + (\bar{\eta} \times \mu_o \bar{H}_i) \times \bar{h}_i] \cdot \bar{n}_s\, ds$$

$$- \int_{V_i} (\nabla \times \bar{\xi} \times \bar{H}_i) \cdot (\nabla \times \bar{\eta} \times \bar{H}_i)\, d\tau \tag{5.2.2.3.3}$$

Now use

$$(\bar{\eta} \times \mu_i \bar{H}_i) \times \bar{h}_i = -\bar{\eta}\,(\mu_i \bar{H}_i \cdot \bar{h}_i) + \mu_i \bar{H}_i\,(\bar{\eta} \cdot \bar{h}_i)$$

and $\bar{H}_i \cdot e\bar{n}_s = o$ to write

$$I = -\int_{V_i} \mu_i\,(\nabla \times \bar{\xi} \times \bar{H}_i) \cdot (\nabla \times \bar{\eta} \times \bar{H}_i)$$

$$- \int_S (\bar{\eta} \cdot \bar{n}_s)\,(p + \mu_o \bar{H}_i \cdot \bar{h}_i)\, d\, s$$

(5.2.2.3.4)

Now, use the force equilibrium condition to write the second term as

$$-\int_S [\frac{\mu_o}{2}[\frac{\partial H_o^{\,2}}{\partial n_s}](\bar{\xi} \cdot \bar{n}_s)(\eta \cdot \bar{n}_s) + (\bar{\eta} \cdot \bar{n}_s)(\mu_o \bar{h}_o \cdot \bar{H}_o)]ds$$

(5.2.2.3.5

We will now manipulate this last term into the form

$$-\int_{V_o} \mu_o\, \bar{h}_o \cdot \bar{h}_o'\, d\,\tau, \qquad (5.2.2.3.6)$$

where h_o' is the perturbation field due to η.

Since $\nabla \times h_o' = o$, we can write $h_o' = \nabla \times \bar{A}'$, $\nabla \times \nabla \times \bar{A}' = o$. So (6)

$$= -\int_{V_o} \mu_o \bar{h}_o \cdot \nabla \times A' \, d\tau = -\int_{V_o} \mu_o \nabla \cdot [\bar{A}' \times \bar{h}_o] \, d\tau$$

(5.2.2.3.7)

$$= -\int_{S+S_o} \mu_o (\bar{A}' \times \bar{h}_o) \cdot n \, ds = +\int_{S+S_o} \mu_o \bar{h}_o \cdot (\bar{A}' \times \bar{n}) ds$$

Now we can rewrite (1.11) in terms of $\bar{A}'$. From the first-order terms of (1.11)

$$\bar{n}_s \times (\bar{e} + \bar{V} \times \bar{H}_o) = o = n_s \times \bar{e} - (\bar{n}_s \cdot \bar{v}) \bar{H}_o$$

$$= -\frac{\partial}{\partial t} [n_s \times A + (\bar{n}_s \cdot \bar{\xi}) H_o]$$

So

$$\bar{n}_s \times \bar{A}' + (\bar{n}_s \cdot \bar{\eta}) \bar{H}_o = o \text{ on } S, \ \bar{n} \times \bar{A}' = o \text{ on } S_o$$

(5.2.2.3.8)

Hence (7) becomes the last term of (5),

$$-\int_S \mu_o (\bar{h}_o \cdot \bar{H}_o)(\bar{\eta} \cdot \bar{n}) \, ds$$

So our final result is

$$\int_{V_i} \bar{\eta} \cdot \bar{F}(\bar{\xi})\, d\tau = -\int_{V_i} \mu_i\, (\nabla \times \bar{\xi} \times \bar{H}_i) \cdot (\nabla \times \bar{\eta} \times \bar{H}_i)\, d\tau$$

$$- \int_S \frac{\mu_o}{2} \frac{\partial H_o^2}{\partial n_s} \cdot (\bar{\xi} \cdot \bar{n}_s)(\bar{\eta} \cdot \bar{n}_s)\, ds$$

$$- \int_{V_o} \mu_o\, \bar{h}_o \cdot \bar{h}_o' d\tau \qquad (5.2.2.3.9)$$

which is symmetric in $\bar{\xi}$ and $\bar{\eta}$. So, the system is self-adjoint.

5.2.2.4 Conclusions

We may now write the potential energy as

$$\psi = \int_{V_i} \frac{\mu_i}{2} |h_i|^2 \, d\tau + \int_S \frac{\mu_o}{4} \frac{\partial H_o^2}{\partial n_s} (\bar{\xi} \cdot \bar{n}_s)^2 \, ds$$

$$+ \int_{V_o} \frac{\mu_o}{2} |h_o|^2 \, d\tau \qquad (5.2.2.4.1)$$

This is clearly real. Thus, if $T = -\omega^2 \int_{V_i} \frac{\rho}{2} |\xi|^2 \, d\tau$ then $\omega^2 = \dfrac{\psi}{\int \rho/2 \, |\xi|^2 \, d\tau}$ is real, and exchange of stabilities applies.

Also, we can say that if

$$\frac{\partial H_o^2}{\partial n_s} \geq o \qquad (5.2.2.4.2)$$

everywhere on the fluid surface S, then $T \geq o$ and $\psi \geq o$ so the total $\frac{dE}{dt} = o$, $E = T + \psi$ implies that the equilibrium is stable. For a closed volume, this is hardly possible. Thus the most dangerous instability is one in which $|h_i|^2$ and $|h_o|^2$ are minimized while $\frac{\partial H_o^2}{\partial n_s} \; |\xi \cdot n|^2$ on S is maximized. The "interchange" mode is that case in which $\bar{h}_i = \bar{h}_o = o$, and we can see that if it occurs it is certainly a stability problem

Note that in(4.1), ψ is a function only of $\overline{\xi}$, since the magnetic fields are completely determined by $\overline{\xi}$. Thus the interpretation of ψ as a potential energy seems well founded.

5.2.2.5 General Form

Here we exhibit the energy principle in its more general form.

Let ϕ, P_o and ρ_o be the equilibrium gravitational potential, pressure, and density, and let $\overline{A}$ represent the perturbation magnetic potential in the vacuum region V_o. Let $P/P_o = (\rho/\rho_o)^{\gamma}$ where P and ρ are the pressure and density. Then

ψ = potential energy of a perturbation

$$= \frac{1}{2} \int_V [\gamma P_o (\nabla \cdot \overline{\xi})^2 + \mu_o (\nabla \times \xi \times \overline{H}_i)^2$$

$$+ (\overline{\xi} \cdot \nabla P_o) (\nabla \cdot \overline{\xi}) - \mu_o \overline{\xi} (\nabla \times \overline{H}_i) \times (\nabla \times \xi \times \overline{H}_i)$$

$$- (\xi \cdot \nabla \phi) (\nabla \cdot \rho_o \overline{\xi})] \, d\tau$$

$$+ \int_{V_o} \frac{\mu_o}{2} | \nabla \times \overline{A} |^2 \, d\tau$$

$$+ \int_S [\nabla P_o + \frac{\mu_o}{2} \nabla |H_i|^2 - \frac{\mu_o}{2} \nabla |H_o|^2] \cdot n_s$$

$$\cdot (\overline{\xi} \cdot \overline{n}_s)^2 \, ds \qquad (5.2.2.5.1)$$

$$T = \text{Kinetic energy} = \int_N \frac{1}{2} \rho_o \left| \frac{\partial \overline{\xi}}{\partial t} \right| d\tau \qquad (5.2.2.5.2)$$

Note that Bernstein points out that the integral over V_i in ψ, for $\nabla\phi = o$, may be rewritten, using $\bar{n}_p = \frac{\nabla P_o}{|\nabla P_o|}$ and $J_o = \nabla \times H_i$, as

$$\frac{1}{2} \int_V [\mu_o |\nabla \times \xi \times \bar{H}_i + (\bar{n}_p \cdot \bar{\xi})(\bar{J}_o \times \bar{n}_p)|^2 + \gamma P_o (\nabla \cdot \bar{\xi})^2$$

$$- 2 \mu_o (\bar{n}_p \cdot \bar{\xi})^2 (J_o \times \bar{n}_p) \cdot (\bar{H}_i \cdot \nabla \bar{n}_p)] \, d\tau$$

(5.2.2.5.3)

The importance of this general form is that it shows the generality of the development in the preceeding chapter,even in the cas of arbitrary internal currents and fields. Hence, we know that the perturbations of the plasma can be represented by an expansion of normal modes, which form a complete orthonormal set over the allowed perturbations.

We will use this result in two ways: first, to write the coupling of the feedback to the plasma in a useful form for stability analysis; and second, to examine the stability of internal modes.

5.2.3 Feedback

5.2.3.1 Linear Feedback

Often, although the basic system of fluids, vacuum, and conducting shell may be unstable, it is possible to increase the stability of the system with some form of feedback control. This may mean applying additional magnetic fields by use of electrodes, volume V_F, which extend through the conducting shell into the vacuum region V_o. The strength of these fields H_F might be determined by some local average of the fluid displacement $\bar{\xi}\cdot\bar{n}_s$, or its time derivative.

We may add these fields to our feedback model by adding terms to the right-hand side of (5.2.2.1.9) to represent the additional magnetic pressure. This leads to an extra term in the energy equation (see Millner and Parker)

$$\frac{dE}{dt} = \int_S \left(\bar{F}_P \cdot \frac{\partial \bar{\xi}}{\partial t}\right) d\,s \qquad (5.2.3.1.1)$$

$$E = T + \psi$$

If this term is of the form $\bar{F}_F = -\mu_o\, \bar{H}_F \cdot \bar{H}_o\, \bar{n}_s$ where

$$\bar{H}_F = \nabla \times \bar{A}_F, \qquad \bar{n}_o \times \bar{A}_F = o \text{ at } s_o$$

$$\bar{n}_s \times \bar{A}_F = o \text{ at } s \qquad (5.2.3.1.2)$$

then we can rewrite the right-hand side of (5.2.3.1.1) using (5.2.2.3.8) as

$$- \int_{s} \mu_o \nabla \bar{H}_F \cdot (\bar{H}_o \frac{\partial \bar{\xi}}{\partial t} \cdot \bar{n}_s) \, d\, s$$

$$= + \int_{s+s_o} \mu_o \bar{H}_F \cdot \frac{\partial \bar{A}}{\partial t} \times \bar{n}_s \, d\, s$$

$$= + \int_{s+s_o} \mu_o \bar{H}_F \times \frac{\partial \bar{A}}{\partial t} \cdot \bar{n}_s \, d\, s$$

$$= + \int_{V_o} \mu_o \nabla \cdot (H_F \times \frac{\partial \bar{A}}{\partial t}) \, d\, \tau$$

$$= + \int_{V_o} [\, \mu_o \frac{\partial \bar{A}}{\partial t} \cdot J_F - \mu_o \bar{H}_F \cdot \frac{\partial \bar{h}_o}{\partial t}] \, d\, \tau \qquad (5.2.3.1.3)$$

The second term of (3) reduces to

$$- \int_{V_o} + \mu_o \frac{\partial \bar{h}_o}{\partial t} \cdot \nabla \times \bar{A}_F \, d\, \tau = \int_{s+s_o} \mu_o \frac{\partial \bar{h}_o}{\partial t} \times \bar{A}_F \cdot \bar{n}_s \, d\, s$$

$$= \int_{s+s_o} \mu_o \frac{\partial \bar{h}_o}{\partial t} \cdot \bar{A}_F \times \bar{n} \, d\, s = o \qquad (5.2.3.1.4)$$

Hence we have

$$\frac{dE}{dt} = + \int_{V_o} \mu_o \frac{\partial \bar{A}}{\partial t} \cdot \bar{J}_F \, d\tau = - \int_{V_F} \bar{e}_o \cdot \bar{J}_F \, d\tau$$

where $\bar{e}_o = \mu_o \frac{\partial A}{\partial t}$ is the electric field due to the perturbation of the fluid and $\bar{J}_F$ is the feedback current.

Note that, for the interchange mode, $\bar{A} = o$ and so the feedback will have no effect!

If we're not worried about the interchange mode, then clearly the thing to do for linear feedback is let

$$\bar{J}_F = -C_1 \bar{A} - C_2 \frac{\partial \bar{A}}{\partial t}, \quad C_1 > o, \; C_2 > o$$

$$= -C_1 \bar{A} + \frac{C_2}{\mu_o} \bar{e} \qquad (5.2.3.1.5)$$

at each point on the feedback electrode volume V_F. The currents are clearly realizeable, since $\nabla \cdot \bar{J}_F = o$ from $\nabla \cdot \bar{A} = o$.

Then we have $E = T + \psi + U$

where

$$U = \int_{V_F} C_1 \frac{\mu_o}{2} |\bar{A}|^2 \, d\tau \qquad (5.2.3.1.6)$$

$$\frac{dE}{dt} = -B, \quad B = C_2 \mu_o \int_{V_F} \left| \frac{\partial \bar{A}}{\partial t} \right|^2 d\tau > o$$

We have used the feedback to make the energy function E more positive for nonzero $\overline{A}$, and created a damping term B which reduces the oscillations of a stable plasma, just as a lossy element does those of a circuit. If all possible perturbations $\overline{\xi}$ of the fluid resulted in a positive definite E and positive B, we can use a Lyapunov stability criterion to say the system was stabilized.

5.2.3.2 Nonlinear Feedback from N Stations

Now we can easily modify our energy arguments to include nonlinear feedback from any number of stations. The integral over the feedback wires V_F (5.2.3.1.4) is simply the sum of integrals over each feedback station, since they are spacially distinct. Let's analyze separately the effects of the currents used for positional feedback and those used for damping (velocity feedback), since the equation is linear.

Let the sensor output be the line integral over the feedback strap ($V_{Fn} = L_n$) of the vector potential.

$$D_n = \int_{L_n} \bar{A} \cdot d\,\bar{\ell} = \text{(Flux linked by the strap due to the perturbation)}$$

(Note that this involves subtracting out the flux due to the feedback currents.) (5.2.3.2.1)

Thus the coupling is given directly by the flux linkage. Then the current J_{Pn} will be a nonlinear function of D_n:

$$\bar{J}_{Pn} = -F_{Pn}(D_n)\frac{d\bar{\ell}}{|d\ell|} \qquad (5.2.3.2.2)$$

Therefore the right-hand side of (5.2.3.1.4) is given by

$$- \frac{dU}{dt} = - \sum_{n=1}^{N} \int_{L_n} \mu_o \frac{\partial \bar{A}}{\partial t} F_{Pn} (D_n) \cdot d\bar{\ell} \tag{5.2.3.2.4}$$

where

$$U = \sum_{n=1}^{N} \int_o^{D_n} F_{Pn}(x) \, d\, x \tag{5.2.3.2.5}$$

The damping part can be easily handled using the alternate form of (5.2.3.1.4), and the same sensor signal. Let

$$\bar{J}_{Vn} = - F_{Vn} \left(\frac{dD_n}{dt} \right) \frac{d\bar{\ell}}{|d\ell|} \tag{5.2.3.2.6}$$

Note that

$$\frac{dD_n}{d\ell} \qquad \int_{Ln} \bar{e} \cdot d\,\ell$$

is just the voltage difference across the feedback strap due to the first-order field of the perturbation. Thus, damping can be obtained by hanging resistors across the feedback straps, giving

$$B = \sum_n \frac{dD_n}{dt} F_n \left(\frac{dD_n}{dt} \right) \tag{5.2.3.2.7}$$

However, this also tells us that velocity feedback is the effect of any slightly conducting material in the vacuum region; for if $\overline{e}$ is approximately the total electric field, then $\overline{J}_f = \sigma \overline{e}$ gives

$$B = \int_{V_F} \sigma |\overline{e}|^2 d\tau \tag{5.2.3.2.8}$$

5.3 Modal Derivations

We now can easily compute all the quantities needed for our stability analysis. However, in order to make the modal form of the analysis explicit, we repeat the last development in terms of modes of the system.

5.3.1 Linear System Plus Feedback

The MHD equations of the (linearized) system allow us to eliminate all variables in favor of $\bar{\xi}$, the displacement, giving

$$\rho \frac{\partial \bar{\xi}}{\partial t^2} = \bar{F}(\bar{\xi}) \tag{5.3.1.1}$$

where $\bar{F}(\bar{\xi})$ is the appropriate linear operator. We dot-multiply (5) with $\bar{\xi}_m(\bar{r})$ to pick out a single mode, and integrate over V_i. Integration of the right-hand side involves use of force equilibrium on the plasma boundary, including the feedback terms. Thus the result is

$$\rho_m \left[\frac{\partial^2 a_m}{\partial t^2} + \omega^2{}_m a_m \right] = \int_S \bar{\xi}_m \cdot (\bar{F}_p + \bar{F}_V)\, dS \tag{5.3.1.2}$$

We may then multiply each mode equation by da_m/dt and sum over all modes to obtain a modal form of (38), using the self-adjointness property to prove that orthogonal modes exist:

$$\frac{dE}{dt} = -B, \quad E = T + \psi + U, \tag{5.3.1.3}$$

where

$$T = \int_V \frac{1}{2} \rho \left|\frac{\partial \bar{\xi}}{dt}\right|^2 d\tau = \sum_m \frac{1}{2} \rho_m \left|\frac{da_m}{dt}\right|^2 > o$$

$$\psi = \sum_m \frac{1}{2} \rho_m \omega_m^2 a_m^2 (\omega_m^2 \text{ defined with no viscosity})$$

$$U = \sum_{n=1}^{N} \int_o^{E_k} F_n(D_n)\, dD_n, \quad E_n = \int_s B_n(\bar{r})\bar{\xi}\cdot n\, dS$$

$$B = \sum_{n=1}^{N} \dot{E}'_n F'_n(\dot{D}'_n) - \sum_{n=1}^{N} \dot{E}_n [F_n(D_n) - F_n(E_n)] + B_V$$

Here the last term, B_V, represents any viscous or resistive effects in the system, with $B_V \geq o$. The first term in B is due to velocity feedback and has the same damping effect as viscosity. Thus, to guarantee that we are not pumping energy into the system, we want to design our sensors and enforcers so that $E'_n(\bar{r}) = D'_n(\bar{r})$. Similarly, looking at the second term in B we want $E_n(\bar{r}) = D_n(\bar{r})$. This tells us that the spatial weighting of the sensing and associated forcing elements should be as similar as possible: $A_n(\bar{r}) = B_n(\bar{r})$. Then we require that our feedback functions be restricted to the first and third quadrants, so

$$\int_0^{D_n} F_n(x)dx > 0 \qquad (5.3.1.4)$$

And $\dot{D}'_n\, F'_n(\dot{D}'_n) > 0$ for all feedback functions F_n.

Since the stability problem is now in the form given in Chapter 4, we will refer to the results quoted there for the rest of the required expressions.

We note at this point that the case of $A_n + B_n$ has not been fully analyzed. Although the system will always be stabilized if the above criteria are met, it may or may not be stable for $A_n + B_n$. The work of Crowley, Dressler, and others with linear systems of low gain implies that the violation of $A_n = B_n$ places an upper limit in the allowable feedback gain, which would imply that the null of a bang-bang system will be unstable.

5.4 Alcator

The proposed Tokamak-type device now under construction at M.I.T. is a possible application of the above-mentioned feedback scheme. It is a toroidal device, with the following approximate projected operating range: (see figure 5.1.1.1)

Major radius $R = 0.5$

Minor radius of plasma $a = 0.13$ m, of shell $R_W = 0.15$ m

Toroidal field $B_Z = 130$ kG $(H_z \cong 10^7$ A/m$)$

Particle density $N_o = 5 \times 10^{20}/m^3$

Mass density $\rho = 8 \times 10^{-7}$ kG/m^3 (protons)

Estimated $q = \dfrac{H_z a}{H_\theta(a) R} \geq q_o = 2.5$ without feedback,

where (r, θ, Z) are consistent with a periodic cylinder model.

The purpose of putting feedback on such a device would be to alter the dynamics so as to lower the value of q consistent with stability of the surface. As long as there is a vacuum region between the conducting shell and the plasma, $q >$ some q_o will be a limitation on the heating current. Ideally, feedback would allow any value of heating current without instability of the surface.

We have a modelling problem in designing this system. Theoretical predictions of growth rates are unreliable and depend critically upon the radial current distribution, which is difficult to measure. Therefore we shall choose a simple uniform current distribution ($J_Z(\bar{r})$= constant), and assume these results to be approximately correct for real situations. We take our modes to be expressed as a Fourier transform of the normal surface perturbation and let the radial variation adapt itself to satisfy the equations of the system. Thus on S,

$$\xi_r(\bar{r}) = \sum_{i=o}^{1} \sum_{j} \sum_{m=o}^{\infty} \sum_{n=-\infty}^{\infty} a_{ijmn}(t)\cos(m\theta + \frac{nZ}{R} + i\,\frac{\pi}{2}),$$

where j represents the possible internal structures possible, and where we have normalized $\bar{\xi}_{ijmn}(\bar{r})$ so that the maximum value of ξ_r on S is unity. We expect the worst situations to occur when the m=-n=1 mode approaches the interchange condition. Therefore our operating point will be assumed to lie near $q \cong +1$ or $H_\theta(a) \cong + H_Z \frac{a}{R} = +2.5 \times 10^6$ A/m.

5.4.1 Design

We refer to the work of Shafranov and others for derivation of the dispersion relation and final form of the modes. (See Appendix D for details of the derivation.) For convenience, we define the following quantities:

$\phi = m + nq$ (a measure of deviation from interchange condition)

$$\gamma^2_{mn} = -\rho \frac{\omega^2_{imn} a^2}{\mu_o H_\theta^2(a)} \quad \text{(growth rate)}$$

For long-wavelength modes (na/R << 1) the following dispersion relation applies:

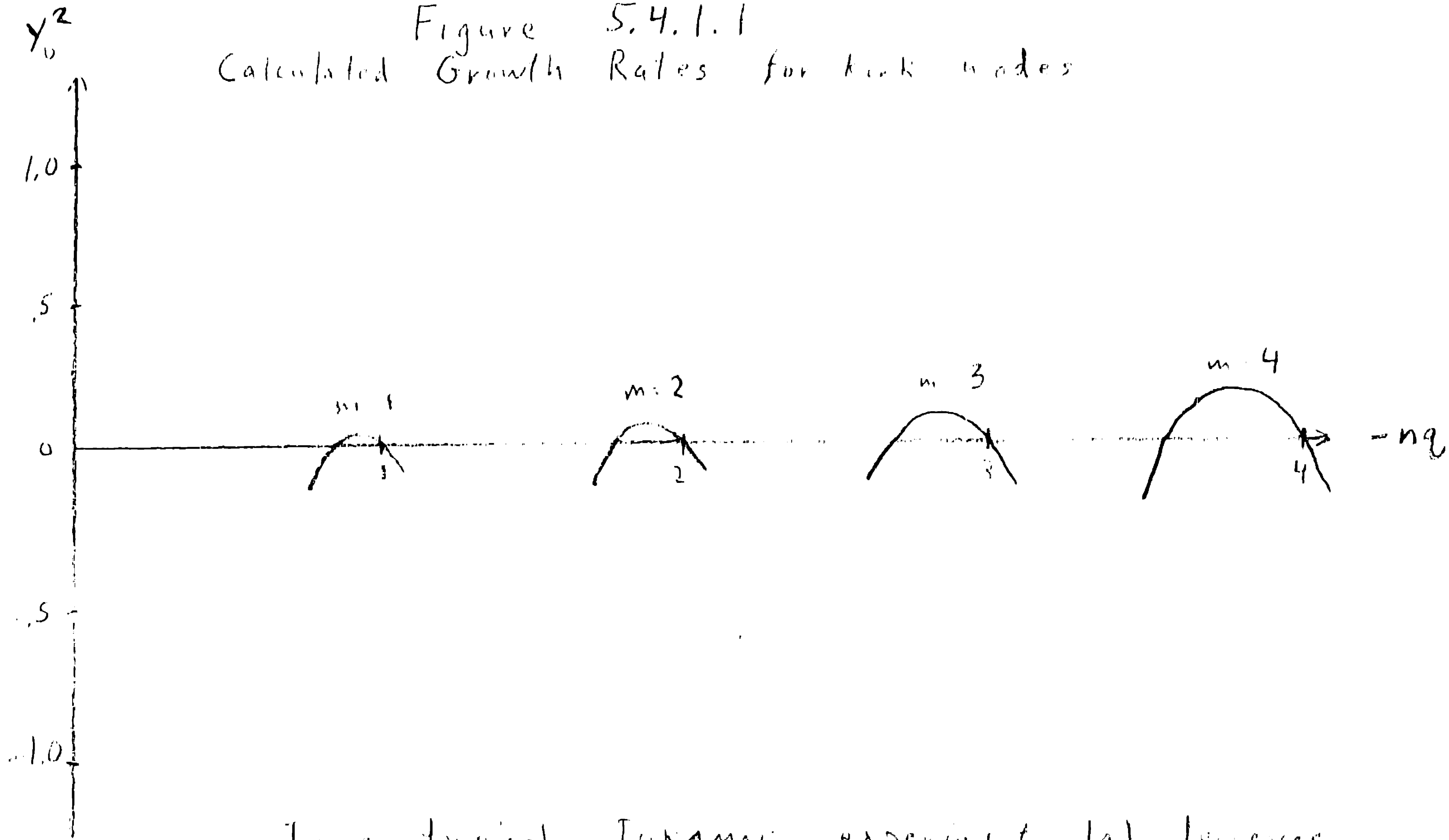
$$\gamma^2_{mn} = 2 \left[\phi - \frac{\phi^2}{1 - \left(\frac{a}{R_W}\right)^{2m}} \right] . \tag{5.4.1.1}$$

This gives a maximum growth rate of

$$\gamma^2 = \frac{1}{2} \left[1 - \left(\frac{a}{R_W}\right)^{2m}\right] \text{ when } \phi = \frac{1}{2} \left[1 - \left(\frac{a}{R_W}\right)^{2m}\right]. \tag{5.4.1.2}$$

Inserting Alcator parameters, this gives $\gamma^2 = \phi \cong 0.13$. Taking $m = n = 1$, we see that $q \cong -0.87$ is in the expected range. Higher m modes would be unstable at higher q but experience has shown them to be less of a problem in terms of confinement. The growth rate, then, is given by

Figure 5.4.1.1
Calculated Growth Rates for kink modes



In a typical Tokamak experiment $|q|$ decreases with time, and modes with $n = 1$, $m = 5, 4, 3, 2, 1$ are seen consecutively.

$$-\omega^2_{i11} = 7 \times 10^{13}\ s^{-2}$$

$$|\omega_{i11}| \cong 8 \times 10^{6}\ s^{-1}.$$

The time $\tau = 1/\omega \cong 1.2 \times 10^{-7}$ s represents the maximum time scale that we can afford to allow in feedback time lag, because of switching, processing, and so forth.

In addition to these modes, there are others (j) with the same m and n, but more complex radial dependence. For a description of these internal modes, see Appendix D, Section 7. In certain ranges of operation near $\phi \cong o$ theory predicts that these modes can combine with the original kink mode to produce an unstable perturbation, with zero surface deflection but unstable internal behavior.

Such internal modes represent a severe limitation on surface-coupled feedback. They represent the limit of the null stability test mentioned above, where only mode amplitudes of the m = n = 1 modes need be considered. If they combine with the kink modes to grow without limit with $\xi_r(a) = o$, eventually such modes will overwhelm the feedback, and enter a new phase of growth dominated by the fast-growing kink. It is quite possible, however, that mechanisms not included in our model will stabilize such modes before they get out of hand.

5.4.2 Approximation of Internal Modes

In general, a complete set of orthonormal functions over the volume V requires a triple infinity of spatial dependences. Thus for each m and n, there could be an infinite number of possible radial functions, each associated with a mode of the plasma. This set is reduced by the restriction that the perturbations satisfy the equations of motion. However, the case we are studying does allow an infinite number of possible radial variations, as is shown in Appendix D. The kink modes are one, the internal modes another set, and the Alfven waves, purely shear, are a third set of solutions. The detailed behavior of internal modes in toroidal geometry is difficult to analyze, since toroidal effects are significant (see Shafranov). Therefore, we make some approximations.

It is much easier to deal with the concepts of feedback stabilization if the mathematics can be simplified to reflect only the key issues. The modes of a plasma column can be broken down into the main kink, with approximate dispersion relation

$$\gamma^2 = 2\left[\phi - \frac{\phi^2}{1 - \left(\frac{a}{R_W}\right)^{2m}} \right] \qquad (5.4.2.1)$$

and the internal modes, plus stable Alfvén waves $Y^2 = -\phi^2$.

As we show in Appendix D, the internal modes approximately obey a dispersion relation

$$Y^2 = \frac{2\phi}{B_j} - \phi^2$$

where B_j is a root of the boundary condition equation

$$J_m \left(j\ K\ a\sqrt{B_j^{\ 2} - 1}\right) = 0$$

This leads to quite small growth rates for internal modes.* We also show in Appendix D that, because of the small coupling of these modes to feedback at the surface, it is likely that

*A plot showing the relative growth rates of the main kink, the internal modes, and the Alfvén waves is given in Figures D.7.1 through D.7.3.

before the plasma seriously threatens to overwhelm the feedback, the internal unstable modes will be dominated by nonlinear mechanisms which would presumably stop their growth.

In toroidal geometry, the most useful stability condition for internal modes is a local one rather than an analysis by modes of the entire system. The energy given by the term integrating over V_i in (5.2.1.3), representing the local energy, must be positive for all perturbations, since the other terms vanish for internal modes. In cylindrical geometry, this constraint reduces to the Shafranov criterion. In toroidal geometry, assuming that magnetic shear dq/dr is small and $dP_o/dr < o$, this leads to the local condition $|q| > 1$ everywhere in the plasma (Mercier). This is first violated on the magnetic axis ($r = o$) for realistic current distributions.

Therefore, we can expect that at sufficiently high heating currents, the interior of the plasma will become turbulent due to unstable internal modes. This will probably be their main effect.

Therefore, if our hypotheses prove correct, there is no reason why feedback control of the form specified would not allow operation of ALCATOR at any desired value of q. For a first experiment, probably the feedback would be arranged to stabilize $m = -n = 1$ modes, and investigate the region near $q = 1/2$ when $m = 1$, $n = -2$ modes might become unstable. There is no data available yet on this range of parameters. However, the heating current would be quite high before such an operating point were reached.

5.4.3 Calculations

We shall assume that the long-wavelength kink modes analyzed first are the only ones present. Then we can use 4.1.2.2.2 and 3 to calculate the magnitude of feedback needed for expected disturbances to lie within the region of stability.

The first thing we notice is that a_m^o becomes very small and makes little contribution to E_o when ω_m^2 becomes large and positive. For $m/n \neq -1$ this is the case, so we can ignore the effect of these modes. So, for a first approximation, we shall assume that $m = -n = 1$ are the only modes that we need consider. If experiment proves this wrong, then modification is clearly possible, but present available results do not contradict this assumption.

We therefore have two unstable modes, from $i = o$ and $i = 1$. This implies two current straps if we can both push and pull on the plasma, or four if we can only push. The first would be an option if we rely on a forcing term $\overline{H}_o \cdot \overline{H}_f$ on S and reverse the current. The second is possible also from a term H_F^2, which can be significant with bang-bang feedback. All other higher terms can be ignored if $|\overline{H}_F| << |\overline{H}_o|$. The first option would appear to require the least current, but, as we shall see, its contribution vanishes at the interchange condition. This aspect will be discussed later.

The key to feasibility of this experiment is the amount of feedback current needed. To find this, we refer to the equation (4.1.2.2.2) relating the feedback force to the region of stability:

$$\rho_m \quad \omega_m^2 \; a_m^o = -\sum_{n=1}^{N} F_n \; A_{nm} \; \frac{D_n^o}{|D_n^o|} \qquad (5.4.3.1)$$

We will take the maximum value of $a_{11}^o = \frac{1}{2}(R_w - a)$ to be 10^{-2}m. To calculate the left-hand side, we require ρ_{11}, which is approximately

$$\rho_{11} \cong \frac{1}{2}\,[2\pi^2 \; a^2 \; R_\rho] \cong 6 \times 10^{-8} \text{ Kg.}$$

Then we notice that if our bang-bang feedback elements cover the surface of the plasma, and push in but don't pull out, then the feedback is of the form

$$F_n = \frac{F}{2}\left(1 + \frac{D_n}{|D_n|}\right)$$

The first term has no effect; it merely adds to the equilibrium pressure. Hence, the effective feedback force is divided by two. Assuming that a mode couples to only two of the four feedback stations, this requires that

$$F_n \, A_{n11} = 8 \times 10^3 \text{ N}$$

(5.4.3.2)

We now wish to express $F_n \, A_{n11}$ in terms of the feedback current I_F. The full expression is

$$F_n \, A_{n11} = \int_S [\; \mu_o \, \overline{H}_o \cdot \overline{H}_F + \frac{1}{2} \mu_o \, H_F^{\,2}] \, \xi_{11} \cdot \overline{n} \, d\, S$$

(5.4.3.3)

We can simplify this expression by using the fact that, for any $\overline{H}_1$ and $\overline{H}_2$,

$$\int_S \mu_o \, \overline{H}_1 \cdot \overline{H}_2 \, \overline{\xi}_m \cdot \overline{n} \, d\, S = \int_{V_o} \overline{A}_{1m} \cdot \overline{J}_2 \, d\, \tau$$

where $\nabla \times \overline{H}_2 = \overline{J}_2$, $\nabla \times \overline{H}_1 = o$, and $\overline{A}_{1m}$ is the vector potential due to the perturbation of $\overline{H}_1$ by $\overline{\xi}_m$. It satisfies boundary conditions that

$$\text{on } S,\; \overline{n} \times \overline{A}_{1m} = -\mu_o \, \overline{H}_1 \, (\overline{\xi}_m \cdot \overline{n})$$

$$\text{on } S_o,\; \overline{n} \times \overline{A}_{1m} = \overline{o}$$

The above identity follows by vector manipulations.

Thus we may rewrite the effective force as

$$F_n \, A_{n11} = \int_{V_o} \bar{A}_{11} \cdot \bar{J}_F + \frac{1}{2} \bar{A}_{F11} \cdot \bar{J}_F \cdot d\gamma$$

(5.4.3.4)

where A_{11} represents the perturbation vector potential attributable to $\bar{\xi}_{11}$ without feedback, and A_{F11} the perturbation of the feedback field alone. We must then place wires so that current will flow along vector potential lines, and so that the current is determined by the flux linked by the wire.

For velocity feedback, our integral becomes $\bar{E}_{11} \cdot \bar{J}_F$, where $\bar{E}_{11}$ is the electric field $-\dfrac{\partial \bar{A}_{11}}{\partial t}$, because of the perturbation, without feedback added. Linear velocity feedback is properly applied by any slightly conducting substance inside the conducting shell, so that its resulting $\partial H_F/\partial t$ creates an electric field much smaller than the perturbation field. Our damping interpretation corresponds to standard energy perturbation results.

We must now estimate the vector potentials $\tilde{A}_{11}$ and $\tilde{A}_{F11}$ at the feedback wires. Since $R_W - a << a$, we may model the vacuum region as a planar geometry. See Appendix E for details. Analysis of A_{11}, the perturbation vacuum potential, shows that it divides into two parts, one irrotational and one solenoidal. The irrotational part produces an electric field with $E = -\nabla\phi$, where for any mode m and n,

$$\phi(r,\theta,z)=\mu_o\omega_{mn} \frac{(\frac{n}{R} H_\theta(a)-\frac{m}{a} H_z) \sinh K(r-R_W)\xi_{rmn}(a)}{K^2 \sinh K(R_W-a)} \tag{5.4.3.5}$$

where

$$K^2 = \left(\frac{n}{R} \right)^2 + \left(\frac{m}{a} \right)^2 .$$

Coupling to this field would have to be electric, and so is relatively small for realistic feedback fields.

The solenoidal part produces the perturbation magnetic field. This vector potential is given by

$$\bar{A}_{11}(r,\theta,z) = \frac{\mu_o\left(\frac{n\hat{\theta}}{R} - \frac{m\hat{z}}{a}\right)(m+nq)H_\theta(a) \sinh K(r-R_W)\xi_{rmn}(a)}{K^2 a \sinh K(R_W - A)} \tag{5.4.3.6}$$

This is the term to which we can couple.

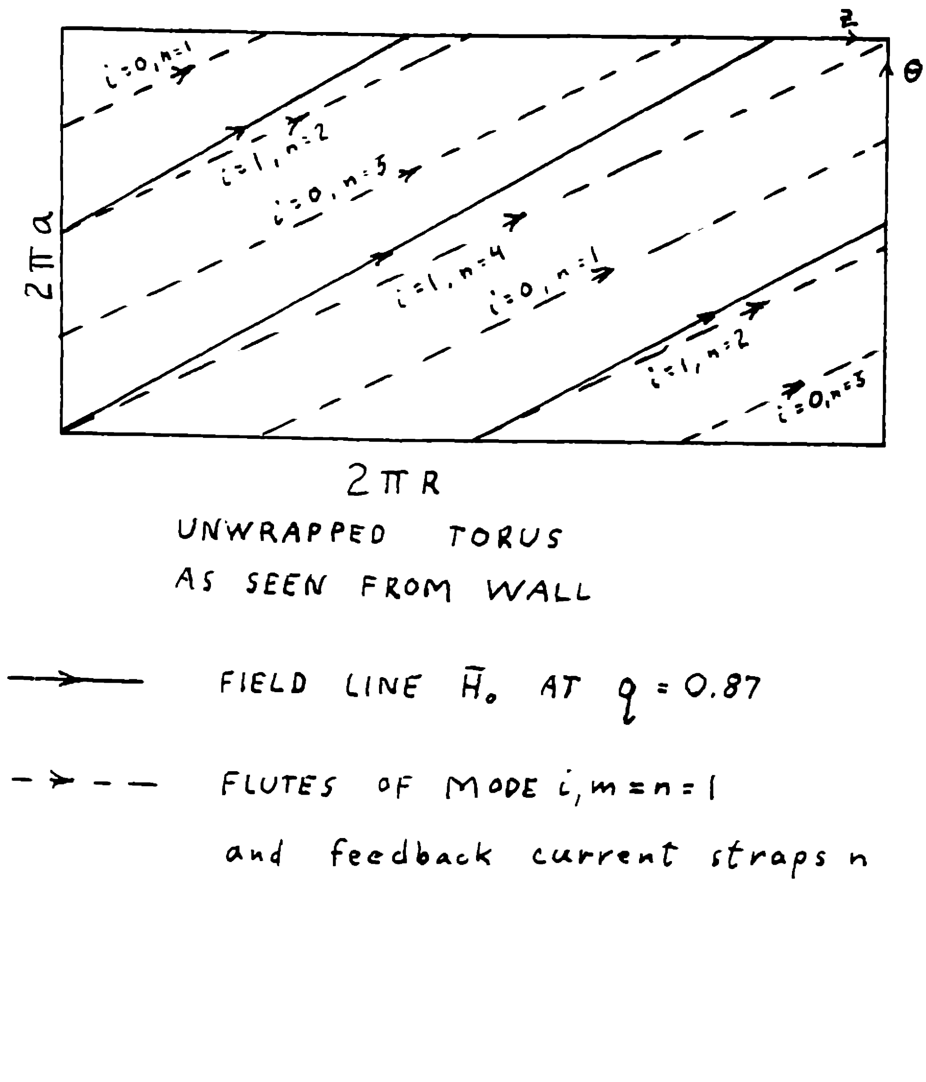
We should notice several things about the potential. First, it points along flutes of the surface perturbation and parallel to the equilibrium surface. Thus, this is the direction in which our feedback current should go, as far from R_W as possible, to maximize the term $\overline{H}_o \cdot \overline{H}_F$ (see Figure 5.4.3.1). Second, the term vanishes when $m + nq = 0$, so such feedback will not affect the interchange modes. Thus, this term will dominate situations far from the interchange situation.

Finally, we note that this term is proportional to $\frac{m}{a} H_\theta(a) + \frac{n}{R} H_Z = \overline{K} \cdot \overline{H}_o$ at $r = a$, where we define the wave vector $\overline{K} = \frac{m}{a}\hat{\theta} + \frac{n}{R}\hat{Z}$. This gives us an easy way to evaluate A_{F11}, the vector potential attributable to the perturbations of the feedback field. We can simply replace the factor with $\overline{K} \cdot \overline{H}_F(a)$. This means that, to maximize the H_F^2 force term, the feedback field should be perpendicular to the flutes, and the feedback currents parallel to them. We therefore see that this geometry is also optimum for feedback contributions of the H_F^2 term. To the extent that $\overline{H}_F(a)$ can be approximated as a constant, this geometry gives

$$\overline{A}_{Fmn} = \frac{\mu_o\left(\frac{n\hat{\theta}}{R} - \frac{m\hat{Z}}{a}\right)(\overline{H}_F(a) \cdot \overline{K}) \sinh K(r-R_W)\, \xi_{rmn}(a)}{K^2 \sinh K(R_W - a)} \tag{5.4.3.7}$$

Figure 5.4.3.1

Feedback Geometry for Alcator



If the total current in the strap is limited, then clearly we want it all to flow in a wire sitting at maximum vector potential, to maximize (4). Let us assume that the $\overline{H}_F \cdot \overline{H}_o$ term dominates. The length of wire is approximately $2\pi R$. This gives a current needed of

$$I_{F_1} = \frac{2F_n A_n}{2\pi R |A_{11}|} \cong 2 \times 10^4 \text{ A.} \tag{5.4.3.8}$$

Now let us assume that the H_F^2 term dominates. This leads to an estimate

$$I_{F_2} \cong \frac{2F_n A_n}{2\pi R A_{F11}} \cong 3 \times 10^4 \text{ A.} \tag{5.4.3.9}$$

The two are quite close. In fact, the value of q at which we assumed we were operating is close to the "break point" between the two extremes. As $q \to 1$, the non-linear term dominates.

When designing the feedback straps, a major consideration will be to minimize the inductance, to avoid arcing during switching transients. This involves maximizing A_{11} while minimizing the integral over V_o of $\frac{1}{2} \mu_o H_F^2$. We can see that a single wire, or impulse of current, would cause very large local magnetic field energy and so is not a good choice. To explore the modification, we assume $\overline{K}_F$ to be distributed in a plane parallel to the equilibrium plasma surface. Its Fourier components produce corresponding components of magnetic field and, since they are orthogonal over the vacuum region, their energies add. Thus, to minimize inductance, we want to distribute the current sinusoidally to match the vector potential and surface perturbation. This is just the distribution of current that occurs on the conducting sheath, a form of linear feedback. Thus the effect of the $\overline{H}_F \cdot \overline{H}_o$ type of feedback is to make the conducting sheath appear arbitrarily close to the plasma, as increasing the gain of a linear feedback system, if such were contemplated. Its maximum effect is that of a wall as $R_W \to a$, and this is the effect of bang-bang on the modes $m = -n = 1$.

Note that the use of H_F^2 feedback lets us do even better, by stabilizing interchange modes that are unaffected by a wall arbitrarily close to the plasma.

We also note that the spreading of feedback currents over a sinusoid results in a decrease in effectiveness because not all the current is at maximum A. This will roughly cancel the addition of terms in our calculation of current, if the current in a strap is equated to the positive half-cycle of a sinusoid. The resulting inductance for such a scheme is approximately 1.3×10^{-7}H.

Assuming the rise time of 1.2 < 10^{-7}s, this gives voltages of 33 kV during switching; difficult but not impossible to work with.

We can use our estimate of the inductance of the feedback straps to get a good picture of the power-output requirements for our current source. We need

Inductance L $\overset{o}{=}$ 1.3 x 10^{-7}

Current I $\overset{o}{=}$ 3 x 10^{4} amps

Switching time $\tau \overset{o}{=}$ 1.2 x 10^{-7} sec.

Thus the energy needed to create the fields is about $\frac{1}{2} LI^2 \overset{o}{=}$ 60 joules giving a power output of 5 x 10^8 watts for 1.2 x 10^{-7} seconds. Thus we will need a very efficient switch. Even assuming a 1% duty cycle and a maximum dissipation of 50 watts, we would need an efficiency $\eta = \frac{\text{Power passed, switching continuously}}{\text{Power handled}}$ of $(1 - 10^{-5})$. This doesn't even consider the problems of steady-state dissipation.

Thus, we would design our feedback so that the currents flow just above flutes in the surface ($\bar{J}_F \times \bar{H}_o \cdot r > 0$). For 1-cm perturbations, we require that approximately 30,000 A be switched in less than a tenth of a microsecond. This would be difficult, but not impossible. Certainly at these currents, linear feedback would be far more difficult. Our experiment would look for increased confinement time, and check assumptions that modes with fast spatial variations will be damped.

In conclusion, we can see that the analysis of nonlinear feedback to continuous systems is quite workable, and can be used with various degrees of approximation. In application to Alcator, it appears feasible to use bang-bang feedback in order to allow lower values of q, while preserving finite separation between plasma and conducting shell. The region of q-operation desired will determine the feedback mode of operation.

5.4.4 Implementation

This section will investigate the feasibility of several schemes for implementation of feedback for use in ALCATOR.

5.4.4.1 Sensors

One way in which the sensing of $D_n(t)$ could be realized would be by use of H-field probes. These would have to be placed and weighted so as to give an output with spatial distribution similar to that of the force generated by the feedback currents. Such schemes have been employed in other feedback experiments.

Other methods would use the feedback straps themselves to sense motions of the plasma, by monitoring the inductance of the straps. As the plasma moved closer to the strap, the inductance should be reduced

$$\frac{\nabla L}{L} \cong \frac{-2K\, e^{-2K(R_W-a)}}{1 - e^{-2K(R_W-a)}}\ \xi$$

or for $K(R_F-a) << 1$ and $\xi = \frac{1}{2}(R_W-a)$

$$\frac{\nabla L}{L} \cong - \frac{\xi}{R_W-a} \cong .5$$

In effect, the inductance is proportional to the volume of vacuum between the plasma surface and the wall near the feedback electrode. By using a high-frequency signal and a tuned circuit, the output could be continuously available. However, noise from the plasma and switching transients could be a serious problem.

Perhaps the best possibility would be to take advantage of the fact that D_n is proportional to the flux linked by the strap. Therefore, by measuring the voltage induced across the strap, not including the feedback-field part, and integrating this voltage to find the flux (with the intention of keeping the flux at its equilibrium value) the output would be available with just the right spacial weighting built in. Integration could be done on a time scale long compared to switching times but short compared to the total experiment (say over several oscillations at expected mode amplitudes) so that long-term drift need not be a problem. Sensing with this method would break down at interchange condition.

5.4.4.2 Processing

The processing involved in this experiment is very simple: the output, and perhaps its derivative, at four independent feedback stations must be amplified, converted to a digital signal, and fed to the switching elements. High-speed digital logic could easily handle this problem with appropriate conversion of the input signal.

5.4.4.3 Switching Elements

The switching elements are the critical ones in the feasibility study. The needed combination of power and speed will be difficult to realize. There is little tradeoff possible, since the most critical parameter is the switching time, which is also the most difficult to achieve. The switching element must switch on and off a current of approximately 30 kiloamps through a load of about .13 microhenries with a rise and fall time of less than .12 microseconds. This requires a capacitor bank with at least 30 kilovolts and at least .13 microfarods capacitance, which can be obtained (Maxwell Capacitor Corp.). This voltage allows fast enough turn-on. The turn-off problem may be prohibitive. It might be accomplished by pulsed feedback, using a rectifier or one-way conducting switch and the L-C resonance time of approximately .13 microseconds. This allows the possibility of merely using the capacitor as an energy storage bank and avoiding any steady-state power drain to the feedback, by reversing the capacitor polarity between pulses.

The required switching time is faster than state-of-the-art systems presently in use or under construction, which all switch on in times approaching one microsecond but cannot switch off once an arc is established in any reasonable time. However, there appears to be no theoretical limitation which rules out such a switch. Lead inductances can be made considerably less than the strap inductances (see, for instance, Les Champs Magnetigues Intenses, Grenoble, 1966, pp. 361-370). Even for conventional coaxial geometry with internal radius about half the external radius, inductance of about 1.3×10^{-7} henries/meter is not too discouraging. (See Pulsed High Magnetic Fields, H. Knoepfel, 1970.) The problem is in finding a fast switching mechanism which can turn off.

Another hopeful point is that the actual measured current profiles in TOKAMAK devices tend to be sharply peaked at r = 0, which gives longer growth times for unstable modes, and so relaxes the switching time requirement. As much as a factor of ten increase is not unlikely, if numerical studies mentioned earlier are applicable. A lesser reduction in required currents would also follow. This would make the requirements similar to those used for SYLLAC.

There appear to be two methods which might prove feasible for current turn-on.

1) Spark-gap methods including Ignitron tubes, involving triggered breakdown in a gap, and using

zero-current point for turn-off. All such tubes currently available appear to require 1 to 10μ-sec. ionization times, making them too slow for this application.

2) Amplifier-chain devices, employing hard vacuum tubes and decoupling transformers, such as those used in SYLLAC.

None of these methods at present give switching times of much less than 1 microsecond. Therefore, the proposed scheme for feedback control of ALCATOR appears to be possible but beyond the state of the art.

5.4.4.4 Comparison with SYLLAC

The best picture of state-of-the-art systems is the Los Alamos Scientific Laboratory's design for feedback control of SYLLAC. They base their philosophy on the fact that no bang-bang switching element is available which allows any choice of current turn-off time. Therefore, they use a high-gain linear amplifier scheme for the feedback.

The sensors are essentially a series of stations around the torus, each consisting of two pairs of light-sensitive elements 90° apart around the minor axis. The plasma in equilibrium casts a beam of light between the two elements of a pair, so that their outputs give a difference signal relating plasma position perturbations.

This signal is amplified, differentiated, and filtered by fast transistorized circuitry and logic elements. The resulting signal is fed to a three-stage vacuum tube amplifier chain with isolating transformers between stages. The output of this chain has a 35 kv swing with 700 amp capacity, and an accumulated delay plus rise time of .25 microseconds, mainly due to the coupling transformers.

This signal then goes to an output transformer to give a secondary current of 4.9 kiloamps at 5 kv. The inductance of the $\ell = o$ feedback windings is .23 microhenries each, and they are driven in pairs in series by the .03 microhenry transformer secondary. This gives a rise time of .49 microseconds, for a total lag of .74 microseconds. Using an estimated growth time of $1/\gamma$ = 1.3 microseconds for SYLLAC parameters, excursions of approximately one centimeter are to be controlled with $\gamma\tau^{o} = .7$. The final configuration is still subject to change, and the inductively loaded amplifier chain is untested. (Source: K. I. Thomassen, consultant to L. A. S. L.)

6.0 Conclusions

The feasibility of nonlinear control of continuous systems must depend upon the characteristics of the systems involved. However, this thesis provides guiding principles for analysis of such control systems, so that the important questions involved can be posed and answered clearly. Most aspects of a practical control system can be analyzed explicitly, and the remaining points are subject to reliable approximation. The advantages and disadvantages of nonlinear control are also clearly visible using these tools.

In the case of feedback control for plasma confinement, it appears that nonlinear control of the bang-bang type is the most feasible form of control. Our present MHD model of the problem does not suggest any theoretical reasons why such control should not be possible. The answers call for switching devices beyond the state of the art. However, a feedback control system for Alcator or a similar device could probably be built given sufficient money and time.

The purpose of this thesis was to reduce the nonlinear continuum control problem to a workable set of approximations, in full knowledge of their limitations.

That has been done, and in the process a number of related problems have been solved. The next step is to build a practical control system based on these principles.

Future controls research could take a number of directions. The dynamical form assumed for a distributed system in this thesis was rather restrictive; the generalization to other important types of systems could be quite rewarding. These might include systems governed by partial differential equations with coefficients which are functions of the dependent variables. The general problem of system control without a complete picture of the state is poorly understood. It may be possible to formulate some type of optimization strategy without complete state estimation in limited situations. Also, the possibilities of scanning controls with finite spatial extent or other variations in the form of control or system will bear investigation. In addition, it may well be possible to deal with many systems using an expansion in terms of a set of orthogonal, but not characteristic, functions. Alternatively, the techniques of measuring modal structure need further development.

For the physicist, there remain numerous types of instabilities which will not respond to the type of feedback presented here. These should provide a rich source of control problems for many years to come.

Appendix A

Energy Loss Mechanisms

A.1 Viscosity

One form which $\bar{\bar{b}}$ might take is that of the viscous force density in a fluid. We can show that this is self-adjoint by showing that $\bar{v} \cdot \bar{F}^V < o$ for all $\bar{v}$.

Let $T_{ij}^{\ V}$ be the viscous stress tensor,

$$F_i^V = \frac{\partial T_{ij}^V}{\partial x_j}$$

We must show that

$$\int_{V_i} v_i \frac{\partial T_{ij}^V}{\partial x_j} \, d\tau < 0 \qquad \text{(A.1.1)}$$

is the viscous dissipation

$$= \int_{V_i} \frac{\partial}{\partial x_j} (v_i T_{ij}^V) \, d\tau - \int_{V_i} T_{ij}^V \frac{\partial v_i}{\partial x_j} \, d\tau \qquad \text{(A.1.2)}$$

$$= \int_S v_i T_{ij}^V \cdot n_j \, dS - \int_{V_i} \Phi_V \, d\tau \qquad \text{(A.1.3)}$$

where

$$T_{ij}^V = 2\mu \dot{e}_{ij} + (\eta - \tfrac{2}{3}\mu)\, \delta_{ij} \, \dot{e}_{kk}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right)$$

$$\int \Phi_V \, d\tau = \int_{V_i} T_{ij}^V \frac{\partial v_i}{\partial x_j} \, d\tau \qquad \text{(A.1.4)}$$

Since $T_{ij} = T_{ji}$, $\int \Phi_v = \int T_{ji} \dfrac{\partial v_i}{\partial x_j} = \int T_{ij} \dfrac{\partial v_j}{\partial x_i}$

so

$$\Phi_v = T^v_{ij} \, \dot{e}_{ij} \tag{A.1.5}$$

$$= 2\mu \, \dot{e}_{ij} \, \dot{e}_{ij} + (\eta - \tfrac{2}{3}\mu) \, \dot{e}_{ij} \, \delta_{ij} \, e_{kk} \tag{A.1.6}$$

$$= 2\mu \left(\dot{e}_{11}^2 + \dot{e}_{12}^2 + \dot{e}_{13}^2 + \dot{e}_{21}^2 + \dot{e}_{22}^2 + \dot{e}_{23}^2 + \dot{e}_{31}^2 + \dot{e}_{32}^2 + \dot{e}_{33}^2 \right) + (\eta - \tfrac{2}{3}\mu)(\dot{e}_{11} + \dot{e}_{22} + \dot{e}_{33})^2 \tag{A.1.7}$$

but $e_{ij} = e_{ji}$ so

$$\Phi_v = 4\mu \left(\dot{e}_{12}^2 + \dot{e}_{23}^2 + \dot{e}_{13}^2 \right) + 2\mu \left(\dot{e}_{11}^2 + \dot{e}_{22}^2 + \dot{e}_{33}^2 \right) + \eta \left(\dot{e}_{11} + \dot{e}_{22} + \dot{e}_{33} \right)^2 - \tfrac{2}{3}\mu \left(\dot{e}_{11}^2 + \dot{e}_{22}^2 + \dot{e}_{33}^2 + 2\dot{e}_{11}\dot{e}_{22} + 2\dot{e}_{22}\dot{e}_{33} + 2\dot{e}_{11}\dot{e}_{33} \right)$$

$$= \tfrac{4}{3}\mu \left(\dot{e}_{11}^2 + \dot{e}_{22}^2 + \dot{e}_{33}^2 - \dot{e}_{11}\dot{e}_{22} - \dot{e}_{22}\dot{e}_{33} - \dot{e}_{11}\dot{e}_{33} \right) + 4\mu \left(\dot{e}_{12}^2 + \dot{e}_{23}^2 + \dot{e}_{13}^2 \right) + \eta \left(\dot{e}_{11} + \dot{e}_{22} + \dot{e}_{33} \right)^2 \tag{A.1.8}$$

The first term in (8) reduces to

$$\frac{2}{3}\mu\left(\dot{e}_{11}^{2} - 2\dot{e}_{11}\dot{e}_{22} + \dot{e}_{22}^{2}\right)$$

$$+\frac{2}{3}\mu\left(\dot{e}_{22}^{2} - 2\dot{e}_{22}\dot{e}_{33} + \dot{e}_{33}^{2}\right)$$

$$+\frac{2}{3}\mu\left(\dot{e}_{11}^{2} - 2\dot{e}_{11}\dot{e}_{33} + \dot{e}_{33}^{2}\right)$$

$$= \frac{2}{3}\mu\left[(\dot{e}_{11} - \dot{e}_{22})^2 + (\dot{e}_{22} - \dot{e}_{33})^2 + (\dot{e}_{11} - \dot{e}_{33})^2\right]$$

So

$$\Phi_v = 4\mu\left(\dot{e}_{12}^{2} + \dot{e}_{23}^{2} + \dot{e}_{13}^{2}\right) + \eta\left(\dot{e}_{11} + \dot{e}_{22} + \dot{e}_{33}\right)^2 + \frac{2}{3}\mu\left[(\dot{e}_{11} - \dot{e}_{22})^2 + (\dot{e}_{22} - \dot{e}_{33})^2 + (\dot{e}_{11} - \dot{e}_{33})^2\right] \qquad \text{(A.1.9)}$$

which is positive for all nonzero $\dot{e}_{ij}$

So Φ_v represents positive dissipation for

$\mu > 0, \quad \eta > 0$

Now let's examine the first term of (3)

$$\int_S v_i T_{ij}^{v} n_j \, d\tau$$

To have a first-order term T_{ij}^{v} at S we require some balancing force by the surface force equilibrium equation. The additional term above is then incorporated into the internal pressure. Thus the presence of viscosity does not change the final form of the energy equation.

A.2 Electrical Conductivity

The addition of viscosity to a fluid system model resulted in the appearance of damping terms in the energy equation. We would expect that a small electrical resistivity in the fluid or other conducting parts, or a small conductivity in the resistive regions, would also have as a first effect the addition of damping terms. The assumption that no other effect would appear is dependent on the usual energy perturbation arguments; namely, that the first-order nonzero fields, currents, fluid displacements, etc. are not significantly changed by these effects. So, only those quantities which vanished in the idealized case such as the electric field in the fluid or parallel to the shell, or the current in the vacuum, would be changed by these effects. The resulting energy loss could be calculated then from direct evaluations of the local power dissipation.

Appendix B

Energy Velocity of a System

The energy velocity of a disturbance may be defined for linear systems as the ratio of time-averages of energy flux to energy, and for quasi-monochromatic signals, is equal to the group velocity (Jeffries and Jeffries). Alternatively, it may be defined in terms of local wave-number and frequency of slowly varying wavetrains (Ramo, et. al.) as the velocity of characteristics of constant wavenumber. The energy between two points moving at that velocity then remains constant in an asymptotic expansion of the perturbation.

We will define the energy velocity as a single quantity representing the rate of change of the "center of energy" of a perturbation. We assume a positive definite local energy density function $w(\bar{r},t)$ which satisfies a conservation relation of the form

$$\frac{\partial w}{\partial t} + \nabla \cdot \bar{S} + P = 0 \qquad \text{(B.1)}$$

for any energy flux $\bar{S}$ and dissipation P. We then define the first moment of the energy density

$$\bar{L}(t) = \int_{\text{Volume}} \bar{r}\, w(\bar{r},t)\, dV \qquad \text{(B.2)}$$

and the total energy

$$E(t) = \int_{Volume} w(\bar{r},t)\, dV \tag{B.3}$$

We define the energy velocity as

$$\bar{V}_e(t) = \frac{d}{dt}\left[\bar{L}/E \right] \tag{B.4}$$

It follows from Equation (1) that, if the dissipation is a linear function of w, for example, $P = w/\tau$, and the disturbance is bounded in spatial extent, then our definition reduces to a ratio of the spatial average energy flux to energy,

$$\bar{V}_e(t) = \frac{1}{E} \int_{Volume} \bar{S}(\bar{r},t)\, dV \tag{B.5}$$

Note that this is true, even for $\tau < 0$, an unstable system.

If the system is linear, homogeneous, and time-invariant, more can be said. We will restrict our notation to one spatial dimension; the generalization presents no problem. Let the physical variables in the system be denoted $Y_n(z,t)$, with

$$Y_n(z,0) = \int_{-\infty}^{\infty} \underline{Y}_n(k)\, e^{-jkz} \frac{dk}{2\pi} \tag{B.6}$$

and $\underline{Y}_n(-k) = \underline{Y}_n^*(k)$ since all variables are real.

If $\omega = \omega(k)$ has more than one branch, then we will assume that the equations describing the system have been used to separate the solutions on different branches of the dispersion relation. (We usually are interested in one branch.) Then, we treat each such solution as a separate $Y_n(z,t)$ and write

$$Y_n(z,t) = \int_{-\infty}^{\infty} \underline{Y}_n(k)\, e^{j[\omega_n(k)t - kz]} \frac{dk}{2\pi} \qquad \text{(B.7)}$$

It is often the case that both the energy density $w(z,t)$ and the dissipation $P(z,t)$ are additive. We may then write

$$w(z,t) = \sum_n a_n Y_n^2(z,t) \qquad a_n > 0$$

$$P(z,t) = \sum \frac{a_n}{\tau_n} Y_n^2(z,t) \qquad \text{(B.8)}$$

If only one branch is excited, the above will, of course, be true in any linear case.

Using Parseval's theorem, we may rewrite Equation (3) in terms of integrals over k instead of over z. By (1) we equate dE/dt to the total dissipation and conclude that

$$\frac{1}{\tau_n} = j\left[\omega_n(k) + \omega_n(-k)\right] \qquad \text{(B.9)}$$

and rewrite

$$E(t) = \sum_n a_n \int_{-\infty}^{\infty} |\underline{Y}_n(k)|^2 e^{-t/\tau_n} \frac{dk}{2\pi} \qquad \text{(B.10)}$$

Note now that $\underline{Y}_n(k)e^{j\omega_n(k)t}$ is the Fourier transform of $Y_n(z,t)$. Writing the transform out and differentiating both sides with respect to k gives the resulting inverse transform

$$z\, Y_n(z,t) = \int_{-\infty}^{\infty} \left[\frac{1}{j} \frac{\partial \underline{Y}_n(k)}{\partial k} + t \frac{\partial \omega_n(k)}{\partial k} Y_n(k) \right] e^{j[\omega_n(k)t - kz]} \frac{dk}{2\pi} \qquad \text{(B.11)}$$

This relation may be used to rewrite Eqs. (2) and (4) as an integral over k. With a little algebra, the terms involving $\partial \underline{Y}_n / \partial k$ cancel, giving the result

$$V_e(t) = \frac{1}{E(t)} \sum_n a_n \int_{-\infty}^{\infty} \frac{\partial \omega_n}{\partial k} |Y_n(k)|^2 e^{-t/\tau_n} \frac{dk}{2\pi} \qquad \text{(B.12)}$$

Note that the imaginary part of $\omega_n(k)$ will make no contribution to V_e, since it is an odd function of k.

If the excitation is on only one branch, or if $\tau_i = \tau$ for all i, then our expression for V_e is time-independent and reduces to a weighted average of the group velocity. Thus, the quasi-monochromatic signal is a limiting case in which $\underline{Y}_n(k)$ is nonzero only in the neighborhood of k_o and the energy velocity is $V_e = \partial\omega/\partial k(k_o)$. The same result is obtained if the group velocity is constant over the range of nonzero $\underline{Y}_n(k)$.

An illustrative example of this approach in a simple context is called for at this point. Consider a transmission line with series and parallel loss per unit length, represented as in Fig. B.1.

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} - Ri$$
$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} - Gv \qquad \text{(B.13)}$$

$$-k^2 = -(j\omega L+R)(j\omega C+G) \qquad \text{(B.14)}$$

$$D(\omega,k) = \omega^2 LC - j\omega(LG+RC) - (k^2+RG) = 0$$

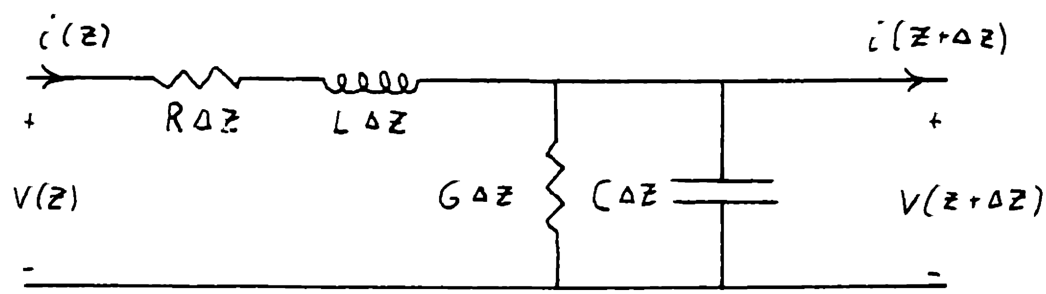
$$\text{(B.15)}$$

$$\omega = \omega_R + j\omega_I$$
$$\omega_R = \pm\sqrt{\frac{k^2}{LC} - \frac{1}{4}\left(\frac{G}{C} - \frac{R}{L}\right)^2}$$
$$\omega_I = \frac{1}{2}\left[\frac{G}{C} + \frac{R}{L}\right] \qquad \text{(B.16)}$$

Choose $W(z,x) = \frac{1}{2} L\, i^2 + \frac{1}{2} C\, V^2$. Assume that we excite the line at one point and observe it some distance away,

Figure B.1

Transmission Line Schematic



so that causlity makes it clear that the currents and voltages split into two modes, only one of which is observed. Thus $w=w(k)$ is unambiguous; we may choose $\frac{\partial \omega_R}{\partial K} > 0$. We can identify $V = Y_1$, $i = Y_2$, $\frac{C}{2} = a_1$, $\frac{L}{2} = a_2$, $\frac{G}{C} + \frac{R}{L} = \frac{1}{\Upsilon}$ and use (12) to find

$$V_e = \frac{\int_{-\infty}^{\infty} \frac{\partial \omega_R}{\partial k} \left[\frac{1}{2} C \,|\underline{V}(k)|^2 + \frac{1}{2} L \,|\underline{I}(k)|^2 \right] \frac{dk}{2\pi}}{\int_{-\infty}^{\infty} \left[\frac{1}{2} C \,|\underline{V}(k)|^2 + \frac{1}{2} L \,|\underline{I}(k)|^2 \right] \frac{dk}{2\pi}}$$

where the underbarred quantities are the Fourier transforms of the initial waveform. Clearly, this is independent of signal when $\frac{\partial \omega_R}{\partial k}$ is independent of k, or $\frac{G}{C} = \frac{R}{L}$, the dispersionless line. We have not learned anything new by this approach, but we have a basic, simple derivation of group velocity in one case, and a way of calculating the energy velocity for a dispersive line in the other.

We might also explore the possibility of redefining (2) as

$$L_j(t) = \int_{-\infty}^{\infty} z^j \, w(z,t) \, dz \qquad \text{(B.17)}$$

and using a form like Equation (4) to define V_{ej}. We find that this involves derivatives $\partial^m\omega/\partial k^m$ up to $m \leq j$. Thus, the series of values of V_{ej} gives information which, for quasi-monochromatic signals, is the equivalent of a Taylor series expansion of $\omega = \omega(k)$. We therefore see that, knowing all such velocities V_{ej} allows reconstruction of the Green's function of the system, as long as $\omega(k)$ is analytic.

Thus our definition, Equation (4), of energy velocity, which in general involves all of the modes of the system, has all the properties we would like it to have. It reduces to other standard definitions in simple cases and can be extended to other moments of the system if desired. More important, it has a firm intuitive basis, and so may serve as a teaching aid in interpreting the signal velocity concept.

Appendix C

The Diffusion Equation

C.1 Analysis

The Lyapunov techniques of the previous chapters can also be applied to processes governed by the diffusion equation. Such a system might become unstable if a growth term, analogous to the electric field term in the spring problem, appears in the dynamics. Suppose, for instance, that the temperature of a long, thin strip of material is fixed at both ends. Let there be an electric current flowing along the strip, heating it to some equilibrium temperature distribution. Now, if an increase in temperature caused a local increase in electrical resistivity, thus resulting in more heat locally generated, then the linearized dynamics would be described by

$$C \frac{\partial \xi}{\partial t} = \gamma \frac{\partial^2 \xi}{\partial z^2} + P\xi \tag{C.1.1}$$

where C is the heat capacity, ξ the temperature variation, γ the thermal conductivity, and P the parameter relating local temperature variation to local heating variation. (This is just the string equation without an inertia term.) Boundary conditions would be for this case $\xi(z=o) = \xi(z=L) = o$.

For the standard linear analysis, we take

$$\xi(z,t) = \sum_{m=1}^{\infty} a_m(t)\, \xi_m(z) \tag{C.1.2}$$

where $\xi_m(z) = \sin K_m z \quad , \quad K_m = \frac{m\pi}{L}$

and $$a_m(t) = C_m\, e^{s_m t} \tag{C.1.3}$$

a_m and C_m purely real.

This gives a dispersion relation

$$c\, s_m = -\gamma K_m^2 + P \tag{C.1.4}$$

with stability determined by the lowest mode,

$$R_e(s_m) < 0 \Rightarrow \qquad \gamma \left(\frac{\pi}{L}\right)^2 > P \tag{C.1.5}$$

An "energy" formulation of the problem can be achieved, as was done previously, by multiplying equation 1 by $\frac{\partial \xi}{\partial t}$ and integrating z = o to L, to give

$$\frac{dE_A}{dt} = -B_A \qquad E_A = \int_0^L \left[\tfrac{1}{2}\gamma\left(\frac{\partial \xi}{\partial z}\right)^2 - \tfrac{1}{2} P \xi^2\right] dz$$
$$B_A = \int_0^L c \left(\frac{\partial \xi}{\partial t}\right)^2 dz \tag{C.1.6}$$

Since $B_A \geq 0$ for all possible ξ, E_A is a Lyapunov function for the system provided that it is positive definite for all possible $\xi(t,z) \neq 0$. This is equivalent to the condition for stability given above in equation 5. The following lemma proves this:

Lemma: If $\gamma(\frac{\pi}{L})^2 > P$, and $\xi(o) = \xi(L) = o$, then

$$E_A = \frac{1}{2}\int_0^L \gamma\left(\frac{\partial \xi}{\partial z}\right)^2 - P\xi^2 \, dz > 0 \qquad \text{for all } \xi \neq o.$$

Proof:

Let equation 2 be used to form a Fourier series for $\xi(t,z)$. Since the terms are mutually orthogonal over the interval [o, L], we may write

$$E_A = \sum_{m=1}^{\infty} \frac{1}{2}\left[\gamma K_m^2 - P\right] a_m^2 \int_0^L \sin^2 K_m z \, dz$$

Clearly this will be positive definite if $\gamma(\frac{\pi}{L})^2 > P$.

We may modify the dynamics by using a feedback heat source $F(t,z)$, perhaps in the form of N feedback stations

$$F(z,t) = -\sum_{n=1}^{N} B_n(z) F_n(D_n(t)) \tag{C.1.7}$$

with sensor dignal

$$D_n(t) = \int_0^L A_n(z)\,\xi(z,t)\,dz = \sum_m A_{nm}\, a_m(t) \tag{C.1.8}$$

Drawing on the previous chapter, we know that best results will be obtained if $B_n(z) = A_n(z)$. Letting $F_n(D_n(t)) = \dfrac{\partial U_n}{\partial D_n}$, we may add the feedback term to the right-hand side of equation 1 and proceed with the energy formulation. This gives $\frac{dE}{dt} = -B$

$$E = E_A + U$$

$$U = \sum_{n=1}^{N} U_n(t) \qquad (C.1.9)$$

If $F_n(D_n)$ is linear with $F_n(D_n) = F_n D_n$, then

$$U_n = \frac{1}{2} F_n D_n^2 \qquad (C.1.10)$$

while if $F_n(D_n)$ is bang-bang with $F_n(D_n) = F_n \frac{D_n}{|D_n|}$

then

$$U_n = F_n |D_n(t)| \qquad (C.1.11)$$

Now, let's interpret our results. First, under what conditions will a "hot spot" develop in the system? This corresponds to an instability in the linearized model, which would grow into some nonlinear final development. Hot spots will occur spontaneously if the system is longer than a critical length $L_c = \pi \sqrt{\frac{\gamma}{P}}$.

Second, how can feedback be used to avoid such instabilities if the system is longer than L_c? This means using $F_n(t)$ to insure that $E > o$ for all possible non-zero $\xi(t,z)$. This is precisely the problem of the last chapter. The results are as follows:

Let M be the largest number m such that $\gamma K_m^2 - P < o$. Then the minimum number N of feedback stations needed to stabilize such a system is M.

The best arrangement of forcing elements, (which in this case might be heaters operated at some bias power) is such that $B_n(z) = A_n(z)$. That is, the heat input has a spatial distribution equal to the sensor distribution. If the sensor is a temperature-sensitive wire resistor and the heater is a wire filament, then they should be interwound.

The best spatial distribution of the two, $A_n(z)$, is that which has maximum coupling to the unstable ($m \leq M$) modes of the system, and minimum coupling to the stable ones ($m > M$). This coupling is measured by the matrix of coupling coefficients.

$$A_{nm} = \int_0^L A_n(z)\, \mathcal{E}_m(z)\, dz, \quad \begin{matrix} n = 1 \text{ to } N \\ m = 1 \text{ to } \infty \end{matrix} \qquad \text{(C.1.12)}$$

We wish to minimize all values of A_{nm} such that $M > M$, and maximize det $\bar{\bar{A}}_M$ where $\bar{\bar{A}}_M$ is the square M x M matrix of coupling coefficents A_{nm}, $1 \leq n,m \leq M$. Also, to minimize the excursions of the system from equilibrium, bang-bang control should be employed.

Once such a control strategy is chosen, we must test two things about it: First, does it result in stability for arbitrarily small perturbations? Second, if it does, what range of perturbations can be controlled?

To answer the first question, the null stability test previously described must be used. To answer the second, the critical value E_o must be calculated using feedback amplitudes F_n, such that all perturbations $\xi(t = o, z)$ lying inside the state-space boundary $E(\xi) = E_o$ will be bounded. The given formulas remain correct.

C.2 Special Properties

It is interesting to note at this point several properties peculiar to the diffusion equation problem. First, since C, the heat capacity is positive, the term B_A does not vanish for any trajectory of $\xi(t,z)$ in state space. Thus the system is asymptotically stable if it is stable at all.

Second, it is possible to interchange the roles of E_A and B_A in our Lyapunov formulation of the problem. Taking equation 1 and multiplying by ξ instead of $\dot{\xi}$, we get

$$\frac{d}{dt}\left[\tfrac{1}{2} C \xi^2\right] = -\gamma\left(\frac{\partial \xi}{\partial z}\right)^2 + P\xi^2 + \gamma \frac{d}{dz}\left[\xi \frac{\partial \xi}{\partial z}\right] - \sum_{n=1}^{N} \xi\, B_n(z)\, F_n(D_n(t)) \qquad \text{(C.2.1)}$$

Integrating over z, we get

$$\frac{dE_B}{dt} = -B_B - B_F \qquad E_B = \int_0^L \tfrac{1}{2} C \xi^2 \, dz$$

$$B_B = \int_0^L \gamma\left(\frac{\partial \xi}{\partial z}\right)^2 - P\xi^2 \, dz$$

$$B_F = \sum_{n=1}^{N} \int_0^L \xi(z,t)\, B_n(z)\, F_n(D_n(t))\, dz \qquad \text{(C.2.2)}$$

If $B_n(z) = A_n(z)$ and $F_n(t)$ is $F_n \frac{D_n}{|D_n|}$ with D_n as given above, then

$$B_F = \sum_{n=1}^{N} F_n |D_n| \qquad (C.2.3)$$

Thus the problem has been shifted from one of making E_A positive definite to an equivalent problem of making $(B_B + B_F)$ positive definite. Either is equally tractable for problems in which $F_n(t)$ depends only on $\xi(t,z)$. However, the first formulation is helpful in interpreting dependence of $F_n(t)$ on $\frac{\partial \xi}{\partial t}$, since such a term becomes part of the decay term B. The second formulation would be useful if the feedback $F_n(t)$ depended on the past history of the perturbation $\int_0^t \xi(t,z)dt$, since such a term would be incorporated into E_B.

It is also interesting to note that the existence of these two ways of formulating the energy problem are dependent for their equivalence on the following identity, which holds for any equation of the form of equation 1 including linear, but not nonlinear feedback:

$$\int_0^L \frac{1}{2} c \frac{\partial^2 \xi}{\partial t^2} \xi \, dz = \int_0^L c \left(\frac{\partial \xi}{\partial t}\right)^2 dz \qquad (C.2.4)$$

C.3 Three-Dimensional Formulation

The thermal conduction problem can be easily generalized to three dimensions. Certainly the boundary condition $\xi = 0$ on all faces of a volume can be achieved with appropriate heat sinks. A material with different thermal conductivities along the orthogonal coordinates is governed by an equation

$$C \frac{\partial \xi}{\partial t} = \gamma_x \frac{\partial^2 \xi}{\partial x^2} + \gamma_y \frac{\partial^2 \xi}{\partial y^2} + \gamma_z \frac{\partial^2 \xi}{\partial z^2} + P \xi \qquad \text{(C.3.1)}$$

The condition for stability of all modes, of the form

$$\xi(t, x, y, z) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} a_{k\ell m}(t) \sin K_k x \sin K_\ell y \sin K_m z$$

where

$$K_k = \frac{k\pi}{L_x} \qquad K_\ell = \frac{\ell \pi}{L_y} \qquad K_m = \frac{m\pi}{L_z} \qquad \text{(C.3.2)}$$

will be determined by the condition

$$\gamma_x \left(\frac{\pi}{L_x}\right)^2 + \gamma_y \left(\frac{\pi}{L_y}\right)^2 + \gamma_z \left(\frac{\pi}{L_z}\right)^2 - P > 0 \qquad \text{(C.3.3)}$$

Violation of this condition might be remedied by using feedback in the form of N heat sources with heat input $-B_n(x,y,z)\ F_n(t)$, where

$$F_n(t) = F_n \frac{D_n}{D_n} \qquad \text{and} \quad D_n(t) = \int_0^{L_x}\int_0^{L_y}\int_0^{L_z} A_n(x,y,z)\,\mathcal{E}(t,x,y,z)\,dx\,dy\,dz \tag{C.3.4}$$

The analysis of such a system proceeds just as did the one-dimensional case, and so will not be described further.

Appendix D

Stability Analysis of a Current-Carrying Plasma Column in a Strong Axial Magnetic Field

I. Introduction

In previous chapters we have described stability analysis techniques for bang-bang feedback to perfectly conducting fluid models of plasmas. We required only a knowledge of the eigenvalues, or growth rates, of the normal modes of the system without feedback.

In this appendix, we will examine the growth rates of plasmas with various current distributions in order to get some idea of the magnitude of the feedback problem involved. We consider a cylindrical geometry of radius a with periodicity in z to represent a toroidal fusion machine. We assume a perfectly conducting shell at radius $R_W > a$, and assume all volume currents flow in the θ-z-directions. All initial value variables are functions only of r, so that by symmetry all normal modes will have dependence:

$$\bar{\xi} = \bar{\xi}(r)\, e^{j(\omega t - m\theta - Kz)} \qquad \text{(D.1.1)}$$

Thus over the unperturbed boundary S of the cylinder of plasma, the dependence of normal displacements for each mode is known.

Unless specified otherwise, all initial values of H, P, ρ, etc. are constant, and $\bar{v}_o = o$. Also, assume $m \geq o$ by symmetry.

II. Transfer relations for a perfectly conducting shell of fluid, $H_\theta = o$, H_z = CONSTANT, $\nabla \cdot \bar{v} = o$, $b < r < a$.

We assume that, inside the fluid,

$$H_{\theta i} = 0 \qquad H_{z_i} = \mathcal{H}_i H_{\theta o} \tag{D.2.1}$$

Since there are no zero-order currents inside, our force equation is, to linear terms,

$$\rho \frac{\partial^2 \bar{\xi}}{\partial t^2} = -\nabla p + \bar{j} \times \bar{B}_i \tag{D.2.2}$$

where
$$\bar{B}_i = \mu_o \hat{z} \mathcal{H}_i H_{\theta o}$$
$$\bar{j} = \nabla \times \bar{h}_i$$
$$\frac{\partial \mu_o \bar{h}_i}{\partial t} = -\nabla \times E_i$$
$$E_i + \bar{v} \times \bar{B}_i = 0$$

giving
$$\bar{h}_i = \nabla \times (\bar{\xi} \times \bar{H}_i) = -jk H_{z_i} \bar{\xi} \tag{D.2.3}$$

Thus
$$\bar{j} \times \bar{B}_i = \mu_o [\nabla \times \nabla \times \bar{\xi} \times \bar{H}_i] \times \bar{H}_i$$
$$= \nabla \left[-\mu_o \left(\tfrac{1}{2} H_i^2 + \bar{h}_i \cdot \bar{H}_i\right)\right] \tag{D.2.4}$$
$$+ \mu_o (\bar{H}_i \cdot \nabla) \bar{H}_i + \mu_o (\bar{H}_i \cdot \nabla) \bar{h}_i + \mu_o (\bar{h}_i \cdot \nabla) \bar{H}_i$$

For our geometry and $\bar{H}_i$, $\nabla(H_i^2) = 0$, $\bar{h}_i \cdot \nabla \bar{H}_i = 0$

$$(\bar{H}_i \cdot \nabla)\bar{h}_i = -K^2 \mathcal{H}_i^2 H_{\theta_0}^2 \bar{\xi}$$

(D.2.5)

Thus we rewrite (2.2) as

$$\left[-\rho\omega^2 + \mu_0 (K \mathcal{H}_i H_{\theta_0})^2\right]\bar{\xi} = -\nabla p^* \qquad (D.2.6)$$

where

$$p^* = p + \tfrac{1}{2}\mu_0 H^2 \qquad (D.2.7)$$

Using $\nabla \cdot \bar{v} = 0$, we take the divergence to get

$$\nabla^2 p^* = 0 \qquad (D.2.8)$$

From the r-component of (2.6), we have

$$\frac{\mu_0 H_{\theta_0}^2}{a^2}\left(Y_0^2 + \phi_i^2\right)\xi_r = -\frac{\partial p^*}{\partial r} \qquad (D.2.9)$$

where

$$Y_0^2 = \frac{-\rho\omega^2 a^2}{\mu_0 H_{\theta_0}^2} \qquad \phi_i^2 = (Ka\,\mathcal{H}_i)^2$$

Thus we choose to express solutions as

$$\xi_r(r) = \xi_{r_a}\left[\frac{J_m'(jKr)\,H_m'(jKb) - J_m'(jKb)\,H_m'(jKr)}{J_m'(jKa)\,H_m'(jKb) - J_m'(jKb)\,H_m'(jKr)}\right]$$

$$+\ \xi_{r_b}\left[\frac{J_m'(jKa)\,H_m'(jKr) - J_m'(jKr)\,H_m'(jKa)}{J_m'(jKa)\,H_m'(jKb) - J_m'(jKb)\,H_m'(jKa)}\right]$$

(D.2.10)

Thus, evaluating the pressure p^* at b and a, we may write transfer relations.

$$\begin{bmatrix} p_a^* \\ p_b^* \end{bmatrix} = -\left(Y_o^2 + \phi_i^2\right)\left(\frac{\mu_o H_{\theta_o}}{a^2}\right)^2 \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \xi_{r_a} \\ \xi_{r_b} \end{bmatrix}$$

(D.2.11)

where

$$A_{11} = \frac{a}{jKa} \frac{J_m(jKa)\,H_m'(jKb) - J_m'(jKb)\,H_m(jKa)}{Den}$$

$$A_{12} = \frac{a}{jKa} \frac{J_m'(jKa)\,H_m(jKa) - J_m(jKa)\,H_m'(jKa)}{Den}$$

$$A_{21} = \frac{b}{jKb} \frac{J_m(jKb)\,H_m'(jKb) - J_m'(jKb)\,H_m(jKb)}{Den}$$

$$A_{22} = \frac{b}{jKb} \frac{J_m'(jKa)\,H_m(jKb) - J_m(jKb)\,H_m'(jKa)}{Den}$$

$$Den = J_m'(jKa)\,H_m(jKb) - J_m'(jKb)\,H_m'(jKa)$$

Special cases:

$$b = 0 \Rightarrow p_a^* = -\left(\frac{\mu_o H_{\theta_o}}{a^2}\right)^2 \left(Y_o^2 + \phi_i^2\right)\left(\frac{a}{jKa}\right) \frac{J_m(jKa)}{J_m'(jKa)}\, \xi_{r_a}$$

(D.2.12)

$b = 0$, $Ka \ll 1$, $m \neq 0 \Rightarrow$

$$p_a^* = -\left(\frac{\mu_o H_{\theta_o}}{a^2}\right)^2 \left(Y_o^2 + \phi_i^2\right) \frac{a}{m}\, \xi_{r_a} \qquad \text{(D.2.13)}$$

$m \neq 0, \quad b \neq 0, \quad Ka << 1 \Rightarrow$

$$A_{11} = \frac{a}{m}\left[\frac{1+\left(\frac{b}{a}\right)^{2m}}{1-\left(\frac{b}{a}\right)^{2m}}\right] \tag{D.2.14}$$

$$A_{12} = -\frac{a}{m}\left(\frac{b}{a}\right)^{m}\left[\frac{2}{1-\left(\frac{b}{a}\right)^{2m}}\right]$$

$$A_{21} = \frac{b}{m}\left(\frac{b}{a}\right)^{m}\left[\frac{2}{1-\left(\frac{b}{a}\right)^{2m}}\right]$$

$$A_{22} = -\frac{b}{m}\left[\frac{1+\left(\frac{b}{a}\right)^{2m}}{1-\left(\frac{b}{a}\right)^{2m}}\right]$$

III. Transfer relations for Uniform Current Density

$$H_{\theta_i} = H_{\theta_0} \frac{\alpha r}{a} \qquad J_0 = \frac{2\alpha H_{\theta_0}}{a}$$

$$H_{z_i} = H_{\theta_0} \mathcal{H}_i \qquad K_b = H_{\theta_0} \alpha \, b/a$$

$$\nabla \cdot \bar{v} = 0 \qquad b \le r \le a \tag{D.3.1}$$

[Note: α = o reduces to previous section.]

Our force equation is now

$$\rho \frac{\partial^2 \bar{\xi}}{\partial t^2} = -\nabla p + \bar{J}_0 \times \bar{b}_i + \bar{j} \times \bar{B}_i \tag{D.3.2}$$

Once again we define $p^* = p + \frac{1}{2} \mu_0 H^2$ and rewrite

$$-\rho \omega^2 \bar{\xi} = -\nabla p^* + \mu_0 \left[\bar{h}_i \cdot \nabla \bar{H}_i + \bar{H}_i \cdot \nabla \bar{h}_i \right] \tag{D.3.3}$$

where
$$\frac{\partial p_0}{\partial r} = -\frac{2\mu_0 \alpha H_{\theta_0}^2 r}{a^2}$$
(D.3.4)

or
$$\frac{\partial p_0}{\partial r} = \frac{-\mu_0 \alpha H_{\theta_0}^2 r}{a^2}$$

for equilibrium. Now we note that

$$\bar{h}_i = \nabla \times (\bar{\xi} \times \bar{H}_i)$$
$$= -j H_{\theta_0} \left(\frac{\alpha m}{a} + k \mathcal{H}_i \right) \bar{\xi} = -j \phi_i H_{\theta_0} \bar{\xi}/a \tag{D.3.5}$$

where
$$\phi_i = \alpha m + k a \mathcal{H}_i \tag{D.3.6}$$

$$\mu_0 \left[\bar{h}_i \cdot \nabla \bar{H}_i + \bar{H}_i \cdot \nabla \bar{h}_i \right] =$$

$$\frac{\mu_0 H_{\theta_0}^2}{a^2} \left[-\phi_i^2 \bar{\xi} + 2j\alpha \phi_i \xi_\theta \hat{r} - 2j\alpha\phi_i \xi_r \hat{\theta} \right] \quad (D.3.7)$$

define $\beta = \dfrac{2\alpha\phi_i}{Y_0^2 + \phi_i^2}$ $\qquad Y_0^2 = \dfrac{-\rho\omega^2 a^2}{\mu_0 H_{\theta_0}^2}$ (D.3.8)

and rewrite

$$\left(\frac{\mu_0 H_{\theta_0}^2}{a^2}\right)(Y_0^2 + \phi_i^2)\,\bar{\xi} = -\nabla p^* + j\beta (Y_0^2 + \phi_i^2)(\xi_\theta \hat{r} - \xi_r \hat{\theta})\left(\frac{\mu_0 H_{\theta_0}^2}{a^2}\right)$$

(D.3.9)

$$\left(\frac{\mu_0 H_{\theta_0}^2}{a^2}\right)(Y_0^2 + \phi_i^2) \begin{bmatrix} 1 & j\beta & 0 \\ -j\beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_r \\ \xi_\theta \\ \xi_z \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial p^*}{\partial r} \\ j\dfrac{m}{r} p^* \\ jK p^* \end{bmatrix}$$

(D.3.10)

Solve the set of equations for the displacement to get

$$\xi_r = \frac{1}{D}\left[-\frac{\partial p^*}{\partial r} - \frac{\beta m}{r} p^* \right]$$

$$\xi_\theta = \frac{1}{D}\left[j\beta \frac{\partial p^*}{\partial r} + j\frac{m}{r} p^* \right]$$

$$\xi_z = \frac{1}{D}\left[(1-\beta^2)\, jK p^* \right]$$

(D.3.11)

where $D = \left(\dfrac{\mu_0 H_{\theta_0}^2}{a^2}\right)(1-\beta^2)(Y_0^2 + \phi_i^2)$ = constant

Now substitute into the equation of state $\nabla \cdot \bar{v} = 0$ to get

$$r\frac{\partial}{\partial r}\left[r\frac{\partial p^*}{\partial r}\right] - p^*\left[K^2r^2(1-B^2)+m^2\right] = 0$$

(D.3.12)

We recognize that this is just $\nabla^2 p^* = o$ in cylindrical coordinates where we replace K^2 by $K^2(1-B^2)$ (a simple stretching of the z-axis by $\frac{1}{\sqrt{1-B^2}}$). Solutions, then, are of the form of Bessel functions of imaginary agreement $j\sqrt{1-B^2}\,Kr = jK'r$.

$$p^*(r) = p_a^*\left[\frac{J_m(jK'r)H_m(jK'b) - J_m(jK'b)H_m(jK'r)}{Den}\right]$$

$$+ p_b^*\left[\frac{J_m(jK'a)H_m(jK'r) - J_m(jK'r)H_m(jK'a)}{Den}\right]$$

$$Den = J_m(jK'a)H_m(jK'b) - J_m(jK'b)\,H_m(jK'a)$$

(D.3.13)

$$\mathcal{E}_r(r) = \frac{-1}{\left(\frac{\mu_o H_{\theta_0}{}^2}{a^2}\right)(Den)(1-B^2)(Y_o{}^2+\phi_i{}^2)} \cdot$$

$$\Big[\, p_a^*\,\Big\{ jK'J_m'(jK'r)H_m(jK'b) - jK'J_m(jK'b)H_m'(jK'r)$$

$$+ \frac{Bm}{r}J_m(jK'r)H_m(jK'b) - \frac{Bm}{r}J_m(jK'b)H_m(jK'r)\Big\}$$

$$+ p_b^*\,\Big\{ jK'J_m(jK'a)H_m'(jK'r) - jK'J_m'(jK'r)H_m(jK'a)$$

$$+ \frac{Bm}{r}J_m(jK'a)H_m(jK'r) - \frac{Bm}{r}J_m(jK'r)H_m(jK'a)\Big\}\Big]$$

(D.3.14)

Thus in matrix form,

$$\begin{bmatrix} \xi_{r_a} \\ \xi_{r_b} \end{bmatrix} = \frac{-1}{\left(\frac{\mu_0 H_{\theta_0}^2}{a^2}\right)(1-\beta^2)(Y_0^2+\phi_i^2)} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} p_a^* \\ p_b^* \end{bmatrix} \quad (D.\ 3.15)$$

$$A_{11} = \frac{1}{a \cdot Den} \cdot \Big[\big(jK'a\, J_m'(jK'a) + \beta m\, J_m(jK'a)\big) H_m(jK'b) - \big(jK'a\, H_m'(jK'a) + \beta m\, H_m(jK'a)\big) J_m(jK'b) \Big]$$

$$A_{12} = \frac{1}{a \cdot Den} \cdot \Big[jK'a\, H_m'(jK'a)\, J_m(jK'a) - jK'a\, J_m'(jK'a)\, H_m(jK'a) \Big]$$

$$A_{21} = \frac{1}{b \cdot Den} \cdot \Big[jK'b\, J_m'(jK'b)\, H_m(jK'b) - jK'b\, H_m'(jK'b)\, J_m(jK'b) \Big]$$

$$A_{22} = \frac{1}{b \cdot Den} \cdot \Big[\big(jK'b\, H_m'(jK'b) + \beta m\, H_m(jK'b)\big) J_m(jK'a) - \big(jK'b\, J_m'(jK'b) + \beta m\, J_m(jK'b)\big) H_m(jK'a) \Big]$$

$$Den = J_m(jK'a)\, H_m(jK'b) - J_m(jK'b)\, H_m(jK'a)$$

$$K' = K\sqrt{1-\beta^2}$$

$$Y_0^2 = -\rho\omega^2 a^2 / \mu_0 H_{\theta_0}^2$$

$$\phi_i^2 = (m\alpha + K a \eta_i)^2$$

$$\beta = \frac{2\alpha\phi_i}{Y_0^2+\phi_i^2}$$

Special cases

b = o $\Rightarrow$

$$\xi_{r_a} = \frac{-1}{a\left(\frac{\mu_o H_{\theta_0}^2}{a^2}\right)\left(\gamma_o^2+\phi_i^2\right)\left(1-\beta^2\right)} \cdot \left[\frac{jK'a\, J_m'(jK'a)}{J_m(jK'a)} + \beta_m\right] \cdot p_a^* \tag{D.3.16}$$

b = o, m $\neq$ o, K'a << 1 $\Rightarrow$

$$\xi_{r_a} = \frac{-m}{a\left(\frac{\mu_o H_{\theta_0}^2}{a^2}\right)\left(\gamma_o^2+\phi_i^2\right)\left(1-\beta^2\right)} \cdot p_a^* \tag{D.3.17}$$

b $\neq$ o, m $\neq$ o, K'a<<1

$$A_{11} = \frac{m}{a}\left[\frac{1+\left(\frac{b}{a}\right)^{2m}}{1-\left(\frac{b}{a}\right)^{2m}} + \beta\right] \tag{D.3.18}$$

$$A_{12} = \frac{m}{a}\left(\frac{b}{a}\right)^{m}\left[\frac{-2}{1-\left(\frac{b}{a}\right)^{2m}}\right]$$

$$A_{21} = \frac{m}{b}\left(\frac{b}{a}\right)^{2m}\left[\frac{2}{1-\left(\frac{b}{a}\right)^{2m}}\right]$$

$$A_{22} = \frac{m}{b}\left[-\frac{1+\left(\frac{b}{a}\right)^{2m}}{1-\left(\frac{b}{a}\right)^{2m}} + \beta\right]$$

IV. Transfer Relations for the Vacuum Region Including Surface

We assume a fixed, perfectly conducting shell at radius R_w. We wish to find the relation between displacements ξ_{ra} and pressure variations p^*_a. This means evaluating the effects of both surface and volume currents at the plasma boundary. We therefore assume external field

$$\bar{H}_o = H_{\theta_o}\left(\frac{a}{r}\hat{\theta} + \mathcal{H}_e\hat{z}\right) \tag{D.4.1}$$

and internal field

$$\bar{H}_i = H_{\theta_o}\left(\frac{\alpha r}{a}\hat{\theta} + \mathcal{H}_i\hat{z}\right) \qquad \frac{\mathcal{H}_i}{\mathcal{H}_e} = \epsilon$$

Also, we note that even if $Ka \ll 1$, it is quite possible in a Tokamak that $KaH_e \sim m$.

Since $J = o$ in the vacuum, $\bar{h} = -\nabla\Phi$, $\nabla^2\Phi = o$. At the plasma surface, $\bar{H}\cdot\bar{n} = o$, so by $\bar{n} = \hat{r} - \frac{1}{r}\frac{\partial\xi_r}{\partial\theta}\hat{\theta} - \frac{\partial\xi_r}{\partial z}\hat{z}$

$$\bar{n} = \bar{r} + j\,\xi_r\left(\frac{m}{r}\hat{\theta} + K\hat{z}\right) \tag{D.4.2}$$

we have $$\bar{h}_r + jH_{\theta_o}\xi_r\left(\frac{m}{r} + K\mathcal{H}_e\right) = 0 \text{ at } r = a \tag{D.4.3}$$

$$\bar{h}_r = 0 \text{ at } r = R_w \tag{D.4.4}$$

Thus

$$\Phi = jH_{\theta_o}\xi_r\left(\frac{m}{r} + K\mathcal{H}_e\right)\left(\frac{a}{jKa}\right)\left[\frac{J_m(jKr)H_m'(jKR_w) - J_m'(jKR_w)H_m(jKr)}{J_m'(jKa)H_m'(jKR_w) - J_m'(jKR_w)H_m'(jKa)}\right] \tag{D.4.5}$$

Magnetic pressure on the plasma is then to first order

$$\tau^M = \frac{1}{2}\mu_0 H^2\Big|_{r=a+\xi} = P_0 + \frac{\mu_0}{2}\frac{\partial}{\partial r}(H_0^2)\Big|_{r=a} \xi_r + \mu_0 (H_{\theta_0} h_{\theta_0} + H_{\theta_0} \mathcal{H}_e h_{z_0}) \qquad \text{(D.4.6)}$$

Omitting equilibrium terms

$$\tau^M = \frac{-\mu_0 H_{\theta_0}^2}{a} \xi_r + j\mu_0 H_{\theta_0} \left(\frac{m}{r} + K\mathcal{H}_e\right) \Phi(a)$$

$$= \frac{-\mu_0 H_{\theta_0}^2}{a} \xi_r + \mu_0 H_{\theta_0}^2 \left(\frac{m}{a} + K\mathcal{H}_e\right)^2 a\, C_0 \qquad \text{(D.4.7)}$$

$$C_0 = \frac{-1}{jKa}\left[\frac{J_m(jKa) H_m'(jKR_w) - J_m'(jKR_w) H_m(jKa)}{J_m'(jKa) H_m'(jKR_w) - J_m'(jKR_w) H_m'(jKa)}\right] > 0 \qquad \text{(D.4.8)}$$

Just inside the plasma surface, $p^*_s = \tau^M$

However, if there are volume currents, then

$$p_a^* + \frac{\mu_0}{2}\frac{\partial}{\partial r}(H_i^2)\Big|_{r=a} \xi_{r_a} + \bar{J}_0 \times \mu_0 \bar{H}_i \cdot \bar{n}\, \xi_r = p_s^* \qquad \text{(D.4.9)}$$

Thus

$$p_a^* = p_s^* + \frac{\mu_0 H_{\theta_0}^2 \alpha^2}{a} \xi_{ra} \qquad \text{(D.4.10)}$$

Thus the final result is

$$p_a^* = \frac{\mu_o H_{\theta_o}^2}{a}\left[\alpha^2 - 1 + C_o \phi_e^2\right] \xi_{r_a} \tag{D.4.11}$$

where

$$\phi_e = m + Ka\mathcal{H}_e$$

and C_o is as given in (4.8)

Special Cases :

$$R_w \to \infty \quad \Rightarrow$$

$$C_o = \frac{-H_m(jKa)}{jKa\,H_m'(jKa)} \geq 0 \tag{D.4.12}$$

$$R_w \to \infty, \; Ka \ll 1, \; m \neq 0 \Rightarrow$$

$$C_o = \frac{1}{m} \tag{D.4.13}$$

$$Ka \ll 1, \; m \neq 0 \quad \Rightarrow$$

$$C_o = \frac{1}{m}\left[\frac{1 + \left(\frac{a}{R_w}\right)^{2m}}{1 - \left(\frac{a}{R_w}\right)^{2m}}\right] \tag{D.4.14}$$

V. Eigenvalues for a Surface-Current Screw Pinch

Assume $\alpha = o$, $H_e \neq o$, $H_i \neq o$, $\nabla \cdot \bar{v} = o$, ρ and P_o constants, $b = o$.

From (2.12)

$$p_a^* = -\left(\frac{\mu_o H_{\theta o}^2}{a^2}\right)\left(Y_o^2 + \phi_i^2\right) a \, \frac{J_m(jKa)}{jKa\, J_m'(jKa)} \xi_{ra} \quad (D.5.1)$$

From (4.11)

$$p_a^* = \frac{\mu_o H_{\theta o}^2}{a}\left[-1 + C_o \phi_e^2\right] \xi_{ra} \quad (D.5.2)$$

Thus

$$Y_o^2 = -\phi_i^2 + \left[\frac{jKa\, J_m'(jKa)}{J_m(jKa)}\right]\left[1 - C_o \phi_e^2\right] \quad (D.5.3)$$

Since $y_o^2 < o$ represents stability, the quantities ϕ_i^2 and ϕ_e^2 represent stabilizing factors, while the term 1 in $[1 - C_o \phi_e^2]$ representing curveature of the external field is destabilizing. In particular, if for the interchange modes defined by $\phi_e = o$, the effect of that term dominates, then as shown by Melcher (1970) no amount of linear feedback will have any effect, and the plasma is unstable.

If we take the limit Ka<<1, m≠o, we have

$$Y_o^2 = -\phi_i^2 + m - \phi_e^2\left[\frac{1 + \left(\frac{a}{R_w}\right)^{2m}}{1 - \left(\frac{a}{R_w}\right)^{2m}}\right] \quad (D.5.4)$$

Substituting $\phi_i^2 = (Ka\mathcal{H}_i)^2 \qquad \phi_e^2 = (m + Ka\mathcal{H}_e)^2$

$$\gamma_o^2 = -\left[(Ka\mathcal{H}_i)^2 - m + \frac{(m + Ka\mathcal{H}_e)^2 \left(1 + \left(\frac{a}{R_w}\right)^{2m}\right)}{1 - \left(\frac{a}{R_w}\right)^{2m}} \right] \tag{D.5.5}$$

Taking $H_i = \varepsilon H_e$, $R_W >> a$

$$\gamma_o^2 = -\left[m(m-1) + 2mKa\mathcal{H}_e + (1+\epsilon^2)(Ka\mathcal{H}_e)^2 \right] \tag{D.5.6}$$

This is the Kruskal-Shafranov stability criterion when m=1:

$$\gamma_o^2 = -2Ka\mathcal{H}_e - (1+\epsilon^2)(Ka\mathcal{H}_e)^2 \tag{D.5.7}$$

with maximum $\gamma_o^2 = 1/1+\epsilon^2$

(D.5.8)

occurring at $Ka\mathcal{H}_e = -1/1+\epsilon^2$

for $K = \frac{n2\pi}{L}$ stability requires $q > \frac{2}{1+\epsilon^2}$

$$q = \frac{2\pi a \mathcal{H}_e}{L}$$

Consider the special case $H_e = H_i = o$.

Note that all modes m > 1 are stable for this approximation, mainly because they are sufficiently far from the interchange direction and so are stabilized by the magnetic fields. This matches the results of Tayler as well as our intuition's (remember that for m=o our expansion is not valid; $y_o^2 > o$).

To examine the effect of R_w consider $\varepsilon = o$ and note that the closer the conducting shell, the more stabilization due to $\phi_e \neq o$ occurs, but the interchange mode ($\phi_e = o$) are unaffected. (See 5.4)

Note that if $\varepsilon = o$, regardless of q there will always be an m such that $\phi_e = o$, unstable. For m=o, we may not expand the Bessel functions as before, and so must consider the more complex analysis of their behavior.

It is clear, however, that for $H_i = H_e = o$

$$\gamma_o^2 = j \frac{Ka\, J_m'(jKa)}{J_m(jKa)}$$ is positive

for all $K > o$. Hence the m = o mode can be stabilized only at the cost of $H_e \neq o$, which moves the interchange direction to higher m, or $H_i \neq o$ which stabilizes modes $K \neq o$ for large enough H_i.

VI. Eigenvalues for Uniform Current Density

$$b = 0 \qquad \nabla \cdot \bar{v} = 0 \qquad H_{\theta_i} = H_{\theta_0} \frac{\alpha r}{a}$$

from (3.16)

$$\xi_{ra} = \left[\frac{-1}{a\left(\frac{\mu_0 H_{\theta_0}^2}{a^2}\right)(1-\beta^2)(Y_0^2 + \phi_i^2)} \right] \left[\frac{jka\sqrt{1-\beta^2}\, J_m'(jka\sqrt{1-\beta^2})}{J_m(jka\sqrt{1-\beta^2})} \right] p_a^* \tag{D.6.1}$$

from (4.11)

$$p_a^* = \frac{\mu_0 H_{\theta_0}^2}{a} \left[\alpha^2 - 1 + C_0 \phi_e^2 \right] \xi_{ra} \tag{D.6.2}$$

Thus

$$Y_0^2 = -\phi_i^2 - \left[\frac{jka\sqrt{1-\beta^2}\, J_m'(jka\sqrt{1-\beta^2})}{J_m(jka\sqrt{1-\beta^2})} \right] \left[\frac{\alpha^2 - 1 + C_0 \phi_e^2}{1-\beta^2} \right] \tag{D.6.3}$$

Here we see the effect of the volume current in terms of α and β: as α increases, more current flows in the volume and less on the surface, and β also increases.

As α increases, the term from the surface curveature (α^2-1) decreases, going to zero as $\alpha = 1$ and the surface current vanishes. However, while this is happening the term $(1-B^2)$ gets smaller and may go negative, representing an instability in the volume of the plasma, where the current is.

To examine this more closely, consider the limit $(m \neq 0)$ $Ka >> 1$, $Ka\sqrt{1-B^2} >> 1$.

$$\gamma_o^2 = -\phi_i^2 - \frac{m}{1-\beta}\left[\alpha^2 - 1 + \frac{\phi_e^2\left(1+\left(\frac{a}{R_w}\right)^{2m}\right)}{m\left(1-\left(\frac{a}{R_w}\right)^{2m}\right)}\right] \quad \text{(D.6.4)}$$

$$1-\beta = \frac{\gamma_o^2 + \phi_i^2 - 2\alpha\phi_i}{\gamma_o^2 + \phi_i^2} \quad \text{(D.6.5)}$$

Substituting and solving for

$$\gamma_o^2 = m(1-\alpha^2) + 2\alpha\phi_i - \frac{\phi_e^2 + \phi_i^2}{1-\left(\frac{a}{R_W}\right)^{2m}} - \frac{\left(\frac{a}{R_w}\right)^{2m}\left(\phi_e^2 - \phi_i^2\right)}{1-\left(\frac{a}{R_w}\right)^{2m}}$$

(D.6.6)

Note that the two destabilizing term show the shift from external to internal factors as α varies from 0 to 1.

Consider the case in which there are no zero-order surface currents; $\alpha = 1$, $\varepsilon = 1$. Then $\phi^2_i = (m + K_a H_e)^2 = \phi_e^2$

$$\gamma_o^2 = 2\left[\phi_e - \frac{\phi_e^2}{1-\left(\frac{a}{R_w}\right)^{2m}}\right] \quad \text{(D.6.7)}$$

This is Shafranov's stability results for Kink modes.

If we define $q_s = \dfrac{2\pi a \mathcal{H}_o}{L} = \dfrac{H_{z_o}\, a}{H_{\theta_o}\left(L/2\pi\right)}$

$$\phi_e^2 = (m + nq_s)^2 \quad \text{(D.6.8)}$$

which matches his notation. If R >> a, this reduces to

$$Y_o^2 = 2\left(\phi_e - \phi_e^2\right) = 2\left[(m+nq) - (m+nq)^2\right] \tag{D.6.9}$$

For any given m this is positive for m <nq< m+1 and parabolic. Thus the m=1 modes can be stabilized by q>1 , but some higher-m modes are always unstable. An examination of the special case of the m=o mode must then be made separately.

Note that the maximum growth rate occurs at m+mq = 1/2 when $y^2_o = \frac{1}{2}$, the same as for α=o, ε=1.

For m=o, we must again consider the full Bessel function structure.

$$Y_o^2 = -(Ka\,q\,\Psi_i)^2 - \frac{j\,Ka\sqrt{1-\beta^2}\;J_o'\left(j\,Ka\sqrt{1-\beta^2}\right)}{(1-\beta^2)\;J_o\left(j\,Ka\sqrt{1-\beta^2}\right)} \tag{D.6.10}$$

Shafranov shows that, for R>>a, Ka<<1 one can determine the minimum value of H_1 assuming stability as a function of α. It varies from 1/2 at α=o to $4/x_1^2$, at α = 1 where x_1= 2.4 is the first zero of $J_o(x)$.

Going back to the expression (D.6.5) and substituting for ϕ_i, ϕ_e we get

$$\begin{aligned} Y_o^2 &= -2m(\alpha^2-1) + 2\alpha\,(m\alpha + Ka\,q\,\Psi_i) - (m\alpha + Ka\,q\,\Psi_i)^2 \\ &\quad - (m + Ka\,q\,\Psi_e)^2\, C_2 \\ C_2 &= \frac{1+\left(\frac{a}{R_w}\right)^{2m}}{1-\left(\frac{a}{R_w}\right)^{2m}} \geq 1 \end{aligned} \tag{D.6.11}$$

$$Y_0^2 = \alpha^2 m(m-1) + 2\alpha K a \mathcal{H}_i (1-m)$$
$$+ \left[m - (Ka\mathcal{H}_i)^2 - C_2 (m + Ka\mathcal{H}_i)^2 \right]$$

(D.6.12)

Notice that, for m=1, Y_o^2 is independent of α.

Let $\mathcal{H}_i = \epsilon \mathcal{H}_e$

for m=1 we have

$$Y_0^2 = 1 - C_2 - 2C_2 Ka\mathcal{H}_e - (C_2 + \epsilon^2)(Ka\mathcal{H}_e)^2$$

(D.6.13)

(Stable for K=o or for $H_e = H_i = o$.)

For stability for all K, we then require

$$-Ka\mathcal{H}_e \gtrless \frac{C_2 \pm \sqrt{\epsilon^2 + C_2(1-\epsilon^2)}}{C_2 + \epsilon^2}$$

(D.6.14)

If R>>a, $C_2 = 1$ and Ka $H_e < \frac{z}{1 + \varepsilon^2}$ or Ka $H_e > o$.

Remembering from Shafranov's notation ($K = \frac{2\pi n}{L}$)

$$n q = Ka\mathcal{H}_e$$

Hence, for m=1, we have for any α the stability requirement

$$q > \frac{2}{1-\epsilon^2}$$

(D.6.15)

For modes m < 1, stability depends upon α. However, we write

$$\gamma_0^2 = m(1-m)\left(\alpha + \frac{Ka\mathcal{H}_i}{m}\right)^2 + m\left[1-\left(\frac{Ka\mathcal{H}_i}{m}\right)^2 - m\,C_2\left(1+\frac{Ka\mathcal{H}_e}{m}\right)^2\right] \qquad \text{(D.6.16)}$$

Clearly, the worst-case $\alpha = \dfrac{-Ka\ H_i}{m}$ for $m \geq 1$ (the interchange direction inside the surface).
Then

$$\gamma_0^2 = \frac{1}{m}\left[m^2 - (nq\epsilon)^2 - C_2(m+nq)^2\right] \qquad \text{(D.6.17)}$$

Note stability for nq=o, since $C_2 \geq 1$

$$\gamma_0^2 = \frac{1}{m}\left[m^2(1-C_2) - 2C_2\,mnq - (C_2+\epsilon^2)(nq)^2\right] \qquad \text{(D.6.18)}$$

for stability, we require

$$-nq \geq \frac{C_2 \pm m\sqrt{C_2^2 + (1-C_2)(C_2+\epsilon_2)}}{C_2+\epsilon^2} \qquad \text{(D.6.19)}$$

for R>>a, $C_2 = 1$, this becomes $-nq > \dfrac{2\,m}{1+\varepsilon^2}$

or

$$-nq < o \qquad \text{(D.6.20)}$$

We may write also

$$Y_o^2 = -\Big[\big((\epsilon n q)^2 + nq\big) + \big(C_2(m+nq)^2 - (m+nq)\big) + \alpha(m-1)(nx + 2nq\epsilon)\Big] \qquad (D.6.21)$$

We may choose $nq \le \frac{1}{\varepsilon^2}$ or $\ge o$ by $|q| \ge \frac{1}{\varepsilon^2}$ but there is no way to make $m + n\, q = \phi_e$ avoid values between the limits $o \le \phi_e \le \frac{1}{C_2}$, because given q we can always find a set of integers (m,n) such that m nq takes on any desired value (except in the "rational surface" case, when q is a ratio of intēgers). However, since $m \ge 1$, $\phi_e = o$ implies $nq \ne o$, so the first term tends to add stability. The third term can be either stabilizing or destabilizing, depending on the size of α/ε . Thus, in order to stabilize all the modes of the plasma, one would want to choose α/ε very different from one. For instance, $\alpha \le 1$, $\varepsilon >> o$, tends to accomplish this, or $\varepsilon << o$, $\alpha >> o$. In general, this tends to make ϕ_i and ϕ_e very different, so that if ϕ_i becomes comparable to 1, then ϕ_e is larger and so $Y_o^2 < o$.

D.7 TOKAMAK Configurations

For realistic considerations of stability of TOKAMAK plasmas, we use the preceeding chapters as an approximate model. We let $L = 2\pi R$ where R is the major radius of the toroidal plasma. This allows direct borrowing of results as long as the effects of this curveature can be ignored.

Since there is no evidence of any equilibrium surface current in such plasmas, we choose $\alpha = \varepsilon = 1$. This gives

$$\phi_i = \phi_e = \phi = m + nq \tag{D.7.1}$$

To keep the problem analytically tractable, we will assume $J_z(r) = J_o$. The dispersion relation then becomes from (D.6.3)

$$Y_o^2 = -\phi^2 - \left[j\,ka\sqrt{1-\beta^2}\,\frac{J_m'(jka\sqrt{1-\beta^2})}{J_m(jka\sqrt{1-\beta^2})} + \beta m \right] \left[\frac{C_o\,\phi^2}{1-\beta^2} \right] \tag{D.7.2}$$

where

$$\beta = \frac{2\phi}{Y_o^2 + \phi^2} \qquad \text{and} \qquad K = \frac{n}{R} \tag{D.7.3}$$

The corresponding mode has a spacial dependence given by

$$\xi_r(\bar{r}) \sim \frac{1}{r} J_m\left(jkr\sqrt{1-\beta^2}\right) e^{-j\left(m\theta + \frac{nz}{R}\right)} \tag{D.7.4}$$

This description of the plasma admits several sets of solutions.

One solution, which is probably the most serious plasma confinement problem, is the kink mode solution: If we assume $K a << 1$ and $B^2 \leq 2$, the dispersion relation can be written as (D.6.7)

$$\gamma_0^2 \doteq 2 \left[\phi - \frac{\phi^2}{1-\left(\frac{a}{R_w}\right)^{2m}} \right] \qquad \text{(D.7.5)}$$

This solution predicts growth rates somewhat higher than those observed. However, Friedberg finds that correction of the current density profile to match experimental results gives numerically calculated growth rates which match experimental results.

For the same values of m and n, however, there are three other sets of solutions, corresponding to larger values of B. As shown by (7.4), these modes have complicated internal structure and small surface perturbation, hence they are referred to as "internal modes". Clarke and Dory point out that these might severely limit feedback stabilization over ranges of small q.

To study these modes, (7.2) can be rearranged by eliminating Y_o^2 in favor of ϕ. Solving for ϕ gives, for $B^2 > 1$,

$$\phi = \frac{2(B^2-1)}{C_o B\left[B_m + Ka\sqrt{B^2-1}\,\dfrac{J_m'(Ka\sqrt{B^2-1})}{J_m(Ka\sqrt{B^2-1})}\right]} \tag{D.7.6}$$

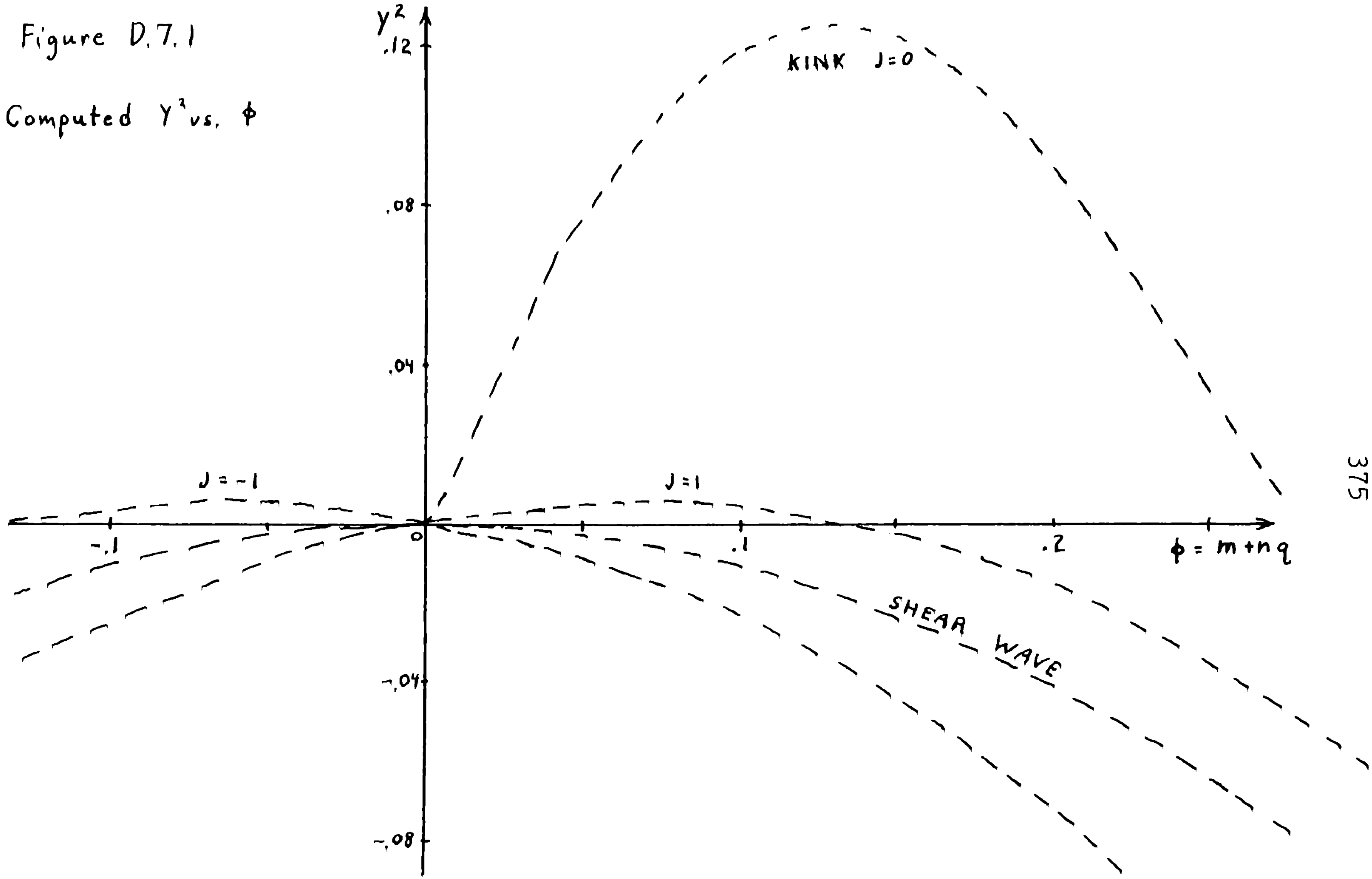
$$Y_o^2 = \frac{2\phi}{B} - \phi^2 \tag{D.7.7}$$

By sweeping B, various solutions for Y_o^2 as a function of ϕ are swept out. Solutions and graphs are given for $1 \le \sqrt{B^2-1}\,Ka \le 11$. Note that $Y_o^2 \ge o$ occurs over the intervals $B > 1$, $o \le \phi \le \frac{2}{B}$, and $B^2 < 1$, $\frac{2}{B} < \phi < o$. Growth rates are almost two orders of magnitude less than the kink, reaching a maximum of $1/B^2$ at $\phi = 1/B$. An additional set of solutions is given by $Y_o^2 = -\phi^2$, $|B| \to \infty$. These are internal (purely shear) waves; they are characterized by $p^*(r) = o$ from (3.16) and $\xi_r(a) = o$ from (4.11). These are always stable.

The dispersion relations for Alfven waves, the kink mode, and the first internal modes are plotted in figure D.7.1. Several internal modes are plotted and compared

Figure D.7.1

Computed Y^2 vs. ϕ



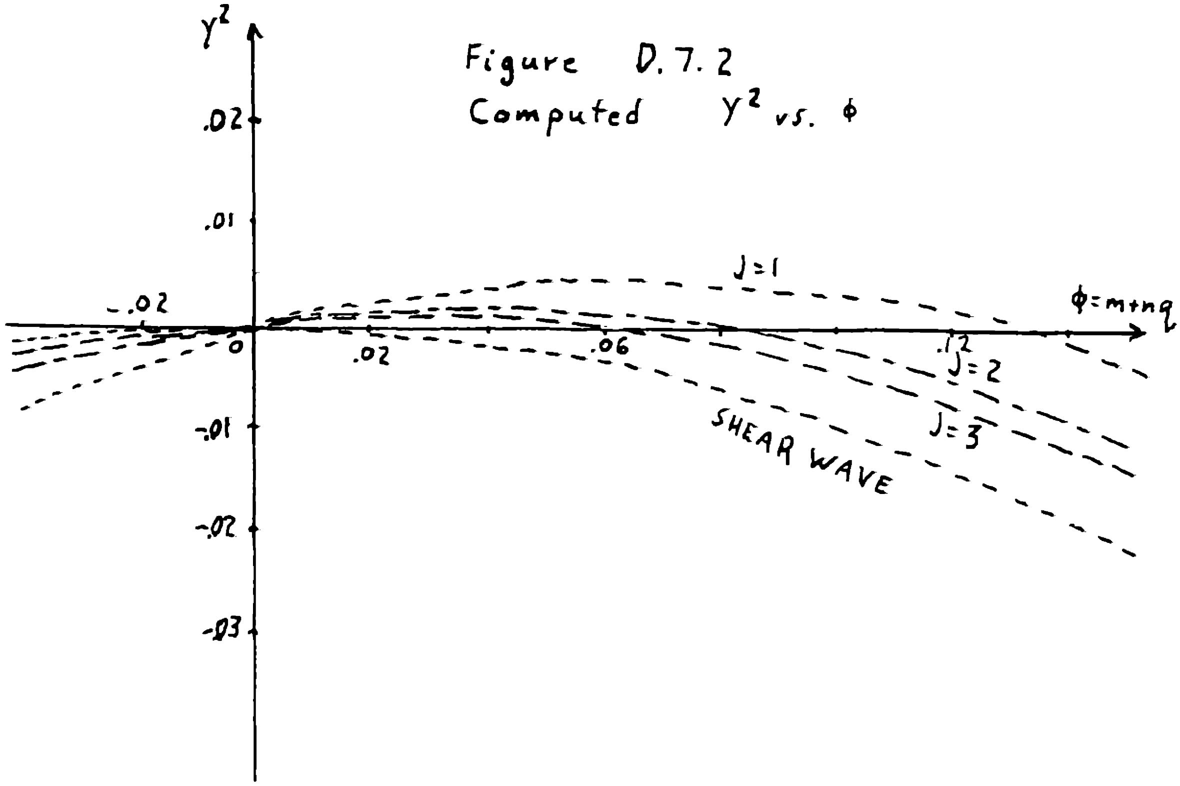


Figure D.7.2
Computed Y^2 vs. ϕ

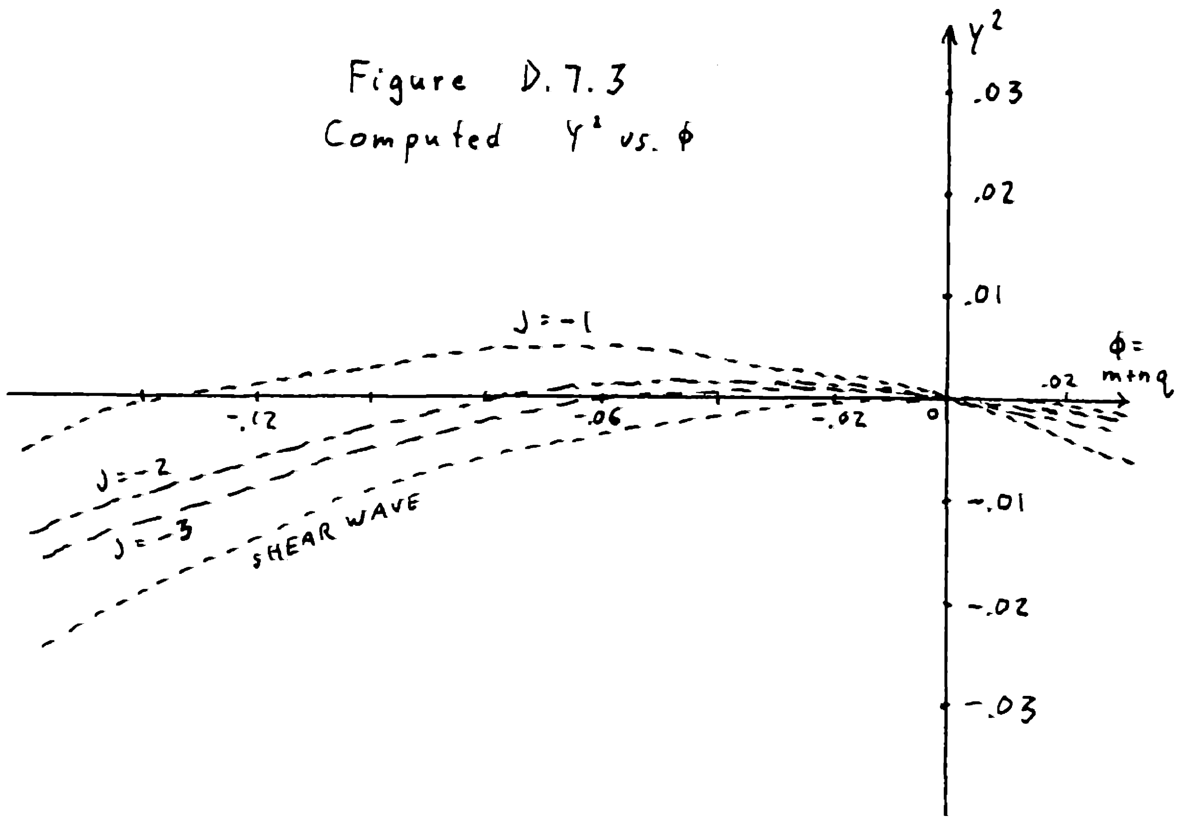


Figure D.7.3
Computed Y^2 vs. ϕ

with Alfven waves in figure D.7.2 for $B > o$, and D.7.3 for $B < o$. Note that the graphs tell us that as q is reduced, the first modes to go unstable will be the internal modes with $B < o$. As $\phi \to o$, an infinite number of them are unstable, although with extremely small growth rates. Then, at $\phi = o$, the kink goes unstable with rapidly increasing growth rate. A second group of unstable internal modes becomes unstable then also, with small growth rates. The closer to $m+nq=o$ one operates, the more modes there are.

A different approach to the analysis of these modes is given in the next section, using energy methods. The energy approach allows examination of effects such as toroidal geometry, which are not easy to handle analytically. However, such an approach sometimes lacks the precision of analytical approximations. We therefore look for a simplified representation of the internal modes using our cylindrical model.

The internal modes can also be approximately represented in a simple dispersion relation, if we assume that the value of B for a given internal mode varies slowly with (m + nq) in the range of $Y^2 \geq o$. Let $B=2\phi/(Y_o^2+\phi^2)$ take on the value it has when $Y^2 = \phi = o$; B_j is the J^{th} root of the equation $J_m(Ka\sqrt{B_j^2-1}) = o$. Then the dispersion relation is essentially parabolic, with $Y^2 = \frac{2\phi}{B_j} - \phi^2$. Note that this is precisely the fixed-boundary problem, and so all wall effects are suppressed in this approximation. $Y^2 = o$ at $\phi = \frac{2}{B_j}$ and $\frac{d(Y^2)}{d\phi} = o$ at $\phi = \frac{1}{B_j}$, $Y^2 = \frac{1}{B_j^2}$.

Table of B_j $m = 1$; $Ka = .260$

j	ARG: $Ka\sqrt{B_j^2-1}$	B_j	$1/B_j$	$2/B_j$	$1/B_j^2$
1	3.8317	14.7	.068	.136	4.6×10^{-3}
2	7.0156	27.0	.037	.074	1.37×10^{-3}
3	10.1735	39.2	.0255	.051	6.5×10^{-4}
-1	3.8317	-14.7	-.068	-.136	4.6×10^{-3}
-2	7.0156	-27.0	-.037	-.024	1.37×10^{-3}
-3	10.1735	-39.2	-.0255	-.051	6.5×10^{-4}

Unstable for $|\phi| < \frac{2}{B_j}$, Max. $Y^2 = \frac{1}{B_j^2}$ at $\phi = \frac{1}{B_j}$

For large B_j, approximately

$$\cos\left(\sqrt{B_j^2 - 1}\ Ka - \frac{3\pi}{4}\right) = o$$

or

$$|B_j|\ Ka - \frac{\pi}{4} \cong j\ \pi$$

$$|B_j| = \frac{\pi\ (j + \frac{1}{4})}{Ka} \cong (4_j + 1)\ \pi$$

How constant is B over the above interval? Worst case: for j = 1, at $Y^2 = o$, $\phi = 5.52 \times 10^{-3}$ B = .14.773 ARG = 3.832 J_1 (ARG) = -2.0×10^{-3} $Y^2 = 7.4 \times 10^{-4}$

In other words, the internal modes do not perturb the surface of the plasma enough to seriously change their dispersion relation from what it would be in a fixed-boundary problem (one in which $\xi_r = o$ at r = a). This tells us that the coupling coefficients A_{nm} are so small for these modes that they will have negligible coupling to the kink modes via the feedback, for determination of stability of the plasma. If the internal modes are stable, then surface feedback can stabilize the entire plasma. If they are unstable, then the system with bang-bang feedback at the surface will behave approximately as an internal mode with boundary condition $\xi_r = o$.

Since it is not feasible to sample or force the inside of the plasma column, we must hope that either the internal modes will be stabilized by some effect not yet accounted for in the model, or that their growth does not grow so large as to overwhelm the feedback system before being limited by internal nonlinear mechanisms.

It therefore appears, based on our cylindrical model with $J_z(r) = J_o$, that stabilization of the surface for m = n = 1 has two main problems:

1) Internal modes at m = n = 1 may result in an internally unstable combination with the main kink, which would have final effects dependent on the nonlinear saturation mechanisms inside the plasma.

2) Modes with m/n = -q for higher m are also unstable with large growth rates.

However, there are two effects which have not been considered here which may eliminate these difficulties. These are the actual current distribution $J_z(r)$ in a Tokamak device, and the effects of nonlinear internal mechanisms which may cause instabilities to saturate.

1) The actual current distribution falls off gradually from r = o to r = a, rather than being constant until r = a. The effects of this have been numerically

investigated, by Friedberg, who finds that the experimentally measured $J_z(r)$ in Scylla IV results in a much lower growth rate γ^2 that matches experimental growth rate values. If the current were all concentrated at the core $r = o$, then Kadowtsev shows that the entire plasma would be stable against all perturbations. A Gaussian current distribution has been investigated by Shafranov (1969) who finds that this has little effect on the $m = n = 1$ mode but stabilizes all modes with $m > 1$. Experimental Tokamak data shows that although oscillations are observed at $n = 1$, $m > 1$ for high values of q, large losses are not observed. Thus we conclude that the current distribution $J_z(r)$ which occurs in Tokamak devices acts to stabilize all modes $m > 1$ which would otherwise be unstable at $m/n = -q$.

2) The effects of nonlinear internal mechanisms, which may cause instabilities to saturate, will be assumed to dominate if predicted plasma displacements become on the order of the column radius a. The work of Yoshikawa, for instance, suggests that the modes with complex spacial structure represent motion of the plasma toward a nearby stable helical equilibrium. Given this information, let us examine the stability of the system in the presence of both the kink and the internal modes. This can be done in two ways.

The hard way is to require that the surface perturbation for $m = -n = 1$ be zero, and solve for the kink mode amplitude in terms of the internal mode amplitudes. Then, substituting into the energy matrix, we use Sylvester's test to determine null stability.

However, as the section on boundary control showed, there is a much easier way. We simply solve for eigenvalues of a new system with boundary condition $\xi_r = o$ at $r = a$. The resulting growth rates are just those of the internal modes with the approximation $B = B_j$!

This simply tells us that the surface perturbations of the internal modes are so small that they couple weakly to the feedback; $A_{nm} << 1$. Then, solving for the kink mode amplitude gives very small values, which are dominated completely in the energy matrix by the energy of the internal modes themselves. Nature has made our system well designed, by making all entries in $\bar{\bar{A}}_\infty$ small, due to the character of the internal modes.

Hence, the plasma with bang-bang feedback at the surface will be unstable only if the internal modes are unstable. It will grow with $\xi_r(a) = o$ until the cancelling kink and internal mode amplitudes become so large that they can overwhelm the feedback, entering a rapid second stage of growth. However, if some nonlinear internal mechanism limits the growth of the internal mode before this point, then the surface will remain stable even if the interior of the plasma becomes turbulent. To examine this question, we look at the internal modes in more detail.

In cylindrical geometry, the maximum growth rate for the first internal mode is approximately $\gamma^2 \cong \frac{1}{B_1^2} = 4.6 \times 10^{-3}$ at $\phi \cong \frac{1}{B_j} = \pm .068$. $J_1(Kr\sqrt{B^2-1})$ has a maximum of .58 at $\frac{r}{a} = .47$. Thus the maximum value of

$$\xi_r = C \frac{J_1(Kr\sqrt{B^2-1})}{(r/a)}$$

is approximately 1.23 C.

At the surface $r_o = a$, however, the more exact dispersion relation gives $\phi = .072$, $B \cong 15.1$, $\gamma^2 \cong 4.3 \times 10^{-3}$, and $Ka\sqrt{B^2-1} \cong 3.92$, and $CJ_1(Ka\sqrt{B^2-1}) = -.0346C$. Therefore, the ratio of surface displacement to maximum internal displacement is approximately .028 for the worst-case internal mode. In order for the cancelling kink and internal mode surface displacements to reach the 1 cm level, the internal displacement of the internal mode would have to be 35 cm; clearly ridiculous. Long before this would happen, we can assume that nonlinear mechanisms would dominate the behavior of the system.

D.8 Energy Method Analysis of Internal Modes

In toroidal geometry, the solution of the equations of motion of the plasma is extremely complex. Rather than using a modal approach, it is easier to directly apply the energy principle to the question of stability. If all modes of a plasma are stable, then for any allowed perturbation the potential energy ψ is positive.

The use of internal modes is in fact simpler to analyze by energy methods than that of modes of the plasma with a free surface. The boundary condition $\xi_r = o$ on the surface eliminates any contribution to the energy due to integrals over S or the vacuum V_o. We are left with only the integral over V_i to evaluate. This is done by Mercier for the case of interest, and the minimum potential energy ψ is found.

Mercier first defines an intrinsic coordinate system which is based on the equilibrium currents and field lines. Magnetic surfaces S_M are surfaces containing flux lines, defined by the parameter F, so that F = o at the magnetic axis. Letting n represent the normal to magnetic surfaces, the potential energy ψ can be transformed into an integration over F and S_M. This gives a local energy for each magnetic surface, so that a local perturbation which is nonzero only in the neighborhood of that surface has a minimum potential energy

$$\psi = \left[4\int_{S_M} \frac{B^2\, dS_m}{|\nabla F|^3}\right]^{-1}\left[\frac{d}{dF}(1/q) + 2\int_{S_M}\frac{\bar{B}\cdot\bar{J}}{|\nabla F|^3}\, dS_M\right] - 2\int_{S_M}\frac{(\bar{J}\times\hat{n})\cdot(\bar{B}\cdot\nabla\hat{n})\, dS_M}{|\nabla F|^3} \qquad \text{(D.8.1)}$$

Here we have let $q(r) = \dfrac{B_\theta R}{B_z r}$, $q' = dq/dr$ and introduce the quantities in terms of the cylindrical case:

$$dF = 2\pi R B_\theta dr \qquad dT = 2\pi R J_\theta dr$$

$$d\bar{F} = 2\pi R B_z dr \qquad d\bar{T} = 2\pi R J_z dr$$

$$\bar{Q} = \bar{J} - \frac{d\bar{T}}{d\bar{F}}\bar{B} \qquad \text{(D.8.2)}$$

$$dV = (2\pi)^2 r R dr \qquad V'' = \frac{d^2V}{d\bar{F}^2}$$

then we may rewrite the potential energy as the balance of two terms:

$$\psi = \left[\frac{1}{2}\frac{d}{dF}\left(\frac{1}{q}\right) + \int_{S_M}\frac{\bar{B}\cdot\bar{Q}}{|\nabla F|^3}\, dS_M\right]^2 + \left[\int_{S_M}\frac{B^2\, dS_M}{|\nabla F|^3}\right]\left[\frac{dP}{dF}\left(\frac{1}{q^2}\right)V'' - \int_{S_M}\frac{|\bar{Q}|^2\, dS_M}{|\nabla F|^3}\right] \qquad \text{(D.8.3)}$$

where the criterion for stability is $\psi \geq 0$.

The crucial role of the specific volume V is thus made clear, for in order to contain an equilibrium plasma $\frac{dP}{dF} < 0$. Hence the specific volume V must have a negative second derivative, making V a minimum, for that term to represent a stabilizing influence. In general, V" will divide into two parts, $V'' = V_o'' + V_p''$ where V_o'' represents the effect of geometry of the currents while V_p'' represents simply the magnetic well complementing the equilibrium pressure.

In cylindrical geometry, if $P' = dP/dr$, $V_p'' = \frac{R \; P'}{r \; B_z^4}$ and so for stability

$$\left[\frac{1}{2} \, q'/q - \frac{P'}{B_z^2} \right]^2 + \frac{r B^2}{R B_\theta^2} P' V_o'' \geq 0 \qquad \text{(D.8.4)}$$

Mercier amplifies this result in the cylindrical case to give the SUYDAM criterion,

$$\frac{1}{4} \left(\frac{q'}{q} \right)^2 + \frac{2}{r} \frac{P'}{B_z^2} \geq 0 \qquad \text{(D.8.5)}$$

which is a local requirement everywhere in the plasma.

The first term of (D.8.4) represents magnetic shear (corrected by a pressure term) in the equilibrium, and is purely stabilizing. (Note that in our model of uniform current density, magnetic shear vanished.) The second is positive for $P' < o$ only if $V_o'' < o$; that is, a magnetic well.

In cylindrical geometry the expression is complicated. However, since internal modes have perturbations near the magnetic axis, an expansion near the magnetic axis gives a tractable expression. The principal result is that, near the magnetic axis,

$$\frac{V_o''(\text{TORUS})}{V_o''(\text{CYLINDER})} = q^2 - 1 \qquad \text{(D.8.6)}$$

Therefore, we can expect that the effects of toroidal geometry will be stabilizing if, on the magnetic axis, $|q| > 1$ and destabilizing if $|q| < 1$. For a realistic current distribution ($dJ_z/dr < o$) $|q|$ will be 2 or 3 at the edge $r = a$ when $|q| = 1$ in the center. Thus, when we approach some critical value of $|q|$ from above, we can expect the plasma to become turbulent, although the surface may remain fixed by use of bang-bang feedback.

Appendix E

Fields in the Vacuum Region Rectangular Geometry

This section examines the structure of the fields due to a mode of the plasma.

At $x=0$, the equilibrium position of the surface of the plasma, S:

$$\bar{n}_S \times \bar{A} = -\bar{H}_o (\bar{\xi} \cdot \bar{n}_S) \tag{E.1}$$

At $x=d$, the conducting shell, S_o:

$$\bar{n}_o \times \bar{A} = 0 \tag{E.2}$$

In the vacuum, $\nabla \times \bar{A} = \bar{h}_o$ and $\nabla \times \bar{h}_o = 0$

Take $\nabla \cdot \bar{A} = 0$ so $\nabla^2 \bar{A} = 0$ (E.3)

Let $\bar{\xi} \cdot \hat{x}$ on S be $Re[\hat{\xi}\, e^{j(\omega t - My - Nz)}]$

$$= |\xi| \cos(\omega t - My - Nz + \phi)$$

Similarly, $\bar{A} = Re[\hat{A}\, e^{j(\omega t - My - Nz)}]$

Assuming periodicity $L_y, L_z \Rightarrow$

$$M = \frac{m 2\pi}{L_y} \qquad N = \frac{n 2\pi}{L_z} \qquad K^2 = M^2 + N^2$$

$$K > 0$$

Let $\bar{H}_o = H_y \hat{y} + H_z \hat{z}$

Then we can find $\hat{A}$:

$$\hat{A}_y = \hat{\xi} H_z \frac{\sinh K(x-d)}{\sinh Kd}$$

$$\hat{A}_z = -\hat{\xi} H_y \frac{\sinh K(x-d)}{\sinh Kd} \tag{E.4}$$

$$\frac{\partial \hat{A}_x}{\partial x} = -j(NH_y - MH_z)\,\hat{\xi}\,\frac{\sinh k(x-d)}{\sinh kd} \qquad (E.5)$$

$$\hat{A}_x = -j\frac{(NH_y - MH_z)}{k}\,\hat{\xi}\,\frac{\cosh k(x-d)}{\sinh kd} \qquad (E.6)$$

This gives $\hat{\bar{e}} = -\mu_o j\omega \hat{\bar{A}} \Rightarrow$

$$\hat{e}_x = -\mu_o \omega \frac{(NH_y - MH_z)}{k}\,\hat{\xi}\,\frac{\cosh k(x-d)}{\sinh kd}$$

$$\hat{e}_y = -j\mu_o \omega H_z\,\hat{\xi}\,\frac{\sinh k(x-d)}{\sinh kd}$$

$$\hat{e}_z = j\mu_o \omega H_y\,\hat{\xi}\,\frac{\sinh k(x-d)}{\sinh kd} \qquad (E.7)$$

The fields are now divided into "magnetic" and "electric" parts. We use subscripts "o" and "s", respectively.

$$\bar{h}_s \equiv 0$$

$$\bar{h} = \bar{h}_o = \nabla \times \bar{A}$$

$$\hat{h}_x = j(MH_y + NH_z)\,\hat{\xi}\,\frac{\sinh k(x-d)}{\sinh kd} \qquad (E.8)$$

$$\hat{h}_y = \frac{M}{k}(MH_y + NH_z)\,\hat{\xi}\,\frac{\cosh k(x-d)}{\sinh kd}$$

$$\hat{h}_z = \frac{N}{k}(MH_y + NH_z)\,\hat{\xi}\,\frac{\cosh k(x-d)}{\sinh kd}$$

So if $MH_y + NH_z = 0$ then $\bar{h} = 0$ (interchange condition)

However, at interchange $\bar{A} \neq 0$, $\bar{e} \neq 0$

$$\bar{e} = \bar{e}_o + \bar{e}_s \qquad \bar{e}_s = -\nabla \phi_s$$

$$\bar{e}_o \to 0 \text{ if } MH_Y + NH_Z \to 0$$

$$\hat{\phi}_s = \mu_o \omega \frac{(NH_Y - MH_Z)}{K^2} \hat{\mathcal{E}} \frac{\sinh K(x-d)}{\sinh Kd} \qquad \text{(E.9)}$$

$\bar{e}_s$, the "electrostatic" field, goes as $\bar{k} \times \bar{H}_o$:

$$\hat{e}_{s_x} = -\mu_o \omega \frac{(NH_Y - MH_Z)}{K} \hat{\mathcal{E}} \frac{\cosh K(x-d)}{\sinh Kd} \qquad \text{(E.10)}$$

$$\hat{e}_{s_Y} = j \mu_o \omega \frac{M}{K^2} (NH_Y - MH_Z) \hat{\mathcal{E}} \frac{\sinh K(x-d)}{\sinh Kd}$$

$$\hat{e}_{s_Z} = j \mu_o \omega \frac{N}{K^2} (NH_Y - MH_Z) \hat{\mathcal{E}} \frac{\sinh K(x-d)}{\sinh Kd}$$

$\bar{e}_o$ is parallel to the plasma and goes as $\bar{K} \cdot \bar{H}_o$

$$\hat{e}_{o_x} = 0 \qquad \text{(E.11)}$$

$$\hat{e}_{o_Y} = -j \mu_o \omega (MH_Y + NH_Z) \frac{N}{K^2} \hat{\mathcal{E}} \frac{\sinh K(x-d)}{\sinh Kd}$$

$$\hat{e}_{o_Z} = j \mu_o \omega (MH_Y + NH_Z) \frac{M}{K^2} \hat{\mathcal{E}} \frac{\sinh K(x-d)}{\sinh Kd}$$

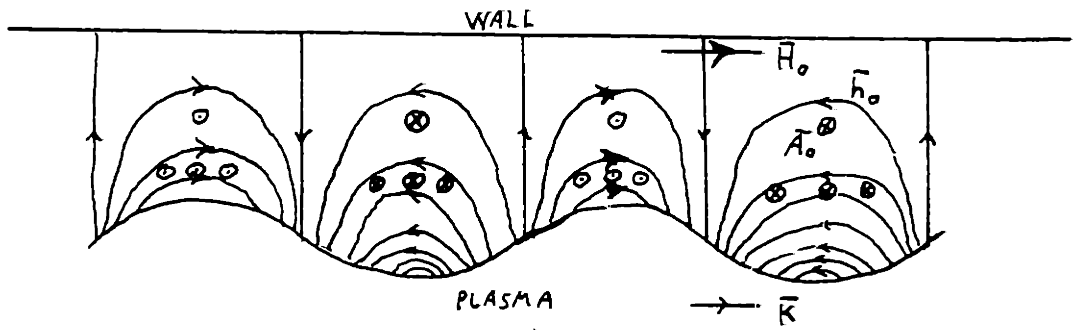
Hence the vector potential splits into 2 parts, $\bar{A} = \bar{A}_s + \bar{A}_o$, each given by

$\bar{A} = \dfrac{\bar{e}}{-\mu_o j \omega}$. Fields are sketched in E.1.

Figure E.1 Field Plots

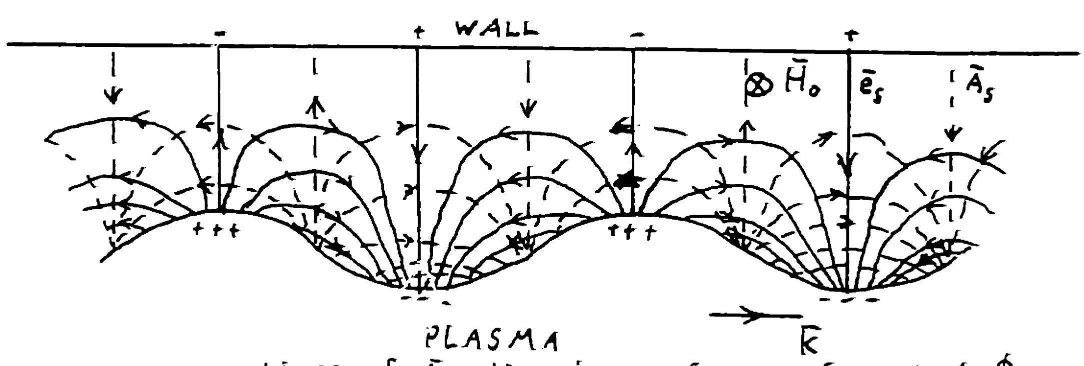
(Without Feedback)

Case I: $\bar{k} \parallel \bar{H}_0$



$\bar{e}_o$ looks like $\bar{A}_o$ shifted $\lambda/4$

Case II: $\bar{k} \perp \bar{H}_0$



Lines of $\bar{A}_s$ lie in surfaces of constant Φ_s

Consider the case of $\mu_o H_F \cdot H_o$ feedback.

I. If $\nabla \cdot \overline{J}_F = o$ then we can not interact via the "static" fields $\overline{e}_s$ and $\overline{A}_s$ because they have zero curl, so integrating them around any closed path in the vacuum is zero. Also, since they have zero parallel components at the wall, bringing currents through the wall will not avoid the problem.

II. Velocity feedback. For $\nabla \cdot J_F = o$, $\overline{J}_F$ not intersecting the wall, interact with $\overline{e}_o$. Place current loop in the y-z plane, having flux of loop in + x direction whenever $\xi_x > o$. This could be done with passive resistive material!

III. Position feedback

So put current in horizontal plane so that its field reinforces the o-order field at peaks in ξ_x, near the plasma (So, flux of loop is opposite the x-directed self-field flux) essentially like putting perfectly conducting coils in horizontal plane, for passive feedback.

In practical terms, suppose we want to design a feedback current system which will have maximum effect on a mode of the form $\xi_x(y,z) = \xi \cos (M_Y + Nz)$ $o < y < L_Y$, $o < z < L_Z$, $o < x < d$ in the form of positional feedback of the $\overline{H}_F \cdot \overline{H}_o$ type, subject to the constraints.

$\nabla \cdot \bar{J}_F = o$

If $< I_{max}$ on each wire, as few wires as possible.

$J_F = o$ for $x < x_F < d$ (We cannot get too close to the plasma.)

Then we will want to put all the available current into a wire which closes on itself and is antiparallel to the largest available $\bar{A}_o$ field line which stays above $x = x_F$. From our results for $\bar{A}_o$, we see that the value of $\bar{A}_o$ is the superposition of two travelling waves, and is of the form

$$\bar{A}_o = (N\hat{y} - M\hat{z}) \frac{(MH_y + NH_z)}{K^2} \frac{\sinh K(x-d)}{\sinh Kd} \xi_x \qquad (E.14)$$

This is clearly maximum at $x = x_F$, $M_Y + Nz = n\pi$ so we will put the wires there. We can lable the wires by their z-intercept at $z = o$, $Y = \frac{n\pi}{M} = \frac{nL_y}{2m}$. The direction of the wires is given by $\hat{u}$ such that $M\hat{y} + N\hat{z} \times \hat{\ell} = o$ or $\hat{u} = + \frac{1}{K}(N\hat{y} - M\hat{z})$, antiparallel to $\bar{A}_o$. Thus, for n = even wires, $n = o$ to $2m - 2$, we have current flowing in the u direction, and in n = odd wires, $n = 1$ to $2m-1$, we have current flowing in the opposite direction, for $\xi > o$. We can even join the ends to make one continuous wire if we wish, although this creates impedance problems.

Or, if the system is periodic in Y and z (a torus) we might make each wire a loop and bring the ends out through the wall close together ($\int \bar{A}_o \cdot d\ell =$ $\int \mu_o \bar{h}_o \cdot \bar{n}\, ds$ = Flux).

Since we are limited to $|I_F| < I_{max}$, we'll get the most force on the plasma by making $|I_F| = I_{max}$ at all times for $\xi \neq o$. See Figure E.2 for a sketch of this arrangement. Thus our prescription for the current through the n^{th} wire, n = o, to 2m-1, is

$$\bar{I}_{F_n} = \hat{u}\,(-1)^n \frac{\xi_x}{|\xi_x|} I_{Max} \quad \text{on} \quad \begin{array}{l} x = x_F \\ \dfrac{2my}{L_Y} + \dfrac{2nz}{L_z} = n \end{array} \tag{E.15}$$

Note that, as a quickcheck, at y = z = o we have for $\xi > o$, $\xi > o$ and n = o so $\bar{I}_F = I_{max}\,\hat{u}$. Suppose $H_z = o$, N = o. Then $\hat{u} = -\hat{z}$ so that, at x = o, $H_F = +\hat{y}$ which is parallel to $\bar{H}_o$ and so pushes down on the bulging $\xi > o$.

For N ≠ o, the result is not so intuitive! The wires are perpendicular to the mode vector $\bar{K}$, not the field $\bar{H}_o$.

We are placing our wires above the expected peaks and valleys of ξ_x for position feedback.

For velocity feedback, we want to maximize $\bar{J}_F \cdot \bar{e}_o$, $\bar{e}_o = -\mu_o \dfrac{\partial \bar{A}_o}{\partial t}$ so we want to place $\bar{J}_F$ parallel to $\bar{e}_o$. The problem is precisely the same, except that we now place our wires above the expected peaks and valleys of $\frac{\partial \xi_x}{\partial t}$. All that remains to be determined is the sign, which comes from the expression for $\bar{e}_o$:

$$\bar{e}_o = (-N\hat{y} + M\hat{z})\mu_0 \frac{(MH_y + NH_z)}{K^2} \frac{\sinh k(x-d)}{\sinh kd} \frac{\partial \xi_x}{\partial t} \tag{E.16}$$

Thus we place the wires in the same positions, and let the direction for $\frac{\partial \xi_x}{\partial t} > 0$ be $\hat{u} = \frac{N\hat{y} - M\hat{z}}{K}$ parallel to $\bar{e}_o$, the same as before! Our prescription is

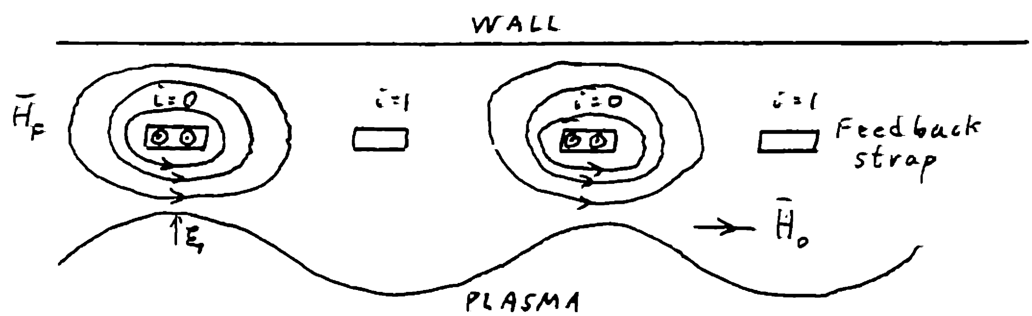
$$\bar{I}_{F_n} = \hat{\mu}(-1)^n \frac{\frac{\partial \xi_x}{\partial t}}{\left|\frac{\partial \xi_x}{\partial t}\right|} I_{max} \quad \text{on} \quad x = x_F, \quad \frac{2my}{L_y} + \frac{2nz}{L_z} = n \tag{E.17}$$

This can be realized by just putting passive resistive material in the vacuum region.

Now consider $\frac{1}{2}\mu_o H_F^2$ feedback force.

$$\frac{dE}{dt} = -\int_S \mu_0 \bar{H}_F \cdot \bar{H}_F \frac{\partial \bar{\xi}}{\partial t} \cdot \bar{n}\, dS \tag{E.18}$$

Figure E.2 Feedback Current Geometry



Let $\bar{A}_1$ = vector potential of self-field of plasma due to $\bar{H}_F$ and $\bar{\xi}$, $\bar{h}_1$

$$\nabla \times \bar{h}_1 = 0 \text{ in } V_0 \qquad \hat{n}_s \times \bar{A}_1 = -\bar{H}_F \, \bar{\xi} \cdot \hat{n}_s \text{ at } S$$

$$\hat{n}_s \times \bar{A}_1 = 0 \text{ at } S_0$$

$$\frac{dE}{dt} = \int_{V_F} \mu_0 \frac{\partial \bar{A}_1}{\partial t} \cdot \bar{J}_F \, d\tau = -\int_{V_F} \bar{e}_1 \cdot \bar{J}_F \, d\tau \qquad \text{E.19}$$

If $\bar{J}_F = J_F y$, $H_F \cong H_F z$ looks like $\bar{H}_0$ locally. So e_{1y} is in phase with $\bar{\xi} \cdot \bar{n}_s$. So the force will be in phase with $-\bar{\xi} \cdot \bar{n}_s$. This is nonzero if $\bar{e} \neq o$. So, choose $\bar{J}_F$ parallel to the flutes $(\bar{J}_F \;\; \bar{K})$ of the mode to be stabilized. Then, since A_1 has a component $< o$ parallel to the ridges, we get $\bar{J}_F \cdot \bar{A} < o$ as desired for position feedback similarly for velocity feedback. Thus the geometry for optimum H_F^2 feedback is the same as for linear feedback!

Appendix F

Proof that Det $\overline{\overline{A}}_N \neq o$ for a Useful Geometry

Many systems obey a periodic boundary condition;

$$\xi(z+1) = \xi(z) \tag{F.1}$$

In such a case, it is useful to know whether a feedback geometry with N feedback stations of the form

$$A_n(z) = \begin{cases} H(z - n/N) & \frac{n}{N} < z < \frac{n+1}{N} \\ 0 & \text{otherwise} \end{cases} \qquad n = 0 \text{ to } N-1 \tag{F.2}$$

will produce independent sampling of the first N modes of the system. A particularly neat proof of this fact is available, using the wave-train analysis approach. We must assume that H(z) is real and has a nonzero components m = o and m = N/2 over the interval [o, 1/N].

Perturbations of the system may be Fourier analyzed

$$\xi(z) = \sum_{m=-\infty}^{\infty} a_m(t)\, e^{-jK_m z} \tag{F.3}$$

where $K_m = 2\pi m$.

We wish to prove that det $\overline{\overline{A}}_N \neq o$ where

$$A_{nm} = \int_0^1 A_n(z)\, e^{-jk_m z}\, dz \tag{F.4}$$

for $n \leq N$, $-N < 2m \leq N$.

We let

$$H_m(K_m) = \int_0^{1/N} H(z)\, e^{-jK_m z}\, dz \tag{F.5}$$

Then we note that by our assumptions on $H(z)$, $H_m \neq o$ for $-N < 2m \leq N$, and that

$$A_{nm} = e^{-j\frac{n}{N}K_m}\, H_m(K_m) \tag{F.6}$$

Now, since $H_m(K_m) \neq o$, it can be factored out of the rows of the determinant $\underline{\underline{A}}_N$, and leaves entries of the form

$$A_{nm} = e^{-j\, n\, m\, 2\pi/N} \tag{F.7}$$

Thus, in any given column m, we have entries forming a power series

$$X_m^r \;, \qquad n = 0 \text{ to } N-1 \tag{F.8}$$

We also know that all values of x_m are distinct, because

$$x_m = e^{-j\, 2\pi m/N} \tag{F.9}$$

which means that the x_m are the N distinct roots of the equation

$$X^N = 1 \tag{F.10}$$

Therefore, our matrix $\bar{A}_N$ is in the form of a Vander Monde matrix;

$$\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ X_1 & X_2 & X_3 & X_4 & & X_N \\ X_1{}^2 & X_2{}^2 & & & & X_N{}^2 \\ X_1{}^3 & X_2{}^3 & & & & X_N{}^3 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ X_1{}^N & X_2{}^N & & & & X_N{}^N \end{vmatrix}$$

with all x_m distinct. Therefore, det $\bar{A}_N \neq o$ and the proof is complete.

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