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In memory of John H. Conway.

John H. Conway liked inventing new sequences, mostly because it is fun.[1] However, there is another reason: simple sequences can lead to very deep mathematics.

One simple but puzzling collection of sequences comes from the infamous Collatz problem. Take a positive integer and iteratively apply the following rule: if a number is odd, triple it and add one; if it is even, halve it:

$$t_{n+1} = \begin{cases} 3t_n + 1 & \text{if } t_n \text{ is odd,} \\ \frac{t_n}{2} & \text{if } t_n \text{ is even.} \end{cases}$$

It is conjectured that the sequences produced by this rule will always reach an infinite cycle of $4, 2, 1, 4, 2, 1, \ldots$.

Paul Erdős is rumored to have said about the Collatz conjecture, "Mathematics is not yet ready for such problems" [4]. John wanted to create a lot more examples of sequences that are similar to those in the Collatz problem [8]. He hoped that this would make mathematics ready for new horizons. At the very least, he wanted to make his sequences as notorious as the Collatz problem.

Did you know that numbers have destinies, as John Conway called them? To have a destiny, a number needs to have a life, or in mathematical terms, destinies are defined with respect to an operation or a function. If the function is the Collatz rule, then every nonnegative integer's conjectured destiny is the cycle described above.

Two numbers have the same destiny with respect to the Collatz rule if the tails of the sequences they generate coincide. Suppose $a(n)$ and $b(n)$ are the two trajectories. Then the numbers $a(0)$ and $b(0)$ that define these sequences have the same destiny if there exist N and M such that for every nonnegative integer j, $a(N+j) = b(M+j)$. In particular, all numbers in the same trajectory $a(n)$ have the same destiny.

Conway tried to invent sequences that produce interesting destinies. One example that is relevant to this paper is the RATS sequence. RATS is an abbreviation for Reverse Add Then Sort. To calculate the next term of the sequence, we reverse the digits of the current term and add the number thus obtained to the current term. We then sort the digits of the

[1]This section is based on Tanya Khovanova's personal recollections of encounters with John Conway.

resulting sum by arranging them in increasing order. For example, to calculate RATS(732), we reverse 732, getting 237, then add 732 and 237, getting 969, then sort the digits. Thus, RATS(732) = 699.

Let's look at the RATS sequence starting with 1: 1, 2, 4, 8, 16, 77, 145, 668, 1345, 6677, 13444, 55778, 133345, 666677, 1333444, 5567777, 12333445, 66666677, 133333444, 556667777, 1233334444, 5566667777, 12333334444, 55666667777, 123333334444, 556666667777, ... We can prove that this sequence is infinite, because its terms fall into a repetitive pattern with an increasing number of digits. John Conway calls the destiny of 1 "the creeper." Conway conjectured that RATS destinies are either the creeper or a cycle [5].

John especially liked tweaking the Fibonacci rule to invent new sequences. He usually called such sequences "Fibs." He said that he tried more than 100 different Fibs to see which of them had exciting destinies. We can naturally extend the term for destinies of numbers to destinies of pairs of numbers that uniquely define a tweaked Fibonacci sequence. John told everyone who would listen about his sequences, and some of his definitions were picked up by his colleagues and studied more thoroughly. We give three examples of such sequences.

The first example is Conway's subprime Fibonacci sequences [6]. Each element in such a sequence depends on the previous two terms. As in the Fibonacci sequence, we first sum the previous two terms. Then there is a tweak: if the result is composite, we divide it by its smallest prime factor. For example, if we start with 0 and 1, we get the sequence 0, 1, 1, 2, 3, 5, 4, 3, 7, and so on. Similar to the Collatz conjecture, there are indications and arguments that subprime Fibonacci sequences, regardless of the initial terms, have to end in a cycle, but this has not yet been proved.

The second example is the n-free Fibonacci sequences [1]. Such a sequence is determined by its two starting terms and an integer $n > 1$. As in all Fibs, each element in such a sequence depends on the previous two terms. As in the Fibonacci sequence, we first sum the previous two terms. But then there is a tweak: if the result is divisible by n, we repeatedly divide it by n until we get a term that is not divisible by n. Thus, except for the two initial terms, the rest of the sequence is n-free, meaning that none of the calculated terms are divisible by n. The behavior of such sequences is very different for different values of n. For example, 2-free Fibonacci sequences end in a cycle of length 1. It is conjectured that all 3-free Fibonacci sequences end in a cycle. It is also conjectured that with probability 1, a 4-free Fibonacci sequence does not cycle. If we look at 5-free Fibonacci sequences, some of them end in a cycle and some do not. For example, consider the Lucas sequence: the sequence that follows the Fibonacci rule but starts with integers 2 and 1. The Lucas sequence begins with 2, 1, 3, 4, 7, 11, and continues growing exponentially. Taken modulo 5, this sequence turns into a cycle 2, 1, 3, 4, 2, 1, and so on. Because this cycle doesn't include zeros, the Lucas sequence is a 5-free Fibonacci sequence.

The third example is a sorted Fibonacci sequence. Each element in such a sequence depends on the previous two terms. As in the Fibonacci sequence, we first sum the previous two terms. But then there is a tweak: we sort the digits of the result in increasing order. For example, suppose the first two terms are 0 and 1 as in the Fibonacci sequence. For a while, the sorted terms continue as in the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, ... However, the next term is not 21, but 12. It is followed by 25, 37, 26 (for the sum of 62), and so on:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 12,\ 25,\ 37,\ 26,\ \ldots.$$

One might continue the calculation and check that we end in a cycle of 120 terms. The largest value in the cycle is 667.

If we start with 1 and 1, 1 and 2, or any other two consecutive numbers from the sequence above, we clearly get the same result. Starting with a different pair of numbers can produce a different destiny. There are known cycles of lengths 3, 8, 24, 96, and 120. For example, you get a cycle of length 3 if you start with 27 and 27.

Conway conjectured that the destinies of all pairs of nonnegative integers under the sorted Fibs rule are cycles. The first observation that supports this conjecture is that the sequences can't grow as fast as the Fibonacci sequences. Each time we sort, the result doesn't increase and almost always decreases, especially for large numbers. Moreover, as soon as a zero appears as a digit in a sum, the result decreases tenfold. As in many other sequences invented by Conway, there is a shaky probabilistic argument supporting the conjecture. If the numbers are sufficiently large, we expect zeros to appear with high probability, decreasing the sequence. The problem with this argument is that the terms of the sequence, with digits being in increasing order, are very structured, and thus are highly nonrandom.

There are many conjectures related to sequences that John Conway invented. Quoting Erdős again: "Mathematics is not yet ready for such problems."

* * *

One of the authors of this paper, Tanya Khovanova, runs a program called PRIMES STEP, which conducts math research with gifted students in grades 6 through 9. The other authors were students in this program.

The project resulting in this paper was done in the 2017–2018 academic year. The topic of our research was base 3/2. We first studied properties of base 3/2, and then we explored various sequences written in that base. Our results are available in a paper posted at arXiv [3].

The students loved the Fibonacci sequence, but translating it to base 3/2 is not very interesting, for we just need to convert every term to base 3/2. This is where Conway's multitude of Fibs sequences comes in handy. The examples of subprime Fibs and n-free Fibs presented above, while beautiful, are not very exciting in base 3/2 for the same reason as for the Fibonacci sequence: the next element depends only on the previous two terms.

The sorted Fibs sequence, however, depends on the base, and it is interesting to see the destinies of pairs of numbers under the sorted Fibs rule in base 3/2. We are following John's tradition of calling our sequences "Fibs" to emphasize that these are not Fibonacci sequences.

In this paper, we study sorted Fibs written in base 3/2. What is base 3/2? How does one even think about a fractional base anyway? Our readers will be familiar, of course, with base 10, but there are many uses for other bases, such as 2, 12, and 60. Base 2, or binary, is useful because there are only two states for each place value, meaning that a number can be represented easily by a series of transistors that are either on or off, and as a result forms the basis for machine languages. Bases 12 and 60 are useful because they have many factors and hence can be divided nicely into smaller increments. We use these bases to partition time.

One way of thinking about how integer bases such as these work, invented by Propp [9] and popularized by Tanton [10], is the idea of exploding dots, which allows a natural extension into fractional bases. A more rigorous discussion of such fractional bases is covered in [2] and also in [7]. We explain exploding dots and base 3/2 in detail below.

In the next section, we give examples of the destinies of pairs of numbers with respect to the sorted Fibs rule. We call the example that starts with 0 and 1 and ends in an infinite sequence the Pinocchio sequence. An example that starts with 2 and 22 ends in a cycle of length 3: 112, 1122, 1122. At the end of this section, we state the main theorem that all the destinies are either the Pinocchio sequence or the above-mentioned 3-cycle.

In the following section, we discuss how to sum two terms in sorted Fibs and what we can say about the previous two terms given the current term. We then turn our attention to the maximum number of twos in two consecutive terms of sorted Fibs and study sequences in which this number is constant. This allows us to finish the proof of the main theorem about destinies in sorted Fibs.

We next describe the rule for reverse sorted Fibs and list their destinies. Each sequence of reverse sorted Fibs ends in one of a series of cycles or in the infinitely growing sequence called the Oihcconip sequence.

We see that sorted Fibs and reverse sorted Fibs have similar destinies to those of RATS sequences. Each of these rules produces one special destiny that grows to infinity and other destinies that cycle. The analogues of the creeper sequence are the Pinocchio and Oihcconip sequences. The good news is that while the destinies of the RATS sequence are only conjectured, in the case of sorted and reverse sorted Fibs, we have a proof.

Exploding Dots and Base 3/2

Propp [9] and Tanton [10] explained the algorithm of exploding dots, in which boxes are arranged in a line. We place N dots in the rightmost box. After that, the dots explode according to a rule that depends on the base. Here is the base-10 rule. Whenever there are ten dots in one box, they explode into one dot, which is placed in the next box to the left. This continues until no boxes with ten or more dots remain. By writing down the resulting number of dots in each box, starting from the left, we get a representation of N in base 10.

We can apply this to other integer bases as well. To write 11 in base 3, we start with 11 dots in the rightmost box, as in Figure 1.

Figure 1: 11 base 3: step 1.

Then each group of three dots in the rightmost box explodes, and one dot per group appears in the box to the left, as in Figure 2.

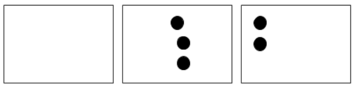

Figure 2: 11 base 3: step 2.

Finally, the three dots in the second box explode into one dot to its left, as shown in Figure 3.

Figure 3: 11 base 3: step 3.

By reading the number of dots in each box from left to right, we see that 11 is written as 102 in base 3. We denote the base-b representation of N by $(N)_b$ and the evaluation of a string of digits w written in base b by $[w]_b$. From our previous example, we have that $(11)_3 = 102$ and $[102]_3 = 11$.

Interestingly, this algorithm can be extended easily from integer bases to fractional bases [10]. We can use a rule in which b dots in one box explode into a dots in the next box to represent integers in base b/a. In this paper, we work in base 3/2, which means that every triplet of dots explodes and creates two new dots in the box to the left. To represent 11 in this base, we use the process shown in Figure 4.

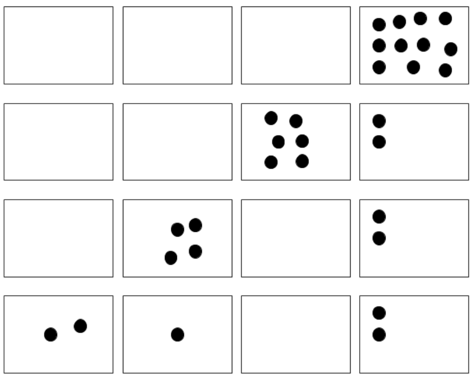

Figure 4: 11 base 3/2.

Using this system of exploding dots, we have that $(11)_{3/2} = 2102$. We can see that every integer is written using the digits 0, 1, and 2.

This representation behaves quite a bit as one would expect from base 3/2. The rightmost box represents $(3/2)^0$, the next $(3/2)^1$, then $(3/2)^2$, and so on. The number $2 \cdot (3/2)^3 + 1 \cdot (3/2)^2 + 0 \cdot (3/2)^1 + 2 \cdot (3/2)^0$ is indeed equal to 11. We can use this just like any other base to represent numbers.

More formally, we express a nonnegative integer N in base 3/2 recursively: $(N)_{3/2} = d_k d_{k-1} \dots d_1 d_0$. The last digit d_0 is the remainder of N modulo 3. The rest of the digits, $d_k d_{k-1} \dots d_1$, are $\left(\frac{2(N-d_0)}{3}\right)_{3/2}$. See [7] for more details.

The first few nonnegative integers are expressed in base 3/2 as follows:

$$0,\ 1,\ 2,\ 20,\ 21,\ 22,\ 210,\ 211,\ 212,\ 2100,\ 2101,\ 2102,\ 2120.$$

There are many fascinating properties of base 3/2. See [10] for those we discuss here and more. For example, integers greater than 1 have 2 as the leftmost digit in their base-3/2 representation. This is true because each carry adds 2 to the previous place. For the same reason, a proper prefix of any integer written in base 3/2 evaluates to an even integer. One might notice that the beginnings of integers are very restricted, but their endings are not.

We see that integers in base 3/2 look like integers in base 3 and base-10 integers written using only the digits 0, 1, and 2. However, if we take a string of digits consisting of 0, 1, and 2 and evaluate it as a number written in base 3/2, we do not always get an integer. As we mentioned before, a string of length greater than 1 starting with the digit 1 cannot be an integer. For example, $[112]_{3/2} = 23/4 = 5.75$. Strings written with the digits 0, 1, and 2 in base 3/2 are called 3/2-integers. They look like integers, but they do not necessarily evaluate to integers.

Examples of Sorted Fibs

We shall now study sorted Fibs in base 3/2. We begin with the sorted Fibs sequence f_n with the same two initial values as in the Fibonacci sequence: $f_0 = 0$ and $f_1 = 1$. To calculate f_{n+1}, we add f_{n-1} and f_n in base 3/2 and sort the digits in increasing order. It follows that the numbers in the sequence are written as several ones followed by several twos.

In contrast to base 10, the sequence is not periodic and grows indefinitely:

$$0,\ 1,\ 1,\ 2,\ 2,\ 12,\ 12,\ 112,\ 112,\ 1112,\ 1112,\ 11112,\ \dots.$$

This sequence plays a special role among sorted Fibs in base 3/2. In recognition of its constant growth, like the nose of Collodi's mendacious marionette, we call this Fibs sequence the Pinocchio sequence.

From now on, we use the notation δ^k to denote a string of k digits in a row that are all δ. If there are only one or two digits in a run, we sometimes drop the exponential notation. The following lemma proves the pattern that can be seen in the Pinocchio sequence above.

Lemma 1. *In the Pinocchio sequence, we have $f_{2k-1} = f_{2k} = 1^{k-2}2$, where $k > 1$.*

Proof. We prove this by induction. The base case is for $k = 2$, in which we have $f_3 = f_4 = 2$. Thus, the base case holds.

To calculate f_{2k+1}, we need to add f_{2k-1} and f_{2k}, which can be represented as $1^{k-2}2$ and $1^{k-2}2$ respectively. The sum is equal to $2^{k-2}4$ before carrying. The digit four is written as 21, so we can replace the 4 with 1 and add 2 to the next digit to the left. After continually carrying 2, we end up with 21^{k-1}. Sorting gives us the desired result.

To calculate f_{2k+2}, we need to add f_{2k} and f_{2k+1}. By the induction hypothesis, $f_{2k+1} = f_{2k} + 10^k = f_{2k-1} + 10^{k-1}$. Using the previous calculation yields $f_{2k} + f_{2k+1} = 21^{k-1} + 10^{k-1} = 201^{k-1}$. After sorting, we get the desired result. □

We are interested in destinies of this sequence depending on the two initial terms. After them, all terms of the sequence are sorted. From now on, we assume that we start with sorted numbers. Here are examples of two starting numbers that end in the same pattern as above: $(1, 1)$, $(2, 112)$, $(1, 12)$. We note that the starting numbers might not be evaluated as integers; they are 3/2-integers.

However, not all starting numbers end in the Pinocchio sequence. Starting with 2 and 22, we get 2, 22, 112, 122, 1122, 122, 122, 112, 1122, 1122, 112, 1122, and so on. The sequence becomes periodic with a period-3 cycle: 112, 1122, 1122.

Our goal is to show that all possible destinies for any starting terms that are 3/2-integers are either the Pinocchio sequence or the 3-cycle above.

Theorem 2. *Every sorted Fibs sequence eventually turns into either the Pinocchio sequence or the 3-cycle 112, 1122, 1122.*

The proof of this theorem spans the next two sections.

Summing Two Terms

When we add two sorted numbers whose digits are in increasing order, the result is also in increasing order before carrying. Thus, we can represent the sum as $1^a 2^b 3^c 4^d$ before we do the carries. The following lemma describes the result after the carries and sorting.

Lemma 3 (Carrying and sorting)**.** *Given the string $1^a 2^b 3^c 4^d$, after performing the carries and sorting, the resulting string is one of the following:*

1. *$a > 0$ and $d > 1$: $1^{c+1}2^d$.*
2. *$a = 0$ and $d > 1$: $1^{c+2}2^{d-1}$.*
3. *$d = 1$: $1^{b+1}2^{c+1}$.*
4. *$c > 0$ and $d = 0$: $1^b 2^c$.*
5. *$c = 0$ and $d = 0$: $1^a 2^b$.*

Proof. We start by assuming $a > 0$ and $d > 1$. After the carries, we get $20^{a-1}20^b1^c2^{d-2}01$. Then, after sorting, we get $1^{c+1}2^d$.

If $a = 0$ and $d > 1$, then after the carries we get $210^b1^c2^{d-2}01$. Then, after sorting, we get $1^{c+2}2^{d-1}$. If $d = 1$, then after the carries we get $20^a1^b2^c1$. When sorted, we get $1^{b+1}2^{c+1}$. If $d = 0$ and $c > 0$, then after the carries we get $20^a1^b2^{c-1}0$. Then after sorting, we get 1^b2^c. Finally, if $c = 0$ and $d = 0$, there are no carries, so sorting gives us the same result: 1^a2^b. □

We previously denoted the terms of the Pinocchio sequence by f_n. We reuse this notation for any sorted Fibs. The term f_n depends on the first two terms of the sequence, which we will specify if needed.

The numbers a, b, c, and d play a big role in the coming proofs. For this reason, we want to associate them with every term of the sequence. That is, a_n, b_n, c_n, and d_n correspond to the sum of f_{n-2} and f_{n-1} before the carries, so we have $f_{n-2} + f_{n-1} = 1^{a_n}2^{b_n}3^{c_n}4^{d_n}$. By assumption, all terms of the sequence are sorted. Let y_n be the number of ones in the nth entry, and let z_n be the number of twos in the nth entry. This gives us

$$f_n = 1^{y_n}2^{z_n}.$$

Integers a_n, b_n, c_n, and d_n give us some information about f_{n-2} and f_{n-1}. For example, we know the minimum of their number of twos:

$$\min\{z_{n-2}, z_{n-1}\} = d.$$

For the maximum of the number of twos there are two possibilities:

$$\max\{z_{n-2}, z_{n-1}\} = c + d \quad \text{and} \quad \max\{z_{n-2}, z_{n-1}\} = b + c + d.$$

The second situation occurs when one of the numbers is 1^c2^d and the other is 1^a2^{b+c+d}.

We can also estimate the total number of digits:

$$c + d \leq \min\{y_{n-2} + z_{n-2}, y_{n-1} + z_{n-1}\} \leq b + c + d$$

and

$$\max\{y_{n-2} + z_{n-2}, y_{n-1} + z_{n-1}\} = a + b + c + d.$$

Every term in a sorted Fibs sequence, except for the first few terms, has at least one 1 and one 2, as the following corollary explains.

Corollary 4. *For a sorted Fibs sequence that starts with sorted strings, if $n \geq 2$, then $z_n > 0$. Also, if $n \geq 4$, then $y_n > 0$.*

Proof. The only case in the list in Lemma 3 in which the resulting number of twos is zero is the last one, when $b = c = d = 0$. This case is impossible, wince we are summing two nonzero numbers, and the last digit before a carry must be greater than 1. Hence for $n \geq 2$, we have $z_n > 0$. When there is at least one 2 in each number, the last digit of their sum is 1, so there must be a 1 in the number for $n \geq 4$. □

We can bound the sequence z_n of the number of twos.

Lemma 5 (Number of twos)**.** *If $n \geq 4$, then $z_n \leq \max\{z_{n-1}, z_{n-2}\}$.*

Proof. If $n \geq 4$, then both f_{n-1} and f_{n-2} have twos. That means that $d_n > 0$. Therefore, from Lemma 3, we have z_n is one of d_n, $d_n - 1$, and $c_n + 1$. In any case, $z_n \leq c_n + d_n$. On the other hand, one of the previous numbers has at least $c_n + d_n$ twos. □

This means that the sequence z_n is bounded.

Maximum Number of Twos in Two Consecutive Terms

We are interested in the eventual behavior of z_n. As often happens in sequences that depend on the two previous terms, it is easier to study not z_n itself, but its maximum value between two consecutive terms.

Let us denote the maximum number of twos in two consecutive terms f_n and f_{n+1} by m_n: $m_n = \max\{z_n, z_{n+1}\}$. From the previous lemma, it follows that $m_{n+1} \leq m_n$ for $n \geq 5$. Since our sequence is infinite, it follows that m_n stabilizes, and since we are interested only in the destinies of pairs of numbers, that is, the eventual behavior of sorted Fibs, we proceed by studying sequences in which m_n is fixed and equal to M. We call such sequences M-stable. As we showed in Lemma 5, every sorted Fibs sequence eventually becomes M-stable.

In the Pinocchio sequence, we have $z_n = 1$ for $n \geq 3$. Therefore, the tail of the Pinocchio sequence is 1-stable. In the 3-cycle, we have $z_{3n} = 1$ and $z_{3n+1} = z_{3n+2} = 2$. In particular, the 3-cycle is 2-stable. It follows that to prove the theorem, it is enough to prove the following facts.

- Every 1-stable sequence eventually turns into a subsequence of the Pinocchio sequence.
- There is no M-stable sequence with $M > 2$.
- Every 2-stable sequence eventually turns into the 3-cycle.

We now prove these facts.

Lemma 6 (1-stable sequences)**.** *Suppose f_n is a 1-stable sequence. Then the sequence f_n for $n > 1$ forms a subsequence of the Pinocchio sequence.*

Proof. It is enough to find two consecutive terms of our sequence that are also consecutive terms in the Pinocchio sequence. Suppose $f_0 = 1^i2$ and $f_1 = 1^j2$. If $i = j$, then f_n is a subsequence of the Pinocchio sequence. If $i \neq j$, then $f_0 + f_1$ before the carry can be represented as $1^{|i-j|}2^{\min\{i,j\}}4$. Thus, by Lemma 3, $f_2 = 1^{\min\{i,j\}+1}2$.

If $i > j$, then $j = \min\{i, j\}$ and $f_2 = 1^{j+1}2$. Hence, starting from $n = 1$, the sequence f_n is a subsequence of the Pinocchio sequence. If $i < j$, then $f_2 = 1^{i+1}2$. If $i + 1 = j$, we hit the Pinocchio sequence. If $i + 1 < j$, we can apply the previous argument to f_1 and f_2 and get into the Pinocchio sequence with the next term.

In all cases, we get into the Pinocchio sequence no later than with f_2. □

Suppose f_n is an M-stable sequence for $M > 1$. Then for any two consecutive terms n and $n+1$, either $z_n = M$ or $z_{n+1} = M$. At least every other value of z_n has to be M. Now we show that we can't have too many M's in a row.

Lemma 7 (Dip in the number of twos)**.** *Suppose f_n is an M-stable sequence for $M > 1$. Then if $z_n = z_{n+1} = M$, then one of z_{n+2}, z_{n+3}, z_{n+4} equals $M - 1$.*

Proof. Suppose $z_n = z_{n+1} = M$. Then when we sum f_n and f_{n+1} without carries, we get $c_{n+2} = 0$ and $d_{n+2} = M$. Now we look at the same cases as in Lemma 3. Cases 3, 4, and 5, where $d \leq 1$, can be excluded.

If we are in case 2, then $z_{n+2} = M - 1$. If we are in case 1, then $f_{n+2} = 12^M$. Now if we sum f_{n+1} and f_{n+2}, we are again in case 1 or 2. If we are in case 2, then $z_{n+3} = M - 1$. If we are in case 1, then $f_{n+3} = 12^M$. Summing f_{n+2} and f_{n+3} gives us 24^M before the carries, which means that $f_{n+4} = 1112^{M-1}$ and $z_{n+4} = M - 1$. □

Lemma 8. *There exists no M-stable sorted Fibs sequences with $M > 2$.*

Proof. By Lemma 7, there exist two consecutive elements f_{n-2} and f_{n-1} in an M-stable sequence such that $z_{n-2} = M$ and $z_{n-1} < M$. As a result, $c_n > 0$ and $d_n = z_{n-1} > 0$. By Lemma 3, this excludes cases 4 and 5. Thus, z_n must equal either $d_n = z_{n-1}$, $d_n - 1 = z_{n-1} - 1$, or $c_n + 1$.

On the other hand, we must have $z_n = M$ to guarantee M-stability. Therefore, $z_n = c_n + 1$, which is possible only in case 3, corresponding to $d_n = 1$. Therefore, $z_{n-1} = 1$. We have shown that if there is an element f_n in the M-stable sequence such that $z_n \neq M$, then $z_n = 1$.

Now we assume that we have terms f_{n-2} and f_{n-1} such that $z_{n-2} = M$ and $z_{n-1} = 1$. We look at the term n. We have $d_n = 1$ and $c_n > 0$. This means that we are in case 3 of Lemma 3. Hence $z_n = c_n + 1 > 1$. We also have $z_n = M$ to guarantee M-stability. Now we calculate the term z_{n+1}. We have $d_{n+1} = 1$ and $c_{n+1} > 0$, which means that we are in case 3 of Lemma 3. Hence $z_{n+1} = c_{n+1} + 1 > 1$. By our discussion above, $z_{n+1} = M$. Then by Lemma 7, there exists $i > n$ such that $z_i = M - 1 > 1$. This contradicts the fact that z_i must be either M or 1. □

We now look into 2-stable sequences.

Lemma 9. *Every 2-stable sorted Fibs sequence eventually turns into the 3-cycle 112, 1122, 1122.*

Proof. By Lemma 7, there exists a term f_n in a 2-stable sequence such that $z_n = 1$. As a result, $z_{n+1} = 2$. Now we calculate the term z_{n+2}. We have $d_{n+2} = 1$ and $c_{n+2} > 0$, which means that we are in case 3 of Lemma 3. Hence $z_{n+2} = c_{n+2} + 1 > 1$. Thus $z_{n+2} = 2$. By Lemma 7, there exists $i > n + 2$ such that $z_i = 1$. We denote by m the smallest such i. In this case, $z_{m-2} = z_{m-1} = 2$ and $z_m = 1$.

Consider the sum $f_{m-2} + f_{m-1}$ before the carries. We have $d_m = 2$ and $c_m = 0$. The fact that $z_m = 1$ means that we are in case 2 of Lemma 3. It follows that $a_m = 0$ and $f_m = 112$. For the next term, we get $b_{m+1} = c_{m+1} = d_{m+1} = 1$. This corresponds to case 3 of Lemma 3, and $f_{m+2} = 1122$. Thus we have gotten into our desired cycle. □

Now we are ready to prove our Theorem 2 about the destinies of sorted Fibs.

Proof of Theorem 2. We have shown that every sorted Fibs sequence eventually becomes M-stable. Then we showed that every 1-stable sequence's destiny is the Pinocchio sequence. After that, we showed that M-stable sequences do not exist if $M > 2$. Finally, we showed that the destiny of every 2-stable sequence is the 3-cycle. This concludes the proof. □

Reverse Sorted Fibs

There are two natural ways to sort the digits of a number: in increasing or decreasing order. Naturally, there is another sequence worth considering.

The reverse sorted Fibs sequence r_n in base 3/2 is defined as follows: to calculate r_{n+1}, we add r_{n-1} and r_n in base 3/2 and sort the digits in decreasing order, ignoring zeros. It follows that after the initial terms, numbers in the sequence are represented by several twos followed by several ones.

We call the sequence that starts like the Fibonacci sequence with $r_0 = 0$ and $r_1 = 1$ the proper reverse sorted Fibs. Here are several terms of the proper reverse sorted Fibs: 0, 1, 1, 2, 2, 21, 21, 221, 2211, 221, 221, 2211, 221, 221, 2211. This sequence becomes cyclic starting from r_7.

We want to study the eventual behavior of the reverse sorted Fibs depending on the starting terms. By computational experiments, we found a series of 3-cycles that such a sequence can turn into:

$$2^k1,\ 2^k1,\ 2^k1^2,$$

where $k > 1$.

We also found a sequence growing indefinitely:

$$2^k1^2,\ 2^k1^2,\ 2^{k+1}1^2,\ 2^{k+1}1^2,\ 2^{k+2}1^2,\ 2^{k+2}1^2,$$

and so on, where $k > 1$. The similarity between the sorted Fibs and the reverse sorted Fibs surprised us. They both have exactly one sequence that grows indefinitely. To emphasize this similarity, we reversed the word Pinocchio and named this growing reverse Fibs sequence the Oihcconip sequence.

We were able to find all the destinies of reverse sorted Fibs sequences, which are summarized in the following theorem.

Theorem 10. *For any two starting numbers, the reverse sorted Fibs sequence always ends in a cycle listed above or the tail of the Oihcconip sequence.*

The proof is done by cases. It is too long to include here, but it is available in our paper [3].

Figure 5: John Conway breaks into Fine Hall.

Concluding Remarks

The photograph in Figure 5 was taken in 2009 and is from Tanya Khovanova's private collection. It shows John's playful nature. In this photo, John wanted to get to his office at Fine Hall, the building hosting Princeton University's math department. It was the weekend and the back door was locked, so John was breaking in. John came prepared: he had all the necessary tools in his pocket, for he had done this many times before. There was another door—the main door—which was open but required an extra one-minute walk to get to it. Plus, breaking into a locked door was way more fun.

Despite their appearance, the above paragraph and picture are both really about mathematics, because for John, it was essential that mathematics be fun.

Indeed, John Conway would have been happy to see young students continuing to explore interesting sequences that exhibit fascinating destinies. The sequences in this paper were inspired by the sequence of Sorted Fibs suggested by John and are a variation of Sorted Fibs in a different base. Curiously, the destinies of these new sequences resemble the destinies of the aforementioned RATS sequence, a different sequence that was also invented by John. The Pinocchio and Oihcconip sequences are analogues of the creeper sequence: a RATS sequence that is not a cycle.

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