STATISTICAL APPROACH FOR DETERMINING THE EXTENT OF AN OIL SPILL OVER A ROUGH SURFACE

by

Jean-Pierre Patureau

Engineer, E.N.S.M.A. (France)

(1971)

Submitted in Partial Fulfillment

of the Requirements for the

Degree of

Master of Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

July, 1972

Signature of Author ....
Department of Mechanical Engineering

Certified by ....
Thesis Supervisor

Accepted by ....
Chairman, Departmental Committee on Graduate Students

STATISTICAL APPROACH FOR DETERMINING THE EXTENT OF AN OIL SPILL OVER A ROUGH SURFACE

by

Jean-Pierre Patrueau

Submitted to the Department of Mechanical Engineering on August 25, 1972, in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The final size of an oil pool when a volume V of oil is released over a rough ice surface has been investigated in a statistical way. The permafrost relief where the spill might occur has been characterized by a parametered family of power density spectra along with the variance of the roughness. A simple concept of "most probable pocket" has been developed, directly connected to the peak of the power spectra. And finally, the two first statistical moments of the oil pool have been related to this "most probable pocket" through very simple relationships, both when the oil spreading phenomenon is one and two-dimensional.

Thesis Supervisor: David P. Hoult

Title: Associate Professor of Mechanical Engineering

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I wish to express my appreciation for his continual interest, guidance and encouragement to Professor David P. Hoult. Also, my sincere thanks are extended to Miss Eileen Gough and Mrs. Sandra Williams for typing the first drafts and final copy of this thesis.

NOTATIONS

$D(a,b;\tau)$	Joint frequency distribution of the values of h at two different points separated by the interval τ .
f	Linear frequency.
f_{no}	Frequency position of the peak in the one-dimensional power spectra.
h	Denotes the variations in altitude of the ice surface; often refereed to as "roughness". $h(x)$ in the one-dimensional case, $h(x,y)$ in the two-dimensional one, such that $\langle h \rangle = 0$.
h^+	Positive part of h } such that $h = h^+ + h^-$
h^-	Negative part of h }
k	Modelling parameter of the power density spectra.
Lmp	"Most probable pocket" length one-dimensional case.
P	Power density spectrum. $P(f)$ one-dimensional $P(\underline{U})$ two-dimensional
$p(\alpha,\theta;x)$	Probability density for h and $\frac{dh}{dx}$ to have the values α and θ at point x .
$\langle Q \rangle$	Ensemble average of the stochastic variable Q .
$\bar{Q}$	Space average of Q along a profilometer trace.
S_{Max}	Final size of the pool (area) in the two-dimensional case.
Smp	Area of the "Most probable pocket": two-dimensional case.

T	Modelling parameter of the power spectra (is equal to Lmp) .
$\underline{U} = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix}$	Frequency vector (components u_1 , u_2) .
$\underline{U}^1_{no} = \begin{vmatrix} u_{no\,1} \\ u_{no\,2} \end{vmatrix}$ $\underline{U}^2_{no} = \begin{vmatrix} u_{no\,1} \\ -u_{no\,2} \end{vmatrix}$	Frequency positions of the peaks in the two-dimensional power density spectra.
V	Total volume of oil released on the ice (one or two-dimensional).
Vmp	Volume of a two-dimensional "most probable pocket".
x_{Max}	Final size of the pool (length) in the one-dimensional case.
$\underline{X} = \begin{vmatrix} x \\ y \end{vmatrix}$	Position vector (components x,y) .
y_{Max}	One of the dimensions of the rectangle approximately limiting the two-dimensional oil pool. Such that $x_{Max}\, y_{Max} = S_{Max}$.
$\delta(h)$	Frequency distribution function of h; assumed to be Gaussian.
λ_x	x-wise dimension of an average two-dimensional pocket.
λ_y	y-wise dimension of an average two-dimensional pocket.
μ	Dissymmetry factor $\mu = \lambda x/\lambda y$.
ω	Circular frequency $= 2\pi f$.

$\psi(\tau)$	Auto-correlation function of h
	one-dimensional
$\psi^{-}(\tau)$	Auto-correlation function of h^{-}
$\Psi(\xi,\eta)$	Two-dimensional auto-correlation of h .
$\Psi^{-}(\xi,\eta)$	Two-dimensional auto-correlation function of h^{-} .
σ_h	Variance of $h = \langle h^2 \rangle$.
σ_s	Variance of S_{Max} .
σ_x	Variance of x_{Max} .
Σmp	"Volume"of a one-dimensional "most probable pocket".
θ	Slope in the $\frac{1}{2}$ power law relationship between the non-dimensional $\langle x_{Max} \rangle$ and σ_x .
Θ	Slope in the $\frac{1}{2}$ power law relationship between the non-dimensional $\langle S_{Max} \rangle$ and σ_s .

I - Introduction

As one seriously thinks of using the crude oil discovered in the Arctic Slope, no matter which means of transportation will be adopted, it appears that some understanding of how oil would spread on the ice is of major importance.

This problem cannot be dealt with as a classical oil spill on a plane surface because the characteristic feature of an Arctic Ocean ice surface is its roughness. Moreover, field observations show that such a surface can be regarded as formed of "pockets" of different sizes, some of them being interconnected. One can easily see that the oil will stop spreading abruptly after having filled a certain number of pockets. The goal of this work is to determine the final size of the pool as a function of the amount of oil released.

It is clear that the geometry of the "pockets" previously introduced is entirely related to the local variations in the ice surface altitude; it can therefore be determined by functions $h(x,y)$ defined in an arbitrary plane coordinate system xy as the elevation of the ice surface above a reference level; most often in the rest of this work, these functions will be referred to as roughness height. A simple way to record information about these variations in the elevation is to take airborne laser profilometer traces†. Such data

† A profiler operating from a plane records the phase delay between transmitted to the ground and reflected laser beam light, thus measuring the altitude variations along straight lines and yielding a random output signal.

were obtained by the U.S. Navy in the Beaufort and Chuckchi Seas. There is no doubt, when examining any of these traces, that the major character of the ice roughness is its randomness (Ref. [3]). Releasing the same amount of oil V at different locations on a randomly rough ice surface would result in different values for the measurement of the oil pool final size. Clearly, we must then formulate our problem in a statistical way, namely: the extent to which oil would spread has to be determined in terms of a certain number of statistical parameters characterizing the ice surface where the spill occurs. An analytical treatment of the profilometer traces can provide us with $\bar{h}$ †, the mean value of the roughness height h, σ_h, the variance of h, equal to $\overline{(h-\bar{h})^2}$, and the auto-correlation function of h. We will see in the course of this work that these quantities are sufficient to give an answer to our problem, provided certain simplifying assumptions are made.

With no more field data than those just introduced, it is unlikely, however, that we should end up with an elaborate and complete description of the oil spreading phenomenon. This paper will only aim at determining the first two statistical moments of the oil pool final size - i.e., its average value and its variance - both in cases when the spreading follows a privileged direction (so called "one-dimensional") and when it doesn't (two-dimensional).

---

† The bar denotes an average over a sample.

Section II will introduce and discuss the basic assumptions necessary to set up a simplified model of the phenomenon, whereas Section III will give a general mathematical expression for these quantities in terms of the field data available, namely: $\bar{h}$, σ_h and the auto-correlation of h.

In Section IV, we will be concerned with applying the previous results to certain ice surfaces where field data is available, thereby involving the modelling of the auto-correlation function of h or, more exactly of its cosine Fourier transform. Finally, in those specific cases examined in Section IV, a simple power law will be derived, relating the first two statistical moments of the final size of the oil pool in a dimensionless form.

II - Basic assumptions

If the ice surface where the spill occurs is being considered as randomly rough, it is clear that we cannot have a complete definition of the function h relevant to that specific spill, which is what would be necessary to yield a single, definite value for the oil pool size in that case.

A way of overcoming this difficulty is to retain a certain number of statistical properties of h met by different possible "realizations"† of the ice surface. These properties are then regarded as "ensemble averaged" over all those realizations and generally consist of a certain number of locally defined statistical moments and correlation functions of h . These quantities provide a certain amount of information common to any ice surface realization that belongs to the ensemble. Obviously, the smaller this number, the more random is the surface and, when the number of such properties goes to infinity, the ice surface tends to be perfectly defined and therefore no longer random. In other words, since we are talking about ensemble averages, all the realizations are becoming identical. The idea is thus to have locally defined functions that give statistical information on a whole ensemble of possible realizations of the ice surface.

In the present case, the only statistical data we have is $\overline{h}$,

---

† A "realization" of the ice surface verifying certain statistical conditions is an actual, observed surface such that it meets the previous conditions.

$\sigma_h = \overline{(h-\bar{h})^2}$ and $\overline{h(P)h(P')}$ where the bar denotes an average over a specific "realization" of the ice surface and where P and P' denote any two different points of that surface. Since these data are "space averaged" and therefore not locally defined, we are lead to make an ergodic hypothesis, assimilating the $\overline{\cdot\cdot}$ and $\langle\cdot\rangle$† averages. Our random surface is now characterized by $\langle h\rangle$, $\sigma_h = \langle[h-\langle h\rangle]^2\rangle$ and $\langle h(P)h(P')\rangle$ but clearly, this is not sufficient since nothing is said about the aspect of the surface.

There appears our second assumption, related to the statistics of the roughness height h: we will suppose h is normally distributed. If $\delta(h)$ denotes this distribution, we have immediately:

$$\delta(h) = \frac{1}{\sqrt{2\pi\sigma_h}} \exp\left\{-\frac{(h-\langle h\rangle)^2}{2\sigma_h}\right\} \tag{1}$$

$\delta(h)dh$ represents the fraction of time that h will be found between h and $h+dh$. It does not seem possible to prove this assumption by any physical argument since the natural processes that make the ice grow are extremely intricate and not thoroughly known. Therefore, we have to verify in each case that this is not too far from the observed distribution; this was done as an example on Fig. 3 where the Gaussian curve $\int_{-\infty}^{h} \delta(\tau)\,d\tau$ is plotted against h and is compared to corresponding data points measured on a real surface. It appears that the Gaussian approximation is in relatively good agreement with observations.

The third assumption concerns the way the oil would fill the "pockets" of which the ice surface is formed and more precisely, up to what level. Since the ice relief is random, the answer to this question is unlikely to be unique. Most probably, a large number of different levels would appear, corresponding to more or less "open" or "closed" pockets. However, to keep the problem feasible, we have to account only for an average uniform level of the oil pool, which is another way to say that all the "pockets" are interconnected. Moreover, it seems a reasonable assumption - at least most simplifying - to say that this uniform level is the same as that of $\langle h \rangle$. No observations exist so far to confirm such a point and the reason why we chose this level is that any other one would have been even more arbitrary and would have lead to more intricate mathematical derivations.

The fourth hypothesis reads as follows: we will always assume throughout this study that the number of "pockets" involved in an oil slick is very large; in other words, the ratio of the volume V released to the mean volume of a pocket is large; this will allow us to give a statistical answer for the final size of the oil pool. This assumption may be considered as valid in so far V is generally large (10,000 to 50,000 m^3 for the crash of a supertanker) and also the ice roughness is such that a big pocket contains many small ones and so on; therefore, even if the slick occurs in a big one, the numbers of pockets involved will always be large.

These four assumptions obviously cannot lead to a sophisticated model since they mainly reflect our lack of information about ice topography in the Arctic Slope. Moreover, the general mathematical results obtained from this model won't be the unique approximate solution to the oil spreading problem since, for instance, another frequency distribution for h could have been selected - log-normal particularly - , or another height of the oil pool level could have been chosen. Nevertheless, the simple model we can thus build provides us with a first understanding of how oil spreads over a statistical surface.

III - Mathematical analysis

The first four subsections will be concerned with a two-dimensional study whereas subsection III-5 will briefly state the results for a one-dimensional case.

III - 1. Problem

We suppose no privileged direction in the ice surface, at least for the scale of the roughness we are interested in.† Therefore, the ice topography is determined by a two-dimensional function $h(x,y)$ describing the variations of the surface altitude with respect to a reference plane. The origin of the coordinates x and y is taken at the point of oil release and the reference level is chosen so that:

$$\langle h(x,y) \rangle = 0 \tag{2}$$

There again, $\langle h(x,y) \rangle$ denotes the ensemble average of $h(x,y)$ where the averaging ensemble consists of the many ice surface realizations that can be observed in the region one considers. From our hypothesis, $\langle h(x,y) \rangle$ does not depend on x and y and therefore has to be identified to the space average $\overline{h(x,y)}$ of $h(x,y)$ over the realizations of the ice surface themselves.

By the same token, $\sigma_h = \langle h^2(x,y) \rangle$ is the variance of $h(x,y)$ calculated over the whole ensemble. Hence, the frequency distribution of $h(x,y)$, assumed to be Gaussian, is independent of any direction

† It has been shown that big pressure ridges do appear with certain definite orientations.

and is given by Eq.(1).

A proper investigation of the actual ice topography can[†] provide us with a two-dimensional auto-correlation function defined as:

$$\Psi(\xi,\eta) = \langle h(x+\xi, y+\eta)\, h(x,y) \rangle \tag{3}$$

We can thus pose the two-dimensional problem in the following terms: in a region of the Arctic Ocean where we know $\sigma_h = \langle h^2(x,y) \rangle$ and $\Psi(\xi,\eta)$, a volume V of oil is released at point $x=y=0$. What will be the total area covered with oil and also, what will be the final geometrical aspect of the pool?

The general mathematical answer will consist of the first two statistical moments of the oil pool size in terms of V, σ_h and $\Psi(\xi,\eta)$, along with a dissymmetry factor expressing the most probable ratio of the longest dimension of the pool to the smallest one.

---

† By building a two-dimensional grid of laser profilometer traces.

III - 2. Dissymmetry of the final oil pool.

As we do not allow any dissymmetry in the frequency distribution of $h(x,y)$, the final shape of the slick will be influenced by the aspect of the "pockets" themselves in the plane xy . In general, the ice surface is not isotropically rough. It follows that if λ_x and λ_y are the ensemble averages of a pocket dimensions in the x and y directions respectively, the ratio μ of λ_x to λ_y can be called "dissymmetry factor" since it provides a rough idea about the final pool aspect.

From Appendix A, it appears that:

$$\mu = \left\{ \frac{\partial^2 \Psi}{\partial \eta^2}(0,0) \Big/ \frac{\partial^2 \Psi}{\partial \xi^2}(0,0) \right\}^{1/2} \tag{4}$$

Now, let us imagine how oil would spread from point of release. Since the volume V is supposed to be very large, the shape of the pockets is not important at the beginning and the slick would propagate circularly. But, after this initial phase, one can assume that the geometrical aspect of the pockets would influence the final shape of the oil slick.

Considering a laser profilometer trace that passes through the origin of the coordinates x and y , it is clear from our ergodic hypothesis that the statistics of the roughness height remain the same all along the trace. It follows that the origin of the coordinates is a geometric center of symmetry for the pool shape. Moreover, in order to simplify further calculations, we assume that the resulting oil pool is rectangular, x_{Max} and y_{Max} being the x-direction and y-direction dimensions respectively.

Therefore, it follows that:

$$\frac{x_{Max}}{y_{Max}} = \mu \neq \text{ unity, in general.}$$

Obviously, when the ice surface is truly isotropic, μ is unity and the final pool is a square in our assumption.

III - 3. First Two Statistical Moments of the Oil Pool Final Size.

As was suggested at the end of the last sub-section, the oil pool is likely to be symmetric with respect to the point of release i.e., the origin of the coordinates. Hence, defining $h^+(x,y)$ and $h^-(x,y)$ such that $h=h^+ + h^-$ and:

$$\begin{aligned} h^-(x,y) &= 0 \quad \text{when} \quad h(x,y) \geqslant 0 \\ h^-(x,y) &= h(x,y) \quad \text{when} \quad h(x,y) < 0 \end{aligned} \tag{5}$$

we can introduce $v(x,y)$ as the volume held by the ice surface - under the third assumption of Section II - within a rectangle of dimensions x and y. Then:

$$v(x,y) = -\int_{-x/2}^{+x/2}\int_{-y/2}^{+y/2} h^-(x,y)\,dx\,dy \tag{6}$$

This is a mathematical definition and, evidently, it does not account for the fact that the oil-air interface must be horizontal when edge pockets are not completely filled up to the average uniform level; this fact is illustrated on the sketch Fig. 1 for the one-dimensional spreading. It follows that the mathematical extent of the oil is only slightly different from the real one: at most within a pocket. Since we made the assumption of a large number of pockets, we can define x_{Max} and y_{Max} as the x-direction and y-direction maximum extents of the final pool when a volume V of oil is being released over a specific realization of the ice surface; namely:

$$V = -\int_{-x_{Max}/2}^{x_{Max}/2} \int_{-y_{Max}/2}^{y_{Max}/2} h^{-}(x,y)\, dx dy \tag{7}$$

Or, introducing the area S_{Max} defined as:

$$S_{Max} = x_{Max}\, y_{Max} \tag{8}$$

Relation (7) becomes:

$$V = -\int_{S_{Max}} h^{-}(x,y)\, dx\, dy \tag{9}$$

As the function h^{-} is everywhere continuous, we can also write relation (9) as:

$$V = -\int_{\langle S_{Max}\rangle} h^{-}(x,y)\, dx\, dy - \left[S_{Max} - \langle S_{Max}\rangle\right] h^{-}(\mu,\rho) \tag{10}$$

where point (μ,ρ) belongs to the surface $(S_{Max} - \langle S_{Max}\rangle)$; $h^{-}(\mu,\rho)$ is exactly the average of h^{-} taken over the surface $(S_{Max} - \langle S_{Max}\rangle)$.

We now assume that:

$$h^{-}(\mu,\rho) = \langle h^{-}(\xi,\eta)\rangle \tag{11}$$

This cannot be justified in general. Therefore, Eq.(11) is more valid as this surface becomes large since, when the averaging sample gets large enough, the average of h^{-} approaches $\overline{h^{-}(\xi,\eta)}$, hence $\langle h^{-}(\xi,\eta)\rangle$

from our ergodic hypothesis. An "a posteriori" verification will thus be possible.

Then, taking the ensemble average of (10) and making use of the previous approximation, we obtain the first statistical moment of the pool size, namely:

$$\langle S_{Max} \rangle = - \frac{V}{\langle h^{-}(\xi,\eta) \rangle} \tag{12}$$

To determine the second moment, let us first notice that σ_S, the variance of S_{Max} and σ_v the variance of $v(\langle x_{Max}\rangle, \langle y_{Max}\rangle)$ are related. By definition:

$$\sigma_S = \langle (S_{Max} - \langle S_{Max} \rangle)^2 \rangle \tag{13}$$

$$\sigma_v = \langle \left[v(\langle x_{Max}\rangle, \langle y_{Max}\rangle) - \langle v(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle \right]^2 \rangle \tag{14}$$

From relations (10) and (11), it comes out that:

$$\langle \left[\int_{\langle S_{Max}\rangle} h^{-}(x,y) - V \right]^2 \rangle = \langle (S_{Max} - \langle S_{Max}\rangle)^2 \rangle \langle h^{-}(\xi,\eta) \rangle^2 \tag{15}$$

Or:

$$\sigma_v = \sigma_S \langle h^{-}(\xi,\eta) \rangle^2 \tag{16}$$

Therefore, computing σ_v in Appendix B, we end up with:

$$\sigma_S = \frac{16}{\langle h^-(\xi,\eta)\rangle_S^2} \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \int_0^{\langle y_{Max}\rangle/\sqrt{2}} \left[\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \xi\right]\left[\frac{\langle y_{Max}\rangle}{\sqrt{2}} - \eta\right]\left[\Psi^-(\xi,\eta) - \frac{\sigma_h}{2\pi}\right] d\xi\, d\eta \tag{17}$$

From all previous calculations, especially relations (12) and (17), it appears that knowing $\langle h^-(\xi,\eta)\rangle$ and $\Psi^-(\xi,\eta)$ enables us to derive the average value $\langle S_{Max}\rangle$ and the standard deviation $\sigma_S^{1/2}$ of the total area covered with oil, in terms of the amount V released. Furthermore, an idea about the dissymmetry in the aspect of the pool can be gotten from coefficient μ . This will constitute our answer to the two-dimensional problem posed in Sub-section III - 1., provided we can relate $\langle h^-(\xi,\eta)\rangle$ and $\Psi^-(\xi,\eta)$ to the available statistical data i.e., σ_h and $\Psi(\xi,\eta)$. This is the purpose of the next sub-section.

III - 4. Determination of $\langle h^{-}(\xi,\eta)\rangle$ and $\Psi^{-}(\xi,\eta)$

We now go into more detail about the distribution function $\delta(h)$ that was first mentioned in the Introduction. This function tells us that the probability h will lie between γ and $\gamma+d\gamma$ is $\delta(\gamma)d\gamma$ and therefore:

$$\int_{-\infty}^{+\infty} \delta(\gamma)d\gamma = 1 \tag{18}$$

Assuming, as in our second hypothesis, that this frequency distribution is Gaussian with mean value 0 implies that h will be negative as many times as it is positive. Since γ is the value taken by a stochastic function h , it is also a stochastic variable. Therefore, as we know its frequency distribution function $\delta(\gamma)$, it is clear that the average of a stochastic quantity Q uniquely related to γ is expressed by:

$$\langle Q\rangle = \int_{-\infty}^{+\infty} Q(\gamma)\,\delta(\gamma)\,d\gamma \tag{19}$$

If now we identify Q to the value of $h^{-}(x,y)$, it follows from relations (5):

$$\begin{aligned} Q &= \gamma \quad \text{when} \quad \gamma \leqslant 0 \\ Q &= 0 \quad \text{when} \quad \gamma > 0 \end{aligned} \tag{20}$$

Then, substituting into Eq.(19) for $\delta(\gamma)$ and $Q(\gamma)$:

$$\langle h^-(\xi,\eta)\rangle = \int_{-\infty}^{0} \frac{\gamma}{\sqrt{2\pi\sigma_h}} \exp\left(-\frac{\gamma^2}{2\sigma_h}\right) d\gamma \tag{21}$$

Or, performing the integration:

$$\langle h^-(\xi,\eta)\rangle = -\sqrt{\frac{\sigma_h}{2\pi}} \tag{22}$$

Now,, to determine $\Psi^-(\xi,\eta)$, we have to extend what we previously did for a one-dimensional distribution of h to a joint two-dimensional frequency distribution. We end up with the following relation (see Appendix C for details):

$$\Psi^-(\xi,\eta) = \left[\frac{(\Psi_0-\Psi)^3}{\pi(\Psi_0+\Psi)}\right]^{1/2} \cdot \left\{ \int_0^{\infty} (u^2-\tfrac{1}{2})\, erf(u) \exp\left(-u^2\frac{\Psi_0-\Psi}{\Psi_0+\Psi}\right) du + \frac{\Psi_0+\Psi}{4\sqrt{\pi}\,\Psi_0} \right\} \tag{23}$$

where we have set for convenience:

$$\begin{aligned} \Psi_0 &= \Psi(0,0) \\ \Psi &= \Psi(\xi,\eta) \end{aligned} \tag{24}$$

It appears (see relation (D7) in Appendix D) that $\Psi^-(0,0) = \frac{1}{2}\Psi(0,0)$ which is consistent with the physical conditions at $\Psi = \Psi_0$ namely:

$$\begin{aligned} \Psi(0,0) = \langle h^2\rangle = \langle (h^+ + h^-)^2\rangle &= \langle h^{+2}\rangle + \langle h^{-2}\rangle \\ + 2\langle h^+ h^-\rangle &= 2\langle h^{-2}\rangle \end{aligned} \tag{25}$$

when assuming the statistics of h^- and h^+ are the same as suggested by a normally distributed h. The product h^+h^- must always be zero as can be seen by Eq.(5).

Furthermore, when ξ or η tends to infinity, $\Psi^-(\xi,\eta)$ tends to $\frac{\sigma_h}{2\pi}$, in compliance with the fact that $h^-(x,y)$ and $h^-(x+\xi,y+\eta)$ are then no longer correlated. Therefore, according to relation (22):

$$\begin{aligned} \langle h^-(x,y)h^-(x+\xi,y+\eta)\rangle &= \langle h^-(x,y)\rangle\langle h^-(x+\xi,y+\eta)\rangle = \langle h^-(\xi,\eta)\rangle^2 \\ &= \sigma_h/2\pi \end{aligned} \tag{26}$$

We have therefore provided a means of deriving the statistical quantities we need in Sub-section III - 3 i.e., $\Psi^-(\xi,\eta)$ and $\langle h^-(\xi,\eta)\rangle$, from the field data generally available, namely: $\sigma_h = \overline{(h-\bar{h})^2}$, the variance of the roughness height and $\Psi(\xi,\eta)$, its auto-correlation function.

III - 5. One-Dimensional Case.

We can imagine that in a particular region, the statistics of the ice roughness are almost one-dimensional; that is to say, the averaging ensemble consists here of traces parallel to this privileged direction we denote by x . Therefore, the ensemble $\langle h \rangle$ and auto-correlation function are only functions of x . And, if $h(x)$ denotes the elevation (positive or negative) above the mean value of h taken along the trace (s-e Fig. 1), we have from our ergodic hypothesis:

$$\begin{aligned} \langle h(x) \rangle &= \overline{h}(x) = 0 \\ \langle h^2(x) \rangle &= \overline{h^2(x)} = \sigma_h \quad \text{variance of } h \end{aligned} \tag{27}$$

and, $\psi(\tau)$ being the auto-correlation function of $h(x)$:

$$\psi(\tau) = \langle h(x)\, h(x+\tau) \rangle = \overline{h(x)\, h(x+\tau)} \tag{28}$$

From the previous considerations, it follows that any actual surface profile parallel to x will be described by the statistical properties (27) and (28).

Now, suppose an oil spill occurs in such a zone, a volume V being released per unit width perpendicular to x ; if the spill is wide enough, we can regard the spreading of oil as one-dimensional in the direction of x . This is not completely unrealistic since narrow water leads bounded by large floes are fairly frequent in the Arctic Slope. Under these conditions, a large oil spill would first spread rapidly on water, thus creating this wide spill (V per unit width) that would further spread over ice almost in a single direction perpendicular to the lead.

We can then put our problem this way: a volume of oil V per unit direction perpendicular to x is released at point $x=0$. What will be the maximum abscissa x_{Max} reached by the oil when it has come to a standstill?

We will just state the one-dimensional results since their derivation is quite similar to that of the two-dimensional study in the previous sub-sections. A more detailed analysis is given in Appendix E. The first statistical moment $\langle x_{Max}\rangle$ of the oil pool final size is found to be:

$$\langle x_{Max}\rangle = -\frac{V}{\langle h^-(\xi)\rangle} \tag{29}$$

whereas the second moment σ_x defined as:

$$\sigma_x = \left\langle \left[x_{Max} - \langle x_{Max}\rangle \right]^2 \right\rangle \tag{30}$$

appears to be:

$$\sigma_x = \frac{4}{\langle h^-\rangle^2} \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \tau\right)\left(\psi^-(\tau) - \frac{\sigma_h}{2\pi}\right) d\tau \tag{31}$$

Here again, $\psi^-(\tau)$ and $\langle h^-\rangle$ are related to $\psi(\tau)$ and σ_h through the relations:

$$\langle h^- \rangle = -\sqrt{\frac{\sigma_h}{2\pi}} \tag{32}$$

$$\psi^-(\tau) = \sqrt{\frac{(\psi_0-\psi)^3}{\pi(\psi_0+\psi)}} \left\{ \int_0^\infty \left(u^2-\frac{1}{2}\right) \mathrm{erf}(u) \exp\left(-u^2\frac{\psi_0-\psi}{\psi_0+\psi}\right) du + \frac{\psi_0+\psi}{4\sqrt{\pi}\,\psi_0} \right\} \tag{33}$$

where

$$\psi_0 = \psi(0)$$
$$\psi = \psi(\tau)$$

III - 6. General Mathematical Results

In this section, we have been able to give an analytical expression for the first two statistical moments of the oil slick final size, resulting from a spill of volume V . Thus, when the oil spreading is either one-dimensional (with a privileged direction) or two-dimensional (no privileged direction), the assumptions made in Section II enabled us to derive the ensemble average value of the pool size and its standard deviation, as well as a "dissymmetry factor" μ characterizing the aspect of the slick (in the two-dimensional case only).

These quantities are all determined in terms of σ_h , the variance of the roughness height h , and the auto-correlation function of h (either $\varphi(\tau)$ or $\Psi(\xi,\eta)$).

Namely:

$$\left.\begin{aligned} \langle x_{Max} \rangle &= \frac{V\sqrt{2\pi}}{\sqrt{\sigma_h}} \\ \sigma_x = \langle [x_{Max} - \langle x_{Max} \rangle]^2 \rangle &= \frac{8\pi}{\sigma_h} \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \left[\frac{\langle x_{Max} \rangle}{\sqrt{2}} - \tau \right] \left[\varphi^-(\tau) - \frac{\sigma_h}{2\pi} \right] d\tau \\ &\text{where } \varphi^-(\tau) \text{ is obtained by equation (C15)} \end{aligned}\right\} \quad \begin{array}{l} \text{1-dim.} \\ \text{case} \\ (34) \end{array}$$

$$
\left.
\begin{aligned}
\langle S_{Max} \rangle &= \frac{V\sqrt{2\pi}}{\sqrt{\sigma_h}} \\
\sigma_S = \langle [S_{Max} - \langle S_{Max}\rangle]^2 \rangle &= \frac{32\pi}{\sigma_h} \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \int_0^{\langle y_{Max}\rangle/\sqrt{2}} \left[\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \xi\right] \cdot \\
&\qquad \left[\frac{\langle y_{Max}\rangle}{\sqrt{2}} - \eta\right]\left[\Psi^{-}(\xi,\eta) - \frac{\sigma_h}{2\pi}\right] d\xi\, d\eta \\
\mu &= \frac{\partial^2 \Psi}{\partial \eta^2}(0,0) \Big/ \frac{\partial^2 \Psi}{\partial \xi^2}(0,0)
\end{aligned}
\right\}
\quad \text{2-dim. case (35)}
$$

where Ψ^{-} is obtained by Eq. (C18) .

However, we have to point out again that these results do not constitute the unique answer to the general problem of oil spreading over an ice surface since the basic assumptions made in Sub-station III-2 are quite limiting. Nevertheless, they can provide us with a first understanding of the possible effects of an oil slick in an Arctic environment.

IV - Application - Sample Calculations

In this section, the general mathematical results of the last section will be applied to a certain number of specific cases of ice topography where the corresponding statistical field data is available.

Both in the one and two-dimensional cases (for privileged and no privileged direction of spreading respectively), the statistics, i.e., the first and second moments of the final size of the oil pool, have been determined in terms of σ_h , the variance of the roughness and ψ , its auto-correlation function. We recall here that h stands for the variations in altitude of the ice surface with mean value 0 .

$$\sigma_h = \langle h^2 \rangle \tag{36}$$

$$\psi(\tau) = \langle h(x)\, h(x+\tau) \rangle \quad \text{in the 1-dim. case} \tag{37}$$

$$\Psi(\xi,\eta) = \langle h(x,y)\, h(x+\xi,y+\eta) \rangle \quad \text{in the 2 dim case} \tag{38}$$

Then, within a certain region of the Arctic Slope that can generally be regarded as large compared to the area of an eventual oil slick, the ice topography obeys the statistics expressed either by Eqs. (36) and (37) or by Eqs. (36) and (38) when the oil spreading phenomenon is one and two-dimensional respectively.

Now, in order to give a numerical answer to a specific oil slick problem, we have to know σ_h and either ψ or Ψ characterizing that part of the Arctic Slope where the spill occurs, and that are the results of field observations. However, the statistical properties of the permafrost relief are generally given in the form of a power density spectrum of the ice surface (Ref. [3]). We must therefore use that kind of field data instead of the auto-correlation function itself.

From the Wiener theorem in generalized harmonic analysis, it appears that the auto-correlation function and the power density spectrum are related by a Fourier cosine transformation. Thus, if P denotes the power density spectrum of h, it follows that:

$$\left.\begin{aligned} \psi(\tau) &= \int_{-\infty}^{+\infty} P(f)\cos 2\pi f\tau \, df \\ P(f) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi(\tau)\cos 2\pi f\tau \, d\tau \\ \psi(0) &= \sigma_h = \int_{-\infty}^{+\infty} P(f)\,df \end{aligned}\right\} \quad (39)$$

in the one-dimensional case

And:

$$\left.\begin{aligned} \Psi(\underline{X}) &= \int_{-\infty}^{+\infty} P(\underline{U})\cos(2\pi\underline{U}\cdot\underline{X})\, d\underline{U} \\ P(\underline{U}) &= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \Psi(\underline{X})\cos(2\pi\underline{U}\cdot\underline{X})\, d\underline{X} \\ \Psi(\underline{0}) &= \sigma_h = \int_{-\infty}^{+\infty} P(\underline{U})\, d\underline{U} \end{aligned}\right\} \quad (40)$$

in the two-dimensional case

where $\underline{U}$ is a frequency vector of components u_1 and u_2 and $\underline{X}$ any vector of the plane xy of components ξ and η .

Therefore, it is clear that the information we need about the ice surface statistics, i.e., σ_h and the auto-correlation function of h , is contained in the power density spectrum P . The problem is now to build a continuous function P fitting the observed power spectrum as well as possible. This is the object of the next sub-section.

IV - 1. Modelling of the Power Spectra

In order to make the mathematical analysis of Section II applicable to any part of the permafrost in the Arctic Ocean, we have to determine the parameters necessary to describe the most important features of the ice surface statistics. A general answer to this modelling problem cannot be given here, even by increasing the number of such parameters since the actual data is far from being complete and reliable. Moreover, another problem is raised by the two-dimensional case of oil spreading since no corresponding field data - two-dimensional functions $\Psi(\xi,\eta)$ - is available to date. Therefore, in order to obtain numerical orders of magnitude, we have assumed that a two-dimensional auto-correlation function $\Psi(\xi,\eta)$ can be regarded as the product of two one-dimensional auto-correlation functions. Namely:

$$\Psi(\xi,\eta) = \sigma_h \, \psi_1(\xi) \, \psi_2(\eta) \tag{41}$$

There is no evidence at all that this is correct and it only appears as a fair and indispensable assumption to treat the two-dimensional case. Therefore, since the latter can be derived from one-dimensional field data, this sub-section will only be concerned with the modelling of the one-dimensional power spectra that directly come out by Fourier analyzing laser profilometer traces such as those mentioned in the Introduction. The two-dimensional model will only be stated following the above assumption.

There is no doubt that modelling $P(f)$ or $\psi(\tau)$ is equivalent since the power density spectrum and the auto-correlation function are uniquely related through a Fourier transformation. In fact, from the few data examined - both ψ and P - , it turns out that the shape of the spectra is quite varying thereby making the choice of parameters difficult, whereas the form of $\psi(\tau)$ seems to remain pretty much the same; it is, therefore, easier and perhaps more reliable to propose a model for $\psi(\tau)$ and then take its Fourier transform. Since this function presents observed damped oscillations around a zero value and starts at $\psi(0) = \sigma_h$ (by definition of ψ), we think that two parameters k and T are sufficient to set up a consistent model. Namely:

$$\psi(\tau) = \frac{\sigma_h}{1 + k\tau^2} \cos 2\pi \frac{\tau}{T} \tag{42}$$

where k is the inverse of a length squared and T has the dimension of a length.

However, as we mentioned at the beginning of this section, this is not the form under which field measurements generally appear. Thus, we have to derive a model for the power spectrum $P(f)$. This can be done by taking the inverse Fourier transform of the model for ψ and evaluating k and T from the power spectra. It follows that:

$$\left.\begin{aligned} P(f) &= \frac{\pi\sigma_h}{\sqrt{k}} \exp\left(-\frac{2\pi f}{\sqrt{k}}\right)\cosh\left(\frac{2\pi}{T\sqrt{k}}\right) \quad : f > 1/T \\ P(f) &= \frac{\pi\sigma_h}{\sqrt{k}} \exp\left(-\frac{2\pi}{T\sqrt{k}}\right)\cosh\left(\frac{2\pi f}{\sqrt{k}}\right) \quad : f < 1/T \\ P(f) &= P(-f) \end{aligned}\right\} \tag{43}$$

There exists a symmetry with respect to the origin and it is therefore sufficient to consider only the positive frequencies for which a peak in the power spectrum appears at $f = 1/T$. In that case, the normalization of the spectrum is such that:

$$\int_0^\infty P(f)\,df = \sigma_h \tag{44}$$

which is equivalent to multiplying $P(f)$ by a factor two in relation (43).

This relation constitutes the two-parameters model[†] we propose for the one-dimensional power spectra. When considering specific ice surface statistics, the parameters k and T can be determined by having the model fit exactly the observed intensity and frequency position of the peak on the power spectrum. In other words, if f_{no} is the observed position of the peak, we have:

$$T = 1/f_{no} \tag{45}$$

and k is the solution of the following transcendental equation:

---

† σ_h is not a parameter in that model since $\int_0^\infty P(f)\,df = \sigma_h$ for any k's and T's .

$$P(f_{no}) = \frac{2\pi\sigma_h}{\sqrt{k}} \exp\left(-\frac{2\pi}{T\sqrt{k}}\right) \cosh\left(\frac{2\pi}{T\sqrt{k}}\right) \tag{46}$$

Figs. 4 to 6 show one-dimensional spectra modelled in this way. Each figure is relevant to different ice surface statistics i.e., different k's and T's; the solid line shows the proposed model whereas the black squares represent the corresponding field measured spectrum. It appears that the decaying of the spectrum is acceptably verified but, unfortunately, the lower frequency region, beyond the peak is not very well matched. As the peaks of the power density spectra play a central role in our analysis, a point pertaining to their exact definition has to be raised here.

As a matter of fact, the behavior of the power spectra for very low frequencies (where the peaks lie most often) remain unclear. These data generally come out by Fourier analyzing field measurements of the ice surface variations in altitude. We must notice that the recording device does not have an infinite analyzing band-width and that in order to remove the variations in altitude of the plane which carries the recorder the data signal is filtered through a high-band pass filter. Therefore, we cannot rely on the information at low frequencies provided by such methods, and it would be necessary to set up a physical argument to account for the general aspect of the power spectra in the low frequencies. This requires a thorough understanding of the ice formation and life and has not been done to date. Our modelling is therefore restricted to such field measured power spectra that exhibit a well defined single predominant peak, almost regardless of their behavior for frequencies less than the peak frequency.

Let us not state the resulting model for a two-dimensional statistics. Following the assumption we made earlier, we write:

$$\Psi(\xi,\eta) = \sigma_h \, \psi_1(\xi)\,\psi_2(\eta) \qquad (47)$$

where ψ_1 and ψ_2 are one-dimensional auto-correlation functions, hence modelled by relation (42) with parameters k_1, T_1 and k_2, T_2 respectively. It follows that the corresponding two-dimensional power spectrum is $P(\underline{U})$ such that:

$$P(\underline{U}) = \sigma_h \, P_1(u_1)\, P_2(u_2)$$

where u_1 and u_2 are the components of the frequency vector $\underline{U}$ and P_1 and P_2 are the two one-dimensional models corresponding to the parameters k_1, T_1 and k_2, T_2 respectively. Namely:

$$\left.\begin{aligned} P_1(u_1) &= \frac{\pi}{\sqrt{k_1}} \exp\left(-\frac{2\pi u_1}{\sqrt{k_1}}\right)\cosh\left(\frac{2\pi}{T_1\sqrt{k_1}}\right) \\ P_2(u_2) &= \frac{\pi}{\sqrt{k_1}} \exp\left(-\frac{2\pi u_2}{\sqrt{k_2}}\right)\cosh\left(\frac{2\pi}{T_2\sqrt{k_2}}\right) \end{aligned} \quad \text{if } u_1, u_2 > 1/T \right\} \qquad (48)$$

$$\left.\begin{aligned} P_1(u_1) &= \frac{\pi}{\sqrt{k_1}} \exp\left(-\frac{2\pi}{T_1\sqrt{k_1}}\right)\cosh\left(\frac{2\pi u_1}{\sqrt{k_1}}\right) \\ P_2(u_2) &= \frac{\pi}{\sqrt{k_2}} \exp\left(-\frac{2\pi}{T_2\sqrt{k_2}}\right)\cosh\left(\frac{2\pi u_2}{\sqrt{k_2}}\right) \end{aligned} \quad \text{if } u_1, u_2 < 1/T \right\} \qquad (49)$$

$$P(-\underline{U}) = P(\underline{U})$$

Here again, $P(\underline{U})$ is symmetric with respect to the origin in the frequency plane. Hence, one needs only consider a half-plane (for instance $u_1 > 0$), normalizing the power density spectrum so that:

$$\int_{u_1 > 0} P(\underline{U})\, d\underline{U} = \sigma_h \tag{50}$$

which is equivalent to multiplying $P(\underline{U})$ as expressed in Eqs. (48 and 49) by a factor 2.

The overall aspect of such a power spectrum is given in Fig. 7. Two peaks appear, corresponding to the two peaks of $P_2(u_2)$. As was said earlier, no comparison with field data is possible due to the lack of the latter.

In this section, a simple two-parameters model has been proposed for the one-dimensional power spectra and is in good agreement with observations in so far as a real predominant peak appears in the measured spectra. The two-dimensional model has been purely hypothesized as being a cross product of two one-dimensional models. The next sub-section will use these models to develop a concept of "most probable pocket" in the ice relief.

IV - 2. Concept of Most Probable Pocket

In this sub-section, both in the one and two-dimensional cases, a "most probable pocket" will be defined as the "most apparent periodical relief" in the ice surface; mathematically, it will be identified with the negative part of the main term in the Fourier series expansion of the ice surface. Moreover, being the most characteristic feature of an ice surface, this most probable pocket will be used to non-dimensionalize our further results.

As soon as we have the power density spectrum $P(\omega)$ of the function h, from harmonic analysis† we know that:

$$h(x) = \int_0^{+\infty} F(\omega) \cos(\omega x)\, d\omega \tag{51}$$

with
$$2\,P(\omega)\,d\omega = \left| F(\omega)\,d\omega \right|^2$$
$$\omega = 2\pi f$$

It is then clear that we can approximate this function by an infinite Fourier cosine series which is nothing more than replacing the surface contained between the power spectrum curve and the frequency axis by an infinite number of rectangles of width $\Delta\omega_n$ and height $P(\omega_n)$ where ω_n is a circular frequency and $P(\omega_n)$ the value of the power spectrum at that frequency.

† See Ref. [4].

Thus:

$$h(x) \simeq \sum_{0}^{\infty} \sqrt{2P(\omega_n)\,\Delta\omega_n}\,\cos(\omega_n x - \varphi_n) \tag{52}$$

in the one-dimensional case

by the same token:

$$h(\underline{X}) \simeq \sum_{0}^{\infty} \sqrt{2P(\underline{U}_n)\,\Delta\underline{U}_n}\,\cos(\underline{U}_n \cdot \underline{X} - \varphi_n) \tag{53}$$

in the two-dimensional case.

In these equations, the "approximately equal" symbol tends to an "equal" symbol when $\Delta\omega_n - \Delta\underline{U}_n$ in the two-dimensional case - tends to zero, that is to say when the sum is replaced by an integral.

The amount of randomness characteristic of any given realization of the ice surface among a particular statistics appears in the set of phases φ_n that are regarded as completely random variables.

Now, in so far as the power spectrum decays fast enough when the frequency increases and decreases from its value at the peak, there is no doubt that the term corresponding to the frequency position of the peak, ω_{no} - $\underline{U}^1_{no}$, $\underline{U}^2_{no}$ † in the two-dimensional case - in the Fourier series is prevailing over the other ones.

Figs. (4 to 6) show that such a fast decaying is confirmed by field data when ω increases, but this is not the case when ω decreases since, as was already mentioned, this low frequency part of the spectra is not quite reliable. However, it seems intuitively reasonable to

† $\underline{U}^1_{no}$, $\underline{U}^2_{no}$ refer here to the two peaks of a two-dimensional spectrum.

expect that the zero frequency values of the spectra have to be zero; in other words, in the Fourier expansion of h, terms with an infinite wave length should have zero amplitude because otherwise, this would mean that there exists points infinitely far from the origin of the coordinates with non-zero roughness height. This is intuitively non-acceptable.

A rigorous argument can only come out from a detailed discussion of the ice relief formation and evolution. This is beyond the scope of this analysis and we must be content with the assumption that a real peak exists in every spectra and its frequency position and intensity explicitely appear on the field data.

Thus, it follows that an even cruder approximation for h would be:

$$h(x) \simeq \sqrt{2\sigma_h}\,\cos\left(\omega_{no}x - \varphi_{no}\right) \tag{54}$$

in the 1-dim. case.

and

$$h(\underline{X}) \cong \sqrt{2\sigma_h}\left[\cos\left(\underline{\omega}^1_{no}\cdot\underline{X} - \varphi^1_{no}\right) + \cos\left(\underline{\omega}^2_{no}\cdot\underline{X} - \varphi^2_{no}\right)\right] \tag{55}$$

in the 2-dim. case

For the one-dimensional case, we can look at Eq. (52) as expressing h in terms of a <u>superposition of sinusoids</u> of period $\frac{2\pi}{\omega_n}$ and intensity

$\sqrt{2P(\omega_n)\Delta\omega_n}$. Each sinusoid will contribute - in proportion to its amplitude - to form the actual aspect of a typical one-dimensional profilometer trace verifying the statistics expressed in the power spectrum P . Then, Eq. (54) tells us that the greatest amplitude sinusoid, and therefore that which most influences the shape of the real pockets, corresponds to the frequency of the peak. Hence the notation "most probable pocket" refers to this particular term of the Fourier expansion of h . This privileged pattern in the ice relief has a length L_{mp} equal to one period T of the sinusoid and a "volume" per unit width Σ_{mp} equal to the area contained between a negative arch of the curve and the axis, namely:

$$L_{mp} = \frac{2\pi}{\omega_{no}} = T \tag{56}$$

$$\Sigma_{mp} = -\int_{\substack{0 \\ \text{negative integrand}}}^{L_{mp}} \sqrt{2\sigma_h}\, \cos(\omega_{no}x)\, dx \tag{57}$$

Or, integrating:

$$\Sigma_{mp} = \frac{\sqrt{2\sigma_h}}{\omega_{no}} = \frac{T\sqrt{2\sigma_h}}{\pi} \tag{58}$$

By the same token, for the two-dimensional case, Eq. (53) is an expansion of the ice surface in terms of sinusoidal surfaces, the most apparent pattern corresponding to the two peaks of frequency vectors

$\underline{U}_{no}^{1} = (u_{no1}, u_{no2})$ and $\underline{U}_{no}^{2} = (u_{no1}, -u_{no2})$ as in Eq. (55), (see Fig. 7). Analogous to the one-dimensional case, the "most probable pocket" has here a surface S_{mp} and a volume V_{mp} corresponding to an increase of $(u_{no1})^{-1}$ in x and $(u_{no2})^{-1}$ in y.

Or more precisely:

$$S_{mp} = \frac{4\pi^2}{u_{no1}\, u_{no2}} \tag{59}$$

$$V_{mp} = - \int_{\substack{\text{1 period in} \\ \text{1 period in} \\ \text{negative integrand}}} \sqrt{2\sigma_h}\left[\cos \underline{U}_{no}^{1} \cdot \underline{X} + \cos \underline{U}_{no}^{2} \cdot \underline{X}\right] d\underline{X} \tag{60}$$

Upon integration, this yields:

$$V_{mp} = \frac{8\sqrt{2\sigma_h}}{u_{no1}\, u_{no2}} \tag{61}$$

Since they express the most important feature of the power density spectra - namely, their peak - , the "most probable pocket" dimensions will provide a good reference to non-dimensionalize our further results. This concept of most probable pocket has also the advantage of giving, in a simple way (through Eqs. (52) and (53), a rough estimate of what the ice surface looks like.

IV - 3. General Mathematical Results of Section III Using the Modelled Spectra of IV -1.

With Eqs. (34) in the one-dimensional case and (35) in the two-dimensional case, we have obtained a relation between the first and second statistical moments of the oil pool final size, the auto-correlation function of h^- and the parameter σ_h of the ice surface statistics. The purpose of this section is to derive more simple relations, using modelled spectra, and non-dimensionalize these results with respect to the "most probable pocket". Thus, it will be shown that in their dimensionless form, the second moment is proportional to the first one through a factor θ only dependent on the non-dimensional group kT^2 (or $k_1k_2T_1^2T_2^2$ in the two-dimensional case) for suitable values of this group.

IV - 3.1. <u>One-Dimensional Case.</u>

We recall here that if $\langle x_{Max} \rangle$ is the final dimension of the oil pool and σ_x its variance, we have, from the mathematical analysis of Section III:

$$\langle x_{Max} \rangle = \frac{V\sqrt{2\pi}}{\sqrt{\sigma_h}} \tag{62}$$

$$\sigma_x = .8\pi \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \tau\right)\left(\frac{\psi^-(\tau)}{\sigma_h} - \frac{1}{2\pi}\right) d\tau \tag{63}$$

$\psi^-(\tau)$ being obtainable from $\psi(\tau)$ defined as:

$$\psi(\tau) = \frac{\sigma_h}{1 + k\tau^2} \cos 2\pi \frac{\tau}{T} \tag{64}$$

Through the relation (see Appendix D).

$$\psi^-(\tau) = \mathcal{F}(\psi) = \frac{1}{2\pi(\psi_0^2 - \psi^2)^{1/2}} \iint_0^\infty ab \exp\left(-\frac{(a^2+b^2)\psi_0 - 2ab\psi}{2(\psi_0^2 - \psi^2)}\right) da\, db \tag{65}$$

We can notice that, for k's and T's such that $\left(\frac{\psi^-(\tau)}{\sigma_h} - \frac{1}{2\pi}\right)$ approaches zero within one period T - or one most probable pocket length L_{mp} - Eq. (63) becomes:

$$\sigma_x = 8\pi \int_0^{L_{mp}} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \tau\right)\left(\frac{\psi^-(\tau)}{\sigma_h} - \frac{1}{2\pi}\right) d\tau \tag{66}$$

Since the integrand in relation (66) is a continuous function of τ , we can define τ_o such that:

$$\sigma_x = 8\pi L_{mp} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \tau_o\right)\left(\frac{\psi^-(\tau_o)}{\sigma_h} - \frac{1}{2\pi}\right) \tag{67}$$

Now, from Appendix D, it is clear that $\left(\frac{\psi^-(\tau)}{\sigma_h} - \frac{1}{2\pi}\right)$ will be positive between $\tau = 0$ and $\tau = \frac{T}{4}$ and negative between $\tau = \frac{T}{4}$ and $\tau = \frac{T}{2}$; hence:

$$0 < \tau_o < T/4 \tag{68}$$

In so far as $\langle x_{Max}\rangle$ is large as compared to $\frac{T}{4}$, we can rewrite Eq. (67):

$$\sigma_x = \frac{8\pi}{\sqrt{2}} L_{mp}\, \alpha \,\langle x_{Max}\rangle \tag{69}$$

with $$\alpha = \frac{\psi^-(\tau_o)}{\sigma_h} - \frac{1}{2\pi} \tag{70}$$

Taking the square root of Eq. (69), it appears that the standard deviation of the oil pool final size is proportional to $\langle x_{Max}\rangle^{1/2}$. This is in accordance with the following physical argument. Let ΔV be the volume of a pocket per unit width, and let us split it into a "most probable pocket" volume plus a fluctuating quantity V':

$$\Delta V = L_{mp} \langle h^- \rangle + V' \quad \dagger \tag{71}$$

If $\sigma_{\Delta V}$ is the variance of ΔV and $\sigma_{V'}$ that of V', we have obviously:

$$\sigma_{\Delta V} = \sigma_{V'} \tag{72}$$

Now the total volume V of a spill is such that:

$$V = N \Delta V \tag{73}$$

Furthermore, all these pockets are statistically independent if we assume that the correlation for h^- goes to zero within an interval L_{mp}.

Therefore

$$\sigma_V = N \sigma_{\Delta V} = N \sigma_{V'} \tag{74}$$

where σ_V is the variance of the volume occupied by N pockets. Since $\langle x_{Max} \rangle$ is always of the form

$$\langle x_{Max} \rangle = N L_{mp} \tag{75}$$

we have from (74):

$$\sigma_V = \langle x_{Max} \rangle \frac{\sigma_{V'}}{L_{mp}} \tag{76}$$

---

† L_{mp} is defined in (56).

$\sigma_x^{1/2} = \left(\frac{\sigma_v}{\langle h^- \rangle^2} \right)^{1/2}$ is then proportional to $\langle x_{Max} \rangle^{1/2}$ as expressed in Eq. (69).

Now, let us show that α defined in relation (70) is independent of $\langle x_{Max} \rangle$ for sufficiently large values of $\langle x_{Max} \rangle$.

Eq. (66) can be rewritten:

$$\alpha = \frac{1}{L_{mp}} \int_0^{L_{mp}} \frac{[\langle x_{Max} \rangle/\sqrt{2} - \tau]}{[\langle x_{Max} \rangle/\sqrt{2} - \tau_0]} \left(\psi^-(\tau)/\sigma_h - \frac{1}{2\pi} \right) d\tau \qquad (77)$$

the ratio $\frac{\langle x_{Max} \rangle/\sqrt{2} - \tau}{\langle x_{Max} \rangle/\sqrt{2} - \tau_0}$ can be regarded as constant in so far $\tau - \tau_0$ remains small as compared to $\langle x_{Max} \rangle$ when τ varies from 0 to L_{mp}. In other words, since τ_0 is of order $L_{mp}/4$, α independent of $\langle x_{Max} \rangle$ <u>is a reasonable approximation when the number of "most probable pockets" involved in the oil spill is large</u> (of order 10 or greater).

We can also show that, when kT^2 exceeds a number to be determined, α only depends on the non-dimensional group kT^2:

From Appendix D, we know that:

$$\psi^-(\tau) = \mathcal{F}(\psi) = \frac{1}{2\pi(\psi_0^2 - \psi^2)^{1/2}} \int_0^{\infty}\!\!\int ab \exp\left(-\frac{(a^2+b^2)\psi_0 - 2ab\psi}{2(\psi_0^2 - \psi^2)} \right) da\, db \qquad (78)$$

Furthermore, it appears that, setting

$$\tau_0 = \gamma L_{mp} = \gamma T \qquad (79)$$

γ is almost a constant within a few percent; this result comes out directly from a large number of numerical computations with different k's and T's and is not surprising since the overall shape of the functions $\left(\frac{\psi^{-}(\tau)}{\sigma_h} - \frac{1}{2\pi}\right)$ does not change very much (see Fig. 9).

Then,

$$\psi(\tau_0) = \frac{\sigma_h}{1 + k\gamma^2 T^2} \cos 2\pi\gamma \tag{80}$$

is only a function of kT^2.

Hence, from Eq. (78)

$$\psi^{-}(\tau_0) = \mathcal{F}(\psi(\tau_0)) = \text{function of } kT^2 \text{ only} \tag{81}$$

We have therefore shown that α depends on the non-dimensional group kT^2 only.

Non-dimensionalizing Eq. (69) with respect to the "most probable pocket" length L_{mp}, we obtain:

$$\frac{\sigma_x^{1/2}}{L_{mp}} = \theta \left(\frac{\langle x_{Max}\rangle}{L_{mp}}\right)^{1/2} \tag{82}$$

with

$$\theta = \sqrt{\frac{8\pi\alpha}{\sqrt{2}}} \tag{83}$$

We now determine the universal curve θ versus kT^2 for k and T's that verify the assumptions we have made in the previous derivations.

The only one concerned with k and T tells us that $\left[\frac{\psi^{-}(\tau)}{\sigma_h} - \frac{1}{2\pi} \right]$ must decay to zero within a period† $T = L_{mp}$. From Appendix D, we know that this is equivalent to $\psi(\tau)$ going to zero within a period T ; if we assume that $\psi(T) = \frac{\psi(0)}{50}$ is a good approximation for $\psi(T)$ equals zero, it follows that:

$$\psi(0)/\psi(T) \geqslant 50 \tag{84}$$

or:

$$kT^2 \geqslant 50 \tag{85}$$

We will therefore compute the curve $\theta(kT^2)$ for $kT^2 \geqslant 50$. This is represented on Fig. 11. We can notice that, when kT^2 goes to infinity, θ goes to zero. This is not surprising since, when $T = \infty$, α equals a constant divided by infinity and when $k = \infty$, α equals zero times a constant; in both cases, θ is therefore zero.

Knowing the k and T of a given ice surface statistics, we are now in a position to obtain the corresponding θ (on Fig. 11) and, with Eq. (82) have got a single relationship between the two first statistical moments of the oil pool size x_{Max} , in their non-dimensional form.

---

† We could have taken the same condition within two or more periods T ; the dependence of α on kT^2 only would not have been changed, but α independent of $\langle x_{Max} \rangle$ could be questioned. However, the decaying to zero within a period seems to be in good accordance with field data.

IV - 3.2. Two-Dimensional Case.

We recall that the two-dimensional size of the oil pool is S_{Max} and its variance σ_S such that:

$$\langle S_{Max} \rangle = \frac{V\sqrt{2\pi}}{\sqrt{\sigma_h}} \tag{86}$$

$$\sigma_S = 32\pi \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \int_0^{\langle y_{Max}\rangle/\sqrt{2}} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \xi\right)\left(\frac{\langle y_{Max}\rangle}{\sqrt{2}} - \eta\right) \cdot \left(\frac{\Psi^-(\xi,\eta)}{\sigma_h} - \frac{1}{2\pi}\right) d\xi\, d\eta \tag{87}$$

where $\Psi^-(\xi,\eta)$ is obtainable from $\Psi(\xi,\eta)$ defined as:

$$\Psi(\xi,\eta) = \frac{\sigma_h}{(1+k_1\xi^2)(1+k_2\eta^2)} \cos 2\pi\frac{\xi}{T_1} \cos 2\pi\frac{\eta}{T_2} \tag{88}$$

Going through the same kind of derivations as we did for the one-dimensional case, and under the same conditions for k's , T's and S_{Max} , we will obtain comparable results.

Thus, for k_1 , k_2 , T_1 and T_2 such that $\Psi^-(\xi,\eta)$ goes to 0 within the area $T_1 \times T_2$ and sufficiently large $\langle S_{Max}\rangle$, we can write

$$\frac{\sigma_S^{1/2}}{S_{mp}} = \Theta\left(\frac{\langle S_{Max}\rangle}{S_{mp}}\right)^{1/2} \tag{89}$$

Θ being only a function of $k_1 k_2 T_1^2 T_2^2$ (if we neglect $k_1 T_1^2 + k_2 T_2^2$ compared to $k_1 k_2 T_1^2 T_2^2$). Θ versus dimensionless group has also been determined on Fig. 13, and one notices that for $\Psi^{-}(\xi,\eta)$ to go to zero within the area $T_1 \times T_2$, $k_1 k_2 T_1^2 T_2^2$ has to be greater than 2500 if we keep the same standards as in the one-dimensional case for the meaning of "goes to zero".

Therefore, knowing k_1, k_2, T_1 and T_2 of the two-dimensional statistics we are considering, we are in a position to obtain the corresponding Θ and thereby, we have got a simple $\frac{1}{2}$ power law relationship between the non-dimensional first and second statistical moments of the oil pool size.

IV.-4. Results.

Having three experimental power density spectra coming from different parts of the Arctic Ocean, we have attempted to insert them in our parametered model described in Section IV - 1. We obtained the following values for the parameters k and T † (see the modelled spectra on Figs. 4 to 6).

(90)

T	k	σ_h	
115 ft.	0.00378 ft.$^{-2}$	2.295 ft.^{2}	1
200 ft.	0.002786 ft.$^{-2}$	1.016 ft.^{2}	2
540 ft.	0.04075 ft.$^{-2}$	0.2033 ft.^{2}	3

Then, on Fig. 10, plotting the relationship between the two first moments of the oil pool size in the one-dimensional case for any of the three above statistics. we have compared the simple $\frac{1}{2}$ power laws given by Eq. (82) with the corresponding θ'_s (solid lines) to points obtained from the general mathematical model - Eq. (34) - for different volumes released V. The agreement is extremely good for any statistics, as well as it is on Fig. 12 for the two-dimensional case; in order to build two-dimensional statistics, we have made the following combinations of one-dimensional statistics: 1 + 1, 2 + 2, 3 + 3 and 1 + 2 of Table (90).

† We notice that kT^2 is always larger than 50 for all three statistics which allows us to use Section IV - 3 analysis.

Therefore, in so far the conditions over k and T is verified ($kT^2 \geqslant 50$ for instance), this $\frac{1}{2}$ power law relationship between the oil pool final size and its standard deviation appears to be a very good approximation to solve our problem, both in the one and two-dimensional case.

It might be interesting to show a rough estimate of what these numerical results can be. Let us take a volume V = 1000 ft.^{3} per foot width in our one-dimensional case; this yields the following figures for each of the statistics of Table (90):

$$(1) \longrightarrow \langle x_{Max} \rangle = 1458 \text{ ft} \qquad \sigma_x^{1/2} = 304 \text{ ft}$$

$$(2) \longrightarrow \langle x_{Max} \rangle = 2487 \text{ ft} \qquad \sigma_x^{1/2} = 480 \text{ ft}$$

$$(3) \longrightarrow \langle x_{Max} \rangle = 5560 \text{ ft} \qquad \sigma_x^{1/2} = 675 \text{ ft}$$

As to the two-dimensional case, let us take a volume $V = 5 \times 10^5 \text{ ft}^3$ which appears to be a reasonable size, even in case of a supertanker crash, then, we obtain for the following combinations of one-dimensional statistics:

	(1) + (1)	(2) + (2)	(3) + (3)	(1) + (2)
$\langle S_{Max} \rangle = \langle x_{Max}\, y_{Max} \rangle$	7.29×10^5 ft^2	1.24×10^6 ft^2	2.78×10^6 ft^2	1.6×10^6 ft^2
$\mu = (y_{Max}/x_{Max})^{-1}$	1	1	1	2.2
$\sigma_S^{1/2}$	37 000 ft^2	72 000 ft^2	116 000 ft^2	146 000 ft^2
S_{mp}	13 200 ft^2	40 000 ft^2	290 000 ft^2	23 000 ft^2

(91)

Such results for σ_x and σ_s provide good evidence that our assumptions of Eqs. (11) and ($E7$) are valid. We recall that in these assumptions, the average of h^- over a length $\sigma_x^{1/2}$ or area $\sigma_s^{1/2}$ was identified to $\overline{h^-(\xi)}$, the average of h^- over the whole realization of the surface, on the ground that $\sigma_x^{1/2}$ or $\sigma_s^{1/2}$ was large enough.

Since the correlation for h approaches zero within one "most probable pocket" dimension, a length L_{mp} - or an area S_{mp} in the two-dimension case - of the ice surface realization can be considered as containing all the statistical characteristics of the entire surface. Therefore, the average value of h^- over a length L_{mp} - or area S_{mp} should not differ very much from $\langle h^-(\xi)\rangle$.

Hence, it comes out that for V such that the standard deviation of the oil pool is much less than a "most probable pocket", the assumptions are no longer verified, thereby making the lower part of the curves σ_x versus $\langle x_{Max}\rangle$ - or σ_s versus $\langle S_{Max}\rangle$ - less trustworthy.

V - Conclusion - Summary - Discussion.

In this study, the two first statistical moments of the oil pool resulting from a spill of volume V have been determined in terms of σ_h , the variance of the roughness height h and the auto-correlation function of h , both when the spreading is one-dimensional (one privileged direction) and when it is two-dimensional (no privileged direction).

These results have then been developed and simplified in compliance with a two-parameter (k, T) model proposed for one-dimensional statistics as long as their power spectrum exhibits a single predominant peak. If $P(f)$ is the modelled spectrum, we have:

$$P(f) = \frac{2\pi\sigma_h}{\sqrt{k}} \exp\left(-\frac{2\pi f}{\sqrt{k}}\right) \cosh\left(\frac{2\pi}{T\sqrt{k}}\right) \quad \text{if} \quad f > 1/T$$
$$P(f) = \frac{2\pi\sigma_h}{\sqrt{k}} \exp\left(-\frac{2\pi}{T\sqrt{k}}\right) \cosh\left(\frac{2\pi f}{\sqrt{k}}\right) \quad \text{if} \quad f < 1/T \tag{92}$$

T is nothing more than the inverse of the frequency position of the peak and k is determined by the intensity of the spectrum at that peak, through the transcendental equation:

$$P\left(\frac{1}{T}\right) = \frac{2\pi\sigma_h}{\sqrt{k}} \exp\left(-\frac{2\pi}{T\sqrt{k}}\right) \cosh\frac{2\pi}{T\sqrt{k}} \tag{93}$$

Two-dimensional statistics have been regarded as characterized by a product of two one-dimensional power density spectra, therefore, their modelling is quite similar to that of the one-dimensional case.

A "most probable pocket" that could be physically defined as the "most apparent periodical relief" in the ice surface has been introduced both in the one and two-dimensional case. Mathamatically, it is identified to the main term in the Fourier series expansion of the ice surface.

If $\frac{1}{T}$ is the frequency position of the peak in a one-dimensional spectrum and (u_{no1}, u_{no2}), $(u_{no1}, -u_{no2})$ the positions of the peaks in a two-dimensional spectrum (see Fig. 7), the dimensions of this "most probable pocket" are its size and volume:

$$\left.\begin{array}{ll} \text{length} & L_{mp} = T \\ \text{"volume"} & \Sigma_{mp} = \dfrac{T\sqrt{2\sigma_h}}{\pi} \end{array}\right\} \quad \text{one dim. case} \tag{94}$$

$$\left.\begin{array}{ll} \text{area} & S_{mp} = T_1 T_2 = \dfrac{4\pi^2}{u_{no1} u_{no2}} \\ \text{volume} & V_{mp} = \dfrac{8\sqrt{2\sigma_h}}{u_{no1}\, u_{no2}} \end{array}\right\} \quad \text{two dim. case} \tag{95}$$

The first statistical moment of the final size of the oil pool - namely x_{Max} in the one-dimensional case or S_{Max} in the two-dimensional one - has been determined and then non-dimensionalized with respect to the "most probable pocket" size:

$$\left.\begin{aligned} \frac{\langle x_{Max}\rangle}{L_{mp}} &= \frac{V}{L_{mp}}\sqrt{\frac{2\pi}{\sigma_h}} = \frac{V}{\Sigma_{mp}}\cdot\frac{2}{\sqrt{\pi}} \\ \frac{\langle S_{Max}\rangle}{S_{mp}} &= \frac{V}{S_{mp}}\sqrt{\frac{2\pi}{\sigma_h}} = \frac{V}{V_{mp}}\cdot\frac{4}{\pi^{3/2}} \end{aligned}\right\} \tag{96}$$

A simple $\frac{1}{2}$ power law relationship between the non-dimensional first and second moments of the pool size has been developed with a slope θ - Θ in the two-dimensional case - , depending only on the dimensionless group kT^2 - $k_1 k_2 T_1^2 T_2^2$ in the two-dimensional case - that characterized the modelled spectra. Namely, if σ_x is the variance of x_{Max} and σ_S that of S_{Max} :

$$\frac{\sigma_x^{1/2}}{L_{mp}} = \theta\left(\frac{\langle x_{Max}\rangle}{L_{mp}}\right)^{1/2} \qquad \text{see Figs. (10) and (11)} \tag{97}$$

$$\frac{\sigma_S^{1/2}}{S_{mp}} = \Theta\left(\frac{\langle S_{Max}\rangle}{S_{mp}}\right)^{1/2} \qquad \text{see Figs. (12) and (13)} \tag{98}$$

Therefore, suppose we are given the volume V of oil released and the parameters which characterize the region of the Arctic Ocean where the spill takes place: k and T in the one-dimensional case or k_1, k_2, T_1 and T_2 in the two-dimensional one, along with the variance σ_h of the roughness height; we are then in a position to predict the final size of the oil pool as well as its standard deviation. Furthermore, in the two-dimensional case, an idea about the dissymmetry of the pool can be gotten from the quantity μ , the ratio of the dimensions of the rectangle limiting the pool. With relation (4):

$$\mu = \left(\frac{(2\pi/T_2)^2 + 2k_2}{(2\pi/T_1)^2 + 2k_1} \right)^{1/2} \tag{99}$$

We have to notice here that this rectangular shape $x_{Max} \times y_{Max}$ surely does not represent the real aspect of an oil pool. The latter is much closer to an ellipse - a circle in the case of a truly isotropic surface - and it follows that the size computed under this assumption can result in a 13% underestimation of $\langle x_{Max} \rangle$ and $\langle y_{Max} \rangle$. The computations relative to an ellipse, although quite intricate, can be performed but, in view of the precision obtained on the statistical field measurements, we did not deem it worthwhile.

The limitations for the use of these power law relationships between the first two moments of the pool size (Eqs. (97) and (98)) are concerned with the value of kT^2 and $\langle x_{Max} \rangle$ - $\langle S_{Max} \rangle$ and $k_1 k_2 T_1^2 T_2^2$ in the two-dimensional case - , namely:

$$kT^2 \geq 50 \quad \text{or} \quad k_1 k_2 T_1^2 T_2^2 \geq 2500 \tag{100}$$

$$\frac{\langle x_{Max} \rangle}{L_{mp}} \, , \, \frac{\langle S_{Max} \rangle}{L_{mp}} \geq 10 \tag{101}$$

These conditions, described in some detail in Sub-section IV -3 do not influence more general results as those of Eqs. (34) and (35) which are valid for any power spectra model.

Also, an important point to understand is the actual behavior of the power spectra at very low frequencies since it has been here purely hypothesized, as was also the two-dimensional statistics, for there is no evidence at all that a two-dimensional power spectrum should be the product of two one-dimensional ones.

To conclude this study, we can say that the previous analysis yields simple and useful results as to the prediction of oil spreading over a rough surface of ice; but, since the modelling of the power density spectra has been developed from a limited quantity of data, a strong experimental ground is required to completely justify this proposed model.

Appendix A

In this appendix, we will determine the "dissymmetry factor" defined as the ratio

$$\mu = \frac{\lambda_x}{\lambda_y} \tag{A1}$$

where λ_x and λ_y are the average dimensions of a pocket in the x and y directions respectively.

Let us consider an actual laser profilometer trace parallel to the x direction. Let $r_x(x)$ denote the elevation of ice above the mean level $\langle h(x) \rangle = 0$. Considering an ensemble average over all the traces parallel to x, we can write:

$$\langle r_x^2(x) \rangle = \sigma_h \tag{A2}$$

And:

$$\langle r_x(x+\xi)\, r_x(x) \rangle = \Psi(\xi, 0) \tag{A3}$$

Using our third hypothesis concerned with the way pockets are filled by oil, we can see that λ_x corresponds to the average distance separating two zeros of the function $r_x(x)$ (see Fig. 2). $r_x(x)$ is normally distributed from our second hypothesis and, since $\frac{dr_x}{dx}$ is linearly related to $r_x(x)$, it also holds a Gaussian distribution. Hence, the vector

$$\begin{Vmatrix} r_x(x) \\ \frac{dr_x}{dx}(x) \end{Vmatrix}$$

is normally distributed.

Therefore, if $p(\alpha, \theta ; x)$ represents the probability that:

$$\begin{aligned} \alpha &\leqslant r_x(x) \leqslant \alpha + d\alpha \\ \theta &\leqslant \frac{dr_x}{dx}(x) \leqslant \theta + d\theta \end{aligned} \tag{A4}$$

we have:†

$$p(\alpha, \theta ; x) = \frac{1}{2\pi} |M|^{-1} \exp\left(-\frac{1}{2} \| \alpha \;\; \theta \| \; M^{-1} \begin{Vmatrix} \alpha \\ \theta \end{Vmatrix} \right) \tag{A5}$$

where:

$$M = \begin{Vmatrix} \Psi(0,0) & 0 \\ 0 & -\frac{\partial^2 \Psi}{\partial \xi^2}(0,0) \end{Vmatrix} \tag{A6}$$

Relation (A5) can easily be reduced to:

$$p(\alpha, \theta ; x) = \frac{(-\Psi_0 \Psi''_{\xi 0})^{-1/2}}{2\pi} \exp\left(-\frac{\alpha^2}{2\Psi_0} + \frac{\theta^2}{2\Psi''_{\xi 0}}\right) \tag{A7}$$

---

† see Ref. [1].

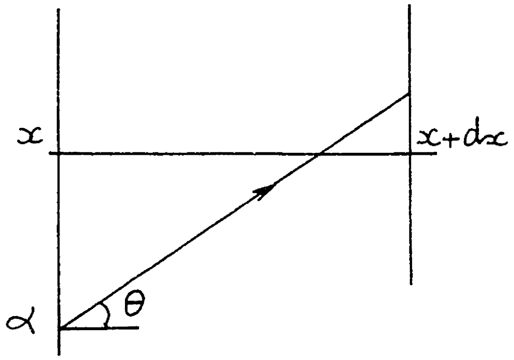
in which we have set for convenience:

$$\Psi''_{\xi_0} = \frac{\partial^2 \Psi}{\partial \xi^2}(0,0)$$

$$\Psi_0 = \Psi(0,0) \tag{A8}$$

Now, let us find the probability that $r_x(x)$ has a zero between x and $x+dx$; in the interval, all the cross-sections of the possible realizations of the ice surface are regarded as straight lines.

At first, we will derive the probability for $r_x(x)$ to have a zero with positive slope. For this to happen, α and θ have to be related by the inequality (clearly shown on the sketch beside):



$$-\theta dx < \alpha < 0 \tag{A9}$$

Then, the probability for $r_x(x)$ to have a zero between x and $x+dx$ with a positive θ is:

$$\int_0^{\infty} d\theta \int_{-\infty}^{0} p(\alpha, \theta; x)\, d\alpha$$

And, the probability density for this happening is:

$$\left(\frac{d}{d(\Delta x)} \int_0^{\infty} d\theta \int_{-\theta\Delta x}^{0} p(\alpha, \theta; x)\, d\alpha \right)_{\Delta x = 0}$$

which reduces to:

$$\left[- \int_0^{\infty} (-\theta)\, p(-\theta\Delta x, \theta ; x)\, d\theta \right]_{\Delta x = 0}$$

or:

$$\int_0^{\infty} \theta\, p(0, \theta ; x)\, d\theta \tag{A10}$$

By the same token, the probability density for $r_x(x)$ to have a zero between x and $x+dx$ with a negative slope is:

$$- \int_{-\infty}^{0} \theta\, p(0, \theta ; x)\, d\theta \tag{A11}$$

Adding relations (A10) and (A11), we obtain $\Pi(x)$, probability density for $r_x(x)$ to have a zero between x and $x+dx$:

$$\Pi(x) = \int_{-\infty}^{+\infty} |\theta|\, p(0, \theta ; x)\, d\theta \tag{A12}$$

Using relation (A5) and integrating, it comes out:

$$\Pi(x) = \frac{1}{\pi} \left(- \frac{\varphi''_{\xi_0}}{\varphi_0} \right)^{1/2} \tag{A13}$$

The integral of $\Pi(x)$ over a unit length gives the number of expected zeros per unit of a x -direction since the integral from $-\infty$ to $+\infty$ obviously gives the total number of zeros.

Therefore, λ_x , the average distance between two zeros of $r_x(x)$, can be approximated by:

$$\lambda_x = \frac{1}{\Pi(x)} = \pi\left(-\frac{\Psi_0}{\Psi''_{\xi 0}}\right)^{1/2} \tag{A14}$$

All that has been done for profilometer traces parallel to the x-direction could be done for traces parallel to the y-direction simply by introducing the function $r_y(y)$, analogous to $r_x(x)$. Going through the same kind of calculations as the previous ones, we would end up with:

$$\lambda_y = \pi\left(-\frac{\Psi_0}{\Psi''_{\eta 0}}\right)^{1/2} \tag{A15}$$

It follows that μ defined in (A1) is given by:

$$\mu = \frac{\lambda_x}{\lambda_y} = \left(\frac{\partial^2\Psi}{\partial\eta^2}(0,0) \Big/ \frac{\partial^2\Psi}{\partial\xi^2}(0,0)\right)^{1/2} \tag{A16}$$

Appendix B

The goal of this appendix is to calculate σ_v defined as:

$$\sigma_v = \left\langle \left[v(\langle x_{Max}\rangle, \langle y_{Max}\rangle) - \langle v(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle \right]^2 \right\rangle \tag{B1}$$

where:

$$v(x,y) = -\int_{-x/2}^{x/2}\int_{-y/2}^{y/2} h^-(u,v)\,du\,dv \tag{B2}$$

Expanding the square in (B1), we obtain:

$$\sigma_v = \langle v^2(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle - \langle 2v(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle \cdot \langle v(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle + \langle v(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle^2 \tag{B3}$$

or:

$$\sigma_v = \langle v^2(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle - \langle v(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle^2 \tag{B4}$$

Making use of (B2), we have:

$$\langle v^2(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle =$$

$$\left\langle \iint_{-\frac{\langle x_{Max}\rangle}{2}}^{\frac{\langle x_{Max}\rangle}{2}} \iint_{-\frac{\langle y_{Max}\rangle}{2}}^{\frac{\langle y_{Max}\rangle}{2}} h^-(x_1,y_1)\,h^-(x_2,y_2)\,dx_1\,dx_2\,dy_1\,dy_2 \right\rangle \tag{B5}$$

Since ensemble averages commute with linear operations, we can write:

$$\langle v^2(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle = \iint_{-\frac{\langle x_{Max}\rangle}{2}}^{\frac{\langle x_{Max}\rangle}{2}} \iint_{-\frac{\langle y_{Max}\rangle}{2}}^{\frac{\langle y_{Max}\rangle}{2}} \langle h^-(x_1,y_1)\, h^-(x_2,y_2)\rangle \, dx_1 dx_2 \, dy_1 dy_2$$

Or, introducing the auto-correlation function $\Psi^-(\xi,\eta)$:

$$\Psi^-(\xi,\eta) = \langle h^-(x+\xi, y+\eta)\, h^-(x,y)\rangle \;,$$

it comes out that:

$$\langle v^2(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle = \iint_{-\frac{\langle x_{Max}\rangle}{2}}^{\frac{\langle x_{Max}\rangle}{2}} \iint_{-\frac{\langle y_{Max}\rangle}{2}}^{\frac{\langle y_{Max}\rangle}{2}} \Psi^-(|x_2-x_1|, |y_2-y_1|) \, dx_1 dx_2 \, dy_1 dy_2 \tag{B6}$$

Introducing the new variables:

$$\begin{aligned} \xi &= |x_2 - x_1| \\ \eta &= |y_2 - y_1| \end{aligned} \tag{B7}$$

we obtain:

$$\langle v^2(\langle x_{Max}\rangle, \langle y_{Max}\rangle)\rangle = 16 \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \int_0^{\langle y_{Max}\rangle/\sqrt{2}} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \xi\right)\left(\frac{\langle y_{Max}\rangle}{\sqrt{2}} - \eta\right) \Psi^-(\xi,\eta)\, d\xi\, d\eta \tag{B8}$$

From relation (B2), it is clear that:

$$\langle v(\langle x_{Max}\rangle,\langle y_{Max}\rangle)\rangle^2 = \langle S_{Max}\rangle^2 \langle h^-(\xi,\eta)\rangle^2 = V^2 \tag{B9}$$

We also notice that:

$$\langle S_{Max}\rangle^2 \langle h^-(\xi,\eta)\rangle^2 = 16 \int_0^{\langle x_{max}\rangle/\sqrt{2}} \int_0^{\langle y_{Max}\rangle/\sqrt{2}} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \xi\right)\left(\frac{\langle y_{Max}\rangle}{\sqrt{2}} - \eta\right) \frac{\sigma_h}{2\pi}\, d\xi\, d\eta$$

Therefore, from (B4), (B8) and (B9), we end up with:

$$\sigma_V = 16 \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \int_0^{\langle y_{Max}\rangle/\sqrt{2}} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \xi\right)\left(\frac{\langle y_{Max}\rangle}{\sqrt{2}} - \eta\right)\left[\Psi^-(\xi,\eta) - \frac{\sigma_h}{2\pi}\right] d\xi\, d\eta \tag{B10}$$

Appendix C

In this appendix, $\psi^{-}(\tau)$ in terms of $\psi(\tau)$ will be derived.

$$\begin{aligned} \psi^{-}(\tau) &= \langle h^{-}(x)\, h^{-}(x+\tau) \rangle \\ \psi(\tau) &= \langle h(x)\, h(x+\tau) \rangle \end{aligned} \qquad \text{(C1)}$$

where $h^{-}(x)$ is defined in relation (5).

At first, we introduce the following notations:

$$\begin{aligned} h(x) &= a & \qquad h^{-}(x) &= a^{-} \\ h(x+\tau) &= b & \qquad h^{-}(x+\tau) &= b^{-} \end{aligned} \qquad \text{(C2)}$$

Let $D(a,b,\tau)$ be the first joint probability density of the couple (a,b) . $D(a,b,\tau)\,da\,db$ represents here the percentage of times that $h(x)$ will lie between a and $a+da$ while $h(x+\tau)$ lies between b and $b+db$.

Then, from our second hypothesis, $D(a,b,\tau)$ is Gaussian with mean value 0 , and can thus be written:

$$D(a,b,\tau) = \frac{1}{2\pi} |M|^{-1} \exp\left(-\frac{1}{2}\, {}^{t}\|r\|\, M^{-1} \|r\|\right) \qquad \text{(C3)}$$

where

$${}^{t}\|r\| = \|\, a \quad b \,\| \qquad \text{(C4)}$$

and M is the second moments matrix:

$$M = \begin{Vmatrix} \langle a^2 \rangle & \langle ab \rangle \\ \langle ab \rangle & \langle b^2 \rangle \end{Vmatrix} \qquad \text{(C5)}$$

From the definition of $\psi(\tau)$, it is obvious that:

$$\langle b^2 \rangle = \langle a^2 \rangle = \psi(0) \qquad \text{(C6)}$$

and

$$\langle ab \rangle = \psi(\tau) \qquad \text{(C7)}$$

in other words, M depends only on τ .

The mean value of a stochastic quantity Q uniquely related to the couple of stochastic variables (a,b) is perfectly defined as soon as we know the joint frequency distribution of the latter i.e., $D(a,b,\tau)$. It is:

$$\langle Q \rangle = \iint_{-\infty}^{+\infty} Q(a,b)\; D(a,b,\tau)\; da\, db \qquad \text{(C8)}$$

Or, when identifying Q with the product $a^- b^-$

$$Q = a^- b^- = \begin{cases} ab & \text{when } a \text{ and } b < 0 \\ 0 & \text{when } a \text{ or } b \geq 0 \end{cases} \qquad \text{(C9)}$$

and, substituting for $D(a,b,\tau)$:

$$\langle a^- b^- \rangle = \psi^-(\tau) = \frac{1}{2\pi(\psi_0^2 - \psi^2)^{1/2}} \cdot \iint\limits_0^\infty ab \exp\left(-\frac{(a^2+b^2)\psi_0 - 2ab\psi}{2(\psi_0^2-\psi^2)}\right) da\,db \tag{C10}$$

in which we have set for convenience:

$$\begin{aligned} \psi &= \psi(\tau) \\ \psi(0) &= \psi_0 \end{aligned} \tag{C11}$$

Introducing the new variables x and y such that:

$$\begin{aligned} a &= x - y \\ b &= x + y \end{aligned} \tag{C12}$$

Eq.(C10) becomes, noticing that the Jacobian of the transformation is 2:

$$\psi^-(\tau) = \frac{2}{\pi(\psi_0^2 - \psi^2)^{1/2}} \int_0^\infty dx \int_0^x (x^2 - y^2) \exp\left(-\frac{x^2}{\psi_0+\psi} - \frac{y^2}{\psi_0-\psi}\right) dy \tag{C13}$$

And, making use of the error function defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{C14}$$

we end up with:

$$\psi^-(\tau) = \left[\frac{(\psi_o-\psi)^3}{\pi(\psi_o+\psi)}\right]^{1/2} \cdot \left\{ \int_0^\infty \left(u^2-\frac{1}{2}\right) \operatorname{erf}(u) \exp\left(-u^2\frac{\psi_o-\psi}{\psi_o+\psi}\right) du + \frac{\psi_o+\psi}{4\sqrt{\pi}\,\psi_o} \right\} \tag{C15}$$

We notice that for the two-dimensional auto-correlation function $\Psi^-(\xi,\eta)$, the result is exactly similar to Eq. (C15).

$$\begin{aligned} \Psi^-(\xi,\eta) &= \langle h^-(x,y)\,h^-(x+\xi,y+\eta)\rangle \\ \Psi(\xi,\eta) &= \langle h(x,y)\,h(x+\xi,y+\eta)\rangle \end{aligned} \tag{C16}$$

With the notation:

$$\begin{aligned} \Psi_o &= \Psi(0,0) \\ \Psi &= \Psi(\xi,\eta) \end{aligned} \tag{C17}$$

we have:

$$\Psi^-(\xi,\eta) = \left[\frac{(\Psi_o-\Psi)^3}{\pi(\Psi_o+\Psi)}\right]^{1/2} \left\{ \int_0^\infty \left(u^2-\frac{1}{2}\right) \operatorname{erf}(u) \exp\left(-u^2\frac{\Psi_o-\Psi}{\Psi_o+\Psi}\right) du + \frac{\Psi_o+\Psi}{4\sqrt{\pi}\,\Psi_o} \right\} \tag{C18}$$

Appendix D

The purpose of this appendix is to examine the behavior of $\psi^{-}(\tau)$.

From relation (C15), we have:

$$\psi^{-}(\tau) = \left[\frac{(\psi_0-\psi)^3}{\pi(\psi_0+\psi)}\right]^{1/2} \left\{ \int_0^{\infty} (u^2-\tfrac{1}{2}) \operatorname{erf}(u) \exp\left(-u^2 \frac{\psi_0-\psi}{\psi_0+\psi}\right) du + \frac{\psi_0+\psi}{4\sqrt{\pi}\,\psi_0} \right\} \tag{D1}$$

We can split the integral into two parts:

$$I = \int_0^{\infty} (u^2-\tfrac{1}{2}) \operatorname{erf}(u) \exp\left(-u^2 \frac{\psi_0-\psi}{\psi_0+\psi}\right) du = I_1 + I_2 \tag{D2}$$

where:

$$\left.\begin{aligned} I_1 &= \int_0^{n} (u^2-\tfrac{1}{2}) \operatorname{erf}(u) \exp\left(-u^2 \frac{\psi_0-\psi}{\psi_0+\psi}\right) du \\ I_2 &= k_n \int_0^{\infty} (u^2-\tfrac{1}{2}) \exp\left(-u^2 \frac{\psi_0+\psi}{\psi_0+\psi}\right) du \end{aligned}\right\} \tag{D3}$$

with:

$$k_n = \frac{\displaystyle\int_n^{\infty} (u^2-\tfrac{1}{2}) \operatorname{erf}(u) \exp\left(-u^2 \frac{\psi_0-\psi}{\psi_0+\psi}\right) du}{\displaystyle\int_n^{\infty} (u^2-\tfrac{1}{2}) \exp\left(-u^2 \frac{\psi_0-\psi}{\psi_0+\psi}\right) du} \tag{D4}$$

k_n is perfectly defined since the two above integrals are convergent.

Since erf(z) as defined in Section 3 approaches unity when z tends to infinity, k_n tends to 1 when n approaches infinity.

$$I_2 = k_n(J+K)$$

where:

$$J = \int_n^\infty u^2 \exp\left(-u^2 \frac{\psi_o-\psi}{\psi_o+\psi}\right) du$$

$$K = -\int_n^\infty \frac{1}{2} \exp\left(-u^2 \frac{\psi_o-\psi}{\psi_o+\psi}\right) du$$

After some algebra, it follows that:

$$I_2 = k_n \left\{ \frac{3}{2} \cdot \frac{\psi_o+\psi}{\psi_o-\psi} \exp\left(-n^2 \frac{\psi_o-\psi}{\psi_o+\psi}\right) + \frac{2\psi}{\psi_o-\psi} \sqrt{\frac{\pi(\psi_o+\psi)}{16(\psi_o-\psi)}} \operatorname{erfc}\left(n\sqrt{\frac{\psi_o-\psi}{\psi_o+\psi}}\right) \right\} \tag{D5}$$

here, erfc(z) is the complementary function of erf(z)

i) $\tau \to 0$

When $\tau = 0$, $\psi = \psi_o$ and therefore relation (A5) becomes:

$$I_2 = k_n \left\{ \frac{3}{2} \cdot \frac{\psi_o+\psi}{\psi_o-\psi} \exp\left(-n^2 \frac{\psi_o-\psi}{\psi_o+\psi}\right) + \frac{2\psi}{\psi_o-\psi} \sqrt{\frac{\pi}{16} \cdot \frac{\psi_o+\psi}{\psi_o-\psi}} \right\}_{\psi=\psi_o} \tag{D6}$$

having n increase to infinity, it follows from (A1), (A2), (A3) and (A6) that the only remaining non zero term is:

$$\psi^-(0) = \left\{ \left[\frac{(\psi_o-\psi)^3}{\pi(\psi_o+\psi)}\right]^{1/2} \left[\frac{2\psi}{\psi_o-\psi} \sqrt{\frac{\pi}{16}} \left(\frac{\psi_o+\psi}{\psi_o-\psi}\right)^{1/2} \right] \right\}_{\psi=\psi_o}$$

or, with $\psi = \psi_o$:

$$\psi^{-}(0) = \frac{\psi_0}{2} = \frac{1}{2}\psi(0) \tag{D7}$$

ii) $\tau \to \infty$

We have to go back here to relation (C10),namely:

$$\psi^{-}(\tau) = \frac{1}{2\pi[\psi_0^2 - \psi^2]^{1/2}} \iint_0^\infty ab \exp\left\{-\frac{(a^2+b^2)\psi_0 - 2ab\psi}{2(\psi_0^2-\psi^2)}\right\} da\,db$$

$\tau \to \infty$ is equivalent to have $\psi \to 0$ since, at infinity, $\psi(\tau)$ is zero.

Hence, integrating the above relation for $\psi = 0$, one easily obtains:

$$\psi^{-}(\infty) = \frac{\psi_0}{2\pi} = \frac{\sigma_h}{2\pi} \tag{D8}$$

which is not surprising since, for an infinite τ, $h^{-}(x)$ and $h^{-}(x + \tau)$ are no more correlated and therefore:

$$\langle h^{-}(x)h^{-}(x+\tau)\rangle = \langle h^{-}(x)\rangle \cdot \langle h^{-}(x+\tau)\rangle = \langle h^{-}(x)\rangle^2 = \frac{\psi_0}{2\pi}$$

The aspect of a typical $\psi^{-}(\tau)$ is sketched on Fig. 8.

iii) application

For numerical computation of $\psi^{-}(\tau)$ from $\psi(\tau)$, it is interesting to notice that:

n = 3 in relation (A3) is highly sufficient since:

$$\mathrm{erf}(3) \simeq 0.99998 \simeq 1$$

implying $k_3 = 1$

Therefore, relation (A1) becomes, for the purpose of numerical calculations:

$$\psi^-(\tau) \simeq \left[\frac{(\psi_0-\psi)^3}{\pi(\psi_0+\psi)}\right]^{1/2} \left\{ \int_0^3 \left(u^2 - \tfrac{1}{2}\right)\mathrm{erf}(u)\exp\left(-u^2\frac{\psi_0-\psi}{\psi_0+\psi}\right)du \right.$$

$$\left. + \frac{\psi_0+\psi}{4\sqrt{\pi}\,\psi_0} + \frac{3(\psi_0+\psi)}{2(\psi_0-\psi)}\exp\left(-9\frac{\psi_0-\psi}{\psi_0+\psi}\right) + \frac{\psi_0}{2(\psi_0-\psi)}\sqrt{\frac{\pi(\psi_0+\psi)}{\psi_0-\psi}}\,\mathrm{erfc}\left(3\sqrt{\frac{\psi_0-\psi}{\psi_0+\psi}}\right)\right\}$$

(D9)

Appendix E

The purpose of this appendix is to give some details on the derivations relative to the mathematical analysis of the one-dimensional case Thus, the first statistical moment $\langle x_{Max} \rangle$ of the pool extent and its variance σ_x will be determined in terms of σ_h , the variance of the roughness height and $\psi(\tau)$ the auto-correlation of the latter.

We introduce the notations $h^+(x)$ and $h^-(x)$ such that

$$h(x) = h^+(x) + h^-(x) \,. \tag{E1}$$

and

$$\begin{array}{lll} h^+(x) = h(x) & \text{when} & h > 0 \\ h^+(x) = 0 & \text{when} & h \leq 0 \\ h^-(x) = h(x) & \text{when} & h < 0 \\ h^-(x) = 0 & \text{when} & h \geq 0 \end{array} \tag{E2}$$

We define $v(x)$, amount of oil per unit width that would fill the pockets between abscissae 0 and x :

$$v(x) = - \int_0^x h^-(\xi)\, d\xi \tag{E3}$$

The extent of the pool, x_{Max} , for a given realization of the ice profile is such that:

$$V = - \int_0^{x_{Max}} h^-(\xi)\, d\xi \tag{E4}$$

If $\langle x_{Max}\rangle$ denotes the ensemble average of x_{Max} , we can always write (E4) as:

$$-V = \int_0^{\langle x_{Max}\rangle} h^-(\xi)\,d\xi + \left(x_{Max} - \langle x_{Max}\rangle\right) h^-(\xi) \tag{E5}$$

since h^- is a continuous function of x ; ξ corresponds here to a point of the profile where:

$$\langle x_{Max}\rangle \leqslant \xi \leqslant x_{Max} \tag{E6}$$

As was done for the two-dimensional case and with the same limitations, we assume that:

$$h^-(\xi) = \langle h^-(\xi)\rangle \tag{E7}$$

(E5) is then written:

$$-V = \int_0^{\langle x_{Max}\rangle} h^-(\xi)\,d\xi + \left(x_{Max} - \langle x_{Max}\rangle\right) \langle h^-(\xi)\rangle \tag{E8}$$

Taking the ensemble ave-age of (E8) and noticing that this kind of average commutes with linear operations:

$$-V = \langle h^-(\xi)\rangle\langle x_{Max}\rangle \tag{E9}$$

Then:

$$\langle x_{Max}\rangle = -\frac{V}{\langle h^-(\xi)\rangle} \tag{E10}$$

To derive the variance σ_x of x_{Max}, defined as:

$$\sigma_x = \langle (x_{Max} - \langle x_{Max} \rangle)^2 \rangle \qquad \text{(E11)}$$

we come back to (E8) and write it as:

$$\left\langle \left[\int_0^{\langle x_{Max} \rangle} h^-(\xi) d\xi + V \right]^2 \right\rangle = \langle (x_{Max} - \langle x_{Max} \rangle)^2 \rangle \langle h^-(\xi) \rangle^2 \qquad \text{(E12)}$$

Or:

$$\sigma_v = \sigma_x \langle h^-(\xi) \rangle^2 \qquad \text{(E13)}$$

where σ_v is the variance of the volume $v(\langle x_{Max} \rangle)$:

$$\sigma_v = \left\langle \left[v(\langle x_{Max} \rangle) - \langle v(\langle x_{Max} \rangle) \rangle \right]^2 \right\rangle \qquad \text{(E14)}$$

Also, expanding (E14):

$$\sigma_v = \langle v^2(\langle x_{Max} \rangle) \rangle - \langle x_{Max} \rangle^2 \langle h^-(\xi) \rangle^2 \qquad \text{(E15)}$$

From (E3), it is clear that

$$\langle v^2(\langle x_{Max} \rangle) \rangle = \left\langle \iint_0^{\langle x_{Max} \rangle} h^-(\xi) h^-(\eta) \, d\xi \, d\eta \right\rangle$$

$$= \iint_0^{\langle x_{Max}\rangle} \langle h^-(\xi)\, h^-(\eta) \rangle \, d\xi \, d\eta$$

$$= \iint_0^{\langle x_{Max}\rangle} \psi^-(|\xi-\eta|) \, d\xi \, d\eta \qquad \text{(E16)}$$

where $\psi^-(\tau)$ is the auto-correlation for h^- .

Making the following change of variables:

$$\tau = |\xi - \eta|$$
$$\theta = |\xi + \eta| \qquad \text{(E17)}$$

We end up with:

$$\langle v^2(\langle x_{Max}\rangle) \rangle = 4 \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \tau \right) \psi^-(\tau) \, d\tau \qquad \text{(E18)}$$

Noticing that:

$$\langle x_{Max}\rangle^2 \langle h^-(\xi)\rangle^2 = 4 \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \tau \right) \frac{\sigma_h}{2\pi} \, d\tau \qquad \text{(E19)}$$

and using (E13) and (E15), we obtain:

$$\sigma_x = 4 \int_0^{\langle x_{Max}\rangle/\sqrt{2}} \left(\frac{\langle x_{Max}\rangle}{\sqrt{2}} - \tau \right) \left(\psi^-(\tau) - \frac{\sigma_h}{2\pi} \right) d\tau \tag{E20}$$

$\psi^-(\tau)$ in terms of $\psi(\tau)$ is determined in Appendix C (Eq. (C15)) while the value of $\langle h^- \rangle$ has already been determined in terms of σ_h in Eq. (22).

REFERENCES

[1] S. O. Rice, "Mathematical analysis of random noise", p. 133 in Selected papers on noise and stochastic processes , edited by Nelson Wax., Dover Publications, New York, 1954.

[2] J. L. Lumley and H. A. Panofsky, "The structure of atmospheric turbulence", Interscience, New York, 1964.

[3] CRREl - Corps of Engineers, Hanover, New Hampshire, "Sev Arctic Environment data package", 1970.

[4] Norbert Wiener, "Generalized harmonic analysis - Tauberian theorems", MIT Press, 1964.

[5] Dr. Hibler, CRREL, Hanover, New Hampshire, personal communication.

LIST OF FIGURES

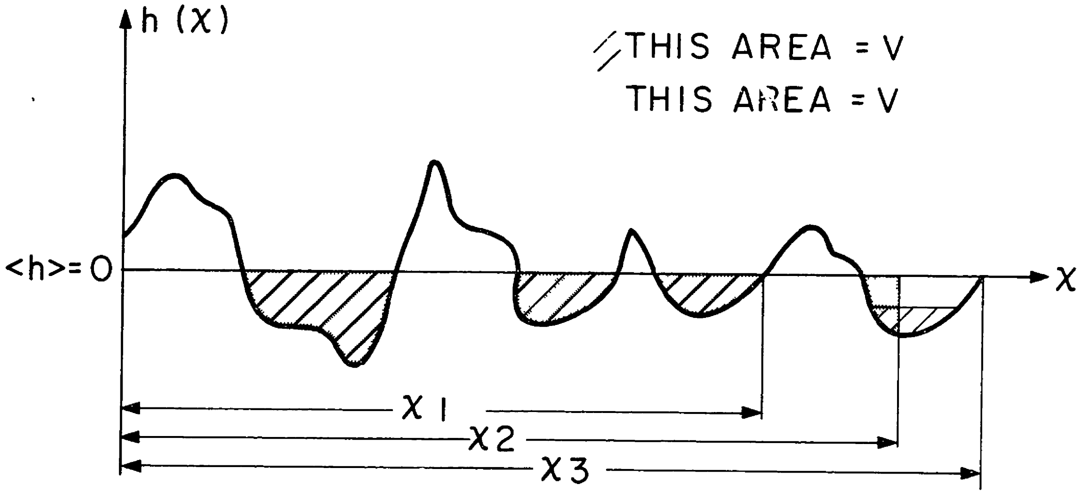


FIG. 1 GENERAL ASPECT OF ONE REALIZATION OF THE PROFILE

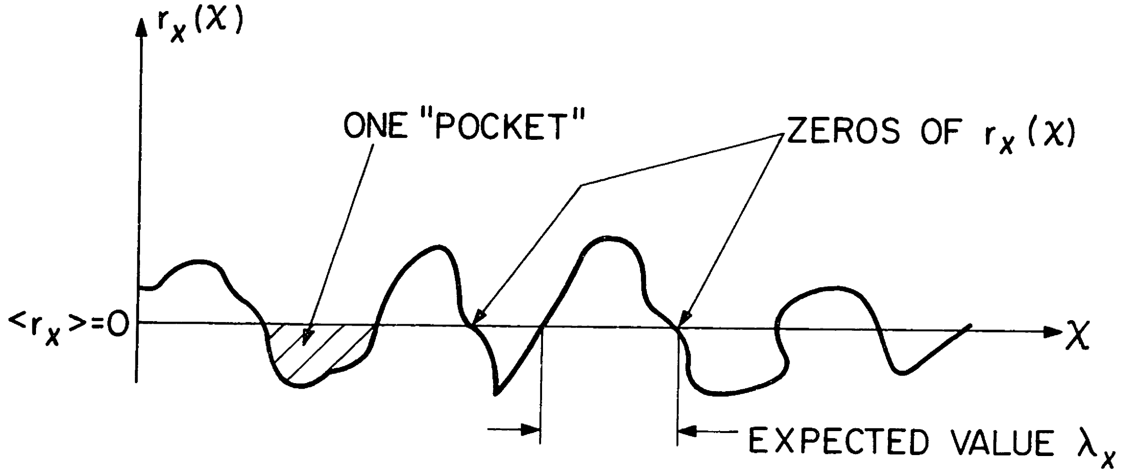


FIG. 2 AVERAGE LONGITUDINAL DIMENSION OF A POCKET

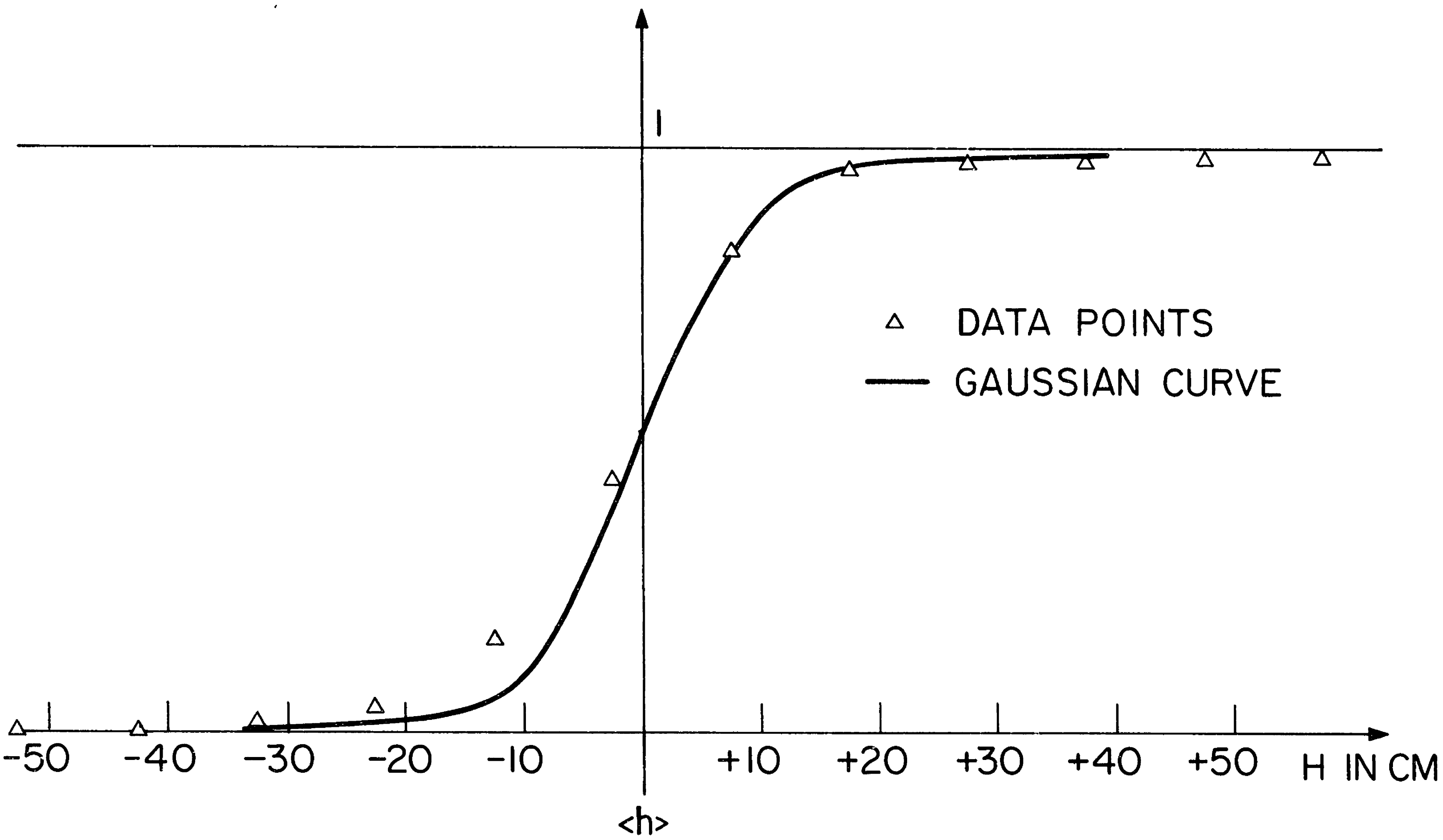


FIG. 3 VERIFICATION FOR h NORMALLY DISTRIBUTED

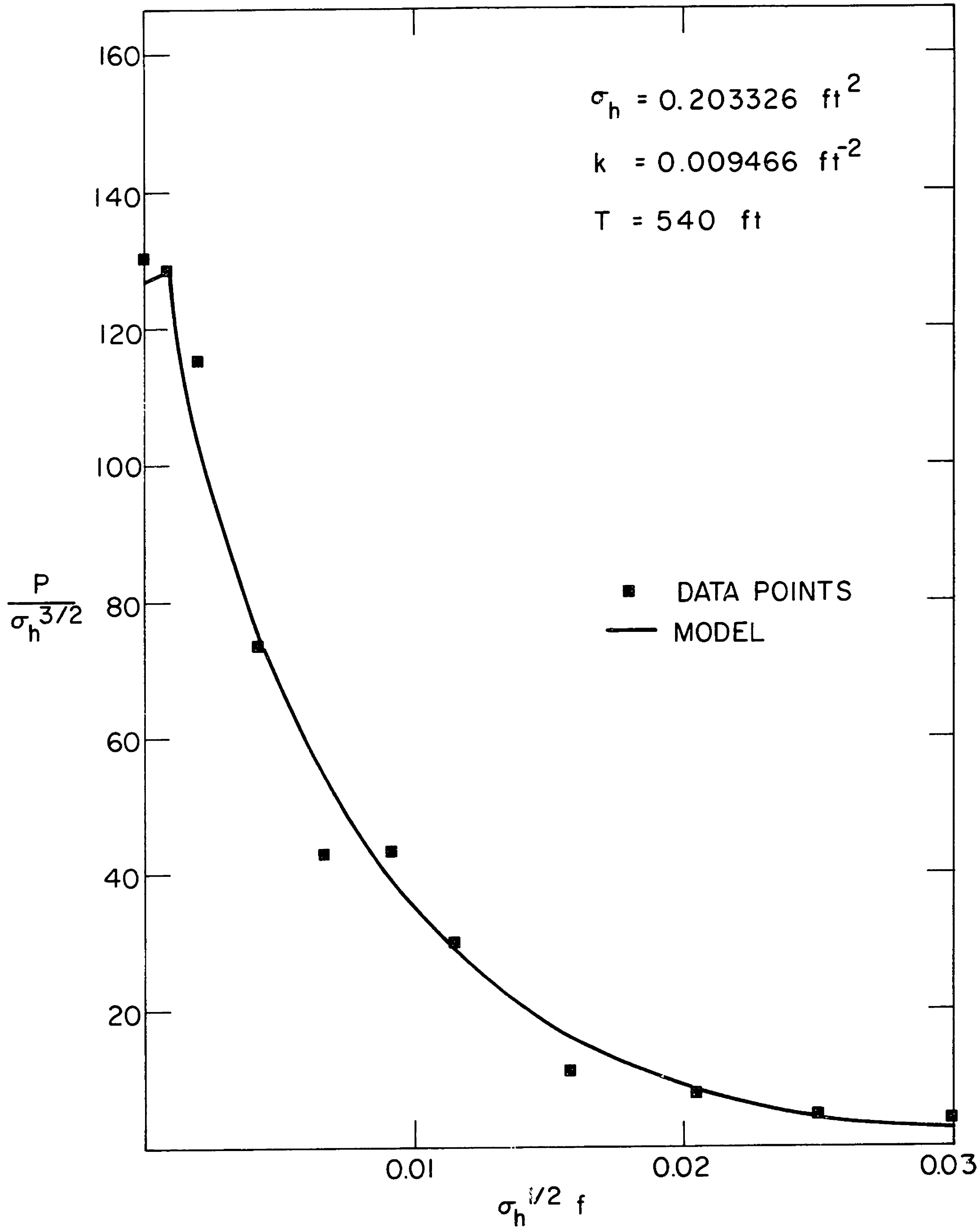


FIG. 4 EXAMPLE OF ONE-DIMENSIONAL POWER SPECTRUM

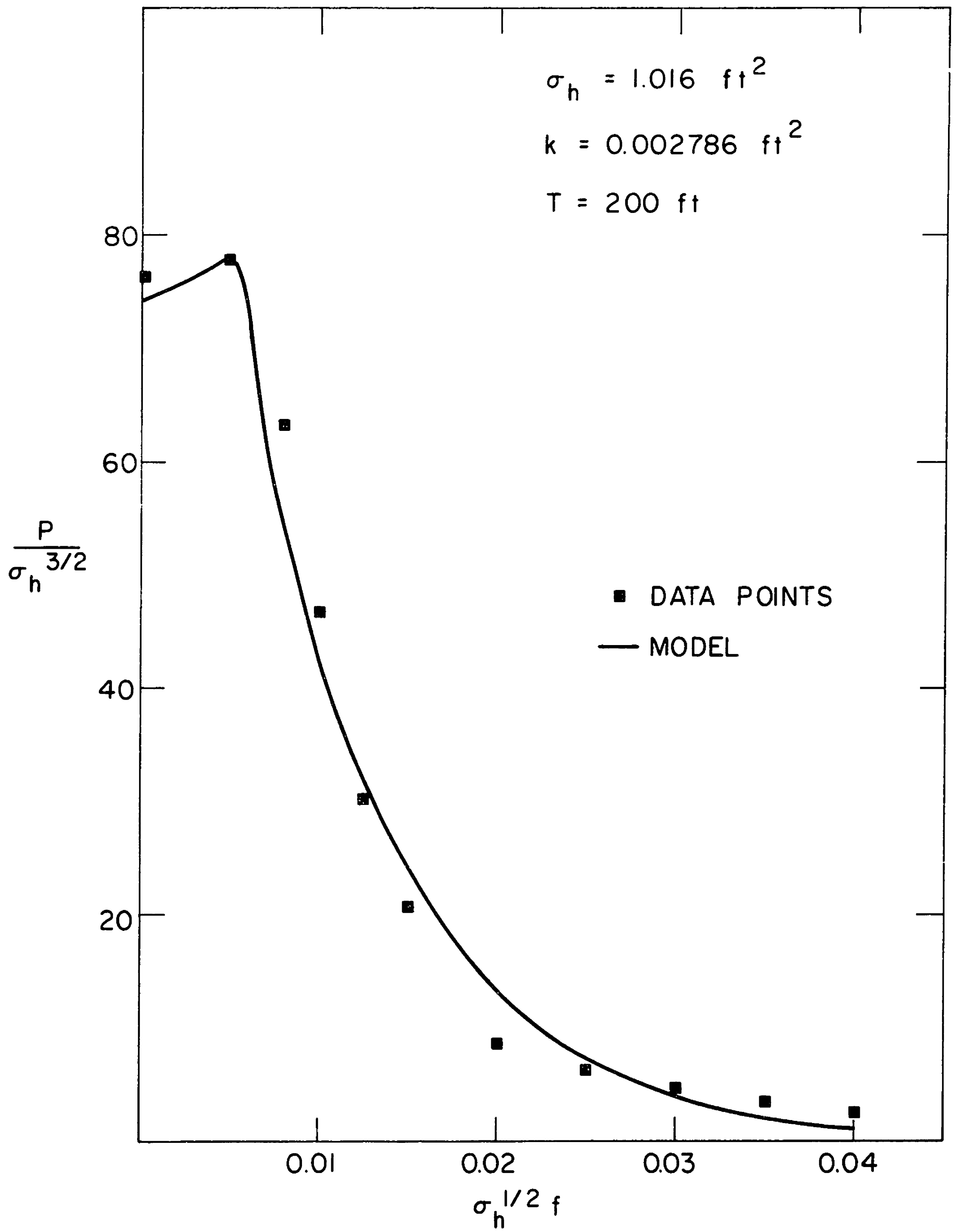


FIG. 5 EXAMPLE OF ONE-DIMENSIONAL POWER SPECTRUM

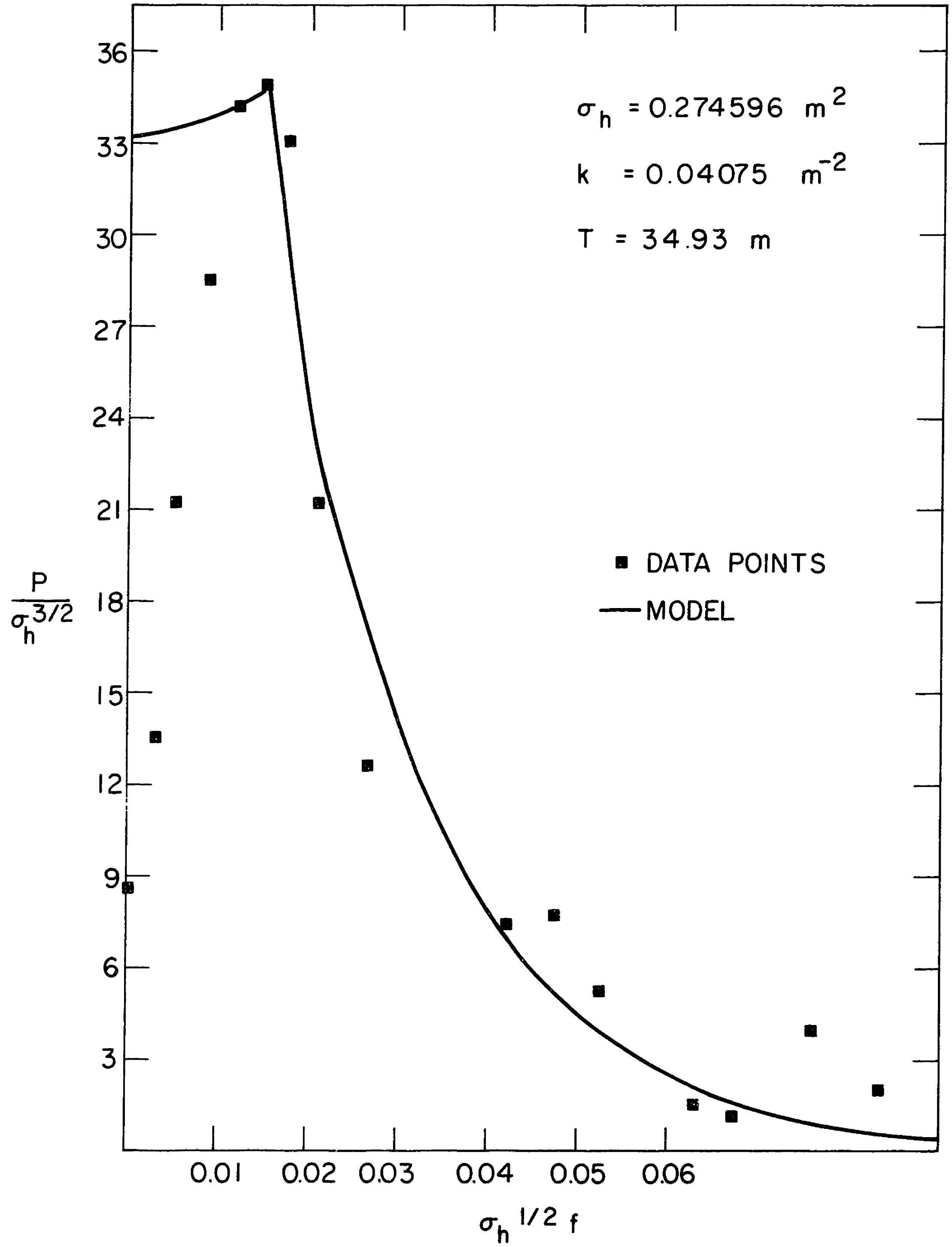


FIG. 6 EXAMPLE OF ONE-DIMENSIONAL POWER SPECTRUM

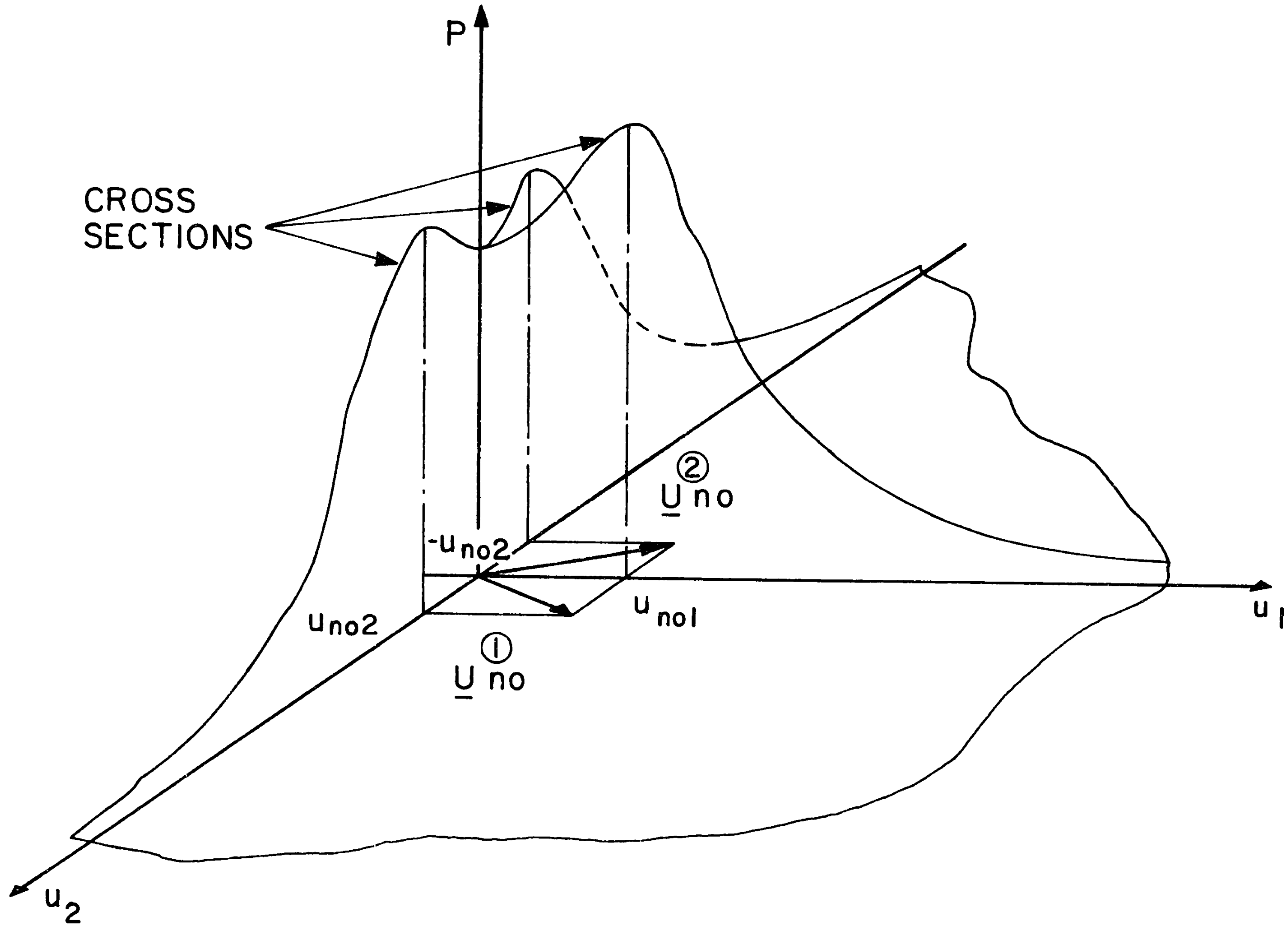


FIG. 7 EXAMPLE OF TWO-DIMENSIONAL POWER SPECTRUM

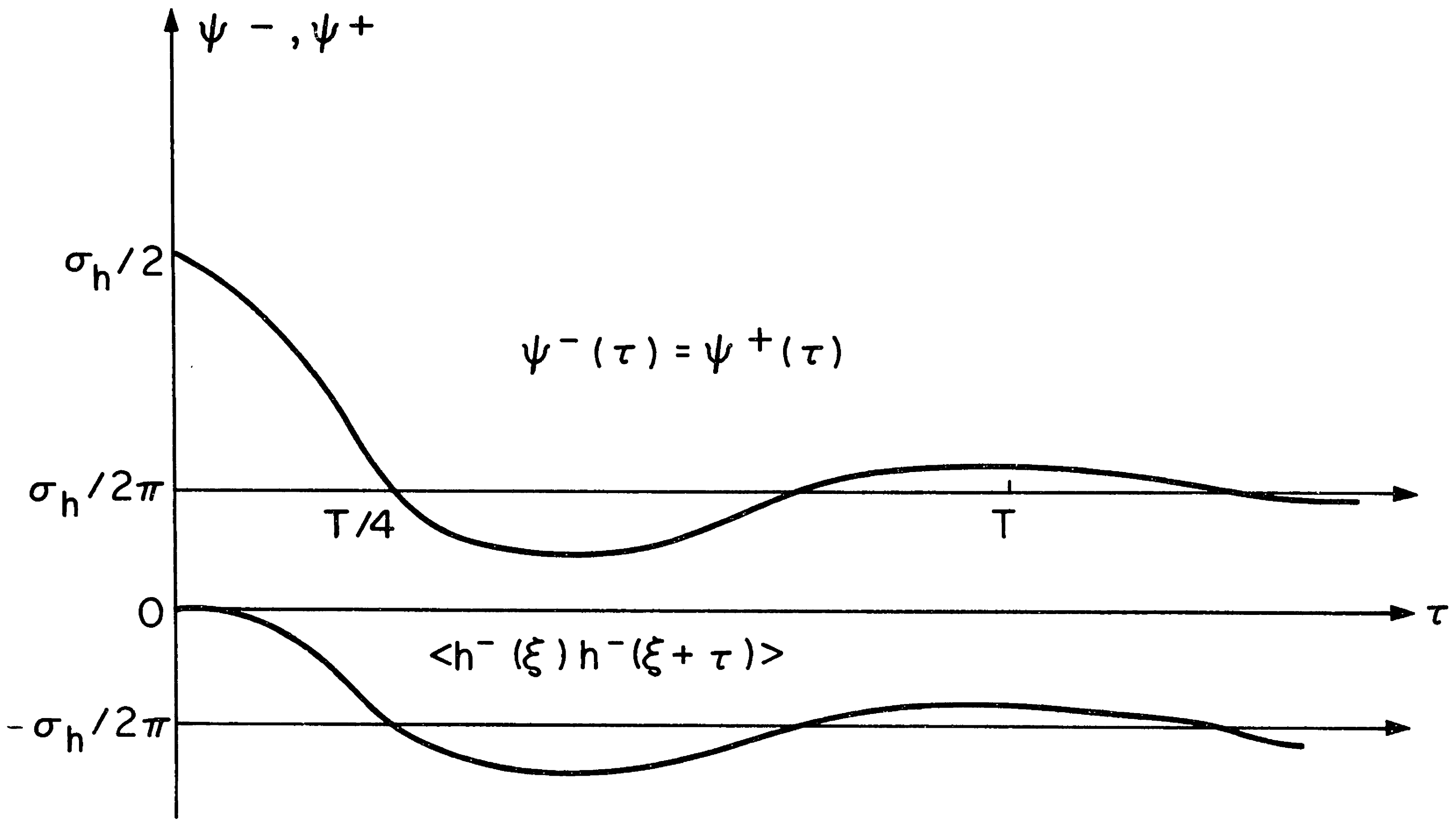


FIG. 8 ASPECT OF THE AUTO-CORRELATION FUNCTION OF h^- AND h^+

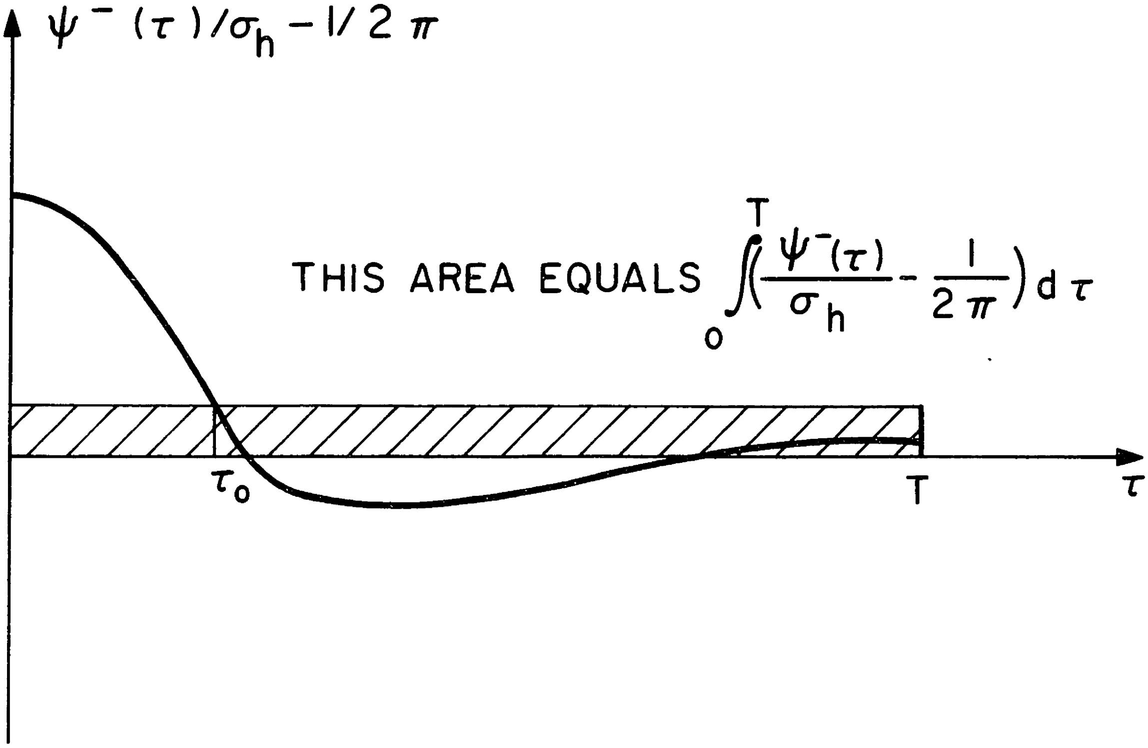


FIG. 9 POSITION OF τ_0 WITH RESPECT TO T (SECTION IV - 3 OF THE TEXT)

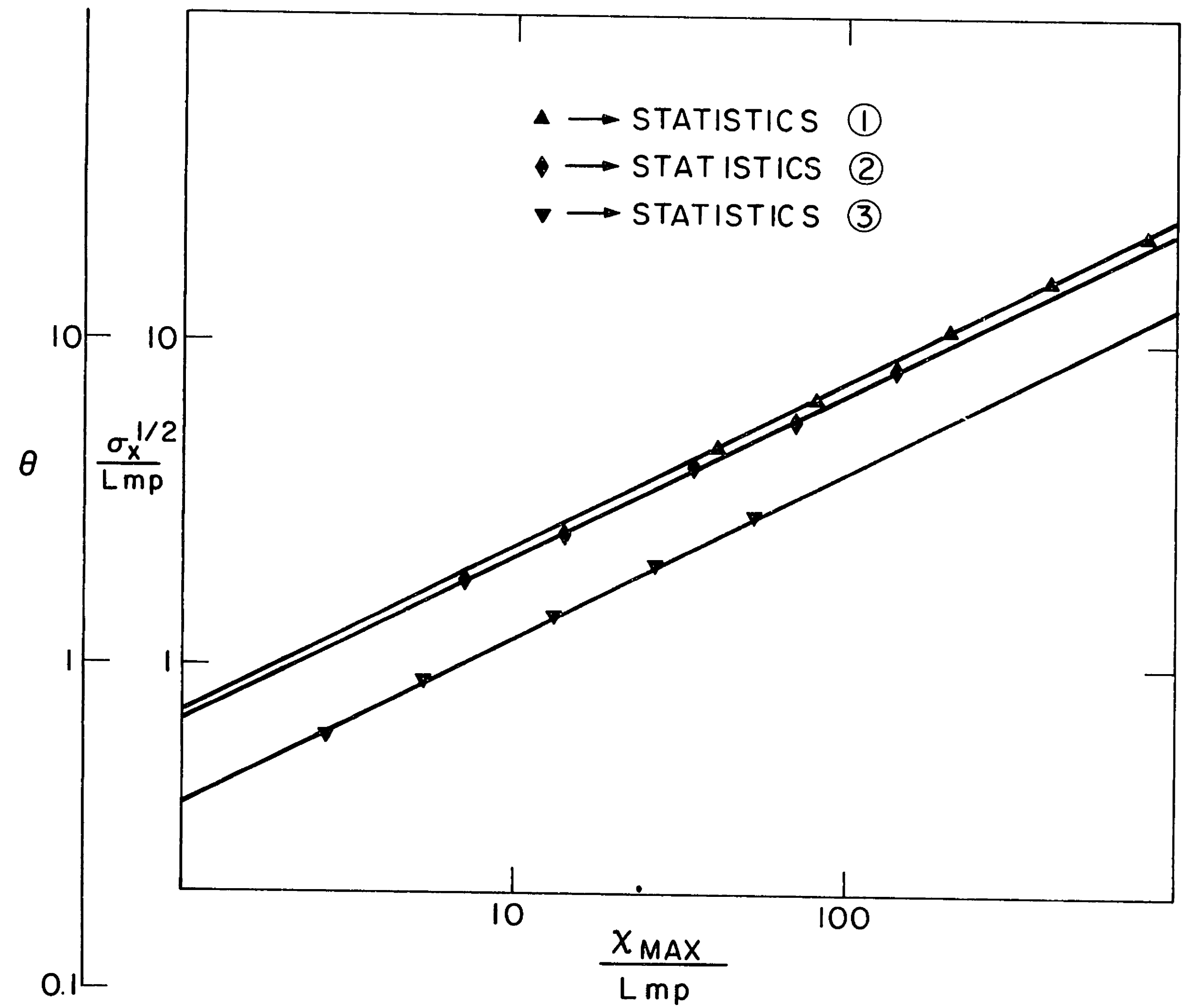


FIG. 10 RELATIONSHIP BETWEEN <X_{MAX}> AND $\sigma_X^{1/2}$.
ONE-DIMENSIONAL CASE

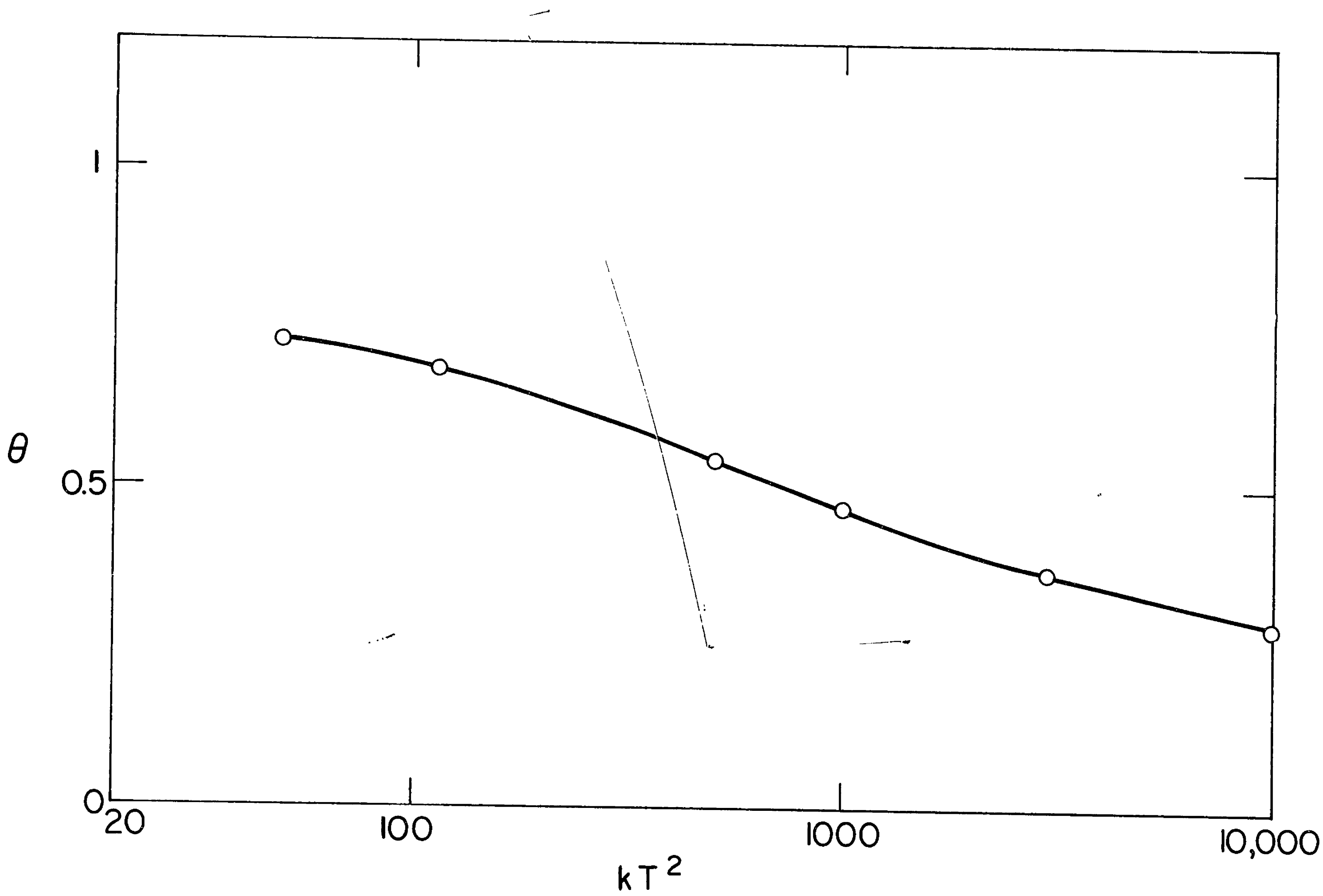


FIG. 11 SLOPE θ VERSUS NON-DIMENSIONAL GROUP kT^2.
ONE DIMENSIONAL CASE

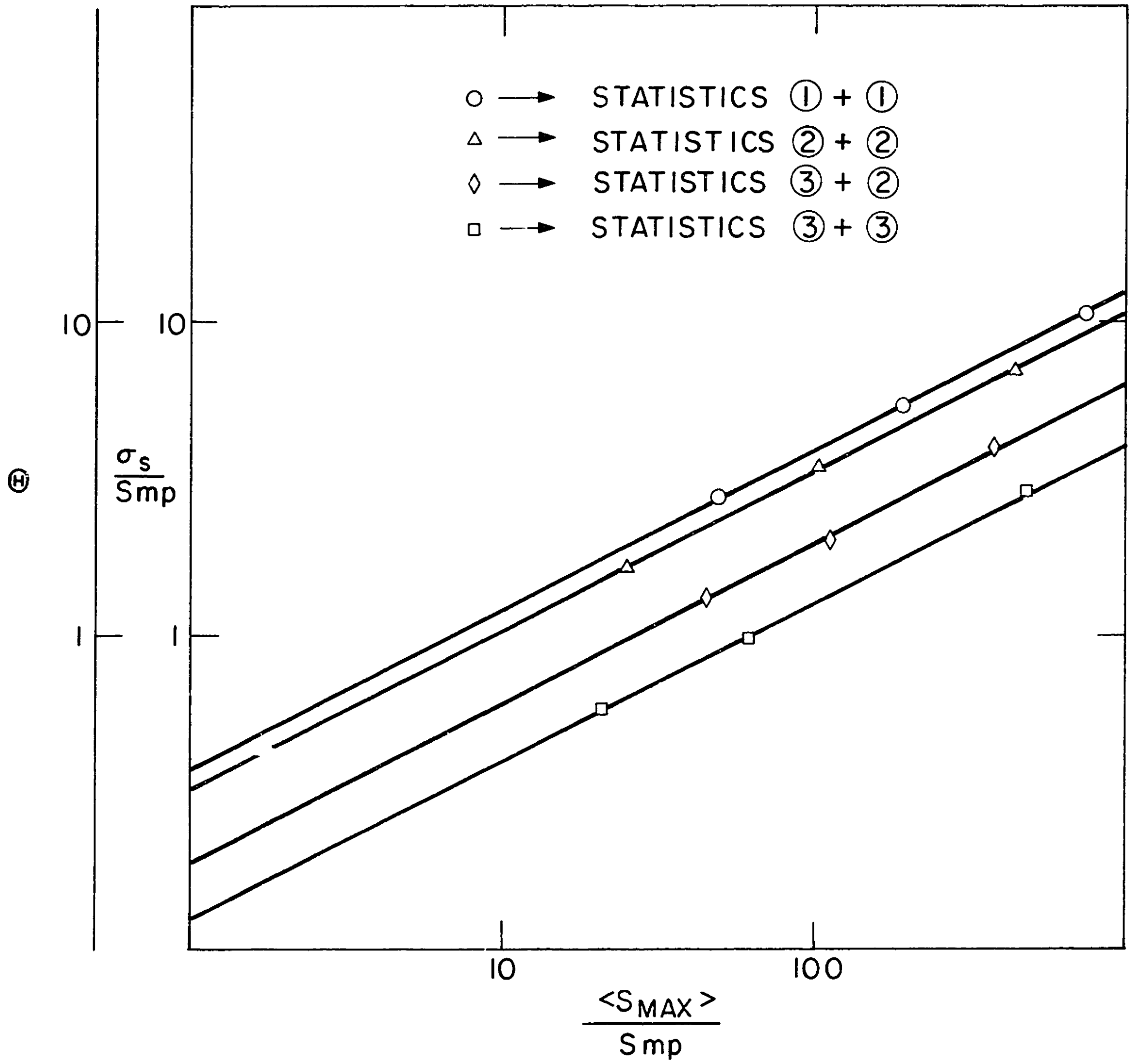


FIG. 12 RELATIONSHIP BETWEEN $<S_{MAX}>$ AND $\sigma_S^{1/2}$
TWO-DIMENSIONAL CASE

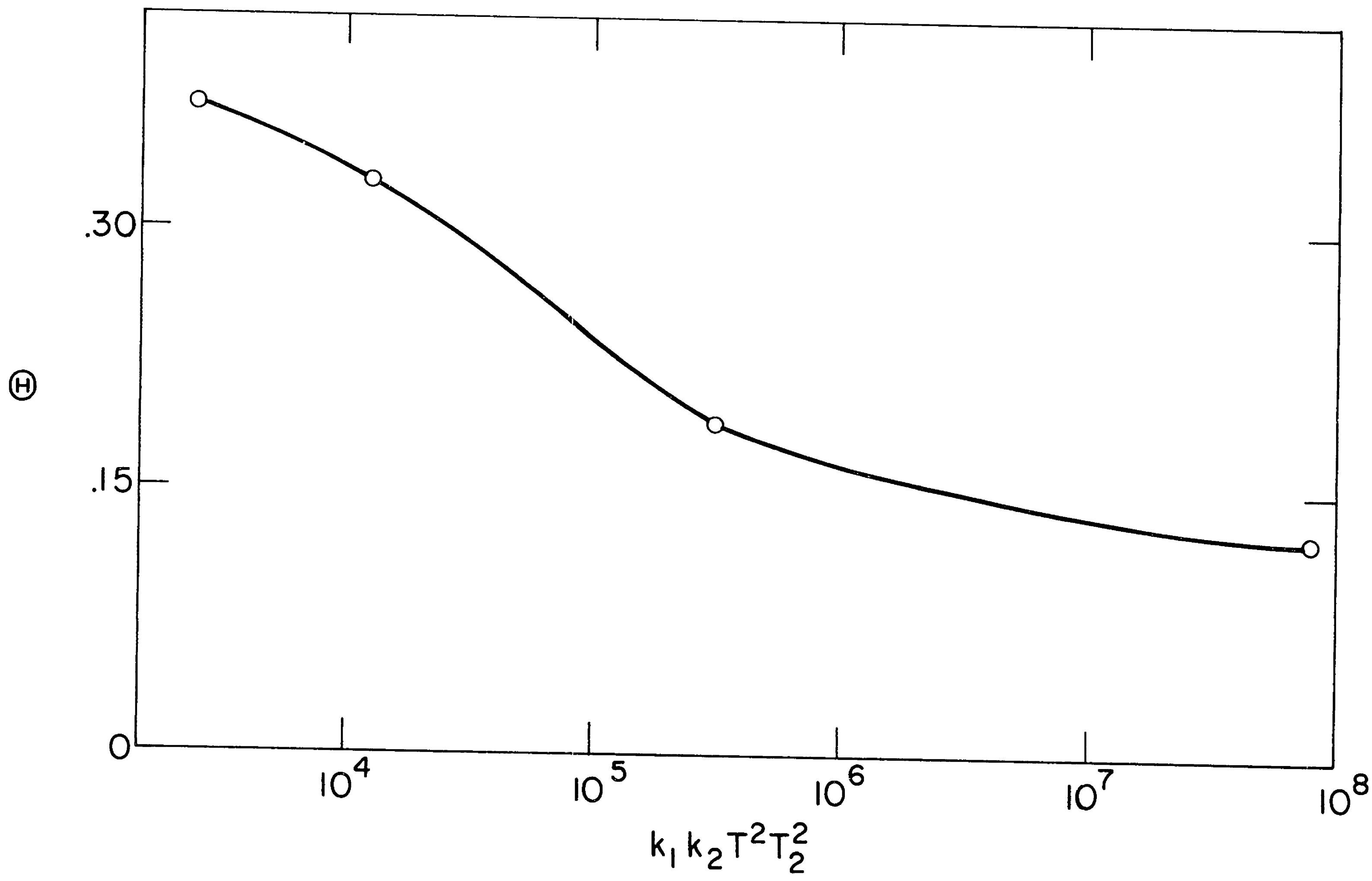


FIG. 13 SLOPE Ⓗ VERSUS NON-DIMENSIONAL GROUP $k_1 k_2 T_1^2 T_2^2$. TWO-DIMENSIONAL CASE