Construction of Tame Types

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To Professor Roger Howe on the occasion of his 70th Birthday

§1 Introduction

Let F be a non archimedean local field and G a reductive group defined over F. To study the category $\mathcal{R} := \mathcal{R}(G)$ of smooth representations of G, Bushnell and Kutzko [BK] formulated the theory of types, which build on the theory of minimal K-types [HM] and a general framework due to Bernstein [Be]. This category $\mathcal{R}$ is decomposed into a product of additive subcategories by the theory of Bernstein center [Be]:

$$\mathcal{R} = \prod_{\mathfrak{s}\in\mathcal{J}} \mathcal{R}^{\mathfrak{s}}(G), \tag{1}$$

where $\mathcal{J}$ is the set of inertial classes of cuspidal pairs. Moreover, each factor $\mathcal{R}^{\mathfrak{s}} := \mathcal{R}^{\mathfrak{s}}(G)$ can not be further decomposed (see [BK] for details). When we have a type in the sense of Bushnell and Kutzko, we have the means to study a finite number of these $\mathcal{R}^{\mathfrak{s}}$. To be effective, we would like to have one type for each single $\mathcal{R}^{\mathfrak{s}}$. The existence and the nature of such types has been a fundamental problem.

In this article, we will give a construction of types for a general reductive p-adic group G. Our method produces types corresponding to a single factor $\mathcal{R}^{\mathfrak{s}}$. The construction here is not new: it is the same construction used in [Yu] for supercuspidal representations, when certain obvious constraints pertaining to supercuspidality are removed. What is new is that an additional constraint must be imposed concerning the embeddings of buildings. For any (tamely ramified twist of a) Levi subgroup G' of G, there is a family of embeddings of the (extended) building $\mathcal{B}(G')$ of G' into that $\mathcal{B}(G)$ of G. This family forms a Euclidean space. The choice of embeddings is unimportant for almost all applications. But it is crucial here that we avoid a certain set of embeddings of measure 0. This generic choice of embeddings allows us to construct our types as covers of supercuspidal types on Levi subgroups in a uniform manner. To prove that this construction indeed yields G-covers and thus types in the sense of Bushnell and Kutzko [BK], we use ideas from the work of [K, MP2].

In §9, we sketch a proof that our construction yields sufficiently many types to study all irreducible admissible representations under a suitable "tameness" hypothesis on G and F.

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§2 Notation and conventions

2.1 We adopt all notation and conventions from [Yu, p. 582]. However, in this paper we do not need to treat base field extensions extensively except in (4.3). Therefore, we work over a fixed non-archimedean local field F, that is, F is either a p-adic field or a function field over finite field. If G is an algebraic group over F, we will denote G's group of rational points also by G for simplicity. This should lead to no confusion.

2.2 Throughout this paper, G is a connected reductive group over F, split over a tamely ramified extension of F. For any maximal F-split torus S, $\Phi(G, S, F)$ denotes the corresponding set of roots in G. For $a \in \Phi(G, S, F)$, let U_a (resp. $\mathfrak{u}_a$) be the root subgroup (resp. root subspace) corresponding to a.

By a Levi subgroup of G, we mean an F-subgroup of G which is a Levi factor of a parabolic F-subgroup of G. By a twisted Levi subgroup G, we mean an F-subgroup G' of G such that $G' \otimes_F \bar{F}$ is a Levi subgroup of $G \otimes_F \bar{F}$.

2.3 We assume that the residue characteristic p of F is not a torsion prime for $\psi(G)^\vee$, the root datum dual to the root datum $\psi(G)$ of $G \otimes_F \bar{F}$. See [Yu, §7] and [St] for the relevant notions. By [St, 2.3], p is not a torsion prime for $\psi(G')^\vee$, for any (twisted) Levi subgroup G' of G. From §7 on, we also assume that p is odd.

2.4 Let $\vec{G} = (G^0, \ldots, G^d)$ be a tamely ramified twisted Levi sequence in G, that is, each G^i is a E-Levi subgroup of G over a tamely ramified finite extension E of F ([Yu, p. 586]). Let M^0 be an F-Levi subgroup of G^0. Let $\mathcal{Z}_{\mathrm{s}}(M^0)$ be the maximal F-split torus of the center $\mathcal{Z}(M^0)$ of M^0. We define M^i to be the centralizer of $\mathcal{Z}_{\mathrm{s}}(M^0)$ in G^i.

Lemma

(a) *M^i is an F-Levi subgroup of G^i.*

(b) *$\vec{M} := (M^0, M^1, \ldots, M^d)$ is a generalized twisted Levi sequence in $M := M^d$ in the sense of [Yu, page 616].*

(c) *$\mathcal{Z}(M^0)/\mathcal{Z}(M^d)$ is F-anisotropic.*

PROOF. (a) follows from [Bo, 20.4]. It then follows that M^i is a twisted Levi subgroup of G^j for $i \leq j$. Therefore, for $i \leq j$, M^i is the centralizer of $\mathcal{Z}(M^i)^\circ$ in G^j, hence is also the centralizer of $\mathcal{Z}(M^i)^\circ$ in M^j. Again by [Bo, 20.4], this implies that M^i is a twisted Levi subgroup of M^j. We have proved (b).

Finally, since $\mathcal{Z}(M^d) \subset \mathcal{Z}(M^0)$, the F-split rank of $\mathcal{Z}(M^d)^\circ$ is smaller than or equal to that of $\mathcal{Z}(M^0)^\circ$. By construction, $\mathcal{Z}(M^d)^\circ \supset \mathcal{Z}_{\mathrm{s}}(M^0)$. Therefore $\mathcal{Z}(M^d)^\circ$ and $\mathcal{Z}(M^0)^\circ$ have the same F-split rank. This proves (c). ∎

Remark We observe that $\vec{M}$ is a tamely ramified twisted Levi sequence in M.

§3 Generic embeddings of buildings

3.1 We recall that, if G' is a tamely ramified twisted Levi subgroup of G, then there exists a family of natural embeddings of buildings $\mathcal{B}(G') \hookrightarrow \mathcal{B}(G)$, which is an affine space under $X_*(\mathcal{Z}_{\mathrm{s}}(G')) \otimes \mathbb{R}$. All these embeddings have the same image. Two embeddings in the same orbit of $X_*(\mathcal{Z}_{\mathrm{s}}(G)) \otimes \mathbb{R}$ can be regarded as the same for most purposes.

3.2 **Definition** Let M be a Levi subgroup of G, $y \in \mathcal{B}(M)$, and $s \in \mathbb{R}$. We say that the embedding $\iota : \mathcal{B}(M) \hookrightarrow \mathcal{B}(G)$ is *(y, s)-generic*, or *s-generic with respect to y*, if $U_{a,\iota(y),s} = U_{a,\iota(y),s+}$ for all $a \in \Phi(G, S, F) \smallsetminus \Phi(M, S, F)$, where S is any maximal F-split torus of M such that $y \in A(M, S, F)$. Here $A(M, S, F)$ is the apartment associated to S in $\mathcal{B}(M)$.

Once an embedding ι is fixed, we will identify $\mathcal{B}(M)$ as a subset of $\mathcal{B}(G)$.

Here, $\{U_{a,\iota(y),r}\}_{r\in\mathbb{R}}$ is the filtration on the root group U_a, $a \in \Phi(G, S, F)$ so that $U_{a,\iota(y),r} = U_a \cap G_{\iota(y),r}$, where $\{G_{\iota(y),r}\}_{r\geq 0}$ is the Moy-Prasad filtration (see [MP1], [MP2]). The following two results illustrate the usefulness of the notion of generic embeddings.

3.3 **Proposition** *Let G, M, y be as above and let $\iota : \mathcal{B}(M) \hookrightarrow \mathcal{B}(G)$ be 0-generic relative to y. Let $P = MU$ be a parabolic F-subgroup of G with Levi factor M. For any smooth representation V of G, the natural map $r_U : V \to V_U$ from V to its Jacquet module induces a bijection*

$$r_U : V^{G_{y,0+}} \to (V_U)^{M_{y,0+}}.$$

This is a reformulation of [MP2, Proposition 6.7]. Note that $y \in \mathcal{B}(M)$ and an embedding $\mathcal{B}(M) \hookrightarrow \mathcal{B}(G)$ is not always 0-generic with respect to y.

3.4 **Remark** We can use generic embeddings to gain some new insight for the result in [Yu, §17]. Indeed, when G' is a Levi subgroup, one observes that the main result in [Yu, Theorem 17.1] is obvious when the embedding $\iota : \mathcal{B}(G') \hookrightarrow \mathcal{B}(G)$ implicitly used is (y, s)-generic, where $s = r/2$. In this case, one can argue directly using the last paragraph of the proof [Yu, Corollary 17.3].

In general, one argues that there is an embedding $\iota_1 : \mathcal{B}(G') \hookrightarrow \mathcal{B}(G)$ close to ι which is (y, s)-generic (see (3.6) below), and $(J, \tilde{\phi}) = (J, \mathrm{ind}_{J_1}^{J}\, \tilde{\phi}_1)$ where $(J_1, \tilde{\phi}_1)$ is constructed in the same way as $(J, \tilde{\phi})$ but using ι_1 in place of ι. Then the theorem follows immediately from the generic case. In fact, this is just rephrasing the proof in [Yu, §17]. Our J_1 is the ad hoc object $J_\vdash$ used there. However, now we view [Yu, Lemma 17.2] as a literal special case of [Yu, Theorem 9.4] by varying the embedding, and we should regard [Yu, Theorem 17.1] in the case of a generic embedding as the essential result.

3.5 We now work in the setting of 2.4. Consider a commutative diagram of embeddings:

$$\{\iota\} : \quad \begin{array}{ccccccc} \mathcal{B}(G^0) & \hookrightarrow & \mathcal{B}(G^1) & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{B}(G^d) \\ \uparrow & & \uparrow & & & & \uparrow \\ \mathcal{B}(M^0) & \hookrightarrow & \mathcal{B}(M^1) & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{B}(M^d) \end{array}$$

To specify such a diagram of embeddings, it suffices to give the image of a fixed $y \in \mathcal{B}(M^0)$ in $\mathcal{B}(M^i)$, $1 \leq i \leq d$ and in $\mathcal{B}(G^i)$, $0 \leq i \leq d$. We will denote this whole diagram by $\{\iota\}$, and we will denote by ι any composite embedding in this diagram (from $\mathcal{B}(M^i)$ to $\mathcal{B}(M^j)$ or $\mathcal{B}(G^j)$, or from $\mathcal{B}(G^i)$ to $\mathcal{B}(G^j)$, for $i \leq j$).

Definition Let $\vec{s} = (s_0, \dots, s_d)$ be a sequence of real numbers, and $y \in \mathcal{B}(M^0)$. We say that $\{\iota\}$ is *$\vec{s}$-generic (relative to y)* if $\iota : \mathcal{B}(M^i) \to \mathcal{B}(G^i)$ is s_i-generic relative to $\iota(y) \in \mathcal{B}(M^i)$ for $0 \leq i \leq d$.

3.6 We now establish the abundance of generic embeddings.

Let $\{\iota\}$ be a commutative diagram of embeddings as in §3.5, and $y \in \mathcal{B}(M^0)$. Denote the image of y in $\mathcal{B}(M^i)$ by y_i, and that in $\mathcal{B}(G^i)$ by z_i, $0 \leq i \leq d$. Let $v \in X_*(\mathcal{Z}_{\mathrm{s}}(M^0)) \otimes \mathbb{R}$. There is a commutative diagram of embeddings, to be denoted by $\{\iota\}_v$, in which the image of y in $\mathcal{B}(M^i)$ is y_i, and that in $\mathcal{B}(G^i)$ is $z_i + v$, $0 \leq i \leq d$.

Lemma *Fix $\vec{s} \in \mathbb{R}^{d+1}$.*

(a) *$\vec{s}$-generic commutative diagrams of embeddings exist.*

(b) *Assume that $G \neq M$. For $0 \leq i \leq d$, let S^i be a maximal F-split torus of M^i. Let $\gamma \in X_*(\mathcal{Z}_{\mathrm{s}}(M^0)) \otimes \mathbb{R}$ be such that $\langle a, \gamma \rangle \neq 0$ for $a \in \Phi(G^i, S^i, F) \smallsetminus \Phi(M^i, S^i, F)$, $0 \leq i \leq d$. Then the set of $t \in \mathbb{R}$ such that the commutative diagram of embeddings $\{\iota\}_{t\gamma}$ is not $\vec{s}$-generic is an infinite discrete subset of $\mathbb{R}$.*

Proof. For $i = 0, 1, \cdots, d$ and $a \in \big(\Phi(G^i, S^i, F) \smallsetminus \Phi(M^i, S^i, F)\big)$, there exist infinite discrete subsets $\Gamma_{i,a}$ of $\mathbb{R}$ such that the set of $v \in V = X_*(\mathcal{Z}_{\mathrm{s}}(M^0)) \otimes \mathbb{R}$ with $\{\iota\}_v$ not $\vec{s}$-generic is the union of hyperplanes in V defined by $a(v) = c$, $c \in \Gamma_{i,a}$. Both statements follow easily from this. ∎

§4 Covers and decompositions

4.1 **Iwahori-type decompositions** Let $P = MU$ be a parabolic F-subgroup of G with Levi factor M, and $\bar{P} = M\bar{U}$ the opposite parabolic. A compact open subgroup K of G is said to *decompose with respect to* $U, M, \bar{U}$ if

$$K = (K \cap U).(K \cap M).(K \cap \bar{U}).$$

4.2 **Covers** Let M be a Levi subgroup of G, K (resp. K_M) a compact open subgroup of G (resp. M), and ρ (resp. ρ_M) an irreducible smooth representation of K (resp. K_M). The pair (K, ρ) is called a G-*cover* of (K_M, ρ_M) if for any opposite pair of parabolic subgroups $P = MU, \bar{P} = M\bar{U}$ with Levi factor M, we have

(i) K decomposes with respect to $(U, M, \bar{U})$.

(ii) $\rho | K_M = \rho_M$ and $K \cap U, K \cap \bar{U} \subset \ker(\rho)$.

(iii) For any smooth representation V of G, the natural map from V to its Jacquet module V_U induces an injection on $V^{(K,\rho)}$, the (K, ρ)-isotypic subspace of V.

This definition is due to Bushnell and Kutzko [BK], although we have used a reformulation given in [Bl, Théorème 1] (see also [GR, §4.1]).

4.3 We now give a useful (probably well-known) class of compact open subgroups with the decomposition property with respect to $(U, M, \bar{U})$.

Fix $\iota : \mathcal{B}(M) \hookrightarrow \mathcal{B}(G)$ and consider $\mathcal{B}(M)$ as a subset of $\mathcal{B}(G)$. Let E/F be a finite Galois extension such that $G \otimes E$ is split. Let T be a maximal torus of M, defined over F and split over E. Let y be a point of $A(M, T, E) \cap \mathcal{B}(G)$. Put $\Phi^0 = \Phi(G, T, E) \cup \{0\}$ and let $f : \Phi^0 \to \tilde{\mathbb{R}}$ be a $\mathrm{Gal}(E/F)$-stable and concave function. Then we can define $G(E)_{y,f}$ and $K = G_{y,f} = G(E)_{y,f} \cap G$ as in [Yu, page 608].

In addition, let $K_+ = G_{y,f_+}$, where f_+ is the concave function $a \mapsto \max(f(a), 0+)$ [Yu, Lemma 13.1 (ii)].

Proposition *Suppose that $f(a) \geq 0$ for all $a \in \Phi^0$ and $f(a) > 0$ for all $a \in \Phi^0 \smallsetminus \Phi(M, T, E)$.*

(a) *$K = G_{y,f}$ and $K_+ = G_{y,f_+}$ decompose with respect to $(U, M, \bar{U})$.*

(b) *Let $\hat{K}_M$ be a compact open subgroup of M containing $K_+ \cap M$ such that $\hat{K}_M$ normalizes K_+. Then $\hat{K} := (K \cap U)\hat{K}_M(K \cap \bar{U})$ is an open compact subgroup, which decomposes with respect to $(U, M, \bar{U})$. Moreover, $\hat{K}/K_+ \simeq \hat{K}_M/(K_+ \cap M)$.*

Proof. (a) It suffices to prove the assertion for $K_E = G(E)_{y,f}$. Indeed, if K_E decomposes with respect to $(U, M, \bar{U})$, and $g \in G_{y,f}$, then $g = um\bar{u}$ for unique elements $u \in K_E \cap U(E), m \in K_E \cap M(E), \bar{u} \in K_E \cap \bar{U}(E)$. For any $\sigma \in \mathrm{Gal}(E/F)$, $um\bar{u} = g = \sigma(g) = \sigma(u)\sigma(m)\sigma(\bar{u})$. By the uniqueness of the decomposition (a consequence of the big cell theorem [Bo, 14.21]), we have $u \in U(E)^{\mathrm{Gal}(E/F)} = U(F)$, etc. It follows that $K = G_{y,f}$ decomposes with respect to $(U, M, \bar{U})$. A similar remark applies to K_+.

We now prove (a) with the additional assumption that $E = F$ and T is split over F. The statement about K_+ is then a special case of [BT1, 6.4.48]. Write $\Phi^0 = \Phi_U \sqcup \Phi_M \sqcup \Phi_{\bar{U}}$, where $\Phi_M = \Phi(M, T, F) \cup \{0\}$, and Φ_U (resp. $\Phi_{\bar{U}}$) is the set of roots for the action of T on $\mathrm{Lie}\, U$ (resp. on $\mathrm{Lie}\, \bar{U}$)). For $H = U, M, \bar{U}$, let

$$f_H(a) = \begin{cases} f(a) & \text{if } a \in \Phi_H, \\ \infty & \text{otherwise.} \end{cases}$$

Then f_H is concave by [Yu, Lemma 13.1 (iv)], and $K^H = G_{y,f_H} \subset K \cap H$.

By [BT1, 6.4.43], K^M normalizes K^U. Therefore, $K^U K^M$ is a subgroup of G, and is the same as $G_{y,f_{UM}}$, where $f_{UM} = \inf(f_U, f_M)$. Again by [BT1, 6.4.43], $K^U K^M \subset K$ normalizes K_+. Therefore, $K^U K^M K_+$ is a subgroup of G. Since $K_+ = K^U(K_+ \cap M)K^{\bar{U}}$ and $K^U(K_+ \cap M) \subset K^U K^M$, we have $K^U K^M K_+ = K^U K^M K^{\bar{U}}$ is a subgroup of G. It follows that this subgroup is $G_{y,f} = K$, and $K^H = K \cap H$ for $H = U, M, \bar{U}$. This proves (a).

(b) Since $\hat{K}_M \subset M$ normalizes U, $\hat{K}_M$ normalizes K^U and $K^U\hat{K}_M$ is a subgroup of G. This subgroup normalizes K_+ since both K^U and $\hat{K}_M$ do. Therefore, $K^U\hat{K}_M K_+$ is a subgroup of G. Clearly, this subgroup is equal to $K^U\hat{K}_M K^{\bar{U}}$.

The natural morphism $\hat{K}_M \to \hat{K}/K_+$ is surjective with kernel $\hat{K}_M \cap K_+ = (K_+ \cap M)$. This gives the asserted isomorphism and finishes the proof of (b). ∎

§5 Heisenberg triples

***5.1* Definition** Let $J \supset J_+$ be compact, open, pro-p subgroups of G, and let $\varphi : J_+ \to \mathbb{C}^\times$ be a smooth character such that $\varphi(J_+) = \mu_p := \{\zeta \in \mathbb{C}^\times : \zeta^p = 1\}$. We say that (J, J_+, φ) is a Heisenberg triple if

(i) J_+ is a normal subgroup of J and J/J_+ is an abelian group of exponent p.
(ii) The commutator subgroup $[J, J_+] \subset \ker(\varphi)$.
(iii) The symplectic pairing $(J/J_+) \times (J/J_+) \to \mu_p$, $(a, b) \mapsto \varphi(aba^{-1}b^{-1})$ is non-degenerate.

Notice that (i) and (ii) imply that the pairing in (iii) is well-defined. It follows that $J/\ker(\varphi)$ is a Heisenberg p-group. Such triples often occur in the representation theory of p-adic groups.

***5.2* Example** We now recall the fundamental Heisenberg triple used in [Yu]. The setting of this example will be in force through the rest of this section. Let (G', G) be a tamely ramified twisted Levi sequence in G, and $y \in \mathcal{B}(G')$. Fix an embedding $\mathcal{B}(G') \hookrightarrow \mathcal{B}(G)$ to identify $\mathcal{B}(G')$ as a subset of $\mathcal{B}(G)$. Let r be a positive real number and $\phi : G'_{y,r:r+} \to \mathbb{C}^\times$ a G-generic character in the sense of [Yu, §9]. Put $s = r/2$, $J = (G', G)_{y,(r,s)}$, $J_+ = (G', G)_{y,(r,s+)}$, and $\varphi : J_+ \to \mathbb{C}^\times$ the character obtained by extending the restriction of ϕ to $G'_{y,r}$ trivially across a subgroup of G that is 'perpendicular' to G' in a suitable sense (see [Yu, §4]). Then (J, J_+, φ) is a Heisenberg triple by [Yu, Lemma 11.1].

5.3 Keep the settings in (5.2). In addition, let M' be a Levi subgroup of G' such that $y \in \mathcal{B}(M')$. Let M be the centralizer in G of $\mathcal{Z}_s(M')$. Then, if we put $(G^0, G^1) = (G', G)$, $(M^0, M^1) := (M', M)$ is a Levi sequence as in 2.4. We also fix a commutative diagram of embeddings of buildings extending $\mathcal{B}(G') \hookrightarrow \mathcal{B}(G)$:

$$\begin{array}{ccc} \mathcal{B}(G') & \hookrightarrow & \mathcal{B}(G) \\ \uparrow & & \uparrow \\ \mathcal{B}(M') & \hookrightarrow & \mathcal{B}(M) \end{array}$$

We treat these embeddings as inclusions.

Lemma *$\phi_M := \phi|M'_{y,r}$ is M-generic of depth r relative to y.*

Proof. By definition, $\phi|G'_{y,r:r+}$ is realized by a G-generic element $X^* \in (\mathrm{Lie}^* \mathcal{Z}(G'))_{-r}$. Let E/F be a finite extension over which M', M, G' and G are all split, and T a maximal E-split torus of M' such that $y \in A(M', T, E)$. Then the genericity of X^* means: $\mathrm{ord}(X^*(H_a)) = -r$ for all $a \in \Phi(G, T, E) \smallsetminus \Phi(G', T, E)$ (recall that $H_a = da^\vee(1)$ and $a^\vee : \mathbb{G}_\mathrm{m} \to T$ is the coroot of a). This is condition **GE1** in [Yu, §8]. By (2.3) and [Yu, Lemma 8.1], **GE2** holds automatically.

Clearly, $\phi_M|M'_{y,r:r+}$ is realized by $X^*_M :=$ the image of X^* under $\mathrm{Lie}^* G' \to \mathrm{Lie}^* M'$. So we have $X^*_M(H_a) = X^*(H_a)$ has valuation $-r$ for all $a \in \Phi(M, T, E) \smallsetminus \Phi(M', T, E) \subset \Phi(G, T, E) \smallsetminus \Phi(G', T, E)$ (notice that, for $a \in \Phi(M, T, E) \subset \Phi(G, T, E)$, H_a is the same whether we consider a as a root of M or of G). Again by (2.3) and [Yu, Lemma 8.1], this shows that ϕ_M is M-generic of depth r relative to y. ∎

5.4 Keep the settings in (5.2) and (5.3). We now give the crucial construction of a new Heisenberg triple needed in this paper. Let

$$\tilde{J} = (M,G)_{y,(r,s)} \cap (G',G)_{y,(r,s)}, \qquad \tilde{J}_+ = (M,G)_{y,(r,s)} \cap (G',G)_{y,(r,s+)}, \qquad \tilde{\varphi} = \varphi|\tilde{J}_+,$$

where φ is the character used in 5.2. By [Yu, Lemma 13.2], the groups $\tilde{J}$ and $\tilde{J}_+$ can be described by concave functions in a suitable way.

Lemma

(a) *$(\tilde{J}, \tilde{J}_+, \tilde{\varphi})$ is a Heisenberg triple.*

(b) *Let $P = MU$ be a parabolic F-subgroup of G with Levi factor M and unipotent radical U. Let $\bar{U}$ be the unipotent radical of the opposite parabolic subgroup. Let W and $\bar{W}$ be the images of $\tilde{J}^U := \tilde{J} \cap U$ and $\tilde{J}^{\bar{U}} := \tilde{J} \cap \bar{U}$ in $\tilde{V} := \tilde{J}/\tilde{J}_+$ respectively. Then W and $\bar{W}$ are maximal isotropic subspaces of $\tilde{V}$, and $\tilde{V} = W + \bar{W}$. In other words, W and $\bar{W}$ form a complete polarization of $\tilde{V}$.*

(c) *Let C be the center of $H := \tilde{J}/\ker(\tilde{\varphi})$. Let $A, \bar{A}$ be the images of $\tilde{J}^U$ and $\tilde{J}^{\bar{U}}$ in H. Then $A \cap C = \bar{A} \cap C = \{1\}$.*

(d) *$(G',G)_{y,(0,s)} \cap U$ normalizes $\tilde{J}^U$.*

Proof. Let (J, J_+, φ) be the triple in 5.2, so that $\tilde{J} \subset J$ and $\tilde{J}_+ = \tilde{J} \cap J_+$. Thus conditions (i) and (ii) in 5.1 follow from the corresponding statements for (J, J_+, φ). It is also clear that $\tilde{\varphi}(\tilde{J}_+) = \mu_p$.

Denote by (J^M, J^M_+, φ^M) the Heisenberg triple obtained by applying the construction of 5.2 to (M', M, y, ϕ_M). Then $J^M \subset J$, $J^M_+ = J^M \cap J_+$. Put $V = J/J_+$, $V_M = J^M/J^M_+$, $\tilde{V} = \tilde{J}/\tilde{J}_+$. Then we have embeddings of $V_M \hookrightarrow V$ and $\tilde{V} \hookrightarrow V$, which are compatible with the symplectic pairings on these spaces. We will regard these embeddings as inclusions.

By [Yu, Lemma 13.3], $J = J^M \tilde{J}$. This implies $V_M + \tilde{V} = V$. We now claim $V_M \perp \tilde{V}$. Since V_M is non-degenerate, this will imply $V = V_M \oplus \tilde{V}$ (orthogonal direct sum of symplectic spaces) and $\tilde{V}$ is non-degenerate. So condition (iii) in 5.1 will follow.

By Proposition 4.3,

$$\tilde{J} = \tilde{J}^U M_{y,r} \tilde{J}^{\bar{U}}.$$

It follows that we have $\tilde{V} = W + \bar{W}$. To prove the claim, it suffices to show that both W and $\bar{W}$ are perpendicular to V_M in V.

Indeed, if $a \in J^M$, $b \in \tilde{J}^U$, then $aba^{-1}b^{-1} \in J_+ \cap \tilde{J}^U \subset \tilde{J}_+ \cap \tilde{J}^U \subset \ker(\varphi)$. This shows $V_M \perp W$. The same argument proves $V_M \perp \bar{W}$. This finishes the proof of (a).

Similarly, if $a, b \in \tilde{J}^U$, then $aba^{-1}b^{-1} \in \tilde{J}_+ \cap \tilde{J}^U \subset \ker(\varphi)$. Therefore W is a totally isotropic subspace of $\tilde{V}$. The same goes for $\bar{W}$. Since $V = W + \bar{W}$, both W and $\bar{W}$ are maximal isotropic subspaces. This proves (b).

(c) is obvious since $\tilde{J}_+ \cap \tilde{J}^U, \tilde{J}_+ \cap \tilde{J}^{\bar{U}} \subset \ker(\varphi)$.

To prove (d), it suffices to prove the analogous statement when F is replaced by a finite, tamely ramified, Galois extension field E. Therefore, we may and do assume that M' is split over F, and that there is a maximal F-split torus T of M' such that $y \in A(M', T, F)$. Then we can write $\tilde{J}^U = G_{y,f}$, $(G',G)_{y,(0,s)} \cap U = G_{y,g}$ for suitable concave functions f, g on $\Phi^0 = \Phi(G,T,F) \cup \{0\}$.

Explicitly, let $v \in X_*(\mathcal{Z}_{\mathrm{s}}(M)) \otimes \mathbb{R}$ be such that $\langle a, v\rangle > 0$ for all roots a of $\mathcal{Z}_{\mathrm{s}}(M)$ on the Lie algebra of U. Then

$$f(a) = \begin{cases} r & \text{if } \langle a, v\rangle > 0, a \in \Phi(G', T, F), \\ s & \text{if } \langle a, v\rangle > 0, a \notin \Phi(G', T, F), \\ \infty & \text{if } \langle a, v\rangle \leq 0; \end{cases} \qquad g(a) = \begin{cases} 0 & \text{if } \langle a, v\rangle > 0, a \in \Phi(G', T, F), \\ s & \text{if } \langle a, v\rangle > 0, a \notin \Phi(G', T, F), \\ \infty & \text{if } \langle a, v\rangle \leq 0; \end{cases}$$

According to [BT1, 6.4.43], it suffices to check

$$f(pa + qb) \leq pf(a) + qg(b)$$

whenever $p, q \in \mathbb{Z}_{>0}$, $a, b, pa + qb \in \Phi^0$. This condition is easily verified and hence (d) is proved. ∎

***5.5* Lemma** *Let H be a finite Heisenberg p-group with center C. Assume that A, $\bar{A}$ are subgroups of H such that $A \cap C = \bar{A} \cap C = \{1\}$, and the image of A and $\bar{A}$ in $V := H/C$ form a complete polarization. Let ψ be a non-trivial character of C and (X, ρ) a complex representation of H such that $\rho|C$ is ψ-isotypic. Let $v \in X^A$ be non-zero. Then $\sum_{b \in \bar{A}} b.v$ is also non-zero.*

Proof. This is [K, Lemma 16.4]. For completeness, we produce a proof here. Assume $v \neq 0$ is fixed by A. For $b \in \bar{A}$, $a \in A$, we have

$$ab.v = aba^{-1}b^{-1}ba.v = \psi(aba^{-1}b^{-1})(b.v).$$

Therefore, $b.v$ is an eigenvector for A for the character $\psi_b : a \mapsto \psi(aba^{-1}b^{-1})$. As these characters are distinct, the list of (non-zero) vectors $\{b.v\}_{b \in \bar{A}}$ is linearly independent. It follows that their sum is non-zero. ∎

§6 Some covers of linear characters

***6.1* Setup** We now work in the setting of 2.4. Assume in addition:

(i) for each $0 \leq i \leq d-1$, we have a quasi-character $\phi_i : G^i \to \mathbb{C}^\times$ such that ϕ_i is G^{i+1}-generic of depth r_i relative to any $x \in \mathcal{B}(G')$; and

(ii) these depths satisfy

$$0 < r_0 < r_1 < \cdots < r_{d-1}; \text{ and}$$

(iii) we have a point $y \in \mathcal{B}(M^0)$ and a commutative diagram of embeddings $\{\iota\}$ as in 3.5, which is $\vec{s}$-generic relative to y, where

$$\vec{s} = (0, s_0, \ldots, s_{d-1}) = \left(0, \frac{r_0}{2}, \frac{r_1}{2}, \ldots, \frac{r_{d-1}}{2}\right)$$

(notice that $s_i = r_i/2$ is the $(i+1)$-st component of $\vec{s}$ while r_i is the i-th component of $\vec{r}$).

We can now form compact subgroups similarly as in [Yu, §3]:

$$\mathcal{K}^i = G^0_{y,0}G^1_{y,s_0}\cdots G^i_{y,s_{i-1}}, \qquad \mathcal{K}^i_+ = G^0_{y,0+}G^1_{y,s_0+}\cdots G^i_{y,s_{i-1}+}.$$

Let $\theta_i : \mathcal{K}^i_+ \to \mathbb{C}^\times$ be the character defined as in [Yu, §4]. Put $\mathcal{K} = \mathcal{K}^d$, $\mathcal{K}_+ = \mathcal{K}^d_+$, $\theta = \theta_d$, $\mathcal{K}^M_+ = \mathcal{K}_+ \cap M$, $\theta_M = \theta|\mathcal{K}^M_+$. We caution the reader that these groups do depend on the choice of $\{\iota\}$, although the dependency is suppressed in the above notation following [Yu]. We will write $\mathcal{K}^i\{\iota\}$, $\mathcal{K}^i_+\{\iota\}$, etc., when we need to make the dependency clear.

Remark 1. In (i), if ϕ_i is G^{i+1}-generic of depth r_i relative to an $y \in \mathcal{B}(G^i)$, it is G^{i+1}-generic of same depth relative to any $y\mathcal{B}(G^i)$. This follows from the definition of the genericity of ϕ_i.

2. Note that this $\mathcal{K}^i$ and K^i in §7.4 are different from the $K^i = G^0_{[y]}G^1_{y,s_0}\cdots G^i_{y,s_{i-1}}$ in [Yu, p591] in general. While $\mathcal{K}^i$ and K^i here are compact, K^i in [Yu] is compact mod center in general.

***6.2* Lemma** *Let $P = MU$ be a parabolic F-subgroup of G with Levi factor M and unipotent radical U, and $\bar{P} = M\bar{U}$ the opposite parabolic. Then we have*

(a) *$\mathcal{K}$ and $\mathcal{K}_+$ decompose with respect to $(U, M, \bar{U})$.*

(b) *The character θ is trivial on $\mathcal{K}_+ \cap U$ and $\mathcal{K}_+ \cap \bar{U}$.*

Proof. (a) follows from Proposition 4.3 (a). We first observe that $\phi_i(U \cap G^i) = \{1\}$ since $U \cap G^i$ lies in the commutator subgroup of G^i. Assertion (b) follows from this and the definition of $\hat{\phi}_i$ and θ_i in [Yu, §4]. ∎

***6.3* Theorem** *The pair $(\mathcal{K}_+, \theta)$ is a G-cover of $(\mathcal{K}^M_+, \theta_M)$.*

Proof. We proceed by induction on d. The argument here is similar to that in [K, Proposition 17.2]: the case $d = 0$ is a reformulation of Proposition 3.3 and the inductive step when $d \geq 1$ follows the method of [MP2], using Lemma 5.5 to play the role of [MP2, Proposition 6.1].

Now assume $d \geq 1$ and also $G \neq M$, since there is nothing to prove if $G = M$. Among the three defining conditions for a cover (4.2), (i) and (ii) are just the preceding lemma. It remains to prove (iii): for any parabolic F-subgroup $P = MU$ of G with Levi factor M, and for any smooth representation V of G, the canonical map from V to its Jacquet module V_U is injective on the $(\mathcal{K}_+, \theta)$-isotypic subspace of V.

Let $v \in V$ be a non-zero $(\mathcal{K}_+, \theta)$-isotypic vector. It suffices to show that $\int_{N_i} g.v\, dg \neq 0$ for an increasing family of open compact subgroups $\{N_i\}$ of U whose union is the whole of U.

Choose $\gamma \in X_*(\mathcal{Z}_{\mathrm{s}}(M^0)) \otimes \mathbb{R}$ such that $\langle a, \gamma\rangle > 0$ for all roots a of $\mathcal{Z}_{\mathrm{s}}(M^0)$ on the Lie algebra of U. For $t \in \mathbb{R}$, form the groups $\mathcal{K}(t) = \mathcal{K}\{\iota\}_{t\gamma}$ and $\mathcal{K}_+(t) = \mathcal{K}_+\{\iota\}_{t\gamma}$ (see 3.6 for the notation). Put $N(t) = \mathcal{K}_+(t) \cap U$ and $\bar{N}(t) = \mathcal{K}_+(t) \cap \bar{U}$. Then $N(t) \subset N(t')$, $\bar{N}(t) \supset \bar{N}(t')$ for $t < t'$ and $\bigcup_{t\in\mathbb{R}} N(t) = U$.

By Lemma 3.6, there is an infinite sequence $\cdots < t_{-1} < t_0 < t_1 < t_2 < \cdots$ such that $t_n \to +\infty$ and $t_{-n} \to -\infty$ as $n \to +\infty$, and $\mathcal{K}_+(t)$ is constant on the open intervals $t_{i-1} < t < t_i$ $(i \in \mathbb{Z})$. Therefore, we will denote by $\mathcal{K}_+(t_{i-1}, t_i)$ the group $\mathcal{K}_+(t)$ for any $t \in (t_{i-1}, t_i)$.

In fact, $M \cap \mathcal{K}_+(t) = \mathcal{K}_+^M$, $M \cap \mathcal{K}(t) = \mathcal{K}^M := \mathcal{K} \cap M$ for all t. For $i \in \mathbb{Z}$, let $N_i = U \cap \mathcal{K}_+(t_{i-1}, t_i)$, $\bar{N}_i = \bar{U} \cap \mathcal{K}_+(t_{i-1}, t_i)$. Then by Proposition 4.3, we have

$$\begin{aligned}\mathcal{K}_+(t_{i-1}, t_i) &= N_i \mathcal{K}_+^M \bar{N}_i, \\ \mathcal{K}_+(t_i) &= N_i \mathcal{K}_+^M \bar{N}_{i+1}, \\ \mathcal{K}(t_i) &= N_{i+1} \mathcal{K}^M \bar{N}_i.\end{aligned}$$

Since $\{\iota\} = \{\iota\}_{0\gamma}$ is $\vec{s}$-generic, the value $t = 0$ lies on one of the open intervals (t_i, t_{i+1}). We may and do asssume that $0 \in (t_0, t_1)$. Now put $v_1 = v$, and for $i \geq 1$, define inductively

$$v_{i+1} = \int_{N_{i+1}} x.v_i \, dx.$$

We make two claims: (i) v_i is a non-zero multiple of $\int_{N_i} x.v \, dx$, and (ii) v_i is $\big(\mathcal{K}_+(t_{i-1}, t_i), \theta(t_{i-1}, t_i)\big)$-isotypic, where $\theta(t_{i-1}, t_i)$ is the character of $\mathcal{K}_+(t_{i-1}, t_i)$ obtained by using the construction of 6.1 but with the embeddings $\{\iota\}_{t\gamma}$, $t \in (t_{i-1}, t_i)$ in place of $\{\iota\}$.

We prove the claims by induction. (i) is a simple consequence of Fubini's theorem and the unimodularity of U.

We have seen that $\mathcal{K}_+(t_i, t_{i+1}) = N_{i+1} \mathcal{K}_+^M \bar{N}_{i+1}$. Since $N_{i+1}, \bar{N}_{i+1} \subset \ker\big(\theta(t_i, t_{i+1})\big)$ by Lemma 6.2, it follows that $\ker\big(\theta(t_i, t_{i+1})\big) = N_{i+1} \ker(\theta_M) \bar{N}_{i+1}$. In particular, $N_{i+1} \ker(\theta_M) \bar{N}_{i+1}$ is a subgroup. Now we prove (ii). By the induction hypothesis, v_i is fixed by $\ker(\theta_M) \bar{N}_i \supset \ker(\theta_M) \bar{N}_{i+1}$. Since $N_{i+1} \ker(\theta_M) \bar{N}_{i+1}$ is a compact subgroup, it follows that v_{i+1} is fixed by this compact subgroup. To finish the proof of (ii), it suffices to show that $\mathcal{K}_+^M$ acts on v_{i+1} via the the character θ_M. Indeed, for $g \in \mathcal{K}_+^M$,

$$g.v_{i+1} = \int_{N_{i+1}} gxg^{-1}.g.v_i \, dx = \int_{N_{i+1}} (gxg^{-1}).\theta_M(g) v_i \, dx = \theta_M(g) v_{i+1}.$$

The last equality is because the compact group $\mathcal{K}_+^M$ normalizes N_{i+1}.

It remains to prove the most important statement:

Lemma $v_{i+1} \neq 0$ *for all* $i \geq 0$.

Proof. We prove this by induction on i. The case $i = 0$ holds by assumption. Let $(\tilde{J}, \tilde{J}_+, \tilde{\varphi})$ be the triple constructed in (5.4) with $(M', M, G', G, \phi, r) = (M^{d-1}, M, G^{d-1}, G, \phi_{d-1}, r_{d-1})$ and the embeddings $\{\iota\}_{t_i\gamma}$. By Lemma 5.4 (d), $N_{i+1} = \mathcal{K}(t_i) \cap U \subset (G', G)_{y,(0,s)} \cap U$ normalizes $\tilde{J} \cap U$, where $(G', G)_{y,(0,s)}$ is formed using the embeddings $\{\iota\}_{t_i\gamma}$. It follows that $(N_{i+1} \cap G')(\tilde{J} \cap U)$ is a subgroup, and in fact

$$(N_{i+1} \cap G')(\tilde{J} \cap U) = N_{i+1}$$

by [Yu, Lemmas 13.3 and 13.4].

It follows that v_{i+1} is a non-zero multiple of

$$\int_{N_{i+1} \cap G'} \int_{\tilde{J} \cap U} y.x.v_i \, dx \, dy$$

We make two more claims: (iii) $v' = \int_{\tilde{J}\cap U} x.v_i\,dx$ is non-zero; (iv) v' is $\big(\mathcal{K}_+(t_{i-1},t_i)\cap G', \theta(t_{i-1},t_i)\big)$-isotypic.

To prove (iii), let W be the $(\tilde{J}_+, \tilde{\varphi})$-isotypic subspace of V. This is naturally a representation of $\tilde{J}$. It is easy to verify that $\tilde{J}_+ \subset \mathcal{K}_+(t_{i-1},t_i)$ and $\theta(t_{i-1},t_i)|\tilde{J}_+ = \tilde{\varphi}$. Therefore, $v_i \in W$ and the integral defining v' can be calculated within the $\tilde{J}$-representation W. The image of $\bar{N}_i \cap \tilde{J}$ in $\tilde{J}/\ker(\tilde{\varphi})$ is the same as that of $\tilde{J}^{\bar{U}}$, and the image of $N_{i+1} \cap \tilde{J}$ in $\tilde{J}/\ker(\tilde{\varphi})$ is the same as that of $\tilde{J}^U$, where $\tilde{J}^U$ and $\tilde{J}^{\bar{U}}$ are as in 5.4. By (ii), $\bar{N}_i$ fixes v_i. This implies $v' \neq 0$ by Lemmas 5.4 and 5.5.

The proof of (iv) is similar to the proof of (ii) above.

Finally, we prove the lemma. We can regard V as a smooth representation of G'. We may also regard v' as a vector in the representation $V \otimes \phi_{d-1}^{-1}$ of G'. The integral $\int_{N_{i+1}\cap G'} y.v'\,dy$ is the same whether we use the action of G' on V or on $V \otimes \phi_{d-1}^{-1}$, since $\phi_{d-1}(N_{i+1} \cap G') = \{1\}$.

We can now apply the setting of this section to $G' = G^{d-1}$, the twisted Levi sequence $(G^0, \dots, G^{d-1})$, the characters $(\phi_0, \dots, \phi_{d-2})$, and the $(0, r_0/2, \dots, r_{d-2}/2)$-generic embeddings $\{\iota\}_{t\gamma}$, where $t \in (t_i, t_{i+1})$. Then we form the group $\mathcal{K}'_+$ and θ' as in 6.1, and these are nothing but $\mathcal{K}'_+ = \mathcal{K}_+(t_{i-1},t_i) \cap G'$, and $\theta' = (\theta(t_{i-1},t_i)|\mathcal{K}'_+) \otimes \phi_{d-1}^{-1}$. Recall that $v' \in V \otimes \phi_{d-1}^{-1}$ is $(\mathcal{K}'_+, \theta')$-isotypic by (iv). Therefore, we can apply the induction hypothesis (for the induction on d) to conclude $v_{i+1} \neq 0$. This proves the lemma, and hence the theorem. ∎

6.4 **Corollary** *Let K be a compact open subgroup of G and ρ an irreducible smooth representation of K. Suppose that (K, ρ) satisfies conditions* (i) *and* (ii) *in (4.2), $K \supset \mathcal{K}_+$, and $\rho|\mathcal{K}_+$ is θ-isotypic. Then (K, ρ) is a G-cover of $(K \cap M, \rho|K \cap M)$.*

Proof. For an irreducible smooth representation V of G, let V^ρ (resp. V^θ) denote the ρ-isotypic (resp. θ-isotypic) component in V. Then, $V^\rho \subset V^\theta$. Since the natural map $V \to V_U$ on V^θ is injective, it is also injective on V^ρ. ∎

§7 Construction of types

From now on we assume that p is odd.

7.1 **Depth-zero datum** We now review the construction of types of depth zero by [MP2]. We define a depth-zero datum to be a triple $\big((G, M), (y, \iota), (K_M, \rho_M)\big)$ such that

- (i) G is a connected reductive group over F and M a Levi subgroup of G.
- (ii) $y \in \mathcal{B}(M)$ is such that $M_{y,0}$ is a maximal parahoric subgroup of M, and $\iota : \mathcal{B}(M) \hookrightarrow \mathcal{B}(G)$ is a 0-generic embedding relative to y.
- (iii) K_M is a compact open subgroup of M containing $M_{y,0}$ as a normal subgroup, and ρ_M is an irreducible smooth representation of K_M such that $\rho_M|M_{y,0}$ contains a cuspidal representation of $M_{y,0:0+}$.

Remark Since $M_{y,0}$ is a normal subgroup of K_M, the restriction $\rho_M|M_{y,0}$ is ρ'-isotypic where ρ' is any irreducible cuspidal representation occurring in $\rho_M|M_{y,0}$. It follows that $\rho_M|M_{y,0+}$ is trivial ($\mathbf{1}$-isotypic).

This datum encodes not only a type of depth 0 in G but also how it arises from a cover, as follows. By [MP6, Proposition 6.8] and [BK, Proposition 5.4], (K_M, ρ_M) is an $\mathfrak{S}$-type where $\mathfrak{S}$ is a finite set of the form $\{[M, \pi_1], \ldots, [M, \pi_n]\}$ with the π_i's irreducible supercuspidal representations of M. Note that *when K_M is the maximal compact subgroup fixing y of M, $\mathfrak{S}$ is a singleton.*

By Proposition 4.3 (b), $K_G := K_M G_{\iota(y),0}$ is a subgroup such that

$$K_G/G_{\iota(y),0+} \simeq K_M/M_{y,0+}.$$

Let ρ_G be the representation of K_G obtained by composing the above isomorphism with ρ_M. Then (K_G, ρ_G) is a G-cover of (K_M, ρ_M). Therefore, by [BK, Theorem 8.3], it is an $\mathfrak{S}(G)$-type, where $\mathfrak{S}(G)$ is defined in [BK, §8].

***7.2* The datum** The datum Σ from which we will construct a type is a 5-tuple

$$\Sigma := \big((\vec{G}, M^0), (y, \{\iota\}), \vec{r}, (K_{M^0}, \rho_{M^0}), \vec{\phi}\big)$$

entirely analogous to that in [Yu, §3], as follows:

D1 $\vec{G} = (G^0, \ldots, G^d)$ is a tamely ramified twisted Levi sequence in G, and M^0 a Levi subgroup of G^0. Unlike [Yu, §3], we impose **no** assumption on $\mathcal{Z}(G^0)/\mathcal{Z}(G)$. We construct a Levi subgroup M of G and a generalized twisted Levi sequence $\vec{M}$ in M as in 2.4.

D2 y is a point in $\mathcal{B}(M^0)$, and $\{\iota\}$ is a commutative diagram of embeddings of buildings as in (3.5), $\vec{s}$-generic relative to y, where $\vec{s} = (0, r_0/2, \ldots, r_{d-1}/2)$.

D3 $\vec{r} = (r_0, \ldots, r_d)$ is a sequence of real numbers satisfying $0 < r_0 < r_1 < \cdots < r_{d-1} \leq r_d$ if $d > 0$, $0 \leq r_0$ if $d = 0$.

D4 $\big(G^0, M^0, y, \iota : \mathcal{B}(M^0) \hookrightarrow \mathcal{B}(G^0), (K_{M^0}, \rho_{M^0})\big)$ is a depth zero datum.

D5 $\vec{\phi} = (\phi_0, \ldots, \phi_d)$, where ϕ_i is a quasi-character of G^i such that ϕ_i is G^{i+1}-generic of depth r_i relative to x for all $x \in \mathcal{B}(G^i)$.

***7.3* Remark** Again, the datum encodes not just the type itself but also how the type arises as a cover. In practice, one may start with a 5-tuple $(\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ similar to [Yu, §3], but with no assumption on $\mathcal{Z}(G^0)/\mathcal{Z}(G)$, and instead of **D4** of [Yu, §3], we assume that $(G^0_{y,0}, \rho)$ is an unrefined minimal K-type of depth 0 in the sense of Moy-Prasad. We then construct M^0 and $\{\iota\}$ as follows. By [MP2, 6.3], to the parahoric subgroup $G^0_{y,0}$ of G^0, we can attach a Levi subgroup M^0 of G^0, unique up to conjugacy by $G^0_{y,0}$. From the construction there, we see that there is an embedding $\iota : \mathcal{B}(M^0) \hookrightarrow \mathcal{B}(G^0)$ whose image contains y, $M^0_{y,0}$ is maximal parahoric, and ι is 0-generic relative to y. One can then extend/modify ι to a family $\{\iota\}$ which is $\vec{s}$-generic by Lemma 3.6.

Of course, there are choices involved in this procedure. Also, in order to get the finest $\mathfrak{S}$-types, i.e., $\mathfrak{S}$-types with $\mathfrak{S} = \{\mathfrak{s}\}$ a singleton, we need to refine the datum ρ a little bit. Eventually we end up with a datum as defined above.

***7.4* The construction** We now put $K^0 = K_{G^0} = K_{M^0} G^0_{y,0} = K_{M^0} \mathcal{K}^0$ and $\rho = \rho_{G^0}$ as in (7.1), and for $i \geq 1$, put

$$K^i = K^0 G^1_{y,s_0} \cdots G^i_{y,s_{i-1}} = K_{M^0} \mathcal{K}^i, \qquad K^i_+ = \mathcal{K}^i_+.$$

Again we remind the reader that these groups depend on the choice of $\{\iota\}$, and this K^i may be different from the one used in [Yu, §3 and §4]. Nevertheless, it is easy to see that the construction in [Yu, §4] can be carried out literally without any modification to give a representation ρ_i for each K^i, $i \geq 0$, with $\rho_0 = \rho$.

Moreover, $\Sigma_M := (\vec{M}, y, \vec{r}, \rho_M, \vec{\phi})$ is a datum for constructing a supercuspidal type in M; see [Yu, Remark 15.4] and the discussions following [Yu, Theorem 15.7]. So we can construct supercuspidal types (K_M^i, ρ_i^M), for each $i \geq 0$, where $K_M^0 = K_{M^0}$,

$$K_M^i = K_{M^0} M_{y,s_0}^1 \cdots M_{y,s_{i-1}}^i, \qquad i \geq 1.$$

Write

$$\mathcal{T}^i := (K^i, \rho^i),\ \mathcal{T} := \mathcal{T}^d = (K^d, \rho^d); \qquad \mathcal{T}_M^i := (K_M^i, \rho_i^M),\ \mathcal{T}_M := \mathcal{T}_M^d = (K_M^d, \rho_d^M).$$

Let $\mathfrak{S}_i$ be the finite set such that $\mathcal{T}_M^i$ is an $\mathfrak{S}_i$-type in M^i. *If K_{M^0} is the fixer of y in M^0, $\mathfrak{S}_i$ is a singleton.*

For $\pi \in \mathcal{R}(G^i)$, we write $\mathcal{T}^i < \pi$ if ρ^i occurs in $\pi | K^i$.

***7.5* Theorem** *For $i \geq 0$, $\mathcal{T}^i$ is a G^i-cover of $\mathcal{T}_M^i$. Hence $\mathcal{T}^i$ is an $\mathfrak{S}_i(G^i)$-type in G^i.*

Proof. The second statement follows from the first and [BK, Theorem 8.3].

Condition (i) in 4.2 follows from Proposition 4.3 (b).

We now verify condition (ii) in 4.2 by induction. The case of $i = 0$ is just the definition. The inductive construction of ρ_i from ρ_{i-1} in [Yu, §4] relies on the Heisenberg triple (J^i, J_+^i, φ_i), where $\varphi_i = \hat{\phi}_{i-1} | J_+^i$. Similarly, to construct ρ_i^M from ρ_{i-1}^M we use an analogous Heisenberg triple $(J_M^i, J_{M,+}^i, \varphi_i^M)$. It follows from the definitions of these objects that $J_M^i = J^i \cap M_i$, $J_{M,+}^i = J_+^i \cap M_i$, and $\varphi_i^M = \varphi_i | J_{M,+}^i$. Moreover, J^i and J_+^i decompose with respect to $(U^i, M^i, \bar{U}^i)$ by Proposition 4.3, where $U^i = U \cap G^i$, $\bar{U}^i = \bar{U} \cap G^i$. Since $\{\iota\}$ is s-generic, we have $J^i \cap U^i = J_+^i \cap U^i$ and $J^i \cap \bar{U}^i = J_+^i \cap \bar{U}^i$. It follows that the inclusion $J_M^i \subset J^i$ induces an isomorphism

$$J_M^i / J_{M,+}^i \simeq J^i / J_+^i.$$

Let $N^i = \ker(\varphi_i)$, $N_M^i = \ker(\varphi_i^M)$. We can verify that the following diagram is commutative:

$$\begin{array}{ccc} J_M^i / N_M^i & \xrightarrow{\sim} & (J_M^i / J_{M,+}^i)^\sharp \\ \downarrow \wr & & \downarrow \wr \\ J^i / N^i & \xrightarrow{\sim} & (J^i / J_+^i)^\sharp, \end{array}$$

where $(J_M^i / J_{M,+}^i)^\sharp$ and $(J^i / J_+^i)^\sharp$ are defined as in [Yu, §10]. The vertical arrows are the isomorphisms induced by inclusion, and the horizontal arrows are the canonical special isomorphisms constructed in [Yu, Proposition 11.4]. In addition, the following diagram is also commutative:

$$\begin{array}{ccc} K_M^{i-1} & \longrightarrow & \mathrm{Sp}(J_M^i / N_M^i) \\ \bigcap & & \downarrow \\ K^{i-1} & \longrightarrow & \mathrm{Sp}(J^i / N^i), \end{array}$$

where the horizonal arrows are induced by conjugations, and the vertical arrow on the right is induced by the isomorphism $J^i_M/J^i_{M,+} \simeq J^i/J^i_+$. It follows from these and the definitions of ρ_i and ρ_i^M that we do have $\rho_i|K^i_M = \rho_i^M$.

Define (K^i_+, θ_i) as in the preceding section. Note that this K^i_+ is identical to the K^i_+ used in [Yu]. By [Yu, Proposition 4.4], $\rho_i|K^i_+$ is θ_i-isotypic. By Proposition 6.2, $K^i_+ \cap U^i, K^i_+ \cap \bar{U}^i \subset \ker(\rho_i)$. Since $K^i_+ \cap U^i = K^i \cap U^i$ and $K^i_+ \cap \bar{U}^i = K^i \cap \bar{U}^i$ by the genericity of $\{\iota\}$, we have proved condition (ii) of 4.2 completely.

Now we see that all hypotheses for Corollary 6.4 are satisfied. The theorem is proved. ∎

§8 Support of Hecke algebras

Let (K^i, ρ_i) be as in §7.4. Let $\check{\rho}_i$ be the contragradient of ρ_i. Then the Hecke algebra $\mathcal{H}(G^i, \rho_i)$ associated to (K^i, ρ_i) is defined as follows:

$$\mathcal{H}(G^i, \rho_i) = \{f \in C_c(G^i, \mathrm{End}(\check{\rho}_i)) \mid f(jgj') = \check{\rho}_i(j)f(g)\check{\rho}_i(j') \text{ for } g \in G^i,\ j, j' \in K^i\}.$$

As in [Yu, §17], we write $\check{\mathcal{H}}(G^i, \rho_i)$ for $\mathcal{H}(G^i, \check{\rho}_i)$. For $g \in G^i$, let $I_g(\rho_i)$ denote the space of intertwining maps $\mathrm{Hom}_{K^i \cap gK^ig^{-1}}({}^g\rho_i, \rho_i)$ where ${}^g\rho_i$ is a representation of gK^ig^{-1} with ${}^g\rho_i(h) = \rho_i(g^{-1}hg)$ for $h \in gK^ig^{-1}$ (see also [Yu, p582]).

***8.1* Theorem**

(a) The support of $\check{\mathcal{H}}(G^i, \rho_i)$ is contained in $K^iG^0K^i$.

(b) For $g \in G^0$, we have

$$I_g(\rho_i) = I_g(\rho_0 \mid K^0) \otimes I_g(\tilde{\phi}_0) \otimes \cdots \otimes I_g(\tilde{\phi}_{i-1})$$

where $I_g(\tilde{\phi}_j)$ is 1-dimensional for $j = 0, \cdots, d-1$.

Again the proof in [Yu, §15] can be carried out without change.

***8.2* Corollary** *The support of $\check{\mathcal{H}}(G^i, \rho_i)$ is contained in $K^iN_{G^0}(M^0)K^i$.*

Proof. By [Mo, Theorem 4.15], for $g \in G^0$, we have $I_g(\rho_0 \mid K^0) \neq 0$ only if $g \in K^0N_{G^0}(M^0)K^0$. Hence, combining with the above theorem, the corollary follows. ∎

§9 Exhaustion

Recall that in [K], it is proved that all supercuspidal representations arise from the construction given in [Yu] under some hypotheses (see [K, §3.4]). In this section, we prove that our construction gives all types parameterizing $\mathcal{R}^\mathfrak{s}$, $\mathfrak{s} \in \mathcal{I}$ (see (1) in §1) under the same hypotheses (Hk), (HB), (HGT) and (H$\mathcal{N}$) (see [K, §3.4] for details). We adopt notation and terminologies from [K].

***9.1* Theorem** *Suppose* (Hk), (HB), (HGT) *and* (H$\mathcal{N}$) *are valid. For each inertial class $\mathfrak{s} \in \mathcal{I}$, there is a datum $\big((\vec{G}, M^0), (y, \{\iota\}), \vec{r}, (K_{M^0}, \rho_{M^0}), \vec{\phi}\big)$ so that (K^d, ρ_d) is an $\mathfrak{s}$-type.*

SKETCH OF THE PROOF. Let $\mathcal{E}^t(G)$ be the set of irreducible smooth tempered representations. Note that for any $\mathfrak{s} = [(M_\mathfrak{s}, \pi_\mathfrak{s})] \in \mathcal{I}$, the Plancherel measure of $\mathcal{E}^t(G) \cap \mathcal{R}^\mathfrak{s}$ is nonzero. Hence, it is enough to show that there are $\pi \in \mathcal{E}^t(G) \cap \mathcal{R}^\mathfrak{s}$ and a datum $\big((\vec{G}, M^0), (y, \{\iota\}), \vec{r}, (K_{M^0}, \rho_{M^0}), \vec{\phi}\big)$ so that (K^d, ρ_d) gives a G-cover of the supercuspidal type of the cuspidal pair $(M_\mathfrak{s}, \pi_\mathfrak{s})$ which supports π. We sketch a proof in several steps below:

(1) Recall from [KM, §4], for a given strongly good datum $(\vec{G}, x, \vec{r}, \vec{\phi})$, upon fixing embeddings of buildings, $\mathcal{B}(G^0) \hookrightarrow \mathcal{B}(G^1) \hookrightarrow \cdots \hookrightarrow \mathcal{B}(G^d)$, one can associate a strongly good type (K^d_{x+}, θ_d) where $K^d_{x+} = G^0_{x,0+} G^1_{x,s_0+} \cdots G^d_{x,s_{d-1}+}$ and θ_d is constructed as in 6.1 (see [KM, §4] for details). Then, by [K, Theorem 11.4], there are $\pi \in \mathcal{E}^t(G) \cap \mathcal{R}^\mathfrak{s}$ and a strongly good type (K^d_{x+}, θ_d) such that $(K^d_{x+}, \theta_d) < \pi$.

(2) Let $V_\pi^{\theta_d}$ be the θ_d isotypic component of V_π. Then, $V_\pi^{\theta_d}$ is stabilized by $G^0_{x,0}$. Let $y \in \mathcal{B}(G^0)$ be such that (i) $G^0_{y,0} \subset G^0_{x,0}$, (ii) $V_\pi^{\theta_d}$ has nontrivial $G^0_{y,0+}$-invariants, and (iii) $G^0_{y,0}$ is minimal satisfying (i) and (ii). Such a y exist since θ_d is trivial on $G^0_{x,0+}$.

(3) Let M^0 be a Levi subgroup of G^0 so that $M^0_{y,0}$ is a maximal parahoric subgroup of M^0, and M the Levi subgroup given by the centralizer of $\mathcal{Z}_\mathrm{s}(M^0)$. Let $P = MU$ be the parabolic subgroup of G so that $(P \cap G^0_{x,0}) G^0_{y,0+} = G^0_{y,0}$. Let $\overline{U}$ be the opposite unipotent radical.

(4) Form $K_\vdash = (\overline{U} M \cap K^d_{x+})(U \cap (G^0_{y,0+} G^1_{x,s_0} \cdots G^d_{x,s_{d-1}}))$, which is defined in [K, §13]. Note that $K_{x+} \subset K_\vdash$ and $\vec{\phi}$ defines a character θ'_d of $K_\vdash$ such that $\theta'_d | K^d_{x+} = \theta_d$. Then, by [K, Corollary 13.12], $(K_\vdash, \theta'_d) < V_\pi$, and $V_\pi^{\theta'_d} \subset V_\pi^{\theta_d}$.

(5) Consider $V_\pi^{\theta'_d}$. Note that $V_\pi^{\theta'_d}$ is stabilized by $K_{y,M} = M^0_{[y]}(M \cap (G^1_{y,s_0} \cdots G^d_{y,s_{d-1}})) = M^0_{[y]} M^1_{y,s_0} \cdots M^d_{y,s_{d-1}}$ where $M^0_{[y]}$ is the stabilizer of the image $[y]$ of y in the reduced building of M and $M^i = M \cap G^i$. Let $\hat{\theta}'_d$ be an irreducible representation of $K_{y,M}$ such that $\hat{\theta}'_d | K_\vdash$ is θ'_d-isotypic as in [K, §13] ($\hat{\theta}'_d$ is denoted by κ in [K, §13]). Then, by [K, Corollary 18.6], there is an irreducible representation τ' of $M^0_{[y]}$ factoring through $Z_M M^0_{y,0+}$ such that $\tau' \otimes \hat{\theta}'_d$ is contained in $V_\pi^{\theta'_d}$. Then, $\tau' | M^0_{y,0}$ induces a cuspidal representation of $M^0_{y,0}/M^0_{y,0+}$ since otherwise, there is a smaller parahoric subgroup $G^0_{z,0} \subset G^0_{x,0}$ with nontrivial $G^0_{z,0+}$ invariants in $V_\pi^{\theta_d}$, which is a contradiction to the choice of $G^0_{y,0}$ in (2).

(6) From the proof of [K, Theorem 19.1], $\pi_M := c\text{-}\mathrm{Ind}_{K_{y,M}}^M \tau' \otimes \hat{\theta}'_d$ is a supercuspidal representation of M associated to a generic datum $(\vec{M}, y, \vec{r}, \vec{\phi}_M, \tau')$ where $\vec{M} = (M^0, M^1, \cdots, M^d)$ and $\vec{\phi}_M = (\phi_0 | M, \cdots, \phi_d | M)$. Moreover, (M, π_M) is equivalent to $\mathfrak{s}$ in $\mathcal{I}$.

(7) Let K_{M^0} be the maximal compact subgroup of $M^0_{[y]}$ and ρ_{M^0} a subrepresentation of τ' when restricted to K_{M^0}. Then, $((\vec{M}, M^0), y, \vec{r}, (K_{M^0}, \rho_{M^0}), \vec{\phi}_M)$ gives a supercuspidal type (K^d_M, ρ^M_d).

(8) Consider $((\vec{G}, M^0), (y, \{\iota\}), \vec{r}, (K_{M^0}, \rho_{M^0}), \vec{\phi})$ where $\{\iota\}$ is $(0, s_0, \cdots, s_{d-1})$-generic. Then, by construction, (K^d, ρ_d) is a cover of (K^d_M, ρ^d_M), hence an $\mathfrak{s}$-type. ∎

Remark 1. The above proof starts with $\pi \in \mathcal{R}^\mathfrak{s}$ and a strongly good type contained in π, proceeds toward nailing down a supercuspidal type (K_M, ρ^M_d) out of the strongly good type, and then finally finds a type as a cover (K^d, ρ_d) of (K_M, ρ^M_d). On the other hand, we note that it is possible to start with a supercuspidal type datum for $\mathfrak{s}$ and work toward getting a cover. However, to nail down $\vec{G}$ in the datum, we find the proof above more efficient.

2. A priori, we can not assume our choices of y in (2) or embeddings $\mathcal{B}(G^0) \hookrightarrow \mathcal{B}(G^1) \hookrightarrow \cdots \hookrightarrow \mathcal{B}(G^d)$ in (1) satisfy any genericity condition. Hence, we still need to work with an auxiliary group $K_\vdash$ in (4) (cf. Remark 3.4).

§10 Equivalence

***10.1* Definition** Let Σ and $\dot\Sigma$ be two data as in §7.2. Let $\mathcal{T} = (K, \rho)$ (resp. $\dot{\mathcal{T}}$) be the type constructed in §7.4 associated to Σ (resp. $\dot\Sigma$).

(i) Define $\mathcal{R}_\mathcal{T}$ to be the category of smooth representations π which are generated by the ρ isotypic components of V_π.

(ii) Let $\dot{\mathcal{T}}$ be the type associated to $\dot\Sigma$. We say that $\mathcal{T}$ and $\dot{\mathcal{T}}$ are equivalent if there is $\mathfrak{S} \subset \mathfrak{I}$ such that $\mathcal{R}^{\mathfrak{S}} = \mathcal{R}_\mathcal{T} = \mathcal{R}_{\dot{\mathcal{T}}}$ where $\mathcal{R}^{\mathfrak{S}} = \prod_{\mathfrak{s} \in \mathfrak{S}} \mathcal{R}^{\mathfrak{s}}$.

From now on, we assume that our data Σ satisfy the hypothesis $\mathrm{C}(\vec{G})$ in [HaMu, §2.6].

***10.2* Theorem** *Let* $\Sigma := \big((\vec{G}, G^0), (y, \{\iota\}), \vec{r}, (K_{G^0}, \rho_{G^0}), \vec\phi\big)$ *and* $\dot\Sigma := \big((\dot{\vec{G}}, \dot G^0), (\dot y, \{\dot\iota\}), \dot{\vec{r}}, (K_{\dot G^0}, \rho_{\dot G^0}), \dot{\vec\phi}\big)$ *be two data such that* Z_{G^0}/Z_G *(resp.* $Z_{\dot G^0}/Z_G$*) is* F*-anisotropic and* K_{G^0} *(resp.* $K_{\dot G^0}$*) is the maximal compact subgroup of* $G^0_{[y]}$ *(resp.* $\dot G^0_{[\dot y]}$*). Let* $\mathcal{T} := (K, \rho)$ *and* $\dot{\mathcal{T}} := (\dot K, \dot\rho)$*. Let* $\phi = \prod_{i=0}^d \phi_i$ *and* $\dot\phi = \prod_{i=0}^{\dot d} \dot\phi_i$ *be characters of* G^0 *and* $\dot G^0$ *respectively. Then,* $\mathcal{T}$ *and* $\dot{\mathcal{T}}$ *are equivalent if and only if there is* $g \in G$ *such that* $gy = \dot y$*,* ${}^gK_{G^0} = K_{\dot G^0}$ *and* ${}^g(\rho_{G^0} \otimes \phi) \simeq (\dot\rho_{\dot G^0} \otimes \dot\phi)$ *as* $K_{\dot G^0}$ *representations.*

Proof. Note that $\mathcal{T}$ and $\dot{\mathcal{T}}$ are supercuspidal types.

Suppose $\mathcal{R}_\mathcal{T} = \mathcal{R}_{\dot{\mathcal{T}}}$. Let $\pi \in \mathcal{R}_\mathcal{T} = \mathcal{R}_{\dot{\mathcal{T}}}$ be irreducible supercuspidal. Then, let $\tilde\rho_{G^0}$ (resp. $\dot{\tilde\rho}_{\dot G^0}$) be a representation of $G^0_{[y]}$ (resp. $\dot G^0_{[\dot y]}$) containing ρ_{G^0} (resp. $\dot\rho_{\dot G^0}$) so that $(\vec{G}, y, \vec{r}, \vec\phi, \tilde\rho_{G^0})$ (resp. $(\dot{\vec{G}}, \dot y, \vec{r}, \dot{\vec\phi}, \dot{\tilde\rho}_{\dot G^0})$) is a supercuspidal datum for π. Then, [HaMu, Theorem 6.7], there is $g \in G$ so that $gy = \dot y$, ${}^gG^0 = \dot G^0$, and ${}^g(\tilde\rho_{G^0} \otimes \phi) \simeq (\tilde\rho_{\dot G^0} \otimes \dot\phi)$ as $\dot G^0_{[\dot y]}$ representations. Since both ${}^g(\rho_{G^0} \otimes \phi)$ and $(\dot\rho_{\dot G^0} \otimes \dot\phi)$ are subrepresentations of $(\tilde\rho_{\dot G^0} \otimes \dot\phi)$ and $K_{\dot G^0}$ is a normal subgroup of $\dot G^0_{[y]}$, there is a $\dot g \in \dot G^0_{[\dot y]}$ so that ${}^{\dot g g}(\rho_{G^0} \otimes \phi) \simeq (\dot\rho_{\dot G^0} \otimes \dot\phi)$.

Conversely, suppose there is $g \in G$ such that $gy = \dot y$, ${}^gK_{G^0} = K_{\dot G^0}$ and ${}^g(\rho_{G^0} \otimes \phi) \simeq (\dot\rho_{\dot G^0} \otimes \dot\phi)$ as $K_{\dot G^0}$ representations. It is enough to show that there is a supercuspidal representation $\pi \in \mathcal{R}_\mathcal{T} \cap \mathcal{R}_{\dot{\mathcal{T}}}$. Let $\tilde\rho_{G^0}$ (resp. $\dot{\tilde\rho}_{\dot G^0}$) be a representation of $G^0_{[y]}$ (resp. $\dot G^0_{[\dot y]}$) containing ρ_{G^0} (resp. $\dot\rho_{\dot G^0}$) so that ${}^g(\tilde\rho_{G^0} \otimes \phi) \simeq (\tilde\rho_{\dot G^0} \otimes \dot\phi)$ as $\dot G^0_{[\dot y]}$ representations. Then, $(\vec{G}, y, \vec{r}, \vec\phi, \tilde\rho_{G^0})$ and $(\dot{\vec{G}}, \dot y, \vec{r}, \dot{\vec\phi}, \dot{\tilde\rho}_{\dot G^0})$ are supercuspidal data, and their associated supercuspidal representations are isomorphic, which are in $\mathcal{R}_\mathcal{T} \cap \mathcal{R}_{\dot{\mathcal{T}}}$. Hence, $\mathcal{R}_\mathcal{T} = \mathcal{R}_{\dot{\mathcal{T}}}$. ∎

***10.3* Theorem** *Let* $\Sigma := \big((\vec{G}, M^0), (y, \{\iota\}), \vec{r}, (K_{M^0}, \rho_{G^0}), \vec\phi\big)$ *and* $\dot\Sigma := \big((\dot{\vec{G}}, \dot M^0), (\dot y, \{\dot\iota\}), \dot{\vec{r}}, (K_{\dot M^0}, \rho_{\dot M^0}), \vec\phi\big)$ *be two data such that* K_{M^0} *(resp.* $K_{\dot M^0}$*) is the maximal compact subgroup of* $M^0_{[y]}$ *(resp.* $\dot M^0_{[\dot y]}$*). Let* $\mathcal{T} := (K, \rho)$ *and* $\dot{\mathcal{T}} := (\dot K, \dot\rho)$*. Let* $\phi = \prod_{i=0}^d (\phi_i | M^0)$ *and* $\dot\phi = \prod_{i=0}^{\dot d} (\dot\phi_i | \dot M^0)$ *be characters of* M^0 *and* $\dot M^0$ *respectively. Then,* $\mathcal{T}$ *and* $\dot{\mathcal{T}}$ *are equivalent if and only if there is* $g \in G$ *such that* ${}^gK_{M^0} = K_{\dot M^0}$ *and* ${}^g(\rho_{M^0} \otimes \phi) \simeq (\dot\rho_{\dot M^0} \otimes \dot\phi)$ *as* $K_{\dot M^0}$ *representations.*

Proof. Let M (resp. $\dot{M}$) be the centralizer of $\mathcal{Z}_s(M^0)$ (resp. $\mathcal{Z}_s(\dot{M}^0)$) in G as in §2.4. Let $\mathfrak{s}$ (resp. $\dot{\mathfrak{s}}$) be the inertial class of (M, π_M) (resp. $(\dot{M}, \dot{\pi}_{\dot{M}})$) where π_M (resp. $\dot{\pi}_{\dot{M}}$) is the supercuspidal representation such that $\mathcal{T}_M < \pi_M$ (resp. $\dot{\mathcal{T}}_{\dot{M}} < \dot{\pi}_{\dot{M}}$). Then, we have $\mathcal{R}_{\mathcal{T}} = \mathcal{R}^{\mathfrak{s}}$ and $\mathcal{R}_{\dot{\mathcal{T}}} = \mathcal{R}^{\dot{\mathfrak{s}}}$.

Suppose $\mathcal{R}_{\mathcal{T}} = \mathcal{R}_{\dot{\mathcal{T}}}$, thus $\mathcal{R}^{\mathfrak{s}} = \mathcal{R}^{\dot{\mathfrak{s}}}$. Then, there is an unramified character χ of M^0 so that ${}^gM = \dot{M}$ and ${}^g\pi_M \simeq \dot{\pi}_{\dot{M}} \otimes \chi$. By Theorem 10.2, there is $\dot{m} \in \dot{M}$ so that ${}^{\dot{m}g}(\rho_{M^0} \otimes \phi) \simeq \rho_{\dot{M}^0} \otimes (\dot{\phi}\chi) = \rho_{\dot{M}^0} \otimes \dot{\phi}$. Since χ is trivial on $\dot{K}$, we have ${}^{\dot{m}g}(\rho_{M^0} \otimes \phi) \simeq = \rho_{\dot{M}^0} \otimes \dot{\phi}$ as representations of $\dot{K}$.

Conversely, suppose there is $g \in G$ such that ${}^gK_{M^0} = K_{\dot{M}^0}$ and ${}^g(\rho_{M^0} \otimes \phi) \simeq (\dot{\rho}_{\dot{M}^0} \otimes \dot{\phi})$. By Theorem 10.2, $\mathcal{R}_{{}^g\mathcal{T}_M} = \mathcal{R}_{\dot{\mathcal{T}}_{\dot{M}}} \subset \mathcal{R}(M)$. Since ${}^g\mathcal{T}$ and $\dot{\mathcal{T}}$ are covers of ${}^g\mathcal{T}_M$ and $\dot{\mathcal{T}}_{\dot{M}}$ respectively, we have $\mathcal{R}_{\mathcal{T}} = \mathcal{R}_{{}^g\mathcal{T}} = \mathcal{R}_{\dot{\mathcal{T}}}$. ∎

Remark In [Ka], Kaletha studied the equivalence of regular representations. His methodology, especially in §3.5, may allow the replacement of the hypothesis $C(\vec{G})$ by a weaker one.

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