ALGEBRAIC SYSTEM THEORY
WITH APPLICATIONS TO DECENTRALIZED CONTROL

by

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ABSTRACT

This research is divided into two major parts. First, a class of linear, finite dimensional, continuous time systems is considered; each system in this class is characterized by the absence of a centralized controller. The concept of decentralized control laws for such systems is then introduced. Four specific classes of decentralized control laws, and the corresponding control objectives are defined; these four classes of control laws are: 1) open loop decentralized control, 2) instantaneous time varying decentralized feedback, 3) instantaneous constant decentralized feedback, and 4) decentralized dynamic compensation. For each of these four control laws sufficient conditions, and in some cases necessary and sufficient conditions, for the controllability of a system are derived. The approach is algebraic in nature, concentrating wherever possible on the underlying algebraic structure of the system in question.

The second portion of the research is motivated by the desire to probe more deeply into the subject of system invariants under decentralized feedback. A module-theoretic characterization of linear, constant systems is developed. The foundation of this theory is Kalman's module-theoretic characterization of linear systems; however, the theory unifies Kalman's work with recently developed polynomial matrix system characterizations. The principal algebraic object in this development is the canonical polynomial matrix, which is shown to play a role in multi-input system theory that is completely analagous to the role played by the minimal annihilator polynomial in single input system theory. A realization theory, based on canonical matrices, is developed; this theory has definite implications towards the realization of infinite dimensional, discrete time systems. Next, it is shown how state feedback may be represented module-theoretically; this representation clearly exhibits the changes in system structure that are attainable via state feedback. Finally, by determining equivalence classes of state feedback laws, the changes in system structure which may be attained by a constrained class of state feedback laws, e.g. decentralized feedback, are determined.

THESIS SUPERVISOR: Sanjoy K. Mitter

TITLE: Professor of Electrical Engineering

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CHAPTER 1

INTRODUCTION

1.1 Introductory Remarks and Summary of Conclusions

In recent years it has been demonstrated that abstract algebra can play a significant role in system theory. By this statement it is meant that algebraic notions such as equivalence relations, congruence relations, partial orderings, universal elements, and canonical factorizations can be interpreted in system-theoretic terms; moreover, such interpretations generally lead to a stronger intuitive feeling, or a better understanding, of systems. Thus, while some may feel that algebra merely introduces abstract nonsense into system theory, most will concede that there is indeed something to be gained by looking at systems algebraically.

It is generally recognized that particular classes of systems may be effectively studied by relating them to particular algebraic categories; usually, if a class of systems is highly structured, then a highly structured algebraic category may be used in studying that class, and quite powerful characterizations of those systems may be expected to result. As examples, the theories of semigroups and monoids have been effectively used in studying automata; the theories of rings and Galois fields enter into coding theory in a natural way; certainly linear algebra and lattices of subspaces are essential in treating linear systems; and it has recently been shown that the structural properties of linear, constant, finite dimensional systems are clearly exhibited in a module-theoretic framework. The success that one has in gaining system-theoretic knowledge by algebraic means is largely dependent on discovering the "right" algebraic category for characterizing the class of systems in question; it is only by discovering the "right" category that full use can be made of its powerful algebraic properties.

In this dissertation we take the philosophy outlined above in studying a particular class of systems, the class of linear, constant, finite dimensional systems with decentralized control. This class of systems is introduced and defined in the beginning of Chapter Two, and our attention is restricted to four classes of decentralized control laws:

1) open loop decentralized control
2) instantaneous time-varying decentralized linear feedback
3) instantaneous constant decentralized linear feedback
4) decentralized linear dynamic compensation

For each of these classes of control laws we also define the control objectives.

The remainder of Chapter Two is involved with determining necessary, sufficient, and both necessary and sufficient, conditions for these various control objectives to be attainable using control laws of the corresponding classes. Throughout this chapter the only algebraic methods that are used are the powerful methods of linear algebra, methods of manipulating subspaces in a lattice of subspaces, and recently developed methods pertaining to (A,B)- invariant and (A,B)- controllability subspaces. The approach that is taken in Chapter Two is to analyze the partitioning of the system state space induced by the various controllable and observable subspaces of the decentralized control agents, and then to determine whether this partitioning is compatible with the particular desired control objective being met. If it is not compatible, then there is still hope that it can be made so by changing the structure of the various controllable and observable subspaces via state feedback.

One of the control objectives that is treated in Chapter Two is the arbitrary allocation of poles via decentralized feedback; in this Chapter several sufficient conditions for decentralized pole allocation are derived. In Chapter Three we continue this discussion and introduce the concept of the invariants of a system under decentralized feedback, that is, those properties of the system that cannot be changed by applying any amount of decentralized feedback. Clearly, if all the invariants can be identified, then it should be possible to characterize, in terms of these invariants, the class of system structures that are attainable via decentralized feedback.

In attempting to identify the invariants of a system under decentralized feedback, the following approach, which is the subject of Chapter Four, was taken. Since recent attempts at deducing the internal structure of a linear system have been highly successful when a module-theoretic framework has been adopted, it was felt that if feedback could be treated module-theoretically then the structural invariants of a system under feedback would become evident.

Therefore, in Chapter Four a method for characterizing a linear system module-theoretically is developed. This characterization uses as a starting point the basic work of Kalman. However, it is shown that each of the module-theoretic aspects of Kalman's theory can be explicitly characterized by an element of the set of canonical polynomial matrices. This adds concreteness to much of Kalman's theory, and demonstrates how the structure of the system may be explicitly expressed in terms of the corresponding canonical matrix. An unforeseen bonus that results from this method of characterizing linear systems is an efficient algorithm for determining a realization of a linear system; moreover, this realization theory is applicable to infinite dimensional, as well as finite dimensional, linear systems.

The remainder of Chapter Four is concerned with determining module-theoretically the structural changes that result in a system when feedback is applied. Since the system structure is expressible in terms of the canonical matrix, the approach that is taken is to determine the changes in the canonical matrix that result when feedback is applied. Then the invariants of the system under feedback may be expressed in terms of those properties of the canonical matrix that cannot be altered by state feedback. By determining equivalence classes of feedback laws, where two feedback laws are said to be equivalent if they result in the same closed loop system structure, the class of system structures attainable via decentralized feedback can then be determined.

Finally, Chapter Four concludes with a discussion of a characterization of (A,B)- invariant and (A,B)- controllability subspaces in terms of canonical matrices.

1.2 Notational Conventions

In this section we briefly summarize the notational conventions that will be adhered to throughout most of the remainder of this dissertation. Additional notations that are germane to a particular chapter or section will be introduced in that chapter or section.

First, a few words about functions, maps, and morphisms. If X and Y are sets, then by the notation

$$f : X \to Y$$

we mean that f is a function from X to Y; X is called the domain of f, and Y is called the codomain of f. To explicitly exhibit the action of f on X, we may also write

$$f : x \mapsto y$$

This notation simply means that $y = f(x)$. For example, the function from the reals to the reals defined by "$y = x^2$" can be represented as

$$f : R \to R$$

$$: x \mapsto x^2$$

If $f : X \to Y$, and if S is a subset of X and T is a subset of Y, then the image of S under f is a subset of Y defined as

$$f(S) = \{f(s) | s \varepsilon S\}$$

and the inverse image of T under f is a subset of X defined as

$$f^{-1}(T) = \{x \varepsilon X | f(x) \varepsilon T\}$$

We also define the image of f as

$$\text{Im } f = f(X)$$

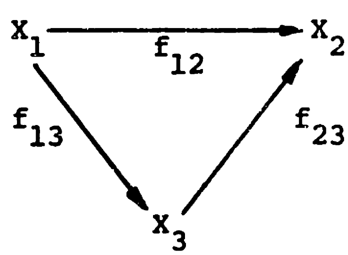
If $f : X \to Y$, we say that f is *surjective* if $\text{Im } f = Y$ and that f is *injective* if f is one-to-one; f is *bijective* if it is both injective and surjective.

If the sets X and Y have additional, and compatible, structure (e.g. as groups, rings, vector spaces, etc.) then we say that $f : X \to Y$ is a *morphism* if it preserves algebraic structure (see [46] or [53]). In this case, if 0 denotes the additive identity of Y, then the *kernel of f* is defined as

$$\text{Ker } f = f^{-1}(\{0\})$$

If X and Y have the structure of K-vector spaces, and also of $K[\lambda]$ - modules, for some field K, then we shall usually (but not always) say that $f : X \to Y$ is a *morphism* if it preserves module structure, and a *map* if it preserves only vector space structure.

By a *commutative diagram* we mean a diagram of the form



where $f_{ij} : X_i \to X_j$, and where

$$f_{23}(f_{12}(x)) = f_{13}(x) \quad \text{, for all } x \in X_1$$

A diagram with more than one "triangle" is commutative if each of the triangular subdiagrams is commutative.

If K is an integer, and if $K \geq 1$, then we define

$$\underline{K} = \{1,2,\ldots K\}$$

If X is a vector space, then the subspaces of X from a <u>lattice</u>; that is, the set of subspaces of X is partially ordered by the inclusion relation:

$$R \subset S \Leftrightarrow (x \in R \Rightarrow x \in S)$$

for R and S subspaces of X. Moreover, we define

$$R \cap S = \{x \mid x \in R \text{ and } x \in S\}$$

and

$$R + S = \{x \mid x = r+s, \text{ for some } r \in R \text{ and } s \in S\}$$

If $R \cap S = 0$, the zero subspace, and if $R + S = T$, then we write

$$T = R \oplus S$$

and say that R and S are independent and that T is the <u>direct sum</u> of R and S. If X is an inner product space, with inner product $\langle x,y \rangle$, then the <u>orthogonal complement</u> of a subspace S is

$$S^{\perp} = \{x \in X \mid \langle x,s \rangle = 0, \text{ for all } s \in S\}$$

It can be easily shown that

$$S \oplus S^{\perp} = X$$

If $A : X \to X$ is a linear map, and if $B \subset X$ is a subspace, then we define

$$AB = \{Ax \mid x \in B\}$$

and

$$\{A \mid B\} = B + AB + A^2B + \ldots$$

Finally, the lattice of subspaces is a <u>modular lattice</u>, since the <u>modular distributive law</u> holds:

$$R \subset T \Rightarrow T \cap (R+S) = R + T \cap S$$

CHAPTER 2

LINEAR SYSTEMS WITH DECENTRALIZED CONTROL

2.1 Introduction

Until fairly recently it has been traditional in system and control theory to consider a dynamical system as being controlled by a single control agent. In such a situation one assumes that all of the available information about the system and its state may be utilized by the single control agent in determining his control strategy. For rather obvious reasons this type of control philosophy is termed "centralized control".

Often, however, it is more reasonable to model the system as being controlled by two or more control agents, each of whom has access to an incomplete information set (i.e. the observations on the system and its state), and each of whom can influence only a portion of the system through his control strategies. Clearly, if there were a higher order control agent, or coordinator, which could receive information from, and send commands to, each of the other control agents, then the overall control of the system would be centralized. In the absence of such a coordinator we shall say that the control of the system is "decentralized".

Decentralized control systems fall into two classes: cooperative and uncooperative. In the former class the individual control agents all strive for the same goal, or at least similar goals; in the latter class the individual agents may be striving for conflicting goals. While there exist several examples of the latter class, e.g. differential games, we shall concern ourselves only with cooperative decentralized control systems.

Examples of cooperative decentralized control systems abound, especially when one considers large scale dynamical systems. Typical examples are: traffic systems, both air and ground; models for macroeconomic systems, where one may conceive of the Congress and the Federal Reserve System as being two uncoordinated control agents [51]; and power distribution systems, where, due to geographical constraints, there can be little coordination between control agents.

In this chapter we shall concern ourselves with a particular class of systems with decentralized control, namely the class of linear, finite dimensional, continuous time systems. Our rationale for restricting attention to this class is as follows. First, due to the powerful mathematical tools that may be applied to linear systems, one can hope to be able to formulate and answer several interesting and meaningful questions concerning the control of these systems. Secondly, many systems with decentralized control may be approximately modeled as belonging to this class. Finally, a fairly thorough theory on the control of linear systems with decentralized control could provide us with an intuitive feeling for the decentralized control of more general systems.

Therefore, we define a linear, finite dimensional, constant system with decentralized control in terms of the following equations:

$$\dot{x} = Ax + \sum_{i \varepsilon \underline{K}} B_i u_i$$

$$y_i = C_i x \quad , i \varepsilon \underline{K} \stackrel{\Delta}{=} \{1,2,\ldots K\}$$

In the above, A is nxn, B_i is nxm_i and C_i is $p_i$xn; $u_i \varepsilon R^{m_i}$ denotes the control applied by, and $y_i \varepsilon R^{p_i}$ the observation made by, control agent i, for each i ε $\underline{K}$. (Consistently throughout this chapter we shall denote a set of integers of the form $\{1,2,\ldots m\}$ by the symbol $\underline{m}$.) In the interests of notational brevity we shall refer to this system as

$$\Sigma = (A,\ B_i,\ C_i,\ i \in \underline{K})$$

with the integers n, and m_i and p_i, $i \in \underline{K}$, being implicit.

If we assume that the control of Σ is initiated at time $t = 0$, then the essence of the decentralization of this control is contained in the following statement:

(2.1-1) For each $i \in \underline{K}$, and each $t \geq 0$, the control value $u_i(t) \in R^{m_i}$ must be expressible as a causal function of the output signal $y_i \in C^{p_i}[0,\infty)$; i.e. for each $i \in \underline{K}$ there exists a family of maps (control laws) $\{F_{i,t}: C^{p_i}[0,t] \to R^{m_i},\ t \geq 0\}$ such that

$$u_i(t) = F_{i,t}[y_i(\tau),\ 0 \leq \tau \leq t],\ \text{for } t \geq 0$$

In order to guarantee that the incorporation of such control laws into Σ will result in y_i being an element of $C^{p_i}[0,\infty)$, for each $i \in \underline{K}$, we also require that the maps $F_{i,t}$ satisfy the following regularity condition:

(2.1-2) The signal u_i defined pointwise as

$$u_i(t) = F_{i,t}[y_i(\tau),\ 0 \leq \tau \leq t],\ \text{for } t \geq 0$$

is at least piecewise continuous in t, for $t \geq 0$, whenever $y_i \in C^{p_i}[0,\infty)$.

Unfortunately, if one allows the control laws to be as general as possible, subject only to (2.1-1) and (2.1-2), then an analysis of the resulting closed loop system appears to be intractable. Therefore, we shall restrict our attention to the following four classes of control laws:

(2.1-3) $F_{i,t}[y(\tau),\ 0 \leq \tau \leq t] = f_i(t,\ y_i(0))$

(2.1-4) $F_{i,t}[y(\tau),\ 0 \leq \tau \leq t] = F_i(t)\ y_i(t)$

(2.1-5) $F_{i,t}[y(\tau),\ 0 \leq \tau \leq t] = F_i\ y_i(t) + G_i\ v_i\ (t)$

(2.1-6) $F_{i,t}[y(\tau),\ 0 \leq \tau \leq t] = F_i(y_i{}'(t) : z_i{}'(t))' + G_i\ v_i(t)$

where z_i satisfies $\dot{z}_i = \hat{A}_i\ z_i + \hat{B}_i\ y_i$

The remainder of this chapter is organized in terms of these four control laws. In Section 2.2 we consider the control law (2.1-3), which is strictly an open loop control law. Our objective in this section will be to establish a theory for the open loop controllability and reachability of Σ. From the results in this section we shall see that a natural objective is to decompose the state space into a direct sum of subspaces, each of which is observed and controlled by a single control agent.

Control law (2.1-4) is considered in Section 2.3. Our goal in this section will be to determine for what pairs of states $(x_o,\ x_1) \in R^n \times R^n$ there exists a $T > 0$ and matrices $F_i(\cdot) \in C^{m_i \times p_i}[0,T]$ such that, when $u_i(t) = F_i(t)\ y_i(t)$, the state of the system is driven from x_o to x_1. It will be shown that recent results in bilinear systems are applicable to this problem.

In Sections 2.4 and 2.5 we consider control law (2.1-5), with the objective being to choose the matricies F_i such that the poles of the resulting closed loop system

$$\hat{\Sigma} = (A + \sum_{j \in \underline{K}} B_j F_j C_j, B_i, C_i, i \in \underline{K})$$

fall at desired locations. As a special case, for example, we may wish to choose the F_i so that $\hat{\Sigma}$ is asymptotically stable. Problems of this nature have been studied by Aoki [2] and, in the discrete time analogue, by McFadden [51], and Morse and Corformat [15]. Our approach will be to consider local (i.e., perturbational) results in Section 2.4, and global results in Section 2.5.

In Section 2.6 we define and derive "generalized observers" that the individual control agents may use to increase their information sets; these are similar to the standard Luenberger observer, with the exception that control agent i does not need to know the controls u_j, for $j \neq i$.

Finally, in Section 2.7 we combine the results of Sections 2.5 and 2.6 to consider control laws of the type (2.1-6), i.e. control laws based on observers or dynamic compensation.

2.2 Open Loop Control of Systems with Decentralized Control

In this section we shall consider the open loop control of systems of the form $\Sigma = (A, B_i, C_i, i \in \underline{K})$, as defined in Section 2.1. Specifically, we shall introduce the concepts of controllability, reachability, and connectedness, all in the open loop sense, as applied to systems with decentralized control. The definitions parallel those in [66]; however, we shall consider only the time invariant case. A discussion of observability and state reconstruction, in the context of decentralized control, is deferred to Section 2.6; if one wishes to assume that each control agent employs a "generalized state reconstructor", then the results of this section are trivially altered by replacing each C_i with $(C_i'; H_i')'$, for some appropriate H_i.

As applied to ordinary linear systems, "controllability" of a system implies the ability to find a control $u_x(\cdot)$ which will drive the system from state x to the zero state in a finite amount of time; moreover, such a control $u_x(\cdot)$ must exist for every $x \varepsilon R^n$. In general, if a system is controllable, then the class of controls which will drive the system x to state zero is strongly dependent on the particular initial state x. However, when the control of the system is decentralized, control agent i cannot measure the entire initial state x, but instead measures $C_i x$; thus the control $u_i(\cdot)$, as applied by control agent i, must be a function of this initial observation $C_i x$.

Similarly, reachability of a system implies the ability to find a control $u_x(\cdot)$ which drives the system from the zero state to state x in a finite amount of time; moreover this must be true for all $x \varepsilon R^n$. It is not immediately clear how this concept changes when the control of the system is decentralized. However, we shall adopt the following convention. We shall assume that agent i is concerned solely with that portion of the state space that can be "seen" through the map C_i; and thus his control $u_i(\cdot)$ is a function of $C_i x$, where x is the target state. Admittedly, this is a somewhat restrictive convention; its rationale is that there is no way that agent i can monitor the effects of his control if it is selected so as to affect a portion of the state space that cannot be seen through C_i. Clearly this restriction is relaxed somewhat if each agent is allowed a"generalized observer."

Finally, a system is connected if for each pair of states $(x_o, x_1) \varepsilon R^n \times R^n$ there exists a control $u_{x_o,x_1}(\cdot)$ which drives the system from the initial state x_o to the target state x_1 in a finite amount of time. Again, when the control is decentralized, we shall insist that the control $u_i(\cdot)$ be a function of $(C_i x_o, C_i x_1)$, for each $i \varepsilon \underline{K}$. These concepts are now summarized formally as

(2.2-1) <u>Definition</u>: Let $\Sigma = (A, B_i, C_i, i \varepsilon \underline{K})$ be a system with decentralized control. Then

(i) Σ is controllable with open loop decentralized controls (or, simply, controllable) if there exists $T < \infty$ and functions $f_i : [0,T] \times R^{p_i} \rightarrow R^{m_i}$ such that the controls

$$u_i(t) = f_i(t, C_i x);\ t \varepsilon [0,T],\ i \varepsilon \underline{K}$$

are at least piecewise continuous in t for all $x \varepsilon R^n$, and such that

$$e^{AT} x + \int_0^T e^{A(T-t)} \sum_{i \varepsilon \underline{K}} B_i f_i(t, C_i x)\, dt = 0,$$

$$\text{for all } x \varepsilon R^n$$

(ii) Σ is reachable with open loop decentralized controls (or, simply, reachable) if there exists $T < \infty$ and functions $g_i : [0,T] \times R^{p_i} \rightarrow R^{m_i}$ such that the controls

$$u_i(t) = g_i(t, C_i x);\ t \varepsilon [0,T],\ i \varepsilon \underline{K}$$

are at least piecewise continuous in t for all $x \varepsilon R^n$, and such that

$$\int_0^T e^{A(T-t)} \sum_{i \varepsilon \underline{K}} B_i g_i(t, C_i x)\, dt = x, \text{ for all } x \varepsilon R^n$$

(iii) Σ is connected with open loop decentralized controls (or, simply, connected) if there exists $T < \infty$ and functions $h_i : [0,T] \times R^{p_i} \times R^{p_i} \to R^{m_i}$ such that the controls

$$u_i(t) = h_i(t, C_i x_o, C_i x); \; t \in [0,T], \; i \in \underline{K}$$

are at least piecewise continuous in t for all $(x_o, x_1) \in R^n \times R^n$, and such that

$$e^{AT} x_o + \int_0^T e^{A(T-t)} \sum_{i \in \underline{K}} B_i h_i(t, C_i x_o, C_i x_1) \, dt = x_1,$$

$$\text{for all } (x_o, x_1) \in R^n \times R^n$$

We shall often refer to the functions f_i, g_i, and h_i as <u>control laws</u>. Of particular interest is the case where the control laws may be chosen to be linear in their second variables:

(2.2-2) <u>Definition</u>: A control law $f : [0,T] \times R^p \to R^m$ is said to be <u>linear</u> if there exists an m x p matrix function $F(\cdot)$ such that

$$f(t, y) = F(t)y; \; t \in [0,T], \; y \in R^p$$

A control law $f : [0,T] \times R^p \times R^p \to R^m$ is said to be <u>linear</u> if there exist two m x p matrix functions, $F_1(\cdot)$ and $F_2(\cdot)$, such that

$$f(t, y_1, y_2) = F_1(t) y_1 + F_2(t) y_2; \; t \in [0,T], \; y_i \in R^p$$

Just as in the centralized control case we have the following result, which allows us to restrict our attention to controllability and reachability:

(2.2-3) <u>Proposition</u>: Σ is connected (connected using linear control laws) (i) if and (ii) only if Σ is both controllable and reachable (controllable and reachable using linear control laws).

Proof: (i) Let Σ be both controllable and reachable, and let T_c and $\{f_i, i \in \underline{K}\}$, and T_r and $\{g_i, i \in \underline{K}\}$ be the respective values of T and the control laws in (2.2-1) (i) and (ii). Define $T = T_c + T_r$ and $h_i : [0,T] \times R^{p_i} \times R^{p_i} \to R^{m_i}$ as

$$h_i(t, C_i x_o, C_i X_1) = \begin{cases} f_i(t, C_i x_o), & 0 \le t \le T_c \\ g_i(t-T_c, C_i x_1), & T_c < t \le T \end{cases}$$

It is then easy to see that these control laws satisfy the condition of (2.2-1) (iii). Clearly, if both f_i and g_i are linear control laws, then so is h_i.

(ii) Clearly, connected implies controllable by considering the special case $x_1 = 0$; similarly, connected implies reachable by taking $x_o = 0$. If the control laws h_i are linear, then so are $f_i(t, C_i x) \triangleq h_i(t, C_i x, 0)$ and $g_i(t, C_i x) \triangleq h_i(t, 0, C_i x)$. ∎

We next derive a set of necessary conditions for Σ to be controllable. These conditions will also be seen to be necessary for Σ to be reachable; thus they are necessary for Σ to be connected.

(2.2-4) __Lemma__: Let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be a system with decentralized control. Then Σ is controllable (reachable) only if

$$\bigcap_{j \in I} \operatorname{Ker} C_j \subset \sum_{j \notin I} \{A|B_j\}$$

for all subsets $I \subset \underline{K}$ (of which there are 2^K), where the intersection of an empty set of subspaces is taken to be R^n, and the sum of an empty set of subspaces is taken to be the zero subspace.

Proof: If Σ is controllable there exist $T < \infty$ and control laws f_i : $[0,T] \times R^{p_i} \to R^{m_i}$ such that

$$\int_0^T e^{-At} \sum_{i\varepsilon K} B_i f_i(t, C_i x)\, dt = -x, \text{ for all } x \varepsilon R^n \tag{2.2-5}$$

From this there immediately follows

$$\int_0^T e^{-At} \sum_{i\varepsilon \underline{K}} B_i f_i(t,0)\, dt = 0 \tag{2.2-6}$$

by evaluating (2.2-5) at $x = 0$.

Now let $I \subset \underline{K}$ be any subset, and take

$$x \in \bigcap_{j\varepsilon I} \operatorname{Ker} C_j$$

From (2.2-5) there then follows

$$\begin{aligned}
-x &= \int_0^T e^{-At}\Big(\sum_{i\varepsilon I} B_i f_i(t, C_i x) + \sum_{i\notin I} B_i f_i(t, C_i x)\Big)\, dt \\
&= \int_0^T e^{-At}\Big(\sum_{i\varepsilon I} B_i f_i(t,0) + \sum_{i\notin I} B_i f_i(t, C_i x)\Big)\, dt \\
&= \int_0^T e^{-At}\Big(\sum_{i\varepsilon \underline{K}} B_i f_i(t,0) + \sum_{i\notin I} B_i (f_i(t,C_i x) - f_i(t,0)\Big)\, dt \\
&= 0 + \int_0^T e^{-At} \sum_{i\notin I} B_i (f_i(t,C_i x) - f_i(t,0))\, dt
\end{aligned}$$

the last line resulting from the use of (2.2-6). But the vector on the

right hand side of the above is necessarily an element of $\sum_{i \notin I} \{A|\mathcal{B}_j\}$.
Thus

$$x \in \bigcap_{j \in I} \text{Ker } C_j \Rightarrow x \in \sum_{j \notin I} \{A|\mathcal{B}_j\}$$

and the lemma is proved for controllability.

If the system is reachable, there exist $T < \infty$ and control laws g_i : $[0,T] \times R^{p_i} \to R^{m_i}$ such that

$$\int_0^T e^{A(T-t)} \sum_{i \in \underline{K}} B_i g_i(t, C_i x)\, dt = x, \text{ for all } x \in R^n$$

The only differences between this equation and (2.2-5) are the presence of an e^{AT} on the left, and the absence of a minus sign on the right. Since neither of these affects the steps in the preceding proof, it is clear that the result is also valid for reachability. ∎

Two special cases of (2.2-4) are now stated as

(2.2-7) Corollary: The system $\Sigma = (A, B_i, C_i, i \in \underline{K})$ is controllable (reachable) only if

(i) $\sum_{i \in \underline{K}} \{A|\mathcal{B}_i\} = R^n$ (controllability of the centralized system)

and

(ii) $\bigcap_{i \in \underline{K}} \text{Ker } C_i = 0$ (complete observations in the centralized system)

We would now like to determine conditions which are both necessary and sufficient for the controllability and reachability of Σ. Unfortunately, it appears to be impossible to find such a set of conditions that can be applied to any system $\Sigma = (A, B_i, C_i, i \in \underline{K})$. Thus, we shall first derive necessary and sufficient conditions for the controllability and reachability of Σ using <u>linear</u> control laws; then we shall consider classes of systems where these conditions apply without the assumption of linear control laws.

We shall need the following result:

(2.2-8) <u>Proposition</u>: Let A be n x n and $B \neq 0$ be n x m, and let $r = \dim \{A|\mathcal{B}\}$. Let G be an n x r matrix whose columns form a basis for $\{A|\mathcal{B}\}$. Then there exists a unique r x m matrix function $\theta(t)$, continuous in t, such that

$$e^{-At} B = G\,\theta(t), \text{ for all } t$$

Moreover, for each $T > 0$, the linear map

$$L : C^m[0,T] \rightarrow R^r$$

$$: u(\cdot) \mapsto \int_0^T \theta(t)\, u(t)\, dt$$

is surjective.

Proof: The existence, uniqueness, and continuity of $\theta(t)$ follow from the facts that $\mathrm{Im}\,(e^{-At} B) \subset \{A|\mathcal{B}\} = \mathcal{G}$ for all t, that G is a basis matrix for $\{A|\mathcal{B}\}$, and that e^{-At} is continuous in t. That L is surjective follows from the facts that G is a basis matrix for $\{A|\mathcal{B}\}$, and that L_1 : $C^m[0,T] \rightarrow \{A|\mathcal{B}\} : u(\cdot) \mapsto \int_0^T e^{-At} B\, u(t)\, dt$ is surjective. ■

(2.2-9) <u>Lemma:</u> Let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be a system with decentralized control. For each $i \in \underline{K}$ let $r_i = \dim \{A|\mathcal{B}_i\}$ and let G_i be an n x r_i matrix whose columns form a basis for $\{A|\mathcal{B}_i\}$. Then Σ is controllable (reachable) with <u>linear</u> control laws if and only if there exist matrices $\phi_i \in R^{r_i \times p_i}$, $i \in \underline{K}$, such that

(2.2-10) $$\sum_{i \in \underline{K}} G_i \phi_i C_i = I$$

Proof: (i) (only if): Let Σ be controllable, and let the linear control laws f_i be given in terms of the matrices $F_i(t)$; thus

$$\int_0^T e^{-At} \sum_{i \in \underline{K}} B_i F_i (t) C_i x \, dt = - x, \text{ for all } x \in R^n$$

For each i

$$\text{Im} (\int_0^T e^{-At} B_i F_i(t) \, dt) \subset \{A|\mathcal{B}_i\} = \mathcal{G}_i$$

Thus there exist ϕ_i such that

$$\int_0^T e^{-At} B_i F_i(t) \, dt = - G_i \phi_i, \; i \in \underline{K}$$

Then

$$- \sum_{i \in \underline{K}} G_i \phi_i C_i x = \sum_{i \in \underline{K}} \int_0^T e^{-At} B_i F_i(t) C_i x \, dt = - x$$

for all $x \in R^n$, i.e.

$$\sum_{i \in \underline{K}} G_i \phi_i C_i = I$$

The proof for reachability is similar, and is therefore omitted.

(ii) (if): Pick the ϕ_i so that (2.2-10) is satisfied. By (2.2-8) there are unique $\theta_i(t) \in C^{r_i \times m_i}[0,\infty)$ such that

$$e^{-At} B_i = G_i \theta_i(t), \text{ for all } t$$

Let $T > 0$ be arbitrary. By (2.2-8) the maps

$$L_i : C^{m_i}[0,T] \to R^{r_i}$$

$$: u(\cdot) \mapsto \int_0^T \theta_i(t)\, u(t)\, dt$$

are surjective; thus there exist matrices $F_i(t)$ such that

$$\int_0^T \theta_i(t)\, F_i\, (t)\, dt = -\phi_i,\ i \in \underline{K}$$

using the $F_i(t)$ so obtained,

$$\int_0^T e^{-At} \sum_{i\in\underline{K}} B_i\, F_i(t)\, C_i\, dt = \sum_{i\in\underline{K}} \int_0^T G_i\, \theta_i(t)\, F_i(t)\, C_i\, dt$$

$$= -\sum_{i\in\underline{K}} G_i\, \phi_i\, C_i$$

$$= -I$$

Thus

$$\int_0^T e^{-At} \sum_{i\in\underline{K}} B_i\, F_i(t)\, C_i\, x\, dt = -x, \text{ for all } x \in R^n$$

The proof for reachability is similar, with the exception that we define the $\theta_i(t)$ to satisfy

$$e^{A(T-t)}\, B_i = G_i\, \theta_i(t)$$

Then, it is easily seen that $u(\cdot) \mapsto \int_0^T \theta_i(t)\, u(t)\, dt$ is surjective; thus define $G_i(t)$ to satisfy

$$\int_0^T \theta_i(t)\, G_i(t)\, dt = \phi_i$$

It is clear that the proof goes through with these minor changes. ■

The result of (2.2-9) can be strengthened somewhat by making use of Kronecker products of matrices. To be definitive, we include

(2.2-11) **Definition**: Let $M \in R^{mxn}$ and $N \in R^{pxq}$ be two arbitrary matrices. Then by the Kronecker product, $M \otimes N$, of M and N we shall mean the following mp x nq matrix:

$$M \otimes N = \begin{pmatrix} M_{11} N & M_{12} N & \cdots & M_{1n} N \\ M_{21} N & M_{22} N & \cdots & M_{2n} N \\ \vdots & & & \\ M_{m1} N & M_{m2} N & \cdots & M_{mn} N \end{pmatrix}$$

(2.2-12) **Theorem**: Let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be a system with decentralized control. Let the integers r_i and the matrices G_i be as defined in (2.2-9). Then Σ is controllable (reachable) with linear control laws if and only if

(2.2-13) $\tilde{e} \in \text{Im}\,(C_1' \otimes G_1;\ C_2' \otimes G_2;\ \ldots\ C_K' \otimes G_K)$

where $\tilde{e} \in R^{n^2}$ is the vector with components

$$\tilde{e}_i = \begin{cases} 1, & \text{if } i = 1, n+2, 2n+3, \ldots n^2 \\ 0, & \text{otherwise} \end{cases}$$

Proof: From (2.2-9) we have only to show that (2.2-13) is satisfied if and only if there exist $\phi_i \in R^{r_i x p_i}$, $i \in \underline{K}$, such that (2.2-10) is satisfied.

To establish this relation, let G, X, C, and Y be matrices of dimensions n x r, r x p, p x n and n x n satisfying

$$Y = G\,X\,C$$

Also, let X and Y be partitioned as

$$X = (x_1;\ x_2;\ \dots\ x_p)$$

$$Y = (y_1;\ y_2;\ \dots\ y_n)$$

and define $\tilde{X} \in R^{rp}$ and $\tilde{Y} \in R^{n^2}$ as

$$\tilde{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$$

$$\tilde{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Then it is a simple matter to verify that

$$\tilde{Y} = (C' \otimes G)\ \tilde{X}$$

Now let $X_i \in R^{r_i x p_i}$, $i \in \underline{K}$, and define Y as

$$Y = \sum_{i \in \underline{K}} G_i\ X_i\ C_i$$

Using the notation of the previous paragraph, it follows that

$$\tilde{Y} = \sum_{i \in \underline{K}} (C_i' \otimes G_i)\ \tilde{X}_i$$

$$= (C_1' \otimes G_1;\ C_2' \otimes G_2;\ \dots\ C_K' \otimes G_K) \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \vdots \\ \tilde{X}_K \end{pmatrix}$$

Thus there exist X_i, $i \in \underline{K}$, such that

$$Y = \sum_{i \in \underline{K}} G_i X_i C_i$$

if and only if

$$\tilde{Y} \in \text{Im}\,(C_1' \otimes G_1; \; \ldots \; C_K' \otimes G_K)$$

Since $\tilde{e}$ is the vector representing the n x n identity matrix, i.e. $\tilde{e} = \tilde{I}$, the theorem follows. ■

While (2.2-12) completely solves the problem of controllability and reachability with linear control laws, it is difficult to interpret in system theoretic terms. That is, we would like to find a result involving the subspaces Ker C_i and $\{A|B_i\}$, $i \in \underline{K}$. In particular, we would like to determine the conditions under which the following occurs: The solution of the controllability problem leads directly to a decomposition of the state space into a direct sum of subspaces, each of which is observed and controlled by a single control agent. The result that we shall prove is the following.

(2.2-14) __Theorem__: Let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be a system with decentralized control. Then, subject to __any__ of the following assumptions

(2.2-15) $$\sum_{i \in \underline{K}} \left(\bigcap_{j \neq i} \text{Ker } C_j \right) = R^n$$

(2.2-16) $$\bigcap_{i \in \underline{K}} \left(\sum_{j \neq i} \{A|B_j\} \right) = 0$$

(2.2-17) $K \leq 2$

necessary and sufficient conditions for Σ to be controllable (reachable) are

(2.2-18) $$\bigcap_{j\epsilon I} \text{Ker } C_j \subset \sum_{j\notin I} \{A|\mathcal{B}_j\}, \text{ for all } I \subseteq \underline{K}$$

Furthermore, in each of these three cases, if (2.2-18) is satisfied, then Σ is controllable (reachable) using linear control laws.

Proof: The necessity of (2.2-18) follows from (2.2-4). The proof for sufficiency is divided into three parts.

Part 1: Assume that (2.2-15) and (2.2-18) are satisfied, and define subspaces S_i as

(2.2-19) $$S_i = \bigcap_{j\neq i} \text{Ker } C_j, \; i \; \epsilon \; \underline{K}$$

We assert that the S_i are independent. Indeed, for each $j\neq i$ it follows from the definition of S_j that $S_j \subset \text{Ker } C_i$; therefore

$$S_i \bigcap \sum_{j\neq i} S_j \subset S_i \bigcap \text{Ker } C_i = \bigcap_{j\epsilon \underline{K}} \text{Ker } C_j = 0$$

where the last equality follows from (2.2-18) with $I = \underline{K}$. Then from (2.2-15),

(2.2-20) $$S_1 \oplus S_2 \oplus \ldots \oplus S_K = R^n$$

Also, as was previously noted,

(2.2-21) $$S_i \bigcap \text{Ker } C_i = 0, \; i \; \epsilon \; \underline{K}$$

and, from (2.2-18), with $I = \underline{K} - \{i\}$, (2.2-22) $\mathcal{S}_i \subset \{A|\mathcal{B}_i\}$, $i \in \underline{K}$

For each $i \in \underline{K}$, determine matrices ϕ_i as follows. Let $d_i = \dim \mathcal{S}_i$, and choose a basis $\{e_{ij}, j \in \underline{d}_i\}$ for $\mathcal{S}_i$. From (2.2-21) it follows that the vectors $\{C_i e_{ij}, j \in \underline{d}_i\}$ are independent; thus there exists an $n \times p_i$ matrix K_i such that

(2.2-23) $K_i C_i e_{ij} = e_{ij}$, $j \in \underline{d}_i$

Moreover, from (2.2-22) it follows that K_i can be chosen so that

(2.2-24) $\operatorname{Im} K_i \subset \{A|\mathcal{B}_i\}$

Now let $r_i = \dim \{A|\mathcal{B}_i\}$, and let G_i be an $n \times r_i$ basis matrix for $\{A|\mathcal{B}_i\}$. From (2.2-24) there is an $r_i \times p_i$ matrix ϕ_i satisfying

(2.2-25) $K_i = G_i \phi_i$

Having determined the matrices ϕ_i, we now note that

$$\sum_{k \in \underline{K}} G_k \phi_k C_k e_{ij} = G_i \phi_i C_i e_{ij} \qquad \text{(from (2.2-19))}$$

$$= K_i C_i e_{ij} \qquad \text{(from (2.2-25))}$$

$$= e_{ij} \qquad \text{(from (2.2-23))}$$

for all $j \in \underline{d}_i$ and $i \in \underline{K}$. But, from (2.2-20) it follows that $\{e_{ij}\}$ is a basis for R^n, whence

$$\sum_{k \in \underline{K}} G_k \phi_k C_k = I$$

and the result follows from (2.2-9).

Part 2: Assume that (2.2-16) and (2.2-18) are satisfied, and define subspaces $\mathcal{R}_i$ as

$$\mathcal{R}_i = \Big(\sum_{j \neq i} \{A|\mathcal{B}_i\} \Big)^{\perp} = \bigcap_{j \neq i} \{A|\mathcal{B}_i\}^{\perp}$$

The $\mathcal{R}_i$ are independent because, from (2.2-18) with $I = \phi$ (the null set),

$$\bigcap_{j \in \underline{K}} \{A|\mathcal{B}_j\}^{\perp} = \Big(\sum_{j \in \underline{K}} \{A|\mathcal{B}_j\} \Big)^{\perp} = (R^n)^{\perp} = 0$$

Thus, from (2.2-16),

$$R^n = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots \oplus \mathcal{R}_K$$

Defining G_i as in part 1, $\bigcap_{j \in \underline{K}} \{A|\mathcal{B}_j\}^{\perp} = 0$ translates to

$$\mathcal{R}_i \cap \text{Ker } G_i' = 0$$

and, from (2.2-18) with $I = \{i\}$,

$$\mathcal{R}_i \subset (\text{Ker } C_i)^{\perp} = \text{Im } C_i'$$

Now, as in part 1, we define $d_i = \dim \mathcal{R}_i$ (in fact $d_i = r_i$) and let $\{e_{ij}, j \in \underline{d}_i\}$ be a basis for $\mathcal{R}_i$. Then $\{G_i' e_{ij}, j \in \underline{d}_i\}$ are independent, so there exists K_i such that

$$K_i G_i' e_{ij} = e_{ij}, \; j \in \underline{d}_i$$

Moreover,

$$\text{Im } K_i = \mathcal{R}_i \subset \text{Im } C_i'$$

Thus there exists ϕ_i' such that

$$K_i = C_i' \phi_i'$$

Finally, having determined the ϕ_i', $i \in \underline{K}$, we verify that

$$\sum_{k \in \underline{K}} C_k' \phi_k' G_k' e_{ij} = C_i' \phi_i' G_i' e_{ij}$$

$$= K_i G_i' e_{ij}$$

$$= e_{ij}$$

for all $j \in \underline{d}_i$, $i \in \underline{K}$. But, $\{e_{ij}\}$ is a basis for R^n, so

$$\sum_{k \in \underline{K}} C_k' \phi_k' G_k' = I$$

and the result follows from (2.2-9).

Part 3: If $K = 1$ and (2.2-18) is satisfied, the result follows trivially, for then $\{A|\mathcal{B}_1\} = R^n$ and C_1 is left invertible. Thus assume that $K = 2$ and that

$$\text{Ker } C_1 \cap \text{Ker } C_2 = 0$$

$$\text{Ker } C_1 \subset \{A|\mathcal{B}_2\}$$

$$\text{Ker } C_2 \subset \{A|\mathcal{B}_1\}$$

(2.2-26) $\{A|\mathcal{B}_1\} + \{A|\mathcal{B}_2\} = R^n$

We shall show that there exist subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ with

(2.2-27) $\text{Ker } C_i \subset \mathcal{N}_i$, $i = 1,2$

and with

$$R^n = \mathcal{N}_1 \oplus \mathcal{N}_2$$

and

$$N_i \subset \{A|\mathcal{B}_j\},\ i \neq j$$

To find the N_i, write

$$\{A|\mathcal{B}_1\} = (\mathrm{Ker}\ C_1 \oplus \mathrm{Ker}\ C_2) \bigcap \{A|\mathcal{B}_1\} \oplus \hat{G}_1$$

for some appropriate $\hat{G}_1$ (not unique, in general). We then have

$$\hat{G}_1 \bigcap (\mathrm{Ker}\ C_1 \oplus \mathrm{Ker}\ C_2) = (\hat{G}_1 \bigcap \{A|\mathcal{B}_1\}) \bigcap (\mathrm{Ker}\ C_1 \oplus \mathrm{Ker}\ C_2)$$

$$= 0$$

so that Ker C_1, Ker C_2, and $\hat{G}_1$ are independent. Define

$$N_2 = \mathrm{Ker}\ C_2 \oplus \hat{G}_1$$

Since both $\hat{G}_1$ and Ker C_2 are subspaces of $\{A|\mathcal{B}_1\}$,

$$\mathrm{Ker}\ C_2 \subset N_2 \subset \{A|\mathcal{B}_1\}$$

as required.

Now write

$$\{A|\mathcal{B}_2\} = (\mathrm{Ker}\ C_1 \oplus N_2) \bigcap \{A|\mathcal{B}_2\} \oplus \hat{G}_2$$

It follows that Ker C_1, N_2 and $\hat{G}_2$ are independent. We define

$$N_1 = \mathrm{Ker}\ C_1 \oplus \hat{G}_2$$

and we have

$$\mathrm{Ker}\ C_1 \subset N_1 \subsetneq \{A|\mathcal{B}_2\}$$

as required.

We must now show that $N_1 \oplus N_2 = R^n$. From (2.2-26), it is enough to show that $N_1 \oplus N_2 \subset \{A|B_i\}$, for i = i,2. But

$$(N_1 \oplus N_2) \cap \{A|B_2\} = (\text{Ker } C_1 \oplus \hat{G}_2 \oplus N_2) \cap \{A|B_2\}$$

$$= \hat{G}_2 \oplus (\text{Ker } C_1 \oplus N_2) \cap \{A|B_2\}$$

$$= \{A|B_2\}$$

whence $\{A|B_2\} \subset N_1 \oplus N_2$. Also

$$(N_1 \oplus N_2) \cap \{A|B_1\} \supset (\text{Ker } C_1 \oplus \text{Ker } C_2 \oplus \hat{G}_1) \cap \{A|B_1\}$$

$$= \hat{G}_1 \oplus (\text{Ker } C_1 \oplus \text{Ker } C_2) \cap \{A|B_1\}$$

$$= \{A|B_1\}$$

whence $\{A|B_1\} \subset N_1 \oplus N_2$.

Finally, from (2.2-27) there exist matrices K_i such that, with $\hat{C}_i = K_i C_i$, Ker $\hat{C}_i = N_i$, for i = 1,2. Thus the system $\hat{\Sigma} = (A, B_i, \hat{C}_i, i=1,2)$ satisfies (2.2-15) and (2.2-18), and it follows that $\hat{\Sigma}$ is controllable and reachable. Letting $\hat{F}_1(t)$ and $\hat{F}_2(t)$ be the matrices in the linear control laws for $\hat{\Sigma}$, it follows that

$$F_i(t) = \hat{F}_i(t) K_i, \ i = 1,2$$

give suitable control laws for Σ. ■

(2.2-28) __Remark__: As was seen in the proof, (2.2-15) together with $\bigcap_{i \in \underline{K}} \text{Ker } C_i = 0$ implies that the state space can be written as a direct sum $S_1 \oplus \ldots \oplus S_K$. For each $i \in \underline{K}$, S_i consists of those states which cannot be seen by agents j, for $j \neq i$; thus, agent i is assigned the task of controlling S_i. This is possible if $S_i \subset \{A|B_i\}$.

(2.2-29) Remark: It is quite easy to show that (2.2-16) implies that the subspaces $\{A|B_i\}$ are independent; then, if (2.2-18) is satisfied when $I = \phi$, the state space can be decomposed as $\{A|B_1\} \oplus \ldots \oplus \{A|B_K\}$. Clearly, control agent i must control $\{A|B_i\}$; and for this to be possible we need $\{A|B_i\} \cap \text{Ker } C_i = 0$. But the latter is true because, from (2.2-18), Ker $C_i =$ $\{A|B_1\} \oplus \ldots \oplus \{A|B_{i-1}\} \oplus \{A|B_{i+1}\} \oplus \ldots \oplus \{A|B_K\}$.

(2.2-30) Remark: It is remarkable that in general the constructive procedure used in part 3 of the above proof will not work when $K > 2$. A more general approach would be to attempt to "increase" each Ker C_i and to "decrease" each $\{A|B_i\}$. That is, we could seek subspaces N_i and W_i satisfying

$$\text{Ker } C_i \subset N_i, \; i \in \underline{K}$$

$$W_i \subset \{A|B_i\}, \; i \in \underline{K}$$

$$\bigcap_{j \in I} N_j \subset \sum_{j \notin I} W_j, \text{ for all } I \subset \underline{K}$$

and at least one of the following:

$$\sum_{i \in \underline{K}} \left(\bigcap_{j \neq i} N_j \right) = R^n$$

or

$$\bigcap_{i\in \underline{K}} \left(\sum_{j\neq i} \mathcal{W}_j \right) = 0$$

However, it is easy to show that a necessary condition for the existence of such subspaces is (2.2-13). Unfortunately, (2.2-18) does not imply (2.2-13), as can be seen from the example in the next remark.

(2.2-31) Remark: That (2.2-18) is not sufficient, when $K > 2$, for the controllability of Σ using linear control laws can be seen from the following example:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad B_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C_1 = (1\ -1\ \ 0\ \ 0), \qquad C_2 = (1\ \ 1\ \ 0\ \ 0)$$

$$C_3 = (1\ \ 0\ \ 0\ \ 0), \qquad C_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is straightforward to show that (2.2-18) is satisfied; however, (2.2-13) is not.

2.3 Linear Systems with Linear, Time Varying, Decentralized Feedback

In this section we consider the control of the system $\Sigma = (A, B_i, C_i, i \in \underline{K})$ using control laws of the following type:

$$u_i(t) = F_i(t)\ y_i(t) = F_i(t)\ C_i\ x(t),\ i \in \underline{K}$$

The control objective will be to drive the system from state x_o at time zero to state x_1 at time T. Defining

$$\hat{A}(t) = A + \sum_{i \in \underline{K}} B_i\ F_i(t)\ C_i$$

and $\phi_{\hat{A}}(t, t_o)$ as the transition matrix for the system $\dot{x} = \hat{A}(t)\ x$, it is clear that x_1 and x_o must be related as

$$x_1 = \phi_{\hat{A}}\ (T,0)\ x_o$$

Thus,in particular, to avoid trivialities we must assume that both x_o and x_1 are nonzero.

(2.3-1) **Definition**: Let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be a system with decentralized control, and let $x_o, x_1 \in R^n$ be two nonzero states. Then we shall say that x_1 is reachable from x_o using time varying decentralized feedback if there exist $T \geq 0$ and $\{F_i(t), i \in \underline{K}\}$ such that

$$x_1 = \phi_{\hat{A}}\ (T,0)\ x_o$$

where $\hat{A}(t) = \sum_{i \in \underline{K}} B_i\ F_i(t)\ C_i$

The problem of determining when x_1 is reachable from x_o can be translated into the framework of some recent results in bilinear systems ([9], [10], and [11]). This is accomplished as follows. Partition each B_i and C_i as

$$B_i = (b_{i1};\ b_{i2};\ \ldots\ b_{i,m_i}),\ i \in \underline{K}$$

$$C_i = \begin{pmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{i,p_i} \end{pmatrix},\ i \in \underline{K}$$

Then if $F_{ijk}(t)$ is the j,k element of $F_i(t)$,

$$B_i\ F_i(t)\ C_i = \sum_{j\in\underline{m_i}} \sum_{k\in\underline{p_i}} F_{ijk}(t)\ b_{ij}\ c_{ik}$$

whence

$$\text{(2.3-2)} \qquad \hat{A}(t) = A + \sum_{i\in\underline{K}} \sum_{j\in\underline{m_i}} \sum_{k\in\underline{p_i}} F_{ikj}(t)\ b_{ij}\ c_{ik}$$

We would now like to determine the set of all transition matrices $\phi_{\hat{A}}(t,0)$ satisfying

$$\dot{\phi}_{\hat{A}}(t,0) = \hat{A}(t)\ \phi_{\hat{A}}(t,0), \qquad \phi_{\hat{A}}(0,0) = I$$

where $\hat{A}(t)$ is given by (2.3-2), and the $F_{ijk}(t)$ are allowed to be any piecewise continuous functions. If this set contains an element of

$$\mathcal{P}_{x_o,x_1} = \{P \in R^{nxn} \mid \det P > 0 \text{ and } P\ x_o = x_1\}$$

then it is clear that x_1 is reachable from x_o. Note that the set $\mathcal{P}_{x_o,x_1}$ is always nonempty if $n \geq 2$, and if both x_0 and x_1 are nonzero.

To determine the set of transition matrices, we first define the Lie algebra generated by A and all the $b_{ij}\ c_{ik}$, $\{A, b_{ij}\ c_{ik}\}_A$, to be the smallest subspace of R^{nxn} which contains A and each $b_{ij}\ c_{ik}$, and which is closed under the Lie bracket operation:

$$[X, Y] = XY - YX; \quad X,Y \in R^{nxn}$$

We also define the Lie group $\{\exp \{A, b_{ij}\ c_{ik}\}_A\}_G$ to be the multiplicative subgroup of R^{nxn} consisting of all finite products of the form $e^{N_1} e^{N_2} \ldots e^{N_\ell}$, for $\ell = 1,2,\ldots,$ where each $N_i \in \{A, b_{ij}\ c_{ik}\}_A$. We now have the following facts:

(2.3-3) __Fact__ ([9]): The set of all transition matrices $\{\phi_{\hat{A}}\ (T,0)\}$ is related to the group $\{\exp \{A, b_{ij}\ c_{ik}\}_A\}_G$ in the following ways:

(i) $\{\phi_{\hat{A}}\ (T,0)\} \subset \{\exp \{A, b_{ij}\ c_{ik}\}_A\}_G$ for all $T \geq 0$.

(ii) If $A = 0$, then $\{\phi_{\hat{A}}\ (T,0)\} = \{\exp \{A, b_{ij}\ c_{ik}\}_A\}_G$ for any $T > 0$.

(iii) If e^{At} is periodic, then

$$\{\phi_{\hat{A}}\ (T,0),\ T \geq 0\} = \{\exp \{A, b_{ij}\ c_{ik}\}_A\}_G$$

We can now state the following result, which follows easily from (2.3-3) and the preceding discussion:

(2.3-4) __Proposition__: Let $n \geq 2$. Then, if e^{At} is periodic, a necessary and sufficient condition for x_1 to be reachable from x_o is that $P_{x_o,x_1} \cap \{\exp \{A, b_{ij}\ c_{ik}\}_A\}_G \neq \phi$.

2.4 Local Results in Pole Allocation via Decentralized Feedback

The contents of this section will be more analytic in nature than algebraic. Specifically, we shall determine how small feedback gains give rise to small perturbations between the open loop and the closed loop poles of a system. Then, if we can find a local right inverse to this relationship, we shall be able to determine small feedback gains which will produce desired changes in the system poles, if these changes are "small enough". If significant changes in the system poles are desired, then one might hope to achieve them by a series of incremental changes in the poles, each incremental change being induced by an additional incremental amount of feedback.

We shall always be dealing with systems defined in terms of real matrices. Thus, we shall necessarily be concerned with changing a set of open loop poles, $\Lambda = \{\lambda_i, i \in \underline{n}\}$, with complex conjugate symmetry ($\lambda \in \Lambda \Rightarrow \bar{\lambda} \in \Lambda$) to a set of closed loop poles, $\hat{\Lambda} = \{\hat{\lambda}_i, i \in \underline{n}\}$ with the same type of symmetry. However, the set of differences, $E = \{\varepsilon_i = \hat{\lambda}_i - \lambda_i, i \in \underline{n}\}$, need not have this symmetry. Partly for this reason, and partly because it is difficult, if not impossible, to functionally express the system poles in terms of the system matrices, we shall concentrate on the changes in the coefficients of the system characteristic polynomial which are induced by small feedback gains. We need only bear in mind the fact that the relation between the system poles and the coefficients of the system characteristic polynomial is a bijection.

We now introduce the concept of local pole assignability.

(2.4-1) Definition: Let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be a system with decentralized control, and let

$$\det(\lambda I - A) = \lambda^n + \sum_{i=0}^{n-1} \lambda^i \alpha_{n-i}$$

be the characteristic polynomial. Then the poles of Σ will be said to be locally assignable via decentralized feedback if there exists $\varepsilon > 0$ such that for each set of real coefficients $\{\hat{\alpha}_i, i \in \underline{n}\}$ satisfying $|\hat{\alpha}_i - \alpha_i| \leq \varepsilon$, $i \in \underline{n}$, there exists at least one set of feedback matrices $\{F_i, i \in \underline{K}\}$ with

$$\det(\lambda I - \hat{A}) = \lambda^n + \sum_{i=0}^{n-1} \lambda^i \hat{\alpha}_{n-i}$$

where $\hat{A} = A + \sum_{i \in \underline{K}} B_i F_i C_i$.

We shall have occasion to use the following result which relates the coefficients of the characteristic polynomial of A to the traces of powers of A.

(2.4-2) Fact: Let A be a real n x n matrix, and let $\{\alpha_i, i \in \underline{n}\}$ denote the coefficients of the characteristic polynomial of A:

$$\det(\lambda I - A) = \lambda^n + \sum_{i=0}^{n-1} \lambda^i \alpha_{n-i}$$

also define $\{s_i, i \in \underline{n}\}$ as

$$s_i = \mathrm{tr}(A^i), \; i \in \underline{n}$$

Then the elements of either of these sets may be uniquely determined from the elements of the other set as

$$\text{(2.4-3)} \qquad \alpha_i = \begin{cases} -s_1, & i = 1 \\ -\frac{1}{i}\left(s_i + \sum_{j=1}^{i-1} s_j\, \alpha_{i-j}\right), & 2 \le i \le n \end{cases}$$

and

$$\text{(2.4-4)} \qquad s_i = \begin{cases} -\alpha_1, & i = 1 \\ -i\,\alpha_i - \sum_{j=1}^{i-1} \alpha_j\, s_{i-j}, & 2 \le i \le n \end{cases}$$

This result, which may be found in [28, p.87], is often referred to as Newton's formula; the quantities s_i are known as Newton's sums. Without rigorously proving the above formulas, it is easy to connect them with some better known results by noting that if $\{\lambda_i,\ i \in \underline{n}\}$ are the eigenvalues of A, then $s_i = \sum_{j \in \underline{n}} \lambda_j^{\,i}$; and

$$\alpha_i = (-1)^i \cdot (\text{sum of all principal } i \times i \text{ minors of } A)$$

$$= (-1)^i \sum_{j_1 < j_2 < \ldots < j_i} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_i}$$

It is then routine, albeit cumbersome, to verify (2.4-3) and (2.4-4).

It is a simple matter to verify that both transformations, $\{s_i\} \mapsto \{\alpha_i\}$ and $\{\alpha_i\} \mapsto \{s_i\}$, are everywhere differentiable to all orders (e.g. both Jacobian matrices are lower triangular with nonzero diagonal elements). Therefore, we can immediately state the following

(2.4-5) <u>Proposition</u>: Let $\Sigma = (A, B_i, C_i,\ i \in \underline{K})$ be a system with decentralized control, and let

$$s_i = \text{tr}\,(A^i),\ i \in \underline{n}$$

be the Newton sums for A. Then the poles of Σ are locally assignable via decentralized feedback if and only if there exists $\varepsilon > 0$ such that for each set of real numbers $\{\hat{s}_i,\ i \in \underline{n}\}$ satisfying $|\hat{s}_i - s_i| \leq \varepsilon$, $i \in \underline{n}$, there exists at least one set of feedback matrices $\{F_i,\ i \in \underline{K}\}$ with

$$\text{tr}\,(\hat{A}^i) = \hat{s}_i,\ i \in \underline{n}$$

where $\hat{A} = A + \sum_{i \in \underline{K}} B_i F_i C_i$.

Our approach will now be to determine the derivative of the map $\{F_i,\ i \in \underline{K}\} \mapsto \{\hat{s}_i,\ i \in \underline{n}\}$. Then, if this derivative is surjective, it will follow that the above map is locally surjective, and thus has a local right inverse. This will provide us with a sufficient condition for local pole assignability. We first prove the following lemma.

(2.4-6) <u>Lemma</u>: Let A and X both be n x n real matrices and define

$$\hat{s}_i(X) = \text{tr}\,[(A + X)^i],\ i \in \underline{n}$$

Then

$$\hat{s}_i(X) = \hat{s}_i(0) + i\,\text{tr}\,(A^{i-1} X) + o(X),\ i \in \underline{n}$$

where $o(X)$ is the small order function with respect to any convenient norm on X.

Proof: Expanding $(A + X)^i$ we get

$$(A + X)^i = A^i + \sum_{j=0}^{i-1} A^j X A^{i-j-1} + \sum_{j=0}^{i-2} \sum_{k=0}^{i-j-2} A^j X A^k X A^{i-j-k-2}$$

$$+ \text{ terms in } X \text{ of order } \geq 3$$

Taking the trace,

$$\mathrm{tr}\ [(A + X)^i] = \mathrm{tr}\ (A^i) + \sum_{j=0}^{i-1} \mathrm{tr}\ (A^j X A^{i-j-1}) + o(X)$$

$$= \mathrm{tr}\ (A^i) + \sum_{j=0}^{i-1} \mathrm{tr}\ (A^{i-1} X) + o(X)$$

$$= \mathrm{tr}\ (A^i) + i\ \mathrm{tr}\ (A^{i-1} X) + o(X)$$

which establishes the lemma. ■

In order to conveniently represent the derivative of the map $\{F_i, i \in \underline{K}\} \mapsto \{\hat{s}_i, i \in \underline{n}\}$, we shall express each F_i (which, recall, is of dimensions $m_i \times p_i$) as

$$F_i = (f_{i1}; f_{i2}; \dots f_{i,p_i})$$

and define the vector $\tilde{f} \in R^q$, $q = \sum_{i \in \underline{K}} m_i\, p_i$, as

$$\tilde{f} = \begin{pmatrix} f_{11} \\ f_{12} \\ \vdots \\ f_{1,p_1} \\ f_{21} \\ \vdots \\ f_{2,p_2} \\ \vdots \\ f_{K,p_K} \end{pmatrix}$$

Then $\{F_i, i \in \underline{K}\} \mapsto \{\hat{s}_i\}$ induces a map $\tilde{f} \mapsto \{\hat{s}_i\}$ in an obvious way. We shall find the derivative of the map $\tilde{f} \mapsto \hat{s}$, where $\hat{s} = (\hat{s}_1, \ldots \hat{s}_n)' \in R^n$.

(2.4-7) **Lemma:** Let $\phi : R^q \to R^n : \tilde{f} \mapsto \hat{s}$ be the map induced by the maps

$$\hat{s}_i = \mathrm{tr}\,[(A + \sum_{k\in\underline{K}} B_k F_k C_k)^i],\ i \in \underline{n}$$

Then the derivative of this map, when evaluated at $\tilde{f} = 0$, is **represented**, with respect to the standard bases in R^q and R^n, by the matrix D:

$$D = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 2 & \ldots & 0 \\ \vdots & & & \\ 0 & .. & 0 & n \end{pmatrix} \begin{pmatrix} c_{11}B_1; & c_{12}B_1; & \ldots & c_{1,p_1}B_1; & c_{21}B_2; & \ldots & c_{K,p_K}B_K \\ c_{11}AB_1; & & \ldots & & & & c_{K,p_K}AB_K \\ \vdots & & & & & & \vdots \\ c_{11}A^{n-1}B_1; & & \ldots & & & & c_{K,p_K}A^{n-1}B_K \end{pmatrix}$$

where $c_{ij}' \in R^n$, and

$$C_i = \begin{pmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{i,p_i} \end{pmatrix},\ i \in \underline{K}$$

Proof: From (2.4-6) it follows that

$$\hat{s}_i = \mathrm{tr}\,(A^i) + i\,\mathrm{tr}\,(A^{i-1} \sum_{k\in\underline{K}} B_k F_k C_k) + o(\sum_{k\in\underline{K}} B_k F_k C_k)$$

Thus

$$\hat{s}_i = \mathrm{tr}\,(A^i) + i \sum_{k\in\underline{K}} \mathrm{tr}\,(C_k A^{i-1} B_k F_k) + o(\tilde{f})$$

But

$$\mathrm{tr}\,(C_k A^{i-1} B_k F_k) = \sum_{j \in \underline{p}_k} c_{kj} A^{i-1} B_k f_{kj}$$

and thus

$$\phi(\tilde{f}) = \phi(0) + D\,\tilde{f} + o(\tilde{f}) \quad \blacksquare$$

An immediate application of the inverse function theorem, e.g. [63, p.35], now provides us with

(2.4-8) **Theorem:** A sufficient condition for the poles of $\Sigma = (A, B_i, C_i, i \in \underline{K})$ to be locally assignable via decentralized feedback is that

$$\mathrm{rank}\ D = n$$

where D is as given in (2.4-7).

Proof: As established in (2.4-5), local pole assignability is equivalent to the local right invertibility of $\phi : R^q \rightarrow R^n$. That is, we must be able to find an $\varepsilon > 0$ and a function $\psi : U \rightarrow R^q$ such that $\phi \circ \psi : U \rightarrow R^n$ is the identity function when its codomain is restricted to U, where

$$U = \{\hat{s} \in R^n \mid |\hat{s}_i - s_i| \leq \varepsilon, i \in \underline{n}\}$$

But, by the inverse function theorem, a sufficient condition for ϕ to have a local right inverse is that D have a right inverse. Clearly the latter is true if and only if the rows of D are independent; since there are n rows in D, the theorem follows. $\blacksquare$

As a direct consequence of (2.4-8), we arrive at our first semi-useful result of this section:

(2.4-9) Corollary: Suppose that $\Sigma = (A, B_i, C_i, i \in \underline{K})$ is stable, but not asymptotically stable. Then a sufficient condition for the existence of feedback matrices $\{F_i, i \in \underline{K}\}$ which render Σ asymptotically stable via decentralized feedback is

$$\text{rank } D = n$$

where D is as given in (2.4-7)

Proof: The hypothesis simply states that A has some eigenvalues on the imaginary axis, but none in the (open) right half plane. Thus, to make Σ asymptotically stable, we merely have to move those eigenvalues that lie on the imaginary axis slightly into the (open) left half plane. This requires only local pole assignability. ■

It should be clear that, since the inverse function theorem does not provide a necessary condition for the existence of a local right inverse to ϕ, the condition rank $D = n$ may fail, and yet the poles of Σ may be locally assignable. We shall attempt shortly to strengthen (2.4-8); however, first we state some necessary conditions for local, and in fact for global, pole assignability via decentralized feedback.

(2.4-10) Proposition: Necessary conditions for the poles of $\Sigma = (A, B_i, C_i, i \in \underline{K})$ to be either locally or globally assignable via decentralized feedback are

(i) $q = \sum_{i \in \underline{K}} m_i p_i \geq n$

(ii) (A, B, C) is both reachable and observable, where

$$B = (B_1; B_2; \dots B_K)$$

and

$$C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_K \end{pmatrix}$$

(iii) For at least one $i \in \underline{K}$, $C_i B_i \neq 0$

Proof: (i) is obvious when one notes that in order to arbitrarily change the n coefficients of the characteristic polynomial, even if these changes are bounded by an arbitrary $\varepsilon > 0$, one needs at least n elements in the vector $\tilde{f}$; i.e. there must be at least n "degrees of freedom" in the matrix $A + \sum_{i \in \underline{K}} B_i F_i C_i$. (ii) follows by noting that decentralized feedback is a special case of state feedback. Since reachability is necessary for arbitrary pole reallocation, even if by arbitrarily small, but nonzero, amounts ([66, Thm. 4], [71]), the reachability part follows; the observability part then follows by duality. Finally, for (iii), we note that if $C_i B_i = 0$ for all $i \in \underline{K}$, then

$$\text{tr}\,(A + \sum_{i \in \underline{K}} B_i F_i C_i) = \text{tr}\,A$$

which simply says that the center of mass of the system poles is invariant under decentralized feedback. ∎

(2.4-11) <u>Corollary</u>: If $\Sigma = (A, B_i, C_i, i \in \underline{K})$ is not stable (asymptotically stable) and if $C_i B_i = 0$ for all $i \in \underline{K}$, then Σ cannot be made stable (asymptotically stable) by decentralized feedback.

Proof: This result follows from the "center of mass" concept in the preceding proof. ■

Finally, if rank D < n, it does not necessarily follow that local assignment of poles via decentralized feedback is impossible. That is, it is quite possible that while the derivative of $\phi : R^q \to R^n$ may be singular when evaluated at $\tilde{f} = 0$, this derivative may be right invertible when evaluated at a small value of $\tilde{f}$. Thus, perturbing the system with a small amount of feedback may result in a full rank D (when calculated from the perturbed system parameters), and thus (2.4-8) may be applied to conclude local pole assignability for the resulting system.

This phenomenon is illustrated by the following example.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C_1 = (1 \quad 0 \quad 0) \quad , \quad C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is easy to check that the necessary conditions of (2.4-10) are all satisfied. Moreover, it is easily seen that for this system any set of poles may be achieved via decentralized feedback; that is, we have global pole assignability. However,

$$D = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 6 & 3 \end{pmatrix}$$

and we can conclude nothing from (2.4-8).

However, if the system is perturbed slightly by letting

$$F_1 = \varepsilon$$

$$F_2 = (0 \quad 0)$$

then

$$A + B_1 F_1 C_1 + B_2 F_2 C_2 = \begin{pmatrix} 1+\varepsilon & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and, if this system matrix is used in calculating D, there results

$$D = \begin{pmatrix} 1 & 0 & 1 \\ 2(1+\varepsilon) & 2 & 2 \\ 3(1+\varepsilon)^2 & 6 & 3 \end{pmatrix}$$

which is easily seen to be nonsingular for any $\varepsilon \neq 0$.

One can now pose the following problem:

(2.4-12) **Problem**: Let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be a system with decentralized control, and define the n x q matrix $\hat{D}(F_1, F_2, \ldots F_K)$ to be the following function of the matrices $F_i \in R^{m_i x p_i}$, for $i \in \underline{K}$:

$$\hat{D}(F_1, \ldots F_K) = \begin{pmatrix} c_{11}B_1 & ; \ldots & c_{1p_1}B_1 ; \ldots & c_{Kp_K}B_K \\ c_{11}\hat{A}B_1 & ; \ldots & c_{1p_1}\hat{A}B_i ; \ldots & c_{Kp_K}\hat{A}B_K \\ \vdots & & \vdots & \\ c_{11}\hat{A}^{n-1}B_1 ; & \ldots & c_{1p_1}\hat{A}^{n-1}B_1 ; \ldots & c_{Kp_K}\hat{A}^{n-1}B_K \end{pmatrix}$$

Where $\hat{A} = A + \sum_{i \varepsilon \underline{K}} B_i F_i C_i$ and $c_{ij}' \varepsilon R^n$ is defined as

$$C_i = \begin{pmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{ip_i} \end{pmatrix} \quad , \text{ for } i \; \varepsilon \; \underline{K}$$

Suppose that rank $\hat{D}(0, 0, \ldots 0) < n$, so that one can conclude nothing from (2.4-8). Under what conditions do there exist "small" values of the F_i such that rank $\hat{D}$ $(F_1, \ldots F_K) = n$?

The approach that one may take in attempting to solve this problem is as follows. One can construct from the function $\hat{D}(F_1, \ldots F_K)$ a polynomial $\Delta(F_{ijk})$ in the q components of the F_i, i ε $\underline{K}$, having the property that $\Delta(F_{ijk}) = 0$ if and only if rank $\hat{D}(F_1, \ldots F_K) < n$. Such a polynomial is

(2.4-13) $\Delta(F_{ijk})$ = sum of the squares of all n x n minors of $\hat{D}(F_1, \ldots F_K)$

Clearly, $\Delta(F_{ijk})$ has the desired property. Then, if $\Delta(F_{ijk})$ is not the zero polynomial (i.e. if $\Delta(F_{ijk})$ is not identically zero), it will follow that the locus of zeros of $\Delta(F_{ijk})$:

$$V = \{\tilde{f} = \begin{pmatrix} F_{111} \\ \vdots \\ F_{1,m_1,p_1} \\ \vdots \\ F_{K,m_K,p_K} \end{pmatrix} \varepsilon\, R^q \mid \Delta(F_{ijk}) = 0\}$$

is a <u>proper</u> algebraic variety ([19, p.2], [26]) in R^q, and consequently, for any point $v \in V$, there exists an arbitrarily small vector $h \in R^q$ such that $v + h \notin V$. This discussion leads directly to the next definition and lemma.

(2.4-14) **Definition:** The matrix function $\hat{D}(F_i, \ldots F_K)$ of (2.4-12) is said to be generically of rank n if whenever rank $\hat{D}(F_1, \ldots F_K) < n$ for a particular choice of the F_i, $i \in \underline{K}$, there exist arbitrarily small matrices $H_i \in R^{m_i x p_i}$, $i \in \underline{K}$, such that rank $\hat{D}(F_1 + H_1, \ldots F_K + H_K) = n$.

(2.4-15) **Lemma:** The matrix function $\hat{D}(F_1, \ldots F_K)$ of (2.4-12) is generically of rank n if and only if the polynomial $\Delta(F_{ijk})$ of (2.4-13) is not the zero polynomial.

These ideas can best be illustrated in terms of the last example. Denoting

$$F_1 = \alpha$$

$$F_2 = (\beta \quad \alpha)$$

we have

$$A + B_1 F_1 C_1 + B_2 F_2 C_2 = \begin{pmatrix} 1+\alpha & 1 & 0 \\ 0 & 1 & 1 \\ 0 & \beta & 1+\alpha \end{pmatrix}$$

and

$$\hat{D}(F_1, F_2) = \begin{pmatrix} 1 & 0 & 1 \\ 1+\alpha & 1 & 1+\gamma \\ (1+\alpha)^2 & 2+\gamma & (1+\gamma)^2 + \beta \end{pmatrix}$$

Then,

$$\Delta(\alpha, \beta, \gamma) = (\beta + \alpha(\gamma - \alpha))^2$$

and, since $\Delta(\alpha, \beta, \gamma)$ is not identically zero, we conclude that for this system $\hat{D}(F_1, F_2)$ is generically of rank 3.

Finally, if (2.4-8) fails, one might wonder if $\hat{D}(F_1, \ldots F_K)$ being generically of rank n is enough to ensure local pole assignability in $\Sigma = (A, B_i, C_i, i \in \underline{K})$. Unfortunately, the answer is "no". This is illustrated by the following example:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad B_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad C_2 = (0 \quad 0 \quad 1)$$

The derivative matrix D is

$$D = \begin{pmatrix} 1 & 0 & 1 \\ 4 & 0 & 4 \\ 12 & 3 & 9 \end{pmatrix}$$

which has rank 2. However, letting

$$F_1 = (\alpha \quad \beta)$$

$$F_2 = \gamma$$

the matrix $\hat{D}(F_1, F_2)$ is

$$\hat{D}(F_1, F_2) = \begin{pmatrix} 1 & 0 & 1 \\ 2+\alpha & 0 & 2+\gamma \\ (2+\alpha)^2 & 1 & (1+\alpha)(3+\gamma) \end{pmatrix}$$

Therefore, since

$$\Delta(\alpha, \beta, \gamma) = (\gamma - \alpha)^2$$

it follows that $\hat{D}(F_1, F_2)$ is generically of rank 3, and one is tempted to hope that the poles of Σ are locally assignable.

However, the Newton sums of $A + B_1 F_1 C_1 + B_2 F_2 C_2$ are

$$s_1 = 4 + \alpha + \gamma$$

$$s_2 = 6 + \alpha^2 + 4\alpha + \gamma^2 + 4\gamma$$

$$s_3 = 10 + \alpha^3 + 6\alpha^2 + 12\alpha + 3\beta + \gamma^3 + 6\gamma^2 + 9\gamma$$

The locus of points (α, β, γ) for which

$$s_2(\alpha, \beta, \gamma) = s_2(0, 0, 0) = 6$$

is a circular cylinder of radius $\sqrt{8}$ whose axis is the line $(-2, \cdot, -2)$. Thus, the maximum value that s_1 can attain, subject to the constraint that $s_2 = 6$, is 4, i.e. the value at the point (0, 0, 0). In short, in this example one cannot increase s_1 while keeping s_2 constant; thus the poles of Σ are not locally assignable.

As a concluding remark we may note that the ideas of this section suggest search techniques (e.g.Newton-Raphson or a gradient method) for finding feedback gains which yield desired closed loop poles. Probably a necessary condition for the well-posedness of such a technique is that $\hat{D}(F_1, \ldots F_K)$ be generically of rank n. Of course this condition implies nothing about the existence of suitable feedback gains, nor does it say anything about the convergence of the search algorithms.

2.5 Pole Allocation via Decentralized Feedback

In this section we continue the discussion of pole allocation via decentralized feedback; however, we shall use a global approach rather than the local approach employed in Section 2.4. The underlying philosophy is to introduce, by decentralized feedback, as much noninteraction into the system Σ as possible, effectively producing a system that is naturally decomposed into independent subsystems. If the i'th control agent can both observe and control the i'th subsystem, then he may apply feedback to arbitrarily reallocate its poles.

We shall make use of the theory of (A, B) - invariant subspaces and (A, B) - controllability subspaces, as developed by Wonham and Morse ([54]-[59], [72], [74], [75]). The results which are germane to our discussion are summarized in Appendix A.

Thus, let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be the system whose poles we wish to relocate. We shall make two assumptions about the observation matrices C_i. The first of these is

$$(2.5\text{-}1) \qquad \bigcap_{i \in \underline{K}} \operatorname{Ker} C_i = 0$$

This simply says that a central control agent having access to all the outputs y_i could uniquely determine the state x without using a dynamic observer. The second assumption is

$$(2.5\text{-}2) \qquad \sum_{i \in \underline{K}} \bigcap_{j \neq i} \operatorname{Ker} C_j = R^n$$

which says that the observation vectors y_i are independent. We shall later be able to relax these assumptions somewhat.

These two assumptions now lead naturally to a decomposition of R^n. Defining the subspaces

$$\mathcal{S}_i = \bigcap_{j \neq i} \operatorname{Ker} C_j, \ i \in \underline{K}$$

it is easy to show (see the proof to (2.2-14)) that the $\mathcal{S}_i$ are independent and that

$$(2.5\text{-}3) \qquad \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \dots \oplus \mathcal{S}_K = R^n$$

Subspace $\mathcal{S}_i$ contains all states that cannot be seen by control agents j, for $j \neq i$; thus it is natural to require that agent i be able to control $\mathcal{S}_i$.

Our approach will be as follows. We shall first attempt to find feedback matrices $F_i \in R^{m_i x p_i}$ which either decouple or triangularly decouple Σ with respect to the $\mathcal{S}_i$, i.e. such that

$$\hat{A}\, S_i \subset S_i,\ i \in \underline{K}$$

or

$$\hat{A}\, S_i \subset S_1 \oplus \ldots \oplus S_i,\ i \in \underline{K}$$

where $\hat{A} = A + \sum_{i \in \underline{K}} B_i F_i C_i$. Then, if certain controllability conditions are met, control agent i may apply additional feedback from y_i to u_i so as to allocate the poles of the system restricted to S_i.

We first define, for each $i \in \underline{K}$, the map $P_i : R^n \to R^n$ to be the projection onto S_i along $\sum_{j \neq i} S_j$; that is, $\mathrm{Im}\ P_i = S_i$ and $\mathrm{Ker}\ P_i = \sum_{j \neq i} S_j$. As a result, the P_i satisfy

$$P_i^{\,2} = P_i\ ,\ i \in \underline{K}$$

$$P_i\, P_j = 0;\ i,\ j \in \underline{K} \text{ and } i \neq j$$

(2.5-4) $$\sum_{i \in \underline{K}} P_i = I$$

Moreover, it also follows that

(2.5-5) <u>Proposition</u>: For each $i \in \underline{K}$ there exists $K_i \in R^{n \times p_i}$ such that $P_i = K_i\, C_i$.

Proof: From the definition of the S_i it follows that $S_j \subset \mathrm{Ker}\ C_i$ for all $j \neq i$, whence

$$\sum_{j \neq i} S_j \subset \mathrm{Ker}\ C_i$$

But then, from (2.5-3)

$$\begin{aligned}\text{Ker } C_i &= R^n \bigcap \text{Ker } C_i \\ &= (S_i + \sum_{j\neq i} S_j) \bigcap \text{Ker } C_i \\ &= S_i \bigcap \text{Ker } C_i + \sum_{j\neq i} S_j \\ &= 0 + \sum_{j\neq i} S_j\end{aligned}$$

using (2.5-1). Thus Ker P_i = Ker C_i, and the result follows. ■

For ease of presentation of the remainder of this section we include

(2.5-6) **Definition:** Let Λ be a set of n complex numbers. Then Λ is said to be symmetric with respect to the subspaces $\{S_i, i \in \underline{K}\}$ if it can be written as a disjoint union: $\Lambda = \bigcup_{i\in \underline{K}} \Lambda_i$, $\Lambda_i \bigcap \Lambda_j = \phi$ for $i \neq j$; and such that, for $i \in \underline{K}$, Λ_i contains $d_i = \dim S_i$ elements and is closed under complex conjugation.

Our first result pertains to decoupling Σ with respect to the subspaces S_i.

(2.5-7) **Lemma:** If, for each i, S_i is an (A, B_i) - controllability subspace, then the poles of Σ can be allocated via decentralized feedback so as to correspond with the n elements of Λ, provided that Λ is symmetric with respect to $\{S_i, i \in \underline{K}\}$.

Proof: Since S_i is an (A, B_i) - controllability subspace, there exist matrices $\tilde{F}_i$ and G_i such that

$$\{A + B_i \tilde{F}_i \mid \text{Im } (B_i G_i)\} = S_i$$

Therefore, letting the matrices K_i be given as in (2.5-5), the feedback laws

$$u_i = \tilde{F}_i K_i y_i + G_i v_i$$

$$= \tilde{F}_i P_i x + G_i v_i$$

produce the following closed loop system:

$$\dot{x} = (A + \sum_{i \varepsilon \underline{K}} B_i \tilde{F}_i P_i) x + \sum_{i \varepsilon \underline{K}} B_i G_i v_i$$

For each i define $x_i = P_i x$; from (2.5-4), $x = \sum_{i \varepsilon \underline{K}} x_i$. Then

$$\dot{x}_i = P_i (A + \sum_{j \varepsilon \underline{K}} B_j F_j P_j) \left(\sum_{k \varepsilon \underline{K}} x_k\right) + \sum_{j \varepsilon \underline{K}} P_i B_j G_j v_j$$

But, $\text{Im} (B_j G_j) \subset S_j$, and

$$(A + \sum_{j \varepsilon \underline{K}} B_j \tilde{F}_j P_j) S_k = (A + B_k \tilde{F}_k) S_k$$

$$\subset S_k$$

Therefore,

$$\dot{x}_i = (A + B_i \tilde{F}_i) x_i + B_i G_i v_i, \; i \in \underline{K}$$

That is, the dynamics for the vectors x_i are uncoupled.

Finally, since $\{A + B_i \tilde{F}_i \mid \text{Im} (B_i G_i)\} = S_i$, there exists a matrix $\hat{F}_i$ such that the spectrum of the restriction of $A + B_i \tilde{F}_i + B_i G_i \hat{F}_i$ to S_i is any desired set of $d_i = \dim S_i$ complex numbers, provided that this set is closed under complex conjugation.

Thus, pick the $\hat{F}_i$ accordingly, define

$$F_i = (\tilde{F}_i + G_i \hat{F}_i) K_i, \; i \in \underline{K}$$

and the proof is complete. ■

While it may happen that Σ can be decoupled as above, the likelihood of this is not very great. Thus we next shall develop a theory of decentralized pole allocation based on the idea of triangularly decoupling the system. We shall need the following lemma:

(2.5-8) **Lemma:** Let $M: R^n \to R^n$ be a map such that, for each $i \in \underline{K}$,

$$M\, S_i \subset S_1 \oplus \dots \oplus S_i$$

Then, for $i \in \underline{K}$ and for $k = 0, 1, \dots n-1$,

(2.5-9) $P_i\, M^k\, P_i = (P_i\, M)^k\, P_i$

(2.5-10) $P_i\, M^k\, x = (P_i\, M)^k\, P_i\, x$, for all $x \in S_1 \oplus \dots \oplus S_i$

Proof: The proof for (2.5-9) is by induction on k; the result is obvious for k = 0 or 1. Suppose the result to be true for $k = \ell - 1$ and write

$$P_i\, M^\ell\, P_i = P_i\, M\, M^{\ell-1}\, P_i = \sum_{j \in \underline{K}} P_i\, M\, P_j\, M^{\ell-1}\, P_i$$

But $\operatorname{Im}(M^{\ell-1}\, P_i) \subset S_1 \oplus \dots \oplus S_i$ by hypothesis; thus

$$P_i\, M^\ell\, P_i = \sum_{j=1}^{i} P_i\, M\, P_j\, M^{\ell-1}\, P_i$$

Likewise, $\operatorname{Im}(M\, P_j) \subset S_1 \oplus \dots \oplus S_j$, and so

$$P_i\, M^\ell\, P_i = P_i\, M\, P_i\, M^{\ell-1}\, P_i$$

$$= P_i\, M\, (P_i\, M)^{\ell-1}\, P_i \qquad \text{(induction)}$$

$$= (P_i\, M)^\ell\, P_i$$

To prove (2.5-10), note that since $x \in S_1 \oplus \dots \oplus S_i$,

$$x = \sum_{j=1}^{i} P_j \, x$$

Therefore

$$P_i \, M^k \, x = \sum_{j=1}^{i} P_i \, M^k \, P_j \, x$$

But $\mathrm{Im}\,(M^k P_j) \subset \mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_j$ by hypothesis, so

$$P_i \, M^k \, x = P_i \, M^k \, P_i \, x$$

$$= (P_i \, M)^k \, P_i \, x$$

where the last step follows from (2.5-9). ■

The next result pertains to the situation when Σ is practically triangularly decoupled to begin with.

(2.5-11) <u>Lemma</u>: If Σ is such that

(2.5-12) $A \, \mathcal{S}_i \subset \mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i, \; i \in \underline{K}$

and

(2.5-13) $P_i \, \{A \mid \mathcal{B}_i \cap (\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i)\} = \mathcal{S}_i, \; i \in \underline{K}$

then the conclusion of (2.5-7) is valid.

Proof: Let G_i be matrices such that

$$\mathrm{Im}\;(B_i \, G_i) = \mathcal{B}_i \cap (\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i), \; i \in \underline{K}$$

and define the controls u_i as

$$u_i = G_i \, v_i, \; i \in \underline{K}$$

Thus, $\dot{x} = Ax + \sum_{j \in \underline{K}} B_j G_j v_j$; and, letting $x_i = P_i x$,

$$\dot{x}_i = P_i \left(Ax + \sum_{j \in \underline{K}} B_j G_j v_j\right)$$

$$= \sum_{j \in \underline{K}} (P_i Ax_j + P_i B_j G_j v_j)$$

In light of (2.5-12), $Ax_j \in S_1 \oplus \ldots \oplus S_j$; moreover by the choice of G_j, $B_j G_j v_j \in S_1 \oplus \ldots \oplus S_j$. Thus

$$\dot{x}_i = \sum_{j=i}^{K} (P_i A x_j + P_i B_j G_j v_j), \; i \in \underline{K}$$

and the system is in triangularly decoupled form.

This structure is retained by feedback of the form

$$v_j = \hat{F}_j K_j y_j = \hat{F}_j P_j x = \hat{F}_j x_j$$

which yields the closed loop system

$$\dot{x}_i = \sum_{j=i}^{K} (P_i A + P_i B_j G_j \hat{F}_j) x_j, \; i \in \underline{K}$$

The set of poles for the above system is then just the union of the spectra of the maps $(P_i A + P_i B_i G_i \hat{F}_i)$ restricted to the corresponding S_i. Thus, for each $i \in \underline{K}$, $d_i = \dim S_i$ arbitrary poles with conjugate symmetry may be assigned to the closed loop system by choosing $\hat{F}_i$ appropriately if

(2.5-14) $\{P_i A \mid \mathrm{Im}\,(P_i B_i G_i)\} = S_i$

However, (2.5-14) follows from the lemma hypothesis if one makes use of (2.5-8). That is, from (2.5-13) it follows that

$$\sum_{k=0}^{n-1} P_i A^k (B_i \cap (S_1 \oplus \ldots \oplus S_i)) = S_i$$

But from hypothesis (2.5-12) and (2.5-10)

$$P_i A^k (\mathcal{B}_i \cap (\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i)) = (P_i A)^k P_i (\mathcal{B}_i \cap (\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i))$$

Thus

$$\{P_i A \mid \operatorname{Im} (P_i B_i G_i)\} = \{P_i A \mid P_i (\mathcal{B}_i \cap (\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i))\}$$

$$= P_i \{A \mid \mathcal{B}_i \cap (\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i)\}$$

$$= \mathcal{S}_i$$

and (2.5-14) follows.

The feedback laws that accomplish all of the above are, of course,

$$v_i = \hat{F}_i K_i y_i$$

or,

$$u_i = G_i \hat{F}_i K_i y_i \qquad \blacksquare$$

Of course, the system hypothesized in (2.5-11) is practically trangularly decoupled to begin with. If this were not the case, then one could attempt to triangularly decouple the system through decentralized feedback. That is, it is now natural to look for matrices $\tilde{F}_i$ such that the matrix

$$\tilde{A} = A + \sum_{i \in \underline{K}} B_i \tilde{F}_i K_i C_i$$

$$= A + \sum_{i \in \underline{K}} B_i \tilde{F}_i P_i$$

satisfies the hypotheses of (2.5-11). The following lemma simplifies this problem.

(2.5-15) **Lemma:** There exist matrices $\tilde{F}_i$ such that, with

$$\tilde{A} = A + \sum_{i \varepsilon \underline{K}} B_i \tilde{F}_i P_i,$$

(2.5-16) $\tilde{A}\, S_i \subset S_1 \oplus \ldots \oplus S_i, \; i \;\varepsilon\; \underline{K}$

(2.5-17) $P_i \; \{\tilde{A} \mid B_i \bigcap (S_1 \oplus \ldots \oplus S_i)\} = S_i, \; i \;\varepsilon\; \underline{K}$

if and only if there exist matrices $\overset{*}{F}_i$ such that

(2.5-18) $(A + B_i \overset{*}{F}_i)\; S_i \subset S_1 \oplus \ldots \oplus S_i, \; i \;\varepsilon\; \underline{K}$ or

(2.5-19) $P_i \; \{A + B_i \overset{*}{F}_i \mid B_i \bigcap (S_1 \oplus \ldots \oplus S_i)\} = S_i, \; i \;\varepsilon\; \underline{K}$

Proof: We show that if (2.5-16) and (2.5-17) are satisfied using matrices $\tilde{F}_i$, then (2.5-18) and (2.5-19) are satisfied when $\overset{*}{F}_i = \tilde{F}_i$. That is, let (2.5-16) be satisfied. Then, since

$$\tilde{A}\, S_i = (A + B_i \tilde{F}_i)\; S_i$$

it is clear that (2.5-18) is satisfied using $\overset{*}{F}_i = \tilde{F}_i$. Next, if (2.5-16) and (2.5-17) are satisfied, using (2.5-8),

$$P_i \; \{\tilde{A} \mid B_i \bigcap (S_1 \oplus \ldots \oplus S_i)\} = \sum_{k=0}^{n-1} P_i \; \tilde{A}^k \; (B_i \bigcap (S_1 \oplus \ldots \oplus S_i))$$

$$= \sum_{k=0}^{n-1} (P_i \; \tilde{A})^k \; P_i \; (B_i \bigcap (S_1 \oplus \ldots \oplus S_i))$$

$$= \sum_{k=0}^{n-1} [P_i (A + B_i \tilde{F}_i)]^k P_i (B_i \bigcap (S_1 \oplus \ldots \oplus S_i))$$

$$= \sum_{k=0}^{n-1} P_i \; (A + B_i \; \tilde{F}_i)^k (B_i \bigcap (S_1 \oplus \ldots \oplus S_i))$$

$$= P_i \{A + B_i \tilde{F}_i \mid B_i \bigcap (S_1 \oplus \ldots \oplus S_i)\}$$

where the second to last line follows since (2.5-18) is satisfied. Therefore, (2.5-19) is satisfied using $\overset{*}{F}_i = \tilde{F}_i$.

Similarly, if (2.5-18) is satisfied for matrices $\overset{*}{F}_i$, then (2.5-16) is satisfied using $\tilde{F}_i = \overset{*}{F}_i$. Then, by reversing the steps above, (2.5-18) and (2.5-19) imply (2.5-17) when $\tilde{F}_i = \overset{*}{F}_i$. ■

The significance of (2.5-15) is that it is now possible to find the matrices $\tilde{F}_i$ <u>independently</u> of one another. As might be expected, the existence of matrices $\overset{*}{F}_i$ satisfying (2.5-18) and (2.5-19) can be expressed in terms of controllability subspaces.

(2.5-20) <u>Lemma</u>: Let $\mathcal{R}_i$ be the maximal (A, B_i)-controllability subspace that is contained in $\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i$. Then, if $\mathcal{S}_i \subset \mathcal{R}_i$, a matrix $\overset{*}{F}_i$ exists which satisfies (2.5-18) and (2.5-19).

Proof: Assuming the lemma hypothesis, it follows that there is a matrix $\overset{*}{F}_i$ such that

$$\mathcal{S}_i \subset \mathcal{R}_i = \{A + B_i \overset{*}{F}_i \mid \mathcal{B}_i \cap \mathcal{R}_i\} \subset \mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i$$

Thus

$$(A + B_i \overset{*}{F}_i)\, \mathcal{S}_i \subset (A + B_i \overset{*}{F}_i)\, \mathcal{R}_i \subset \mathcal{R}_i \subset \mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i$$

and (2.5-18) is satisfied. Moreover, since $\mathcal{R}_i \subset \mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i$,

$$\mathcal{R}_i = \{A + B_i \overset{*}{F}_i \mid \mathcal{B}_i \cap \mathcal{R}_i\} \subset \{A + B_i \overset{*}{F}_i \mid \mathcal{B}_i \cap (\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i)\}$$

so that

$$\mathcal{S}_i \subset \{A + B_i \overset{*}{F}_i \mid \mathcal{B}_i \cap (\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_i)\}$$

Therefore, (2.5-19) is satisfied. ■

The result of the preceding lemmas may now be summarized as a theorem stating sufficient conditions for pole assignment via decentralized feedback. We use the characterization of maximal controllability subspaces stated in Appendix A. It should be noted that the conditions of this theorem are less restrictive than those of (2.5-7).

(2.5-21) __Theorem:__ For each $i \in \underline{K}$, let R_i be the maximal (A, B_i)-controllability subspace contained in $S_1 \oplus \ldots \oplus S_i$. If, for each i,

$$S_i \subset R_i$$

then the poles of Σ can be allocated via decentralized feedback so as to correspond with the n elements of Λ, provided that Λ is symmetric with respect to $\{S_i, i \in \underline{K}\}$.

The R_i can be computed as

$$R_i = \{A + B_i \tilde{F}_i \mid B_i \bigcap V_i\}$$

In the above, $V_i = V_i^{(d_1+\ldots+d_i)}$, where

$$V_i^{(0)} = S_1 \oplus \ldots \oplus S_i$$

and

$$V_i^{(k)} = (S_1 \oplus \ldots \oplus S_i) \bigcap A^{-1}(B_i + V_i^{(k-1)})$$

The matrix $\tilde{F}_i$ is any matrix for which

$$(A + B_i \tilde{F}_i) V_i \subset V_i$$

Since there is nothing special about the order in which the subspaces

S_i are considered, we immediately arrive at the following corollary:

(2.5-21) __Corollary__: If, for some permutation $\sigma : \underline{K} \to \underline{K}$,

$$S_i \subset R_i, \; i \in \underline{K}$$

where R_i is the maximal (A, B_i)-controllability subspace satisfying

$$R_i \subset \sum_{j:\sigma(j) \leq \sigma(i)} S_j, \; i \in \underline{K}$$

then the conclusion of (2.5-21) is valid.

This is the most general result that will be obtained in this section. However, in Section 2.7 we shall consider the situation where each control agent uses a generalized state reconstructor in order to increase his information set. The effect of using observers is equivalent to decreasing the sizes of the subspaces Ker C_i; thus (2.5-2) will no longer be a valid assumption. However, we shall continue to make an assumption similar to (2.5-1), namely that

$$\bigcap_{i \in \underline{K}} (\text{Ker } C_i \bigcap \text{Ker } H_i) = 0$$

where $z_i = H_i x$ represents the additional information due to the observer.

2.6 __Generalized Observability and State Reconstructors__

We now address the question of observability in the system $\Sigma = (A, B_i, C_i, i \in \underline{K})$. Our reasons for undertaking this task are twofold. First, a theory of observability will tend to round out the controllability-reachability discussion in Section 2.2. Secondly, a theory of observability and state reconstruction can be used to generalize the results on pole allocation via decentralized feedback; that is, if it can be shown that control agent i can asymptotically reconstruct a linear function of the state, e.g. $H_i x$, then the matrices C_i in Sections 2.4 and 2.5 can be replaced by the matrices $\binom{C_i}{H_i}$.

We shall begin the discussion by defining the set of states which are indistinguishable by agent i.

(2.6-1) <u>Definition</u>: Let $\Sigma = (A, B_i, C_i, i \varepsilon \underline{K})$ be a system with decentralized control. Then two states $x_1, x_2 \varepsilon R^n$ will be said to be indistinguishable by agent i, with respect to the input $u_i \varepsilon C^{m_i}[0,\infty)$, if there exist two sets of control inputs $\{u_{1j} \varepsilon C^{m_j}[0,\infty), j \neq i\}$ and $\{u_{2j} \varepsilon C^{m_j}[0,\infty), j \neq i\}$ such that

$$C_i[e^{At} x_1 + \int_0^t e^{A(t-\tau)}(B_i u_i(\tau) + \sum_{j\neq i} B_j u_{1j}(\tau))\, d\tau] =$$

$$C_i[e^{At}x_2 + \int_0^t e^{A(t-\tau)}(B_i u_i(\tau) + \sum_{j\neq i} B_j u_{2j}(\tau) d\tau], \text{ for all } t \geq 0$$

That is, x_1 and x_2 are indistinguishable with respect to u_i if control agents j, for $j \neq i$, can "mask" the differences, as they appear to agent i, between these two states. Two immediate results which require no proof are:

(2.6-2) <u>Proposition</u>: States x_1 and x_2 are indistinguishable by agent i, with respect to u_i, if and only if they are indistinguishable by agent i, with respect to the zero input; i.e., the choice of u_i cannot affect the ability to distinguish between

x_1 and x_2.

(2.6-3) **Proposition**: States x_1 and x_2 are indistinguishable by agent i (with respect to any $u_i \in C^{m_i}[0,\infty)$) if and only if the difference $x_1 - x_2$ is indistinguishable from the zero state by agent i.

Those states which are indistinguishable from zero are given a special name:

(2.6-4) **Definition**: The set of states that cannot be distinguished from the zero state by agent i is called the set of unobservable states of agent i, and is denoted as $\mathcal{U}_i$.

Again we have a simple result:

(2.6-5) **Proposition**: For each $i \in \underline{K}$, $\mathcal{U}_i$ is a subspace of R^n. Two states, x_1 and x_2, are indistinguishable by agent i if and only if $x_1 \equiv x_2$ (modulo $\mathcal{U}_i$), that is, x_1 and x_2 are in the same coset of $\mathcal{U}_i$.

(2.6-6) **Remark**: It is important to note that in (2.6-1) we require that u_{1j} and u_{2j} be elements of $C^{m_j}[0,\infty)$, for $j \neq i$. This restriction to continuous functions plays a crucial role in the determination of $\mathcal{U}_i$. Indeed, we shall see that there are states which are "effectively unobservable," but which are <u>not</u> in $\mathcal{U}_i$; this will be seen to be due to the fact that $C^{m_j}[0,\infty)$ is not a complete vector space.

We shall now determine the subspaces $\mathcal{U}_i$. In order to ease the notation, we shall consider the system

$$\dot{x} = Ax + Bu \quad + Dv$$

$$y = Cx$$

where $u \in C^m[0,\infty)$ is the input applied by agent i, y is the observation made by agent i, and $v \in C^\ell[0,\infty)$ represents the inputs applied by agents j, for $j \neq i$. We denote the space of unobservable states as $\mathcal{U}$.

(2.6-7) Lemma: The subspace $\mathcal{U}$ is the maximal (A, D)-invariant subspace contained in Ker C. That is, $\mathcal{U} = \mathcal{U}^{(n)}$, where

$$\mathcal{U}^{(0)} = \text{Ker } C$$

and

$$\mathcal{U}^{(k)} = \text{Ker } C \bigcap A^{-1} (\mathcal{U}^{(k-1)} + \mathcal{D}), \; k \in \underline{n}$$

Proof: A state x is an element of $\mathcal{U}$ if and only if there exists a $v \in C^\ell[0,\infty)$ such that

$$C e^{At} x = \int_0^t C e^{A(t-\tau)} D v(\tau) d\tau, \text{ for all } t \geq 0$$

That is, there must be a $v \in C^\ell [0,\infty)$ such that the solution to

(2.6-8) $\dot{z} = Az + Dv, \; z(0) = 0$

satisfies

$$C z(t) = C e^{At} x, \text{ for all } t \geq 0$$

But this can be true if and only if

$$w(t) \triangleq z(t) - e^{At} x \in \text{Ker } C, \text{ for all } t \geq 0$$

The vector w is easily seen to satisfy

(2.6-9) $\dot{w} = Aw + Dv,\ w(0) = -x$

where the initial condition arises from the requirement that $z(0) = 0$. Thus, $x \in \mathcal{U}$ if and only if there exists $v \in C^{\ell}[0,\infty)$ such that the solution to (2.6-9) satisfies

(2.6-10) $w(t) \in \text{Ker } C$, for all $t \geq 0$

Clearly, since $w(0) = -x$, (2.6-10) implies that

(2.6-11) $\mathcal{U} \subset \text{Ker } C$

We now assert that, if $x \in \mathcal{U}$ and if w satisfies (2.6-9) and (2.6-10), then

(2.6-12) $w(t) \in \mathcal{U}$, for all $t \geq 0$

For, select any $t_1 > 0$. Then the solution to

$$\dot{\hat{w}} = A\hat{w} + D\hat{v},\ \hat{w}(0) = w(t_1)$$

where $\hat{v}(t) = v(t + t_1)$, satisfies

$$\hat{w}(t) = w(t + t_1) \in \text{Ker } C, \text{ for } t \geq 0$$

But the only such solutions must arise, by definition, from initial conditions in $\mathcal{U}$. Thus $w(t_1) \in \mathcal{U}$.

From (2.6-12) there follows

$$\dot{w} = Aw + Dv \in \mathcal{U}, \text{ for all } t \geq 0$$

In particular, at t = 0

$$-Ax + Dv(0) \in \mathcal{U}$$

Since $x \in \mathcal{U}$ is arbitrary, this implies that

$$A\,\mathcal{U} \subset \mathcal{U} + \mathcal{D}$$

that is, $\mathcal{U}$ is an (A,D)-invariant subspace. Recalling (2.6-11), it is also seen that any (A,D)-invariant subspace of Ker C is a space of unobservable states. Thus, the sum of all (A, D)-invariant subspaces of Ker C, i.e. the unique maximal one, is a space of unobservable states. Since $\mathcal{U}$ must be such a subspace, it follows that $\mathcal{U}$ is the maximal (A, D)-invariant subspace of Ker C, proving the lemma. ■

(2.6-13) Remark: The preceding proof depends in a crucial way on the requirement that $v \in C^{\ell}[0,\infty]$. That is, if v were not necessarily continuous, e.g. if v contained a delta function at t = 0, then one could not claim in (2.6-8) that z(0) = 0. Indeed, we shall shortly see that the subspace of "effectively unobservable" states is in general larger than $\mathcal{U}$. The characterization of $\mathcal{U}$ given by (2.6-7) was proved in a slightly different manner in [5], but the authors failed to recognize the "effectively unobservable" phenomenon.

(2.6-14) Remark: For the special case where D = 0, the sequence $\{\mathcal{U}^{(k)}\}$ is just

$$\mathcal{U}^{(0)} = \text{Ker } C$$

$$\mathcal{U}^{(1)} = \text{Ker } C \bigcap A^{-1} \text{ Ker } C = \text{Ker } C \bigcap \text{Ker } (CA)$$

$$\vdots$$

$$\mathcal{U} = \mathcal{U}^{(n)} = \bigcap_{i \in \underline{n}} \text{Ker } (CA^{i-1})$$

as expected.

We have alluded several times to "effectively unobservable" states. We shall develop the notion of such states by determining how one would compute the "observable" portion of an initial state. Thus, let $\mathcal{O}$ be any subspace of R^n satisfying

$$\mathcal{U} \oplus \mathcal{O} = R^n$$

We shall call $\mathcal{O}$ the observable subspace; and, for any $x \in R^n$, if $x = x_u + x_o$ where $x_u \in \mathcal{U}$ and $x_o \in \mathcal{O}$, we shall call x_u the unobservable part of x, and x_o the observable part of x. It is therefore implied that, by observing

$$(2.6\text{-}15) \quad y(t) = C\, e^{At}\, x_o + C\, e^{At}\, x_u + \int_0^t C\, e^{A(t-\tau)}\, Dv(\tau)\, d\tau$$

for $t \in [0,T]$ $(T > 0)$, one could actually calculate x_o, regardless of what $x_u \in \mathcal{U}$ and $v \in C^{\ell}[0,T]$ happened to be.

The relation (2.6-15) is a linear map

$$L : \mathcal{O} \times \mathcal{U} \times C^{\ell}[0,T] \to C^{p}[0,T]$$

$$: (x_o, x_u, v) \mapsto L_1\, x_o + L_2\, x_u + L_3\, v$$

The map L_1 is clearly injective; for if it were not, there would be an unobservable element in $\mathcal{O}$, contradicting $\mathcal{O} \bigcap \mathcal{U} = 0$. Thus one might expect to be able to find a linear map $L_4 : C^p[0,T] \to \mathcal{O}$ such that the composition $L_4 \bullet L$ is the projection on $\mathcal{O}$ along $\mathcal{U} \times C^\ell[0,T]$, i.e. such that

$$L_4 \bullet L : (x_o, x_u, v) \mapsto x_o$$

But, in order for such an L_4 to exist, it would need to satisfy $\text{Ker } L_4 \supset \text{Im } L_2$, $\text{Ker } L_4 \supset \text{Im } L_3$, and $(\text{Ker } L_4) \bigcap \text{Im } L_1 = 0$. Since, by the definition of $\mathcal{U}$, $\text{Im } L_2 \subset \text{Im } L_3$, these requirements are just that

$$\text{Im } L_1 \bigcap \text{Ker } L_4 = 0$$

$$\text{Im } L_3 \subset \text{Ker } L_4$$

However, in order to find such an L_4, particularly if we wished that L_4 be represented as

$$\text{(2.6-16)} \quad L_4 : y(t) \mapsto \int_0^T \hat{L}_4(t)\, y(t)\, dt$$

for some n x p integrable matrix function $\hat{L}_4(\cdot)$, we would need

$$\text{(2.6-17)} \quad (\text{closure of Im } L_1) \bigcap (\text{closure of Im } L_3) = 0$$

Unfortunately, (2.6-17) is seldom true. Since $\mathcal{O}$ is finite dimensional, it follows that the closure of $\text{Im } L_1$ is just $\text{Im } L_1$; but, since $C^\ell[0,T]$ is not finite dimensional, the closure of $\text{Im } L_3$ may be larger than $\text{Im } L_3$.

In fact, define the subspace $\hat{\mathcal{U}}$ as follows

$$\text{(2.6-18)} \quad \hat{\mathcal{U}} = \mathcal{V}^{(n)}$$

where

$$\mathcal{V}^{(0)} = \mathcal{D} \bigcap \text{Ker } C$$

$$\mathcal{V}^{(k)} = (\mathcal{D} + A\, \mathcal{V}^{(k-1)}) \bigcap \text{Ker } C,\ k \in \underline{n}$$

It is easily seen that $\mathcal{V}^{(k)} \supset \mathcal{V}^{(k-1)}$, for $k \in \underline{n}$; therefore, $\hat{\mathcal{U}}$ is the smallest subspace satisfying

$$\hat{\mathcal{U}} = (\mathcal{D} + A\,\hat{\mathcal{U}}) \bigcap \text{Ker } C$$

We shall demonstrate that for each $x \in \hat{\mathcal{U}}$ there exists a generalized function (i.e. a distribution, [37, Ch.1]) v_x such that

$$\text{(2.6-19)} \quad C\,e^{At}\,x = \int_0^{t^+} C\,e^{A\tau}\,D\,v_x\,(t-\tau)\,d\tau, \quad \text{for all } t \gtrsim 0$$

Then, since v_x can be approximated arbitrarily closely (in the distributional sense) by an element of $C^{\ell}[0,T]$, it follows that $C\,e^{At}\,x \in C^p[0,T]$ is in the closure of Im L_3. In particular, if $\{v_i \in C^{\ell}[0,T]\}$ is a sequence converging in the distributional sense to v_x, and if $\{y_i \in C^p[0,T]\}$ is defined as

$$y_i(t) = \int_0^t C\,e^{A\tau}\,D\,v_i\,(t-\tau)\,d\tau, \quad t \in [0,T]$$

then with $L_4 : C^p[0,T] \to \mathcal{O}$ given by (2.6-16), the sequence $\{L_4\,y_i\}$ will converge to $L_4(C\,e^{At}\,x)$. Thus, each $x \in \hat{\mathcal{U}}$ is "effectively unobservable."

To demonstrate the existence of v_x, let $x \in \hat{\mathcal{U}}$. From (2.6-18) there exist sequences $\{x_i,\ 0 \le i \le n\}$ and $\{u_i,\ 0 \le i \le n\}$ such that $x_o = 0$, $x_i \in \mathcal{V}^{(i-1)}$ for $i \in \underline{n}$,

$$x_i = A\,x_{i-1} + D\,u_{i-1}, \ i \in \underline{n}$$

and

$$x = A\,x_n + D\,u_n$$

Now let δ denote the Dirac delta function; and define $\delta^{(i)}$, $0 \le i \le n$, to be the ith (distributional) derivative of δ. Let

$$v_x = \sum_{i=0}^{n} u_{n-i}\,\delta^{(i)}$$

Then, using

$$\int_0^{t^+} C\,e^{A\tau}\,D\,\delta^{(i)}\,(t-\tau)\,d\tau = C\,e^{At}\,A^i\,D + \sum_{j=0}^{i-1} C\,A^{i-j-1}D\,\delta^{(j)}(t), t \ge 0$$

and the fact that $x_i \in \mathcal{V}^{(i-1)} \subset$ Ker C, (2.6-19) is easily verified.

(2.6-20) <u>Definition</u>: we shall call the subspace $\mathcal{U} + \hat{\mathcal{U}}$, where $\hat{\mathcal{U}}$ is given by (2.6-18), the "effectively unobservable" subspace.

(2.6-21) <u>Remark</u>: We can now decompose the state space as

$$R^n = (\mathcal{U} + \hat{\mathcal{U}}) \oplus \hat{\mathcal{O}}$$

Although it is not obvious now that $C\, e^{At}\, x \notin$ (closure of Im L_3) for all $x \in \hat{\mathcal{O}}$, this fact will become apparent after we prove (2.6

(2.6-22) <u>Remark</u>: If one wished to determine a linear map $\hat{L} : C^p[0,T]$ that produced the observable part of the initial state (i.e., the part in $\hat{\mathcal{O}}$) one could procede as follows. Let $\{e_i\}$ be a basi for $\hat{\mathcal{O}}$ and define $f_i \in C^p[0,T]$ as $f_i(t) = C\, e^{At}\, e_i$; let $g_i \in C^p[0,T]$ be the result of projecting f_i orthogonally onto th closure of Im L_3. Then, if $x_o \in \hat{\mathcal{O}}$ is the observable part of the initial state, and if $x_o = \sum_{i=1}^{d} \alpha_i\, e_i$, where $d = \dim \hat{\mathcal{O}}$,

$$(\alpha_1, \ldots \alpha_d)' = M^{-1}\left(\int_0^T (f_1(t)-g_1(t))'\, y(t)dt, \ldots \int_0^T (f_d(t)-g_d(t))'\, y(t)\right.$$

where

$$M_{ij} = \int_0^T (f_i(t)-g_i(t))'\ f_j(t)\ dt$$

However, this is difficult to carry out, because it is not clear how one characterizes the closure of Im L_3. Fortunately, we shall not have to do this, as we shall instead develop the theor of generalized asymptotic state reconstructors.

We now place the foregoing in the framework of systems with decentral-ized control.

(2.6-23) <u>Theorem</u>: Let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be a system with decentralized control, and define the matrices

$$\hat{B}_i = (B_1, \ldots B_{i-1}, B_{i+1}, \ldots B_K), \; i \in \underline{K}$$

Then, the unobservable subspace, $\mathcal{U}_i$, and the effectively unobservable subspace, $\mathcal{U}_i + \hat{\mathcal{U}}_i$, of agent i are given as

$$\mathcal{U}_i = \mathcal{U}_i^{(n)}, \; \mathcal{U}_i + \hat{\mathcal{U}}_i = \mathcal{U}_i^{(n)} + \mathcal{V}_i^{(n)}$$

where

$$\mathcal{U}_i^{(0)} = \text{Ker } C_i, \; \mathcal{U}_i^{(k)} = \text{Ker } C_i \bigcap A^{-1} (\mathcal{U}_i^{(k-1)} + \hat{\mathcal{B}}_i), \; k \varepsilon \underline{n}$$

and

$$\mathcal{V}_i^{(0)} = \hat{\mathcal{B}}_i \bigcap \text{Ker } C_i, \; \mathcal{V}_i^{(k)} = \text{Ker } C_i \bigcap (\hat{\mathcal{B}}_i + A\, \mathcal{V}_i^{(k-1)}), \; k$$

We now go back to the system $\dot{x} = Ax + Bu + Dv$, $y = Cx$ and determine how one would asymptotically reconstruct the observable part of the state. The approach is somewhat lacking in rigour, but the end result will justi[fy] the means.

Suppose, to begin, that both u and v are known, and that (A,C) is observable. Then, there exists a matrix K such that (A-KC) is stable; thus if

(2.6-24) $\dot{z} = (A - KC)\, z + Ky + Bu + Dv$

it follows, by easy calculations ([48]), that $z(t) - x(t) \to 0$ as $t \to \infty$. Moreover, the rapidity with which the error $e = z - x$ goes to zero can be adjusted by assigning the eigenvalues of $A - KC$ (possible by observabilit[y]

However, since the input v is not available, we try the following. Suppose H is an r x n matrix that satisfies

(2.6-25) $\mathcal{D} \subset \text{Ker } H$

Also assume that for any polynomial $\alpha(\lambda)$, of degree r and with real coefficients, there exist an r x r matrix F and an n x r matrix K such that

(2.6-26) $H(A - KC) = FH$

and

(2.6-27) $\det(\lambda I - F) = \alpha(\lambda)$

One can then multiply (2.6-24) on the left by H, and the following results from (2.6-25) and (2.6-26):

(2.6-28) $\frac{d}{dt}(Hz) = F H z + H K y + H B u$

Since, by (2.6-27), we can find F and K so that F is stable, it follows that (2.6-28) describes an asymptotic reconstructor of the quantity Hx.

It is clear that, in the interests of reconstructing as much of x as possible without knowledge of v, H should satisfy (2.6-25) as tightly as possible. Moreover, it is easily seen that if a matrix H_1 has the above desired properties then so does H_2, if $\operatorname{Ker} H_1 = \operatorname{Ker} H_2$. We now state

(2.6-29) <u>Definition</u>: Given the system $\Sigma' : \dot{x} = Ax + Bu + Dv$, $y = Cx$, with v unknown; a subspace $\mathcal{S}$ will be called unreconstructible if, for all matrices H satisfying (2.6-25), (2.6-26), and (2.6-27) (the latter two for arbitrary polynomials of the appropriate degree), $\mathcal{S} \subset \operatorname{Ker} H$. <u>The</u> unreconstructible subspace is defined as the largest subspace that is unreconstructible.

(2.6-30) <u>Lemma</u>: Given the system Σ' of (2.6-29), the unreconstructible subspace is $\overset{*}{H}^{\perp}$, where $\overset{*}{H}$ is the largest (A', C')-controllability subspace contained in $\mathcal{D}^{\perp}$.

Proof: Let H be a matrix satisfying (2.6-25) - (2.6-27), and define H = Im H'. From (2.6-26),

$$(A' - C' K') H' = H' F'$$

so H is (A', C')-invariant. From (2.6-27) we conclude that the spectrum of the restriction of (A' - K' C') to H can be arbitrarily assigned (subject to conjugate symmetry) by choice of K'; thus H must be an (A', C')-controllability subspace.

The unreconstructibile subspace clearly must be in the orthogonal complement of H, for <u>all</u> (A', C')-controllability subspaces H for which $\mathcal{D} \subset$ Ker H. But this latter is true if and only if $(\text{Ker H})^{\perp} = H \subset \mathcal{D}^{\perp}$. Thus the unreconstructibile subspace is precisely $\overset{*}{H}^{\perp}$, where $\overset{*}{H}$ is the largest (A', C')-controllability subspace of $\mathcal{D}^{\perp}$. ■

Of course, there are some states in $\overset{*}{H}^{\perp}$ that can be measured directly through C; if H is such that Im H' = $\overset{*}{H}$, then the states that cannot be "seen" are those in Ker C $\bigcap$ Ker H. We now show that these states are precisely the states in $U + \hat{U}$.

(2.6-31) <u>Theorem</u>: Given the system Σ' : $\dot{x} = Ax + Bu + Dv$, $y = Cx$, with v unknown; let U, $\hat{U}$, and $\overset{*}{H}$ be as given in (2.6-7), (2.6-18), and (2.6-30). Then

$$U + \hat{U} = \text{Ker } C \bigcap \overset{*}{H}^{\perp}$$

Proof: $\overset{*}{H}$, the maximal (A', C')-controllability subspace of $\mathcal{D}^{\perp}$, can be expressed as (see Appendix A)

$$\overset{*}{H} = W \bigcap N^{\perp}$$

where W is the maximal (A', C')-invariant subspace of $\mathcal{D}^{\perp}$, and N is the maximal (A, D)-invariant subspace of Ker C. But, from (2.6-7), $N = U$; therefore

$$\text{Ker } C \bigcap \overset{*}{H}{}^{\perp} = \text{Ker } C \bigcap (W^{\perp} + U)$$

$$= U + \text{Ker } C \bigcap W^{\perp}$$

since $U \subset \text{Ker } C$.

But, defining $C = \text{Im } C'$, $W = W^{(n)}$ where

$$W^{(0)} = \mathcal{D}^{\perp}$$

$$W^{(k)} = \mathcal{D}^{\perp} \bigcap A^{-1} (W^{(k-1)} + C),\ k \in \underline{n}$$

Therefore, $W^{\perp} = W^{(n)\perp}$ where, since $C^{\perp} = \text{Ker } C$,

$$W^{(0)\perp} = \mathcal{D}$$

$$W^{(k)\perp} = \mathcal{D} + A\,(W^{(k-1)\perp} \bigcap \text{Ker } C),\ k \in \underline{n}$$

Comparing this with (2.6-18), we see that

$$W^{\perp} \bigcap \text{Ker } C = \hat{U} \quad \blacksquare$$

(2.6-32) Remark: There are situations where the subspace $U + \hat{U}$ is the zero subspace. For example, it is easy to show that $U = 0$ if and only if the system $\dot{x} = (A + DF)\,x + Dv$, $y = Cx$ is observable for all F. Also, $\hat{U} = 0$ if and only if $\mathcal{D} \bigcap \text{Ker } C = 0$; this will happen, for example, if $C = D'$.

We now return to the system $\Sigma = (A, B_i, C_i, i \in \underline{K})$ to summarize the above results.

(2.6-33) **Theorem**: Let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be a system with decentralized control. Then, through the use of generalized state reconstructors, the observation matrix C_i of agent i can be replaced by the matrix $\binom{C_i}{H_i}$, which satisfies

$$\text{Ker} \binom{C_i}{H_i} = \mathcal{U}_i + \hat{\mathcal{U}}_i, \; i \in \underline{K}$$

where $\mathcal{U}_i$ and $\hat{\mathcal{U}}_i$ are as given in (2.6-23).

Finally, we note that control agent i can use a minimal order state reconstructor whose output, $\hat{z}_i$, asymptotically approaches the quantity $\hat{H}_i x$, where

$$\text{Ker} \begin{pmatrix} C_i \\ \hat{H}_i \end{pmatrix} = \text{Ker } C_i \bigcap \text{Ker } \hat{H}_i = \mathcal{U}_i + \hat{\mathcal{U}}_i$$

and

$$\text{Ker } C_i + \text{Ker } \hat{H}_i = R^n$$

the latter equation implying that the rows of $\begin{pmatrix} C_i \\ \hat{H}_i \end{pmatrix}$ are independent. This follows from the next lemma, the proof of which is a simple variation of [72]:

(2.6-34) **Lemma**: Let $\overset{*}{\mathcal{H}}$ be the maximal (A', C')-controllability subspace contained in $\mathcal{D}^{\perp}$, let $\mathcal{C} = \text{Im } C'$, and let $d_1 = \dim \overset{*}{\mathcal{H}}$ and $d_2 = \dim (\overset{*}{\mathcal{H}} \bigcap \mathcal{C})$. Then, if $d_1 > d_2$, for every set Λ of $d_1 - d_2$

complex numbers, closed under complex congugation, there exist a subspace $\hat{H}$ and a matrix K' such that

$$\overset{*}{H} = \hat{H} \oplus \overset{*}{H} \cap C$$

$$(A' - C' K') \hat{H} \subset \hat{H}$$

$$\Lambda = \text{spectrum of } A' + C' K' \text{ restricted to } \hat{H}$$

The integer $d_1 - d_2$ is the dimension of a minimal order state reconstructor, whose output asymptotically reconstructs $\hat{H}x$, where $\text{Im } \hat{H}' = \hat{H}$.

This gives us our final result:

(2.6-35) <u>Theorem</u>: In the system $\Sigma = (A, B_i, C_i, i \in \underline{K})$, the dimensions of the minimal order state reconstructors that validate the conclusion of (2.6-33) are

$$n_i = \dim (\text{Ker } C_i) - \dim (U_i + \hat{U}_i), i \in \underline{K}$$

Proof: From (2.6-34), the dimension of a minimal order state reconstructor for agent i is

$$n_i = \dim (\overset{*}{H}_i) - \dim (\overset{*}{H}_i \cap C_i)$$

$$= \dim (\overset{*}{H}_i + C_i) - \dim (C_i)$$

$$= n - \dim (\overset{*}{H}_i + C_i)^{\perp} - \dim (C_i)$$

$$= n - \dim (\overset{*}{H}_i{}^{\perp} \cap \text{Ker } C_i) - \dim (C_i)$$

$$= n - \dim (U_i + \hat{U}_i) - \dim (C_i)$$

$$= \dim(\operatorname{Ker} C_i) - \dim(\mathcal{U}_i + \hat{\mathcal{U}}_i)$$

where the second to last line follows from (2.6-31). ∎

2.7 Pole Allocation with Increased Information Sets

In this section we shall investigate the consequences of removing the assumptions (2.5-1) and (2.5-2). Our reasons for doing this are as follows. First, while (2.5-1) may be a reasonable assumption if the order, n, of $\Sigma = (A, B_i, C_i, i \in \underline{K})$ is not very large (especially if $n \cong K$), this assumption becomes more unreasonable as n increases. That is, for large scale systems one would expect that there would be a nontrivial subspace of states that could not be directly observed (i.e., without the use of a dynamic observer) by any of the control agents. Thus, in place of (2.5-1) we shall assume in this section that

(2.7-1) $$\bigcap_{i \in \underline{K}} (\mathcal{U}_i + \hat{\mathcal{U}}_i) = 0$$

where $\mathcal{U}_i + \hat{\mathcal{U}}_i$ is the effectively unobservable subspace of agent i defined in (2.6-23). That is, we shall assume in this section that each agent uses a generalized state reconstructor, as described in Section 2.6.

As to (2.5-2), this assumption implies that the control agents make independent observations on the system. While this assumption is perhaps reasonable when the control agents do not use dynamic observers, the assumption of independent observations when dynamic observers are used is highly questionable. Thus in this section it will be generally assumed that

(2.7-2) $\sum_{i \in \underline{K}} \bigcap_{j \neq i} (\mathcal{U}_j + \hat{\mathcal{U}}_j) \neq R^n$

We shall attempt to make use of (2.5-21) in determining sufficient conditions for arbitrary pole allocation via decentralized dynamic compensation, i.e. decentralized feedback from the observations y_i and the outputs of the generalized observers. However, since (2.5-21) depends crucially on the assumption (2.5-2), the analogue of which we are not making in this section, we shall take the following approach. We shall attempt to determine subspaces $\mathcal{N}_i$, $i \in \underline{K}$, satisfying

(2.7-3) $\mathcal{N}_i \supset \mathcal{U}_i + \hat{\mathcal{U}}_i, \ i \in \underline{K}$

(2.7-4) $\bigcap_{i \in \underline{K}} \mathcal{N}_i = 0$

(2.7-5) $\sum_{i \in \underline{K}} \bigcap_{j \neq i} \mathcal{N}_j = R^n$

Then we shall apply (2.5-21), using

(2.7-6) $S_i \triangleq \bigcap_{j \neq i} \mathcal{N}_j, \ i \in \underline{K}$

The significance of subspace $\mathcal{N}_i$ is as follows. By using a state reconstructor, control agent i can asymptotically measure the quantity

$$\begin{pmatrix} y_i \\ z_i \end{pmatrix} = \begin{pmatrix} C_i \\ H_i \end{pmatrix} x$$

where Ker $\begin{pmatrix} C_i \\ H_i \end{pmatrix} = \mathcal{U}_i + \hat{\mathcal{U}}_i$. However, he is not constrained to using all of this information. Since, for any $\mathcal{N}_i \supset \mathcal{U}_i + \hat{\mathcal{U}}_i$ there exists a matrix M_i such that

$$\text{Ker } (M_i \begin{pmatrix} C_i \\ H_i \end{pmatrix}) = N_i$$

it is clear that control agent i may increase his unobservable subspace to N_i if he simply uses the observations

$$w_i = M_i \begin{pmatrix} y_i \\ z_i \end{pmatrix}$$

Thus, we immediately arrive at the following result:

(2.7-7) Proposition: Let $\Sigma = (A, B_i, C_i, i \in \underline{K})$ be a system with decentralized control, and let the effectively unobservable subspaces $U_i + \hat{U}_i$ be as given in (2.6-21). If there exist subspaces N_i, $i \in \underline{K}$, satisfying (2.7-3) through (2.7-5); and if the subspaces S_i, given by (2.7-6), satisfy the conditions of the corollary (2.5-21); then the poles of the closed loop system resulting from decentralized dynamic feedback in Σ may be allocated as follows. The poles of each generalized state reconstructor (which may be taken to be of minimal order, as in (2.6-35)), may be independently allocated subject to conjugate symmetry; the remaining n poles may be arbitrarily allocated so as to correspond with the set Λ, where Λ is any set of n elements, symmetric with respect to $\{S_i, i \in \underline{K}\}$.

Proof: For each $i \in \underline{K}$, let $\hat{H}_i$ be as given in (2.6-34); let $\hat{H}_i$ be $n_i \times n$. Then, for any monic polynomial $\alpha_i(\lambda)$, with real coefficients and of degree n_i, there exist matrices K_i and $\hat{A}_i$ such that $\hat{H}_i(A - K_i C_i) = \hat{A}_i \hat{H}_i$, and $\det(\lambda I - \hat{A}_i) = \alpha_i(\lambda)$. Let

$$\dot{\hat{z}}_i = \hat{A}_i \hat{z}_i + \hat{H}_i K_i y_i + \hat{H}_i B_i u_i$$

Then it is easy to show that the quantity

$$e_i \triangleq \hat{z}_i - \hat{H}_i x$$

satisfies

$$\dot{e}_i = \hat{A}_i e_i$$

From the construction of $\hat{H}_i$, $\text{Ker} \begin{pmatrix} C_i \\ \hat{H}_i \end{pmatrix} = U_i + \hat{U}_i$. Choose the matrix M_i

such that

$$\text{Ker}\ (M_i \begin{pmatrix} C_i \\ \hat{H}_i \end{pmatrix}) = N_i$$

From (2.5-21) there exist matrices F_i, $i \in \underline{K}$, such that the set of eigenvalues of

$$\hat{A} = A + \sum_{i \in \underline{K}} B_i F_i M_i \begin{pmatrix} C_i \\ \hat{H}_i \end{pmatrix}$$

is Λ. Thus, define the feedback control laws as

$$u_i = F_i M_{i1} y_i + F_i M_{i2} \hat{z}_i$$

where M_i is partitioned, conformably with $\begin{pmatrix} C_i \\ \hat{H}_i \end{pmatrix}$, as

$$M_i = (M_{i1}; M_{i2})$$

Using the components of x and e_i, $i \in \underline{K}$, as state variables for the closed loop system, we see that, since $\hat{z}_i = e_i + \hat{H}_i x$,

$$\frac{d}{dt}\begin{pmatrix} x \\ e_1 \\ \vdots \\ e_K \end{pmatrix} \begin{pmatrix} \hat{A} & B_1F_1M_{12} & B_2F_2M_{22} & \cdots & B_KF_KM_{K2} \\ 0 & \hat{A}_1 & 0 & & 0 \\ & 0 & A_2 & & \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \hat{A}_K \end{pmatrix} \begin{pmatrix} x \\ e_1 \\ \vdots \\ e_K \end{pmatrix}$$

The poles of the closed loop system are as promised. ■

Thus the problem reduces to a search for suitable subspaces N_i. It is a fairly simple matter to increase the subspaces $U_i + \hat{U}_i$ to subspaces N_i such that (2.7-3) through (2.7-5) are satisfied. For example, one could let

$$N_1 = U_1 + \hat{U}_1$$

and, for i = 2,...K, choose $N_i \supset U_i + \hat{U}_i$ such that

$$\sum_{j=1}^{i} (U_j + \hat{U}_j)^{\perp} = \left(\sum_{j=1}^{i-1} (U_j + \hat{U}_j)^{\perp} \right) \oplus N_i^{\perp}$$

Since, from (2.7-1), $\sum_{i \epsilon \underline{K}} (U_i + \hat{U}_i)^{\perp} = R^n$, this process is guaranteed to produce a set of subspaces N_i satisfying (2.7-3) through (2.7-5). However, we must attempt to choose the N_i so that the subspaces S_i, given by (2.7-6), satisfy the requirements of (2.5-21). This, unfortunately is not a simple task.

The problem of determining sufficient conditions for the existence of appropriate subspaces N_i, i ε $\underline{K}$, remains unsolved. However, we can state the following result, the philosophy of which is to first find (A, B_i)-controllability subspaces R_i, and then check to see if these subspaces are compatible, via (2.5-21), with subspaces N_i satisfying (2.7-3) through (2.7-5):

(2.7-8) __Theorem__: Assume that K = 2 and that

$$\{A \mid B_1\} + \{A \mid B_2\} = R^n$$

Let R_i be an (A, B_i)-controllability subspace, i = 1, 2, such that

$$R_1 + R_2 = R^n$$

(this is possible by the first assumption). Then the poles of Σ = (A, B_i, C_i, i = 1, 2) may be allocated via decentralized dynamic feedback if

(2.7-9) $(\mathcal{U}_1 + \hat{\mathcal{U}}_1) \subset \mathcal{R}_2$

(2.7-10) $(\mathcal{U}_2 + \hat{\mathcal{U}}_2) \subset \mathcal{R}_1$

(2.7-11) $(\mathcal{U}_1 + \hat{\mathcal{U}}_1) \bigcap \mathcal{R}_1 = 0$

where $\{\mathcal{U}_i + \hat{\mathcal{U}}_i,\ i = 1, 2\}$ are the effectively unobservable subspaces given in (2.6-21).

Proof: For the case $K = 2$, $S_1 = N_2$ and $S_2 = N_1$; thus we must find N_1 and N_2 such that

$$N_2 = \mathcal{R}_1$$

(2.7-12) $N_2 \oplus N_1 = \mathcal{R}_i + \mathcal{R}_2$

(2.7-13) $N_1 \subset \mathcal{R}_2$

and

(2.7-14) $N_i \supset \mathcal{U}_i + \hat{\mathcal{U}}_i,\ i = 1, 2$

Clearly, $N_2 = \mathcal{R}_1$ satisfies (2.7-14).

Next noce that, from (2.7-9) and (2.7-11), the subspaces $\mathcal{R}_1 \bigcap \mathcal{R}_2$ and $\mathcal{R}_2 \bigcap (\mathcal{U}_1 + \hat{\mathcal{U}}_1) = (\mathcal{U}_1 + \hat{\mathcal{U}}_1)$ are independent. Thus write

$$\mathcal{R}_2 = \mathcal{R}_2 \bigcap \mathcal{R}_1 \oplus \mathcal{R}_2 \bigcap (\mathcal{U}_1 + \hat{\mathcal{U}}_1) \oplus \hat{S}_2$$

$$= \mathcal{R}_2 \bigcap \mathcal{R}_1 \oplus (\mathcal{U}_1 + \hat{\mathcal{U}}_1) \oplus \hat{S}_2$$

and define

$$N_1 = (\mathcal{U}_1 + \hat{\mathcal{U}}_1) \oplus \hat{S}_2$$

Clearly, (2.7-14) is satisfied, as is (2.7-13). Moreover,

$$N_2 \oplus N_1 = R_1 + (U_1 + \hat{U}_1) + \hat{S}_2$$

$$= R_1 + R_2 \bigcap R_1 + (U_1 + \hat{U}_1) + \hat{S}_2$$

$$= R_1 + R_2$$

to satisfy (2.7-12).

We have thus found subspaces N_i such that $N_1 \bigcap N_2 = 0$, and, with $S_i = \bigcap_{j \neq i} N_j$,

$$S_i \oplus S_2 = R^n$$

$$S_i \subset R_i \subset S_1 + S_i, \; i = 1, 2$$

Thus (2.5-21) may be applied. (Note that the controllability subspaces R_i need not be maximal for (2.5-21) to be valid). ■

(2.7-15) **Remark:** It would appear at first glance that one could perform such a construction when $K > 2$. It can be fairly easily shown that subspaces $\{N_i, \; i \in \underline{K}\}$ exist such that, with $S_i = \bigcap_{j \neq i} N_j$,

$$N_i \supset U_i + \hat{U}_i, \; i \in \underline{K}$$

$$S_1 \oplus \cdots \oplus S_i = R_1 \oplus \cdots \oplus R_i, \; i \in \underline{K}$$

if and only if

$$\sum_{j=1}^{i-1} R_j \supset \left(\sum_{j=1}^{i} R_j\right) \bigcap (U_i + \hat{U}_i), \text{ for } i \in \underline{K}$$

However, we also need

$$S_i \subset R_i, \; i \in \underline{K}$$

and it is this requirement that causes difficulties when $K > 2$.

(2.7-16) Remark: Unfortunately, there is presently no way to determine the existence of controllability subspaces $\mathcal{R}_i$ satisfying (2.7-9) through (2.7-11). Thus, if this theorem is to be applied, one may have to consider a large number of pairs of controllability subspaces $(\mathcal{R}_1, \mathcal{R}_2)$.

CHAPTER 3

FURTHER DISCUSSION OF POLE ALLOCATION AND INTRODUCTION TO FEEDBACK INVARIANTS

3.1 Discussion of Pole Allocation Results

The problem of pole allocation via decentralized feedback may be approached from two points of view. In this section we shall elaborate on these two points of view, discuss the sufficient conditions derived in Sections 2.5 and 2.7, and propose a second method for more nearly completely solving this problem.

In formulating the problem of pole allocation via decentralized feedback, we stated in Chapter 2 that our goals are to establish necessary and sufficient conditions for the existence of feedback matrices $\{F_i, i \in \underline{K}\}$ such that the set of eigenvalues of $A + \sum_{i \in \underline{K}} B_i F_i C_i$ corresponds with a desired set Λ of n numbers. However, there is a secondary issue which we have not mentioned up to this point. This issue involves the actual determination of the matrices F_i. Since F_i represents feedback applied by control agent i, one might interpret the phrase "decentralized control" to imply that each F_i should be computable by control agent i alone. That is, there are actually two levels involved in the control of $\Sigma = (A, B_i, C_i, i \in \underline{K})$. The higher of the two levels is the determination of suitable feedback matrices, the lower level involves utilizing the feedback matrices in decentralized control laws. A strict interpretation of "decentralized control" would be to require both of these levels to be decentralized.

If the above interpretation is adhered to, then it should be clear that Σ must be at least triangularly decoupled, and that this decoupled structure must remain invariant under decentralized feedback. Only if this is true can it be guaranteed that agent i can compute his feedback matrix F_i, so as to allocate poles in his subsystem, without knowledge of either the objectives or the feedback matrices of control agents j, for $j \neq i$. Thus the structure of the system must be such that

(3.1-1) $\bigcap_{i \in \underline{K}} \text{Ker } C_i = 0$

(3.1-2) $\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_K = R^n$, where $\mathcal{S}_i = \bigcap_{j \neq i} \text{Ker } C_j$

$P_i\{A|\mathcal{B}_i\} = \mathcal{S}_i$, and

$\{A \mid \mathcal{B}_i\} \subset \sum_{j:\sigma(j) \leq \sigma(i)} \mathcal{S}_j$, for some permutation $\sigma:\underline{K} \to \underline{K}$

If Σ has this structure, then it is easy to verify that Σ retains this structure when arbitrary decentralized feedback is applied. Control agent i then simply chooses F_i so that the eigenvalues of $P_i(A + B_i F_i C_i)$, restricted to $\mathcal{S}_i$, are as desired, where P_i is the projection on $\mathcal{S}_i$ along Ker C_i.

While the above system structure is the only one where pole allocation via decentralized feedback in the strict sense is possible, this structure is not very interesting. Thus, we may relax the strict interpretation of decentralized feedback in the following manner. We shall still demand that each F_i be computable independently of the others; however, we shall allow the control agents to have a common objective: first decouple the system as much as possible, and then allocate poles while retaining the decoupled structure. This is in fact the situation treated in (2.5-21). We have seen that if, for each i, there is an (A, B_i)-controllability subspace $\mathcal{R}_i$ such that

$$S_i \subset R_i \subset \sum_{j:\sigma(j)\leq\sigma(i)} S_i$$

then the decoupling and pole assignment are possible. Note, however, that, in spite of the fact that the F_i's which produce the desired decoupled structure may be determined independently, implicit in the determination of each F_i is knowledge of the permutation $\sigma : \underline{K} \rightarrow \underline{K}$, which determines the nesting, or chain, structure in the triangular decoupled system. Moreover, if agent i fails in decoupling the effects of his control from agent j, for some j where $\sigma(j) > \sigma(i)$, since agent j need not decouple the effects of his control from agent i, the result will be that these two agents remain coupled; and subsequent attempts at assigning poles will fail.

One can next consider a class of decentralized feedback problems where triangular decoupling is a primary objective, but where the feedback matrices F_i which achieve this objective need not be determined independently. However, once the decoupled structure is achieved, the additional feedback matrices necessary for arbitrary pole allocation can be determined independently.

As a result of (2.5-15) we know that, under the assumptions (3.1-1) and (3.1-2), if the system can be triangularly decoupled with decentralized feedback, then this decoupling can be can be achieved by solving for each F_i independently. However, if assumption (3.1-2) is dropped (as, for example, when the unobservable subspaces are decreased through the use of generalized observers) one can no longer make this claim. In the absence of (3.1-2) it is extremely difficult to determine necessary and sufficient conditions for the existence of matrices $\{F_i, i \in \underline{K}\}$ which achieve a desired triangularly decoupled structure.

One approach that can be taken at this point is to attempt to modify the theories of (A, B)-invariant and controllability subspaces in order to accommodate decentralized feedback. A result in this direction is the following, which extends the theory of (A, B)-invariant subspaces.

(3.1-3) <u>Proposition:</u> Let $\mathcal{V} \subset R^n$ be an arbitrary subspace. Then, there exist matrices F_1 and F_2 such that

$$(A + B_1 F_1 C_1 + B_2 F_2 C_2) \mathcal{V} \subset \mathcal{V}$$

if and only if

(3.1-4) $A \mathcal{V} \subset \mathcal{V} + \mathcal{B}_1 + \mathcal{B}_2$

(3.1-5) $A(\mathcal{V} \bigcap \text{Ker } C_1) \subset \mathcal{V} + \mathcal{B}_2$

(3.1-6) $A(\mathcal{V} \bigcap \text{Ker } C_2) \subset \mathcal{V} + \mathcal{B}_1$

(3.1-7) $A(\mathcal{V} \bigcap \text{Ker } C_1 \bigcap \text{Ker } C_2) \subset \mathcal{V}$

Proof: For necessity, take $v \in \mathcal{V}$. Then

$$(A + B_1 F_1 C_1 + B_2 F_2 C_2)\, v = \hat{v} \in \mathcal{V}$$

by assumption. Thus

$$Av = \hat{v} - B_1 F_1 C_1 v - B_2 F_2 C_2 v \in \mathcal{V} + \mathcal{B}_1 + \mathcal{B}_2$$

to establish (3.1-4). Moreover, if $v \in \mathcal{V} \bigcap \text{Ker } C_1$,

$$Av = \hat{v} - B_2 F_2 C_2 v \in \mathcal{V} + \mathcal{B}_2$$

which proves (3.1-5); (3.1-6) and (3.1-7) follow similarly.

For sufficiency, we write

$$\mathcal{V} \bigcap \text{Ker } C_1 = \mathcal{V} \bigcap \text{Ker } C_1 \bigcap \text{Ker } C_2 \oplus \hat{\mathcal{V}}_1$$

$$\mathcal{V} \bigcap \text{Ker } C_2 = \mathcal{V} \bigcap \text{Ker } C_1 \bigcap \text{Ker } C_2 \oplus \hat{\mathcal{V}}_2$$

We note that

$$\hat{\mathcal{V}}_1 \bigcap \text{Ker } C_2 = (\hat{\mathcal{V}}_1 \bigcap (\mathcal{V} \bigcap \text{Ker } C_1)) \bigcap \text{Ker } C_2 = 0$$

and similarly,

$$\hat{\mathcal{V}}_2 \bigcap \text{Ker } C_1 = 0$$

Therefore,

$$\hat{\mathcal{V}}_1 \bigcap (\mathcal{V} \bigcap \text{Ker } C_1 \bigcap \text{Ker } C_2 \oplus \hat{\mathcal{V}}_2) = \hat{\mathcal{V}}_1 \bigcap \mathcal{V} \bigcap \text{Ker } C_2 = 0$$

and we have

$$\mathcal{V} \bigcap \text{Ker } C_1 + \mathcal{V} \bigcap \text{Ker } C_2 = \mathcal{V} \bigcap \text{Ker } C_1 \bigcap \text{Ker } C_2 \oplus \hat{\mathcal{V}}_1 \oplus \hat{\mathcal{V}}_2$$

Now write

$$\mathcal{V} = \mathcal{V} \bigcap \text{Ker } C_1 \bigcap \text{Ker } C_2 \oplus \hat{\mathcal{V}}_1 \oplus \hat{\mathcal{V}}_2 \oplus \hat{\mathcal{V}}$$

Let $\{e_i\}$,$\{\hat{v}_{1i}\}$, $\{\hat{v}_{2i}\}$, and $\{\hat{v}_i\}$ be bases for $\mathcal{V} \bigcap \text{Ker } C_1 \bigcap \text{Ker } C_2$, $\hat{\mathcal{V}}_1$, $\hat{\mathcal{V}}_2$, and $\hat{\mathcal{V}}$, respectively. From (3.1-4) to (3.1-7) we know that

$A\, e_i \in \mathcal{V}$, for all i

$A\, \hat{v}_{1i} - B_2\, u_{1i} \in \mathcal{V}$; some u_{1i}, for all i

$A\, \hat{v}_{2i} - B_1\, u_{2i} \in \mathcal{V}$; some u_{2i}, for all i

$$A\,\hat{v}_i - B_1\,u_{3i} - B_2\,u_{4i} \,\varepsilon\, V;\ \text{some } u_{3i},\ u_{4i},\ \text{for all } i$$

We would like to find F_1, F_2 such that

$$(A + B_1\,F_1\,C_1 + B_2\,F_2\,C_2)\ v\ \varepsilon\ V$$

for every $v\ \varepsilon\ V$; or, equivalently, when v is any one of the above basis elements.

We first note that

$$(A + B_1\,F_1\,C_1 + B_2\,F_2\,C_2)\ e_i = A\,e_i\ \varepsilon\ V$$

for all matrices F_1 and F_2. Now consider the vectors

$$\{C_1\,\hat{v}_{2i}\} \bigcup \{C_1\,\hat{v}_i\}$$

These vectors are independent because

$$\begin{aligned} \text{Ker } C_1 \bigcap (\hat{V}_2 \oplus \hat{V}) &= \text{Ker } C_1 \bigcap V \bigcap (\hat{V}_2 \oplus \hat{V}) \\ &= (V \bigcap \text{Ker } C_1 \bigcap \text{Ker } C_2 \oplus \hat{V}_1) \bigcap (\hat{V}_2 \oplus \hat{V}) \\ &= 0 \end{aligned}$$

Similarly, the vectors

$$\{C_2\,\hat{v}_{1i}\} \bigcup \{C_2\,\hat{v}_i\}$$

are independent. Now define F_1 and F_2 so that

$$F_1\,C_1\,\hat{v}_{2i} = -\,u_{2i},\ F_1\,C_1\,\hat{v}_i = -\,u_{3i}$$

and

$$F_2\,C_2\,\hat{v}_{1i} = -\,u_{1i}, \quad F_2\,C_2\,\hat{v}_i = -\,u_{4i}$$

Using these matrices we now have

$$(A + B_1 F_1 C_1 + B_2 F_2 C_2)\, e_i \in \mathcal{V}$$

$$(A + B_1 F_1 C_1 + B_2 F_2 C_2)\, \hat{v}_{1i} = A\, \hat{v}_{1i} - B_2\, u_{1i} \in \mathcal{V}$$

$$(A + B_1 F_1 C_1 + B_2 F_2 C_2)\, \hat{v}_{2i} = A\, \hat{v}_{2i} - B_1\, u_{2i} \in \mathcal{V}$$

$$(A + B_1 F_1 C_1 + B_2 F_2 C_2)\, \hat{v}_{i} = A\, \hat{v}_{i} - B_1\, u_{3i} - B_2\, u_{4i} \in \mathcal{V} \qquad \square$$

Having determined the analogue of an (A,B)-invariant subspace, we now look at controllability subspaces.

(3.1-8) Proposition: Let $\mathcal{R} \subset R^n$ be an arbitrary subspace. Then $\mathcal{R}$ is a controllability subspace under decentralized feedback with two control agents, i.e. there exist F_1 and F_2 such that

$$\mathcal{R} = \{A + B_1 F_1 C_1 + B_2 F_2 C_2 \mid \mathcal{R} \cap (\mathcal{B}_1 + \mathcal{B}_2)\}$$

if and only if (3.1-4) through (3.1-7) are satisfied and, in addition, $\mathcal{R} = \mathcal{R}_n$ where

$$\mathcal{R}_o = 0, \quad \mathcal{R}_i = \mathcal{R} \cap (\mathcal{B}_1 + \mathcal{B}_2 + A\, \mathcal{R}_{i-1}),\ i \in \underline{n}$$

(3.1-9) Remark: It is unfortunate that (3.1-3) cannot be extended to accommodate more than two decentralized feedback controllers. The difficulties involved are quite similar to those noted in (2.2-30). It is, however, easy to show that a necessary condition for the existence of $\{F_i,\ i \in \underline{K}\}$ such that

$$(A + \sum_{i \in \underline{K}} B_i F_i C_i)\, \mathcal{V} \subset \mathcal{V}$$

is

$$A\,(V \cap \bigcap_{j\in I} \mathrm{Ker}\, C_j) \subset V + \sum_{j\notin I} B_j$$

for all subsets $I \subset \underline{K}$

(3.1-10) <u>Remark</u>: The result (3.1-8) is not as useful as it might appear. What one would like to have is a characterization of subspaces of the form

(3.1-11) $$R = \{A + B_1 F_1 C_1 + B_2 F_2 C_2 \mid B_1 \cap R\}$$

Such a subspace would then be controllable by agent 1. It can be shown that R satisfies (3.1-11) if and only if (3.1-4) through (3.1-7) are satisfied and, in addition, there exists an F_2 such that $R = \hat{R}_n$ where

$$\hat{R}_o = 0, \quad \hat{R}_i = R \cap (B_1 + (A + B_2 F_2 C_2)\hat{R}_{i-1}),\ i \in \underline{n}.$$

The dependence on F_2 makes this an unsatisfactory characterization.

A second detraction from (3.1-8) is that, even though R might be such a controllability subspace, in general one cannot allocate the eigenvalues of $A + B_1 F_1 C_1 + B_2 F_2 C_2$ restricted to R. As an example, suppose $C_1 = C_2 = 0$ and $R = \{A \mid (B_1 + B_2) \cap R\}$.

In the foregoing we have outlined a hierarchy of approaches to the pole allocation problem, all of which take as their primary objective the achievement of a triangularly decoupled system. However, this approach clearly places undue emphasis on the decoupling aspect, particularly if the control agents are allowed to collaborate in the determination of the feedback matrices F_i. That is, if we allow the F_i to be determined simultaneously then we are "simply" faced with the problem of selecting the F_i such that

$$\det(\lambda I - A \sum_{i \in \underline{K}} B_i F_i C_i) = \alpha(\lambda)$$

where $\alpha(\lambda)$ is the desired closed loop characteristic polynomial.

If we write

$$A + \sum_{i \in \underline{K}} B_i F_i C_i = A + (B_1 ; \dots B_K) \begin{pmatrix} F_1 & 0 & \dots & 0 \\ 0 & F_2 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & F_K \end{pmatrix} \begin{pmatrix} C_1 \\ \vdots \\ C_K \end{pmatrix}$$

then the problem of pole assignment via decentralized control can be phrased as a problem of pole assignment via output feedback, where the feedback matrix {outputs} $\mapsto$ {inputs} is constrained to a particular block diagonal form. If the matrix

$$\begin{pmatrix} C_1 \\ \vdots \\ C_K \end{pmatrix}$$

is n x n and nonsingular (which is essentially the case under assumptions (3.1-1) and (3.1-2)) then the problem becomes a state feedback problem where the feedback matrix satisfies the block diagonal constraint.

Problems where the feedback matrices satisfy certain constraints are in general unsolved. However, we shall attempt to formulate an approach to these problems, as introduced in the next section, in Chapter Fcur.

3.2 Introduction to Feedback Invariants

In Section 3.1 we noted that the problem of pole allocation via decentralized control can be formulated in terms of finding a feedback matrix F, of dimensions $(\sum_{i\varepsilon \underline{K}} m_i) \times (\sum_{i\varepsilon \underline{K}} p_i)$, such that

$$\det \left(\lambda I - A - (B_1;\ldots B_K)\, F \begin{pmatrix} C_1 \\ \vdots \\ C_K \end{pmatrix}\right) = \alpha(\lambda)$$

where F is constrained to be of the form

$$F = \begin{pmatrix} F_1 & 0 & \cdots & 0 \\ 0 & F_2 & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & F_K \end{pmatrix}, \quad (F_i \text{ is } m_i \times p_i)$$

Clearly, one could also represent decentralized feedback as a highly constrained class of state feedback.

Thus, in Chapter Four we shall develop a method for algebraically characterizing state feedback, and for determining the changes in system structure that can be induced by the use of state feedback. We shall also see that, in certain cases, feedback laws that are constrained to be of a particular class can be accommodated in this framework. Thus, we can hope to derive further results pertaining to decentralized feedback.

We shall formulate our ideas in terms of invariants. This concept is as follows (see also [53], [60], [75]). Let X be a set, and let E be an equivalence relation on X; thus, for all x, y, z ε X,

$$x \; E \; x$$

$$x \; E \; y \Longleftrightarrow y \; E \; x$$

$$x \; E \; y \text{ and } y \; E \; z \Rightarrow x \; E \; z$$

It then follows that X may be naturally decomposed as the disjoint union of equivalence classes under E:

$$\text{(3.2-1)} \qquad X = \dot{\bigcup_{\alpha}} X_{\alpha}$$

where each X_{α} is "generated" by an element $x_{\alpha} \in X$:

$$X_{\alpha} = \{y \in X \mid y \; E \; x_{\alpha}\}$$

A function $f : X \to S$ is called an <u>invariant of E</u> if

$$x \; E \; y \Rightarrow f(x) = f(y)$$

That is, an invariant of E simply must be constant over each of the sets X_{α}. A <u>complete system of invariants</u> is a list of functions $f_i : X \to S_i$, each of which is an invariant of E, and where

$$f_i(x) = f_i(y), \text{ for all } i \Rightarrow x \; E \; y$$

A system of invariants of E, $\{f_i : X \to S_i, \; i \in \underline{I}\}$, is said to be <u>independent</u> if there exists no $i \in \underline{I}$ with the property that

$$f_j(x) = f_j(y), \text{ for all } j \in \underline{I} - \{i\} \Rightarrow f_i(x) = f_i(y)$$

Thus, if $\{f_i : X \to S_i\}$ is a complete set of invariants of E, knowledge of the values $\{f_i(x)\}$ is the same as knowing the equivalence class to which x belongs. A complete set of independent invariants is, in a sense, a minimal complete set of invariants.

Finally, if X is decomposed as in (3.2-1), the set

$$C = \{x_\alpha\}$$

is called a set of <u>canonical forms</u>. The significance of C is that for every $x \in X$, there exists exactly one canonical form x_α such that $x \mathrel{E} x_\alpha$. Thus, if $\{f_i : X \to S_i\}$ is a complete set of invariants, knowledge of $\{f_i(x)\}$ allows us to uniquely determine a canonical form x_α such that $x \mathrel{E} x_\alpha$.

These ideas will now be placed in the context of system theory. Let (A,B) denote the system governed by

$$\dot{x} = Ax + Bu: \quad A \in R^{nxn}, \quad B \in R^{nxm}$$

Let GL(k) denote the group of k x k invertible matrices, and let $\underline{F} = R^{mxn}$. Then, by introducing state feedback and coordinate transformations in the input and state vector spaces, we can produce a new system described by

$$\dot{z} = T(A + B\,F)\,T^{-1}\,z + T\,B\,G\,v$$

where $T \in GL(n)$, $G \in GL(m)$, and $F \in \underline{F}$. We shall now define a relation, $\sim$, on the set of pairs (A,B) as follows:

$$(A,B) \sim (\hat{A},\hat{B}) \iff (\hat{A},\hat{B}) = (T(A + B\,F)\,T^{-1},\; T\,B\,G)$$

$$\text{for some } T \in GL(n),\ G \in GL(m),\ F \in \underline{F}$$

It is a straightforward exercise to verify that $\sim$ is an equivalence relation.

Thus, it is now of interest to determine a complete system of independent invariants of $\sim$, as well as a set of canonical forms. Having accomplished this, we shall be able to identify all properties of the pair (A, B) that <u>cannot</u> be changed by introducing feedback and coordinate transformations; any such property must be an invariant of $\sim$, and as such must be expressible in terms of the complete system of independent invariants.

The problem of determining a complete set of independent invariants of $\sim$, and a set of canonical forms $\{(A_\alpha, B_\alpha)\}$, has been completely solved ([42], [60], [75]) for the case where the pairs (A, B) are constrained so that

(3.2-2) $\{A \mid \mathcal{B}\} = R^n$

The solution is as follows.

(3.2-3) <u>Proposition</u>: For each pair $(A, B) \in R^{n\times n}$ subject to (3.2-2), determine two sets of integers $\{r_i, i \geq 0\}$ and $\{\nu_i, i \in \underline{m}\}$ as follows. Define

$$r_o = \text{rank } B$$

$$r_i = \text{rank } (B; \ldots A^i B) - \text{rank } (B; \ldots A^{i-1} B), \; i \geq 1$$

and

$$\nu_i = \# \{r_j \mid r_j \geq i\}, \quad i \in \underline{m}$$

where # denotes cardinality.

Then, $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_m \geq 0$, $\sum_{i\in\underline{m}} \nu_i = n$, and the maps $(A,B) \mapsto \nu_i$, $i \in \underline{m}$, are a complete set of independent invariants for $\sim$. For each pair (A,B), there exists a unique canonical pair (A_c, B_c) with $(A,B) \sim (A_c,B_c)$. The canonical pair is specified in terms of the integers $\{\nu_i, i \in \underline{m}\}$ as

$$A_c = \begin{pmatrix} H_{\nu_1} & 0 & \dots & 0 \\ 0 & H_{\nu_2} & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & H_{\nu_{m'}} \end{pmatrix}$$

$$B_c = \begin{pmatrix} h_{\nu_1} & 0 & \dots & 0 & 0 \\ 0 & h_{\nu_2} & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & h_{\nu_{m'}} & \underbrace{0}_{m-m' \text{ columns}} \end{pmatrix}$$

where $m' = \max \{i \mid \nu_i > 0\}$ and

$$H_k = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (k \times k)$$

$$h_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad (k \times 1)$$

(3.2-4) Remark: When (3.2-2) is not satisfied, a complete set of invariants can be found as follows. Let $\bar{A} : \{A \mid \mathcal{B}\} \to \{A \mid \mathcal{B}\}$ be the restriction of $A : R^n \to R^n$, and $\bar{B} : R^m \to \{A \mid \mathcal{B}\}$ the restriction of $B : R^m \to R^n$. Let $\{\bar{\nu}_i, i \in \underline{m}\}$ be determined from

$(\bar{A}, \bar{B})$ as in (3.2-3). Also, for $P : R^n \to R^n/\{A \mid \mathcal{B}\} \triangleq \mathcal{X}$, let $\hat{A} : \mathcal{X} \to \mathcal{X}$ be the map induced by $A : R^n \to R^n$; i.e., $\hat{A}$ satisfies $\hat{A}\, P = P\, A$. Finally, let $\{\psi_i(\lambda),\ i \in \underline{n}\}$ be the invariant polynomials of $\hat{A}$, ordered so that $\psi_{i+1}(\lambda)$ divides $\psi_i(\lambda)$ for $1 \leq i \leq n - 1$, and with $\psi_i(\lambda) \triangleq 1$ for $\dim(\mathcal{X}) < i \leq n$.

Then, a complete set of independent invariants for $\sim$ is the set of maps $(A, B) \mapsto \bar{\nu}_i$, $i \in \underline{m}$, together with the maps $(A, B) \mapsto \hat{\psi}_i(\lambda)$, $i \in \underline{n}$.

One of the important results arising from (3.2-3) is

(3.2-5) Theorem: (Rosenbrock, [61]): Given $(A, B) \in R^{n\times n} \times R^{n\times m}$ such that $\{A \mid \mathcal{B}\} = R^n$, there exists a matrix $F \in R^{m\times n}$ such that the invariant polynomials of $A + B\,F$ are identical to the polynomials of $\{\psi_i(\lambda),\ i \in \underline{n}\}$ if

$$\psi_{i+1}(\lambda) \text{ divides } \psi_i(\lambda), \text{ for } 1 \leq i \leq n - 1$$

and

$$\sum_{j=1}^{i} \deg \psi_j \geq \sum_{j=1}^{i} \nu_j, \quad \text{for } i \in \underline{m}$$

where $\{\nu_i,\ i \in \underline{m}\}$ are determined as in (3.2-3).

We shall prove (3.2-3) and (3.2-5) in Chapter Four, where a module theoretic characterization of linear systems is developed. It will be seen there that the problem of obtaining a complete set of independent invariants when the feedback is constrained to be of a particular class (e.g. as in the case of decentralized feedback) remains unsolved. However, it will be possible to determine a characterization of system structures attainable via decentralized feedback.

CHAPTER 4

A CHARACTERIZATION OF LINEAR SYSTEMS VIA POLYNOMIAL MATRICES AND MODULE THEORY

4.1 Introduction

Techniques for solving problems in the area of linear, constant, finite dimensional systems usually may be classified as belonging to one of the following classes: transform techniques, where an external description of the system is given by a transfer matrix; state space techniques, where the internal dynamics of the system are exhibited explicitly by a vector differential equation; polynomial matrix techniques, where the system is described by the "system matrix;" and abstract algebraic techniques, where the system description is in terms of the canonical factorization of the input-output map. Any one of the above classes of techniques may be more appropriate, in solving a particular problem, than the others. For example, state space techniques are appropriate for treating the decoupling problem; polynomial matrix techniques have been applied by Rosenbrock ([61]) and Wolovich ([67] - [70]) in problems of model matching; algebraic techniques have been used by Kalman ([38], [39], [41], [43]), Arbib and Zeiger ([3], [76]), and Giveon and Zalcstein ([31]) in determining the structural properties of a system from an external description.

In this chapter we shall develop a new method for characterizing linear systems. This characterization is mostly algebraic in nature, building on the module - theoretic characterization of linear systems developed by Kalman; however, polynomial matrices, transfer matrices, and dynamical system properties play a significant role in the development. Thus, the contents of this chapter may be viewed somewhat as a unification of the previously mentioned classes of linear system techniques.

More importantly, however, this characterization will provide us with a new and powerful way to treat the subject of feedback invariants; also, new results pertaining to controllability subspaces may be obtained fairly easily in this new framework. Thus, the abstract algebraic approach to linear system theory will be seen to have a significantly wider scope than has been apparent from previously published works.

The remainder of this chapter is organized as follows. Section 4.2 is devoted to a review of Kalman's module-theoretic characterization of linear systems. In Section 4.3, a new method of canonically characterizing submodules of a free module, quotient modules, and canonical projections will be presented; these characterizations will be seen to be in terms of polynomial matrices of a particular class, which we shall denote as canonical matrices. In Section 4.4 we shall develop a characterization of linear systems in terms of canonical polynomial matrices; the appropriateness of this characterization for problems in realization and system structure will be demonstrated. In Section 4.5 the theory is extended to incorporate state feedback; the subjects of feedback invariants and controllability subspace characterizations will be presented. Finally, in Section 4.6 we explore the subject of feedback invariants when the feedback is constrained to be of a particular class, as in decentralized feedback.

4.2 Review of Linear System Theory via Modules

This section is intended as both an introduction and a review of the algebraic characterization of linear systems via module theory. The approach and notation closely follow that in [38, Ch.10]. References that will provide the required background in algebra are [53, Ch. 1-10], [46], and to some extent [21]; most of the commonly used concepts and results are summarized in Appendix B. References that illustrate the varied ways that abstract algebra can be used to characterize systems are the works of: Arbib and Zeiger, [3], [76]; Giveon and Zalcstein, [31]; and Kalman, [39], [41].

In what follows we shall always be considering systems that are linear, constant, and described in discrete time. A very similar characterization of linear, constant, continuous time systems has been presented by Kalman and Hautus, [43]; and Bensoussan, Delfour, and Mitter [6]. However, since we are interested in the underlying algebraic structure of a system, and since for each continuous time finite dimensional system there is a discrete time system with the identical structure, we shall not lose any generality in restricting ourselves to discrete time systems. Also, in this chapter the systems will always be defined over the real numbers, R; however, as we are considering discrete time systems, practically all the results remain valid if R is replaced by an arbitrary field K.

Thus, let Σ denote a linear, constant, discrete time, causal system. We shall think of Σ as processing strings of input values taken from R^m, and from the input strings producing strings of output values in R^p. Because of the assumptions on Σ, this system is completely characterized by an input-output map f_Σ

$$f_\Sigma: (R^m)^* \to R^p$$

where

$$(R^m)^* = \{(\cdots,0,0,u_{-k},u_{-k+1},\cdots u_0) \mid k \in Z_+ = \text{nonnegative integers, and each } u_i \in R^m\}$$

Thus, $(R^m)^*$ is just the set of all semi-infinite sequences $(\cdots,u_{-2},u_{-1},u_0)$ of vectors in R^m for which only a finite number of the u_i are nonzero. The interpretation that one should give to a sequence $(\cdots 0,0,u_{-k},\cdots u_0) \in (R^m)^*$ is that of a string of control input values which starts at time $t = -k$ and terminates at time $t = 0$. Each such sequence $u^* \in (R^m)^*$ produces an output value (in R^p) at time $t = 1$; it is in terms of this output value that we define the action of f_Σ on u^*:

$$f_\Sigma: \quad (R^m)^* \to R^p$$

$$: \quad u^* \mapsto y_1 = \text{output value at time } t = 1 \text{ resulting from input string } u^*$$

We also define the extended output space $(R^p)^{**}$ to be the set of all semi-infinite sequences of vectors in R^p:

$$(R^p)^{**} = \{(y_1,y_2,y_3,\cdots) \mid \text{each } y_i \in R^p\}$$

and the extended input-output map f_Σ^* as

$$f_\Sigma^* : \quad (R^m)^* \to (R^p)^{**}$$

$$: \quad u^* \mapsto y^{**} = (y_1,y_2,y_3,\cdots)$$

where, for each $i \geq 1$,

$$y_i = \text{output value at time } t = i \text{ resulting from input string } u^*$$

It is clear that the map f_Σ^* uniquely determines the map f_Σ; we shall shortly see that f_Σ uniquely determines f_Σ^* when Σ is linear and constant.

The spaces $R^p,(R^m)^*$ and $(R^p)^{**}$ admit the structure of R-vector spaces, where addition and multiplication by scalars in the latter two are defined in the obvious way: If $u^* = (\cdots 0,u_{-k},\cdots u_0) \in (R^m)^*$ and $v^* =$

$(\cdots 0, v_{-\ell}, \cdots v_0) \in (R^m)^*$, where (say) $k \geq \ell$, then

$$u^* + v^* = (\cdots 0, u_{-k}, \cdots u_{-\ell-1}, u_{-\ell} + v_{-\ell}, \cdots u_0 + v_0)$$

$$au^* = (\cdots 0, au_{-k}, \cdots au_0), \text{ for all } a \in R$$

Also, if $y^{**} = (y_1, y_2, \cdots) \in (R^p)^{**}$ and $w^{**} = (w_1, w_2, \cdots) \in (R^p)^{**}$ then

$$y^{**} + w^{**} = (y_1 + w_1, y_2 + w_2, \cdots)$$

$$ay^{**} = (ay_1, ay_2, \cdots), \text{ for all } a \in R$$

We also define the left shift operators σ_* and σ_{**} as

(4.2-1) $\quad \sigma_*: \ (R^m)^* \to (R^m)^*$

$\quad : \ (\cdots 0, u_{-k}, \cdots, u_0) \mapsto (\cdots 0, u_{-k}, \cdots u_0, 0)$

and

(4.2-2) $\quad \sigma_{**}: \ (R^p)^{**} \to (R^p)^{**}$

$\quad : \ (y_1, y_2, y_3, \cdots) \mapsto (y_2, y_3, \cdots)$

These operators are easily shown to be R-linear maps.

We can now define linear, constant, discrete time systems.

(4.2-3) Definition: Σ is a linear, constant, discrete time system if

$$f^*_\Sigma (a\sigma_*^k u^* + b\sigma_*^\ell v^*) = a\sigma^k_{**} f^*_\Sigma(u^*) + b\sigma^\ell_{**} f^*_\Sigma(v^*)$$

for all $a,b \in R$; $u^*, v^* \in (R^m)^*$; and $k, \ell \geq 0$.

Clearly, if Σ is linear and constant, then $f_\Sigma: \ (R^m)^* \to R^p$ is an R-linear map. The following result indicates that Σ is completely characterized by f_Σ.

(4.2-4) Proposition: Let Σ be a linear, constant, discrete time system with input space $(R^m)^*$ and output space $(R^p)^{**}$.

Then

(i) f^*_Σ is determined by f_Σ as

$$f^*_\Sigma : \ (R^m)^* \to (R^p)^{**}$$

$: \quad u^* \mapsto (f_\Sigma(u^*), f_\Sigma(\sigma_* u^*), f_\Sigma(\sigma_*^2 u^*), \cdots)$

(ii) If $\{e_i, i \in \underline{m}\}$ is the standard basis for R^m, and if matrices $G_j \in R^{p \times m}$ are defined as

$$G_j = (f_\Sigma(e^*_{1j}); \cdots; f_\Sigma(e^*_{mj})), \; j \geq 0 \tag{4.2-5}$$

where

$$e^*_{i0} = (\cdots 0, 0, e_i)$$

and

$$e^*_{i,j+1} = \sigma_* e^*_{ij} \;, \text{ for } j \geq 0$$

then f_Σ is represented as

$$f_\Sigma : (\cdots 0, 0, u_{-k}, \cdots u_0) \mapsto \sum_{j=0}^{k} G_j u_{-j}$$

(iii) With $\{G_j, j \geq 0\}$ as given by (4.2-5), the map f^*_Σ : $(R^m)^* \to (R^p)^{**}$ is represented as

$$f^*_\Sigma : \quad (\cdots 0, 0, u_{-k}, \cdots u_0) \mapsto (y_1, y_2, y_3, \cdots)$$

where

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} G_0 & G_1 & \cdots & G_k \\ G_1 & G_2 & \cdots & G_{k+1} \\ G_2 & G_3 & \cdots & G_{k+2} \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} u_0 \\ u_{-1} \\ \vdots \\ u_{-k} \end{bmatrix}$$

(iv) If $(\cdots 0, 0, u_{t_0}, u_{t_0+1}, \cdots)$ is an arbitrary (possibly semi-infinite) input string which starts at time $t = t_0$, then the resulting output string $(\cdots y_{t_0-1}, y_{t_0}, y_{t_0+1}, \cdots)$ is given by

(4.2-6) $y_j = f_\Sigma(\cdots,u_{j-2},u_{j-1})$, for all j

(4.2-7) Remark: The sequence $\{G_j, j \geq 0\}$ is known as the pulse response of Σ. The infinite array

$$H = \begin{pmatrix} G_0 & G_1 & G_2 & \cdot & \cdot & \cdot \\ G_1 & G_2 & G_3 & \cdot & \cdot & \cdot \\ G_2 & G_3 & G_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \end{pmatrix}$$

is called the Hankel matrix associated with Σ.

(4.2-8) Remark: From (4.2-4), part (iv), we immediately see that our definition of linear, constant, discrete time systems has the notion of causality already "built in." That is, if the input sequence is $(\cdots 0,0,u_{t_0},u_{t_0+1},\cdots)$, then from (4.2-6),

$$y_j = f_\Sigma(\cdots 0,0,0) = 0 \text{ , for all } j \leq t_0$$

since f_Σ is a linear map.

(4.2-9) Remark: If Σ is a system described by the equations

$$x_{t+1} = Ax_t + Bu_t,\ x_{-\infty} = 0$$

$$y_t = Cx_t$$

Then it is easily seen that Σ is linear and constant according to the definition (4.2-3), and that the pulse response $\{G_j, j \geq 0\}$ is given by

$$G_j = CA^jB,\ j \geq 0$$

The above characterization of Σ may now be translated into the language of modules as follows. In place of the space of input sequences, $(R^m)^*$, we substitute the free $R[\lambda]$ - module $R^m[\lambda]$, where $R[\lambda]$ is the ring of polynomials in the indeterminate λ with coefficients in R, and $R^m[\lambda]$ is the module of polynominal m-vectors in the indeterminate λ with coefficients in R^m. The correspondence between $(R^m)^*$ and $R^m[\lambda]$ is as follows. Each element $(\cdots 0, u_{-k}, \cdots u_0) \in (R^m)^*$ determines an element $u(\lambda) \in R^m[\lambda]$ by the rule

$$u(\lambda) = \sum_{i=0}^{k} \lambda^i u_{-i}$$

This establishes an R-linear isomorphism between $(R^m)^*$ and $R^m[\lambda]$. Defining multiplication by λ in $R^m[\lambda]$ as

$$\lambda \cdot \sum_{i=0}^{k} \lambda^i u_{-i} = \sum_{i=1}^{k+1} \lambda^i u_{-i+1}$$

it is easily seen that the action of λ on $R^m[\lambda]$ is equivalent to the action of σ_* on $(R^m)^*$, defined in (4.2-1).

Similarly, we establish an R-linear isomorphism between $(R^p)^{**}$ and $R^p[[\lambda^{-1}]]$, the $R[\lambda]$-module of formal vector power series in the indeterminate λ^{-1} with coefficients in R^p.

This isomophism is just

$$(R^p)^{**} \xrightarrow{\sim} R^p[[\lambda^{-1}]]$$

$$(y_1, y_2, \cdots) \mapsto \sum_{i=1}^{\infty} \lambda^{-i} y_i = y(\lambda)$$

We define multiplication by λ in $R^p[[\lambda^{-1}]]$ as

$$\lambda \cdot \sum_{i=1}^{\infty} \lambda^{-i} y_i = \sum_{i=1}^{\infty} \lambda^{-i} y_{i+1}$$

That is, to compute $\lambda y(\lambda)$, for $y(\lambda) \in R^p[[\lambda^{-1}]]$, simply formally multiply λ and $y(\lambda)$, and then delete any nonnegative powers of λ. It is easy to see that the action of λ on $R^p[[\lambda^{-1}]]$ is the same as the action of σ_{**} on $(R^p)^{**}$, as defined in (4.2-2).

Clearly, the input-output map f_Σ: $(R^m)^* \to R^p$ and the extended input-output map f^*_Σ : $(R^m)^* \to (R^p)^{**}$ uniquely determine corresponding maps $\hat{f}_\Sigma$: $R^m[\lambda] \to R^p$ and $\hat{f}^*_\Sigma$: $R^m[\lambda] \to R^p[[\lambda^{-1}]]$. For ease of notation, we shall call the former

$$f_\Sigma: \; R^m[\lambda] \to R^p$$

and the latter

$$f^*_\Sigma: \; R^m[\lambda] \to R^p[[\lambda^{-1}]]$$

Then, in module-theoretic terms, (4.2-3) becomes

(4.2-10) <u>Proposition</u>: Σ is a linear, constant, discrete time system if and only if f^*_Σ: $R^m[\lambda] \to R^p[[\lambda^{-1}]]$ is a morphism of $R[\lambda]$-modules.

Proof: From (4.2-3), Σ is linear and constant if and only if

$$f^*_\Sigma(a\lambda^k u(\lambda) + b\lambda^\ell v(\lambda)) = a\lambda^k f^*_\Sigma(u(\lambda)) + b\lambda^\ell f^*_\Sigma(v(\lambda))$$

for all $a, b \in R$; $u(\lambda)$, $v(\lambda) \in R^m[\lambda]$; and $k, \ell \geq 0$. Thus, Σ is linear and constant if and only if

$$f^*_\Sigma(\alpha(\lambda)u(\lambda) + \beta(\lambda)v(\lambda)) = \alpha(\lambda)f^*_\Sigma(u(\lambda)) + \beta(\lambda)\, f^*_\Sigma(v(\lambda))$$

for all $\alpha(\lambda), \beta(\lambda) \in R[\lambda]$ and all $u(\lambda), v(\lambda) \in R^m[\lambda]$. ■

(4.2-11) <u>Definition</u>: If Σ is a linear, constant, discrete time system, then the map f^*_Σ: $R^m[\lambda] \to R^p[[\lambda^{-1}]]$ will be called the <u>input-output morphism</u> of Σ.

We have already seen that the extended input-output map f^*_Σ: $(R^m)^* \to (R^p)^{**}$ can be represented in terms of the Hankel matrix of Σ. We now give an equivalent representation of the input-output morphism.

(4.2-12) **Proposition:** Let Σ be a linear, constant, discrete time system, and let $f^*_\Sigma: R^m[\lambda] \to R^p[[\lambda^{-1}]]$ be the input-output morphism. Then f^*_Σ is represented as

$$f^*_\Sigma: \; R^m[\lambda] \to R^p[[\lambda^{-1}]]$$

$$: \; u(\lambda) \mapsto [H(\lambda)u(\lambda)]$$

where, if $\{G_j, j \geq 0\}$ is as given in (4.2-5), $H(\lambda) \in R^{p\times m}[[\lambda^{-1}]]$ is

$$H(\lambda) = \sum_{j=1}^{\infty} \lambda^{-j} G_{j-i}$$

and where the product $[H(\lambda)u(\lambda)]$ is interpreted as the formal product $H(\lambda)u(\lambda)$ with all nonnegative powers of λ deleted.

Proof: From (4.2-4), part (iii), it follows that if

$$u(\lambda) = \sum_{i=0}^{k} \lambda^{i} u_{-i}$$

then

$$f^*_\Sigma(u(\lambda)) = y(\lambda) = \sum_{i=1}^{\infty} \lambda^{-i} y_i$$

where

(4.2-13) $$y_i = \sum_{j=0}^{k} G_{j+i-1} u_{-j}, \text{ for } i \geq 1$$

But, formal multiplication of $H(\lambda)$ by $u(\lambda)$ gives

$$H(\lambda)u(\lambda) = \left(\sum_{j=1}^{\infty} \lambda^{-j} G_{j-1} \right) \cdot \left(\sum_{i=0}^{k} \lambda^{i} u_{-i} \right)$$

$$= \sum_{j=-k+1}^{0} \lambda^{-j} \sum_{i=1-j}^{k} G_{j+i-1} u_{-i}$$

$$+ \sum_{j=1}^{\infty} \lambda^{-j} \sum_{i=0}^{k} G_{j+i-1} u_{-i}$$

Therefore, deleting the nonnegative powers of λ,

$$[H(\lambda)u(\lambda)] = \sum_{j=1}^{\infty} \lambda^{-j} \sum_{i=0}^{k} G_{j+i-1} u_{-i}$$

$$= \sum_{j=1}^{\infty} \lambda^{-j} y_j$$

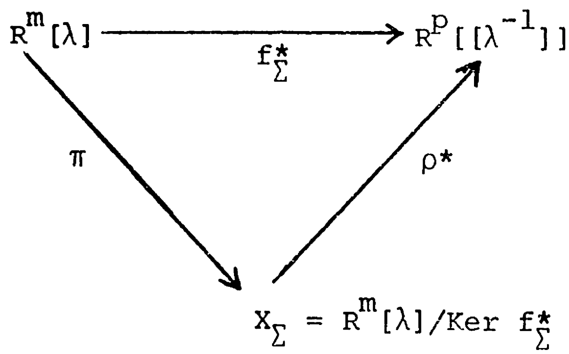
$$= y(\lambda)$$

where the second to last line follows from (4.2-13). ■

(4.2-14) <u>Remark</u>: This proposition illustrates the algebraic significance of $H(\lambda)$, which is really just the familiar transfer matrix written as a formal power series in λ^{-1}.

The input-output morphism f_Σ^*: $R^m[\lambda] \to R^p[[\lambda^{-1}]]$ has provided us with a module-theoretic, <u>external</u> description of Σ. Aside from the notational simplicity that results from expressing inputs as polynominal m-vectors, and outputs as power series p-vectors, this external description of Σ is identical to that provided by the extended input-output map, f_Σ^*: $(R^m)^* \to (R^p)^{**}$. However, the power of the module theoretic characterization will soon become apparent when we deduce an internal description of Σ from the input-output morphism.

We procede as follows. Let X_Σ be the quotient $R[\lambda]$-module $R^m[\lambda]/\mathrm{Ker}\ f_\Sigma^*$; X_Σ has the interpretation of being the set of Nerode equivalence classes in $R^m[\lambda]$ (see [38]). Let π: $R^m[\lambda] \to X_\Sigma$ denote the canonical projection, and let ρ^*: $X_\Sigma \to R^p[[\lambda^{-1}]]$ be chosen so that the following diagram commutes:



(4.2-15)

Such a ρ^* clearly exists because of the definition of X_Σ; moreover, π is an epimorphism (i.e. surjective) and ρ^* is unique and a monomophism (i.e., injective).

By a <u>realization</u> of Σ we shall mean a quadruple (X,A,B,C) where X is an R-vector space, A: $X \to X$ is an R-linear endomorphism, B: $R^m \to X$ and C: $X \to R^p$ are R-linear maps, and such that

$$CA^iB = G_i \quad , \; i \geq 0$$

where $\{G_i, i \geq 0\}$ is as given in (4.2-5). By a <u>canonical</u> realization we shall mean a realization that is both reachable and observable, i.e. such that

$$\{A|B\} = X$$

$$\bigcap_i \mathrm{Ker}(CA^i) = 0$$

By a <u>finite dimensional</u> realization we shall mean a realization where the dimension of X, as an R-vector space, is finite. It is well known ([8],[38]) that every canonical, finite dimensional realization is <u>minimal</u>, i.e. there exists no other realization where the dimension of $\mathcal{X}$ is smaller.

We can now prove the following.

(4.2-16) **Theorem:** Let Σ be a linear, constant, discrete time system, and let f_Σ^* be the input-output morphism; let f_Σ^* be canonically factored as in (4.2-15). Then, an abstract realization Σ can be obtained as follows.

(i) Define the R-vector space $\mathcal{X}$ to be X_Σ, i.e. the underlying set X_Σ together with the action of R on X_Σ (implicit by the action of $R[\lambda]$ on X_Σ).

(ii) Define the endomorphism $A: \mathcal{X} \to \mathcal{X}$ as

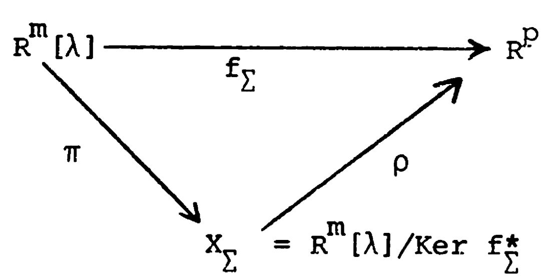
$$A: \quad x \mapsto \lambda x$$

This is well defined, because each $x \in \mathcal{X}$ is also an element of the $R[\lambda]$-module X_Σ.

(iii) Define $\imath: R^m \to R^m[\lambda]$ as the insertion of R^m into $R^m[\lambda]$ (this is well defined as an R-linear map) and define $B: R^m \to \mathcal{X}$ as

$$B = \pi \circ \imath$$

(iv) Clearly, Ker $f_\Sigma^* \subset$ Ker f_Σ; thus there exists an R-linear map $\rho: X_\Sigma \to R^p$ such that the following diagram commutes



Thus, define $C: \mathcal{X} \to R^p$ as

$$C = \rho$$

Proof: We must show that, for the (X,A,B,C) defined above,

$$CA^jB = G_j \, , \; j \geq 0$$

But,

$$CA^jB = \rho\lambda^j\pi\iota$$

$$= \rho\pi\lambda^j\iota \qquad \text{(since } \pi \text{ is a } R[\lambda]\text{-morphism)}$$

$$= f_\Sigma\lambda^j\iota \qquad \text{(definition of } \rho)$$

If e_k is the k'th standard basis vector of R^m, then $\lambda^j\iota e_k$ is the polynomial m-vector that represents the input sequence e^*_{kj} (see (4.2-4), part (ii)). Thus,

$$CA^jB\, e_k = f_\Sigma(e^*_{kj})$$

$$= \text{k'th column of } G_j \text{ (from 4.2-5))}$$

Therefore,

$$CA^jB\, e_k = G_j\, e_k \, , \; k \in \underline{m}$$

whence

$$CA^jB = G_j \quad , \quad \text{for all } j \geq 0 \qquad \blacksquare$$

It is important to determine when the system Σ can be realized as a finite dimensional system. The following theorem answers this question.

(4.2-17) Theorem: Let Σ be a linear, constant, discrete time system and let the input-output morphism f^*_Σ be canonically factored as in (4.2-15). Then,

(i) There exists a finite dimensional realization of Σ if and only if the quotient module X_Σ is a torsion module.

(ii) The realization defined in (4.2-16) is always canonical (i.e. both reachable and observable).

(iii) If X_Σ is torsion, the realization of (4.2-16) is minimal.

Proof: (i) If $(\mathcal{X}, A,B,C)$ is a finite dimensional realization of dimension n, by the Cayley-Hamilton theorem there is an $\alpha(\lambda) \in R[\lambda]$ of degree n such that $\alpha(A) = 0$; if $\alpha(\lambda) = \sum_{i=0}^{n} \lambda^i \alpha_i$, this implies that

$$\sum_{i=0}^{n} G_{i+k}\alpha_i = \sum_{i=0}^{n} CA^{i+k}B\alpha_i = CA^k\alpha(A)B = 0, \text{ for all } k \geq 0$$

Let $u \in R^m$ be any vector; if $y(\lambda) = f_\Sigma^*(\alpha(\lambda)u)$, then from the proof to (4.2-12),

$$y(\lambda) = \sum_{j=1}^{\infty} \lambda^{-j} \sum_{i=0}^{n} G_{j+i-1}\alpha_i u = 0$$

Since the elements of R^m are generators for $R^m[\lambda]$, it now follows that

$$f_\Sigma^*(\alpha(\lambda)u(\lambda)) = 0 \text{ , for all } u(\lambda) \in R^m[\lambda]$$

Thus, $\alpha(\lambda)R^m[\lambda] \subset \text{Ker } f_\Sigma^*$, and it follows that

$$\alpha(\lambda)\ x = 0 \text{ , for all } x \in X_\Sigma$$

i.e., X_Σ is torsion.

Conversely, suppose that X_Σ is torsion, and consider the realization of (4.2-16). We shall derive a simple bound on dim $(\mathcal{X})$. Indeed, if $\alpha(\lambda)$ is the minimal annihilator of X_Σ and if $\partial\alpha(\lambda) = d$, then, for all $u \in R^m$,

$$\pi(\alpha(\lambda)u) = \alpha(\lambda)\pi(u) = 0$$

whence it follows inductively that

$$\pi(\lambda^{d+k}u) \in R\text{-linear span of } \{\pi(\lambda^i u),\ 0 \leq i \leq d-1\}, \text{ for } k \geq 0$$

Since $\pi: R^m[\lambda] \to X_\Sigma$ is surjective, it then follows that a basis for $\mathcal{X}$ can be selected from the vectors $\{\pi(\lambda^i e_k);\ 0 \leq i \leq d-1,\ k \in \underline{m}\}$ where $\{e_k, k \in \underline{m}\}$ is any basis for R^m. Therefore,

$$\dim(X) \leq md < \infty$$

(ii) To show that the realization of (4.2-16) is reachable, note that

$$\{A|B\} \triangleq \sum_{j\geq 0} A^j \operatorname{Im} B$$

$$= \sum_{j\geq 0} \lambda^j \operatorname{Im}(\pi \circ \iota)$$

$$= \pi \sum_{j\geq 0} \operatorname{Im}(\lambda^j \circ \iota)$$

$$= \pi R^m[\lambda]$$

$$= X$$

since π is surjective. To show that this realization is observable, let $x \in \bigcap_i \operatorname{Ker}(CA^i)$. But $x = \pi(u(\lambda))$ for some $u(\lambda) \in R^m[\lambda]$. Thus

$$\rho\lambda^i\pi(u(\lambda)) = \rho\pi(\lambda^i u(\lambda)) = f_\Sigma(\lambda^i u(\lambda)) = 0, \text{ for all } i \geq 0$$

whence, using (4.2-4) part (i),

$$f_\Sigma^*(u(\lambda)) = 0$$

Thus $u(\lambda) \in \operatorname{Ker} f_\Sigma^*$, $x = 0$, and the realization is observable.

(iii) This part follows from (i) and the fact that finite dimensional canonical realizations are minimal. ■

While the realization of (4.2-16) is canonical, it suffers from the fact that it is abstract, i.e. X and A,B, and C are abstract quantities. It will be the purpose of the next section to develop a method for concretely representing these quantities. Thus a complete representation theory of linear, constant systems, including a theory for concrete realization, will evolve.

4.3 Characterizations of Modules and Morphisms via Canonical Polynomial Matrices

In Section 4.2 we have seen that, if Σ is a linear, constant, discrete time system with input-output morphism f^*_Σ: $R^m[\lambda] \to R^p[[\lambda^{-1}]]$, then the internal structure of any canonical (i.e. both reachable and observable) realization of Σ may be completely deduced from the canonical factorization.

$$\begin{array}{ccc} R^m[\lambda] & \xrightarrow{\quad f^*_\Sigma \quad} & R^p[[\lambda^{-1}]] \\ & {\scriptstyle \pi}\searrow \qquad \nearrow{\scriptstyle \rho^*} & \\ & X_\Sigma = R^m[\lambda]/\text{Ker } f^*_\Sigma & \end{array} \tag{4.3-1}$$

Unfortunately, while (4.3-1) provides us with a complete characterization of Σ (modulo unobservable and unreachable portions), it is too abstract to be of much practical use. What is needed, therefore, is a method of characterizing the morphisms π and ρ^*, and the state module X_Σ, all in concrete terms.

This is trivially possible for the case where m = 1. For then the input space is just $R[\lambda]$, and Ker $f^*_\Sigma \subset R[\lambda]$, being a submodule of $R[\lambda]$, is simply an ideal in $R[\lambda]$. But, since $R[\lambda]$ is a principal ideal domain, it follows that there exists a unique monic polynomial $t(\lambda) \in R[\lambda]$ such that

$$\text{Ker } f^*_\Sigma = (t(\lambda))$$

where $(t(\lambda))$ denotes the ideal generated by $t(\lambda)$, i.e. all polynomial multiples of $t(\lambda)$. The state module X_Σ is then just

$$X_\Sigma = R[\lambda]/(t(\lambda))$$

i.e. X_Σ is isomorphic to the ring of polynomials modulo $t(\lambda)$. Moreover, the canonical projection π: $R[\lambda] \to X_\Sigma$ is basically just the operation "reduce $u(\lambda) \in R[\lambda]$ modulo $t(\lambda)$."

X_Σ and π: $R[\lambda] \to X_\Sigma$ may now be represented in concrete terms as follows. Let the degree of $t(\lambda)$ be n, i.e.

$$t(\lambda) = \lambda^n + \lambda^{n-1}\, t_{n-1} + \cdots + t_0$$

Thus, each element of X_Σ, i.e. each coset of $(t(\lambda))$, has a unique polynomial representative of degree less than n; also, every polynomial of degree less than n determines a coset of $(t(\lambda))$. It follows that X_Σ is R-isomorphic to the n-dimensional R-vector space of all polynomials (in $R[\lambda]$) of degree less than n:

$$X_\Sigma \cong \mathcal{X} \triangleq \{x(\lambda) \in R[\lambda] \mid \partial x(\lambda) < n\} \quad \text{(as R-vector spaces)}$$

We can now represent the action of λ on X_Σ by an endomorphism A: $\mathcal{X} \to \mathcal{X}$, as follows. The set of vectors $\{e_i, i \in \underline{n}\}$, where

$$e_i = \lambda^{i-1}, \ i \in \underline{n}$$

clearly forms a basis for $\mathcal{X}$; thus the set $\{e_i + (t(\lambda)), i \in \underline{n}\}$ forms a basis for X_Σ, taken as an R-vector space. However,

$$\lambda(e_i + (t(\lambda))) = (e_{i+1} + (t(\lambda))), \text{ for } 1 \le i \le n-1$$

and, since $\lambda^n \equiv -(\lambda^{n-1} t_{n-1} + \cdots + t_0)$ (mod $t(\lambda)$),

$$\lambda(e_n + (t(\lambda))) = -\sum_{i=1}^{n} t_{i-1}\, (e_i + (t(\lambda)))$$

Thus, the action of λ on X_Σ is represented by $A\colon \mathcal{X} \to \mathcal{X}$ which, in the basis $\{e_i, i \in \underline{n}\}$, is represented by the matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -t_0 \\ 1 & 0 & & 0 & -t_1 \\ 0 & 1 & & \cdot & \cdot \\ 0 & 0 & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \\ \cdot & \cdot & & 0 & -t_{n-2} \\ 0 & 0 & \cdots & 1 & -t_{n-1} \end{bmatrix} \tag{4.3-2}$$

The canonical projection $\pi\colon\ R[\lambda] \to X_\Sigma$ is just

$$\pi\colon u(\lambda) \mapsto [u(\lambda)] + (t(\lambda))$$

where $[u(\lambda)]$ denotes the unique polynomial of degree less than n such that

$$[u(\lambda)] \equiv u(\lambda) \quad (\text{mod } t(\lambda))$$

Thus, if $\psi\colon X_\Sigma \to \mathcal{X}$ is the R-isomorphism outlined above, it is easy to verify that $\pi\colon\ R[\lambda] \to X_\Sigma$ may be represented by the following R-linear map:

$$\psi\circ\pi\colon\ R[\lambda] \to \mathcal{X}$$

$$:\ u(\lambda) \mapsto (1,\lambda,\cdots,\lambda^{n-1})u(A)b$$

where, if $u(\lambda) = \sum_{i=0}^{k} \lambda^i u_i$, then $u(A) = \sum_{i=0}^{k} A^i u_i$, and where

$$b = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{4.3-3}$$

(Note that we are thinking of elements in $\mathcal{X}$ as being polynomials of degree less than n.)

Finally, the morphism $\rho^*: X_\Sigma \to R^p[[\lambda^{-1}]]$ can be represented by the R-linear map

$$\rho^*\psi^{-1}: X \to R^p[[\lambda^{-1}]]$$

$$: \sum_{i \in \underline{n}} \alpha_i e_i \mapsto \sum_{i \in \underline{n}} \alpha_i f^*_\Sigma(\lambda^{i-1})$$

Defining the map $c: X \to R^p$ as

$$c: \sum_{i \in \underline{n}} \alpha_i e_i \mapsto \sum_{i \in \underline{n}} \alpha_i f_\Sigma(\lambda^{i-1})$$

one can easily verify that (A,b,c) is a realization of Σ, where A and b are as in (4.3-2) and (4.3-3).

Thus, for single input systems, it is fairly easy to represent X_Σ, π, and ρ^* in terms of a vector space and R-linear maps.

In the remainder of this section, and in the next, we shall show that all of the above can be generalized to the multi-input case. Our method of attack will be to generalize the concept of a monic polynomial generating an ideal to that of a "canonical matrix" generating a submodule of $R^m[\lambda]$.

4.3.1 Canonical Matrices and Free Submodules

Motivated by the preceding discussion, we now consider the following problem: Given a submodule $M \subset R^m[\lambda]$, how can one "canonically" represent M, $R^m[\lambda]/M$, and the canonical projection $\pi: R^m[\lambda] \to R^m[\lambda]/M$? (For applications, we have in mind the case where $M = \text{Ker } f^*_\Sigma$, for some input-output morphism f^*_Σ.) Guided by the relative simplicity of the situation in the case where m = 1, we shall try to find a polynomial matrix $T(\lambda)$ with the following properties:

(4.3-4) The columns of $T(\lambda)$ form a set of free generators for M.

(4.3-5) Elements of $R^m[\lambda]$ may be uniquely "reduced modulo $T(\lambda)$."

Property (4.3-4) will allow us to represent M in a canonical manner, while (4.3-5) will canonically exhibit the quotient module $R^m[\lambda]/M$ and the projection π: $R^m[\lambda] \to R^m[\lambda]/M$. These objectives are certainly attainable when m = 1, for then we pick $T(\lambda) \in R[\lambda]$ such that $T(\lambda)$ is monic, and $M = (T(\lambda))$. We shall see that these objectives remain attainable when m > 1.

In what follows, we shall always be concerned with $R[\lambda]$-modules, and with matrices with elements in $R[\lambda]$. However, the results remain valid if $R[\lambda]$ is replaced by $K[\lambda]$, where K is an arbitrary field. Moreover, most of the results remain valid if $R[\lambda]$ is replaced by any Euclidean ring, the main difference in the Euclidean ring case being that there is no notion of "monic," as there is for polynomials.

To establish notation, we include the following

(4.3-6) <u>Definition</u>: (i) If $\alpha(\lambda) \in R[\lambda]$, we denote the degree of $\alpha(\lambda)$ as $\partial\alpha(\lambda)$, or simply $\partial\alpha$; if $\alpha(\lambda) = 0$, we define $\partial\alpha(\lambda) \triangleq -\infty$.

(ii) If $u(\lambda) \in R^m[\lambda]$, we define $\partial u(\lambda) \triangleq \max_{i \in \underline{m}} \partial u_i$, where $u_i(\lambda)$ is the i'th component of $u(\lambda)$; we may also abbreviate $\partial u(\lambda)$ to ∂u.

(iii) If $T(\lambda) \in R^{m\times r}[\lambda]$, define $\partial T(\lambda) = \max_{i\in\underline{m}, j\in\underline{r}} \partial t_{ij}$, where $t_{ij}(\lambda)$ is the i,j'th element of $T(\lambda)$; again, $\partial T(\lambda)$ may be shortened to ∂T.

Next, we introduce the concept of a canonical decomposition of a submodule $M \subset R^m[\lambda]$.

(4.3-7) <u>Definition</u>: Let $M \subset R^m[\lambda]$ be a submodule of rank $r > 0$. We shall say that there exists a canonical decomposition of M if there exists an integer $\alpha > 0$ and a chain of submodules

(4.3-8) $$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_\alpha = M$$

such that, if integers $\{\eta_i, i\varepsilon\underline{\alpha}\}$ are defined as

(4.3-9) $$\eta_i = \min \{\partial u | u(\lambda)\varepsilon M,\ u(\lambda)\notin M_{i-1}\}\ ,\ \text{for } i\varepsilon\underline{\alpha}$$

then

(4.3-10) $$M_i = \sum_{u(\lambda)\varepsilon S_i} R[\lambda]u(\lambda) = \text{submodule generated by } S_i,\ \text{for } i\varepsilon\underline{\alpha}$$

where the subsets $S_i \subset M$ are defined as

(4.3-11) $$S_i = \{u(\lambda)\varepsilon M\ | \partial u \le \eta_i\}\ ,\ \text{for } i\varepsilon\underline{\alpha}$$

If all of the above is true, then we shall call (4.3-8) a <u>canonical decomposition</u> of M.

(4.3-12) <u>Proposition</u>: Every submodule $M \subset R^m[\lambda]$ of rank $r > 0$ has a unique canonical decomposition.

Proof: Suppose there exists a canonical decomposition of M. Then, for each $i\varepsilon\underline{\alpha}$, the triple (η_i, S_i, M_i) is uniquely determined by the submodule M_{i-1}. Since M_0 is defined to be the zero submodule, it follows by induction that the set of triples $\{(\eta_i, S_i, M_i), i\varepsilon\underline{\alpha}\}$ is unique. Thus, if there exists a canonical decomposition of M, it is necessarily unique.

To demonstrate the existence of a canonical decomposition of M, suppose that, for some integer $k > 0$, we have found, recursively via (4.3-9)-(4.3-11), a set of triples $\{(\eta_i, S_i, M_i),\ i\varepsilon\underline{k}\}$ such that

(4.3-13) $$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k \subset M$$

This is certainly possible when k = 1, because rank $M > 0$ implies $M \neq 0$. There are now two possibilities to consider: either $M_k = M$, or $M_k \subsetneq M$.

In the former case, we have a canonical decomposition of M, with $\alpha = k$.

In the latter case, (4.3-9)-(4.3-11) produce a triple $(\eta_{k+1}, S_{k+1}, M_{k+1})$; moreover, from the definition of M_k, it follows that $\eta_{k+1} > \eta_k$, that $S_{k+1} \supsetneq S_k$, and that

$$M_k \subsetneq M_{k+1} \subset M$$

Thus, the chain in (4.3-13) can be extended.

We now proceed resursively, determinnng the triples (η_i, S_i, M_i) until at some point $M_k = M$. This must be true for some finite integer k; otherwise we would have a strictly increasing infinite chain of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k \subsetneq \cdots \subset M$$

contradicting the ascending chain condition (See Appendix B). ■

Proposition (4.3-12) will ultimately form the basis for a procedure for canonically characterizing any nonzero submodule $M \subset R^m[\lambda]$. However, we must first consider several special classes of polynomial matrices. We begin with the following definitions:

(4.3-14) Definition: Let $T(\lambda) = (t_1(\lambda); t_2(\lambda); \cdots t_r(\lambda))$ be an mxr polynomial matrix; and let $\nu_i = \partial t_i$, for $i \epsilon \underline{r}$, and $\nu = \partial T = \max_{i \epsilon \underline{r}} \nu_i$. Then

(i) $T(\lambda)$ is regular if m = r, if $\nu \geq 0$, and if T_ν is non-singular, where T_ν is the coefficient of λ^ν in $T(\lambda)$.

(ii) $T(\lambda)$ is column proper if $\nu_i \geq 0$, for all $i \epsilon \underline{r}$, and if

T_h has full column rank, where T_h is the mxr constant matrix whose i'th column is the coefficient of λ^{ν_i} in $t_i(\lambda)$.

(iii) $T(\lambda)$ is <u>row proper</u> if $T'(\lambda)$ is column proper.

Regular, row proper, and column proper matrices enjoy certain properties that will prove useful in our development. We begin with

(4.3-15) <u>Lemma</u>: Let $T(\lambda)$ be a regular mxm matrix of degree ν. Then each $u(\lambda)\varepsilon R^m[\lambda]$ may be uniquely written as

$$u(\lambda) = T(\lambda)q(\lambda) + r(\lambda)$$

where $q(\lambda)$, $r(\lambda)\varepsilon R^m[\lambda]$, and where $\partial r < \nu$. Moreover, for this unique pair, $\partial q = \partial u - \nu$ if $\partial u \geq \nu$; otherwise $q(\lambda) = 0$.

Proof: This is a standard result; see, e.g., [28, p.77]. ■

The operation of determining the "remainder" $r(\lambda)$ upon "dividing" $u(\lambda)$ by $T(\lambda)$ will be referred to as "reducing $u(\lambda)$ modulo $T(\lambda)$." This concept can be generalized to the case where $T(\lambda)$ is row proper, as follows.

(4.3-16) <u>Lemma</u>: Let $T(\lambda)$ be a row proper mxm matrix whose i'th row is of degree ν_i, $i\varepsilon\underline{m}$. Then any $u(\lambda)\varepsilon R^m[\lambda]$ can be uniquely reduced modulo $T(\lambda)$ in the following sense. For every $u(\lambda)\varepsilon R^m[\lambda]$ there exist unique $q(\lambda)$, $r(\lambda)\varepsilon R^m[\lambda]$ such that

(i) $u(\lambda) = T(\lambda)\,q(\lambda) + r(\lambda)$

(ii) $\partial r_i < \nu_i$, for each $i\varepsilon\underline{m}$

Moreover, if $\eta = \max_{i\varepsilon\underline{m}} (\partial u_i - \nu_i)$, then $\partial q = \eta$ when $\eta \geq 0$; otherwise $q(\lambda) = 0$.

Proof: Define $\nu = \max_{i \in \underline{m}} \nu_i = \partial T$, and let $\Lambda(\lambda)$ be the diagonal mxm matrix such that

$$\Lambda_{ii} = \lambda^{\nu-\nu_i}, \ i \in \underline{m}$$

Then it is easy to see that $\Lambda(\lambda)T(\lambda)$ is regular and of degree ν. Let $u(\lambda) \in R^m[\lambda]$ be arbitrary, and reduce $\Lambda(\lambda)u(\lambda)$ modulo $\Lambda(\lambda)T(\lambda)$:

$$\Lambda(\lambda)u(\lambda) = \Lambda(\lambda)T(\lambda)q(\lambda) + \hat{r}(\lambda)$$

for unique $q(\lambda)$, $\hat{r}(\lambda) \in R^m[\lambda]$ subject to $\partial\hat{r} < \nu$. Now define

$$r(\lambda) = u(\lambda) - T(\lambda)q(\lambda)$$

so that (i) is satisfied. But then

$$\Lambda(\lambda)r(\lambda) = \hat{r}(\lambda)$$

and, since $\partial\hat{r} < \nu$, it follows that (ii) is satisfied. Moreover, $\partial(\Lambda u) = \max_{i \in \underline{m}} (\partial u_i + \nu - \nu_i) = \nu + \eta$; thus, from (4.3-15), $\partial q = \nu + \eta - \nu$ if $\eta \geq 0$ and $q(\lambda) = 0$ if $\eta < 0$.

This establishes the existence of $q(\lambda)$ and $r(\lambda)$ with the desired properties. But, if $q^*(\lambda)$ and $r^*(\lambda)$ are any elements of $R^m[\lambda]$ satisfying (i) and (ii) then

$$\Lambda(\lambda)u(\lambda) = \Lambda(\lambda)T(\lambda)q^*(\lambda) + \Lambda(\lambda)r^*(\lambda)$$

and, moreover, $\partial(\Lambda r^*) < \nu$. But then, by the uniqueness part of (4.3-15), it follows that $q(\lambda) = q^*(\lambda)$ and that

$$\Lambda(\lambda)r(\lambda) = \Lambda(\lambda)r^*(\lambda)$$

But, since $\Lambda(\lambda)$ is not a zero divisor in the ring of mxm polynomial matrices, it then follows that $r(\lambda) = r^*(\lambda)$. This establishes the uniqueness of $q(\lambda)$ and $r(\lambda)$. ■

Column proper matrices will prove useful because of the following property.

(4.3-17) Lemma: Let $T(\lambda)$ be a column proper $m\times r$ matrix, whose i'th column is of degree ν_i, for $i \in \underline{r}$. Then, for all $q(\lambda) \in R^r[\lambda]$,

$$\partial(T(\lambda)q(\lambda)) = \max_{i\in\underline{r}} (\partial q_i + \nu_i)$$

where $q_i(\lambda)$ is the i'th element of $q(\lambda)$.

Proof: The result follows trivially when $q(\lambda) = 0$. We next observe that if $q(\lambda) \neq 0$, then $T(\lambda)q(\lambda) = \sum_{i\in\underline{r}} t_i(\lambda)q_i(\lambda)$, where $t_i(\lambda)$ is column i of $T(\lambda)$. Thus

$$\partial(T(\lambda)q(\lambda)) \leq \max_{i\in\underline{r}} (\partial t_i + \partial q_i) = \max_{i\in\underline{r}} (\partial q_i + \nu_i)$$

Now take $q(\lambda) \neq 0$; and define $\eta \triangleq \max_{i\in\underline{r}} (\partial q_i + \nu_i)$, and $d_i \triangleq \partial q_i$ for $i\in\underline{r}$. It follows, since each $\nu_i \geq 0$, that $\eta \geq 0$ and

$$T(\lambda)q(\lambda) = \lambda^\eta \sum_{i:d_i+\nu_i=\eta} t_{i,\nu_i} q_{i,d_i} + v(\lambda)$$

where $t_{i,j}$ is the coefficient of λ^j in $t_i(\lambda)$, $q_{i,j}$ is the coefficient of λ^j in $q_i(\lambda)$ (if $q_i(\lambda) \neq 0$), and $\partial v(\lambda) < \eta$. But, since $T(\lambda)$ is column proper, $(t_{1,\nu_1}, \cdots t_{r,\nu_r})$ has full column rank. Therefore, since at least one of the q_{i,d_i} is nonzero, the coefficient of λ^η in $T(\lambda)q(\lambda)$ is nonzero. This completes the proof. ■

This last result has a useful corollary:

(4.3-18) Corollary: Let $T(\lambda)$ be a column proper $m\times r$ matrix, whose i'th column, $t_i(\lambda)$, is of degree ν_i. Then

(i) The columns of $T(\lambda)$ are a set of free generators for a submodule $M \subset R^m[\lambda]$.

(ii) If $u(\lambda) \in M$ is a nonzero vector polynomial of degree η, then $u(\lambda)$ must be of the form

$$u(\lambda) = \sum_{i:\nu_i \leq \eta} t_i(\lambda) q_i(\lambda)$$

for unique $q_i(\lambda) \in R[\lambda]$ such that $\partial q_i \leq \eta - \nu_i$.

Proof: (i) Consider the set $\{t_i(\lambda), i \in \underline{r}\}$. These elements are free generators of a submodule of $R^m[\lambda]$ if $q(\lambda) \in R^r[\lambda]$ and $q(\lambda) \neq 0$ imply $T(\lambda)q(\lambda) \neq 0$. But, from (4.3-17),

$$\partial(T(\lambda)q(\lambda)) = \max_{i \in \underline{r}} (\partial q_i + \nu_i) \geq \min_{i \in \underline{r}} \nu_i \geq 0$$

since at least one $q_i(\lambda)$ is nonzero. Therefore, $T(\lambda)q(\lambda) \neq 0$. Clearly, M is the submodule generated by $\{t_i(\lambda), i \in \underline{r}\}$.

(ii) Next, if $u(\lambda) \in M$, because $\{t_i(\lambda), i \in \underline{r}\}$ is a set of free generators for M, there exist unique $q_i(\lambda) \in R[\lambda]$ such that

$$u(\lambda) = \sum_{i \in \underline{r}} t_i(\lambda) q_i(\lambda)$$

But then, from (4.3-17), $\partial u = \max_{i \in \underline{r}} (\partial q_i + \nu_i)$. Therefore, it follows that

$$q_i(\lambda) = 0, \text{ if } \nu_i > \partial u$$

and that

$$\partial q_i \leq \partial u - \nu_i, \text{ if } \nu_i \leq \partial u \quad \blacksquare$$

As in (4.3-18), we shall frequently wish to refer to a submodule generated by the columns of a polynomial matrix. Thus, we now establish the following convention:

(4.3-19) Definition: Let $T(\lambda) = (t_1(\lambda) \vdots \cdots t_r(\lambda))$ be an arbitrary $m \times r$ polynomial matrix. Then, by $\operatorname{Im} T(\lambda)$ we shall mean the submodule of $R^m[\lambda]$ generated by $\{t_i(\lambda), i \in \underline{r}\}$:

$$\operatorname{Im} T(\lambda) \triangleq \{u(\lambda) \in R^m[\lambda] \mid u(\lambda) = \sum_{i \in \underline{r}} t_i(\lambda) q_i(\lambda), \text{ for some } q_i(\lambda) \in R[\lambda]\}$$

We now introduce a fourth class of polynomial matrices, canonical matrices. Such matrices will be seen to share certain of the desirable properties of both column proper and row proper matrices.

(4.3-20) Definition: Let $T(\lambda)$ be an mxr polynomial matrix, where $r \leq m$. Then $T(\lambda)$ is a canonical matrix if there exist two sets of integers $\{\nu_i, i \in \underline{r}\}$ and $\{m_i, i \in \underline{r}\}$ such that

(i) $m_i \in \underline{m}$ for each i

(ii) $\nu_i \geq 0$, for each i

(iii) $i \neq j$ implies $m_i \neq m_j$

(iv) $i < j$ implies $\nu_i \leq \nu_j$

(v) $i < j$ and $\nu_i = \nu_j$ imply $m_i < m_j$

(vi) $t_{m_i,i}(\lambda)$ is monic and of degree ν_i, for each $i \in \underline{r}$

(vii) $i \neq j$ implies $\partial t_{m_i,j} < \nu_i$

(viii) $j < m_i$ implies $\partial t_{j,i} < \nu_i$

(ix) $k \neq i$ and $\nu_k = \nu_i$ implies $\partial t_{m_k,i} < \nu_i$

(x) $t_{j,i} \leq \nu_i$, for all $i \in \underline{r}$ and $j \in \underline{m}$

(4.3-21) Remark: Although (4.3-20) may appear somewhat formidable, these conditions may be interpreted as follows. First, (vi), (viii), and (x) imply that the sets $\{\nu_i\}$ and $\{m_i\}$ may be determined as

$$\nu_i = \text{degree of column } i \text{ of } T(\lambda), \text{ for } i \in \underline{r}$$

$$m_i = \min \{j \mid \partial t_{j,i} = \nu_i\}, \text{ for } i \in \underline{r}$$

Having determined these integers, one then checks that $t_{m_i,i}(\lambda)$ is monic; that $0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_r$; that the m_i are distinct integers; and that the degree of each element in row m_i of $T(\lambda)$, except $t_{m_i,i}(\lambda)$, is less than ν_i. The remaining conditions must be checked only when the integers ν_i are not distinct.

It sometimes will be convenient to partition a canonical matrix, thus grouping columns of the same degree. Therefore, we define:

(4.3-22) <u>Definition</u>: Let $T(\lambda) = (t,(\lambda);\cdots t_r(\lambda))$ be an mxr canonical matrix. Then, by a <u>canonical partitioning</u> of $T(\lambda)$ we shall mean a partitioning

$$T(\lambda) = (T_1(\lambda);T_2(\lambda);\cdots T_\alpha(\lambda))$$

such that $\partial T_1 < \partial T_2 < \cdots < \partial T_\alpha$ and such that, for each $i\varepsilon\underline{\alpha}$, all the columns of $T_i(\lambda)$ are of the same degree. Thus, α equals the number of <u>distinct</u> integers in $\{\nu_i, i\varepsilon\underline{r}\}$.

The significance of canonical matrices will have been established when we shall have proved (4.3-33). This theorem will prove that for every nonzero submodule $M \subset R^m[\lambda]$ there exists an mxr canonical matrix $T(\lambda)$ such that $M = \text{Im}\, T(\lambda)$; that is, the columns of $T(\lambda)$ are a "canonical" set of free generators for M. The first result relating canonical matrices, canonical partitionings, and canonical decompositions of submodules is the following.

(4.3-23) <u>Lemma</u>: Let $T(\lambda)$ be an mxr canonical matrix, and let $(T_1(\lambda);\cdots;T_\alpha(\lambda))$ be a canonical partitioning of $T(\lambda)$.

Define $\mathcal{M} \subset R^m[\lambda]$ as: $\mathcal{M} = \text{Im } T(\lambda)$. Then

(i) rank $\mathcal{M} = r$

(ii) The unique canonical decomposition of $\mathcal{M}$ is

$$0 = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \cdots \subsetneq \mathcal{M}_\alpha = \mathcal{M}$$

where

$$\mathcal{M}_i \triangleq \text{Im}(T_1(\lambda);\cdots;T_i(\lambda)),\ \text{for } i\varepsilon\underline{\alpha} \tag{4.3-25}$$

Proof: For (i), it is easily seen that $T(\lambda)$ is column proper (the matrix T_h in (4.3-14) (ii) can be made to have an rxr lower triangular submatrix, by a permutation of its rows). Thus, from (4.3-18) (i) it follows that the columns of $T(\lambda)$ are free generators for $\mathcal{M}$, whence rank $\mathcal{M} = r$.

For (ii), first note that the $\mathcal{M}_i$, as defined in (4.3-24), satisfy (4.3-8). Thus, it remains to show that they also satisfy (4.3-9) - (4.3-11).

Suppose that $T_i(\lambda)$ is $m \times r_i$, for $i\varepsilon\underline{\alpha}$. We first show that the η_i, as defined in (4.3-9), satisfy

$$\eta_i = \partial T_i(\lambda),\ i\varepsilon\underline{\alpha} \tag{4.3-25}$$

Indeed, it is certainly true that $\eta_i \leq \partial T_i$; for, since the columns of $T(\lambda)$ are free, each column of $T_i(\lambda)$ is an element of $\mathcal{M}$, but it is not an element of $\mathcal{M}_{i-1}$. To show that $\eta_i \geq \partial T_i$, note that if $u(\lambda) \in \mathcal{M}$ and $u(\lambda) \notin \mathcal{M}_{i-1}$ then

$$u(\lambda) = \sum_{j\varepsilon\underline{\alpha}} T_j(\lambda) q_j(\lambda)$$

for unique $q_j(\lambda) \in R^{r_j}[\lambda]$, where at least one $q_j(\lambda)$ for $j \geq i$ is nonzero. But then, from (4.3-17), since $T(\lambda)$ is column proper,

$$\partial u(\lambda) = \max_{j\varepsilon\underline{\alpha}} (\partial q_j + \partial T_j)$$

$$\geq \max_{j \geq i} \; (\partial q_j + \partial T_j)$$

$$\geq \partial T_i$$

where the last line follows from the facts that $\partial T_i < \partial T_{i+1} < \cdots < \partial T_\alpha$ and that at least one element of $\{q_j, j \geq i\}$ is nonzero. Thus, (4.3-25) follows.

Next, from (4.3-18) (ii) the subset S_i is just

$$S_i = \{u(\lambda) \varepsilon M \mid \partial u \leq \eta_i\}$$
$$= \{u(\lambda) = \sum_{j \leq i} T_j(\lambda) q_j(\lambda) \mid q_j(\lambda) \varepsilon R^{r_j}[\lambda] \text{ and } \partial q_j \leq \eta_i - \eta_j\}$$

Therefore, M_i, as defined in (4.3-24), is just

$$M = \mathrm{Im}(T_1(\lambda); \cdots ; T_i(\lambda))$$

$$= \sum_{u(\lambda) \varepsilon S_i} R[\lambda] u(\lambda) \quad , \quad i \varepsilon \underline{\alpha}$$

in agreement with (4.3-10), and the proof of (ii) is complete. ■

We shall eventually prove what is essentially the converse to (4.3-23), namely: For every nonzero submodule $M \subset R^m[\lambda]$ there exists a unique canonical matrix $T(\lambda)$ such that $M = \mathrm{Im}\, T(\lambda)$. However, first we must prove the following result which establishes what is perhaps the most important property of canonical matrices.

(4.3-26) **Theorem:** Let $T(\lambda)$ be an mxn canonical matrix and let $\{\nu_i, i\varepsilon \underline{n}\}$ and $\{m_i, i\varepsilon \underline{n}\}$ be the associated sets of integers (uniquely determined by $T(\lambda)$, from (4.3-21)). They any $u(\lambda) \varepsilon R^m[\lambda]$ can be uniquely reduced modulo $T(\lambda)$ in the following sense. For any $u(\lambda) \varepsilon R^m[\lambda]$ there exist unique $q(\lambda) \varepsilon R^n[\lambda]$

and $r(\lambda)\varepsilon R^m[\lambda]$ such that

(i) $u(\lambda) = T(\lambda)q(\lambda) + r(\lambda)$

and

(ii) $\partial r_{m_i} < \nu_i$, for $i\varepsilon \underline{n}$

Moreover, for this unique pair $(q(\lambda),r(\lambda))$,

(iii) $\partial r(\lambda) \leq \partial u(\lambda)$

and

(iv) $\partial q_i \leq d - \nu_i$ if $\nu_i \leq d$; otherwise $q_i(\lambda) = 0$

where

$$d = \max_{i\varepsilon \underline{n}} \partial u_{m_i} \leq \partial u(\lambda)$$

and where $q_i(\lambda)$ is the i'th element of $q(\lambda)$.

Proof: Let P be an mxm permutation matrix such that $(Px)_i = x_{m_i}$, for all $x\varepsilon R^m$ and $i\varepsilon \underline{n}$, and define the nxn and (m-n)xn matrices $\hat{T}(\lambda)$ and $\tilde{T}(\lambda)$ as

$$\begin{pmatrix} \hat{T}(\lambda) \\ \tilde{T}(\lambda) \end{pmatrix} = PT(\lambda)$$

Let $u(\lambda) \varepsilon R^m[\lambda]$ be arbitrary, and define $\hat{u}(\lambda) \varepsilon R^n[\lambda]$ and $\tilde{u}(\lambda) \varepsilon R^{m-n}[\lambda]$ as

$$\begin{pmatrix} \hat{u}(\lambda) \\ \tilde{u}(\lambda) \end{pmatrix} = Pu(\lambda)$$

It is easy to see that $\hat{T}(\lambda)$ is row proper, and that the i'th row of $\hat{T}(\lambda)$ is of degree ν_i. Therefore, by (4.3-16), there exist unique $q(\lambda)$, $\hat{r}(\lambda) \varepsilon R^n[\lambda]$ such that

$$\hat{u}(\lambda) = \hat{T}(\lambda)q(\lambda) + \hat{r}(\lambda)$$

and

(4.3-27) $\partial\hat{r}_i < \nu_i$, for $i \in \underline{n}$

Now define

$$\tilde{r}(\lambda) = \tilde{u}(\lambda) - \tilde{T}(\lambda)q(\lambda)$$

and set

$$r(\lambda) = P^{-1}\begin{pmatrix}\hat{r}(\lambda)\\ \tilde{r}(\lambda)\end{pmatrix}$$

The $q(\lambda)$ and $r(\lambda)$ so determined clearly satisfy (i) and (ii). Also, if (ii) is to be satisfied, then (4.3-27) must be satisfied. Thus, the uniqueness of $q(\lambda)$ and $\hat{r}(\lambda)$ implies the uniqueness of $q(\lambda)$ and $r(\lambda)$.

Demonstrating the validity of (iii) and (iv) is somewhat more difficult. To proceed methodically, we must define several new vectors and matrices. Thus, let the canonical partitioning of $T(\lambda)$ be

$$T(\lambda) = (T_1(\lambda);\ T_2(\lambda);\cdots;T_\alpha(\lambda))$$

where $T_i(\lambda)$ is $m\mathrm{x}n_i$,

$$\partial T_i(\lambda) = \eta_i \quad , i \in \underline{\alpha}$$

and

$$\eta_1 < \eta_2 < \cdots < \eta_\alpha$$

For notational simplicity, define

$$\ell_i = \sum_{j\leq i} n_j \ , \text{ for } \ i \in \underline{\alpha}$$

Now define, for each $i \in \underline{\alpha}$, the following matrices and vectors. $\hat{P}_i(\ell_i\mathrm{x}m)$ and $\tilde{P}_i$ $((m-\ell_i)\mathrm{x}m)$ are defined so that

$$P = \begin{pmatrix}\hat{P}_i\\ \tilde{P}_i\end{pmatrix} \begin{matrix}\}\ell_i \text{ rows}\\ \\ \end{matrix}$$

where P is the permutation matrix in the first half of the proof. $\hat{T}_i(\lambda)$ $(\ell_i x \ell_i)$ and $\tilde{T}_i(\lambda)$ $((m-\ell_i)x\ell_i)$ are defined as

$$\hat{T}_i(\lambda) = \hat{P}_i(T_1(\lambda);\cdots;\ T_i(\lambda))$$

and

$$\tilde{T}_i(\lambda) = \tilde{P}_i(T_1(\lambda);\cdots T_i(\lambda))$$

Note that $\hat{T}_\alpha(\lambda) = \hat{T}(\lambda)$ and $\tilde{T}_\alpha(\lambda) = \tilde{T}(\lambda)$. Next, fix $u(\lambda) \varepsilon R^m[\lambda]$, and let $q(\lambda)$ and $r(\lambda)$ be the unique vectors such that

$$u(\lambda) = T(\lambda)q(\lambda) + r(\lambda);\quad \partial r_{m_i} < \nu_i$$

Define

$$\hat{v}_i(\lambda) = \hat{P}_i\ u(\lambda)$$
$$\tilde{v}_i(\lambda) = \tilde{P}_i\ u(\lambda)$$
$$\hat{s}_i(\lambda) = \hat{P}_i\ r(\lambda)$$
$$\tilde{s}_i(\lambda) = \tilde{P}_i\ r(\lambda)$$
$$\hat{q}_\alpha(\lambda) = q(\lambda)$$

and, for $1 \leq i \leq \alpha-1$, define $\hat{q}_i(\lambda) \varepsilon R^{\ell_i}[\lambda]$ and $\tilde{q}_i(\lambda) \varepsilon R^{\eta_{i+1}}[\lambda]$ so that

$$\hat{q}_{i+1}(\lambda) = \begin{pmatrix} \hat{q}_i(\lambda) \\ \tilde{q}_i(\lambda) \end{pmatrix} \begin{matrix} \}\ \ell_i \text{ elements} \\ \}\ \eta_{i+1} \text{ elements} \end{matrix}$$

There are now two cases we must consider: (a) $\partial\hat{v}_i < \eta_i$ for all $i \varepsilon \underline{\alpha}$, and (b) $\partial v_i \geq \eta_i$ for some $i \varepsilon \underline{\alpha}$. In the former case, it follows from the definition of the η_i that

$$\partial u_{m_i} < \nu_i\quad, \text{ for all } i \varepsilon \underline{n}$$

Therefore, since in this case $u(\lambda)$ satisfies the degree requirements that $r(\lambda)$ must satisfy, it follows that $r(\lambda) = u(\lambda)$, i.e. the unique reduction of $u(\lambda)$ modulo $T(\lambda)$ is just

$$u(\lambda) = T(\lambda)\cdot 0 + u(\lambda)$$

Thus, $q(\lambda) = 0$, which satisfies (iv); and $r(\lambda) = u(\lambda)$, which satisfies (iii).

In case (b), define

$$k \triangleq \max \{i \mid \partial\hat{v}_i \geq \eta_i\}$$

Again, we have two cases to consider: (bl) $k = \alpha$, and (b2) $k < \alpha$. In the former case it follows that

$$d \triangleq \max_{i\in\underline{n}} \partial u_{m_i}$$

$$= \partial\hat{v}_\alpha$$

$$\geq \eta_\alpha$$

$$> \partial\hat{s}_\alpha \qquad (4.3\text{-}28)$$

where the last inequality follows from the facts that $r(\lambda)$ satisfies (ii), and that $\nu_i \leq \eta_\alpha$ for all $i \in \underline{n}$. Thus

$$\partial(\hat{v}_\alpha(\lambda)-\hat{s}_\alpha(\lambda)) = d$$

and, since

$$\hat{v}_\alpha(\lambda)-\hat{s}_\alpha(\lambda) = \hat{T}_\alpha(\lambda)\hat{q}_\alpha(\lambda)$$

and $\hat{T}_\alpha(\lambda)$ is column proper (easily proven), it follows from (4.3-18) (ii) that

$$\partial q_i \leq d-\nu_i \text{ , for } i \in \underline{n}$$

This proves (iv). Finally, because the degree of column i of $\tilde{T}_\alpha(\lambda)$ is not greater than ν_i,

$$\partial(\tilde{T}_\alpha(\lambda)\hat{q}_\alpha(\lambda)) \leq d$$

Thus, since

$$r(\lambda) = P^{-1}\begin{pmatrix}\hat{s}_\alpha(\lambda)\\ \tilde{s}_\alpha(\lambda)\end{pmatrix} = P^{-1}\begin{pmatrix}\hat{s}_\alpha(\lambda)\\ \tilde{v}_\alpha(\lambda)\ -\tilde{T}_\alpha(\lambda)\hat{q}_\alpha(\lambda)\end{pmatrix}$$

it follows, using (4.3-18), that

$$\begin{aligned}\partial r(\lambda) &\le \max(\partial\hat{s}_\alpha, \max(\partial\tilde{v}_\alpha, \partial(\tilde{T}_\alpha(\lambda)\hat{q}_\alpha(\lambda))))\\ &\le \max\ (d, \partial\tilde{v}_\alpha)\\ &= \partial u(\lambda)\end{aligned}$$

which proves (iii).

We now consider case (b2). In this situation we have

$$\eta_i > \partial\hat{v}_i \quad , \quad \text{for } k+1 \le i \le \alpha$$

and

$$\eta_k \le \partial v_k$$

We shall prove by induction that

(4.3-20) $\qquad \hat{v}_i(\lambda) = \hat{T}_i(\lambda)\hat{q}_i(\lambda) + \hat{s}_i(\lambda)$, for $k \le i \le \alpha$

and that

(4.3-30) $\qquad \tilde{q}_i(\lambda) = 0$, for $k \le i \le \alpha-1$

We shall accomplish this by proving

(4.3-31) $\quad \hat{v}_i(\lambda) = \hat{T}_i(\lambda)\hat{q}_i(\lambda) + \hat{s}_i(\lambda)$ <u>and</u> $\partial\hat{v}_i < \eta_i$ imply $q_{i-1}(\lambda) = 0$

and

(4.3-32) $\quad \tilde{q}_i(\lambda) = 0$ <u>and</u> $\hat{v}_{i+1}(\lambda) = \hat{T}_{i+1}(\lambda)\hat{q}_{i+1}(\lambda) + \hat{s}_{i+1}(\lambda)$

imply $\quad \hat{v}_i(\lambda) = \hat{T}_i(\lambda)\hat{q}_i(\lambda) + \hat{s}_i(\lambda)$

Then, since the hypotheses for (4.3-31) are satisfied when $i = \alpha$, induction will take over.

To prove (4.3-31) we note that, from condition (ii), $\partial\hat{s}_i < \eta_i$ for all $i \in \underline{\alpha}$. Consequently, assuming the hypotheses of (4.3-31),

$$\partial(\hat{T}_i(\lambda)\hat{q}_i(\lambda)) = \partial(\hat{v}_i(\lambda)-\hat{s}_i(\lambda)) < \eta_i$$

Then, since $\hat{T}_i(\lambda)$ is column proper, (4.3-18) tells us that $\tilde{q}_{i-1}(\lambda) = 0$, proving (4.3-31). The proof of (4.3-32) is immediate when one recalls the definitions of $\hat{v}_i(\lambda)$, $\hat{s}_i(\lambda)$, $\hat{T}_i(\lambda)$, $\hat{q}_i(\lambda)$ and $\tilde{q}_i(\lambda)$. Therefore, (4.3-29) and (4.3-30) have been proved.

From (4.3-30) it immediately follows that condition (iv) is satisfied for $\ell_k+1 \leq i \leq n$; thus it remains to prove (iv) for $i \in \underline{\ell}_k$. But, from the definitions of d and k,

$$d \geq \partial\hat{v}_k \geq \eta_k$$

Also, since condition (ii) is met,

$$\eta_k > \partial\hat{s}_k$$

Therefore, from (4.3-29) with i = k,

$$\begin{aligned}\partial(\hat{T}_k(\lambda)\hat{q}_k(\lambda)) &= \partial(\hat{v}_k(\lambda)-\hat{s}_k(\lambda))\\ &= \partial\hat{v}_k\\ &\leq d\end{aligned}$$

Since $T_k(\lambda)$ is column proper, it now follows from (4.3-18) that

$$\partial q_i \leq d-\nu_i \quad , \text{ for all } i \in \underline{\ell}_k$$

thus completing the proof for (iv).

Finally, to prove (iii) in this case we note that

$$\partial(\tilde{T}_k(\lambda)\hat{q}_k(\lambda)) \leq d$$

and that

$$r(\lambda) = P^{-1}\begin{pmatrix}\hat{s}_k(\lambda)\\ \tilde{s}_k(\lambda)\end{pmatrix} = P^{-1}\begin{pmatrix}\hat{s}_k(\lambda)\\ \tilde{v}_k(\lambda)-\tilde{T}_k(\lambda)\hat{q}_k(\lambda)\end{pmatrix}$$

Therefore, $\partial r \leq \max(d, \partial\tilde{v}_k) = \partial u$; this proves (iii). ∎

The next result, which states that for every nonzero submodule $M \subset R^m[\lambda]$ there exists a unique canonical matrix $T(\lambda)$ such that $M = \operatorname{Im} T(\lambda)$, will serve as a second indication of the importance of canonical matrices.

(4.3-33) **Theorem:** Let $M \subset R^m[\lambda]$ be a submodule of rank $r > 0$, and let

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\alpha = M$$

be the unique canonical decomposition of M. Then there exists a mxr canonical matrix $T(\lambda)$ such that $M = \operatorname{Im} T(\lambda)$. Moreover, the canonical partitioning of $T(\lambda)$:

(4.3-34) $$T(\lambda) = (T_1(\lambda);\cdots;T_\beta(\lambda))$$

is such that $\alpha = \beta$ and

(4.3-35) $$M_i = \operatorname{Im}(T_1(\lambda);\cdots;T_i(\lambda)), \text{ for all } i \in \underline{\alpha}$$

Proof: We first note that, in consideration of (4.3-23), if there exists a canonical $T(\lambda)$ such that $\operatorname{Im}T(\lambda) = M$, then it necessarily follows that: (i) rank M = rank $T(\lambda)$, whence $T(\lambda)$ is mxr; and (ii) the canonical partitioning, (4.3-34), of $T(\lambda)$ is such that $\alpha = \beta$, and such that (4.3-35) is satisfied.

Therefore, to prove the theorem, it is sufficient to prove the following:

(4.3-36) There exists a unique canonical matrix $T_1(\lambda)$ such that $\operatorname{Im}T_1(\lambda) = M_1$.

and

(4.3-37) If $k < \alpha$, and if $(T_1(\lambda);\cdots;T_k(\lambda))$ is a canonically partitioned canonical matrix such that $\operatorname{Im}(T_1(\lambda);\cdots;T_k(\lambda)) = M_k$, then there exists a unique canonical matrix $T_{k+1}(\lambda)$ such that

$$\mathrm{Im}(T_1(\lambda);\cdots;T_{k+1}(\lambda)) = M_{k+1}$$

To prove (4.3-36), we proceed as follows. Let η_1, S_1, and M_1 be as defined in (4.3-9)-(4.3-11). The subset $S_1 \subset M$ is clearly closed under R-linear combinations; thus S_1 is an R-vector space. Moreover, S_1 is finite dimensional; indeed, since $S_1 \subset R^m[\lambda]$, $\dim_R(S_1) \leq m(\eta_1+1)$.

Now let $V_1 \subset R^m$ be the subset consisting of all coefficients of λ^{η_1} in $u(\lambda)$, for $u(\lambda) \in S_1$; since $\partial u \leq \eta_1$ for all $u(\lambda) \in S_1$, V_1 can be defined as

$$V_1 = \{v \in R^m | \partial(u(\lambda)-\lambda^{\eta_1}v) < \eta_1 \text{ for some } u(\lambda) \in S_1\}$$

Clearly, V_1 is an R-vector space, and $\dim(V_1) = d_1 \leq m$, for some d_1.

Next, consider the function

$$\phi_1: \quad S_1 \to V_1$$

$$: u(\lambda) \mapsto v = \text{coefficient of } \lambda^{\eta_1} \text{ in } u(\lambda)$$

It is easy to verify that ϕ_1 is an R-linear map. Moreover, $\mathrm{Ker}\phi_1 = 0$ because, by the definition of η_1, if $\phi_1 u(\lambda) = 0$ then $\partial u < \eta_1$, so that $u(\lambda) \in M_0 = 0$. Therefore, ϕ_1 establishes an R-linear isomorphism from S_1 to V_1.

We next determine a canonical basis for V_1 as follows. Let B_1 be any $m \times d_1$ matrix whose columns are a basis for V_1, and define integers $m_{1,i}$, $i \in \underline{d}_1$, as

$$m_{1,1} = \min\{k | \text{row } k \text{ of } B_1 \text{ is nonzero}\}$$

$$m_{1,i} = \min\{k > m_{1,i-1} | \text{row } k \text{ of } B_1 \text{ is independent of rows } 1,2,\cdots k-1 \text{ of } B_1\},$$
$$\text{for } 2 \leq i \leq d_1$$

By a standard argument, it is easy to show that the set $\{m_{1,i}, i \in \underline{d}_1\}$ is independent of the choice of basis matrix B_1. Then by a standard Gaussian reduction on the columns of B_1, there exists a unique matrix E_1 whose columns are a basis for V_1, and for which

$$(E_1)_{i,j} = \begin{cases} 1, \text{ if } i = m_{1,j} \\ 0; \text{ if } i < m_{1,j}, \text{ or if } i = m_{1,k} \text{ for some } k \neq j \\ \text{nothing in particular, otherwise} \end{cases}$$

Now carry the columns of E_1 back through ϕ_1^{-1} to obtain d_1 elements of S_1; form these vector polynomials into the $m \times d_1$ matrix $T_1(\lambda)$:

$$T_1(\lambda) = \phi_1^{-1} E_1$$

Since ϕ_1 is an isomorphism from S_1 to V_1, it follows that the columns of $T_1(\lambda)$ are R-linearly independent. Moreover, since each column of $T_1(\lambda)$ is a vector polynomial of degree η_1, it follows that $T_1(\lambda)$ is column proper; thus, by (4.3-18), these columns are free generators for the submodule $\text{Im}T_1(\lambda)$. But, since every element of M_1 is an $R[\lambda]$-linear combination of elements of S_1, and since any element of S_1 is an R-linear combination of the columns of $T_1(\lambda)$, it follows that $\text{Im}T_1(\lambda) = M_1$.

Thus, we have found an $m \times d_1$ matrix $T_1(\lambda)$ whose columns are free generators for M_1. Because of the particular structure of the coefficient of λ^{η_1} in $T_1(\lambda)$ (which is, of course, just E_1), it is clear that $T_1(\lambda)$ is a canonical matrix. Also, note that, from (4.3-23),

$$\text{rank } M_1 = d_1$$

If $\hat{T}_1(\lambda)$ were another canonical matrix such that $\text{Im}\hat{T}_1(\lambda) = M_1$ then it follows from (4.3-23) that $\hat{T}_1(\lambda)$ must be of dimensions $m \times d_1$, and that every column of $\hat{T}_1(\lambda)$ must be of degree η_1. Therefore, since $\text{Im}\hat{T}_1(\lambda) \subset M$, every column of $\hat{T}_1(\lambda)$ is an element of S_1. The coefficient of λ^{η_1} in $\hat{T}_1(\lambda)$ is

$$\hat{E}_1 = \emptyset_1 \hat{T}_1(\lambda)$$

and, since $\hat{T}_1(\lambda)$ is canonical, there must exist integers $\hat{m}_{1,i}$, $i \in \underline{d}_1$, such that $\hat{m}_{1,1} < \hat{m}_{1,2} < \cdots < \hat{m}_{1,d_1}$, and

$$(\hat{E}_1)_{i,j} = \begin{cases} 1, \text{ if } i = \hat{m}_{1,j} \\ 0; \text{ if } i < \hat{m}_{1,j}, \text{ or if } i = \hat{m}_{1,k} \text{ for some } k \neq j \end{cases}$$

But then, since $\emptyset_1 \hat{T}_1(\lambda) = \hat{E}_1$ must be a basis matrix for V_1, it follows that $\hat{m}_{1,i} = m_{1,i}$ for $i \in \underline{d}_1$, and that $\hat{E}_1 = E_1$. Consequently

$$\hat{T}_1(\lambda) = \emptyset^{-1}\hat{E}_1 = \emptyset_1^{-1} E_1 = T_1(\lambda)$$

Thus $T_1(\lambda)$, as determined above is the unique canonical matrix such that $\text{Im}T_1(\lambda) = M_1$. This completes the proof for (4.3-36).

To prove (4.3-37), since $k < \alpha$ the quantities η_{k+1}, S_{k+1}, and M_{k+1} are well defined by (4.3-9)-(4.3-11). The subset $S_{k+1} \subset M$ is clearly an R-vector space; however, it is too large to be treated in the manner that we treated S_1. Therefore, we now define a map ψ_{k+1}: $S_{k+1} \to R^m[\lambda]$ so that, for each $u(\lambda) \in S_{k+1}$, $\psi_{k+1}u(\lambda)$ is the unique remainder upon reducing $u(\lambda)$ modulo the canonical matrix $(T_1(\lambda);\cdots;T_k(\lambda))$. (See (4.3-39)). That is,

$$u(\lambda) = (T_1(\lambda);\cdots;T_k(\lambda))q(\lambda) + \psi_{k+1}u(\lambda)$$

In this modulo reduction, by (4.3-26) (iii),

$$\partial(\psi_{k+1}u(\lambda)) \leq \partial u(\lambda)$$

Therefore, if $u(\lambda) \in S_{k+1}$, then $\psi_{k+1}\, u(\lambda) \in S_{k+1}$; moreover, ψ_{k+1} is easily seen to be R-linear. Thus ψ_{k+1} is an R-linear endomorphism

$$\psi_{k+1}: \; S_{k+1} \to S_{k+1}$$

Now define the subspace $\hat{S}_{k+1} \subset S_{k+1}$ as

$$\hat{S}_{k+1} = \psi_{k+1} \; S_{k+1}$$

One can easily verify that $\psi_{k+1}: \; S_{k+1} \to S_{k+1}$ is the projection on $\hat{S}_{k+1}$ along $M_k \bigcap S_{k+1}$. That is, the unique modulo reduction of $\hat{u}(\lambda) \in \hat{S}_{k+1}$ is

$$\hat{u}(\lambda) = (T_1(\lambda);\cdots;T_k(\lambda))\cdot 0 + \hat{u}(\lambda)$$

while $\psi_{k+1}u(\lambda) = 0$ if and only if

$$u(\lambda) \in \mathrm{Im}(T_1(\lambda);\cdots;T_k(\lambda))\bigcap S_{k+1} = M_k \bigcap S_{k+1}$$

Therefore,

$$S_{k+1} = S_{k+1} \bigcap M_k \quad \oplus \quad \hat{S}_{k+1} \tag{4.3-38}$$

Every element in $\hat{S}_{k+1}$ is either a polynomial vector of degree η_{k+1}, or the zero vector. This is because, for all $u(\lambda) \in S_{k+1}$, $\partial(\psi_{k+1}u(\lambda)) \leq \eta_{k+1}$; and if $\partial(\psi_{k+1}u(\lambda)) < \eta_{k+1}$ then, by the definition of η_{k+1}, $\psi_{k+1}u(\lambda) \in M_k$ so that $\psi_{k+1}u(\lambda) \in \hat{S}_{k+1} \bigcap M_k = 0$. Therefore, $\hat{S}_{k+1}$ is isomorphic to the R-vector space $V_{k+1} \subset R^m$ consisting of the coefficients of $\lambda^{\eta_{k+1}}$ in elements of $\hat{S}_{k+1}$. Just as in considering S_1, we define the R-linear isomorphism $\phi_{k+1}: \; \hat{S}_{k+1} \to V_{k+1}: \; \hat{u}(\lambda) \mapsto$ coefficient of $\lambda^{\eta_{k+1}}$ in $\hat{u}(\lambda)$; the unique matrix E_{k+1} whose columns are a canonical basis for V_{k+1}; and the canonical matrix $T_{k+1}(\lambda) = \phi_{k+1}^{-1}\, E_{k+1}$. $T_{k+1}(\lambda)$ is

$m \times d_{k+1}$, where $d_{k+1} = \dim(\mathcal{V}_{k+1})$. Also, $T_{k+1}(\lambda)$ is the unique canonical matrix such that

$$\mathrm{Im}T_{k+1}(\lambda) = \text{submodule generated by } \hat{S}_{k+1}$$

Finally, (4.3-38) implies that $\hat{S}_{k+1}$ and $S_{k+1} \cap M_k$ together generate M_{k+1}, so that since $S_{k+1} \cap M_k \subset \mathrm{Im}(T_1(\lambda);\cdots;T_k(\lambda))$,

$$\mathrm{Im}(T_1(\lambda);\cdots;T_{k+1}(\lambda)) = M_{k+1}$$

By construction of $T_{k+1}(\lambda)$, $\psi_{k+1}T_{k+1}(\lambda) = T_{k+1}(\lambda)$. This, together with the facts that $T_{k+1}(\lambda)$ is a canonical matrix and that $\eta_{k+1} > \eta_k$, implies that $(T_1(\lambda);\cdots;T_{k+1}(\lambda))$ is canonical.

If $\hat{T}_{k+1}(\lambda)$ is another matrix such that $(T_1(\lambda);\cdots;T_k(\lambda);\hat{T}_{k+1})\lambda))$ is canonical and such that $M_{k+1} = \mathrm{Im}(T_1(\lambda);\cdots;T_k(\lambda);\hat{T}_{k+1}(\lambda))$, then it follows from (4.3-23) that $\hat{T}_{k+1}(\lambda)$ is $m \times d_{k+1}$ and that each column of $\hat{T}_{k+1}(\lambda)$ is of degree η_{k+1}. Also, it must be true that $\psi_{k+1}\hat{T}_{k+1}(\lambda) = \hat{T}_{k+1}(\lambda)$. But then, $\hat{T}_{k+1}(\lambda)$ must be a canonical matrix such that $\mathrm{Im}\hat{T}_{k+1}(\lambda)$ is the submodule generated by $\hat{S}_{k+1}$. But $T_{k+1}(\lambda)$ is the only such canonical matrix, whence

$$\hat{T}_{k+1}(\lambda) = T_{k+1}(\lambda)$$

This completes the proof of (4.3-37). ■

(4.3-39) <u>Remark</u>: Note that, if $T(\lambda)$ is an $m \times r$ canonical matrix, then the integers $\{\nu_i, i \in \underline{r}$ and $\{m_i, i \in \underline{r}\}$ may be determined from $T(\lambda)$ as in (4.3-21). Thus, the R-linear maps ψ_k: $S_k \to S_k$ in the above proof are well defined in terms of the matrices $(T_1(\lambda);\cdots;T_{k-1}(\lambda))$.

(4.3-40) <u>Remark</u>: In the above proof, we have produced what might be called a canonical direct sum decomposition of M:

$$M = M_1 \oplus \hat{M}_2 \oplus \cdots \oplus \hat{M}_\alpha$$

where

$$M_1 = \sum_{u(\lambda) \varepsilon S_1} R[\lambda]u(\lambda)$$

$$\hat{M}_i = \sum_{u(\lambda) \varepsilon S_i} R[\lambda]\psi_i u(\lambda) \quad , \text{ for } \quad 2 \leq i \leq \alpha$$

As a consequence of (4.3-23) and (4.3-33) we now state:

(4.3-41) Corollary: Let $M \subset R^m[\lambda]$ be a submodule of rank $r > 0$. Then, in the canonical decomposition of M:

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\alpha = M$$

the integer α is no larger than r.

Proof: (4.3-33) asserts the existence of a unique canonical matrix $T(\lambda) = (T_1(\lambda);\cdots;T_\alpha(\lambda))$ such that $\text{Im}T(\lambda) = M$; while (4.3-23) claims that rank $T(\lambda) = r$. Clearly, rank $T(\lambda) \geq \alpha$, since the columns of $T(\lambda)$ are free. ■

The main drawback in the proof to (4.3-33) is that it is not particularly constructive. That is, it is an existence proof, but it is not clear how one would actually determine the integers η_i and the vector spaces S_i. We shall see in Section 4.4 that in the case where $M = \text{Ker } f^*_\Sigma$, for some input-output morphism $f^*_\Sigma : R^m[\lambda] \to R^p[[\lambda^{-1}]]$, there is an easy method for computing the corresponding canonical matrix $T(\lambda)$ from the Hankel matrix associated with f^*_Σ. The next series of results derives a method for determining a canonical $T(\lambda)$ such that $\text{Im}T(\lambda) = M$, where M is the image submodule of an arbitrary mxn polynomial matrix. Thus, these results pertain to "canonicalizing" an arbitrary polynomial matrix.

We shall soon state an algorithm for performing this task; however, in order to formulate the algorithm in concise language, we shall need the following definitions.

(4.3-42) <u>Definition</u>: (i) Let $I \subset \underline{n}$ be an arbitrary subset. Then, by $\#I$ we shall mean the cardinality of I (so that $0 \leq \#I \leq n$).

(ii) Let $M(\lambda)$ be an mxn matrix, and let $I \subset \underline{n}$. If $\#I = k > 0$, and if the elements of I are $i_1 < i_2 < \cdots < i_k$, then by $M_I(\lambda)$ we shall mean the mxk matrix whose j'th column is the i_j'th column of $M(\lambda)$:

$$M_I(\lambda) = (m_{i_1}(\lambda);m_{i_2}(\lambda);\cdots;m_{i_k}(\lambda))$$

If $\#I = 0$, then $M_I(\lambda)$ is the "mx0 nonexistent matrix." Thus, by this convention, if I and J are two arbitrary sets such that $I \cap J = \emptyset$ and $I \cup J = \underline{n}$, then the matrix $(M_I(\lambda);M_J(\lambda))$ is just $M(\lambda)P$ for some permutation matrix P.

(4.3-43) <u>Algorithm</u>: Let $D(\lambda)$ be an arbitrary nonzero mxn polynomial matrix. Perform the following operations, in sequence unless specified otherwise:

Step 1 -

(i) Set $i = 1$

(ii) Denote column j of $D(\lambda)$ as $d_j(\lambda)$, and define:

$$\gamma_0 = \min\{\gamma \mid \gamma \geq 0 \text{ and } \gamma = \partial d_j, \text{ for some } j\in\underline{n}\}$$

$$s_0 = \min\{j\in\underline{n} \mid \partial d_j = \gamma_0\}$$

$$L_0 = \underline{n}-\{s_0\}$$

(iii) Determine $\alpha \in R$ so that $d_{s_0}(\lambda)\alpha$ is a canonical mx1 matrix (See (4.3-44)) and define

$$A^{(1)}(\lambda) = d_{s_0}(\lambda)\alpha$$

$$r_1 = 1$$

(iv) If n = 1, go to Step 4

(v) If n > 1, define

$$B^{(1)}(\lambda) = D_{L_0}(\lambda)$$

Step 2-

(i) At this point $A^{(i)}(\lambda)$ is canonical (See (4.3-46)). Define $C^{(i)}(\lambda)$ to be the $m\times(n-r_i)$ matrix whose columns are the columns of $B^{(i)}(\lambda)$ reduced modulo $A^{(i)}(\lambda)$ (See (4.3-45)).

(ii) If $C^{(i)}(\lambda) = 0$, go to Step 4.

Step 3-

(i) Denote column j of $C^{(i)}(\lambda)$ as $c_j^{(i)}(\lambda)$, and define:

$$\gamma_i = \min\ \{\gamma \mid \gamma \geq 0 \text{ and } \gamma = \partial c_j^{(i)},\ \text{for some } j\ \varepsilon\ \underline{n-r_i}\}$$

$$s_i = \min\ \{j\ \varepsilon\ \underline{n-r_i} \mid \partial\ c_j^{(i)} = \gamma_i\}$$

$$L_i = \underline{n-r_i} - \{s_i\} = \{1,2,\cdots,n-r_i\} - \{s_i\}$$

(ii) Denote column j of $A^{(i)}(\lambda)$ as $a_j^{(i)}(\lambda)$, and determine

$$I_i = \{j\ \varepsilon\ \underline{r_i} \mid \partial a_j^{(i)} < \gamma_i\}$$

$$J_i = \{j\ \varepsilon\ \underline{r_i} \mid \partial a_j^{(i)} = \gamma_i\}$$

$$K_i = \{j\ \varepsilon\ \underline{r_i} \mid \partial a_j^{(i)} > \gamma_i\}$$

(iii) Determine a constant, square matrix, P_i, such that the matrix

$(A_{J_i}^{(i)}(\lambda);\ c_{s_i}^{(i)}(\lambda))P_i$ is canonical (See (4.3-46))

(iv) Define:

$$A^{(i+1)}(\lambda) = (A_{I_i}^{(i)}(\lambda);\ (A_{J_i}^{(i)}(\lambda); c_{s_i}^{(i)}(\lambda))P_i)$$

$$r_{i+1} = \#I_i + \#J_i + 1$$

(v) If $r_{i+1} = n$, increment i by 1 and go to Step 4.

(iv) If $r_{i+1} < n$, define

$$B^{(i+1)}(\lambda) = (A_{K_i}^{(i)}(\lambda);\ C_{L_i}^{(i)}(\lambda))$$

Increment i by 1, and go to Step 2.

Step 4 - Upon arriving here, $r_i = \text{rank } D(\lambda)$, and $A^{(i)}(\lambda)$ is the unique canonical $m \times r_i$ matrix such that

$$\text{Im}D(\lambda) = \text{Im}A^{(i)}(\lambda)$$

(4.3-44) **Remark**: Clearly, if $d(\lambda) \in R^m[\lambda]$ is nonzero, there exists a unique $\alpha \in R$ such that $d(\lambda)\alpha$ is a canonical mx1 matrix. For, if $\partial d = \nu$, if $k = \min \{i \in \underline{m} | \partial d_i = \nu\}$, and if $d_k(\lambda) = a\lambda^\nu + b\lambda^{\nu-1} + \cdots + c$, then $\alpha = 1/a$ will make $d(\lambda)\alpha$ canonical; also, it is easy to see that this is true for no other value of α. This explains Step 1 - (iii) of (4.3-43).

(4.3-45) **Remark**: As noted in (4.3-39), an mxn canonical matrix uniquely determines the integers $\{\nu_i, i \in \underline{n}\}$ and $\{m_i, i \in \underline{n}\}$ in terms of which a modulo reduction is specified.

We now must show that (4.3-43) does indeed produce, in a finite number of operations, a canonical matrix whose columns are free generators for $\text{Im}D(\lambda)$. This is the subject of the following theorem.

(4.3-46) Theorem: All of the operations in Algorithm (4.3-43) are well defined. In particular, each of the matrices $A^{(i)}(\lambda)$ is canonical; and, for each i there exists a unique, square, constant, nonsingular matrix P_i which satisfies the requirements of Step 3-(iii).

The operations defined recursively by (4.3-43) are equivalent to a sequence of elementary operations on the columns of $D(\lambda)$. Moreover, this sequence is finite; in particular, the number N of iterations through Step 3 of (4.3-43) is bounded by

$$N \leq r(n\nu + n - r + 1) - 1$$

where $\nu = \partial T(\lambda)$ and $r = \text{rank } T(\lambda)$.

Proof: It is clear from (4.3-44) that $A^{(1)}(\lambda)$ is canonical. We now show that if $A^{(i)}(\lambda)$ is canonical, if $r_i < n$, and if $C^{(i)}(\lambda) \neq 0$; then there exists a unique P_i to satisfy the requirements of Step 3-(iii), and $A^{(i+1)}(\lambda)$ will be canonical.

Thus, assume $A^{(i)}(\lambda)$ canonical, $r_i < n$, and $C^{(i)}(\lambda) \neq 0$. Since $C^{(i)}(\lambda)$ is obtained by reducing $B^{(i)}(\lambda)$ modulo $A^{(i)}(\lambda)$, it follows that each column of $C^{(i)}(\lambda)$ would remain unchanged if it were to be reduced modulo $A^{(i)}(\lambda)$. Since any element of Im $A^{(i)}(\lambda)$ can be reduced to 0 modulo $A^{(i)}(\lambda)$, it follows that no nonzero column of $C^{(i)}(\lambda)$ is an element of Im $A^{(i)}(\lambda)$; in particular,

$$c_{s_i}^{(i)}(\lambda) \notin \text{Im } A^{(i)}(\lambda)$$

If the subset J_i is the null set, then it is trivial to find P_i so that $(A^{(i)}_{J_i}(\lambda);\ c^{(i)}_{s_i}(\lambda))P_i$ is canonical. For in this case, $(A^{(i)}_{J_i}(\lambda);c^{(i)}_{s_i}(\lambda))$ is mxl, and $P_i \in R$ can be found as α was in Step 1 (iii); this P_i is clearly unique.

If $J_i \neq \emptyset$, then since $c^{(i)}_{s_i}(\lambda) \notin \text{Im } A^{(i)}(\lambda)$ and $\text{Im } A^{(i)}_{J_i}(k) \subset \text{Im } A^{(i)}(\lambda)$, it follows that

(4.3-47) $$c^{(i)}_{s_i}(\lambda) \not\subset \text{Im } A^{(i)}_{J_i}(\lambda)$$

Because $c^{(i)}_{s_i}(\lambda)$ and the columns of $A^{(i)}_{J_i}(\lambda)$ are all of degree γ_i, (4.3-47) implies that the coefficient of λ^{γ_i} in $(A^{(i)}_{J_i}(\lambda);c^{(i)}_{s_i}(\lambda))$ has full column rank. Thus, as in the proof to (4.3-36), we can find a nonsingular, constant matrix P_i such that $(A^{(i)}_{J_i}(\lambda);c_{s_i}(\lambda))P_i$ is canonical; moreover, since this P_i must "canonicalize" the coefficient of λ^{γ_i} in $(A^{(i)}_{J_i}(\lambda);c^{(i)}_{s_i}(\lambda))$ (to the form consisting of mostly 1's and 0's) it is clear that P_i is unique.

Since $A^{(i)}(\lambda)$ is canonical, it follows that $A^{(i)}_{J_i}(\lambda)$ cannot be further reduced modulo $A^{(i)}_{I_i}(\lambda)$; also, since $c^{(i)}_{s_i}(\lambda)$ is the result of reducing $b^{(i)}_{s_i}(\lambda)$ modulo $A^{(i)}(\lambda)$, it follows that $c^{(i)}_{s_i}(\lambda)$ cannot be further reduced modulo $A^{(i)}_{I_i}(\lambda)$. Therefore, $(A^{(i)}_{J_i}(\lambda);c^{(i)}_{s_i}(\lambda))P_i$ cannot be further reduced modulo $A^{(i)}_{I_i}(\lambda)$. This, together with the facts that $A^{(i)}_{I_i}(\lambda)$ and $(A^{(i)}_{J_i}(\lambda);\ c^{(i)}_{s_i}(\lambda))P_i$ are canonical, and that

$$\partial A^{(i)}_{I_i}(\lambda) < \partial((A^{(i)}_{J_i}(\lambda);c^{(i)}_{s_i}(\lambda))P_i)$$

implies that

$$A^{(i+1)}(\lambda) = (A^{(i)}_{I_i}(\lambda);(A^{(i)}_{J_i}(\lambda);c^{(i)}_{s_i}(\lambda))P_i)$$

is canonical.

To show that the operations of (4.3-43) are equivalent to a sequence of column operations on $D(\lambda)$, we proceed as follows.

Clearly, the matrix $(A^{(1)}(\lambda);B^{(1)}(\lambda))$ is related to $D(\lambda)$ as

$$(A^{(1)}(\lambda);B^{(1)}(\lambda)) = D(\lambda)M_1$$

where M_1 is nxn, constant, and nonsingular. The action of M_1, on $D(\lambda)$ can be represented by a permutation of the columns, followed by multiplication of column 1 by a scalar.

Also, for each i, since $C^{(i)}(\lambda)$ is $B^{(i)}(\lambda)$ reduced modulo $A^{(i)}(\lambda)$,

$$C^{(i)}(\lambda) = B^{(i)}(\lambda) - A^{(i)}(\lambda)\ Q_i(\lambda)$$

for some $Q^{(i)}(\lambda)$. Therefore,

$$\begin{aligned}(A^{(i)}(\lambda);\ C^{(i)}(\lambda)) &= (A^{(i)}(\lambda);\ B^{(i)}(\lambda)) \begin{pmatrix} I & -Q_i(\lambda) \\ 0 & I \end{pmatrix} \\ &= (A^{(i)}(\lambda);\ B^{(i)}(\lambda))\ U_i(\lambda)\end{aligned}$$

where $U_i(\lambda)$ is n x n and unimodular. The action of $U_i(\lambda)$ can be represented as a series of elementary operations on the columns of $(A^{(i)}(\lambda);\ B^{(i)}(\lambda))$.

Finally, for each i,

$$(A^{(i+1)}(\lambda);\ B^{(i+1)}(\lambda)) = (A^{(i)}(\lambda);\ C^{(i)}(\lambda))M_{i+1}$$

for a constant, nonsingular M_{i+1} (also representable by elementary column operations). This proves our claim that each $(A^{(i)}(\lambda);\ B^{(i)}(\lambda))$ and each $(A^{(i)}(\lambda);\ C^{(i)}(\lambda))$ can be obtained from $D(\lambda)$ by column operations.

We must now show that (4.3-43) does indeed produce, in a finite number of operations, a canonical matrix whose image is the submodule Im $D(\lambda)$. From the preceding paragraphs it follows that

$$\text{Im}(A^{(i)}(\lambda);\ B^{(i)}(\lambda)) = \text{Im}(A^{(i)}(\lambda);\ C^{(i)}(\lambda)) = \text{Im } D(\lambda)$$

for each i. Thus, if it ever happens that either $r_i = n$ or $C^{(i)}(\lambda) = 0$, then, for that value of i, $\text{Im } A^{(i)}(\lambda) = \text{Im } D(\lambda)$. Since $A^{(i)}(\lambda)$ is always canonical, it follows that if Step 4 is ever reached, then, for that value of i, $A^{(i)}(\lambda)$ is the sought-after canonical matrix.

We now show that Step 4 is reached after only a finite number of iterations through Step 3. We assume that $\partial T(\lambda) = \nu$ and that rank $T(\lambda) = r$, and argue as follows.

For every i, the degree of every nonzero column of $B^{(i)}(\lambda)$ is greater than or equal to $\partial A^{(i)}$. Thus, after having completed Step 2(i), there are two possibilities: (a) $\gamma_i \geq \partial A^{(i)}$, and (b) $\gamma_i < \partial A^{(i)}$. In case (a)

$$r_{i+1} = r_i + 1$$

while in case (b),

$$1 \leq r_{i+1} \leq r_i$$

The degrees of the columns of $B^{(i)}(\lambda)$ cannot increase when these columns are reduced modulo the canonical matrix $A^{(i)}(\lambda)$. Therefore, since we are operating on a total of n columns, (n-r) of which will eventually be reduced to the zero vector, and r of which can be reduced to polynomial vectors of nonnegative degree, it follows that case (b) can occur <u>at most</u> a total of

$$r\nu + (n-r)(\nu+1) = n\nu + n - r$$

times.

Moreover, case (a) can occur for at most r-1 consecutive values of i before either case (b) occurs or Step 4 is reached. For, if case (a) occurs r times in succession, then after the last of these, r_i will necessarily be larger than r, contradicting Im $A^{(i)}(\lambda) \subset$ Im $D(\lambda)$.

Depicting the successive cases for successive values of i by a sequence of a's and b's:

a a a b a a a a b b a a b a ... b a a

we see that in every allowable sequence there are at most $(n\nu + n - r)$ b's, and at most r - 1 a's separating any two successive b's. Thus, the total length, N, of each allowable sequence is bounded as

$$N \leq r(n\nu+n-r) + r - 1 = r(n\nu+n-r+1) - 1 \quad \blacksquare$$

(4.3-48) __Remark__: Examination of the algorithm (4.3-43) reveals that it is really just a generalization of the familiar Euclidean algorithm for computing the greatest common divisor of a set of polynomials in $R[\lambda]$. Thus, we may refer to (4.3-43) as the "generalized Euclidean algorithm".

In Section 4.3.3 we shall show that there is a natural partial ordering in the set of m x n canonical matrices (for $1 \leq n \leq m$), and that this partial ordering induces a lattice isomorphism between the set of such canonical matrices and the lattice of submodules of $R^m[\lambda]$. However, we now proceed into Section 4.3.2, where we develop explicit representations for the quotient module $R^m[\lambda]/M$ and the canonical projection $\pi : R^m[\lambda] \rightarrow R^m[\lambda]/M$, for any submodule $M \subset R^m[\lambda]$.

4.3.2 Representations for Quotient Modules and Canonical Projections in Terms of Canonical Matrices

In this section we obtain explicit characterizations for the quotient module $R^m[\lambda]/M$ and the canonical projection $\pi : R^m[\lambda] \to R^m[\lambda]/M$, where M is any submodule of $R^m[\lambda]$. The characterizations will be in terms of the unique canonical matrix $T(\lambda)$ such that $\text{Im } T(\lambda) = M$. It is well known (see Appendix B) that, if rank $M = r$, then

$$R^m[\lambda]/M \cong T \oplus F$$

where T is a torsion module, isomorphic to the torsion submodule of $R^m[\lambda]/M$, and F is a free module of rank $m - r$. We shall obtain explicit characterizations for both F and T.

We begin with

(4.3-49) **Theorem**: Let $M \subset R^m[\lambda]$ be an arbitrary submodule of rank $r > 0$, and let $T(\lambda)$ be the unique canonical matrix such that $M = \text{Im } T(\lambda)$. Then there exist a torsion module X and an R-linear map $\theta : X \to R^{m-r}[\lambda]$ such that the quotient module $R^m[\lambda]/M$ is isomorphic to the Cartesian product $X \times R^{m-r}[\lambda]$, first made into an R-vector space in the obvious way (componentwise addition and scalar multiplication), and then made into an $R[\lambda]$-module by defining

$$\lambda \cdot (x, v(\lambda)) = (\lambda x, \lambda v(\lambda) - \theta x); \text{ for } x \in X,\ v(\lambda) \in R^{m-r}[\lambda]$$

X and θ are determined from $T(\lambda)$ as follows. If $\{\nu_i\}$ and $\{m_i\}$ are the sets of integers associated with $T(\lambda)$, define $\hat{T}(\lambda)$ $(r \times r)$ and $\tilde{T}(\lambda)$ $((m-r) \times r)$ as in the proof to (4.3-26). Then define

(4.3-50) $$X \triangleq R^r[\lambda]/\text{Im}\,\hat{T}(\lambda) \cong \{\bar{x}(\lambda) \in R^r[\lambda] \mid \partial\bar{x}_i < \nu_i\}$$

and

(4.3-51) $$\theta : X \to R^{m-r}[\lambda]$$

$$: \bar{x}(\lambda) + \text{Im}\,\hat{T}(\lambda) \mapsto \tilde{T}(\lambda)\, x_h$$

where $\partial\bar{x}_i < \nu_i$, for $i \in \underline{r}$, and $x_h \in R^r$ is the vector whose ith element is the coefficient of $\lambda^{\nu_i - 1}$ in $\bar{x}_i(\lambda)$.

Proof: Elements of $R^m[\lambda]/M$ are cosets of the form $u(\lambda) + M$, with $u(\lambda) \in R^m[\lambda]$. For each $u(\lambda) \in R^m[\lambda]$ there is, from (4.3-26), a unique $\bar{u}(\lambda) \in R^m[\lambda]$ such that

$$u(\lambda) + M = \bar{u}(\lambda) + M$$

and

$$\partial\bar{u}_{m_i} < \nu_i\ , \text{ for } i \in \underline{r}$$

($\bar{u}(\lambda)$ is the remainder resulting from reducing any element of $u(\lambda) + M$ modulo $T(\lambda)$). Thus, the following sets are bijective:

$$R^m[\lambda]/M \cong \{\bar{u}(\lambda) \in R^m[\lambda] \mid \partial\bar{u}_{m_i} < \nu_i\ , i \in \underline{r}\}$$

$$\cong \{\bar{x}(\lambda) \in R^r[\lambda] \mid \partial\bar{x}_i < \nu_i\ , i \in \underline{r}\} \times R^{m-r}[\lambda]$$

By the same argument, X, as defined above, and $\{\bar{x}(\lambda) \in R^r[\lambda] \mid \partial\bar{x}_i < \nu_i\}$ are set-isomorphic. Thus the set isomorphism

$$R^m[\lambda]/M \cong X \times R^{m-r}[\lambda]$$

is established. Further, the above is easily seen to be an isomorphism of R-vector spaces.

It is clear that X, as defined in (4.3-50), is an $R[\lambda]$-module; that it is a torsion module may be demonstrated as follows. Defining

$$\psi(\lambda) \triangleq \det \hat{T}(\lambda)$$

(nonzero because $\hat{T}(\lambda)$ is column proper) and letting $C(\lambda)$ be the $r \times r$ matrix of $(r-1) \times (r-1)$ cofactors of $\hat{T}(\lambda)$, it is clear that

$$\hat{T}(\lambda)\ C(\lambda) = \psi(\lambda) I$$

Now, if $x(\lambda) + \text{Im}\ \hat{T}(\lambda) \in R^r[\lambda]/\text{Im}\ \hat{T}(\lambda)$, then

$$\begin{aligned}\psi(\lambda)(x(\lambda) + \text{Im}\ \hat{T}(\lambda)) &= \psi(\lambda)\ x(\lambda) + \text{Im}\ \hat{T}(\lambda) \\ &= \hat{T}(\lambda)\ C(\lambda)\ x(\lambda) + \text{Im}\ \hat{T}(\lambda) \\ &= 0 + \text{Im}\ \hat{T}(\lambda)\end{aligned}$$

whence it follows that X is torsion.

Finally, if $\bar{u}(\lambda)$ is the canonical representative of $u(\lambda) + M$, then the canonical representative of $\lambda(u(\lambda) + M)$ is found by reducing $\lambda\bar{u}(\lambda)$ modulo $T(\lambda)$. Since $\bar{u}(\lambda)$ can be written uniquely as

$$\bar{u}(\lambda) = P^{-1} \begin{pmatrix} \bar{x}(\lambda) \\ v(\lambda) \end{pmatrix}$$

where $\bar{x}(\lambda) \in R^r[\lambda]$, $\partial\bar{x}_i < \nu_i$, $v(\lambda) \in R^{m-r}[\lambda]$, and P is the permutation matrix in the proof to (4.3-26), it follows that reducing $\lambda\bar{u}(\lambda)$ modulo $T(\lambda)$ is equivalent to reducing

$$P\lambda\bar{u}(\lambda) = \begin{pmatrix} \lambda\,\bar{x}(\lambda) \\ \lambda\, v(\lambda) \end{pmatrix}$$

modulo

$$PT(\lambda) = \begin{pmatrix} \hat{T}(\lambda) \\ \tilde{T}(\lambda) \end{pmatrix}$$

Denoting the remainder upon reducing $\lambda\overline{u}(\lambda)$ modulo $T(\lambda)$ as $[\lambda\overline{u}(\lambda)]$, and the remainder upon reducing $\lambda\overline{x}(\lambda)$ modulo $\hat{T}(\lambda)$ as $[\lambda\overline{x}(\lambda)]$, it is easy to see that

$$P[\lambda\overline{u}(\lambda)] = \begin{pmatrix} [\lambda\overline{x}(\lambda)] \\ \lambda v(\lambda) - \tilde{T}(\lambda)\overline{x}_h \end{pmatrix}$$

This establishes the module isomorphism between $R^m[\lambda]/M$ and $X \times R^{m-r}[\lambda]$. It is easy to see that the map $\theta : X \to R^{m-r}[\lambda]$, defined in (4.3-51), is R-linear. ■

Since any finitely generated $R[\lambda]$-torsion module can be represented in terms of a finite dimensional R-vector space X and an endomorphism $A : X \to X$, we immediately arrive at the following corollary.

(4.3-52) <u>Corollary</u>: If $M \subset R^m[\lambda]$ is a submodule of rank $r > 0$, if $T(\lambda)$ is the canonical matrix such that $\text{Im } T(\lambda) = M$, and if $\{\nu_i\}$ and $\{m_i\}$ are the sets of integers associated with $T(\lambda)$, then there exists an R-vector space X of dimension $n = \sum_{i \in \underline{r}} \nu_i$ such that $R^m[\lambda]/M$ and $X \times R^{m-r}[\lambda]$ are isomorphic as R-vector spaces.

Furthermore, there exist R-linear maps $A : X \to X$ and $H : X \to R^r$ such that, if the action of λ on $X \times R^{m-r}[\lambda]$ is defined as

$$\lambda(x, v(\lambda)) \triangleq (Ax, \lambda v(\lambda) - \tilde{T}(\lambda)Hx)$$

then $R^m[\lambda]/M$ and $X \times R^{m-r}[\lambda]$ are isomorphic as $R[\lambda]$-modules. In the above, $\tilde{T}(\lambda)$ is as given in (4.3-49).

With respect to a "canonical basis" in X, and the standard basis in R^r, the matrices for A and H are given in terms of the integers ν_i and the elements of $\hat{T}(\lambda)$ (as in (4.3-49) as follows:

(4.3-53)
$$h_{i,j} = \begin{cases} 1, \text{ if } j = \nu_1 + \nu_2 + \ldots + \nu_i \\ 0, \text{ otherwise} \end{cases} \quad i \in \underline{r},\ j \in \underline{n}$$

(4.3-54)
$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & & \\ \vdots & \vdots & & \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{pmatrix}$$

where A_{ij} is $\nu_i \times \nu_j$ and, if

$$\hat{T}_{ij}(\lambda) = \begin{cases} \lambda^{\nu_i} - \sum_{k \in \underline{\nu}_i} a_{iik} \lambda^{k-1}, \text{ for } j = i \\ - \sum_{k \in \underline{\nu}_i} a_{ijk} \lambda^{k-1}, \text{ for } j \neq i \end{cases}$$

then

$$A_{ii} = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_{ii1} \\ 1 & 0 & & 0 & a_{ii2} \\ 0 & 1 & & \vdots & \vdots \\ 0 & 0 & & & \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & 1 & a_{ii\nu_i} \end{bmatrix}$$

and

$$A_{ij} = \begin{pmatrix} 0 & 0 \ldots 0 & a_{ij1} \\ 0 & 0 \quad\; 0 & a_{ij2} \\ \vdots & \vdots \quad\; \vdots & \vdots \\ 0 & \underbrace{0 \quad\; 0}_{\nu_j - 1 \text{ columns}} & a_{ij\nu_i} \end{pmatrix}, \text{ for } i \neq j$$

Proof: Define $\mathcal{X}$ to be the $R[\lambda]$-module X, taken as an R-vector space. Define the R-linear endomosphism $A : \mathcal{X} \to \mathcal{X}$ so that the action of A on $\mathcal{X}$ is the same as the action of λ on X. Next, define $H : \mathcal{X} \to R^r$ as

$$H : \bar{x}(\lambda) + \text{Im } \hat{T}(\lambda) \to x_h$$

where $\partial\bar{x}_i < \nu_i$, for $i \in \underline{r}$, and $x_h \in R^r$ is the vector whose ith element is the coefficient of $\lambda^{\nu_i - 1}$ in $\bar{x}_i(\lambda)$. Clearly, $\tilde{T}(\lambda)H = \theta$, where θ is given by (4.3-51); thus the induced module structure of $\mathcal{X} \times R^{m-r}[\lambda]$ is identical to that of $X \times R^{m-r}[\lambda]$.

To demonstrate that $\dim(\mathcal{X}) = \sum_{i \in \underline{r}} \nu_i$, we construct a basis for $\mathcal{X}$ as follows. Recall that each element $\bar{x}(\lambda) + \text{Im } \hat{T}(\lambda) \in X$ has a canonical representative $\bar{x}(\lambda)$, where $\partial\bar{x}_i < \nu_i$. It is clear that each such representative can be expressed as an R-linear combination of the vectors $\{v_{i,j} = \lambda^{j-1} e_i \; ; \; j \in \underline{\nu_i} \; , \; i \in \underline{r}\}$ where $\{e_i, \; i \in \underline{r}\}$ is the standard basis for R^r; moreover, by (4.3-18), the only R-linear combination of $\{v_{ij}\}$ that is an element of $\text{Im } \hat{T}(\lambda)$ is the zero linear combination. Thus, $\{b_{i,j} \; ; \; j \in \underline{\nu_i} \; , \; i \in \underline{r}\}$, defined as

$$b_{i,j} = v_{i,j} + \text{Im } \hat{T}(\lambda) \; ; \; j \in \underline{\nu_i} \; , \; i \in \underline{r}$$

is a set of basis vectors for $\mathcal{X}$. This verifies the claim that $\dim(\mathcal{X}) = \sum_{i \in \underline{r}} \nu_i$.

To determine the matrices for H and A, we take the basis $\{b_{ij}\}$ in the order $b_{1,1}, b_{1,2}, \ldots b_{1,\nu_1}, b_{2,1}, \ldots b_{r,\nu_r}$.

The map H must produce the vector $x_h \in R^r$ from $x(\lambda) + \text{Im}\,\hat{T}(\lambda) \in X$. But the ith element of x_h is just the coefficient of b_{i,ν_i} in the representation of $x(\lambda) + \text{Im}\,\hat{T}(\lambda)$ in the basis $\{b_{i,j}\}$. This verifies (4.3-53).

To determine the matrix for A, we need only determine the action of λ on each $b_{i,j}$. Thus:

$$A\, b_{i,j} \triangleq \lambda\, b_{i,j} = b_{i,j+1}, \text{ for } 1 \le j \le \nu_i - 1, \text{ and } i \in \underline{r}.$$

Moreover, $A\, b_{i,\nu_i} \triangleq \lambda\, b_{i,\nu_i}$ may be found by reducing $\lambda v_{i,\nu_i}$ modulo $\hat{T}(\lambda)$. The result is

$$\lambda v_{i,\nu_i} = \hat{T}(\lambda)\, e_i + [\lambda v_{i,\nu_i}]$$

whence

$$[\lambda v_{i,\nu_i}] = \lambda v_{i,\nu_i} - \hat{T}(\lambda)\, e_i$$

$$= \lambda^{\nu_i} e_i - \begin{pmatrix} \hat{T}_{1i}(\lambda) \\ \hat{T}_{2i}(\lambda) \\ \vdots \\ \hat{T}_{ri}(\lambda) \end{pmatrix}$$

$$= \begin{bmatrix} \sum_{k \in \underline{\nu}_1} a_{1ik}\, \lambda^{k-1} \\ \sum_{k \in \underline{\nu}_2} a_{2ik}\, \lambda^{k-1} \\ \vdots \\ \sum_{k \in \underline{\nu}_r} a_{rik}\, \lambda^{k-1} \end{bmatrix}$$

Therefore,

$$A\, b_{i,\nu_i} = \sum_{k \in \underline{\nu}_1} a_{1ik}\, b_{1,k} + \ldots + \sum_{k \in \underline{\nu}_r} a_{rik}\, b_{r,k}$$

to verify (4.3-54), and the proof is complete. ∎

(4.3-55) **Remark:** In (4.3-52) we have used only one of many possible bases for X. Clearly, another basis would, in general, result in different matrices for A and H; however, the maps A and H are invariant under choice of basis in X. In Section 4.5 we shall consider a second basis for X, one that will allow an easy treatment of state feedback in linear systems.

The torsion module X in (4.3-49) is not, in general, isomorphic to the torsion submodule of $R^m[\lambda]/M$. This is due to the coupling between X and $R^{m-r}[\lambda]$ from the linear map θ. We next prove several results relating the structure of X to that of the torsion submodule of $R^m[\lambda]/M$.

(4.3-56) **Lemma:** Assume that $n = \sum_{i \in \underline{r}} \nu_i > 0$, i.e. that $\partial T(\lambda) > 0$. Also, denote that torsion submodule of $X \times R^{m-r}[\lambda]$ by T:

$$T = \{(x,v(\lambda)) \in X \times R^{m-r}[\lambda] \,|\, \alpha(\lambda)(x,v(\lambda)) = 0 \text{ for some nonzero } \alpha(\lambda) \in R[\lambda]\}$$

Then

(i) There exists no element in T of the form $(0,v(\lambda))$, for $v(\lambda) \neq 0$.

(ii) If $\tilde{T}(\lambda) = 0$, then $T = \{(x,0) \,|\, x \in X\}$.

(iii) If rank $\tilde{T}(\lambda) = r$, then there exists no element in T of the form $(x,0)$, for $x \neq 0$.

Proof: (i) Assume that $(0,v(\lambda)$ ε $\mathcal{T}$. Then there is a nonzero $\alpha(\lambda)$ ε $R[\lambda]$ such that $\alpha(\lambda)(0,v(\lambda)) = 0$. But clearly, $\alpha(\lambda)(0,v(\lambda)) = (0,\alpha(\lambda)v(\lambda))$. Thus $(0,v(\lambda))$ ε $\mathcal{T}$ if and only if $\alpha(\lambda)$ $v(\lambda) = 0$; i.e. if and only if $v(\lambda) = 0$.

(ii) Let $\tilde{T}(\lambda) = 0$. Then for all $(x,v(\lambda))$ ε $X \times R^{m-r}[\lambda]$ and $\alpha(\lambda)$ ε $R[\lambda]$,

$$\alpha(\lambda)(x,v(\lambda)) = (\alpha(\lambda)x,\alpha(\lambda)v(\lambda))$$

Therefore, if $(x,v(\lambda))$ ε $\mathcal{T}$, then $v(\lambda)$ must be zero; i.e.,

$$\mathcal{T} \subset \{(x,0)\,|\,x \;\varepsilon\; X\}$$

On the other hand, X is a finitely generated torsion module, so there is an $\alpha(\lambda)$ ε $R[\lambda]$ such that $\alpha(\lambda)x = 0$, for all x ε X. Consequently, $\alpha(\lambda)(x,0) = 0$ for all x ε X, and

$$\mathcal{T} \supset \{(x,0)\,|\,x \;\varepsilon\; X\}$$

(iii) We first show that, using the representation of (4.3-52), if $\alpha(\lambda)$ ε $R[\lambda]$ is given by

$$\alpha(\lambda) = \lambda^k a_k + \lambda^{k-1} a_{k-1} + \dots + a_0$$

then

$$\alpha(\lambda)(x,0) = (\alpha(A)x,\; -\tilde{T}(\lambda)H \sum_{j=0}^{k-1} \lambda^j \alpha^{(j)}(A)x) \tag{4.3-57}$$

where

$$\alpha^{(j)}(\lambda) = a_k\lambda^{k-j-1} + a_{k-1}\lambda^{k-j-2} + \dots + a_{j+1}\,, \quad \text{for } 0 \le j \le k-1 \tag{4.3-58}$$

This follows from the fact that

$$\lambda^i(x,0) = (A^i x,\; -\sum_{j=0}^{i-1} \lambda^j \tilde{T}(\lambda)H A^{i-j-1}x)\,, \text{ for } i \ge 1$$

as can be easily shown by continued use of

$$\lambda(x,v(\lambda)) = (Ax,\ \lambda v(\lambda) - \tilde{T}(\lambda)\ Hx)$$

Now suppose that $(x,0) \in T$ and that rank $\tilde{T}(\lambda) = r$. From the former, there is a nonzero $\alpha(\lambda) \in R[\lambda]$ such that $\alpha(\lambda)(x,0) = 0$. Clearly, we can assume that $\partial\alpha \geq n$; for if $\partial\alpha < n$, then $\lambda^{n-\partial\alpha}\ \alpha(\lambda)$ is of degree n, and it also annihilates $(x,0)$.

From (4.3-57) with $k = \partial\alpha$,

$$(\alpha(A)x,\ -\tilde{T}(\lambda)\ H \sum_{j=0}^{k-1} \lambda^j\ \alpha^{(j)}(A)x) = 0$$

and in particular, since rank $\tilde{T}(\lambda) = r$,

$$H\ \alpha^{(j)}(A)\ x = 0\ ,\ \text{for } 0 \leq j \leq k-1 \geq n-1$$

From this it follows, since $a_k \neq 0$, that

$$H\ A^j\ x = 0\ ,\ \text{for } 0 \leq j \leq n-1$$

However, it is easy to check that the pair (H,A) is an observable pair, whence it follows that $x = 0$. ■

The module structures of X and T are more explicitly related by the next result.

(4.3-59) <u>Theorem</u>: T is isomorphic (as an $R[\lambda]$-module) to a submodule of X. Thus, in particular,

(i) If the invariant factors of X are denoted by $\{\psi_i(\lambda)\}$, where $\psi_{i+1}(\lambda)$ divides $\psi_i(\lambda)$, and those of T are denoted by $\{\phi_i(\lambda)\}$, where $\phi_{i+1}(\lambda)$ divides $\phi_i(\lambda)$, then $\phi_i(\lambda)$ divides $\psi_i(\lambda)$, for all i.

(ii) The dimension of $\mathcal{T}$ as an R-vector space is no larger than $n = \sum_{i \in \underline{r}} \nu_i$.

Proof: Define the subset $\mathcal{S} \subset X$ as

$$\mathcal{S} = \{x \in X \mid (x, v(\lambda)) \in \mathcal{T}, \text{ for some } v(\lambda) \in R^{m-r}[\lambda]\}$$

Clearly, $\mathcal{S}$ is nonempty, since $0 \in \mathcal{S}$. Now suppose that $x \in \mathcal{S}$, and that both $(x, v_1(\lambda)) \in \mathcal{T}$ and $(x, v_2(\lambda)) \in \mathcal{T}$. But then,

$$(0, v_1(\lambda) - v_2(\lambda)) = (x, v_1(\lambda)) - (x, v_2(\lambda)) \in \mathcal{T}$$

and from (4.3-56)(i) it follows that $v_1(\lambda) = v_2(\lambda)$.

Thus, if $x \in \mathcal{S}$, there is a <u>unique</u> $v_x(\lambda) \in R^{m-r}[\lambda]$ such that $(x, v_x(\lambda)) \in \mathcal{T}$. Now define the set-theoretic map

$$\Psi : \mathcal{S} \to \mathcal{T}$$

$$: x \mapsto (x, v_x(\lambda))$$

Ψ is clearly a bijection. We shall now show that $\mathcal{S}$ is a submodule of X and that Ψ is a module isomorphism.

Let $x, y \in \mathcal{S}$. Then, because $\mathcal{T}$ is a submodule,

(4.3-60) $$\alpha(\lambda)(x, v_x(\lambda)) + \beta(\lambda)(y, v_y(\lambda)) \in \mathcal{T}$$

for arbitrary $\alpha(\lambda), \beta(\lambda) \in R[\lambda]$. But, if $\partial\alpha(\lambda) = k$, it is easy to verify that

$$\alpha(\lambda)(x, v_x(\lambda)) = (\alpha(\lambda)x, \alpha(\lambda)v_x(\lambda) - \sum_{j=0}^{k-1} \lambda^j \theta\alpha^{(j)}(\lambda)x)$$

where the $\alpha^{(j)}(\lambda) \in R[\lambda]$ are as in (4.3-58). Similarly, if $\partial\beta(\lambda) = \ell$, then

$$\beta(\lambda)(y, v_y(\lambda)) = (\beta(\lambda)y, \beta(\lambda)v_y(\lambda) - \sum_{j=0}^{\ell-1} \lambda^j \theta\beta^{(j)}(\lambda)y)$$

Thus, defining

$$z(\lambda) \triangleq \alpha(\lambda)\ x(\lambda) + \beta(\lambda)\ y(\lambda)$$

and

$$v_z(\lambda) \triangleq \alpha(\lambda) v_x(\lambda) + \beta(\lambda) v_z(\lambda) - \sum_{j=0}^{k-1} \lambda^j \theta\alpha^{(j)}(\lambda) x - \sum_{j=0}^{\ell-1} \lambda^j \theta\beta^{(j)}(\lambda) y$$

it immediately follows from (4.3-60) that

$$(z(\lambda),\ v_z(\lambda)) \in T$$

Therefore, S is a submodule of X. Moreover,

$$\begin{aligned} \Psi(\alpha(\lambda)\ x + \beta(\lambda) y) &= (z(\lambda),\ v_z(\lambda)) \\ &= \alpha(\lambda)(x, v_x(\lambda)) + \beta(\lambda)(y, v_y(\lambda)) \\ &= \alpha(\lambda)\Psi(\lambda) + \beta(\lambda)\Psi(y) \end{aligned}$$

whence it follows that $\Psi : S \to T$ is a module isomorphism.

Statements (i) and (ii) now follow immediately. ■

(4.3-61) Remark: If we wished to characterize T by a vector space-endomorphism pair (V,F), we could do so by choosing V to be the A-invariant subspace of X whose elements are the same as those of $S \subset X$, and by defining $F : V \to V$ to be the restriction of A to V.

As a corollary to (4.3-59) we now prove the following result, which may be used to determine the submodule $S \subset X$.

(4.3-62) Corollary: Let $S \subset X$ be given as

$$S = \{x \in X \mid (x, v(\lambda)) \in T, \text{ for some } v(\lambda) \in R^{m-r}[\lambda]$$

and for each $x \in X$, let $\mu_x(\lambda)$ denote the (monic) minimal annihilator of x. Then

(i) If $x \in S$, the minimal annihilator of $(x, v_x(\lambda)) \in T$ is $\mu_x(\lambda)$.

(ii) $x \in S$ if and only if $\mu_x(\lambda)$ is a divisor of each of the m-r components of

$$\left(\sum_{j=0}^{k-1} \lambda^j \theta\, \mu_x^{(j)}(\lambda) x\right) \in R^{m-r}[\lambda]$$

where, if

$$\mu_x(\lambda) = \lambda^k + \lambda^{k-1} \gamma_{k-1} + \ldots + \lambda\gamma_1 + \gamma_0$$

the polynomials $\mu_x^{(j)}(\lambda)$ are defined as

$$\mu_x^{(j)}(\lambda) = \lambda^{k-j-1} + \gamma_{k-1} \lambda^{k-j-2} + \ldots + \gamma_{j+1} ,$$

$$\text{for } 0 \leq j \leq k-1$$

(iii) If $x \in S$, then

$$v_x(\lambda) = \frac{1}{\mu_x(\lambda)} \sum_{j=0}^{k-1} \lambda^j \theta\, \mu_x^{(j)}(\lambda)\, x$$

Proof: (i) Let $x \in S$, and let the (monic) minimal annihilator of $(x, v_x(\lambda))$ be $\mu(\lambda)$. Then, since $\Psi : S \to T$ of the proof to (4.3-59) is an isomorphism,

$$0 = \Psi(0) = \Psi(\mu_x(\lambda)x) = \mu_x(\lambda)(x, v_x(\lambda))$$

so that $\mu_x(\lambda)$ is a multiple of $\mu(\lambda)$. But also,

$$0 = \Psi^{-1}(0) = \Psi^{-1}(\mu(\lambda)(x, v_x(\lambda))) = \mu(\lambda)\ x$$

so that $\mu(\lambda)$ is a multiple of $\mu_x(\lambda)$. Since both $\mu(\lambda)$ and $\mu_x(\lambda)$ are taken to be monic, it follows that $\mu(\lambda) = \mu_x(\lambda)$.

(ii) It now follows that $x \in S$ if and only if there exists $v_x(\lambda) \in R^{m-r}[\lambda]$ such that

$$0 = \mu_x(\lambda)(x, v_x(\lambda)) = (\mu_x(\lambda)x, \mu_x(\lambda)v_x(\lambda) - \sum_{j=0}^{k-1} \lambda^j \theta\, \mu_x^{(j)}(\lambda)\; x)$$

Thus, $x \in S$ if and only if

$$\mu_x(\lambda)v_x(\lambda) = \sum_{j=0}^{k-1} \lambda^j \theta\, \mu_x^{(j)}(\lambda)\; x$$

for some $v_x(\lambda) \in R^{m-r}[\lambda]$, and the result follows.

(iii) This result is now obvious. ■

Results (4.3-56) and (4.3-59) provide us with revealing, albeit incomplete, characterizations of the torsion submodule of $R^m[\lambda]/M$; while (4.3-62), in theory, provides a method for actually determining T. However, in order to apply (4.3-62), one has to determine $\mu_x(\lambda)$ for every $x \in X$. While this is possible to accomplish, it is quite difficult. (It amounts to calculating the Jordan form of the matrix A in (4.3-54).) We therefore next consider an algorithmic method for determining both the torsion and free submodules of $R^m[\lambda]/M$.

(4.3-63) <u>Lemma</u>: Let $M \subset R^m[\lambda]$ be a submodule of rank $r > 0$, and let $T(\lambda)$ be the unique $m \times r$ canonical matrix such that $\text{Im } T(\lambda) = M$. Define unique $r \times r$ canonical matrices $T_1(\lambda)$ and $T_2(\lambda)$ such that

$$\text{Im } T_1(\lambda) = \text{Im } T'(\lambda)$$

$$\text{Im } T_2(\lambda) = \text{Im } T_1'(\lambda)$$

Then

$$R^m[\lambda]/\mathcal{M} \cong (R^r[\lambda]/\text{Im } T_2(\lambda)) \oplus R^{m-r}[\lambda]$$

so that

$$\mathcal{T} \cong R^r[\lambda]/\text{Im } T_2[\lambda]$$

where $\mathcal{T}$ is the torsion submodule of $X \times R^{m-r}[\lambda]$.

Proof: From (4.3-46) it follows that $T_1(\lambda)$ is obtained from $T'(\lambda)$ by elementary column operations, as is $T_2(\lambda)$ from $T_1{}'(\lambda)$. Thus, there exist unimodular matrices $Q_1(\lambda)$ (m x m) and $Q_2(\lambda)$ (r x r) such that

$$(T_1(\lambda);\ 0) = T'(\lambda)\ Q_1{}'(\lambda)$$

$$T_2(\lambda) = T_1{}'(\lambda)\ Q_2(\lambda)$$

Thus,

$$T(\lambda) = Q_1^{-1}(\lambda) \begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix} Q_2^{-1}(\lambda)$$

Since $Q_1(\lambda)$ and $Q_2(\lambda)$ represent isomorphisms ($R^m[\lambda] \tilde{\rightarrow} R^m[\lambda]$ and $R^r[\lambda] \tilde{\rightarrow} R^r[\lambda]$, respectively), it is now clear that

$$R^m[\lambda]/\mathcal{M} = R^m[\lambda]/\text{Im } T(\lambda)$$

$$\cong R^m[\lambda]/\text{Im} \begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix}$$

where the isomorphism is explicitly given by

$$u(\lambda) + \mathcal{M} \mapsto Q_1(\lambda)u(\lambda) + \text{Im} \begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix}$$

But, from (4.3-56)(ii), it follows that

$$R^m[\lambda]/\mathrm{Im}\begin{pmatrix}T_2(\lambda)\\0\end{pmatrix} \cong (R^r[\lambda]/\mathrm{Im}\ T_2(\lambda)) \oplus R^{m-r}[\lambda]$$

which completes the proof. ■

We can now use (4.3-63) to explicitly characterize the free and torsion submodules of $X \times R^{m-r}[\lambda]$:

(4.3-64) __Theorem__: Let $Q_1(\lambda)$ be the unimodular $m \times m$ matrix in the proof to (4.3-63), and let P be the $m \times m$ permutation matrix of (4.3-49) (i.e., $PT(\lambda) = \binom{\hat{T}(\lambda)}{\tilde{T}(\lambda)}$)). Partition $Q_1^{-1}(\lambda)$ as

$$Q_1^{-1}(\lambda) = (L_1(\lambda); L_2(\lambda)) \ , (L_1(\lambda) \text{ is } m \times r)$$

Further, let $\{\tilde{\nu}_i\}$ be the degree integers associated with the canonical matrix $T_2(\lambda)$ of (4.3-63). Then

(i) $L_1(\lambda)$ is unique

(ii) An internal direct sum decomposition of $X \times R^{m-r}[\lambda]$ ("internal" meaning that both direct summands are actually submodules of $X \times R^{m-r}[\lambda]$, and not just isomorphic to submodules of $X \times R^{m-r}[\lambda]$) into its free and torsion submodules is

$$X \times R^{m-r}[\lambda] = \mathcal{T} \oplus \mathcal{F}$$

where

(4.3-65) $$\mathcal{F} = \{(\bar{x}(\lambda)+\mathrm{Im}\hat{T}(\lambda), v(\lambda)) \mid \binom{\bar{x}(\lambda)}{v(\lambda)} \equiv PL_2(\lambda)z(\lambda) \ (\mathrm{mod}\binom{\hat{T}(\lambda)}{\tilde{T}(\lambda)}) \text{ and } \partial\bar{x}_i < \nu_i \text{ for } i \in \underline{r}, \text{ for some } z(\lambda) \in R^{m-r}[\lambda]\}$$

and

$$(4.3\text{-}66) \qquad \mathcal{T} = \left\{(x, v_x(\lambda)) \,\middle|\, x = x(\lambda) + \operatorname{Im}\hat{T}(\lambda), \text{ and } \begin{pmatrix} x(\lambda) \\ v_x(\lambda) \end{pmatrix} = PL_1(\lambda)\bar{y}(\lambda), \right.$$

$$\left. \text{for some } \bar{y}(\lambda) \in R^r[\lambda] \text{ such that } \partial\bar{y}_i < \tilde{\nu}_i \,,\; i \in \underline{r}\right\}$$

Moreover, in the expression for $\mathcal{T}$, each of the polynomial vectors $x(\lambda)$ automatically satisfies $\partial x_i < \nu_i$, $i \in \underline{r}$; thus each $x(\lambda)$ is automatically a canonical representative for the coset $x(\lambda) + \operatorname{Im}\hat{T}(\lambda)$.

Proof: (i) From the definition of $Q_1(\lambda)$,

$$T(\lambda) = Q_1^{-1}(\lambda) \begin{pmatrix} T_1'(\lambda) \\ 0 \end{pmatrix}$$

so that

$$T'(\lambda) = (T_1(\lambda);\ 0) \begin{pmatrix} L_1'(\lambda) \\ L_2'(\lambda) \end{pmatrix} = T_1(\lambda)\ L_1'(\lambda)$$

Since $T_1(\lambda)$ is canonical, its columns are free generators for $\operatorname{Im} T_1(\lambda)$, and it follows that $L_1'(\lambda)$ is unique.

(ii) We have seen from (4.3-63) that $R^m[\lambda]/\mathcal{M}$ and $R^m[\lambda]/\operatorname{Im}\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix}$ are isomorphic; this isomorphism is given by

$$u(\lambda) + \mathcal{M} \mapsto Q_1(\lambda)\ u(\lambda) + \operatorname{Im}\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix}$$

But also, from (4.3-49), $R^m[\lambda]/\mathcal{M}$ and $X \times R^{m-r}[\lambda]$ are isomorphic, with the isomorphism given by

$$u(\lambda) + \mathcal{M} \mapsto Pu(\lambda) + \operatorname{Im}\begin{pmatrix} \hat{T}(\lambda) \\ \tilde{T}(\lambda) \end{pmatrix} = \begin{pmatrix} \bar{x}(\lambda) \\ v(\lambda) \end{pmatrix} + \operatorname{Im}\begin{pmatrix} \hat{T}(\lambda) \\ \tilde{T}(\lambda) \end{pmatrix}$$

$$\mapsto (\bar{x}(\lambda) + \operatorname{Im}\hat{T}(\lambda),\ v(\lambda))$$

where $\partial\bar{x}_i < \nu_i$ for $i \in \underline{r}$. Therefore, we have the following isomorphism:

$$\Gamma : R^m[\lambda]/\mathrm{Im}\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix} \tilde{\to} X \times R^{m-r}[\lambda]$$

$$: \binom{y(\lambda)}{z(\lambda)} + \mathrm{Im}\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix} \mapsto (\bar{x}(\lambda) + \mathrm{Im}\, \hat{T}(\lambda),\ v(\lambda))$$

where $\partial \bar{x}_i < \nu_i$ for $i \in \underline{r}$, and

$$\binom{\bar{x}(\lambda)}{v(\lambda)} \equiv P\, Q_1^{-1}(\lambda)\binom{y(\lambda)}{z(\lambda)} (\mathrm{mod}\binom{\hat{T}(\lambda)}{\tilde{T}(\lambda)})$$

$$\equiv P(L_1(\lambda)y(\lambda) + L_2(\lambda)z(\lambda))(\mathrm{mod}\binom{\hat{T}(\lambda)}{\tilde{T}(\lambda)})$$

It is easily seen that the torsion submodule of $R^m[\lambda]/\mathrm{Im}\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix}$ is just

$$\hat{T} = \{\binom{y(\lambda)}{0} + \mathrm{Im}\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix} \mid y(\lambda) \in R^r[\lambda]\}$$

and, a complementary free submodule of $R^m[\lambda]/\mathrm{Im}\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix}$ is

$$\hat{F} = \{\binom{0}{z(\lambda)} + \mathrm{Im}\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix} \mid z(\lambda) \in R^{m-r}[\lambda]\}$$

(there is no <u>unique</u> free submodule, in general). Therefore, it follows that complementary torsion and free submodules in $X \times R^{m-r}[\lambda]$ are the isomorphic images of the above ones in $R^m[\lambda]/\mathrm{Im}\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix}$:

(4.3-67) $$T = \Gamma\hat{T}$$

$$= \{(\bar{x}(\lambda) + \mathrm{Im}\, \hat{T}(\lambda), v(\lambda)) \mid \binom{\bar{x}(\lambda)}{v(\lambda)} \equiv PL_1(\lambda)y(\lambda)(\mathrm{mod}\binom{T(\lambda)}{T(\lambda)}),$$

$$\text{for some } y(\lambda) \in R^r[\lambda], \text{ and } \partial\bar{x}_i < \nu_i \text{ for } i \in \underline{r}\}$$

and

(4.3-68) $\mathcal{F} = \Gamma\hat{\mathcal{F}}$

$$= \{(\bar{x}(\lambda) + \text{Im } \hat{T}(\lambda), v(\lambda)) \mid \begin{pmatrix}\bar{x}(\lambda)\\ v(\lambda)\end{pmatrix} = PL_2(\lambda)z(\lambda)(\text{mod}\begin{pmatrix}\hat{T}(\lambda)\\ \tilde{T}(\lambda)\end{pmatrix}),$$

$$\text{for some } z(\lambda) \in R^{m-r}[\lambda], \text{ and } \partial\bar{x}_i < \nu_i \text{ for } i \in \underline{r}\}$$

Expressions (4.3-65) and (4.3-68) are in agreement; to show that (4.3-66) and (4.3-67) are also compatible, and that the concluding statement of (4.3-64) is valid, we need to argue as follows.

Since $T_2(\lambda)$ is a canonical matrix with degree integers $\{\tilde{\nu}_i, i \in \underline{r}\}$, it follows that each element of $\mathcal{T}$ can be expressed in the form

$$\begin{pmatrix}\bar{y}(\lambda)\\ 0\end{pmatrix} + \text{Im}\begin{pmatrix}T_2(\lambda)\\ 0\end{pmatrix}; \text{ for } \partial\bar{y}_i < \tilde{\nu}_i \text{ , } i \in \underline{r}$$

Thus, to show that (4.3-66) and (4.3-67) are in agreement, it is enough to show that

$$\begin{pmatrix}x(\lambda)\\ v_x(\lambda)\end{pmatrix} = PL_1(\lambda)\ \bar{y}(\lambda) \quad \underline{\text{and}} \quad \partial\bar{y}_i < \tilde{\nu}_i \text{ , } i \in \underline{r} \text{ imply } \partial x_i < \nu_i, \ i \in r$$

But this is true if

$$\partial(PL_1(\lambda))_{i,j} \leq \nu_i - \tilde{\nu}_j \text{ ; for } i,j \in \underline{r}$$

or, equivalently, if

(4.3-69) $$\partial(L_1(\lambda))_{m_i,j} \leq \nu_i - \tilde{\nu}_j \text{ ; for } i,j \in \underline{r}$$

To demonstrate (4.3-69), let $\{\hat{\nu}_i, i \in \underline{r}\}$ be the degree integers for the canonical matrix $T_1(\lambda)$ (of (4.3-63)). Then, since column m_i of $T'(\lambda)$ is of degree ν_i, and since $T'(\lambda) = T_1(\lambda)\ L_1'(\lambda)$, it follows from (4.3-18) that

$$\partial(L_1'(\lambda))_{j,m_i} \leq \nu_i - \hat{\nu}_j$$

Next, we observe that the transformation that canonicalizes $T_1'(\lambda)$:

$$T_2(\lambda) = T_1'(\lambda)\ Q_2(\lambda)$$

is accomplished by a <u>constant</u> matrix Q_2 which can be factored into a permutation matrix and an upper triangular matrix. Thus it follows easily that $\hat{\nu}_i = \tilde{\nu}_i$, for $i \in \underline{r}$, and (4.3-69) is verified. ∎

(4.3-70) <u>Remark</u>: Note that there is no <u>unique</u> free submodule of $X \times R^{m-r}[\lambda]$, unless $\partial T_2(\lambda) = 0$.

(4.3-71) <u>Remark</u>: The following informal observation results from (4.3-62) and (4.3-63). The isomorphisms

$$\mathcal{T} \cong \mathcal{S} \subset \mathcal{X} = R^r[\lambda]/\text{Im}\ \hat{T}(\lambda)$$

and

$$\mathcal{T} \cong R^r[\lambda]/\text{Im}\ T_2(\lambda)$$

imply that the invariant polynomials of $T_2(\lambda)$, $\{\phi_i(\lambda)\}$, and ~~those of $\hat{T}(\lambda)$, $\{\psi_i(\lambda)\}$, satisfy~~

$$\phi_i(\lambda) \text{ divides } \psi_i(\lambda), \text{ for } i \in \underline{r}$$

Having characterized the quotient module $R^m[\lambda]/\mathcal{M}$ in several ways, we now turn our attention to the canonical projection $\pi : R^m[\lambda] \to R^m[\lambda]/\mathcal{M}$. It is clear that this projection induces two morphisms, $\hat{\pi} : R^m[\lambda] \to X \times R^{m-r}[\lambda]$ (or, equivalently, $\hat{\pi} : R^m[\lambda] \to \mathcal{X} \times R^{m-r}[\lambda]$) and $\tilde{\pi} : R^m[\lambda] \to (R^r[\lambda]/\text{Im}\ T_2(\lambda)) \oplus R^{m-r}[\lambda]$, where the codomains of $\hat{\pi}$ and $\tilde{\pi}$ are defined in (4.3-49) and (4.3-63). We shall obtain explicit expressions for these two induced morphisms.

Clearly, the morphism $\hat{\pi} : R^m[\lambda] \to X \times R^{m-r}[\lambda]$ induced by the projection $\pi : R^m[\lambda] \to R^m[\lambda]/M$ is given by

$$\hat{\pi} : u(\lambda) \mapsto (x, v(\lambda))$$

where

$$x = \bar{x}(\lambda) + \operatorname{Im} \hat{T}(\lambda)$$

and where $\bar{x}(\lambda) \in R^r(\lambda)$ and $v(\lambda) \in R^{m-r}[\lambda]$ satisfy

$$\partial \bar{x}_i < \nu_i \ , \ i \in \underline{r}$$

and

$$P^{-1}\binom{\bar{x}(\lambda)}{v(\lambda)} \equiv u(\lambda) \ \ (\text{mod } T(\lambda))$$

In the above, of course, $T(\lambda)$ is the unique $m \times r$ canonical matrix such that $\operatorname{Im} T(\lambda) = M$, $\{\nu_i\}$ and $\{m_i\}$ are the associated sets of integers, P is an $m \times m$ permutation matrix such that $(Px)_i = x_{m_i}$ for $i \in \underline{r}$, and $PT(\lambda) = \binom{\hat{T}(\lambda)}{\tilde{T}(\lambda)}$.

While the above characterization of $\hat{\pi}$ is certainly correct, it is an algorithmic characterization in the sense that $u(\lambda)$ must be reduced modulo $T(\lambda)$ in order to determine x and $v(\lambda)$. It therefore is logical to seek a "closed form" expression for $\hat{\pi}$.

In order to achieve a closed form expression for $\hat{\pi} : R^m[\lambda] \to X \times R^{m-r}[\lambda]$, we shall replace the codomain of $\hat{\pi}$ by $\mathcal{X} \times R^{m-r}[\lambda]$ ($X \times R^{m-r}[\lambda]$ and $\mathcal{X} \times R^{m-r}[\lambda]$ are really the "same" module), and seek a matrix representation of $\hat{\pi} : R^m[\lambda] \to \mathcal{X} \times R^{m-r}[\lambda]$. The concept of a matrix representation of this morphism is defined as

(4.3-72) <u>Definition</u>: Let $\{v_{i,j} + \text{Im}\,\hat{T}(\lambda);\ j \in \underline{\nu_i}\ ,\ i \in \underline{r}\}$ be the basis of X (as in the proof to (4.3-52)) where

$$v_{i,j} = \lambda^{j-1} e_i\ ;\ j \in \underline{\nu_i}\ ,\ i \in \underline{r}\ \text{(not defined if } \nu_i = 0)$$

and where $\{e_i,\ i \in \underline{r}\}$ is the standard basis for R^r. Also, let $\{\hat{e}_i\ ,\ i \in \underline{m-r}\}$ be the standard basis for R^{m-r}; thus $\{\hat{e}_i\}$ is a set of free generators for $R^{m-r}[\lambda]$. Thus, for each $(x,v(\lambda)) \in X \times R^{m-r}[\lambda]$ there exist unique $\alpha_{i,j} \in R$, $j \in \underline{\nu_i}$ and $i \in \underline{r}$, and $\beta_i(\lambda) \in R[\lambda]$, $i \in \underline{m-r}$ such that

(4.3-72) $$(x,v(\lambda)) = \Big(\sum_{i\in\underline{r}} \sum_{j\in\underline{\nu_i}} \alpha_{i,j}(v_{i,j} + \text{Im}\,\hat{T}(\lambda)),\ \sum_{i\in\underline{m-r}} \beta_i(\lambda)\hat{e}_i\Big)$$

Then, if $u(\lambda) = (\sum_{i=0}^{k} \lambda^i u_i) \in R^m[\lambda]$, and if

$$\hat{\pi}\, u(\lambda) = (x,v(\lambda))$$

a <u>matrix representation</u> of $\hat{\pi} : R^m[\lambda] \to X \times R^{m-r}[\lambda]$ is a representation of the R-linear map

$$u(\lambda) \mapsto \left(\begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1,\nu_1} \\ \alpha_{21} \\ \vdots \\ \alpha_{r,\nu_r} \end{bmatrix} ,\ \begin{bmatrix} \beta_1(\lambda) \\ \beta_2(\lambda) \\ \vdots \\ \beta_{m-r}(\lambda) \end{bmatrix} \right) \in R^n \times R^{m-r}[\lambda]$$

where $n = \sum_{i\in\underline{r}} \nu_i$.

(4.3-73) **Remark**: Suppose that $\nu_i > 0$ for all $i \in \underline{r}$. If $\alpha \in R^n$ denotes the vector of the α_{ij}'s, and $\beta(\lambda) \in R^{m-r}[\lambda]$ denotes the polynomial vector of the $\beta_i(\lambda)$'s, then (4.3-72) may be written as

$$(x,v(\lambda)) = (S(\lambda)\alpha + \text{Im}\,\hat{T}(\lambda),\ \beta(\lambda))$$

where $S(\lambda)$ is the following r x n matrix:

$$S(\lambda)=\begin{pmatrix} 1 & \lambda\ldots\lambda^{\nu_1-1} & 0 & 0\ldots 0 & 0\ldots 0\ldots 0 \\ 0 & 0\ldots 0 & 1 & \lambda\ldots\lambda^{\nu_2-1} & 0\ldots 0\ldots 0 \\ \vdots & & & & \\ 0 & 0\ldots & & & 0\ 1\ \lambda\ldots\lambda^{\nu_r-1} \end{pmatrix}$$

Therefore, a matrix representation for $\hat{\pi} : R^m[\lambda] \to X \times R^{m-r}[\lambda]$ will provide us with a closed form expression for reducing any $u(\lambda)$ modulo $T(\lambda)$. The unique remainder will be just

$$r(\lambda) = P^{-1}\binom{S(\lambda)\alpha}{\beta(\lambda)}$$

where $(\alpha,\beta(\lambda))$ is determined from $\hat{\pi}\,u(\lambda)$ as in (4.3-72).

In the case where $\nu_1 = \ldots = \nu_{r_1} = 0$, the elements m_1, $m_2,\ldots m_{r_1}$ of any remainder must be zero. Thus, in this case the result of reducing $u(\lambda)$ modulo $T(\lambda)$ is just

$$r(\lambda) = P^{-1}\begin{pmatrix} 0 \\ S(\lambda) \\ \beta(\lambda) \end{pmatrix} \begin{matrix} \}r_1 \text{ elements} \\ \\ \\ \end{matrix}$$

where now $S(\lambda)$ is $(r-r_1) \times n$, the r_1 rows corresponding to zero ν_i's having been deleted.

In determining the matrix representation for $\hat{\pi} : \mathcal{X} \times R^{m-r}[\lambda]$, we shall make use of

(4.3-74) __Lemma:__ Let the insertion of R^m into $R^m[\lambda]$ be denoted by $\iota : R^m \to R^m[\lambda]$. Then if

$$u(\lambda) = (\sum_{j=0}^{k} \lambda^j u_j) \in R^m[\lambda]$$

the element $\hat{\pi}\, u(\lambda) \in \mathcal{X} \times R^{m-r}[\lambda]$ may be computed as

$$\hat{\pi}\, u(\lambda) = \sum_{j=0}^{k} \lambda^j (\hat{\pi}\cdot\iota)(u_j)$$

Proof: This follows easily from the fact that $\hat{\pi}$ is an $R[\lambda]$-module morphism. ■

The significance of (4.3-74) is that it is now possible to "build up" the morphism $\hat{\pi}$ from the R-linear map $(\hat{\pi}\cdot\iota) : R^m \to \mathcal{X} \times R^{m-r}[\lambda]$. We next determine the matrix representation for this R-linear map.

(4.3-75) __Lemma:__ The matrix representation for the R-linear map $(\hat{\pi}\ \iota) : R^m \to \mathcal{X} \times R^{m-r}[\lambda]$ is

$$(\hat{\pi}\cdot\iota) : u \mapsto (\alpha, \beta(\lambda))$$

where $\alpha \in R^n$ and $\beta(\lambda) \in R^{m-r}[\lambda]$ are given by

$$\alpha = B\ u$$

and

$$\beta(\lambda) = D\ u$$

where $B \in R^{n \times m}$ and $D \in R^{(m-r) \times m}$ are defined as follows.

Let $T(\lambda)$, the canonical matrix associated with M, be partitioned canonically as

$$T(\lambda) = (T_1(\lambda);\ T_2(\lambda);\ \ldots\ T_\alpha(\lambda))$$

where $T_i(\lambda)$ is $m \times r_i$ and $\partial T_i(\lambda) = \eta_i$. Let P be the $m \times m$ permutation matrix such that $PT(\lambda) = \begin{pmatrix} \hat{T}(\lambda) \\ \tilde{T}(\lambda) \end{pmatrix}$, and define p_i to be row i of P.

Then, <u>when $\eta_1 > 0$,</u>

(4.3-76)
$$B = \begin{bmatrix} \left.\begin{matrix} p_1 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} \nu_1 \text{ rows} \\ \left.\begin{matrix} p_2 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} \nu_2 \text{ rows} \\ \begin{matrix} p_3 \\ 0 \\ \vdots \end{matrix} \quad \vdots \\ \left.\begin{matrix} p_r \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} \nu_r \text{ rows} \end{bmatrix}$$

and

(4.3-77)
$$D = \begin{pmatrix} p_{r+1} \\ p_{r+2} \\ \vdots \\ p_m \end{pmatrix}$$

Otherwise, when $\eta_1 = 0$, B and D are given in terms of P and $T_1(\lambda)$ as follows. Partition the (constant) matrix PT_1 as

$$P\,T_1 = \begin{pmatrix} \hat{T}_1 \\ \tilde{T}_1 \end{pmatrix} \}\, r_1 \text{ rows}$$

and define $\tilde{t}_{1i}$ to be row i of $\tilde{T}_1$. Also partition P as

$$P = \begin{pmatrix} \hat{P}_1 \\ \tilde{P}_1 \end{pmatrix} \}\, r_1 \text{ rows}$$

and define $\tilde{p}_{1i}$ to be row i of $\tilde{P}_1$. Then

$$B = \begin{bmatrix} \tilde{p}_{11} - \tilde{t}_{11}\,\hat{P}_1 \\ 0 \\ \vdots \\ 0 \\ \tilde{p}_{12} - \tilde{t}_{12}\,\hat{P}_1 \\ 0 \\ \vdots \\ 0 \\ \tilde{p}_{13} - \tilde{t}_{13}\,\hat{P}_1 \\ 0 \\ \vdots \\ \tilde{p}_{1,r-r_1} - \tilde{t}_{1,r-r_1}\,\hat{P}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} \nu_{r_1+1} \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} \nu_{r_1+2} \text{ rows} \\ \vdots \\ \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} \nu_{r} \text{ rows} \end{matrix} \tag{4.3-78}$$

and

$$D = \begin{pmatrix} \tilde{p}_{1,r-r_1+1} - \tilde{t}_{1,r-r_1+1}\,\hat{P}_1 \\ \vdots \\ \tilde{p}_{1,m-r_1} - \tilde{t}_{1,m-r_1}\,\hat{P}_1 \end{pmatrix} \tag{4.3-79}$$

Proof: To determine the matrix representation for $\hat{\pi}\cdot\iota$, it suffices to consider an arbitrary $u \in R^m$, reduce P u modulo $PT(\lambda) = \binom{\hat{T}(\lambda)}{\tilde{T}(\lambda)}$, and then express the resulting remainder in terms of α and $\beta(\lambda)$, as in (4.3-73). Thus, it is enough to show that the modulo reduction of P u is

$$P\,u = PT(\lambda)\,q(\lambda) + \begin{pmatrix} 0 \\ S(\lambda)\,B\,u \\ Du \end{pmatrix} \begin{matrix} \}r_1 \text{ elements if } \eta_1 = 0 \\ \\ \\ \end{matrix}$$

where $S(\lambda)$ is as in (4.3-73).

To this end, first consider the case $\eta_1 > 0$. Then it follows from (4.3-26) that, since $\partial(Pu) = 0$, $q(\lambda) = 0$. Thus the remainder upon reducing Pu modulo $PT(\lambda)$ is just Pu. But then

$$\binom{S(\lambda)\alpha}{\beta(\lambda)} = Pu$$

and it easily follows that $\alpha = Bu$ and $\beta(\lambda) = Du$, where B and D are as in (4.3-76) and (4.3-77).

On the other hand, if $\eta_1 = 0$, then $q(\lambda) \neq 0$. In fact, since $T_1(\lambda)$ is canonical and of degree zero, it follows that

$$P\,T_1(\lambda) = \binom{I}{\tilde{T}_1} \}r_1 \text{ rows}$$

whence it easily follows from (4.3-26) that

$$q(\lambda) = \begin{pmatrix} \hat{P}_1 u \\ 0 \end{pmatrix} \begin{matrix} \}r_1 \text{ elements} \\ \\ \end{matrix}$$

Therefore, the remainder upon reducing Pu modulo $PT(\lambda)$ is

$$\begin{pmatrix} 0 \\ S(\lambda)\alpha \\ \beta(\lambda) \end{pmatrix} = Pu - \binom{I}{\tilde{T}_1} \hat{P}_1 u$$

$$= \begin{pmatrix} 0 \\ \tilde{P}_1 u - \tilde{T}_1 \hat{P}_1 u \end{pmatrix} \begin{matrix} \} r_1 \text{ elements} \\ \\ \end{matrix}$$

Consequently, α and $\beta(\lambda)$ must satisfy

$$\binom{S(\lambda)\alpha}{\beta(\lambda)} = \tilde{P}_1 u - \tilde{T}_1 \hat{P}_1 u$$

and it is easily seen that $\alpha = Bu$, $\beta(\lambda) = Du$, for B and D given by (4.3-78) and (4.3-79). ■

We now use (4.3-74) and (4.3-75) to deduce the matrix representation for $\hat{\pi} : R^m[\lambda] \to X \times R^{m-r}[\lambda]$.

(4.3-80) **Theorem:** The matrix representation for the $R[\lambda]$-morphism $\hat{\pi} : R^m[\lambda] \to X \times R^{m-r}[\lambda]$ is

$$\hat{\pi} : u(\lambda) \mapsto (\alpha, \beta(\lambda))$$

where $\alpha \in R^n$ and $\beta(\lambda) \in R^{m-r}[\lambda]$ are as follows. If

$$u(\lambda) = \begin{pmatrix} u_1(\lambda) \\ u_2(\lambda) \\ \vdots \\ u_m(\lambda) \end{pmatrix} \in R^m[\lambda]$$

then

$$\alpha = \sum_{\ell \in \underline{m}} u_\ell(A)\, b_\ell$$

and

$$\beta(\lambda) = Du(\lambda) - \tilde{T}(\lambda)H \sum_{j=0}^{k-1} \lambda^j \sum_{\ell \varepsilon \underline{m}} u_\ell^{(j)}(A)\ b_\ell$$

In the above, $k = \partial u$; A is the n x n matrix defined in (4.3-54); b_ℓ is column ℓ of B, as defined in (4.3-76) and (4.3-78); D is as defined in (4.3-77) and (4.3-79); H is as defined in (4.3-53); and $\tilde{T}(\lambda)$ is the (m-r) x r matrix such that $\binom{\hat{T}(\lambda)}{\tilde{T}(\lambda)} = PT(\lambda)$. The polynomials $u_\ell^{(j)}(\lambda) \in R[\lambda]$ are defined as $u_\ell^{(j)}(\lambda) = \lambda^{k-j-1}u_{\ell,k} + \lambda^{k-j-2}u_{\ell,k-1} + \ldots + u_{\ell,j+1}$, $0 \leq j \leq k-1$

where

$$u_\ell(\lambda) = \lambda^k\ u_{\ell,k} + \lambda^{k-1}\ u_{\ell,k-1} + \ldots + u_{\ell,0}$$

Proof: If $v \in R^m$, then by (4.3-75), $v \to (Bv,Dv)$. Therefore, if

$$v(\lambda) = \sum_{i=0}^{k} \lambda^i\ v_i \in R^m[\lambda]$$

it follows from (4.3-74) that

$$v(\lambda) \mapsto \sum_{i=0}^{k} \lambda^i \cdot (Bv_i, Dv_i)$$

But, from (4.3-52), the action of λ^i on a pair $(\alpha,\beta(\lambda))$ is

$$\lambda^i \cdot (\alpha,\beta(\lambda)) = (A^i\alpha, \lambda^i\beta(\lambda) - \tilde{T}(\lambda)H\sum_{j=0}^{i-1} \lambda^j A^{i-j-1}\alpha)$$

Consequently,

$$v(\lambda) \mapsto \sum_{i=0}^{k} (A^i Bv_i, \lambda^i Dv_i - \tilde{T}(\lambda)H\sum_{j=0}^{i-1} \lambda^j A^{i-j-1} Bv_i)$$

$$= (\sum_{i=0}^{k} A^i Bv_i, \sum_{i=0}^{k} \lambda^i Dv_i - \tilde{T}(\lambda)H\sum_{j=0}^{k-1} \lambda^j \sum_{i=j+1}^{k} A^{i-j-1} Bv_i)$$

Finally, if

$$\sum_{i=0}^{k} \lambda^i v_i = \begin{pmatrix} u_1(\lambda) \\ u_2(\lambda) \\ \vdots \\ u_m(\lambda) \end{pmatrix}$$

then it is easily seen that

$$\sum_{i=0}^{k} A^i B v_i = \sum_{\ell \epsilon \underline{m}} \sum_{i=0}^{k} u_{\ell,i} \lambda^i b_\ell$$

$$= \sum_{\ell \epsilon \underline{m}} u_\ell(A) b_\ell$$

Similarly,

$$\sum_{i=j+1}^{k} A^{i-j-1} B v_i = \sum_{\ell \epsilon \underline{m}} \sum_{i=j+1}^{k} u_{\ell,i} A^i b_\ell$$

$$= \sum_{\ell \epsilon \underline{m}} u_\ell^{(j)}(A) b_\ell$$

and the proof is complete. ■

(4.3-81) __Remark__: It now follows from (4.3-73) that the closed form expression for the remainder resulting from reducing $u(\lambda)$ modulo $T(\lambda)$ is

$$r(\lambda) = P^{-1} \begin{pmatrix} 0 \\ S(\lambda) \sum_{\ell \epsilon \underline{m}} u_\ell(A) b_\ell \\ Du(\lambda) - \tilde{T}(\lambda) H \sum_{j=0}^{k-1} \lambda^j \sum_{\ell \epsilon \underline{m}} u_\ell^{(j)}(A) b_\ell \end{pmatrix} \begin{matrix} \}r_1 \text{ elements} \\ \\ \\ \end{matrix}$$

where $\nu_1 = \ldots = \nu_{r_1} = 0$.

The matrix representation for the morphism $\pi: R^m[\lambda] \to (R^r[\lambda]/\text{Im } T_2(\lambda)) \oplus R^{m-r}[\lambda]$ induced by the projection $\pi : R^m[\lambda] \to R^m[\lambda]/M$ is defined similarly; we shall merely sketch through the development.

Let $\bar{P}$ be the r x r permutation matrix such that $(\bar{P}x)_i = x_{\tilde{m}_i}$, where $\{\tilde{m}_i, i \in \underline{r}\}$ are the row integers associated with $T_2(\lambda)$. Then $\bar{P}T_2(\lambda)$ is a canonical matrix (r x r) with associated degree integers $\{\tilde{\nu}_i, i \in \underline{r}\}$, and an $\tilde{n}$ x $\tilde{n}$ matrix $\tilde{A}(\tilde{n} = \sum_{i \in \underline{r}} \tilde{\nu}_i)$ may be defined from the elements of $\bar{P}\, T_2(\lambda)$ just as A was defined from the elements of $\hat{T}(\lambda)$ (in (4.3-52)). Defining a basis $\{\tilde{v}_{i,j} + \text{Im } T_2(\lambda);\ j \in \underline{\tilde{\nu}_i},\ i \in \underline{r}\}$ in $R^r[\lambda]/\text{Im } T_2(\lambda)$ with

$$\tilde{v}_{i,j} = \lambda^{j-1} e_i \ ;\ j \in \underline{\tilde{\nu}_i}\ ,\ i \in \underline{r} \text{ (not defined if } \tilde{\nu}_i = 0)$$

it then follows that elements $(x,v(\lambda)) \in (R^r[\lambda]/\text{Im } T_2(\lambda)) \oplus R^{m-r}[\lambda]$ may be represented by a pair $(\tilde{\alpha},\tilde{\beta}(\lambda))$, $\tilde{\alpha} \in R^{\tilde{n}}$ and $\tilde{\beta}(\lambda) \in R^{m-r}[\lambda]$; moreover, the action of λ on such a pair is

$$\lambda(\tilde{\alpha},\tilde{\beta}(\lambda)) = (\tilde{A}\tilde{\alpha},\lambda\beta(\lambda))$$

Finally, we define an $\tilde{n}$ x m matrix $\tilde{B}$ from the m x r canonical matrix $\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix}$ and the m x m permutation matrix $\begin{pmatrix} \bar{P} & 0 \\ 0 & I \end{pmatrix}$ just as B was determined in (4.3-75) from $T(\lambda)$ and P.

(4.3-82) **Theorem:** Let $Q_1(\lambda)$ be the m x m unimodular matrix determined in the proof to (4.3-63); let

$$Q_1(\lambda) = \sum_{i=0}^{k} \lambda^i Q_i = \begin{pmatrix} Q_{11}(\lambda) \\ Q_{12}(\lambda) \end{pmatrix} \}\text{r rows}$$

Also, let the $\tilde{n}$ x $\tilde{n}$ and $\tilde{n}$ x m matrices $\tilde{A}$ and $\tilde{B}$ be determined as in the preceding paragraph. Then the matrix representation for $\tilde{\pi} : R^m[\lambda] \to (R^r[\lambda]/\text{Im } T_2(\lambda)) \oplus R^{m-r}[\lambda]$ is

$$u(\lambda) \mapsto \left(\sum_{\ell\in\underline{m}} u_\ell(\tilde{A})\left(\sum_{j=0}^{k} \tilde{A}^i \tilde{B} q_{i,\ell}\right), Q_{12}(\lambda)\ u(\lambda)\right)$$

where

$$Q_i = (q_{i,1}; q_{i,2}; \ldots q_{i,m}) , i \in \underline{k}$$

Proof: Apply (4.3-80), first noting that

$$\tilde{\pi} : R^m[\lambda] \to (R^r[\lambda]/\text{Im } T_2(\lambda)) \oplus R^{m-r}[\lambda]$$

$$: u(\lambda) \to Q_1(\lambda)\ u(\lambda) + \text{Im}\begin{pmatrix} T_2(\lambda) \\ 0 \end{pmatrix} \quad \blacksquare$$

This concludes the treatment of explicit characterizations of $R^m[\lambda]/M$ and $\pi : R^m[\lambda] \to R^m[\lambda]/M$. In Section 4.4 we shall see that these characterizations lend themselves readily to system representations.

4.3.3 The Lattice of Canonical Matrices

In (4.3-33) we proved that for every nonzero submodule $M \subset R^m[\lambda]$ of rank r there exists a unique m x r canonical matrix $T(\lambda)$ such that $\text{Im } T(\lambda) = M$. This establishes a map $\hat{T} : \hat{L} \to \hat{C}$ where

$$\hat{L} = \{M \subset R^m[\lambda] \mid M \text{ is a nonzero submodule}\}$$

and

$$\hat{C} = \{T(\lambda) \mid T(\lambda) \text{ is an m x r canonical matrix, for some } 1 \le r \le m\}$$

Clearly, $\hat{T}$ is injective; also, it is easy to see that $\text{Im}\,\hat{T} = \hat{C}$. Thus $\hat{T}$ is a bijection.

If we augment $\hat{L}$ and $\hat{C}$ by adding the zero submodule to $\hat{L}$:

(4.3-83) $\mathcal{L} = \hat{\mathcal{L}} \bigcup \{0\}$

$= \{\mathcal{M} \subset R^m[\lambda] \,|\, \mathcal{M} \text{ is a submodule}\}$

and the m x 1 zero matrix to $\mathcal{C}$:

(4.3-84) $\mathcal{C} = \hat{\mathcal{C}} \bigcup \{\text{m x 1 zero matrix}\}$

$= \{T(\lambda) \,|\, \text{either } T(\lambda) = 0(m \times 1), \text{ or } T(\lambda) \text{ is an } m \times r$ canonical matrix, for some $1 \leq r \leq m\}$

it is clear that $\hat{T}$ extends to a bijection

$$T : \mathcal{L} \tilde{\rightarrow} \mathcal{C}$$

In this section we shall show that $\mathcal{C}$ admits the structure of a lattice (e.g., [53, Chapter 14]) and that T is a lattice isomorphism. We begin with the following definitions:

(4.3-85) <u>Definition</u>: Let $A(\lambda)$ and $B(\lambda)$ be two polynomial matrices. Then

(i) If there exists a polynomial matrix $C(\lambda)$ such that $A(\lambda) = B(\lambda)\, C(\lambda)$, then $B(\lambda)$ is called a left divisor (l.d.) of $A(\lambda)$, and $A(\lambda)$ is called a right multiple (r.m.) of $B(\lambda)$. When such a $C(\lambda)$ exists, we also say that $B'(\lambda)$ is a right divisor (r.d.) of $A'(\lambda)$, and $A'(\lambda)$ is a left multiple (l.m.) of $B'(\lambda)$.

(ii) If $A(\lambda)$ and $B(\lambda)$ have the same number of rows, then a greatest common left divisor (g.c.l.d.) of $A(\lambda)$ and $B(\lambda)$ is any polynomial matrix $D(\lambda)$ which is a l.d. of both $A(\lambda)$ and $B(\lambda)$, and which is a r.m. of every other common l.d. of both $A(\lambda)$ and $B(\lambda)$.

(iii) If $A(\lambda)$ and $B(\lambda)$ have the same number of rows, then a least common right multiple (l.c.r.m.) of $A(\lambda)$ and $B(\lambda)$ is any polynomial matrix $M(\lambda)$ which is a r.m. of both $A(\lambda)$ and $B(\lambda)$, and which is a l.d. of every other common r.m. of $A(\lambda)$ and $B(\lambda)$.

(iv) Similarly, we can define greatest common right divisors and least common left multiples.

We now wish to show that for every pair of matrices in $\mathcal{C}$ there exists a unique g.c.l.d. and a unique l.c.r.m.. This we do by proving

(4.3-86) Lemma: Let $T_1(\lambda)$ and $T_2(\lambda)$ be two arbitrary elements of $\mathcal{C}$, where $\mathcal{C}$ is given by (4.3-84). Then

(i) There exists exactly one polynomial matrix, $T_3(\lambda)$, which is both a g.c.l.d. of $T_1(\lambda)$ and $T_2(\lambda)$, and an element of $\mathcal{C}$. In fact, $T_3(\lambda)$ is the unique element of $\mathcal{C}$ such that

$$\text{Im } T_3(\lambda) = \text{Im}(T_1(\lambda);\ T_2(\lambda))$$

(ii) There exists exactly one polynomial matrix, $T_4(\lambda)$, which is both a l.c.r.m. of $T_1(\lambda)$ and $T_2(\lambda)$, and an element of $\mathcal{C}$. This matrix is the unique element of $\mathcal{C}$ such that

$$\text{Im } T_4(\lambda) = \text{Im } T_1(\lambda) \bigcap \text{Im } T_2(\lambda)$$

Proof: (i) From (4.3-85) it is easily seen that $D(\lambda)$ is a common left divisor of $T_1(\lambda)$ and $T_2(\lambda)$ if and only if

$$\text{Im } T_i(\lambda) \subset \text{Im } D(\lambda); \text{ for } i = 1,2$$

Clearly, $D(\lambda) = T_3(\lambda)$ satisfies these relations. Moreover, for any $D(\lambda)$ satisfying these relations,

$$\text{Im } T_3(\lambda) \subset \text{Im } D(\lambda)$$

whence $T_3(\lambda)$ is a g.c.l.d. of $T_1(\lambda)$ and $T_2(\lambda)$. If there is another element of $\mathcal{C}$, say $\hat{T}_3(\lambda)$, that is also a g.c.l.d. of $T_1(\lambda)$ and $T_2(\lambda)$, then it follows that Im $\hat{T}_3(\lambda)$ = Im $T_3(\lambda)$, whence $\hat{T}_3(\lambda) = T_3(\lambda)$.

(ii) Similarly, $M(\lambda)$ is a common right multiple of $T_1(\lambda)$ and $T_2(\lambda)$ if and only if

$$\text{Im } M(\lambda) \subset \text{Im } T_i(\lambda) \quad ; \ i = 1,2$$

$M(\lambda) = T_4(\lambda)$ satisfies these relations; and any other $M(\lambda)$ satisfying them must satisfy

$$\text{Im } M(\lambda) \subset \text{Im } T_4(\lambda)$$

Thus $M(\lambda)$ is a l.c.r.m. of $T_1(\lambda)$ and $T_2(\lambda)$. The uniqueness of $M(\lambda)$ follows easily. ■

We now define the operations of "meet" and "join" in $\mathcal{C}$:

(4.3-87) <u>Definition</u>: For any two elements $T_1(\lambda)$ and $T_2(\lambda)$ of $\mathcal{C}$, we establish the following notation:

(i) $T_1(\lambda) \leq T_2(\lambda)$ if and only if $T_2(\lambda)$ is a left divisor of $T_1(\lambda)$.

(ii) The unique element $T_3(\lambda) \in \mathcal{C}$ satisfying $\text{Im}\, T_3(\lambda) = \text{Im}(T_1(\lambda);\, T_2(\lambda))$ is defined as

$$T_3(\lambda) = T_1(\lambda) \vee T_2(\lambda)$$

(iii) The unique element $T_4(\lambda) \in \mathcal{C}$ satisfying $\text{Im}\, T_4(\lambda) = \text{Im}\, T_1(\lambda) \cap \text{Im}\, T_2(\lambda)$ is defined as

$$T_4(\lambda) = T_1(\lambda) \wedge T_2(\lambda)$$

We can now state the following result, which establishes $\mathcal{C}$ as a lattice isomorphic to L:

(4.3-88) Theorem: $\mathcal{C}$ admits the structure of a lattice, partially ordered by the relation $\leq$, and with the binary operations meet, $\wedge$, and join, $\vee$, as defined in (4.3-87). Moreover, the bijection $T : L \to \mathcal{C}$ is a lattice isomorphism, where the lattice structure in L is the obvious structure: partially ordered by $\subset$, and with the binary operations $\cap$ (meet) and $+$ (join).

Proof: One needs only to verify that $M_1 \subset M_2$ implies $T(M_1) \leq T(M_2)$, that $T(M_1+M_2) = T(M_1) \vee T(M_2)$, and that $T(M_1 \cap M_2) = T(M_1) \wedge T(M_2)$ for all M_1, $M_2 \in L$. But these are obvious. ∎

(4.3-89) Remark: If we wished to compute the elements $T_1(\lambda) \vee T_2(\lambda)$ and $T_1(\lambda) \wedge T_2(\lambda)$, we could proceed as follows. First, if $Q(\lambda)$ is a unimodular matrix such that $(T_1(\lambda);\, T_2(\lambda))\, Q(\lambda)$ is canonical, then it is clear that

$$T_1(\lambda) \vee T_2(\lambda) = (T_1(\lambda);\, T_2(\lambda))\, Q(\lambda)$$

Note that $T_1(\lambda) \bigvee T_2(\lambda)$ can be computed via algorithm (4.3-43).

Computing $T_1(\lambda) \bigwedge T_2(\lambda)$ is only slightly more difficult. Essentially, we must find a $T_4(\lambda) \in \mathcal{C}$ such that

$$\text{Im } T_4(\lambda) = \text{Im } T_1(\lambda) \bigcap \text{Im } T_2(\lambda)$$

But if $T_1(\lambda)$ is $m \times r_1$ and $T_2(\lambda)$ is $m \times r_2$, then $u(\lambda) \in \text{Im } T_1(\lambda) \bigcap \text{Im } T_2(\lambda)$ if and only if there exist $x(\lambda) \in R^{r_1}[\lambda]$, $y(\lambda) \in R^{r_2}[\lambda]$ such that

$$u(\lambda) = T_1(\lambda)\ x(\lambda) = T_2(\lambda)\ y(\lambda)$$

Thus, to find all $u(\lambda) \in \text{Im } T_1(\lambda) \bigcap \text{Im } T_2(\lambda)$, it suffices to find all $\binom{x(\lambda)}{y(\lambda)} \in \text{Ker}(T_1(\lambda);\ -T_2(\lambda))$ and use:

$$\text{Im } T_1(\lambda) \bigcap \text{Im } T_2(\lambda) = \{T_1(\lambda)x(\lambda) \mid \binom{x(\lambda)}{-y(\lambda)} \in \text{Ker}(T_1(\lambda);\ T_2(\lambda)) \text{ for some } y(\lambda) \in R^{r_2}[\lambda]\}$$

It is easily shown that $\text{Ker}(T_1(\lambda);\ T_2(\lambda))$ is a free submodule of $R^{r_1+r_2}[\lambda]$. Moreover, if $Q(\lambda)$ is a unimodular matrix that canonicalizes $(T_1(\lambda);\ T_2(\lambda))$, and if $Q(\lambda)$ is partitioned as

$$Q(\lambda) = \begin{pmatrix} Q_{11}(\lambda) & Q_{12}(\lambda) \\ Q_{21}(\lambda) & \underbrace{Q_{22}(\lambda)}_{r_1 + r_2 - \text{rank}(T_1(\lambda);\ T_2(\lambda)) \text{ columns}} \end{pmatrix} \begin{matrix} \}r_1 \text{ rows} \\ \}r_2 \text{ rows} \end{matrix}$$

then a simple argument reveals that the columns of

$$\begin{pmatrix} Q_{12}(\lambda) \\ Q_{22}(\lambda) \end{pmatrix}$$

are free generators for $\text{Ker}(T_1(\lambda);\ T_2(\lambda))$. Therefore,

$$\text{Im } T_1(\lambda) \bigcap \text{Im } T_2(\lambda) = \text{Im}(T_1(\lambda)\ Q_{12}(\lambda))$$

Therefore, it follows that one way to compute $T_1(\lambda) \bigwedge T_2(\lambda)$ is to canonicalize $(T_1(\lambda);\ T_2(\lambda))$, and in the process, compute $Q(\lambda)$. Then $T_1(\lambda)\ Q_{12}(\lambda)$ is canonicalized to produce $T_1(\lambda) \bigwedge T_2(\lambda)$. This method requires two uses of algorithm (4.3-43).

We next derive a condition that must be satisfied if two canonical matrices in $\mathcal{C}$ are to be related as $T(\lambda) \leq \hat{T}(\lambda)$.

(4.3-90) <u>Lemma</u>: Let $T(\lambda)$, $\hat{T}(\lambda) \in \mathcal{C}$; and let the ranks and the sets of degree integers be r and $\hat{r}$, and $\{\nu_i,\ i \in \underline{r}\}$ and $\{\hat{\nu}_i,\ i \in \underline{\hat{r}}\}$, respectively. Then $T(\lambda) \leq \hat{T}(\lambda)$ only if

(i) $\qquad r \leq \hat{r}$

(ii) $\qquad \nu_i \geq \hat{\nu}_i$, for $i \in \underline{r}$

Proof: Since $T(\lambda) \leq \hat{T}(\lambda)$ if and only if $\text{Im } T(\lambda) \subset \text{Im } \hat{T}(\lambda)$, (i) follows trivially,

To prove (ii), note that $T(\lambda) \leq \hat{T}(\lambda)$ implies

(4.3-91) $$T(\lambda) = \hat{T}(\lambda)\ Q(\lambda)$$

for some $Q(\lambda) \in R^{\hat{r} \times \hat{r}}[\lambda]$. Now suppose that $\nu_k < \hat{\nu}_k$ for some $k \in \underline{r}$. Then, since the integers $\{\nu_i\}$ satisfy

$$\nu_i \leq \nu_k\ , \text{ for all } i \in \underline{k}$$

it follows that

$$\nu_i < \hat{\nu}_k\ , \text{ for all } i \in \underline{k}$$

Let $\{t_i(\lambda)\}$, $\{\hat{t}_i(\lambda)\}$, and $\{q_i(\lambda)\}$ be the columns of $T(\lambda)$, $\hat{T}(\lambda)$, and $Q(\lambda)$, respectively, in (4.3-91). Then, from (4.3-91),

$$t_i(\lambda) = \hat{T}(\lambda)\, q_i(\lambda) \quad , \; i \in \underline{k}$$

But, since $\nu_i < \hat{\nu}_k$, it follows from (4.3-18) that

$$t_i(\lambda) \in \mathrm{Im}(\hat{t}_1(\lambda); \hat{t}_2(\lambda); \ldots \hat{t}_{k-1}(\lambda)) \quad , \text{ for all } i \in \underline{k}$$

whence it follows that

$$\mathrm{rank}(t_1(\lambda); t_2(\lambda); \ldots t_k(\lambda)) \leq k-1$$

in contradiction to the fact that $T(\lambda)$ is canonical. ∎

A second result, which in a sense complements (4.3-90) is the following

(4.3-91) **Lemma**: Let $T(\lambda)$, $\hat{T}(\Lambda) \in \mathcal{C}$; and let the ranks and the sets of degree integers be r and $\hat{r}$, and $\{\nu_i, \; i \in \underline{r}\}$ and $\{\hat{\nu}_i, \; i \in \underline{\hat{r}}\}$, respectively. Then, if $r = \hat{r}$, if $\nu_i = \hat{\nu}_i$ for $i \in \underline{r}$, and if $T(\lambda) \leq \hat{T}(\lambda)$, it necessarily follows that $T(\lambda) = \hat{T}(\lambda)$.

Proof: Since $r = \hat{r}$ and $T(\lambda) \leq \hat{T}(\lambda)$, there is an $r \times r$ $Q(\lambda)$ such that $T(\lambda) = \hat{T}(\lambda)\, Q(\lambda)$. Since $\nu_i = \hat{\nu}_i$ for all $i \in \underline{r}$, it follows that the canonical partitioning for $T(\lambda)$ is conformable with that of $\hat{T}(\lambda)$; partitioning $Q(\lambda)$ conformably with these canonical partitionings, we have

$$(T_1(\lambda); \ldots; T_\alpha(\lambda)) = (\hat{T}_1(\lambda); \ldots; \hat{T}_\alpha(\lambda)) \begin{pmatrix} Q_{11}(\lambda) & \cdots & Q_{1\alpha}(\lambda) \\ \vdots & & \\ Q_{\alpha 1}(\lambda) & \cdots & Q_{\alpha\alpha}(\lambda) \end{pmatrix}$$

But then, using (4.3-18),

$$Q_{ij}(\lambda) = 0, \text{ if } \partial T_j < \partial \hat{T}_i$$

and

$$\partial Q_{ii} = 0, \text{ for all } i \in \underline{\alpha}$$

Thus it follows that det $Q(\lambda)$ is a constant. But det $Q(\lambda) \neq 0$, since rank $T(\lambda)$ = rank $\hat{T}(\lambda)$. Thus $Q(\lambda)$ is unimodular, and consequently Im $T(\lambda)$ = Im $\hat{T}(\lambda)$. But this implies that $T(\lambda) = \hat{T}(\lambda)$, and the proof is complete. ∎

In (4.3-87) we defined the partial ordering $\leq$ and the binary operations and in the lattice C. We now extend these definitions to include all polynomial matrices with m rows.

(4.3-92) <u>Definition</u>: Let P_m denote the set of polynomial matrices with m rows:

$$P_m = \{P(\lambda) \,|\, P(\lambda) \in R^{m \times s}[\lambda] \text{ for some } s \geq 1\}$$

Then we define the relation $\leq$ on P_m and the functions $\bigwedge: P_m \times P_m \rightarrow C$ and $\bigvee: P_m \times P_m \rightarrow C$ as

(i) $P_1(\lambda) \leq P_2(\lambda)$ if and only if $T(\text{Im } P_1(\lambda)) \leq T(\text{Im } P_2(\lambda))$

(ii) $P_i(\lambda) \bigwedge P_2(\lambda) \triangleq T(\text{Im } P_1(\lambda)) \bigwedge T(\text{Im } P_2(\lambda))$

(iii) $P_1(\lambda) \bigvee P_2(\lambda) \triangleq T(\text{Im } P_1(\lambda)) \bigvee T(\text{Im } P_2(\lambda))$

where $T : L \tilde{\rightarrow} C$ is the lattice isomorphism.

Our rationale for extending these definitions to P_m is that $P_1(\lambda) \bigvee P_2(\lambda)$ gives us a canonical g.c.l.d. of $P_1(\lambda)$ and $P_2(\lambda)$, while $P_1(\lambda) \bigwedge P_2(\lambda)$ gives us their canonical l.c.r.m.. We now can also introduce the ideas of relative primeness of polynomial matrices.

(4.3-93) <u>Definition</u>: Two matrices $P_1(\lambda)$, $P_2(\lambda) \in \mathcal{P}_m$ are said to be relatively left prime if every g.c.l.d. of $P_1(\lambda)$ and $P_2(\lambda)$ is a unimodular m x m matrix. Two matrices $P_1(\lambda)$ and $P_2(\lambda)$, where $P_1'(\lambda)$, $P_2'(\lambda) \in \mathcal{P}_m$, are said to be relatively right prime if every g.c.r.d. of $P_1(\lambda)$ and $P_2(\lambda)$ is a unimodular m x m matrix.

Relative primeness can be easily determined as follows:

(4.3-94) <u>Theorem</u>: Let $P_1(\lambda)$, $P_2(\lambda) \in \mathcal{P}_m$. Then the following are equivalent:

(i) $P_1(\lambda)$ and $P_2(\lambda)$ are relatively left prime

(ii) $P_1'(\lambda)$ and $P_2'(\lambda)$ are relatively right prime

(iii) $P_1(\lambda) \bigvee P_2(\lambda) = I$ (m x m identity matrix)

(iv) $\text{Im } P_1(\lambda) + \text{Im } P_2(\lambda) = R^m[\lambda]$

(v) If $P_i(\lambda) \in R^{mxq_i^2}[\lambda]$ for $i = 1,2$, then $\text{Im}\begin{pmatrix} P_1'(\lambda) \\ P_2'(\lambda) \end{pmatrix}$ is a direct summand of $R^{q_1+q_2}[\lambda]$.

Proof: The equivalence of (i) and (ii) follows from (4.3-93), while the equivalence of (iii) and (iv) follows from (4.3-92) and (4.3-88). Also, (i) implies (iii) because $P_1(\lambda) \bigvee P_2(\lambda)$ must be unimodular and canonical; while (iii) implies (i) since any g.c.l.d. must divide $P_1(\lambda) \bigvee P_2(\lambda)$, i.e. if $Q(\lambda)$ is a g.c.l.d. of $P_1(\lambda)$ and $P_2(\lambda)$ then $Q(\lambda)\ P(\lambda) = I$, whence $Q(\lambda)$ is unimodular.

We now show that (iii) implies (v). Indeed, from the definition of $P_1(\lambda) \bigvee P_2(\lambda)$, there exists a unimodular $Q(\lambda)$ such that

$$(P_1(\lambda);\ P_2(\lambda))\ Q(\lambda) = (I;\ 0)$$

Thus

$$\begin{pmatrix} P_1'(\lambda) \\ P_2'(\lambda) \end{pmatrix} = Q'^{-1}(\lambda) \binom{I}{0}$$

That is, $\begin{pmatrix} P_1'(\lambda) \\ P_2'(\lambda) \end{pmatrix}$ consists of the first m columns of $Q'^{-1}(\lambda)$. But, since $Q'^{-1}(\lambda)$ is unimodular, these columns are m of (q_1+q_2) free generators for $R^{q_1+q_2}[\lambda]$, and (v) follows.

Conversely, if Im $\begin{pmatrix} P_1'(\lambda) \\ P_2'(\lambda) \end{pmatrix}$ is a direct summand of $R^{q_1+q_2}[\lambda]$, there exists a $(q_1+q_2)\times(q_1+q_2-m)$ matrix $P(\lambda)$ such that $\left(\begin{array}{c|c} P_1'(\lambda) & \\ & P(\lambda) \\ P_2'(\lambda) & \end{array}\right)$ is unimodular. But then there exists a $Q(\lambda)$ such that

$$Q(\lambda) \begin{pmatrix} P_1'(\lambda) \\ P_2'(\lambda) \end{pmatrix} = I$$

and it follows that (v) implies (iii). ■

Throughout the remainder of this section we shall assume that $T(\lambda)$ is an m x m canonical matrix. It thus follows from (4.3-52) that the quotient module $R^m[\lambda]/\text{Im } T(\lambda)$ may be represented by a pair (X,A) where X is an R-vector space, of dimension $n = \sum_{i\in\underline{m}} \nu_i$, defined as

$$X \triangleq R^m[\lambda]/\text{Im } T(\lambda) \quad \text{(as R-vector spaces)}$$

and where $A : X \to X$ is the endomorphism induced by the action of λ on $R^m[\lambda]/\text{Im } T(\lambda)$:

$$A : X \to X$$

$$: x \mapsto \lambda x$$

As was seen in (4.3-52), the matrix for A, with respect to a "canonical" basis in X is given by (4.3-54). We shall show how the action of A on X can be deduced from an examination of a sublattice of C. We first prove:

(4.3-95) <u>Lemma</u>: The lattice of A-invariant subspaces in $\mathcal{X}$ is isomorphic to the lattice of submodules M satisfying $\text{Im } T(\lambda) \subset M \subset R^m[\lambda]$. This lattice isomorphism is explicitly given by: $M \mapsto M/\text{Im } T(\lambda) = S \subset \mathcal{X}$.

Proof: It is clear that a subspace $S \subset \mathcal{X}$ (as an R-vector space) is A-invariant if and only if $AS \subset S$; but this is true if and only if $\lambda S \subset S$ (thinking of S as a subset of the $R[\lambda]$-module $R^m[\lambda]/\text{Im } T(\lambda)$). Thus S is A-invariant if and only if S is a submodule. Clearly, the submodules of $R^m[\lambda]/\text{Im } T(\lambda)$ are lattice-isomorphic to the set of submodules M, where $\text{Im } T(\lambda) \subset M \subset R^m[\lambda]$, and the proof is complete. ■

This leads immediately to the next result.

(4.3-96) <u>Corollary</u>: The lattice of A-invariant subspaces in $\mathcal{X}$ is isomorphic to the following sublattice of $\mathcal{C}$:

$$[T(\lambda), I] \triangleq \{C(\lambda) \in \mathcal{C} \mid T(\lambda) \leq C(\lambda) \leq I\}$$

The explicit isomorphism is $C(\lambda) \mapsto \text{Im } C(\lambda)/\text{Im } T(\lambda) \subset \mathcal{X}$.

Proof: Apply (4.3-80) and (4.3-95). ■

One of the more important subclasses of the class of A-invariant subspaces in $\mathcal{X}$ is the subclass of <u>cyclic</u> A-invariant subspaces. Clearly, this subclass is not a sublattice of the lattice of A-invariant subspaces; however, it can be very easily represented in terms of the canonical matrix $T(\lambda)$:

(4.3-97) <u>Lemma:</u> Let $x \in \mathcal{X}$, and let $\{A|x\}$ denote the cyclic A-invariant subspace of $\mathcal{X}$ generated by x. Then there exists $u(\lambda) \in R^m[\lambda]$ such that

$$\mathrm{Im}(T(\lambda)\bigvee u(\lambda))/\mathrm{Im}\ T(\lambda) = \{A|\mathcal{X}\}$$

Therefore, the following sets are bijective:

$$\{S \subset X | S \text{ is cyclic}\} \cong \{T(\lambda)\bigvee u(\lambda) | u(\lambda) \in R^m[\lambda]\}$$

Proof: Letting $\pi : R^m[\lambda] \to R^m[\lambda]/\mathrm{Im}\ T(\lambda) = X$ denote the canonical projection, since π is surjective there exists $u(\lambda) \in R^m[\lambda]$ such that $\pi\ u(\lambda) = x$. Consequently, $\pi\lambda^i u(\lambda) = A^i x$, for $i \geq 0$; thus it follows that

$$\begin{aligned}\mathrm{Im}(T(\lambda)\bigvee u(\lambda))/\mathrm{Im}\ T(\lambda) &= (\mathrm{Im}\ T(\lambda) + \mathrm{Im}\ u(\lambda))/\mathrm{Im}\ T(\lambda) \\ &= R\text{-linear span of } (x, Ax, A^2x, \ldots) \\ &= \{A|\mathcal{X}\}\end{aligned}$$

Clearly, the $u(\lambda)$ selected above is not unique; $u(\lambda) + T(\lambda)\ q(\lambda)$ will also do, for all $q(\lambda) \in R^m[\lambda]$ (as will $v(\lambda)$, so long as $\pi v(\lambda)$ is a generator for $\{A|\mathcal{X}\}$). However, it follows from (4.3-96) that each of these $u(\lambda)$'s results in the same canonical matrix $T(\lambda)\bigvee u(\lambda)$. This proves the second statement. ■

We recall that, given a cyclic subspace S of X, the <u>minimal annihilator</u> of S is the monic polynomial $\psi(\lambda)$ of least degree such that $\psi(A)s = 0$ for all $s \in S$. We now relate the concept of minimal annihilators to canonical matrices.

(4.3-98) <u>Lemma</u>: Let $\{A|\mathcal{X}\}$ be a cyclic A-invariant subspace of X, and let $u(\lambda) \in R^m[\lambda]$ be such that

(4.3-99) $$\mathrm{Im}(T(\lambda)\bigvee u(\lambda))/\mathrm{Im}\ T(\lambda) = \{A|\mathcal{X}\}$$

Then $\psi(\lambda)$, the minimal annihilator of $\{A|\chi\}$ is the unique monic polynomial satisfying

$$\psi(\lambda)(u(\lambda)\wedge I) = u(\lambda)\wedge T(\lambda)$$

Proof: It is clear that, if x is any generator of $\{A|\chi\}$, then $\psi(\lambda)$ is just the minimal annihilator of x. If $u(\lambda) \in R^m[\lambda]$ is chosen so as to satisfy (4.3-99), then $\pi u(\lambda)$ must be a generator of $\{A|\chi\}$; thus $\psi(\lambda)$ is the monic polynomial of least degree such that

$$\psi(\lambda)\ \pi u(\lambda) = 0$$

or, equivalently, such that

$$\psi(\lambda)\ u(\lambda) \in \text{Ker}\ \pi = \text{Im}\ T(\lambda)$$

for any $u(\lambda)$ satisfying (4.3-99). Thus, $\psi(\lambda)$ is the monic polynomial of minimal degree such that

$$\psi(\lambda)\ u(\lambda) \in \text{Im}\ T(\lambda) \bigcap \text{Im}\ u(\lambda) = \text{Im}(T(\lambda)\wedge u(\lambda))$$

and the proof is complete. ∎

(4.3-100) Remark: Both $T(\lambda)\vee u(\lambda)$ and $\psi(\lambda)$ can be determined from a single application of algorithm (4.3-43). That is, if $Q(\lambda)$ is an $(m+1)\times(m+1)$ unimodular matrix that canonicalizes $(T(\lambda);\ u(\lambda))$:

$$(T(\lambda);\ u(\lambda))\ Q(\lambda) = (T(\lambda)\vee u(\lambda);\ 0)$$

and if the last column of $Q(\lambda)$ is denoted by $\begin{pmatrix} q_1(\lambda) \\ q_2(\lambda) \end{pmatrix}$, where $q_2(\lambda) \in R[\lambda]$, then it is easily seen that

$$\text{Ker}(T(\lambda);\ u(\lambda)) = \text{Im}\begin{pmatrix} q_1(\lambda) \\ q_2(\lambda) \end{pmatrix}$$

Consequently, $q_2(\lambda)$, after it has been normalized to make it monic, is the sought-after $\psi(\lambda)$.

(4.3-101) <u>Remark</u>: It should be clear that, if $\hat{T}(\lambda) = T(\lambda) \bigvee u(\lambda)$, then a matrix $\hat{A}$ may be obtained from $T(\lambda)$ exactly as A was obtained from $T(\lambda)$ (in (4.3-52)). Clearly, this matrix $\hat{A}$ represents the map induced by A in the quotient space $\mathcal{X}/\{A|\mathcal{X}\}$, where

$$\{A|\mathcal{X}\} = \mathrm{Im}(T(\lambda) \bigvee u(\lambda))/\mathrm{Im}\ T(\lambda)$$

One can now continue, and find an $\hat{A}$-invariant subspace of $\mathcal{X}/\{A|\mathcal{X}\}$, the minimal annihilator of this subspace, a new canonical matrix $\tilde{T}(\lambda) = \hat{T}(\lambda) \bigvee v(\lambda)$, and a new matrix $\tilde{A}$. In this manner a cyclic decomposition of $\mathcal{X}$ will result (although not necessarily a canonical one).

In Section 4.5 we shall apply these and similar ideas to determine new ways of characterizing A-invariant, (A,B)-invariant, and (A,B)-controllability subspaces associated with linear systems.

4.4 Module-Theoretic Characterizations of Linear Multi-Input Systems via Polynomial Matrices

In this section we shall apply the results of Section 4.3 to obtain characterizations of multi-input linear systems. The approach will be analogous to that in the prologue to Section 4.3, where a characterization of single input systems was obtained. By representing the kernel of the input-output morphism in terms of a canonical matrix, a representation theory for linear systems will evolve. It will also be seen that we need not restrict ourselves to finite dimensional systems, since representations and realizations for infinite dimensional, discrete time systems may be obtained as easily as in the finite dimensional case.

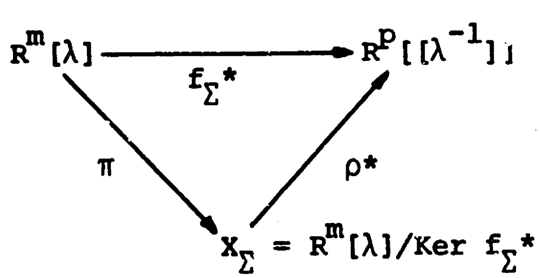
4.4.1 Representation of the State Module and Related Morphisms: Realization

We now return to the notation of Section 4.2. That is, let Σ be a linear, discrete time, constant system characterized by the input-output morphism:

$$f_\Sigma^* : R^m[\lambda] \to R^p[[\lambda^{-1}]]$$

and let the canonical factorization of f_Σ^* be

(4.4-1)



We shall assume initially that rank Ker $f_\Sigma^* = r$, where $0 \leq r \leq m$; later we shall restrict r to $1 \leq r \leq m$, and finally, to $r = m$.

In the case where $r = 0$, it follows that Ker f_Σ^* is the zero submodule. Thus, in (4.4-1) we may take as the representation of Σ:

$$X_\Sigma = R^m[\lambda]$$

$$\pi = \text{identity morphism on } R^m[\lambda]$$

$$\rho^* = f_\Sigma^*$$

A realization of Σ,$(\mathcal{X}, A, B, C)$, follows easily. That is, define

$$\mathcal{X} = R^m[\lambda] \quad \text{(as an R-vector space)}$$

$$A : \mathcal{X} \to \mathcal{X}$$

$$: u(\lambda) \mapsto \lambda u(\lambda)$$

$$B : R^m \to \mathcal{X}$$

$$: v \mapsto \imath(v) = v$$

where $\imath : R^m \to R^m[\lambda]$ is the insertion, and

$$C : \mathcal{X} \to R^p$$

$$: x \mapsto f_\Sigma(x)$$

where $f_\Sigma : R^m[\lambda] \to R^p$ is the input-output map, as defined in Section 4.2. From (4.2-16) and (4.2-17) it follows that $(\mathcal{X}, A, B, C)$ is indeed a canonical realization of Σ.

In the less trivial case where $r > 0$, we know from Section 4.3 that there is a unique $m \times r$ canonical matrix $T(\lambda)$ such that

$$\text{Im } T(\lambda) = \text{Ker } f_\Sigma^*$$

Letting, as usual, $\{\nu_i, i \in \underline{r}\}$ and $\{m_i, i \in \underline{r}\}$ be the sets of degree integers and row integers, we define an m x m permutation matrix P so that $(Px)_i = x_{m_i}$ for all $i \in \underline{r}$ and $x \in R^m$; and we define $\hat{T}(\lambda)(r \times r)$ and $\tilde{T}(\lambda)$ $((m-r) \times r)$ so that

$$P\,T(\lambda) = \binom{\hat{T}(\lambda)}{\tilde{T}(\lambda)}$$

It now follows from (4.3-49) that X_Σ may be characterized by $X \times R^{m-r}[\lambda]$, where X is the quotient module

$$X = R^r[\lambda]/\mathrm{Im}\ \hat{T}(\lambda)$$

However, as was seen in (4.3-52), X is isomorphic, as an R-vector space, to R^n, where

$$n = \sum_{i \in \underline{r}} \nu_i$$

Define the isomorphism $\phi : X \tilde{\rightarrow} R^n$ to be the coordinate isomorphism with respect to the basis $\{\lambda^j e_i + \mathrm{Im}\ \hat{T}(\lambda);\ j \in \underline{\nu_i},\ i \in \underline{r}\}$ in X; that is,

$$\phi : \sum_{i,j} x_{i,j}(\lambda^j e_i + \mathrm{Im}\ \hat{T}(\lambda)) \mapsto \begin{bmatrix} x_{11} \\ \vdots \\ x_{1,\nu_1} \\ x_{21} \\ \vdots \\ x_{r,\nu_r} \end{bmatrix} \in R^n$$

Then, with the matrices A, H, B, and D defined as in (4.3-54), (4.3-53), (4.3-76) or (4.3-78), and (4.3-77) or (4.3-79), respectively, the projection $\pi : R^m[\lambda] \rightarrow X_\Sigma$ may be characterized by

(4.4-2) $$\hat{\pi} : R^m[\lambda] \rightarrow R^n \times R^{m-r}[\lambda]$$

$$: u(\lambda) \mapsto (\sum_{\ell \in \underline{m}} u_\ell(A) b_\ell,\ Du(\lambda) - \tilde{T}(\lambda) H \sum_{j=0}^{k-1} \lambda^j \sum_{\ell \in \underline{m}} u_\ell^{(j)}(A) b_\ell)$$

where $\partial u(\lambda) = k$, and

$$u(\lambda) = \begin{pmatrix} u_1(\lambda) \\ u_2(\lambda) \\ \vdots \\ u_m(\lambda) \end{pmatrix}$$

Therefore, if

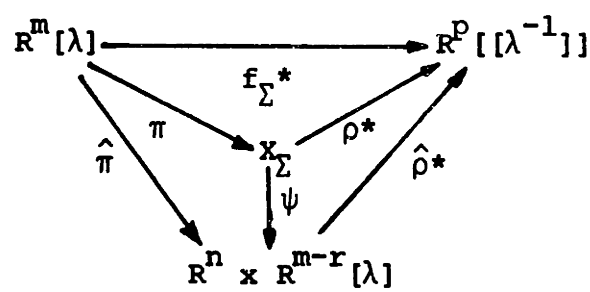
$$\psi : X_\Sigma \tilde{\rightarrow} R^n \times R^{m-r}[\lambda]$$

is the isomorphism relating X_Σ and $R^n \times R^{m-r}[\lambda]$, and if $\hat{\rho}^* : R^n \rightarrow R^{m-r}[\lambda]$ is defined as

(4.4-3) $$\hat{\rho}^* = \rho^* \psi^{-1}$$

it follows that Σ may be characterized by either the top or the middle route in the commutative diagram:

(4.4-4)



Clearly, the bottom route in (4.4-4) would provide the more explicit characterization of Σ if we could determine an explicit characterization of $\hat{\rho}^*$.

This is accomplished as follows. Each element $(x,v(\lambda)) \in R^n \times R^{m-r}[\lambda]$ is the image under $\hat{\pi}$ of some element $u(\lambda) \in R^m[\lambda]$; we need only find such an element, and define $\hat{\rho}^*(x, v(\lambda)) = f_\Sigma^*(u(\lambda))$. But, from the remark (4.3-81) it follows easily that the element

(4.4-5) $$u(\lambda) = P^{-1}\begin{bmatrix} 0 \\ S(\lambda)x \\ v(\lambda) \end{bmatrix} \begin{matrix} \}\#\text{zero rows} = \#\text{zero } \nu_i\text{'s} \\ \\ \\ \end{matrix}$$

satisfies

$$\hat{\pi}(u(\lambda)) = (x,v(\lambda))$$

In (4.4-5),

$$S(\lambda) = \begin{pmatrix} 1 & \ldots\lambda^{\nu_i-1} & 0 & 0 \ldots 0 \ldots & & & & 0 \\ 0 & 0 \ldots 0 & 1 & \lambda \quad \lambda^{\nu_2-1} & 0 \ldots & & & 0 \\ \\ 0 & 0 & \ldots & & 0 & 1 & \lambda\ldots\lambda^{\nu_r-1} \end{pmatrix}$$

where row i of $S(\lambda)$ is deleted if $\nu_i = 0$. Thus we have:

$$\hat{\rho}^*(x,v(\lambda)) = f_\Sigma^*(P^{-1}\begin{pmatrix} 0 \\ S(\lambda)x \\ v(\lambda) \end{pmatrix})$$

We can obtain a more explicit expression for $\hat{\rho}^*$ by determining a realization of Σ, which we now proceed to do.

Define the quadruple (X, F, G, C) as follows.

(4.4-6) $$X = R^n \times R^{m-r}[\lambda] \quad \text{(as an R-vector space)}$$

(4.4-7) $$F : X \to X$$

$$: (x,v(\lambda)) \to (Ax,\lambda v(\lambda) - \tilde{T}(\lambda)Hx)$$

where A, H, and $\tilde{T}(\lambda)$ are as defined in (4.3-52). Also,

(4.4-8) $$G : R^m \to X$$

$$: u \to (Bu, Du)$$

where B and D are as defined in (4.3-75). Finally,

(4.4-9) $$C : X \to R^p$$

$$: (x, v(\lambda)) \to f_\Sigma(P^{-1}\begin{pmatrix} 0 \\ S(\lambda)x \\ v(\lambda) \end{pmatrix})$$

where $f_\Sigma : R^m[\lambda] \to R^p$ is the R-linear input-output map of Σ.

A matrix representation of $K : X \to R^p$ is obtained via

(4.4-10) <u>Lemma</u>: Assume that $\partial T(\lambda) > 0$. Let $\{G_i \in R^{p \times m}, i \geq 0\}$ be the pulse response of Σ, as defined in Section 4.2; and let $P \in R^{m \times m}$ be the permutation matrix such that $PT(\lambda) = \begin{pmatrix}\hat{T}(\lambda) \\ \tilde{T}(\lambda)\end{pmatrix}$. Also, define $\tilde{r}$, $\hat{r}$, and the set $\{\hat{\nu}_i, i \in \underline{\hat{r}}\}$ as

$$\tilde{r} = \min\{i | \nu_i > 0\} - 1$$

$$\hat{r} = r - \tilde{r}$$

$$\hat{\nu}_i = \nu_{i+\tilde{r}}, \text{ for } i \in \underline{\hat{r}}$$

For each $k \geq 0$, define

$$\hat{g}_{k,i} = \text{column } \tilde{r} + i \text{ of } (G_k P^{-1}), \text{ for } 1 \leq i \leq m-\hat{r}$$

and

$$\hat{G}_k = (\hat{g}_{k,\hat{r}+1};\ \hat{g}_{k,\hat{r}+2};\ \ldots\ \hat{g}_{k,m-\tilde{r}}) \in R^{p \times (m-r)}$$

Finally, define $C \in R^{p \times n}$ as

$$C = (\hat{g}_{0,1};\ \hat{g}_{1,1};\ \hat{g}_{\hat{\nu}_1-1,1};\ \hat{g}_{0,2};\dots\hat{g}_{\hat{\nu}_2-1,2};\dots\hat{g}_{\hat{\nu}_{\hat{r}}-1,\hat{r}})$$

Then, the R-linear map $K : \mathcal{X} \to R^p$ of (4.4-9) is given by

(4.4-11) $$K : (x,v(\lambda)) \mapsto C\,x + \sum_{i=0}^{k} \hat{G}_i\, v_i$$

where

$$v(\lambda) = \sum_{i=0}^{k} \lambda^i\, v_i$$

Proof: From (4.4-9) and (4.2-4)(ii) it follows that if

$$\begin{pmatrix} 0 \\ S(\lambda)x \\ v(\lambda) \end{pmatrix} = \sum_{i=0}^{\ell} \lambda^i\, u_i = u(\lambda)$$

then

$$K(x,v(\lambda)) = \sum_{i=0}^{\ell} G_i\, P^{-1}\, u_i$$

Thus, we have only to determine the relation between $(x,v(\lambda))$ and the u_i's. But, since $S(\lambda)$ is just

$$S(\lambda) = \begin{pmatrix} 1 & \dots\lambda^{\hat{\nu}_1-1} & 0 & 0 & 0\ \dots & 0 \\ 0 & 0 & 0 & 1\dots\lambda^{\hat{\nu}_2-1} & 0\ \dots & 0 \\ \vdots & & & & & \\ 0 & \dots & & & & 1\dots\lambda^{\hat{\nu}_r-1} \end{pmatrix}$$

it is easily seen that

$$u_i = \begin{pmatrix} 0 \\ w_{i,1} \\ w_{i,2} \\ \vdots \\ w_{i,\hat{r}} \\ v_i \end{pmatrix} \begin{matrix} \}\tilde{r} \text{ elements} \\ \\ \hat{r} \text{ elements} \\ \\ \\ \}m-r \text{ elements} \end{matrix}$$

where

$$w_{i,j} = \begin{cases} 0, \text{ if } i \geq \hat{\nu}_j \\ x_{1+i+\sum_{q<j} \hat{\nu}_q}, \text{ if } i < \hat{\nu}_j \end{cases}$$

Therefore,

$$\begin{aligned} K(x,v(\lambda)) &= \hat{g}_{0,1}\, x_1 + \hat{g}_{0,2}\, x_{\hat{\nu}_1+1} + \ldots + \hat{g}_{0,\hat{r}}\, x_{\hat{\nu}_1+\ldots+\nu_{\hat{r}-1}+1} \\ &+ \hat{g}_{1,1}\, x_2 + \ldots + \hat{g}_{1,\hat{r}}\, x_{\hat{\nu}_1+\ldots+\hat{\nu}_{\hat{r}-1}+2} \\ &+ \ldots + \hat{g}_{\hat{\nu}_{\hat{r}}-1,\hat{r}}\, x_n \\ &+ \sum_{i=0}^{k} \hat{G}_i\, v_i \\ &= C\, x + \sum_{i=0}^{k} \hat{G}_i\, v_i \end{aligned}$$

and the proof is complete. ■

We now state

(4.4-12) **Theorem:** The quadruple ($\mathcal{X}$, F, J, K) defined in (4.4-6), (4.4-7), (4.4-8), and (4.4-10) is a canonical realization of Σ. Moreover, Σ is a finite dimensional system if and only if r=m.

Proof: The first statement is an immediate consequence of (4.2-16) and (4.2-17) if one notes that $\mathcal{X}$ may be endowed with the structure of an $R[\lambda]$-module where the action of λ on $\mathcal{X}$ is the same as the action of F on $\mathcal{X}$ (see (4.3-49)); that J is just the composition $\hat{\pi} \circ \iota$ (see (4.3-75)); and that $K \circ \hat{\pi} = f_\Sigma$.

Clearly, Σ is finite dimensional (by (4.2-17)) if and only if $R^m[\lambda]/\text{Ker } f_\Sigma^*$ is a torsion module. But this is true if and only if $r = m$. ∎

(4.4-13) Remark: If $r < m$ there is a free submodule of $R^m[\lambda]/\text{Ker } f_\Sigma^*$, and consequently Σ has an infinite dimensional part. In the above realization of Σ we have represented the infinite-dimensional component of the state vector by $v(\lambda) \in R^{m-r}[\lambda]$. Clearly, we could equally as well represent this component by an infinite vector

$$v(\lambda) = \sum_{i=0}^{k} \lambda^i v_i \mapsto \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_k \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

Using this representation for $v(\lambda)$, a block diagram of the realization of (4.4-12) is as given in Fig. 4-1. In this diagram,

$$\tilde{T}(\lambda) = \sum_i \lambda^i \tilde{T}_i$$

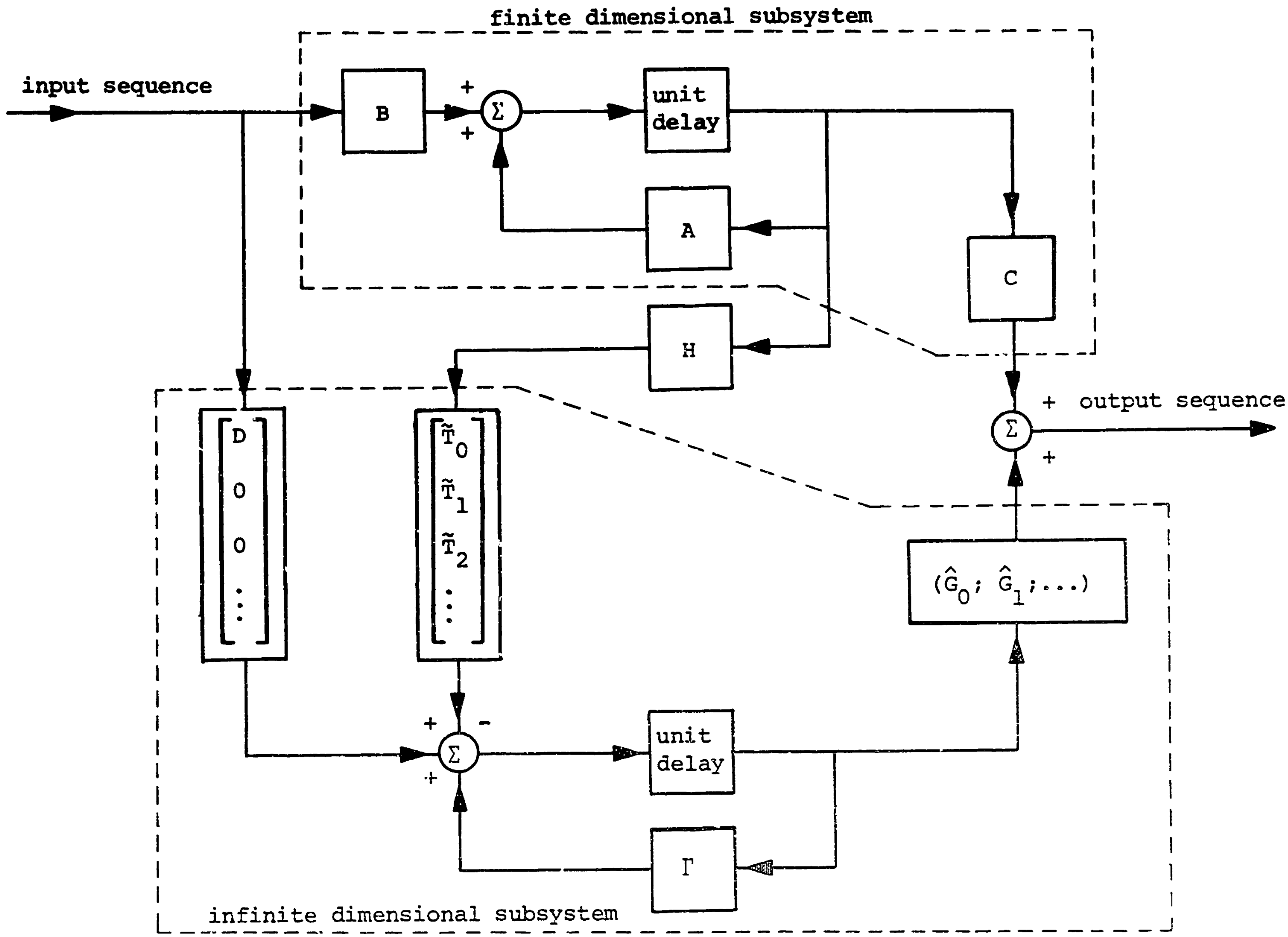


Figure 4-1: Realization of Σ via (4.4-12)

and Γ is the following infinite matrix:

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & \\ I & 0 & 0 & \cdots \\ 0 & I & 0 & \\ 0 & 0 & I & \\ & \vdots & & \end{bmatrix}$$

where each I is the $(m - r) \times (m - r)$ identity matrix.

(4.4-14) Remark: In (4.4-10) (and by implication in (4.4-12)) we have assumed that $\partial T(\lambda) > 0$. If this were not the case then the finite dimensional subsystem in Fig. 4-1 would be of dimension zero (i.e., effectively not present).

(4.4-15) Remark: In (4.3-63) we obtained a second representation of the quotient module $R^m[\lambda]/\text{Im } T(\lambda)$ as

$$(R^r[\lambda]/\text{Im } T_2(\lambda)) \oplus R^{m-r}[\lambda]$$

and in (4.3-82) we derived a characterization of the corresponding morphism $\tilde{\pi} : R^m[\lambda] \to (R^r[\lambda]/\text{Im } T_2(\lambda)) \oplus R^{m-r}[\lambda]$. These results can be used to obtain a second canonical realization of Σ. When Σ is finite dimensional, this realization will be essentially identical to the one given above. However, if Σ is infinite dimensional, this second realization may well have a lower order finite dimensional subsystem than the realization of Fig. 4-1. This is due to the fact that $S \subset X$ (see (4.3-59)) may be a proper submodule of X. We shall not derive this second realization here, as the determination of the state-to-output map is fairly messy.

4.4.2 Determination of $T(\lambda)$ fron the Hankel Matrix

In the preceding section we have seen that a canonical representation (and thus also a canonical realization) of a linear, constant, discrete time system Σ may be obtained as follows. The structure of the state module X and the input-to-state morphism $\hat{\pi} : R^m[\lambda] \to X$ can be essentially "read off" the canonical matrix $T(\lambda)$, which satisfies

(4.4-16) $$\text{Im } T(\lambda) = \text{Ker } f_\Sigma^*$$

Then, the output-to-state morphism $\hat{\rho}^* : X \to R^p[[\lambda^{-1}]]$ can be determined quite easily from the pulse response $\{G_i, i \geq 0\}$ of Σ. We shall now show that the canonical matrix $T(\lambda)$ may also be determined from the pulse response. It will thus follow, as should be expected, that a complete internal description of Σ (modulo uncontrollable and unobservable parts) may be deduced from $\{G_i, i \geq 0\}$.

We define the Hankel matrix H associated with Σ as in (4.2-7). The finite Hankel matrices $H_{s,q}$, we define as

(4.4-17) $$H_{s,q} = \begin{bmatrix} G_0 & G_1 & G_2 \cdots & G_q \\ G_1 & G_2 & G_3 \cdots & G_{q+1} \\ G_2 & G_3 & G_4 \cdots & G_{q+2} \\ \vdots & & & \\ G_s & G_{s+1} & G_{s+2} \cdots & G_{s+q} \end{bmatrix} \quad ; \; s,q \geq 0$$

Since Ker $f_{\Sigma}*$ is certainly a submodule of $R^m[\lambda]$, and since we have already established, in Section 4.3, the existence and uniqueness of a canonical matrix $T(\lambda)$ satisfying (4.4-17), we have only to relate elements in Ker $f_{\Sigma}*$ to certain properties of the pulse response $\{G_i, i \geq 0\}$. The key, albeit simple, result is

(4.4-18) Lemma: Let $u(\lambda) \in R^m[\lambda]$, and let

$$u(\lambda) = \sum_{i=0}^{k} \lambda^i u_i$$

Then, $u(\lambda) \in$ Ker $f_{\Sigma}*$ if and only if

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_k \end{pmatrix} \in \text{Ker } \mathcal{H}_{s,k}$$

for all $s \geq 0$.

Proof: This is a simple application of (4.2-5)(iii). ■

Thus, each element $u(\lambda) \in$ Ker $f_{\Sigma}*$ determines a linear dependence among the columns of $\mathcal{H}$; and, conversely, each such linear dependence determines an element of Ker $f_{\Sigma}*$. Presumably, the columns of $T(\lambda)$, being "canonical" elements of Ker $f_{\Sigma}*$, correspond to "canonical" linear dependences among the columns of $\mathcal{H}$. This is, in fact, the case.

To explore these canonical linear dependences, we define vectors $y_{i,j,s} \in R^{(s+1)p}$, $i \in \underline{m}$, $j \geq 0$, $s \geq 0$, as

$$(4.4\text{-}19) \quad (y_{1,j,s};\ y_{2,j,s};\ldots\ y_{m,j,s}) = \begin{bmatrix} G_j \\ G_{j+1} \\ G_{j+2} \\ \vdots \\ G_{j+s} \end{bmatrix};\ j,s \geq 0$$

Also, as we recall, the ith column of $T(\lambda)$, $t_i(\lambda)$, is of the form (using (4.3-20)):

$$(4.4\text{-}20) \quad t_i(\lambda) = \lambda^{\nu_i} e_{m_i} + \sum_{j=1}^{m_i} \sum_{k=0}^{\nu_i-1} \lambda^k t_{jik} e_j + \sum_{j=m_i+1}^{m} \sum_{k=0}^{\nu_i} \lambda^k t_{jik} e_j$$

where $\{e_i,\ i \in \underline{m}\}$ is the standard basis for R^m. Moreover, in (4.4-20),

$$(4.4\text{-}21) \quad t_{jik} = 0,\ \text{if } j = m_\ell \text{ and } k \geq \nu_\ell \text{ for some } \ell \in \underline{r} \text{ such that } \nu_\ell \leq \nu_i$$

We can now state

(4.4-22) **Lemma**: Let $t_i(\lambda)$ be column i of $T(\lambda)$, as in (4.4-20). Then the following linear dependence exists among the columns of H:

$$(4.4\text{-}23) \quad y_{m_i,\ \nu_i,s} = -\sum_{j=1}^{m_i} \sum_{k=0}^{\nu_i-1} t_{jik}\, y_{j,k,s} - \sum_{j=m_i+1}^{m} \sum_{k=0}^{\nu_i} t_{jik}\, y_{j,k,s}$$

for all $s \geq 0$ and all $i \in \underline{r}$.

Proof: This follows immediately from (4.4-18) and (4.4-19), and the fact that $t_i(\lambda) \in \text{Ker } f_\Sigma^*$. ■

It is now apparent that it is possible to determine the canonical matrix $T(\lambda)$ from an examination of the columns of H. That is,

(4.4-24) **Theorem**: (i) For each $s \geq 0$, define the infinite array Y_s as

$$Y_s = (y_{m,0,s};\ y_{m-1,0,s};\dots y_{1,0,s};\ y_{m,1,s};\dots y_{1,1,s};\ y_{m,2,s};\dots)$$

(note the special ordering of the columns).

(ii) For each column $y_{j,k,s}$ of Y_s, define the antecedents of $y_{j,k,s}$ to be those columns to the left of $y_{j,k,s}$ in Y_s.

(iii) Define a column $y_{j,k,s}$ to be regular if

$$y_{j,k,s} \notin \text{R-linear span of antecedents of } y_{j,k,s}$$

(iv) For $i \in \underline{m}$ define

$$\kappa_i = \min\{k \mid y_{i,k,s} \in \text{R-linear span of antecedents of } y_{i,k,s}, \text{ for all } s \geq 0\}$$

if this set is nonvoid, and

$$\kappa_i = +\infty, \text{ otherwise}$$

Then

$$\kappa_i = \begin{cases} \nu_j, & \text{if } i = m_j \text{ for some } j \in \underline{r} \\ +\infty, & \text{if } i \notin \{m_k,\ k \in \underline{r}\} \end{cases} \tag{4.4-25}$$

Moreover, if $j = m_i$, the regular antecedents of $y_{j,\kappa_j,s}$ are $\{y_{\ell,k,s} \mid (\ell,k) \in I_{j,s}\}$, where $I_{j,s+1} \supset I_{j,s}$ for all $s \geq 0$ and

$$\bigcup_{s\geq 0} I_{j,s} = \{(\ell,k) \mid k < \min(\kappa_j,\kappa_\ell), \text{ or } k < \min(\kappa_j+1,\kappa_\ell) \ \underline{\text{and}}\ j < \ell \leq m\} \tag{4.4-26}$$

and the linear dependence of $y_{j,\kappa_j,s}$ on its regular antecedents, for all $s \geq 0$, is uniquely given by (4.4-23), subject to the constraint (4.4-21).

Proof: We first note that if $\kappa_i = +\infty$ for all $i \in \underline{m}$, then there exists no R-linear dependence amongst the columns of $\mathcal{H}$. Consequently, from (4.4-18) it follows that Ker $f_\Sigma^* = 0$. Thus $r = 0$, and the theorem follows trivially.

Thus, assume that $\kappa_j < \infty$ for some $j \in \underline{m}$. Then, from the observation that the regularity of $y_{\ell,k,s}$ implies that of $y_{\ell,k,s+1}$, it follows that $I_{j,s+1} \supset I_{j,s}$. Thus, we may define

$$I_{j,\infty} \triangleq \bigcup_{s \geq 0} I_{j,s}$$

Observation of the structure of $\mathcal{H}$ now reveals that $(\ell,k) \notin I_{j,\infty}$ implies $(\ell,k+1) \notin I_{j,\infty}$. This, together with the facts that $k = \kappa_\ell$ implies $(\ell,k) \notin I_{j,\infty}$, while $k < \min(\kappa_\ell,\kappa_j)$ (or $k < \min(\kappa_\ell,\kappa_j+1)$ and $j < \ell \leq m$) implies $(\ell,k) \in I_{j,\infty}$, demonstrates the validity of (4.4-26).

It now follows that for every j such that $\kappa_j < \infty$ there exists an equation of the form

$$y_{j,\kappa_j,s} = \sum_{(\ell,k)\in I_{j,\infty}} a_{\ell,j,k}\, y_{\ell,k,s}\ , \text{ for all } s \geq 0$$

Each such equation determines, via (4.4-18), an element $u_j(\lambda) \in$ Ker f_Σ^* of the form

$$u_j(\lambda) = \lambda^{\kappa_j} e_j - \sum_{(\ell,k)\in I_{j,\infty}} a_{\ell,j,k}\, \lambda^k\, e_\ell$$

If there are q values of j such that $\kappa_j < \infty$, an $m \times q$ matrix $U(\lambda)$ may be formed with the $u_j(\lambda)$'s as columns. Moreover, due to the nature of the sets $I_{j,\infty}$, it is easily seen that the $u_j(\lambda)$'s may be ordered so that the resulting $U(\lambda)$ is a canonical matrix. Note that $u_j(\lambda) \in$ Ker f_Σ^* for all q values of j implies

$$\operatorname{Im} U(\lambda) \subset \operatorname{Ker} f_{\Sigma}^*$$

Thus, if

$$U(\lambda) = (u_{j_1}(\lambda);\ u_{j_2}(\lambda);\ldots\ u_{j_q}(\lambda))$$

then (4.3-90) implies that

$$q \leq r$$

and

(4.4-27) $$\kappa_{j_i} \geq \nu_i\ ,\ \text{for } i \in \underline{q}$$

On the other hand, (4.4-23) implies that $\kappa_{m_i} < \infty$ for all $i \in \underline{r}$, whence, the m_i's being distinct, it follows that

$$r \leq q$$

Thus $r = q$, and the sets $\{m_i,\ i \in \underline{r}\}$ and $\{j_i,\ i \in \underline{q}\}$ are identical. Finally, (4.4-23) implies that

$$\kappa_{m_i} \geq \nu_i\ ,\ \text{for } i \in \underline{r}$$

This, together with (4.4-27) implies that the sets $\{\kappa_j,\ j \in \underline{q}\}$ and $\{\nu_i,\ i \in \underline{r}\}$ are identical. Consequently, applying (4.3-91), it follows that $T(\lambda) = U(\lambda)$, which completes the proof. ■

(4.4-28) Remark: This construction is basically that used by Popov ([60]). The differences are that, while Popov operated on the matrix $(B;\ AB;\ldots\ A^{n-1}B)$, we are operating on the Hankel matrix. Also, we are allowing for the possibility of an infinite dimensional system $(r < m)$.

There are two basic problems with (4.4-24). The first of these is that, in order to determine κ_i, we must check the regularity of each

$y_{i,k,s}$ for all $s \geq 0$, until we find the first value of k such that $y_{i,k,s}$ is not regular, for all $s \geq 0$. Clearly, this is not very practical; however, in general it is necessary, because there are situations where $y_{i,k,s+1}$ is regular, while $y_{i,k,s}$ is not.

The second shortcoming of (4.4-24) is that there is no practical way to finally decide that the value of a particular κ_i is $+\infty$. Essentially, one has to exhaustively verify that $\kappa_i \neq 0$, $\kappa_i \neq 1$, $\kappa_i \neq 2, \ldots$; and each of these verifications has the difficulties sketched out in the preceding paragraph.

In the special case where it is known that $r = m$ and that $\sum_{i \in \underline{m}} \nu_i \leq N$, we can make use of the following result:

(4.4-29) **Theorem**: If it is known a priori that $r = m$ and that $\sum_{i \in \underline{m}} \nu_i \leq N$, then $T(\lambda)$ may be computed as follows. Write:

$$Y_{N-1} = (y_{m,0,N-1}; \ldots\ y_{1,0,N-1};\ y_{m,1,N-1}; \ldots y_{1,1,N-1};\ y_{m,2,N-1}; \ldots)$$

Determine the regularity or non-regularity of each $y_{j,k,N-1}$, as in (4.4-24), and define

$$\kappa_i = \min\{k \mid y_{i,k,N-1} \in \text{R-linear span of antecedents of } y_{i,k,N-1}\},$$

for $i \in \underline{m}$

Then, for each $i \in \underline{m}$, there exist unique constants $a_{\ell,i,k} \in R$ such that

(4.4-30)
$$y_{i,\kappa_i,N-1} = \sum_{\ell=1}^{i} \sum_{k=0}^{\min(\kappa_i,\kappa_\ell)-1} a_{\ell,i,k}\, y_{\ell,k,N-1} + \sum_{\ell=i+1}^{m} \sum_{k=0}^{\min(\kappa_i,\kappa_\ell-1)} a_{\ell,i,k}\, y_{\ell,k,N-1}$$

$y_{i,k,s}$ for all $s \geq 0$, until we find the first value of k such that $y_{i,k,s}$ is <u>not</u> regular, for all $s \geq 0$. Clearly, this is not very practical; however, in general it is necessary, because there are situations where $y_{i,k,s+1}$ is regular, while $y_{i,k,s}$ is not.

The second shortcoming of (4.4-24) is that there is no practical way to finally decide that the value of a particular κ_i is $+\infty$. Essentially, one has to exhaustively verify that $\kappa_i \neq 0$, $\kappa_i \neq 1$, $\kappa_i \neq 2, \ldots$; and each of these verifications has the difficulties sketched out in the preceding paragraph.

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(4.4-29) <u>Theorem</u>: If it is known <u>a priori</u> that $r = m$ and that $\sum_{i \in \underline{m}} \nu_i \leq N$, then $T(\lambda)$ may be computed as follows. Write:

$$Y_{N-1} = (y_{m,0,N-1}; \ldots\ y_{1,0,N-1};\ y_{m,1,N-1}; \ldots y_{1,1,N-1};\ y_{m,2,N-1}; \ldots)$$

Determine the regularity or non-regularity of each $y_{j,k,N-1}$, as in (4.4-24), and define

$$\kappa_i = \min\{k | y_{i,k,N-1} \in \text{R-linear span of antecedents of } y_{i,k,N-1}\},$$

$$\text{for } i \in \underline{m}$$

Then, for each $i \in \underline{m}$, there exist unique constants $a_{\ell,i,k} \in R$ such that

$$(4.4\text{-}30) \qquad y_{i,\kappa_i,N-1} = \sum_{\ell=1}^{i} \sum_{k=0}^{\min(\kappa_i,\kappa_\ell)-1} a_{\ell,i,k}\, y_{\ell,k,N-1}$$

$$+ \sum_{\ell=i+1}^{m} \sum_{k=0}^{\min(\kappa_i,\kappa_\ell-1)} a_{\ell,i,k}\, y_{\ell,k,N-1}$$

Define vector polynomials $u_i(\lambda) \in R^m[\lambda]$ as

$$u_i(\lambda) = \lambda^{\kappa_i} e_i - \sum_{\ell=0}^{i} \sum_{k=0}^{\min(\kappa_i-\kappa_\ell)-1} a_{\ell,i,k} \lambda^k e_\ell$$

$$- \sum_{\ell=i+1}^{m} \sum_{k=0}^{\min(\kappa_i,\kappa_\ell-1)} a_{\ell,i,k} \lambda^k e_\ell$$

Then there exists a unique permutation $\sigma : \underline{m} \to \underline{m}$ such that

$$T(\lambda) = (u_{\sigma(1)}(\lambda);\ u_{\sigma(2)}(\lambda); \ldots\ u_{\sigma(m)}(\lambda))$$

Proof: The result follows immediately from (4.4-24) if it can be shown that $y_{j,k,N-1}$ is regular if and only if $y_{j,k,s}$ is regular for some $s \geq 0$, and that (4.4-30) implies

$$y_{i,\kappa_i,s} = \sum_{\ell=1}^{i} \sum_{k=0}^{\min(\kappa_i,\kappa_\ell)-1} a_{\ell,i,k}\, y_{\ell,k,s}$$

$$+ \sum_{\ell=i+1}^{m} \sum_{k=0}^{\min(\kappa_i,\kappa_\ell-1)} a_{\ell,i,k}\, y_{\ell,k,s}\ , \text{ for } s \geq 0$$

This we accomplish as follows.

Since $\sum_{i\in\underline{m}} \nu_i \triangleq n \leq N$, from (4.4-12) there exist A, B, and C (of dimensions n x n, n x m, and p x n) such that

$$C\, A^i\, B = G_i\ ,\ i \geq 0$$

Consequently, defining the m x m permutation matrix P as

$$P = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ \cdot & \cdot & & \\ \cdot & \cdot & & 0 \\ \cdot & \cdot & & \vdots \\ 0 & 1 & & \cdot \\ 1 & 0 & & 0 \end{bmatrix}$$

it follows that

$$Y_s = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^s \end{bmatrix} (BP;\ ABP;\ A^2BP;\ldots)$$

Moreover, since the realization (R^n, A, B, C) is canonical, it follows that

$$\text{Ker} \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^s \end{bmatrix} = 0 \ , \text{ for all } s \geq n-1$$

Therefore, for all $s \geq N-1 \geq n-1$, the linear dependences among the columns of Y_s are identical to those among the columns of $(BP;\ ABP;\ldots)$, and the assertions follow. ■

(4.4-31) Remark: It is unfortunate that there appears to be no way to base a finite algorithm on (4.4-24) in the general case where $r < m$ (and thus, when the associated system is infinite dimensional). That this is the case can be seen from the following example.

Let $\{G_i,\ i \geq 0\}$ be a pulse response such that

$$\lim_{s \to \infty} \text{rank } H_{s,q} = m(q+1) \ , \text{ for all } q \geq 0$$

where $H_{s,q}$ is as in (4.4-18). Clearly, then, $\text{Ker } f_\Sigma{}^* = 0$. However, this is also the case when the pulse response is $\{\tilde{G}_i, i \geq 0\}$, where

$$\tilde{G}_i = \begin{cases} 0, & 0 \leq i \leq M-1 \\ G_{i-M}, & i \geq M \end{cases}$$

although a casual observation of the first M elements of $\{\tilde{G}_i, i \geq 0\}$ might lead one to conclude that Ker $f_{\Sigma}* = R^m[\lambda]$.

In the next section we shall develop an alternative method of computing $T(\lambda)$; this method will be applicable when Σ is finite dimensional, and when the transfer matrix $H(\lambda)$ is known.

4.4.3 Determination of $T(\lambda)$ from the Transfer Matrix

In this section we derive an algorithmic method for computing $T(\lambda)$ in the case where Σ is finite dimensional and where the transfer matrix, $H(\lambda)$, of Σ is known. In fact, we shall do more than this; we shall derive a canonical representation of $H(\lambda)$ as

$$H(\lambda) = N(\lambda)\, T^{-1}(\lambda)$$

where $N(\lambda)$ is a polynomial matrix, and $T^{-1}(\lambda)$ is the inverse of $T(\lambda)$ (in the field of quotients, $R(\lambda)$). This result is by no means new; it has been derived several different ways by Rosenbock ([61]) and Wolovich and Falb ([67]-[70]). However, the derivation here emphasizes the module-theoretic nature of the result.

In (4.2-12) we introduced the transfer matrix $H(\lambda)$ as an element of $R^{p \times m}[[\lambda^{-1}]]$; and the action of $H(\lambda)$ on an element $u(\lambda) \in R^m[\lambda]$ was defined as

$$H(\lambda) : u(\lambda) \mapsto [H(\lambda)\, u(\lambda)]$$

where $[H(\lambda)\ u(\lambda)]$ is the result of formally multiplying $H(\lambda)$ and $u(\lambda)$, and then deleting all coefficients of nonnegative powers of λ. We now show that when Σ is finite dimensional the formal power series for $H(\lambda)$ may be "summed" to yield an element of $R^{p \times m}(\lambda)$, i.e. a rational p x m matrix.

(4.4-32) Lemma: Let Σ be a finite dimensional, linear, discrete time system, and let $H(\lambda) \in R^{p \times m}[[\lambda^{-1}]]$ denote the transfer matrix of Σ. Then $H(\lambda)$ admits the following representation as an element of $R^{p \times m}(\lambda)$:

(4.4-33)
$$H(\lambda) = \frac{1}{\psi(\lambda)} L(\lambda)$$

where $\psi(\lambda) \in R[\lambda]$, and $L(\lambda) \in R^{p \times m}[\lambda]$. (By (4.4-33) we mean that the formal power series for $H(\lambda)$ is identical to that obtained by multiplying $L(\lambda)$ times the formal power series for $1/\psi(\lambda)$.)

Proof: From (4.2-17)(iii) it follows that the finite dimensionality of Σ implies that the state module X_Σ is torsion. Let $\psi(\lambda)$ be the minimal annihilator of X_Σ. Then, since

$$f_\Sigma{*}(\psi(\lambda)\ u(\lambda)) = 0\ , \text{ for all } u(\lambda) \in R^m[\lambda]$$

it follows, in particular, that

$$f_\Sigma{*}(\psi(\lambda) e_i) = [H(\lambda)\ \psi(\lambda)\ e_i] = 0\ , \text{ for } i \in \underline{m}$$

where $\{e_i,\ i \in \underline{m}\}$ is the standard basis for R^m. In other words,

$$H(\lambda)\ \psi(\lambda)\ e_i \in R^p[\lambda]\ , \text{ for } i \in \underline{m}$$

But this simply says that each column of $H(\lambda)$ can be written as $1/\psi(\lambda)$ times an element of $R^p[\lambda]$, and (4.4-33) follows. ■

(4.4-34) Remark: One can also easily prove that $H(\lambda)$ has the representation of (4.4-33) only if Σ is finite dimensional. Moreover, it is easily seen that $\psi(\lambda)$, the minimal annihilator of X_Σ, is just the least common denominator of the elements of $H(\lambda) \in R^{p \times m}(\lambda)$.

(4.4-35) Lemma: The rational matrix $H(\lambda)$ in (4.4-33) is always proper, i.e. the degree of each column of $L(\lambda)$ is strictly less than $\partial\psi(\lambda)$.

Proof: This is an immediate consequence of the fact that elements of $H(\lambda)$ are also elements of $R[[\lambda^{-1}]]$. ■

(4.4-36) Remark: (4.4-35) corresponds to the fact that in our setup causality is "built into" Σ from the start (c.f. (4.2-8)).

Our next result is the desired representation of $H(\lambda)$ in terms of the canonical matrix $T(\lambda)$:

(4.4-37) Theorem: Let Σ be a finite dimensional linear system and let $H(\lambda) \in R^{p \times m}(\lambda)$ be the transfer matrix. Then there exists a unique $N(\lambda) \in R^{p \times m}[\lambda]$ such that

(4.4-38) $$H(\lambda) = N(\lambda)\, T^{-1}(\lambda)$$

where $T(\lambda)$ is the canonical matrix such that $\mathrm{Im}\, T(\lambda) = \mathrm{Ker}\, f_\Sigma{}^*$. Moreover, if $\{\nu_i,\ i \in \underline{m}\}$ denotes the set of degree integers of $T(\lambda)$, then

(i) ∂(column i of $N(\lambda)$) $< \nu_i$, for $i \in \underline{m}$

(ii) $N(\lambda)$ and $T(\lambda)$ are relatively right prime.

Proof: From the definitions of $H(\lambda)$ and $T(\lambda)$, it follows that $[H(\lambda)\ T(\lambda)] = 0$, i.e. that $H(\lambda)\ T(\lambda) \in R^{p \times m}[\lambda]$. Thus, define

$$N(\lambda) = H(\lambda)\ T(\lambda)$$

Clearly, this $N(\lambda)$ is the only element of $R^{p \times m}[\lambda]$ satisfying (4.4-38).

To prove the remainder of the theorem, we make use of (4.4-33), noting that the pair $(T(\lambda), N(\lambda))$ must satisfy

$$L(\lambda)\ T(\lambda) = \psi(\lambda)\ N(\lambda)$$

or equivalently,

$$\mathrm{Im}\begin{pmatrix} T(\lambda) \\ N(\lambda) \end{pmatrix} \subset \mathrm{Ker}(L(\lambda);\ -\psi(\lambda)I)$$

where $(L(\lambda);\ -\psi(\lambda)I)$ is interpreted as a morphism: $R^{m+p}[\lambda] \to R^p[\lambda]$. On the other hand, if $\begin{pmatrix} u(\lambda) \\ y(\lambda) \end{pmatrix} \in \mathrm{Ker}(L(\lambda);\ -\psi(\lambda)I)$, then

$$L(\lambda)\ u(\lambda) = \psi(\lambda)\ y(\lambda)$$

so that

$$\frac{1}{\psi(\lambda)} L(\lambda)\ u(\lambda) = H(\lambda)\ u(\lambda) = y(\lambda) \in R^p[\lambda]$$

Thus, $u(\lambda) \in \mathrm{Ker}\ f_\Sigma{}^* = \mathrm{Im}\ T(\lambda)$ and it follows that

$$\begin{pmatrix} u(\lambda) \\ y(\lambda) \end{pmatrix} \in \mathrm{Im}\begin{pmatrix} T(\lambda) \\ H(\lambda)T(\lambda) \end{pmatrix} = \mathrm{Im}\begin{pmatrix} T(\lambda) \\ N(\lambda) \end{pmatrix}$$

Consequently,

$$\mathrm{Im}\begin{pmatrix} T(\lambda) \\ N(\lambda) \end{pmatrix} = \mathrm{Ker}(L(\lambda);\ -\psi(\lambda)I) \qquad (4.4\text{-}39)$$

Since $H(\lambda)$ is a proper rational matrix, it is easy to see that column i of $N(\lambda) = H(\lambda)\ T(\lambda)$ is of smaller degree than column i of $T(\lambda)$; this establishes (ii). Moreover, it implies that $\binom{T(\lambda)}{N(\lambda)}$ is the unique canonical matrix satisfying (4.4-39).

Finally, $\mathrm{Ker}(L(\lambda);\ -\psi(\lambda)I)$, being the kernel of a morphism from $R^{m+p}[\lambda]$ to $R^{p}[\lambda]$, is a direct summand of $R^{m+p}[\lambda]$ (see Appendix B). Thus, using (4.3-94), it now follows that $T(\lambda)$ and $N(\lambda)$ are relatively right prime. ■

We now have a simple method of determining both $T(\lambda)$ and $N(\lambda)$ from the transfer matrix $H(\lambda)$:

(4.4-40) <u>Theorem</u>: Let $H(\lambda)$ be written as

$$H(\lambda) = \frac{1}{\psi(\lambda)} L(\lambda)$$

where $L(\lambda) \in R^{p \times m}[\lambda]$ and where $\psi(\lambda)$ is the least common denominator of the elements in $H(\lambda)$ (this latter is not necessary). Let $\hat{T}(\lambda)$ be the p x p canonical matrix such that

$$\mathrm{Im}\ \hat{T}(\lambda) = \mathrm{Im}(L(\lambda);\ -\psi(\lambda)I)$$

and let $Q(\lambda)$ be an (m+p)x(m+p) unimodular matrix that canonicalizes $(L(\lambda);\ -\psi(\lambda)I)$:

$$(\hat{T}(\lambda);\ C) = (L(\lambda);\ -\psi(\lambda)I)\ Q(\lambda)$$

($Q(\lambda)$ can be found via (4.3-43)). Partition $Q(\lambda)$ as

$$Q(\lambda) = (Q_1(\lambda);\ \underbrace{Q_2(\lambda)}_{m\ \text{columns}})$$

Then $\binom{T(\lambda)}{N(\lambda)}$ is the unique canonical matrix such that

$$\operatorname{Im}\binom{T(\lambda)}{N(\lambda)} = \operatorname{Im} Q_2(\lambda)$$

Proof: It is clear that $\operatorname{Im} Q_2(\lambda) \subset \operatorname{Ker}(L(\lambda);\ -\psi(\lambda)I)$. On the other hand, let $x(\lambda) \in \operatorname{Ker}(L(\lambda);\ -\psi(\lambda)I)$; since $Q(\lambda)$ is unimodular, $x(\lambda) = Q_1(\lambda)x_1(\lambda) + Q_2(\lambda)\ x_2(\lambda)$, for unique $x_1(\lambda) \in R^p[\lambda]$ and $x_2(\lambda) \in R^m[\lambda]$. But, since $x(\lambda) \in \operatorname{Ker}(L(\lambda);\ -\psi(\lambda)I)$,

$$\begin{aligned}(L(\lambda);\ -\psi(\lambda)I)\ x(\lambda) &= (L(\lambda);\ -\psi(\lambda)I)(Q_1(\lambda);\ Q_2(\lambda))\begin{pmatrix}x_1(\lambda)\\ x_2(\lambda)\end{pmatrix}\\ &= (\hat{T}(\lambda);\ 0)\begin{pmatrix}x_1(\lambda)\\ x_2(\lambda)\end{pmatrix}\\ &= \hat{T}(\lambda)\ x_1(\lambda)\\ &= 0\end{aligned}$$

whence it follows, $\hat{T}(\lambda)$ being canonical , that $x_1(\lambda) = 0$. Thus $x(\lambda) = Q_2(\lambda)\ x_2(\lambda)$, and so $\operatorname{Ker}(L(\lambda);\ -\psi(\lambda)I) = \operatorname{Im} Q_2(\lambda)$.

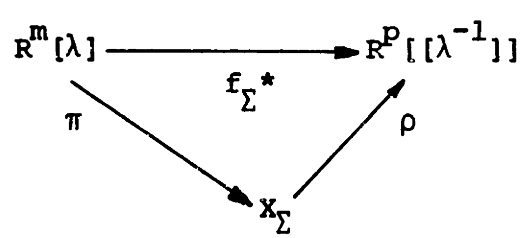
Finally, from the proof to (4.4-37), $\operatorname{Ker}(L(\lambda);\ -\psi(\lambda)I) = \operatorname{Im}\binom{T(\lambda)}{N(\lambda)}$, and the result follows. ■

4.5 Module-Theoretic Treatment of Feedback

In this section we develop a module-theoretic method for determining the effects of feedback on a linear, finite dimensional, discrete time system. We shall attempt to determine the possible changes in system structure that can be attained using feedback. We shall also determine equivalence classes of feedback laws, where two feedback laws are said to be equivalent if they result in identical closed loop system structures.

In the preceding section it has been demonstrated that two of the quantities which algebraically characterize a system, namely the state module X_Σ and the input-to-state morphism $\pi : R^m[\lambda] \rightarrow X_\Sigma$, are completely specified in terms of the unique canonical matrix $T(\lambda)$ satisfying $\mathrm{Im}\ T(\lambda) = \mathrm{Ker}\ f_\Sigma^*$; moreover, the state-to-output morphism $\rho^* : X_\Sigma \rightarrow R^p[[\lambda^{-1}]]$ is specified by $T(\lambda)$ and a few numbers from the pulse response $\{G_i,\ i \geq 0\}$. Thus, one philosophy that we could take in this section would be to determine the changes that result in Ker f_Σ^* when linear feedback is applied around the system.

However, since observability is not necessarily preserved under state feedback, Ker f_Σ^* may undergo drastic changes when small amounts of feedback are applied (If observability is lost, then Ker f_Σ^* must become "larger"). Therefore, in order to restrict our attention to a more well-posed problem, we shall consider the canonical factorization of f_Σ^*:



and determine the changes that result in the input-to-state map $\pi : R^m[\lambda] \rightarrow X_\Sigma$ when feedback is applied. Since reachability is always preserved under feedback, the resulting input-to-state map will always be surjective, and thus this problem is well-posed. Of course, not only will π change when feedback is applied, but changes in the module structure of X_Σ

will also result. Indeed, it is mostly these changes that will be of interest to us.

4.5.1 The Changes in the State Module and the Input-to-State Map Induced by State Feedback

In this section we shall develop a method for module-theoretically characterizing the changes that result in the input-to-state map π and in the structure of the state module X_Σ, when a linear, instantaneous state feedback law is incorporated into Σ. We shall see that, by appropriately modifying the framework by which one treats Σ module-theoretically, simple characterizations of the input-to-state map and the state module for the closed loop system, in terms of a new canonical matrix, will result.

Initially, we are at somewhat of a disadvantage, with respect to treating feedback, when the standard module-theoretic characterization (i.e. the one described in Section 4.2) of Σ is used. This is because each $u(\lambda) \in R^m[\lambda]$ represents an input whose last nonzero value occurs no later than time $t = 0$; while $\pi u(\lambda)$ represents the state of the system <u>after</u> this last nonzero value has been received, i.e. the state at time $t = 1$. Clearly, if we are to incorporate state feedback into Σ, we must develop some method for representing the sequence of states at times $t = \ldots, -2, -1, 0, 1$ induced by a particular $u(\lambda) \in R^m[\lambda]$.

This can be accomplished by borrowing some ideas from automata theory. We recall that, if ω_1 and ω_2 are two strings of elements from a set S:

$$\omega_i = (s_{i1}, s_{i2}, \ldots s_{i,k_i}); \; i = 1,2$$

where each $s_{ij} \in S$, then the concatenation of ω_1 and ω_2 is defined as

$$\omega_1\omega_2 = (s_{11}, s_{12}, \ldots s_{1,k_1}, s_{21}, s_{22}, \ldots s_{2,k_2})$$

(See [38, Chapter 6]). In automata theory, one uses the fact that the set of all such strings is a monoid (with identity element equal to the null string); then, by determining the action of the machine on an input string ωs, where ω is a string and $s \in S$, one can deduce much of the internal structure of the machine.

With respect to linear, constant, discrete time systems, where the inputs are taken to be elements of $R^m[\lambda]$, the operation of concatenating $u_1(\lambda)$ and $u_2(\lambda)$ results in the input $\lambda^{(1+\partial u_2)} u_1(\lambda) + u_2(\lambda)$. Since we shall be principally concerned with concatenating an element $u(\lambda) \in R^m[\lambda]$ with an element $v \in R^m \subset R^m[\lambda]$, we now define

(4.5-1) <u>Definition</u>: The <u>concatenation operator</u> for the input module $R^m[\lambda]$ is an R-linear map $\gamma : R^m[\lambda] \times R^m \to R^m[\lambda]$ defined as

$$\gamma : R^m[\lambda] \times R^m \to R^m[\lambda]$$

$$: (u(\lambda), v) \to \lambda u(\lambda) + v$$

Using this definition, one can easily see that, if

(4.5-2) $$u(\lambda) = \sum_{i=0}^{k} \lambda^i u_i \in R^m[\lambda]$$

then $u(\lambda)$ may be represented in terms of the sequence $(u_0, u_1, \ldots u_k)$ and γ as

$$u(\lambda) = \gamma(\gamma(\gamma\ldots\gamma(0, u_k), u_{k-1}), u_{k-2}), \ldots u_0)$$

Alternatively, we can define a sequence of elements of $R^m[\lambda]$ as

$$u^{(k)}(\lambda) = 0$$

$$\begin{aligned} u^{(j)}(\lambda) &= \gamma(u^{(j+1)}(\lambda),\ u_{j+1}) \\ &= \lambda u^{(j+1)}(\lambda) + u_{j+1} \end{aligned} \qquad \text{for } 0 \le j \le k-1$$

and it will then follow that

$$u(\lambda) = \gamma(u^{(0)}(\lambda),\ u_0) = \lambda u^{(0)}(\lambda) + u_0$$

When $u(\lambda)$ is given by (4.5-2), then $u^{(j)}(\lambda)$, as in (4.3-3) is clearly just

(4.5-4) $$u^{(j)}(\lambda) = \lambda^{k-j-1} u_k + \lambda^{k-j-2} u_{k-1} + \ldots + u_{j+1},\quad 0 \le j \le k-1$$

Thus, $u^{(j)}(\lambda)$ is the element of $R^m[\lambda]$ which represents the total input string as seen by the system just <u>prior</u> to time $t = -j$. Clearly then, the state at time $t = -j$ is just $\pi u^{(j)}(\lambda)$. This motivates our next definition.

(4.5-5) <u>Definition</u>: Let Σ be a linear, constant, discrete time, finite dimensional system, and let X_Σ be the state module and $\pi : R^m[\lambda] \to X_\Sigma$, the input-to-state map. Then,

(i) By a <u>linear</u>, <u>instantaneous</u>, <u>state feedback law</u> we shall mean an R-linear map $F : X_\Sigma \to R^m$.

(ii) Associated with each such F is the <u>concatenation-with-feedback</u> operator $\gamma_F : R^m[\lambda] \times R^m \to R^m[\lambda]$. γ_F is an R-linear map, and is defined as

$$\gamma_F : R^m[\lambda] \times R^m \to R^m[\lambda]$$

$$: (u(\lambda),\ v) \to \lambda u(\lambda) + v + F\pi u(\lambda)$$

Thus, the concatenation-with-feedback operator describes in a completely natural way the manner in which the sequence of inputs, as seen by the system at successively later instants of time, is built up. It should be clear that this definition of feedback is completely consistent with the ordinary "dynamical" notion of state feedback, i.e. defining

$$u_k = F\, x_k + v_k$$

in the system

$$x_{k+1} = A\, x_k + B\, u_k$$

We next define, in a completely natural way, the input-to-state map induced by a particular feedback law.

(4.5-6) **Definition**: Let Σ be a linear, constant, discrete time, finite dimensional system with state module X_Σ, and let $F : X_\Sigma \to R^m$ be a linear feedback law. Then the input-to-state map induced by F is an R-linear map $\hat{\pi}_F : R^m[\lambda] \to X_\Sigma$ defined as

$$\hat{\pi}_F u(\lambda) = \pi(\gamma_F(u_F^{(0)}(\lambda),\ u_0))$$

where, if $u(\lambda) = \sum_{i=0}^{k} \lambda^i\, u_i$, the sequence $\{u_F^{(j)}(\lambda),\ 0 \le j \le k\}$ is defined as

$$u_F^{(k)}(\lambda) = 0$$

$$u_F^{(j)}(\lambda) = \gamma_F(u_F^{(j+1)}(\lambda),\ u_{j+1}),\ \text{for } 0 \le j \le k-1$$

(4.5-7) **Remark**: It should be clear that, if $u(\lambda)$ is the input to the closed loop system, then the input, as seen by Σ (within the "loop") is just $\gamma_F(u_F^{(0)}(\lambda),\ u_0)$. Thus, $\hat{\pi}_F\, u(\lambda)$ is indeed the state to which Σ will be driven by the input $u(\lambda)$.

We can now give a simple characterization of $\hat{\pi}_F$ in terms of π and F:

(4.5-8) Lemma: Let Σ and F be as in (4.5-6). Also, redefine the state module X_Σ as X. Then

(i) The input-to-state map induced by F is explicitly given by

$$\hat{\pi}_F : R^m[\lambda] \rightarrow X$$

$$: u(\lambda) \rightarrow \pi(u(\lambda+F\pi))$$

where, if $u(\lambda) = \sum_{i=0}^{k} \lambda^i u_i$, then $u(\lambda+F\pi) = \sum_{i=0}^{k} (\lambda+F\pi)^i u_i$.

(ii) There exists an $R[\lambda]$-module X_F, R-isomorphic to X,

$$\psi : X \tilde{\rightarrow} X_F \quad \text{(as R-vector spaces)}$$

such that the R-linear map $\pi_F : R^m[\lambda] \rightarrow X_F$, defined as

$$\pi_F = \psi\hat{\pi}_F$$

is a morphism of $R[\lambda]$-modules.

The module X_F is defined as follows. Let the underlying set and the R-vector space structure of X_F be those of X, and define the action of λ on X_F as

$$\lambda : X_F \rightarrow X_F$$

$$: x \rightarrow \psi(\lambda+\pi F)\ \psi^{-1}x$$

Proof: (i) Let $u(\lambda) = \sum_{i=0}^{k} \lambda^i u_i$. Then the corresponding sequence $\{u_F^{(j)}(\lambda),\ 0 \le j \le k\}$ satisfies

$$u_F^{(k)}(\lambda) = 0$$

$$u_F^{(j)}(\lambda) = \lambda u_F^{(j+1)}(\lambda) + u_{j+1} + F\pi u_F^{(j+1)}(\lambda)$$

$$= (\lambda+F\pi)\, u_F^{(j+1)}(\lambda) + u_{j+1}$$

Therefore,

$$u_F^{(0)}(\lambda) = (\lambda+F\pi)^{k-1} u_k + (\lambda+F\pi)^{k-2} u_{k-1} + \ldots + u_1$$

Since $\hat{\pi}_F u(\lambda) = \pi(\gamma_F(u_F^{(0)}(\lambda), u_0))$, it now follows that

$$\hat{\pi}_F u(\lambda) = \pi((\lambda+F\pi)^k u_k + \ldots + u_0)$$

$$= \pi(u(\lambda+F\pi))$$

as claimed.

(ii) We first show that $\hat{\pi}_F : R^m[\lambda] \to X$ is generally not an $R[\lambda]$-morphism. This follows from the fact that

$$\hat{\pi}_F(\lambda u(\lambda)) = \pi[(\lambda+F\pi)\, u(\lambda+F\pi)]$$

$$= \lambda\pi(u(\lambda+F\pi)) + \pi F\pi(u(\lambda+F\pi))$$

$$= (\lambda+\pi F)\, \pi(u(\lambda+F\pi))$$

$$= (\lambda+\pi F)\, \hat{\pi}_F\, u(\lambda)$$

It will be shown in (4.5-17) that $\hat{\pi}_F : R^m[\lambda] \to X$ is surjective; thus, so long as $\pi F \neq 0$, $\hat{\pi}_F : R^m[\lambda] \to X$ is not an $R[\lambda]$-morphism.

However, since

Proof: (i) Let $u(\lambda) = \sum_{i=0}^{k} \lambda^i u_i$. Then the corresponding sequence $\{u_F^{(j)}(\lambda),\ 0 \le j \le k\}$ satisfies

$$u_F^{(k)}(\lambda) = 0$$

$$u_F^{(j)}(\lambda) = \lambda u_F^{(j+1)}(\lambda) + u_{j+1} + F\pi u_F^{(j+1)}(\lambda)$$

$$= (\lambda+F\pi)\ u_F^{(j+1)}(\lambda) + u_{j+1}$$

Therefore,

$$u_F^{(0)}(\lambda) = (\lambda+F\pi)^{k-1}\ u_k + (\lambda+F\pi)^{k-2}\ u_{k-1}+\ldots+ u_1$$

Since $\hat{\pi}_F u(\lambda) = \pi(\gamma_F(u_F^{(0)}(\lambda),\ u_0))$, it now follows that

$$\hat{\pi}_F u(\lambda) = \pi((\lambda+F\pi)^k\ u_k +\ldots+ u_0)$$

$$= \pi(u(\lambda+F\pi))$$

as claimed.

(ii) We first show that $\hat{\pi}_F : R^m[\lambda] \to X$ is generally not an $R[\lambda]$-morphism. This follows from the fact that

$$\hat{\pi}_F(\lambda u(\lambda)) = \pi[(\lambda+F\pi)\ u(\lambda+F\pi)]$$

$$= \lambda\pi(u(\lambda+F\pi)) + \pi F\pi(u(\lambda+F\pi))$$

$$= (\lambda+\pi F)\ \pi(u(\lambda+F\pi))$$

$$= (\lambda+\pi F)\ \hat{\pi}_F\ u(\lambda)$$

It will be shown in (4.5-17) that $\hat{\pi}_F : R^m[\lambda] \to X$ is surjective; thus, so long as $\pi F \neq 0$, $\hat{\pi}_F : R^m[\lambda] \to X$ is not an $R[\lambda]$-morphism.

However, since

$$\begin{aligned}\pi_F(\lambda u(\lambda)) &= \psi\hat{\pi}_F(\lambda u(\lambda)) \\ &= \psi(\lambda+\pi F)\ \hat{\pi}_F u(\lambda) \\ &= \psi(\lambda+\pi F)\ \psi^{-1}\psi\hat{\pi}_F u(\lambda) \\ &= \lambda\pi_F u(\lambda)\end{aligned}$$

it follows that $\pi_F : R^m[\lambda] \to X_F$ is an $R[\lambda]$-morphism. This completes the proof. ■

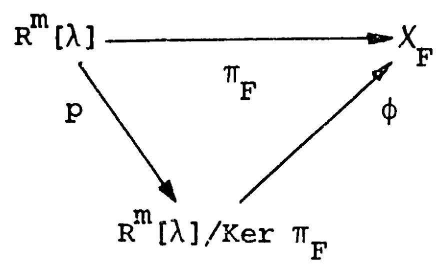
(4.5-10) Remark: It is essential that we interpret $R^m[\lambda]$ as a left $R[\lambda]$ module, in order for (4.5-9) to be valid. That is we must think of $u(\lambda)$ as $\sum_{i=0}^{k} \lambda^i u_i$, and not as $\sum_{i=0}^{k} u_i \lambda^i$. This is due to the fact that in the expression $u(\lambda+F\pi) = \sum_{i=0}^{k} (\lambda+F\pi)^i u_i$, the operator $(\lambda+F\pi)^i$ must appear to the left of the corresponding u_i.

(4.5-11) Remark: The method of redefining the action of λ on X to get a new module X_F is simply one of many ways that one can get a new module from an old one. A similar construction was used by Kalman (c.f. [39, Thm. 5.11]).

We shall shortly develop a method for canonically characterizing both X_F and $\pi_F : R^m[\lambda] \to X_F$. To accomplish these characterizations, we shall determine a unique canonical matrix $T_F(\lambda)$ such that

$$\text{Im } T_F(\lambda) = \text{Ker } \pi_F \subset R^m[\lambda]$$

Then we shall demonstrate that $\pi_F : R^m[\lambda] \to X_F$ is surjective, so that in the canonical factorization of π_F



the morphism ϕ is an isomorphism. It will then follow that the module structure of X_F can be completely "read off" the canonical matrix $T_F(\ )$, as in (4.3-52).

In order to accomplish the above goals, we must determine an inverse to the R-linear map

$$L_F : R^m[\lambda] \to R^m[\lambda]$$

$$: u(\lambda) \to u(\lambda+F\pi)$$

For then, since $\pi_F : R^m[\lambda] \to X_F$ is given by

$$\pi_F = \psi\pi L_F$$

it follows that, ψ being an R-isomorphism,

$$\text{Ker } \pi_F = L_F^{-1} \text{ Ker } \pi$$

$$= L_F^{-1} \text{Im } T(\lambda)$$

At first glance it might appear that $v(\lambda) = u(\lambda+F\pi)$ if and only if $v(\lambda-F\pi) = u(\lambda)$. However, closer inspection reveals that this conjecture is fallacious.

(4.5-12) <u>Lemma</u>: Consider the state module X and the input-to-state map $\pi : R^m[\lambda] \to X$. Let $\dim(X) = n$ (as an R-vector space) and let $\{q_i, i \in \underline{n}\}$ be a basis for X (again, as an R-vector space). Define the free $R[\lambda]$-module generated by $\{q_i\}$ as $X[\lambda]$:

$$\mathcal{X}[\lambda] = \{\sum_{i=0}^{k} \lambda^i x_i \,|\, x_i \in \text{R-vector space } \mathcal{X}, \text{ and } k \in Z_+\}$$

Finally, define the $R[\lambda]$-morphism $\underline{F} : \mathcal{X}[\lambda] \to R^m[\lambda]$ as the natural extension of $F : \mathcal{X} \to R^m$, i.e.

$$\underline{F} : \mathcal{X}[\lambda] \to R^m[\lambda]$$

$$: \sum_{i=0}^{k} \lambda^i x_i \to \sum_{i=0}^{k} \lambda^i F x_i$$

Then

(i) The R-vector space $\mathcal{X} \times \mathcal{X}[\lambda]$ can be made into an $R[\lambda]$-module by defining

$$\lambda(q, x(\lambda)) = (\lambda q, \lambda x(\lambda) + q)$$

(ii) The input-to-state map $\pi : R^m[\lambda] \to \mathcal{X}$ extends to an $R[\lambda]$-morphism

$$\pi^* : R^m[\lambda] \to \mathcal{X} \times \mathcal{X}[\lambda]$$

$$: u(\lambda) \to (\pi u(\lambda), \underline{\pi} u(\lambda))$$

where $\underline{\pi} : R^m[\lambda] \to \mathcal{X}[\lambda]$ is defined as follows. For $u(\lambda) = \sum_{i=0}^{k} \lambda^i u_i \in R^m[\lambda]$, define

$$\underline{\pi}\, u(\lambda) = \sum_{j=0}^{k-1} \lambda^j \pi u^{(j)}(\lambda)$$

where $\{u^{(j)}(\lambda), 0 \leq j \leq k-1\}$ are as in (4.5-4).

(iii) For every $u(\lambda) \in R^m[\lambda]$ there exists a unique $v(\lambda) \in R^m[\lambda]$ such that

$$u(\lambda) = v(\lambda + F\pi)$$

Moreover, $v(\lambda)$ is given by

(4.5-13) $$v(\lambda) = u(\lambda) - \underline{F}\,\underline{\pi}\,u(\lambda)$$

(It is suggested that the reader read remark (4.5-15) before proceeding to the proof of (4.5-12.)

Proof: (i) Since any R-vector space V can be made into an $R[\lambda]$-module simply by defining the action of λ on V to be that of a given R-endomorphism on V, it is clear that we can define $\lambda(q, x(\lambda))$ as $(\lambda q, \lambda x(\lambda) + q)$.

(ii) It is clear that $\pi : R^m[\lambda] \to X$ and $\underline{\pi} : R^m[\lambda] \to X[\lambda]$ are R-linear. Thus, to verify that $\pi^* : R^m[\lambda] \to X \times X[\lambda]$ is an $R[\lambda]$-morphism, it is enough to show that

$$\pi^*(\lambda u(\lambda)) = \lambda\pi^*\, u(\lambda)$$

Since $\pi : R^m[\lambda] \to X$ is an $R[\lambda]$-morphism, $\pi(\lambda u(\lambda)) = \lambda\pi u(\lambda)$. Also, it is easily seen that

$$[\lambda u]^{(j)}(\lambda) = \begin{cases} u^{(j-1)}(\lambda), & \text{for } 1 \leq j \leq k \\ u(\lambda), & \text{for } j = 0 \end{cases}$$

Therefore,

$$\begin{aligned} \pi^*(\lambda u(\lambda)) &= (\pi(\lambda u(\lambda)), \sum_{j=0}^{k} \lambda^j\, \pi([\lambda u]^{(j)}(\lambda))) \\ &= (\lambda\pi u(\lambda), \sum_{j=1}^{k} \lambda^j\, \pi u^{(j-1)}(\lambda) + \pi u(\lambda)) \\ &= \lambda(\pi u(\lambda), \sum_{j=0}^{k-1} \lambda^j\, \pi u^{(j)}(\lambda)) \\ &= \lambda\pi^*\, u(\lambda) \end{aligned}$$

whence π^* is an $R[\lambda]$-morphism.

(iii) We first prove that the R-linear map $(\lambda+F\pi)^i : R^m \to R^m[\lambda]$ satisfies

(4.5-14) $$(\lambda+F\pi)^i = \lambda^i + \sum_{\ell=0}^{i-1} \lambda^\ell F\pi(\lambda+F\pi)^{i-\ell-1}, \text{ for } i \geq 1$$

Indeed (4.5-14) is trivially satisfied when i = 1. Suppose it is satisfied when i = k - 1; then

$$(\lambda+F\pi)^k = (\lambda+F\pi)^{k-1}(\lambda+F\pi)$$

$$= (\lambda^{k-1} + \sum_{\ell=0}^{k-2} \lambda^\ell F\pi(\lambda+F\pi)^{k-\ell-2})(\lambda+F\pi)$$

$$= \lambda^k + \lambda^{k-1} F\pi + \sum_{\ell=0}^{k-2} \lambda^\ell F\pi(\lambda+F\pi)^{k-\ell-1}$$

$$= \lambda^k + \sum_{\ell=0}^{k-1} \lambda^\ell F\pi(\lambda+F\pi)^{k-\ell-1}$$

and the proof of (4.5-14) follows by induction.

Now suppose that there exist $v(\lambda)$, $w(\lambda) \in R^m[\lambda]$ such that, for a given $u(\lambda) \in R^m[\lambda]$,

$$v(\lambda+F\pi) = w(\lambda+F\pi) = u(\lambda)$$

But then, from (4.5-14) it follows that $\partial v = \partial u = \partial w$. Therefore, if $\partial u = k$, and if

$$v(\lambda) = \sum_{i=0}^{k} \lambda^i v_i$$

$$w(\lambda) = \sum_{i=0}^{k} \lambda^i w_i$$

it follows from $v(\lambda+F\pi) = w(\lambda+F\pi)$ and (4.5-14) that

$$(v_0-w_0) + \sum_{i=1}^{k} (\lambda^i + \sum_{\ell=0}^{i-1} \lambda^\ell F\pi(\lambda+F\pi)^{i-\ell-1})(v_i-w_i) = 0$$

But this implies that $v_i = w_i$ for all $0 \le i \le k$. Consequently, if there exists $v(\lambda) \in R^m[\lambda]$ such that $v(\lambda+F\pi) = u(\lambda)$, then $v(\lambda)$ is unique.

Now let $v(\lambda)$ be as defined in (4.5-13). It then follows that

$$v(\lambda+F\pi) = u(\lambda+F\pi) - \sum_{j=0}^{k-1} (\lambda+F\pi)^j F\pi u^{(j)}(\lambda)$$

$$= \sum_{i=0}^{k} (\lambda+F\pi)^i u_i - \sum_{j=0}^{k-1} (\lambda+F\pi)^j F\pi \sum_{\ell=0}^{k-j-1} \lambda^\ell u_{j+\ell+1}$$

Analogously to the manner in which (4.5-14) was proved, one can also easily prove that

$$(\lambda+F\pi)^i = \lambda^i + \sum_{\ell=0}^{i-1} (\lambda+F\pi)^\ell F\pi\lambda^{i-\ell-1}, \text{ for } i \ge 1$$

Therefore,

$$v(\lambda+F\pi) = u_0 + \sum_{i=1}^{k} (\lambda^i + \sum_{\ell=0}^{i-1} (\lambda+F\pi)^\ell F\pi\lambda^{i-\ell-1}) u_i$$

$$- \sum_{j=0}^{k-1} (\lambda+F\pi)^j F\pi \sum_{\ell=0}^{k-j-1} \lambda^\ell u_{j+\ell+1}$$

$$= u(\lambda) + \sum_{i=1}^{k} \sum_{\ell=0}^{i-1} (\lambda+F\pi)^\ell F\pi\lambda^{i-\ell-1} u_i$$

$$- \sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} (\lambda+F\pi)^j F\pi\lambda^\ell u_{j+\ell+1}$$

$$= u(\lambda)$$

and the proof is complete.

(4.5-15) **Remark**: One should think of elements of $X[\lambda]$ as being **formal** vector polynomials of the form $\sum_{i=0}^{k} \lambda^i x_i$, where each

$x_i \in \mathcal{X}$ is to be treated as a vector in the R-vector space $\mathcal{X}$. Thus, there may be cause for some initial confusion, because the quantity λx can have two meanings: if we interpret x as a module element then $\lambda x \in \mathcal{X}$ is the result of acting on x with λ; if we interpret x is a vector then $\lambda x \in \mathcal{X}[\lambda]$ is a polynomial vector of degree one.

(4.5-16) Remark: The action of λ on the module $\mathcal{X} \times \mathcal{X}[\lambda]$ suggests via (4.3-52) that we could represent $\mathcal{X} \times \mathcal{X}[\lambda]$ as

$$\mathcal{X} \times \mathcal{X}[\lambda] \cong R^{2n}[\lambda]/\mathrm{Im}\begin{pmatrix} I-A \\ -I \end{pmatrix}$$

where $n = \dim(\mathcal{X})$ and $A : \mathcal{X} \to \mathcal{X}$ is the R-linear endomorphism whose action on $\mathcal{X}$ is the same as the action of λ on $\mathcal{X}$.

As an immediate consequence of (4.5-12), we now have:

(4.5-17) Corollary: Let $\pi_F : R^m[\lambda] \to \mathcal{X}_F$ be the input-to-state morphism as defined in (4.5-8). Then π_F is surjective.

Proof: Since $\pi_F = \psi\hat{\pi}_F$, and ψ is an R-isomorphism, it follows that π_F is surjective if and only if $\hat{\pi}_F$ is surjective. Thus, let $x \in \mathcal{X}$, and choose $u(\lambda) \in R^m[\lambda]$ so that

$$\pi u(\lambda) = x$$

(This is possible because $\pi : R^m[\lambda] \to \mathcal{X}$ is surjective.) But then, from (4.5-12),

$$\begin{aligned} \hat{\pi}_F(u(\lambda) - \underline{F}\,\underline{\pi}\, u(\lambda)) &= \pi u(\lambda) \\ &= x \end{aligned}$$

Thus $\hat{\pi}_F$ is surjective, and the proof is complete.

(4.5-18) **Remark:** This last result simply says, in module-theoretic terms, that reachability is preserved under state feedback.

We now have the desired module-theoretic characterization of X_F and $\pi_F : R^m[\lambda] \to X_F$:

(4.5-19) **Theorem:** Let $T(\lambda) = (t_1(\lambda); t_2(\lambda); \ldots t_m(\lambda))$ be the canonical matrix such that $\text{Im } T(\lambda) = \text{Ker } f_\Sigma^* = \text{Ker } \pi$, and let $\{\nu_i, i \in \underline{m}\}$ be the associated degree integers. Then

(i) The m x m matrix $U(\lambda)$, defined as

$$U(\lambda) = T(\lambda) - \underline{F}\,\underline{\pi}\,T(\lambda) \tag{4.5-20}$$

has the property that its columns are free generators for Ker π_F. In (4.5-20), $\underline{F}\,\underline{\pi}\,T(\lambda)$ is an m x m matrix whose ith column is $\underline{F}\,\underline{\pi}\,t_i(\lambda)$.

(ii) If $U(\lambda) = (u_1(\lambda), \ldots u_m(\lambda))$, then $\partial u_i = \partial t_i = \nu_i$, for $i \in \underline{m}$. Moreover, if $T_F(\lambda)$ is the unique canonical matrix such that $\text{Im } T_F(\lambda) = \text{Im } U(\lambda)$, and if

$$T_F(\lambda) = (t_{F,1}(\lambda); \ldots t_{F,m}(\lambda))$$

then

$$t_{F,1}(\lambda) = u_1(\lambda), \text{ and} \tag{4.5-21}$$

$$t_{F,i}(\lambda) \equiv u_i(\lambda)(\text{mod}(t_{F,1}(\lambda); \ldots t_{F,i-1}(\lambda))),$$

$$\text{for } 2 \leq i \leq m$$

moreover, the sets of degree and row integers for $T_F(\lambda)$ are identical to those for $T(\lambda)$.

(iii) The state module X_F is $R[\lambda]$-isomorphic to $R^m[\lambda]/\text{Im } T_F(\lambda)$, and representations for X_F and $\pi_F : R^m[\lambda] \to X_F$ may be obtained from $T_F(\lambda)$ as in (4.3-52) and (4.3-80).

Proof: (i) Since $t_i(\lambda) \in \text{Ker } \pi$, it follows from (4.5-12)(iii) that

$$u_i(\lambda+F\pi) = t_i(\lambda) \in \text{Ker } \pi$$

Thus $\pi_F\, u_i(\lambda) = \psi\pi(u_i(\lambda+F\pi)) = \psi\pi t_i(\lambda) = 0$, and it follows that

$$\text{Im } U(\lambda) \subset \text{Ker } \pi_F$$

On the other hand, if $u(\lambda) \in \text{Ker } \pi_F$, then $u(\lambda+F\pi) \in \text{Ker } \pi$, so that

$$u(\lambda+F\pi) = T(\lambda)\; q(\lambda)$$

for some $q(\lambda) \in R^m[\lambda]$. But then, from (4.5-12) (iii),

$$\begin{aligned} u(\lambda) &= T(\lambda)\; q(\lambda) - \underline{F}\; \underline{\pi}\; T(\lambda)\; q(\lambda) \\ &= U(\lambda)\; q(\lambda) \end{aligned}$$

so that

$$\text{Ker } \pi_F \subset \text{Im } U(\lambda)$$

Since $\pi_F : R^m[\lambda] \to X_F$ is surjective, $X_F \cong R^m[\lambda]/\text{Ker } \pi_F$; and thus the fact that X_F is torsion implies that rank Ker $\pi_F = m$. That is, the m columns of $U(\lambda)$ are indeed <u>free</u> generators for Ker π_F.

(ii) From the definitions of $\underline{F}$ and $\underline{\pi}$ in (4.5-12) it is easy to see that $\partial(\underline{F}\; \underline{\pi}\; u(\lambda)) < \partial u(\lambda)$ for all $u(\lambda) \in R^m[\lambda]$. Thus, $\partial u_i = \partial t_i$, as claimed. Moreover, the coefficient of λ^{ν_i} in $t_i(\lambda)$ is identical to the coefficient of λ^{ν_i} in $u_i(\lambda)$. Thus, to canonicalize $U(\lambda)$, we only need to change the low order coefficients in each $u_i(\lambda)$. Since

$\nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$, it then follows that $U(\lambda)$ may be canonicalized in the manner indicated by (4.3-21). Since the high order coefficient of $t_{F,i}(\lambda)$ is not changed in this process, it follows that the row and degree integers remain invariant.

(iii) We have already observed above that the fact that π_F is surjective implies that $X_F \cong R^m[\lambda]/\mathrm{Im}\ T_F(\lambda)$.

While (4.5-19) completely describes, in module-theoretic terms, the changes that the state model and the input-to-state map undergo when state feedback is applied, it is not as concrete a result as one might hope to derive. The lack of concreteness is due to the fact that X and X_F are abstract vector spaces, and that $F : X \rightarrow R^m$ is just a linear map. That is, the next logical step is to determine a convenient basis in X and to represent F as a matrix with respect to this basis. This we shall do in the next section.

4.5.2 System Invariants and Equivalence Classes of Feedback Laws

In Section 4.3.2 we saw that there exists a "canonical" basis in the state module X such that, with respect to this basis the matrices for the maps $A : X \rightarrow X$ and $B : R^m \rightarrow X$ have the simple structures as given in (4.3-54) and (4.3-76). In this section we shall see that there is a second "canonical basis in X, one which allows a simple treatment of the effects of state feedback on the system. We shall be able to determine complete sets of system invariants under state feedback, and under state feedback and coordinate transformations in the input space; these results will be identical to those obtained by Popov ([60]), Kalman ([42]), Brunovsky ([13]), and Wonham and Morse ([75]); the first nearly complete treatment is due to Luenberger ([50]). We shall also determine equivalence classes of feedback laws, where two feedback laws are equivalent if and only if they result in identical input-to-state structures.

Our primary motivation in looking for a canonical basis in X is to find a basis which, when used as a set of free generators for $X[\lambda]$, results in the n x m polynomial matrix representing $\underline{\pi}\, T(\lambda)$ having a particularly simple form. If this can be achieved, then by also expressing F in this basis, the expression (4.5-20) for $U(\lambda)$ will be quite easy to interpret.

However, the columns of $\underline{\pi}\, T(\lambda)$ are simply

$$\underline{\pi}\, t_i(\lambda) \quad , \text{ for } i \in \underline{m}$$

and, from (4.5-12) it follows that

$$\underline{\pi}\, t_i(\lambda) = \sum_{j=0}^{\nu_i - 1} \lambda^j \, \pi t_i^{(j)}(\lambda)$$

where the polynomial vectors $t_i^{(j)}(\lambda)$, $0 \le j \le \nu_i-1$, are obtained from $t_i(\lambda)$ as the polynomial vectors $u^{(j)}(\lambda)$ were obtained from $u(\lambda)$ in (4.5-4). What could be more natural than to use the elements $\{\pi t_i^{(j)}(\lambda);$ $0 \le j \le \nu_i-1,\ i \in \underline{m}\}$ as a basis in X?

(4.5-22) **Theorem:** (Assume that $\nu_i > 0$ for all i). The set of elements $\{\pi t_i^{(j)};\ 0 \le j \le \nu_i-1,\ i \in \underline{m}\}$ is a basis for (the R-vector space) X. Moreover, with respect to this basis the endomorphism $A : X \to X : x \to \lambda x$ is represented by a matrix A of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & & A_{2m} \\ \vdots & & & \\ A_{m1} & A_{m2} & & A_{mm} \end{pmatrix}$$

where A_{ij} is $\nu_i \times \nu_j$, and where

$$A_{ii} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & 0 & 1 \\ x & x & & x & x \end{pmatrix} \tag{4.5-23}$$

and

$$A_{ij} = \begin{pmatrix} 0 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ x & x & x \end{pmatrix} \quad , \text{ for } i \neq j$$

where the x's denote possible nonzero elements, also, with respect to this basis and the standard basis in R^m, the R-linear map $\pi \cdot i : R^m \to X$ is represented by an $n \times m$ matrix B satisfying

$$(4.5\text{-}24) \qquad BP^{-1} = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & 0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & 0 \\ x & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & 0 \\ x & x & 1 & 0 \cdots & 0 \\ 0 & 0 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ x & x & \cdots & x & 1 \end{bmatrix} \begin{matrix} \left.\right\} \nu_1 \text{ rows} \\ \left.\right\} \nu_2 \text{ rows} \\ \\ \left.\right\} \nu_m \text{ rows} \end{matrix}$$

where P is the unique permutation matrix satisfying $(Px)_i = x_{m_i}$, for all $i \in \underline{m}$.

Proof: We first show that the set $\{ \pi t_i^{(j)}(\lambda) \}$ is a basis for X. Indeed, suppose that there are numbers $a_{ij} \in R$ such that

$$\sum_{i \in \underline{m}} \sum_{j=0}^{\nu_i - 1} a_{ij} \, \pi t_i^{(j)}(\lambda) = 0$$

But this can be true only if

$$u(\lambda) \triangleq \sum_{i,j} a_{ij} \, t_i^{(j)}(\lambda) \in \operatorname{Ker} \pi$$

That is, the unique reduction of $u(\lambda)$ modulo $T(\lambda)$ is

$$u(\lambda) = T(\lambda)q(\lambda) + 0$$

However, from the definition of the $t_i^{(j)}(\lambda)$, it follows that $\partial u_{m_i} < \nu_i$, for all $i \in \underline{m}$; thus the unique reduction of $u(\lambda)$ modulo $T(\lambda)$ is

$$u(\lambda) = T(\lambda) \cdot 0 + u(\lambda)$$

It thus follows that

$$\sum_{i,j} a_{i,j}\, t_i^{(j)}(\lambda) = 0$$

Finally, it is easily seen that the $t_i^{(j)}(\lambda)$ are R-linearly independent; thus $a_{i,j} = 0$ for all i,j , and the proof of the R-independence of the $\pi t_i^{(j)}(\lambda)$ is established.

We now show that the matrix for A, with respect to the basis $\{ \pi t_1^{(0)}(\lambda), \pi t_1^{(1)}(\lambda), \ldots \pi t_1^{(\nu_1-1)}(\lambda), \ldots \pi t_m^{(\nu_m-1)}(\lambda)\}$, is as given in (4.5-23). To this end, we note that

$$\lambda t_i^{(j)}(\lambda) = \begin{cases} t_i^{(j-1)}(\lambda) - t_{i,j} \;, & 1 \leq j \leq \nu_i-1 \\ \\ t_i(\lambda) - t_{i,0} \;, & j = 0 \end{cases}$$

where $t_{i,j}$ is the coefficient of λ^j in $t_i(\lambda)$. Therefore,

$$A\pi t_i^{(j)}(\lambda) = \begin{cases} \pi t_i^{(j-1)}(\lambda) - \pi t_{i,j} \;, & 1 \leq j \leq \nu_i-1 \\ \\ -\pi t_{i,0} \;, & j = 0 \end{cases}$$

We now note that, as a consequence of $T(\lambda)$ being canonical, the mxm matrix

$$T_h \triangleq (t_{1,\nu_1};\ t_{2,\nu_2};\ \ldots\ t_{m,\nu_m})$$

is nonsingular. Let $c_{i,j} \varepsilon R^m$ be the unique vector satisfying

$$t_{i,j} = T_h\, c_{i,j} \;;\ \text{for } 0 \leq j \leq \nu_i-1,\ i \varepsilon \underline{m}$$

But then

$$\begin{aligned} \pi t_{i,j} &= \pi T_h\, c_{i,j} \\ &= (\pi t_1^{(\nu_1-1)}(\lambda);\ \ldots\ \pi t_m^{(\nu_m-1)}(\lambda))\, c_{i,j} \end{aligned}$$

Consequently,

$$A\pi t_i^{(j)}(\lambda) = \begin{cases} \pi t_i^{(j-1)}(\lambda) + \text{linear combination of the } \pi t_i^{(\nu_i-1)}(\lambda), & 1 \le j \le \nu_i - 1 \\ 0 + \text{linear combination of the } \pi t_i^{(\nu_i-1)}(\lambda), & j = 0 \end{cases}$$

Thus the matrix for A is as given in (4.5-23); moreover, we have derived expressions for the x's in (4.5-23).

To verify (4.5-24), we note that the i'th column of B must represent the vector $\pi e_i \in \mathcal{X}$. But

$$(e_1;\ e_2;\ \dots\ e_m) = T_h\, T_h^{-1}$$

so that

$$(\pi e_1;\ \dots\ \pi e_m) = \pi\, T_h\, T_h^{-1}$$

$$= (\pi t_1^{(\nu_1-1)}(\lambda);\ \dots\ \pi t_m^{(\nu_m-1)}(\lambda))\, T_h^{-1}$$

Thus the matrix for B is

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & .\ 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & .. & 1 \end{bmatrix} T_h^{-1}$$

Finally, we note that PT_h is lower triangular, with 1's on the diagonal; thus $T_h^{-1}\, P^{-1}$ is also lower triangular with 1's on the diagonal, and BP^{-1} has the form of (4.5-24). ■

(4.5-25) **Remark**: In (4.5-22) we have assumed that each ν_i is strictly positive. However, we can easily generalize to the case where one or more of the ν_i's is zero.

In the case where $\nu_i = 0$ for all $i \in \underline{m}$, it follows that $T(\lambda) = I$; consequently, $f_\Sigma{}^*$ is the zero morphism, X has dimension zero, and A and B are meaningless quantities.

In the less trivial case where $\nu_1 = \ldots = \nu_k = 0$ and $\nu_{k+1} > 0$ for some $1 \leq k < m$, it is easily seen that

$$\{\pi t_i{}^{(j)}(\lambda);\ 0 \leq j \leq \nu_i - 1,\ k+1 \leq i \leq m\ \}$$

is a basis for X, and that the matrix for A with respect to this basis is as given in (4.5-22), if one adopts the convention that $A_{i,j}$ is to be deleted if $\nu_i = 0$ or $\nu_j = 0$. The matrix for B is almost as easily obtained. Clearly, now

$$T_h = (t_1(\lambda); \ldots\ t_k(\lambda);\ \hat{T}_h)$$

for some $m \times (m-k)$ matrix $\hat{T}_h$. Thus

$$\begin{aligned} \pi(e_1; \ldots\ e_m) &= \pi T_h\ T_h{}^{-1} \\ &= \pi(t_1(\lambda); \ldots\ t_k(\lambda);\ \hat{T}_h)\ T_h{}^{-1} \\ &= (0; \ldots\ 0;\ \pi t_{k+1}{}^{(\nu_{k+1}-1)}; \ldots\ \pi t_m{}^{(\nu_m-1)})\, T_h^{-1} \end{aligned}$$

and the matrix for B is of the form

$$B = \begin{bmatrix} 0\ldots0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0\ldots0 & 0 & 0 & \ldots & 0 \\ 0\ldots0 & 1 & 0 & \ldots & 0 \\ 0\ldots0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0\ldots0 & 0 & 0 & \ldots & 0 \\ 0\ldots0 & 0 & 1 & 0\ . & 0 \\ 0\ldots0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0\ldots0 & 0 & 0 & \ldots & 1 \end{bmatrix} T_h{}^{-1}$$

and it is immediately seen that BP^{-1} is of the form:

$$BP^{-1} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ x & x & \dots & x & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ x & x & \dots & x & x & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ x & x & \dots & & & & & x & 1 \end{bmatrix} \begin{matrix} \left.\right\} \nu_{k+1} \text{ rows} \\ \left.\right\} \nu_{k+2} \text{ rows} \\ \\ \end{matrix}$$

$$\underbrace{\qquad\qquad}_{k \text{ columns}} \quad \underbrace{\qquad\qquad}_{m-k \text{ columns}}$$

We can now write a more concrete expression for (4.5-20):

(4.5-26) __Lemma__: Let the R-linear map $F: X \to R^m$ be represented with respect to the basis $\{\pi t_i^{(j)}(\lambda)\}$ in X and the standard basis in R^m by the mxn matrix F. Then, the mxm polynomial matrix $U(\lambda)$, defined in (4.5-20), is given by

$$U(\lambda) = T(\lambda) - F\,S(\lambda)$$

where $S(\lambda)$ is the following nxm polynomial matrix:

$$S(\lambda) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \lambda & 0 & & \\ \vdots & \vdots & & \\ \lambda^{\nu_1-1} & 0 & & \\ 0 & 1 & & \\ 0 & \lambda & & \\ \vdots & \vdots & & \\ 0 & \lambda^{\nu_2-1} & & \\ 0 & 0 & & \\ \vdots & \vdots & & \\ 0 & 0 & \dots & \lambda^{\nu_m-1} \end{bmatrix}$$

(If $\nu_i = 0$, then column i of $S(\lambda)$ is zero.)

Proof: If $\nu_i \neq 0$, then

$$\underline{\pi t}_i(\lambda) = \sum_{j=0}^{\nu_i - 1} \lambda^j \, \pi t_i^{(j)}(\lambda)$$

by definition. On the other hand, if $\nu_i = 0$, then $\underline{\pi t}_i(\lambda) = 0$. ■

We have seen how the structure of the state module and the input-to-state morphism can be changed by applying state feedback; clearly, these structural changes are reflected in the changes in the canonical matrix. Thus, by determining all possible canonical matrices that can be achieved by state feedback, we shall be able to examine the invariants of the input-to-state structure.

(4.5-27) Theorem: A complete set of invariants, under state feedback, of the input-to-state structure of Σ consists of the sets $\{ \nu_i, i\epsilon\underline{m} \}$, $\{ m_i, i\epsilon\underline{m} \}$, and $\{ t_{j,i,\nu_i}, j > m_i \}$, where t_{j,i,ν_i} is the coefficient of λ^{ν_i} in $t_{j,i}(\lambda)$.

Proof: We have shown in the proof to (4.5-19) that these sets are invariant under state feedback; that is, the values of the elements in these sets are common to both canonical matrices $T(\lambda)$ and $T_F(\lambda)$. On the other hand, it is clear from (4.5-26) that all other parameters in $T(\lambda)$ can be changed by state feedback. Thus, these sets form a complete set of invariants. But clearly, if any of these parameters is changed, the remaining parameters still define an input-to-state structure; but this structure will be different from the original one. Thus these sets form a independent set of invariants also. ■

We can also determine the invariants of the input-to-state structure when coordinate transformations on the space of input values is allowed.

(4.5-28) __Theorem__: A complete set of independent invariants, under state feedback and coordinate transformations in the space of input values, of the input-to-state structure of Σ consists of the set $\{ \nu_i, i \in \underline{m} \}$.

Proof: If $T(\lambda)$ and $\hat{T}(\lambda)$ are two canonical matrices with the same sets of degree integers, and if T_h and $\hat{T}_h$ are constructed from $T(\lambda)$ and $\hat{T}(\lambda)$ as in the proof to (4.5-22), then it is easily seen that there exists a matrix F such that $T(\lambda) = T_h \hat{T}_h^{-1} \hat{T}(\lambda) - F\, S(\lambda)$. Thus, with transformations in the input space, the sets $\{ m_i, i \in \underline{m} \}$ and $\{ t_{j,i,\nu_i}, j > m_i \}$ are no longer invariant. On the other hand, it is easy to see that, if G is nonsingular, the degree integers of $GT(\lambda)$ are identical to those of $T(\lambda)$. Since the set $\{\nu_i, i \in \underline{m}\}$ is invariant under feedback, the proof follows. ∎

We now define the concept of equivalent feedback laws.

(4.5-29) __Definition__: Two feedback laws $F_i : X \to R^m$, $i=1,2$, are said to be equivalent with respect to Σ if the two corresponding canonical matrices $T_{F_i}(\lambda)$ are identical; if F_1 and F_2 are equivalent with respect to Σ, we write $F_1 \sim F_2$.

Thus, the feedback laws F_1 and F_2 are equivalent if they result in the same input-to-state structures. The following result provides us with a simple test to determine when two feedback laws are equivalent.

(4.5-30) __Theorem__: Let the canonical partitioning of $T(\lambda)$ be

$$T(\lambda) = (T_1(\lambda); T_2(\lambda); \ldots \ T_\alpha(\lambda))$$

and let $S(\lambda)$ be partitioned conformably with $T(\lambda)$ as

$$S(\lambda) = (S_1(\lambda); S_2(\lambda); \ldots \ S_\alpha(\lambda))$$

Then, two feedback laws, F and $\hat{F}$, are equivalent if and only if their corresponding matrices (with respect to the standard basis in R^m and the canonical basis in X given by (4.5-22)) satisfy

$$F\,S_1(\lambda) = \hat{F}\,S_1(\lambda)$$

and

$$(F - \hat{F})\,S_i(\lambda) \equiv 0 \;(\bmod\; \tilde{T}_{i-1}(\lambda)\;)\;,\; i > 1$$

where $\tilde{T}_{i-1}(\lambda)$ is the canonical matrix such that

$$\mathrm{Im}\,\tilde{T}_{i-1}(\lambda) = \mathrm{Im}((T_1(\lambda);\ldots\,T_{i-1}(\lambda))-F(S_1(\lambda);\ldots S_{i-1}(\lambda)))$$

Proof: This is an immediate application of (4.5-21).

4.5.3 (A,B)-Invariant and (A,B)-Controllability Subspaces: Module-Theoretic Implications

We have seen in Section 4.3.3 that the A-invariant subspaces of the state module X can be characterized by the lattice of left divisors of the canonical matrix $T(\lambda)$. Since the canonical matrix undergoes well-defined changes when state feedback is applied, it would seem that these changes would induce changes in the invariant subspace structure of the state module. That is, one might hope that the canonical matrix would provide us with a parameterization of (A,B)-invariant subspaces.

On the surface, this seems reasonable enough. That is, since

$$\mathrm{Im}\;T_F(\lambda) = \mathrm{Im}(\;T(\lambda) - F\,S(\lambda))$$

we have only to somehow determine the left divisors of $T(\lambda) - F\,S(\lambda)$. However, there are two major difficulties. First, there appears to be no effective method for determining left divisors of a polynomial matrix. Secondly, and perhaps more importantly, the basis $\{\pi t_i^{(j)}(\lambda)\}$ does not remain invariant under all posible state feedback laws (It does remain invariant under a nontrivial class of feedback laws, a class which allows for every possible input-to-state structure, subject only to the constraints imposed by the feedback invariants.). Of course, if the basis does not remain invariant, it will be difficult to relate subspaces in X with those in X_F.

Surprisingly, it turns out to be easier to parameterize controllability subspaces than (A,B)-invariant subspaces. The reason is as follows. One can easily show that, for every $u(\lambda) \varepsilon$ Ker $f_\Sigma{}^*$, the coefficients of the corresponding $\underline{\pi}u(\lambda) \varepsilon X[\lambda]$ span a controllability subspace. It then becomes a fairly simple matter to characterize controllability subspaces in terms of $\underline{\pi}$ Ker $f_\Sigma{}^*$.

No more will be said about (A,B)-invariant and (A,B)-controllability subspaces, because the originally hoped-for result (in terms of left divisors of $T_F(\lambda)$) is apparently inaccessible.

4.6 Implications Towards Decentralized Control

At this point we seem somewhat far afield from our original objective: to say something definitive about the system structures that can be achieved by decentralized feedback. The result that appears to be the most nearly relevant to the decentralized control problem is (4.5-30), the result which pertains to equivalence classes of feedback laws, each of which gives rise to an identical input-to-state system structure. Since decentralized feedback can be thought of as a highly constrained form of state feedback, it is, in theory, a simple matter to determine if the equivalence class of feedback laws corresponding to the desired system structure contains a feedback law constrained according to the rules of decentralized control. Unfortunately, the method for determining when two feedback laws are equivalent is algorithmic; thus it is difficult, if not impossible, to characterize, in a closed form, equivalence classes of feedback laws.

However, there is hope that several of the unanswered questions in Section 2.7 may find their solution in an efficient characterization of controllability subspaces. As mentioned in Section 4.5.3, there is promise for such a characterization. It might then be possible to determine controllability subspaces satisfying conditions such as (2.7-9)-(2.7-11).

CHAPTER 5

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

An attempt will be made in this chapter to summarize the contributions of this dissertation and to speculate on possible areas of extension.

In Chapter Two we have treated, in a fair amount of depth, a class of linear systems with decentralized control. Permeated throughout many of the results was the general philosophy that required the individual control agents' controllable and observable subspaces to be compatible in some sense. This compatibility was usually just sufficient to allow each individual control agent to operate independently, without adversely affecting the others. Thus, in the case of open loop decentralized control, it was required that each control agent be able to control that subspace which could not be seen by all the other control agents; while in the case of decentralized pole allocation, a desirable system configuration, one which in some cases could be produced by the right feedback, was a triangularly decoupled system.

One interesting aspect of Chapter Two was the subject of generalized observers, i.e. observers to be used when all the system inputs are not known. It was seen that such generalized observers could be used to increase each control agent's information set. Unfortunately, too much information seems to be as much a curse as too little; and once the control agents increased their information sets as much as possible, they were forced to discard some of their newly gained information in the interests of overall compatibility.

It is unfortunate that the results of Section 2.7 are not very strong. Particularly bothersome is the inability to generalize from the two controller case to the many controller case. It is probably the area covered by Section 2.7, involving the combined use of observers and control law, that warrants the most effort in the future.

As introduced in Chapter Three, the rationale behind indulging in the algebraic system theory of Chapter Four was to attempt to get a handle on the subject of invariants under decentralized feedback, and thereby learn something about the system structures that can be achieved via decentralized feedback. As should be obvious, this attempt resulted mostly in failure; however, the promised characterizations of controllability subspaces could be quite useful in tackling a variety of problems, some in the area of decentralized control.

In spite of falling short of the original goal, it is felt that the contents of Chapter Four are quite valuable. It is interesting that so much of the structure of a system is exhibited by its associated canonical matrix.

There are at least three distinct areas associated with the contents of Chapter Four that warrant further attention. The first of these is a thorough investigation into the <u>real</u> implications towards representations of infinite dimensional systems; moreover, the applicability of a similar representation for infinite dimensional continuous time systems should be looked into. There are obviously several aspects of conventional infinite dimensional system representation that are not considered in our module framework (see [27]).

Considerable time and effort went into trying to prove Rosenbrock's Theorem (3.2-5) in a module-theoretic setting; this warrants further effort. Moreover, a result pertaining to a characterization of the feedback matrices that achieve the goals of Rosenbrock's Theorem could prove quite useful in applications such as output, or decentralized, feedback.

Finally, it should be possible to simultaneously characterize a system by its associated canonical matrix, and the dual system by its associated canonical matrix, in such a way as to achieve a more symmetric characterization of a system (e.g. one which is not heavily biased towards the input-to-state aspect of the system). Such a characterization, for example, could prove useful in determining complete sets of invariants under output feedback.

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APPENDIX A

(A,B)-INVARIANT AND (A,B)-CONTROLLABILITY SUBSPACES

In this appendix are outlined some of the principal results pertaining to the geometric structures of linear, finite dimensional systems, and the ways in which these structures may be changed via state feedback. The theories of (A,B)-controllability subspaces and (A,B)-invariant subspaces were developed by Wonham and Morse, to be used in the solution to the decoupling problem; see references [54]-[59], [72], [74], [75]. The concept of (A,B)-invariant subspaces was also developed by Basile and Marro, independently of Wonham and Morse (they use the term "controlled invariant subspace" for (A,B)-invariant subspace); see references [4], [5].

Throughout this appendix, A, B, and C denote linear maps:

$$A : R^n \to R^n$$

$$B : R^m \to R^n$$

$$C : R^n \to R^p$$

and may be thought of as defining a system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

(A-1) <u>Definition</u>: A subspace $\mathcal{S} \subset R^n$ is said to be A-invariant if $A\mathcal{S} \subset \mathcal{S}$. If $\mathcal{S}$ is A-invariant, then the endomorphism $A : R^n \to R^n$ may be restricted to an endomorphism $\bar{A} : \mathcal{S} \to \mathcal{S} : s \to As$; this is well defined, because $As \in \mathcal{S}$ for all $s \in \mathcal{S}$. We sometimes denote the restriction of A to $\mathcal{S}$ as $A|\mathcal{S}$.

It is easily seen that the set of A-invariant subspaces of R^n is a sublattice of the lattice of subspaces of R^n. That is, if S_1 and S_2 are A-invariant, then so are $S_1 + S_2$ and $S_1 \bigcap S_2$. The structure of the lattice of A-invariant subspaces is completely determined from the Jordan normal form of A; in the particular case where A has n distinct real eigenvalues, there are precisely 2^n distinct A-invariant subspaces, each of which can be written as a direct sum of one-dimensional eigenspaces. For more details on the structure of this lattice, see reference [12].

Given two subspaces $\mathcal{D}$ and N of R^n, the following sets of subspaces are easily shown to be sublattices of the lattice of A-invariant subspaces:

$$L_1 = \{S \mid AS \subset S, \text{ and } S \supset \mathcal{D}\}$$

$$L_2 \quad \{S \mid AS \subset S, \text{ and } S \subset N\}$$

Thus, the following result follows easily:

(A-2) **Proposition:** (i) L_1 contains a unique minimal element, $\hat{S}(\mathcal{D})$, equal to

$$\hat{S}(\mathcal{D}) = \bigcap_{S \varepsilon L_1} S = \sum_{i \varepsilon \underline{n}} A^{i-1}\mathcal{D}$$

(ii) L_2 contains a unique maximal element, $\tilde{S}(N)$, equal to

$$\tilde{S}(\mathcal{D}) = \sum_{S \varepsilon L_2} S = \bigcap_{i \varepsilon \underline{n}} A^{-(i-1)}N$$

It is easily seen that, if $B : R^m \rightarrow R^n$ is the input-to-state map, then $\hat{S}(\text{Im } B)$ is simply the reachable subspace; while if $C : R^n \rightarrow R^p$ is the state-to-output map, then $\tilde{S}(\text{Ker } C)$ is the unobservable subspace.

In certain control situations one would like to decrease the reachable subspace; for example, we might wish that the output $z = Hx$ be unaffected by the input u. One way to accomplish this is to replace the map B by the map BG, where rank $G < m$, thus essentially removing the effect of one or more input channels on the system. The resulting reachable subspace will then be

$$\mathcal{R} = \sum_{i \in \underline{n}} A^{i-1} \operatorname{Im}(BG) \triangleq \{A | \operatorname{Im}(BG)\}$$

It is easily seen that $\mathcal{R}$ is A-invariant, and that

(A-3) $$\mathcal{R} = \{A | \mathcal{B} \cap \mathcal{R}\}$$

(A-4) <u>Definition</u>: A subspace $\mathcal{R}$ satisfying (A-3) is said to be a controllable subspace.

It can be easily shown that the sum of two controllable subspaces is a controllable subspace, but their intersection need not be a controllable subspace. Thus, the set of all controllable subspaces of the pair (A,B) is a <u>join semilattice</u>. The following result is an easy consequence of (A-2).

(A-5) <u>Proposition</u>: The maximal (A,B)-controllable subspace contained in the subspace $\mathcal{N}$ is $\tilde{\mathcal{R}}(\mathcal{N})$, where

$$\tilde{\mathcal{R}}(\mathcal{N}) = \sum_{i \in \underline{n}} A^{i-1} \mathcal{B} \cap \left(\bigcap_{j \in \underline{n}} A^{-(i-1)} \mathcal{N} \right)$$

The basic idea behind (A,B)-invariant and (A,B)-controllability subspaces is to expand the sets of A-invariant and (A,B)-controllable subspaces by allowing A to be replaced by A+BF, for some $F \in R^{m \times n}$. This corresponds to using state feedback, of the form $u = Fx$, in the system

$\dot{x} = Ax + Bu$; the objective of the feedback is to modify the original structure of invariant and controllable subspaces. We therefore define

(A-6) **Definition**: Let $A : R^n \to R^n$ and $B : R^m \to R^n$. Then

(i) A subspace $\mathcal{V} \subset R^n$ is said to be (A,B)-invariant if there exists an $F : R^n \to R^m$ such that

$$(A+BF)\mathcal{V} \subset \mathcal{V}$$

(ii) A subspace $\mathcal{R} \subset R^n$ is said to be an (A,B)-controllability subspace if there exists an $F : R^n \to R^m$ such that

$$\{A+BF | \mathcal{B} \cap \mathcal{R}\} = \mathcal{R}$$

(A-7) **Remark**: It is clear that every (A,B)-controllability subspace is (A,B)-invariant, and that, if $\mathcal{R}$ is a controllability subspace, then

$$\mathcal{R} = \{A+BF | \mathrm{Im}(BG)\}$$

where $\mathrm{Im}(BG) = \mathcal{B} \cap \mathcal{R}$. Thus controllability subspaces allow us more freedom in restricting the reachable subspace to a desired subspace than do controllable subspaces.

It is a fairly trivial exercise to show that

(A-8) **Proposition**: $\mathcal{V}$ is an (A,B)-invariant subspace if and only if $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$.

We shall also make use of

(A-9) **Definition**: The class of maps $F : R^n \to R^m$ for which $(A+BF)\mathcal{V} \subset \mathcal{V}$, for a given $\mathcal{V} \subset R^n$, is denoted as $\underline{F}(\mathcal{V})$ (thus, by (A-8), $\underline{F}(\mathcal{V}) \neq \phi$ if and only if $A\mathcal{V} \subset \mathcal{B} + \mathcal{V}$).

The following are equivalent characterizations of (A,B)-controllability subspaces:

(A-10) <u>Theorem</u>: A subspace $\mathcal{R} \subset R^n$ is an (A,B)-controllability subspace if and only if one of the following (equivalent) sets of conditions is satisfied:

(i) $A\,\mathcal{R} \subset \mathcal{R} + \mathcal{B}$ <u>and</u> $\mathcal{R} = \{A+BF|\mathcal{B} \cap \mathcal{R}\}$ for some $F \in \underline{F}(\mathcal{R})$. (The choice of F is not important - if it works for one $F \in \underline{F}(\mathcal{R})$, then it works for all $F \in \underline{F}(\mathcal{R})$.)

(ii) $A\,\mathcal{R} \subset \mathcal{R} + \mathcal{B}$ <u>and</u> $\mathcal{R} = \mathcal{R}_n$ where $\mathcal{R}_0 = 0$, and $\mathcal{R}_i = (A\mathcal{R}_{i-1}+\mathcal{B}) \cap \mathcal{R}$ for $i \in \underline{n}$.

One can easily demonstrate that the sum of two (A,B)-invariant subspaces is (A,B)-invariant, and that the sum of two (A,B)-controllability subspaces is an (A,B)-controllability subspace; however, this is generally not the case with intersections. Therefore, the sets of (A,B)-invariant and (A,B)-controllability subspaces are both <u>join-semilattices</u> (as was the set of (A,B)-controllable subspaces). It follows that, for any subspace $\mathcal{N} \subset R^n$, there is a unique maximal (A,B)-invariant subspace $\mathcal{V}^* \subset \mathcal{N}$, and a unique maximal (A,B)-controllability subspace $\mathcal{R}^* \subset \mathcal{N}$. These subspaces are characterized as follows

(A-11) <u>Theorem</u>: Let $\mathcal{N} \subset R^n$ be a subspace, and let $\mathcal{V}^*$ (resp. $\mathcal{R}^*$) be the unique maximal (A,B)-invariant (resp. (A,B)-controllability) subspace contained in $\mathcal{N}$. Then

(i) $\mathcal{V}^* = \mathcal{V}_n$, where

$$\mathcal{V}_0 = \mathcal{N}, \text{ and } \mathcal{V}_i = \mathcal{N} \cap A^{-1}(\mathcal{V}_{i-1}+\mathcal{B}) \text{ for } i \in \underline{n}$$

(ii) $\mathcal{R}^* = \{A+BF|\mathcal{B} \bigcap \mathcal{V}^*\}$, for all $F \in \underline{F}(\mathcal{V}^*)$

An alternative, and sometimes quite useful representation of $\mathcal{R}^*$ has recently been derived by Morse ([56]):

(A-12) **Theorem:** Let $\mathcal{N} = \text{Ker } C$, for some $C : R^n \to R^p$. Then $\mathcal{R}^*$, the maximal (A,B)-controllability subspace contained in $\mathcal{N}$, is given by

$$\mathcal{R}^* = \mathcal{V}^* \bigcap \mathcal{W}^{*\perp}$$

where "$\perp$" denotes orthogonal complement, $\mathcal{V}^*$ is the maximal (A,B)-invariant subspace contained in $\mathcal{N}$, and $\mathcal{W}^*$ is the maximal (A',C')-invariant subspace contained in Ker B'.

APPENDIX B

RINGS AND MODULES

In this appendix we summarize some of the aspects of ring and module theory that will prove useful in treating linear system theory from an algebraic point of view. Our treatment is by no means exhaustive, particularly as we shall concentrate on results pertaining to $K[\lambda]$ and modules over $K[\lambda]$. For a more complete picture, the reader is referred to the texts by MacLane and Birkhoff ([53]), Lang ([46]), and Gray ([32]), and the paper by Flanders ([21]).

It is assumed that the reader is at least vaguely familiar with the concept of a ring. For completeness, we include

(B-1) <u>Definition</u>: A ring R (with unit) is a set R together with two binary operations: addition, denoted by +, and multiplication, denoted by $\cdot$, which satisfies the following axioms

(i) R is an abelian group under addition; i.e. the operation + is associative and commutative, there is a unique element $0 \in R$ such that $r + 0 = r$ for all $r \in R$, and for every $r \in R$ there exists a unique element $\hat{r} \in R$ such that $r + \hat{r} = 0$ (we denote $\hat{r}$ as $-r$, and $r + \hat{r}$ as $r - r$).

(ii) R is a monoid under multiplication; i.e. the operation $\cdot$ is associative and there is a unique element $1 \in R$ such that $r \cdot 1 = 1 \cdot r = r$ for all $r \in R$

(iii) multiplication distributes over addition, i.e. $r \cdot (s+t) = r \cdot s + r \cdot t$, and $(s+t) \cdot r = s \cdot r + t \cdot r$ for all $r,s,t \in R$.

We shall usually drop the $\cdot$ denoting multiplication, and write $r \cdot s$ simply as rs. We note that, from the definition of 0 and the distributivity of $\cdot$ over +,

$$rr+0 = rr = r(r+0) = rr+r0$$

whence it follows that $r0 = 0$ (and, similarly, that $0r = 0$) for all $r \in R$. We shall soon be restricting attention to special classes of rings. We therefore define

(B-2) Definition: Let R be a ring. Then R is

(i) commutative, if multiplication in R is commutative

(ii) an integral domain, if $R \neq \{0\}$ and if R contains no zero divisors (i.e. there exist no nonzero elements $r,s \in R$ such that $rs = 0$)

As in all other aspects of algebra, the concept of a morphism is extremely important:

(B-3) Definition: Let R and S be rings. Then a map $\phi : R \to S$ is a ring morphism if

(i) $\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$, for all $r_1, r_2 \in R$

(ii) $\phi(r_1 r_2) = \phi(r_1)\, \phi(r_2)$, for all $r_1, r_2 \in R$

(iii) $\phi(1) = 1'$, where 1 and 1' are the unit elements in R and S, respectively

The image and the kernel of ϕ are defined as

$$\text{Im}\ \phi = \{s \in S \mid s = \phi(r) \text{ for some } r \in R\}$$

$$\text{Ker}\ \phi = \{r \in S \mid \phi(r) = 0 \in S\}$$

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The kernel of a morphism $\phi : R \to S$ is an ideal, a subset of R with a particular type of structure:

(B-4) Definition: Let R be a ring and let $A \subset R$ be a subset. Then A is a (two-sided) ideal in R if

(i) $a_1 - a_2 \in A$, for all $a_1, a_2 \in A$

(ii) $a\,r \in A$ and $r\,a \in A$, for all $a \in A$ and $r \in R$

It follows easily that if A and B are ideals in R, then so are $A \cap B$ and $A + B$, defined as

$$A \cap B = \{r \in R \mid r \in A \text{ and } r \in B\}$$

$$A + B = \{r \in R \mid r = a + b, \text{ for some } a \in A \text{ and } b \in B\}$$

Thus, the set of ideals in R has the structure of a lattice, partially ordered by inclusion. Also, given an ideal A in R, we can consider the set of cosets of A, defined as

$$R/A = \{r + A \mid r \in R\}$$

where

$$r + A = \{r + a \mid a \in A\}, \text{ for each } r \in R$$

This set admits a ring structure, and a useful property, as indicated below.

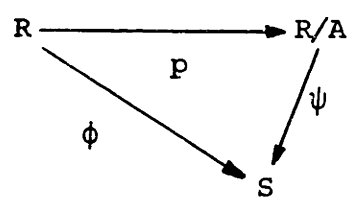
(B-5) Proposition: Let A be an ideal in R, and let R/A be the set of cosets of A. Then

(i) There are unique operations of addition and multiplication in R/A which make R/A a ring, and which make the canonical projection $p : R \to R/A : r \to r + A$ a ring morphism. These operations are simply

$$(r_1 + A) + (r_2 + A) = (r_1 + r_2) + A$$

$$(r_1 + A)(r_2 + A) = (r_1 r_2) + A$$

(ii) Let $\phi : R \to S$ be a ring morphism, and let Ker ϕ A. Then ϕ can be "factored through" R/A in the sense that there exists a unique ring morphism $\psi : R/A \to S$ such that the following diagram commutes:



where $p : R \to R/A$ is the canonical projection : $r \to r + A$.

(B-6) **Definition**: The ring R/A, as described in (B-5)(i), is known as the quotient ring, or the residue class ring.

In Chapter Four of this thesis, we shall be principally concerned with a particular class of rings, which we now define.

(B-7) **Definition**: (i) Let R be a ring, and let A be an ideal in R. Then A is said to be a principal ideal if A consists of all multiples of a ring element a, i.e. if

$$A = Ra \triangleq (a) \triangleq \{ra \mid r \in R\}, \text{ for some } a \in A$$

If A = (a), then we say that a is a generator for A.

(ii) If R is an integral domain, and if every ideal in R is a principal ideal, then we say that R is a principal ideal domain (abbreviated as p.i.d.)

The concept of divisibility enters nicely into the subject of principal ideal domains:

(B-8) <u>Definition</u>: (i) If $a,d \in R$ are such that $a = dc$ for some $c \in R$, then we say d divides a, and write: $d|a$.

(ii) Let $a,b,d \in R$. Then we say that d is a greatest common divisor of a and b if $d|a$ and $d|b$, and whenever $c \in R$, $c|a$, and $c|b$, then $c|d$.

(iii) We say that a and b are relatively prime if the greatest common divisor of a and b is invertible (i.e. if there exists $d^{-1} \in R$ such that $d \cdot d^{-1} = 1$).

(B-9) <u>Proposition</u>: Let R be a p.i.d., and let $a,b \in R$. Then there exists a greatest common divisor, d, of a and b; moreover

$$(d) = (a) + (b)$$

Therefore, a and b are relatively prime if and only if

$$(a) + (b) = R$$

(B-10) <u>Definition</u>: Let K be a field (e.g. the real numbers). Then, by $K[\lambda]$ we shall mean the set of polynomials of the form $x(\lambda) = x_0 + \lambda x_1 + \ldots + \lambda^m x_m$, where each $x_i \in K$, and where $m \in Z_+$ = nonnegative integers. If $x(\lambda) = x_0 + \ldots + \lambda^m x_m \in K[\lambda]$, we define the degree of $x(\lambda)$ to be $\partial x(\lambda) \triangleq \max\{i | x_i \neq 0\}$.

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We now have:

(B-11) **Proposition:** $K[\lambda]$, together with the ordinary addition and multiplication operations for polynomials, is a principal ideal domain.

(B-12) **Remark:** Given $x(\lambda)$, $y(\lambda) \in K[\lambda]$, there exist unique $q(\lambda)$, $r(\lambda) \in K[\lambda]$ such that

$$x(\lambda) = y(\lambda)\ q(\lambda) + r(\lambda)$$

and

$$\partial r(\lambda) < \partial y(\lambda)$$

It is then easily seen that the cosets $x(\lambda) + (y(\lambda))$ and $r(\lambda) + (y(\lambda))$ are identical elements in the quotient ring $K[\lambda]/(y(\lambda))$. Thus, it is seen that the subset of $K[\lambda]$, $\{r(\lambda) \in K[\lambda] | \partial r(\lambda) < \partial y(\lambda)\}$ is set-isomorphic to $K[\lambda]/(y(\lambda))$, and the operation of reducing $x(\lambda)$ modulo $y(\lambda)$ (i.e. obtaining $r(\lambda)$) is a representation of the canonical projection $p : K[\lambda] \rightarrow K[\lambda]/(y(\lambda))$.

Just as the concept of a ring is a generalization of the concept of a field, we can generalize the concept of a vector space over a field to that of a module over a ring. Thus, we next define

(B-13) **Definition:** Let R be a ring. Then, a (left) R-module A is an additive abelian group (see (B-1)(i)) together with a function $R \times A \rightarrow A : (r,a) \rightarrow ra$ subject to the following axioms:

(i) $r(a+b) = ra+rb$; for all $r \in R$, and $a,b \in A$

(ii) $(r+s)a = ra+sa$; for all $r,s \in R$, and $a \in A$

(iii) $(rs)a = r(sa)$; for all $r,s \in R$, and $a \in A$

(iv) $1a = a$; for all $a \in A$

(B-14) <u>Remark</u>: In the above, we say that A is a left R-module because the action of R on A is written with the ring element on the left. However, one does not need to distinguish between left and right R-modules when R is a commutative ring, as will be the case of interest to us.

It should be clear that the ring R acts on the module A in a manner similiar to the action of a field K on a K-vector space. For this reason, we often refer to R as the ring of scalars, and to the action of R on A as scalar multiplication. The concept of a K-linear map between two K-vector spaces is generalized to:

(B-15) <u>Definition</u>: Let A and B be two R-modules, and let $F : A \to B$ be a map. Then F is a <u>morphism of R-modules</u> (or R-morphism) if

$$f(ra+sb) = rf(a) + sf(b)$$

for all $r,s \in R$, and $a,b \in A$

The analogue of a subspace of a K-vector space is a submodule of an R-module, which we define as

(B-16) <u>Definition</u>: Let A be an R-module, and let $S \subset A$ be a subset. Then S is a submodule of A if and only if

$$r_1 s_1 + r_2 s_2 \in S$$

for all $r_1, r_2 \in R$ and all $s_1, s_2 \in S$.

Given two submodules, S_1 and S_2, of an R-module A, the subsets

$$S_1 \cap S_2 = \{a \in A | a \in S_1 \text{ and } a \in S_2\}$$

$$S_1 + S_2 = \{a \in A | a = s_1 + s_2, \text{ for some } s_1 \in S_1 \text{ and } s_2 \in S_2\}$$

are easily seen to be submodules of A. Thus the set of submodules of A is a lattice, partially ordered by inclusion. If $a \in A$ is an arbitrary element, then it is easily seen that $Ra \triangleq \{ra | r \in R\}$ is a submodule of A. This observation can be generalized to:

(B-17) Definition: Let A be an R-module. Then

(i) If there exist elements $a_1, a_2, \ldots a_n \in A$ such that

$$A = Ra_1 + Ra_2 + \ldots + Ra_n$$

then A is said to be a module of finite type; the elements a_i are said to be a finite set of generators of A.

(ii) If A is of finite type, then the smallest integer n for which there exists a set of generators $\{a_1, a_2, \ldots a_n\}$ for A, is called the rank of A.

(iii) If A is of finite type, and if rank A = 1, then A is said to be a cyclic module.

One of the objectives that one usually has when treating a particular module is to decompose it into cyclic submodules. If this can be accomplished, then much of the module structure can be represented in terms of

(B-18) Proposition: Let R be a commutative ring, and let A be a cyclic R-module; let $a \in A$ be a generator of A, so that

$$A = Ra$$

Then, the set $I = \{r \in R | ra = 0\}$ is an ideal in R, and the quotient ring R/I is an R-module isomorphic to A.

If $S \subset A$ is a submodule, then we can construct the set of cosets of S:

$$A/S = \{a + S | a \in A\}$$

where

$$a + S = \{a + s | s \in S\}, \text{ for each } a \in A$$

The set A/S can be made into an R-module by defining

$$(a_1 + S) + (a_2 + S) = (a_1 + a_2) + S; \text{ for } a_1, a_2 \in A$$

$$r(a + S) = (ra) + S; \text{ for } r \in R, a \in A$$

The next result demonstrates how submodules and quotient modules may be effectively used:

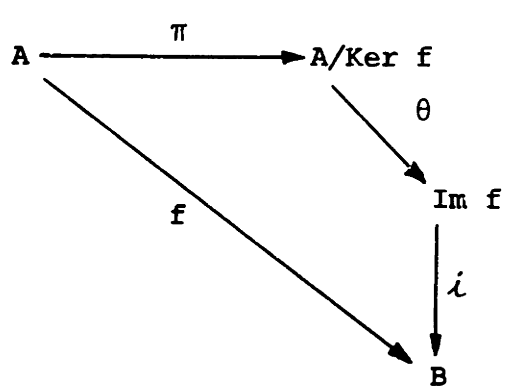
(B-19) Proposition: Let $f : A \to B$ be a morphism of R-modules. Then

(i) Ker $f \subset A$ and Im $f \subset B$ are submodules, where

$$\text{Ker } f = \{a \in A | f(a) = 0\}$$

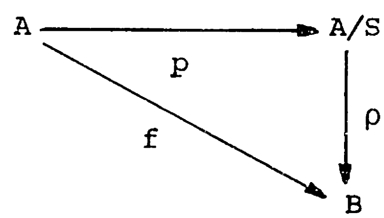
$$\text{Im } f = \{b \in B | b = f(a), \text{ for some } a \in A\}$$

(ii) f may be canonically factored as



where π is the canonical projection ($\pi : a \mapsto a + \text{Ker } f$), θ is an isomorphism ($\theta : (a+\text{Ker } f) \mapsto f(a)$), and i is the insertion of Im f into B ($i : f(a) \mapsto f(a)$).

(iii) If Ker f $\supset$ S, for some submodule $S \subset A$, then there exists a unique R-morphism $\rho : A/S \to B$ such that the following diagram commutes:



where $p : A \to A/S : a \mapsto a + S$.

We very often wish to make new modules out of old modules. One way to do this is as follows:

(B-20) **Definition:** Let A and B be two R-modules. Then

(i) $A \oplus B \triangleq \{(a,b) | a \in A, b \in B\}$ is a module if one defines $r_1(a_1,b_1) + r_2(a_2,b_2) = (r_1a_1 + r_2a_2, r_1b_1 + r_2b_2)$ for all $a_i \in A$, $b_i \in B$, $r_i \in R$. We then call $A \oplus B$ the direct sum of A and B (or, sometimes, the biproduct of A and B).

(ii) If there is an isomorphism $\theta : A \oplus B \xrightarrow{\sim} C$, for a third R-module C, then we write: $C \cong A \oplus B$. Thus, $A \oplus B$ is a "model" for C which exhibits some of the basic internal structure of C.

We also have

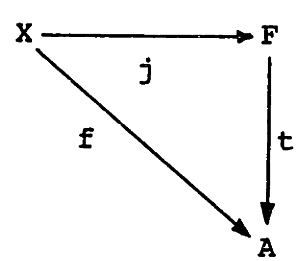
(B-21) <u>Proposition</u>: Let A_1 and A_2 be submodules of an R-module B such that $A_1 \cap A_2 = 0$, and $A_1 \oplus A_2 = B$. Then

$$B \cong A_1 \oplus A_2$$

When this is the case, we usually write $B = A_1 \oplus A_2$; and we say that B is the (internal) direct sum of A_1 and A_2, and that A_1 and A_2 are <u>direct</u> <u>summands</u> of B.

(B-22) <u>Remark</u>: One of the major differences between R-modules and K-vector spaces is the fact that, whereas every subspace of a K-vector space V is a direct summand of V, not every submodule of an R-module A is a direct summand of A. We shall soon see one important case where one can say with certainty that a submodule is a direct summand.

(B-23) <u>Definition</u>: Let F be an R-module, let $X \subset F$ be a subset, and let $j : X \to F$ be the insertion (set-theoretic). Then we say that F is a <u>free</u> module on X, and that X is a set of <u>free</u> generators for F, if, for every map $f : X \to A$ (A is an R-module) there exists a unique R-morphism $t : F \to A$ such that the following diagram commutes:



That is, t is uniquely determined by its values on elements of X.

It is easy to see that the free generators of a free module perform the same function as elements of a basis for a K-vector space. Consequently, any element $a \in F$ may be uniquely written as an R-linear combination of elements in X.

The set $K^m[\lambda]$ is one example of a free $K[\lambda]$-module. We define $K^m[\lambda]$ as

$$K^m[\lambda] = \{\sum_{i=0}^{k} \lambda^i u_i | k \in Z_+ \text{ and each } u_i \in K^m\}$$

That is, $K^m[\lambda]$ is the set of polynomial m-vectors. Addition in $K^m[\lambda]$ and multiplication by an element of $K[\lambda]$ are defined exactly as one defines the analogous operations in the K-vector space K^m (i.e. componentwise). Any basis of K^m will serve as a set of free generators for $K^m[\lambda]$, as will the columns of any m x m polynomial matrix $Q(\lambda)$ satisfying det $Q(\lambda)$ = nonzero constant (such matrices are said to be unimodular).

A class of modules which are radically different from free modules is the class of torsion modules, which we now define. We shall assume in this definition, and in the remainder of this appendix, that R is a principal ideal domain; a good example to think of is $K[\lambda]$.

(B-24) Definition: (i) Let R be a p.i.d., and let A be an R-module. Define

$$T = \{a \in A | ra = 0, \text{ for some nonzero } r \in R\}$$

It is easy to verify that T is a submodule of A; T is called the torsion submodule of A.

(ii) If T = A, then A is called a torsion module.

(iii) If A is a torsion module, then the subset of R

$$I = \{r \in R | ra = 0, \text{ for all } a \in A\}$$

is easily seen to be an ideal in R. Since R is a p.i.d., I is principal, i.e. $I = (\mu)$. We then say that μ is the <u>minimal annihilator</u> of A.

Three properties that modules defined over principal ideal domains enjoy are as follows:

(B-25) <u>Proposition</u>: Let R be a p.i.d., and let A be an R-module of finite type. Then

(i) If A is a free module, then every submodule $B \subset A$ is free, and rank $B \leq$ rank A.

(ii) A satisfies the <u>ascending chain condition</u>: For every chain of submodules

$$S_1 \subset S_2 \subset \ldots \subset A$$

there is an integer N such that $S_i = S_N$, for all $i \geq N$.

(iii) A admits the following canonical decomposition:

(B-26)
$$A \cong C_1 \oplus C_2 \oplus \ldots \oplus C_r \oplus F$$

where, for each $1 \leq i \leq r$, C_i is a cyclic torsion module with minimal annihilator μ_i, F is a free module and

$$\mu_{i+1} | \mu_i, \text{ for } 1 \leq i \leq r-1$$

Moreover, the decomposition in (B-26) is unique up to isomorphism, in the sense that r, $\{\mu_i, i \in \underline{r}\}$, and rank F are all uniquely determined by A. The set $\{\mu_i, i \in \underline{r}\}$

is called the set of <u>invariant factors</u> for the torsion submodule of A.

As a final result in this appendix, we consider the following situation. Let A and F be R-modules, where R is a p.i.d. and F is free and finitely generated; also, let $g : A \to F$ be a module morphism. Then, as in (B-19), g can be factored as

$$A \xrightarrow{\pi} A/\text{Ker } g \xrightarrow{\theta} \text{Im } g \xrightarrow{i} F, \qquad g = i \circ \theta \circ \pi$$

Since Im $g \subset F$, it follows that Im g is free; but θ is an isomorphism, so A/Ker g is also free, and of rank $r < \infty$.

Let $\{x_i + \text{Ker } g,\ i \in \underline{r}\}$ be a set of free generators for A/Ker g. Then there exists a unique morphism $\psi : A/\text{Ker } g \to A$ such that

$$\psi(x_i + \text{Ker } g) = x_i,\ i \in \underline{r}$$

It is easily seen now that

$$(\pi \circ \psi)(a + \text{Ker } g) = a + \text{Ker } g, \text{ for all } a \in A$$

That is, ψ is a right inverse for π.

We wish to show that

(B-27) $$A = \text{Ker } g \oplus \text{Im } \psi$$

Since π is a left inverse for ψ, it follows that Ker $\pi \cap$ Im $\psi = 0$, whence Ker $g \cap$ Im $\psi = 0$. Moreover, for any $a \in A$,

$$a = a - \psi\pi a + \psi\pi a$$

Clearly, $\psi\pi a \in \text{Im}\ \psi$; while $\pi(a - \psi\pi a) = \pi a - \pi a = 0$ implies $a - \psi\pi a \in \text{Ker}\ g$. This demonstrates that $A = \text{Im}\ \psi + \text{Ker}\ g$, and (B-27) follows. We summarize the above as

(B-28) Proposition: Let R be a p.i.d., and let $g : A \to F$ be an R-morphism, where F is free and finitely generated. Then, there exists a submodule $S \subset A$, isomorphic to $A/\text{Ker}\ g$, such that

$$A = \text{Ker}\ g \oplus S$$

BIOGRAPHICAL NOTE

Adrian E. Eckberg, Jr. was born on July 11, 1945 in Providence, Rhode Island. He was graduated from Weymouth High School, Weymouth, Massachusetts, in June, 1963.

Mr. Eckberg entered Massachusetts Institute of Technology in September, 1963, and majored in Electrical Engineering. He was admitted as a graduate student in the Department of Electrical Engineering at M.I.T. in February, 1967. He worked in the area of communication theory, and received the S.B., S.M., and E.E. degrees simultaneously in June, 1970, after having submitted a thesis entitled "Optimal Estimation of VLF Atmospheric Noise."

Mr. Eckberg has been an instructor in the Department of Electrical Engineering, M.I.T. since June, 1969. He has served as a consultant for the Raytheon Co. and M.I.T. Lincoln Laboratory, working in the areas of digital signal processing and fourth generation air traffic control systems. He is a member of Sigma Xi, Eta Kappa Nu, and Tau Beta Pi.