STRUCTURAL ASPECTS OF SYSTEM IDENTIFICATION

by

Keith Glover

B.Sc. (Eng.) Imperial College of Science and Technology,

London University
1967

E.E., M.S. E.E., Massachusetts Institute of Technology
1971

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August 1973

Signature of Author _____
Department of Electrical Engineering

Certified by _____
Thesis Supervisor

Accepted by _____
Chairman, Departmental Committee on Graduate Students

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Submitted to the Department of Electrical Engineering on August 10, 1973, in partial fulfillment of the requirements for the Degreee of Doctor of Philosophy.

ABSTRACT

The problem of identifying linear dynamical systems is studied. The approach taken is to consider structural and deterministic properties of linear systems that have an impact on stochastic identification algorithms. In particular we consider the parametrization of linear systems so that the identification problem is well-posed. (i.e. there is a unique solution and all systems in appropriate class can be represented). Firstly canonical forms for the matrix triple (A,B,C) under the transformation group, $(A,B,C) \overset{T}{\rightarrow} (TAT^{-1},TB,CT^{-1})$, (where $T \in GL(n)$) are discussed and it is shown that numerical difficulties can occur. Then an alternate set of parametrizations which do not have these difficulties are given with an associated realization algorithm. It is then assumed that a parametrization of the system matrices has been established from a priori knowledge of the system, and the question is considered of when the unknown parameters of this system can be identified from input/output observations. It is assumed that the transfer function can be asymptotically identified, and the conditions are derived for the local, global and partial identifiability of the parametrization. Then it is shown that, with the right formulation identifiability in the presence of feedback can be treated in the same way. Similarly the identifiability of parametrizations of systems driven by unobserved white noise is considered using the results from the theory of spectral factorization. Finally the problems associated with parametrizations admitting multiple representations of nonminimal systems are explored. This leads to a study of the geometrical properties of minimal and nonminimal systems (e.g. the codimension of the set of nonminimal systems in the parameter space).

THESIS SUPERVISOR: Professor Jan C. Willems

ACKNOWLEDGEMENTS

It is with great pleasure that I acknowledge the following individuals and organizations:

Professor Jan C. Willems (thesis supervisor) for his stimulating supervision and encouragement. His theoretical insight has been invaluable.

Professor Sanjoy K. Mitter (thesis reader) for many helpful discussions and his patient interest in my work.

Professors Michael Athans and Fred C. Schweppe (thesis readers) for the comments and practical insight they contributed.

Professor Roger W. Brockett of Harvard for several stimulating discussions and original ideas.

My friends and colleagues in the Electronic Systems Laboratory and elsewhere in M.I.T. for many beneficial and enjoyable discussions.

Ms. Linda Kowach for deciphering and typing the manuscript.

Mr. Arthur Giodani for the drafting.

Finally my wife for her tolerance and encouragement.

I am indebted to the Kennedy Memorial Trust for supporting me during the first two years of my graduate studies, and the Electrical Engineering Department at M.I.T. for support during the third year as a teaching assistant.

This research was carried out in the Electronic Systems Laboratory with full support for the final year extended by NASA Ames Research Center under grant NGL-22-009-124.

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NOTATION

I_n	nxn identity matrix
$O_{n,m}$	nxm zero matrix
$GL(n)$	general linear group = $\{T \in R^{nxn} \mid \det T \neq 0\}$
R^n	n-dimensional Euclidean space
$\mathbb{C}$	complex plane
$'$	A' denotes the transpose of the matrix A
$<\cdot,\cdot>$	inner product
$N(\cdot)$	null space
$R(\cdot)$	range space
$N_\varepsilon(\hat{x}) = \{x \in X \mid \|x - \hat{x}\| < \varepsilon\}$	i.e. an ε-neighborhood
C^k	k-times continuously differentiable functions
$O(h^k)$	satisfies $\lim_{h \to 0} \frac{O(h^k)}{h^{k-1}} = 0$
$\|\cdot\|$	Euclidean norm
$\|\cdot\|$	magnitude
$\delta(\tau)$	Dirac delta function
δ_{ij}	Kronecker delta = $\begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$
$\circledast$	Kronecker product

CHAPTER 1

INTRODUCTION

In order to apply the considerable advances of modern control theory it is required to have an accurate system model. Indeed in situations where accurate models exist (e.g. many aerospace problems) practical applications of modern control theory have been very successful. However in many applications accurate system models are not known a priori but must be deduced from observations of the system in operation. This is the so-called identification problem. The lack of adequate system models is perhaps the greatest single obstacle to applying modern control and filtering techniques, especially now that applications are being attempted in areas other than aerospace where system models are less well-understood (e.g. chemical processes, power systems, socio-economic systems). Identification is not only of use for subsequent control and filtering, but is often an end in itself, for example to determine whether a new piece of machinery is performing to specification, or to check the condition of an operating machine.

In this thesis we are solely concerned with the identification of linear systems, since they are a widely applicable class of systems and lend themselves to a tractable mathematical treatment. System identification algorithms can be thought of as being of two types, namely off-line and on-line. Off-line algorithms are generally given a finite set of input-output data and from this data give estimates of the system parameters. On the other hand, on-line algorithms receive

input-output data pairs and update their parameter estimate after each additional data pair, under the restriction that the complexity of the algorithm does not increase with time. An on-line algorithm is therefore a restriction of an off-line algorithm but has the advantage that it can observe a system over an arbitrarily long time interval with an essentially constant computational effort per unit time.

The more popular on-line algorithms are, stochastic approximation (Albert and Gardner (1967), Tsypkin (1972)), least squares (Åström and Eykhoff (1971)), and model reference (Whitaker (1958)), and perhaps the most famous off-line technique is Åström's maximum likelihood method (Åström and Bohlin (1966)) together with some correlation techniques (Mehra (1971)) and instrumental variable methods (Wong and Polak (1967)). The division into on- and off-line techniques is somewhat arbitrary in that it depends on the implementation, and essentially similar algorithms may be implemented both on- and off-line. An excellent survey of identification has been given by Åström and Eykhoff (1971) and contains some 230 references to which the reader is referred for further background material.

Desirable properties of identification algorithms are:

(N = number of sample points)

i) unbiased parameter estimates as $N \to \infty$

ii) efficient parameter estimates (i.e., the error covariance is close to the theoretical minimum)

iii) limited computational requirements at each N. (i.e., fast convergence if an iterative scheme is used)

Property (i) is fairly essential for any schemes but is not in fact satisfied for the classical least squares method in all but the

most elementary systems (see Åström and Eykhoff (1971)). Properties (ii) and (iii) are generally mutually exclusive, with for example the maximum likelihood method satisfying (ii) but not (iii), and stochastic approximation method satisfying (iii) but not (ii).

Essentially any identification algorithm can be considered as the minimization (by some numerical method) of some cost function (which depends on the system parameters and the observation) over the unknown system parameters. Such algorithms have two aspects, firstly a stochastic aspect which depends entirely on the choice of cost function and will determine properties (i) and (ii) above, and secondly a deterministic aspect that determines (iii) above and indeed whether there is a unique solution to the minimization problem. The study of the stochastic aspects is the area where most work on identification has been done, and this essentially involves producing new cost functions that have superior properties. This problem has been studied extensively in the statistics, econometrics and time-series analysis literature (see for example Box and Jenkins (1970)). Much of this work begins with scalar difference (or differential) equations representing the system, and does not take a state space point of view.

For scalar input or scalar output systems many of the structural problems of linear systems are not manifest because there is a natural parametrization (i.e. standard controllable or standard observable form). However for multivariable systems there are significant parametrization problems and the study of the parametrization of linear systems forms the main body of this research. Firstly suppose one

wishes to model an unknown system with a state space model of a certain dimension, then what parametrization is appropriate for identification? Clearly an arbitrary parametrization, for example the system matrices being completely free, may not be suitable because there are many distinct state space realizations of a particular input/output response. This question is considered in Chapter 2 where canonical forms are discussed and their shortcomings in this context are examined. A preferable set of parametrizations for identification are then given with an associated realization algorithm.

When standard parametrizations are used there is no immediate physical interpretation of the states or system parameters, whereas in many applications state equations can be derived where there is a natural interpretation of the states and coefficients but some of the numerical values of the coefficients will be unknown. In such cases the system matrices will be parametrized by the unknown parameters, and a natural question is whether a particular set of unknown parameters can be identified. This problem is considered in Chapter 3 where local, global and partial identifiability and identifiability in the presence of feedback, given input/output observations are considered and straightforward conditions are derived.

When a system is driven by an unobserved white noise process and the output is observed then the identification problem is more difficult. This is the spectral factorization problem which has its origins with Wiener and Masani (1958). Chapter 4 gives some background material and derives conditions for local identifiability under these conditions.

In Chapter 5 the problems associated with nonminimal systems are considered. Most system parametrizations admit multiple representations of nonminimal systems and hence at such systems the identification problem does not have a unique solution. Therefore the geometrical properties of the minimal and nonminimal systems in the parameter space is important and is considered in Chapter 5, where it is shown that at least for single input/single output systems difficulties with nonminimal systems are likely to be encountered.

CHAPTER 2

CANONICAL FORMS FOR IDENTIFICATION

2.1 Introduction

In this chapter we first examine the need in identification for parametrizations of linear dynamical systems. Then two examples of parametrizations which are canonical forms are given, including some new results on these. Some disadvantages of using canonical forms are then given by way of an illustrative example, and an alternate parametrization is proposed which avoids these difficulties. Finally a new realization algorithm is given which is computationally efficient, numerically robust and gives the resulting matrices in a nice form.

Consider the linear continuous or discrete time dynamical systems,

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad y(t) = Cx(t)$$

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k)$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$.

Suppose that it is desired to identify such a system from (possibly noisy) observations of u and y. Assume that only the input/output properties of this system can be identified. For example,

i) The Markov parameters, $H_k = C A^k B$, $k = 0,1,2,..$

ii) The transfer function, $G(s) = C(Is - A)^{-1}B$.

However, in many applications it is desired to identify a state space realization of a system, so that modern state space design and

filtering techniques can be applied directly. It will be shown in Chapter 3 that in applications with sufficient a priori information a natural state space realization is available from physical considerations. However in the present chapter we will assume that little or no a priori information is available, except perhaps the McMillan degree (or order) of the system (see Brockett (1970)).

A knowledge of the Markov parameters or the transfer function of a system does not induce a unique state space realizations. (Finding a state space realization of a given input/output response is called the realization problem, see Ho and Kalman (1970)). Indeed all minimal realizations of a particular transfer function are related by a similarity transformation, T, as follows:

Fact 2.1 (see Brockett (1970)).

If the triples (A_i,B_i,C_i), $i = 1,2$, represent controllable and observable systems (i.e. minimal) then

$$C_1(Is - A_1)^{-1}B_1 = C_2(Is - A_2)^{-1}B_2, \ \forall\, s \in \mathbb{C}\,.$$

if and only if there exists $T \in GL(n)$ such that

$$T A_1 T^{-1} = A_2$$

$$T B_1 = B_2$$

$$C_1 T^{-1} = C_2$$

■

Therefore there are infinitely many equivalent realizations of a particular input/output response. Hence in any identification algorithm which would be minimizing some external cost function over the system parameters, if the complete A,B and C matrices are left as free

parameters there will not be a unique solution. This implies that these minimization algorithms could become ill-posed and in any case they cannot be expected to converge to the same solution for different conditions. Therefore it is necessary to restrict the system matrices to a subset so that there is a unique solution, for example the A,B and C matrices could be parametrized in some way, where,

Definition 2.1: A parametrization of a topological space S is a C^1 mapping from $\Omega \subset R^q$ into S. ■

Canonical forms are one possible solution to the uniqueness problem, and are now introduced.

The relation E given by $(A_1,B_1,C_1)E(A_2,B_2,C_2)$ if there exists $T \in GL(n)$ such that,

$$T A_1 T^{-1} = A_2, \quad T B_1 = B_2, \quad C_1 T^{-1}, \quad C_2,$$

is an equivalence relation, which is identical to equivalence of the transfer functions if the systems are minimal, (Fact 2.1), but for nonminimal systems equivalence of transfer functions does not necessarily imply E-equivalence. The set of systems equivalent to a particular system is termed an orbit in the parameter space (A,B,C). A canonical form is then a subset of the parameter space which intersects each orbit exactly once, or more precisely;(see MacLane and Birkhoff (1967)).

Definition 2.2: Let E be an equivalence relation on the set S, then a canonical form for S under E is a subset $C \subset S$ such that

1) $\forall\, s \in S$ there exists $c \in C$ such that $s\ E\ c$.

and 2) $c_1, c_2 \in C$, with $c_1\ E\ c_2 \Longrightarrow c_1 = c_2$. ■

The choice of the set S for the present problem is of importance and there are four natural candidates. Firstly S could be all matrix triples (A,B,C) which are both controllable and observable, but this set is very difficult to parametrize (see Chapter 5). Secondly S could be the arbitrary matrices (A,B,C), but although this is a well-behaved space it introduces unnecessary complications which are avoided if one chooses S to be the set of controllable (or observable) matrix triples (A,B,C). This latter set is in fact chosen since it includes all the minimal systems, in which we are primarily interested, and is technically most tractable.

Canonical forms may therefore be particularly useful in identification since they overcome the nonuniqueness problems of realization theory. To illustrate these points we now give some simple examples.

Example 2.1

Consider the scalar system (a,b,c), n = m = p = 1, with $b \neq 0$ (i.e. controllable). Then the orbits are given by $b = t\hat{b}$, $c = t^{-1}\hat{c}$, $a = \hat{a}$, with $t \neq 0$ (see Figure 2.1). A canonical form for this case is $a \in \mathbb{R}$, $c \in \mathbb{R}$, $b = 1$ and corresponds to the vertical line given in Figure 2.1. It will be noted that the canonical form intersects each orbit once and no orbit twice, as required. ■

Note that the canonical form in Example 2.1 is a very simple subset of the parameter space, that this is not always the case is illustrated in the following example.

Example 2.2

Consider the system n = 1, m = p = 2, $\left(a, (b_1, b_2), \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right)$,

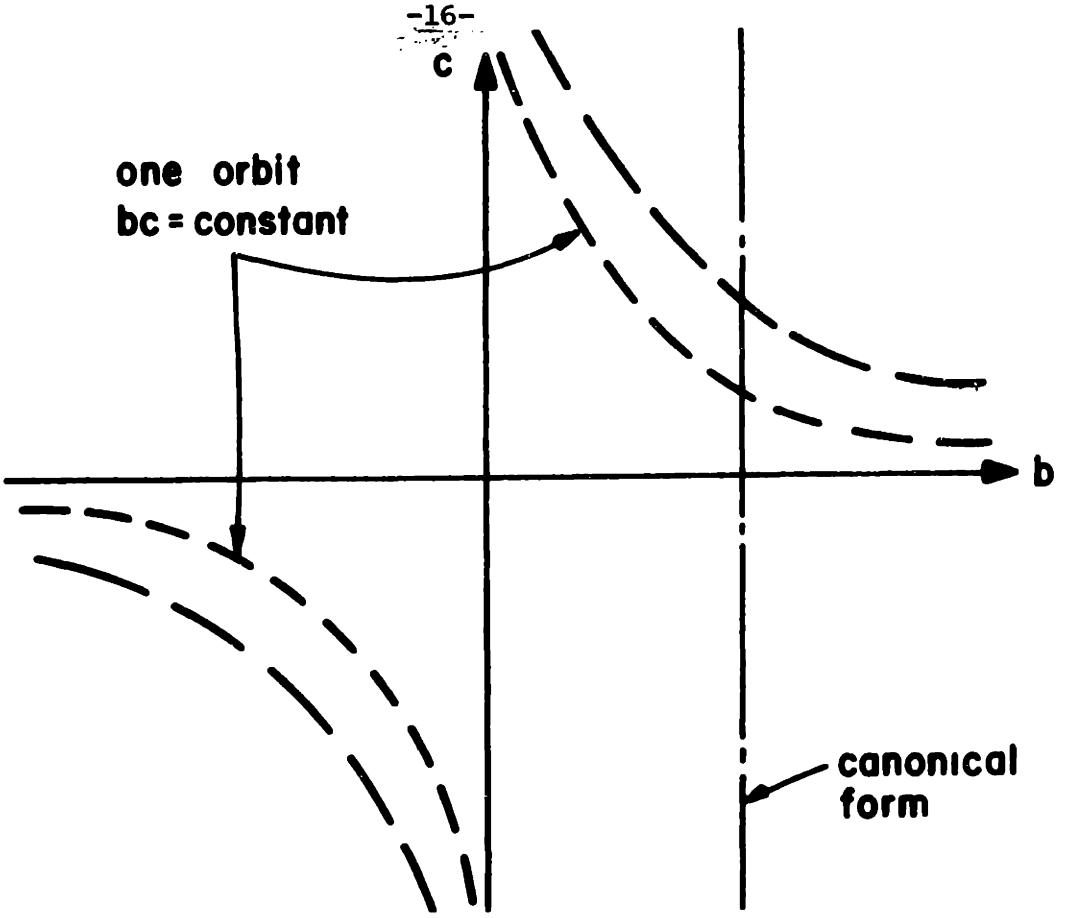

Figure 2.1: Orbits in Parameter Space for a Scalar System

with $(b_1, b_2) \neq (0, 0)$. Now the orbits are given by

$$a = \text{constant}$$

$$(b_1, b_2) = t(\hat{b}_1, \hat{b}_2)$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{t} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \end{pmatrix} \quad , \text{ where } t \in R \text{ and } t \neq 0.$$

In this case a canonical form is given by the set,

$$\left\{ \left(a, (1, b_2), \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right) \;\middle|\; a, b_2, c_1, c_2 \in R \right\}$$

$$\cup \left\{ \left(a, (0, 1), \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right) \;\middle|\; a, c_1, c_2 \in R \right\}$$

The projections of the orbits into (b_1, b_2) space are given in Figure 2.2.

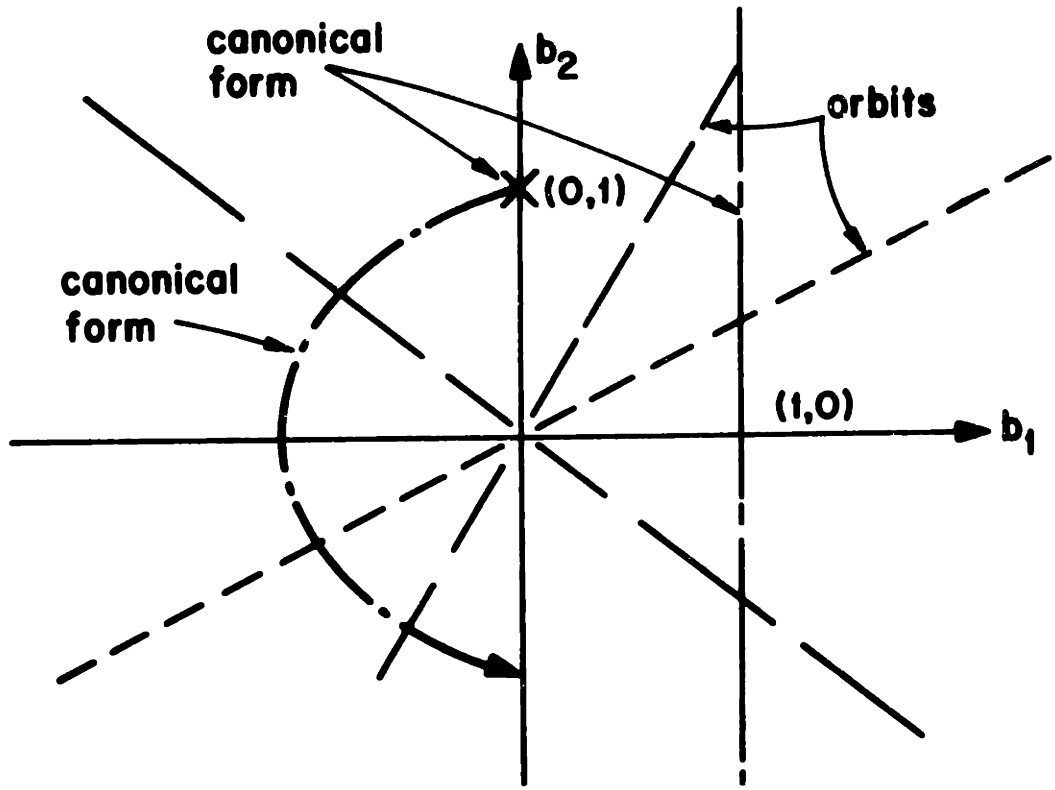

Figure 2.2

Note that this canonical form is not a connected subset of the parameter space. Although a connected subset which is a canonical form is possible, for example the half circle given by,

$$\left\{ \left(a, (\sin\theta, \cos\theta), \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right) \middle| \; a, c_1, c_2 \in R \; , \theta \in [0,\pi) \right\}$$

the resulting canonical form is not particularly satisfactory since some continuous paths in the parameter space are represented by discontinuous paths in the canonical form. ■

As illustrated by the above examples if a canonical form is known for systems of a particular dimension, then the identification problem will have a unique solution if the system matrices are restricted to be in the canonical form.

Example 2.1 is a special case of the standard controllable form for single input systems, i.e.

$$A = \left[\begin{array}{c|c} 0_{n-1,1} & I_{n-1} \\ \hline \multicolumn{2}{c}{\alpha'} \end{array}\right], \quad B = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix}, \quad C = \text{free.}$$

with $\alpha \in R^n$.

However as illustrated by example 2.2 such a nice canonical form is not always possible. In general for the multi input/multi output systems there does not exist a single parametrization of the system matrices that is also a canonical form under the similarity transformation. Canonical forms that consist of a family of parametrizations have been derived by many authors (e.g. Popov (1972), Mayne (1972), Luenberger (1967)), with probably the nicest derivation given by Popov, which is described in the next section. Related results for transfer function are in Rosenbrock (1970).

2.2 Popov's Canonical Form

In this section we summarize the canonical form for multivariable linear dynamical systems given implicitly in Popov (1972). The canonical forms rely on finding a complete set of independent invariants for the pair (A,B), assumed controllable, under the transformation $(A,B) \xrightarrow{T} (T A T^{-1}, T B)$. An invariant is a property of a system which does not change under the transformation. Completeness of a set of invariants means that the set of invariants for any

particular system is sufficient to specify its orbit. Independence means that for every set of values for the invariants there exists a system with these invariants.

Consider the matrix,

$$W = [b_1, b_2..b_m, Ab_1, Ab_2, \ldots\ldots\ldots, A^{n-1} b_1, A^{n-1} b_2, .., A^{n-1} b_m]$$

(where b_i = i^{th} column of B), which will have rank n by the controllability assumption.

Definition 2.3: The i^{th} Kronecker invariant (or index), n_i, is the smallest positive integer such that the vector $A^{n_i} b_i$ is a linear combination of its antecedants (i.e. vectors $A^k b_j$ such that $km + j < n_i m + j$). ■

It can be shown that the set of vectors,

$$P = [b_1, Ab_1, \ldots, A^{n_1-1} b_1, \ldots\ldots\ldots, b_m, Ab_m, \ldots., A^{n_m-1} b_m]$$

which are called regular by Popov, are independent and are in fact the first set of n independent vectors that occur in the matrix W, when moving from left to right. Clearly,

$$n_1 + n_2 + \ldots\ldots\ldots + n_m = n.$$

It is also shown that every non-regular vector is a linear combination of its regular antecedants. The following theorem is the main result in Popov (1972).

Theorem 2.2: A complete set of independent invariants for the pair (A,B), under the transformation $(A,B) \overset{T}{\rightarrow} (T A T^{-1}, T B)$ where $T \in GL(n)$,

are the Kronecker invariants, $(n_1, n_2, \ldots, n_m)$, and the real numbers α_{ijk} defined implicitely (but uniquely) as follows;

$$A^{n_i} b_i = \sum_{j=1}^{i-1} \sum_{k=0}^{\min(n_i, n_j - 1)} \alpha_{ijk} A^k b_j$$

$$+ \sum_{j=i}^{m} \sum_{k=0}^{\min(n_i, n_j) - 1} \alpha_{ijk} A^k b_j \quad \blacksquare$$

Now using the above set of invariants a canonical form can be derived as follows.

Corollary 2.3: A canonical form for the controllable pair (A,B) under the transformation $(A,B) \xrightarrow{T} (T A T^{-1}, T B)$ is given by the following family of parametrizations one for each set of indices, $(n_1, n_2 \ldots n_m)$, such that $n_1 + n_2 \ldots + n_m = n$ and $n_i \geq 0$, $\forall i$.

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \cdots A_{1m} \\ A_{21} & \\ \vdots & \vdots \\ A_{m1} & \cdots\cdots\cdots A_{mm} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_{11} & B_{12} \cdots B_{1m} \\ B_{21} & \\ \vdots & \vdots \\ B_{m1} & \cdots\cdots\cdots B_{mm} \end{bmatrix}$$

where

$$A_{ji} \underset{(n_j \times n_i)}{=} \begin{cases} \left[\begin{array}{c|c} 0_{1,n_i-1} & \begin{array}{c} \alpha_{iio} \\ \\ \alpha_{iil} \\ \vdots \\ \alpha_{ii}{}_{n_i-1} \end{array} \\ I_{n_i-1} & \end{array}\right] & \text{for } j = i. \\ \left[\begin{array}{c|c} & \begin{array}{c} \alpha_{ijo} \\ \\ \alpha_{ijl} \\ \vdots \\ \alpha_{ijk} \\ 0 \\ \vdots \\ 0 \end{array} \\ 0_{n_j,n_i-1} & \end{array}\right], & \begin{array}{l} \text{for } j \neq i \\ k = \min(n_i, n_j-1), \\ \qquad j < i \\ k = \min(n_i, n_j)-1, \\ \qquad j > i \end{array} \end{cases}$$

$$B_{ji} \underset{(n_j \times 1)}{=} \begin{cases} 0_{n_j,1} & j \neq i \qquad n_i \neq 0. \\ \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} & j = i \qquad n_i \neq 0. \end{cases}$$

If rank B < m then $n_i = 0$ for some i and

$$B_{ji} \underset{(n_j \times 1)}{=} \begin{cases} \begin{bmatrix} \alpha_{ijo} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} & j < i, \quad n_j \neq 0, \quad n_i = 0 \\ 0_{n_j,1} & j \geq i. \end{cases}$$

■

Notice that the canonical form consists of a finite number of parametrizations, one for each set of Kronecker indices. If rank B = m is assumed, there are in fact $\binom{n-1}{m-1}$ sets of possible Kronecker indices and hence $\binom{n-1}{m-1}$ different parametrizations are required to make up the canonical form. Clearly if rank B < m is possible a larger number of parametrizations are required.

The transformation that is of interest in identification is

$$(A,B,C) \overset{T}{\rightarrow} (T A T^{-1}, T B, C T^{-1}).$$

A canonical form under this transformation with (A,B) controllable is given by A and B in the form of Corollary 2.3 with C completely free. This is a canonical form because given any minimal triple (A,B,C) there is a unique transformation (= P^{-1}, see Theorem 2.4) taking the pair (A,B) to the canonical form, and hence the addition of the C matrix does not alter this. Note that this canonical form represents non-minimal input/output responses in a nonunique way, which is because input/output equivalent nonminimal systems are not necessarily related by a nonsingular matrix as above.

2.3 Other Canonical Forms

In this section a slightly different approach to canonical forms is reported, and is essentially that given in Luenberger (1967), that is transformations are derived which bring arbitrary matrices (A,B) into special forms.

It is shown in Luenberger (1967) that if the pair (A,B) is controllable then there exists an ordered set of integers, $\kappa = (k_1, k_2, \ldots, k_m)$, such that,

$$k_1 + k_2 + \ldots + k_m = n$$

and $\det P(A,B,\kappa) \neq 0$

where $P(A,B,\kappa) = [b_1, Ab_1, \ldots, A^{k_1-1}b_1, \ldots, b_m, Ab_m, \ldots, A^{k_m-1}b_m]$

Now if we assume that such a set of integers, κ, are given we can state the following result.

<u>Theorem 2.4:</u> Given the set $\kappa = (k_1, k_2 \ldots, k_m)$, and under the assumption on (A,B) that $\det P(A,B,\kappa) \neq 0$, the following forms, $(\tilde{A},\tilde{B})$, constitute a canonical form for (A,B) under the transformation $(A,B) \overset{T}{\rightarrow} (T A T^{-1}, T B)$.

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \cdots A_{1m} \\ A_{21} & \\ \vdots & \\ A_{m1} & \cdots A_{mm} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_{11} & B_{12} \cdots B_{1m} \\ B_{21} & \vdots \\ \vdots & \\ B_{m1} & \cdots B_{mm} \end{bmatrix}$$

$$A_{ji}_{(k_j \times k_i)} = \begin{cases} \left[\begin{array}{c|c} 0_{1,k_i-1} & \alpha_{iio} \\ \hline & \vdots \\ I_{k_i-1} & \alpha_{iik_i-1} \end{array}\right] & j = i \\ \left[\begin{array}{c|c} & \alpha_{ijo} \\ & \vdots \\ 0_{k_j,k_i-1} & \alpha_{ijk_j-1} \end{array}\right] & j \neq i \end{cases}$$

$$B_{ji} \underset{(k_j \times 1)}{=} \begin{cases} \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0_{k_i,1} \end{bmatrix} & j = i \\ & \\ \begin{bmatrix} \alpha_{ijo} \\ \alpha_{ij1} \\ \cdot \\ \cdot \\ \alpha_{ijk_j-1} \end{bmatrix} & j \neq i,\ k_i = 0 \end{cases} \qquad j \neq i,\ k_i \neq 0$$

Further the transformation, T, taking any (A,B) to this form is given by $T = P^{-1}(A,B,K)$.

Proof: That the transformation $T = P^{-1}$ takes any (A,B), satisfying $\det P(A,B,K) \neq 0$, to the above form is stated in Luenberger (1967) and is easy to verify by straightforward manipulation. It is also easy to demonstrate that for any pair $(\tilde{A},\tilde{B})$ in the above form, $P(\tilde{A},\tilde{B},K) = I_n$. Now suppose two systems in the above form are equivalent, say, $\tilde{A}_1 = T\,\tilde{A}_2\,T^{-1}$, $\tilde{B}_1 = T\,\tilde{B}_2$. Then,

$$I_n = P(\tilde{A}_1, \tilde{B}_1, K) = P(\tilde{A}_2, \tilde{B}_2, K)$$

$$\therefore I = T\,P(\tilde{A}_1, \tilde{B}_1, K)$$

$$\therefore T = I$$

and $(\tilde{A}_1, \tilde{B}_1) = (\tilde{A}_2, \tilde{B}_2)$.

That is no two equivalent systems have distinct representations. ■

Notice that if we were not restricted to a particular set κ this would not be a canonical form, because a particular system may have several sets κ_i for which $\det(A,B,\kappa_i) \neq 0$ which is why the more complicated forms of Popov etc. are required.

Also observe that if the indices $\kappa_i = n_i$, $i = 1,..,m$, the Kronecker indices, then the resulting form will be the same as that in Corollary 2.3, since the appropriate entries will be zero.

Now a different canonical form will be derived (also from Luenberger (1967)) where the transformation is more complex, as follows.

Define: $$\sigma_i = \sum_{j=1}^{i} \kappa_i$$

$$e_i = \text{the } \sigma_i^{\text{th}} \text{ row of } P^{-1}(A,B,\kappa),\ i = 1,\ldots,n.$$

$$T = \begin{bmatrix} e_1 \\ e_1A \\ \vdots \\ e_1A^{k_1-1} \\ e_2 \\ \vdots \\ e_m \\ e_mA \\ \vdots \\ e_mA^{k_m-1} \end{bmatrix}$$

$$\tilde{A} = T A T^{-1}$$

$$\tilde{B} = T B.$$

Theorem 2.5: Using the above definitions the forms of $\tilde{A}$ and $\tilde{B}$ are as follows, (assume $\kappa_i > 0 \ \forall i$),

$$\tilde{A} = \begin{bmatrix} A_{11} & \cdot & \cdot & A_{1m} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ A_{m1} & \cdot & \cdot & A_{mm} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_{11} & \cdot & \cdot & B_{1m} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ B_{m1} & \cdot & \cdot & B_{mm} \end{bmatrix}$$

where,

$$A_{ij} \ (k_i \times k_j) = \begin{cases} \left[\begin{array}{c|c} 0_{k_i-1,1} & I_{k_i-1} \\ \hline \gamma_{ii0} \cdot \cdot \cdot \gamma_{iik_i-1} & \end{array}\right] & j = i \\ \left[\begin{array}{c} 0_{k_i-1,k_j} \\ \hline \gamma_{ij0} \ \gamma_{ij1} \cdot \cdot \cdot \gamma_{ijk_j-1} \end{array}\right] & j \neq i, \ k_j \leq k_i \\ \left[\begin{array}{c} 0_{k_i-1,k_j} \\ \hline \gamma_{ij0} \cdot \cdot \gamma_{ijk_i-1} 0 \cdot \cdot 0 \end{array}\right] & j \neq i, \ k_j > k_i \end{cases}$$

$$B_{ji} \ (k_j \times 1) = \begin{cases} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} & j = i \\ 0_{k_i,1} & j \neq i, \ k_i \geq k_j \\ \begin{bmatrix} 0 \\ \cdot \\ 0 \\ \gamma_{ijk_i} \\ \vdots \\ \gamma_{ijk_j-1} \end{bmatrix} & j \neq i, \ k_i < k_j. \end{cases}$$

Further if $(k_i,\ldots,k_m)$ are chosen to be the Kronecker invariants, then for $i \neq j$.

$$B_{ij} = \begin{cases} \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \gamma_{ijk_i-1} \end{bmatrix} & \text{if } k_j < k_i \text{ and } j > i. \\ 0_{k_i,1} & \text{otherwise} \end{cases}$$

and this latter form is canonical.

Proof:

1) Structure of $\tilde{A}$

We have that $T A = \tilde{A} T$

i.e.

$$\begin{bmatrix} e_1A \\ e_1A^2 \\ \vdots \\ e_1A^{k_1} \\ \vdots \\ e_mA \\ \vdots \\ e_mA^{k_m} \end{bmatrix} = \tilde{A} \begin{bmatrix} e_1 \\ e_1A \\ \vdots \\ e_mA^{k_m-1} \end{bmatrix}$$

Since T is non-singular (see Luenberger (1967), Appendix II),

the j^{th} row of $\tilde{A}$ = [0 0 ..... 0 1 0 ..... 0]

(if $j \neq \sigma_i$ for any i) $(j+1)^{th}$ position.

Let the σ_i^{th} rows of $\tilde{A}$ = $[\gamma_{i1o}\cdots\gamma_{i1k_1-1}, \gamma_{i2o}\cdots\cdots\cdots\gamma_{imk_m-1}]$

Then $\{\gamma_{ijk}\}$ are specified uniquely by the equation,

$$(*) \qquad e_i A^{k_i} = \sum_{j=1}^{m} \sum_{k=0}^{k_j-1} \gamma_{ijk} A^k b_j \qquad i = 1,2,\ldots m$$

It just remains to show that,

$$\gamma_{ijk} = 0 \qquad \text{for } j \text{ such that } k_j > k_i$$

$$\text{and } k = k_i, k_i + 1, \ldots, k_j - 1.$$

This is easily proved inductively using the following fact.

Fact: If for some s satisfying $k_j - 2 \geq s \geq k_i$,

$$\gamma_{ijk} = 0 \qquad \forall\ j \text{ such that } k_j \geq s + 2$$

$$\text{and } k \text{ such that } k_j - 1 \geq k \geq s + 1$$

then $\gamma_{ijk} = 0\ \forall$ j such that $k_j \geq s + 1$,

$$\text{and } k \text{ such that } k_j - 1 \geq k \geq s.$$

Proof: Let $\ell \in \{1,\ldots,m\}$ be any integer such that $k_\ell \geq s + 1$. Take the inner product of equation (*) with $A^{k_\ell - s - 1} b_\ell$ to give

$$(**) \quad e_i A^{k_i + k_\ell - s - 1} b_\ell = \gamma_{i\ell s}\, e_\ell A^{k_\ell - 1} b_\ell + \sum_{k=0}^{s-1} \gamma_{i\ell k} A^{k + k_\ell - s - 1} b_\ell$$

$$+ \sum_{\substack{j=1 \\ j \neq \ell}}^{m} \sum_{k=0}^{\min(s, k_j - 1)} \gamma_{ijk} A^{k + k_\ell - s - 1} b_\ell$$

Now by definition of e_i as the σ_i^{th} row of P^{-1} we have

$$e_i A^k b_j = \begin{cases} 1 & \text{if } i = j, \quad k = k_i - 1 \\ 0 & \text{if } i = j, \quad k = 0,1,\ldots,k_i - 2 \\ 0 & \text{if } i \neq j, \quad k = 0,1,\ldots k_j - 1. \end{cases}$$

Therefore in equation (**) since $i \neq \ell$ ($k_\ell \geq s + 1 \geq k_i + 1$), we have

LHS = 0 since $k_i + k_\ell - s - 1 \leq k_\ell - 1$

first summation = 0 since $k + k_\ell - s - 1 \leq k_\ell - 2$

second summation = 0 since $k + k_\ell - s - 1 \leq k_\ell - 1$

$$e_\ell A^{k_\ell - 1} b_\ell = 1$$

Therefore $\gamma_{i\ell s} = 0$ and this can be repeated for all j such that $k_j \geq s + 1$ proving the result.

2) Structure of $\tilde{B}$

We have $\tilde{B} = T B$ and therefore

$$B_{ij} = \begin{bmatrix} e_i \\ e_i A \\ \vdots \\ e_i A^{k_i - 1} \end{bmatrix} b_j$$

and using the values for $e_i A^k b_j$ given previously the result follows easily.

If $(k_1, k_2, \ldots, k_m)$ are the Kronecker indices and j is such that $k_j < k_i$, then $A^k b_j$ is not a regular vector if $k = k_j, k_j + 1, .., k_i - 1$, and is therefore a linear combination of its regular antecedants. (see

section 2.2 for definitions). Therefore $e_i A^k b_j = 0$ unless $A^{k_i-1} b_i$ is an antecedant of $A^k b_j$, which occurs only if $k = k_i - 1$ and $j > i$, thus proving the result.

3) Canonical Form

That the above form is canonical follows from the observations that for any set of Kronecker invariants, (i) the above form has a representation of any (A,B) with these invariants, and (ii) this form and Popov's canonical form have the same number of real-valued free parameters.

Note that unless all the Kronecker indices are equal there will always be some zeros in the σ_i^{th} rows of the A matrix. This canonical form is preferable to that of Popov when the effect of feedback is being studied, since feedback will only alter the σ_i^{th} rows of A. However note that when the k_i are not the Kronecker indices the parametrization is not as simple as that of Theorem 2.4.

A canonical form for the controllable triple (A,B,C) under the transformation $(A,B,C) \xrightarrow{T} (T A T^{-1}, T B, C T^{-1})$ is again given by (A,B) in the above form with C free.

2.4 Remarks on using Canonical Forms for Identification

Two canonical forms for linear dynamical systems have been derived in sections 2.2 and 2.3. Both of these have parametrized the pair (A,B) under the assumption of controllability, and made the C matrix arbitrary, dual canonical forms can be found if the pair (A',C') is parametrized in the same way assuming observability, with the B matrix arbitrary.

For single input systems the canonical form given in section 2.3 is the well-known standard controllable form, and for single output systems the standard observable form is given by Popov's canonical form for the pair (A',C'). In both these cases a single canonical parametrization is required, as given by the previous canonical forms.

The canonical forms include exactly one parametrization with nm + np degrees of freedom, the so called generic case, and this parametrization can represent "almost all" systems of order n. (This occurs when the first n columns of [B, AB,..., A^{n-1} B] are independent). The other parametrizations which will have fewer degrees of freedom are then necessary to represent the boundary of the generic parametrization, and since this boundary is geometrically unpleasant many extra parametrizations may be required.

One may be tempted to suggest ignoring the lower order parametrizations since they have measure zero in some sense, but this is a fallacious argument for two reasons. Firstly it is analagous to saying that almost all square matrices are invertible, so ignore singular matrices, which is clearly a numerically ill-advised step. Secondly the non-generic systems are bound to occur in some natural situations, for

example the following example of two systems connected in parallel.

Example 2.3: Suppose a system is composed of the parallel connection of two subsystems with distinct inputs,

i.e. $\dot{x}_1 = A_1x_1 + B_1u_1, \quad \dot{x}_2 = A_2x_2 + B_2u_2$

$$y = y_1 + y_2 = C_1x_1 + C_2x_2$$

where $x_i \in R^{n^{(i)}}$, $u_i \in R^{m^{(i)}}$ $\quad i = 1,2.$

Then the composite system has

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

and the controllability matrix,

$$W = \begin{bmatrix} B_1 & 0 & A_1B_1 & 0 & \cdots & A_1^{n-1}B_1 & 0 \\ 0 & B_2 & 0 & A_2B_2 & 0 & A_2^{n-1} & B_2 \end{bmatrix}$$

where $n = n^{(1)} + n^{(2)}$.

Now in many cases the first n columns of W are dependent, regardless of the Kronecker indices of the subsystems.

e.g. for $n^{(1)} = 3$, $n^{(2)} = 1$ $m^{(1)} = 1$, $m^{(2)} = 1$.

Then the fourth column of W is dependent on the second, so the system cannot be represented by the generic parametrization. ■

In some sense such cases are not likely to occur because we have assumed zero coupling between the states of the subsystems, which

will occur almost never if one takes the usual measure on the real line. However such situations often occur in practical composite systems and cannot be ignored.

A major disadvantage of using canonical forms for identification is that the realization of the systems close to the boundary of a particular parametrization become numerically ill-posed, and this seems to be inherent in canonical forms of this type. Some difficulties of this type are illustrated by the following example.

Example 2.4: Let $n = 3$, $m = 2$, $p = 2$. For the Popov canonical form, there are two parametrizations for this case if B is assumed to have rank 2.

1) $n_1 = 2, n_2 = 1, \quad A_1 = \left[\begin{array}{cc|c} 0 & \alpha_{110} & \alpha_{210} \\ 1 & \alpha_{111} & \alpha_{211} \\ \hline 0 & \alpha_{120} & \alpha_{220} \end{array}\right], \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

2) $n_1 = 1, n_2 = 2, \quad A_2 = \left[\begin{array}{c|cc} \alpha_{110} & 0 & \alpha_{210} \\ \hline \alpha_{120} & 0 & \alpha_{220} \\ \alpha_{121} & 1 & \alpha_{221} \end{array}\right], \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

with C arbitrary in both cases.

Now suppose we wish to realize the transfer function,

$$G(s) = \begin{bmatrix} s^{-1} & 0 \\ \varepsilon s^{-2} & s^{-2} \end{bmatrix}, \qquad \text{with } 0 < \varepsilon << 1$$

The realization in the canonical form will be,

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & \varepsilon^{-1} \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \end{bmatrix}$$

and as $\varepsilon \to 0$ the ε^{-1} in A_1 tends to infinity compensated by the $\varepsilon \to 0$ in the C matrix. Hence very small errors in identifying the (2,2) element of the matrix C will give large errors in the transfer function. ■

The above undesirable behaviour is not due to some pathalogical property of the system which remains of order 3 for all real-valued ε, but is entirely due to the parametrizations we have chosen for the system. Therefore if one wishes to use a canonical form, and the system being identified is close to, but not on, the boundary of one parametrization, then there are two possible courses of action. Firstly one can assume the system is in fact on the boundary and use a parametrization with fewer degrees of freedom, with the inherent loss in possible accuracy. Secondly one could use the correct parametrization and endure the numerical difficulties indicated above. The former approach is probably preferable, bearing in mind that tne data will in general be imperfect. An approach similar to this has been suggested by Weinert and Anton (1972) and Tse and Weinert (1973).

Notice that in order to determine the Kronecker indices of (A,B), it is necessary to verify that certain vectors are dependent, unless we have the generic case. Such a test is very difficult in a statistical setting because all determinants will in general be non zero, and some threshold will have to be established. However the independence

of a set of vectors is a relatively easy and well-posed statistical problem, as will be illustrated in the next section. Therefore it is relatively easy to find a set of integers $\kappa = (k_1, k_2, \ldots, k_n)$ summing to n such that the matrix $P(A,B,\kappa)$ is clearly non-singular, where

$$P(A,B,\kappa) = [b_1, Ab_1, \ldots, A^{k_1-1}b_1, \ldots, b_m, Ab_m, \ldots, A^{k_m-1}b_m].$$

This latter observation is the basis for an alternate approach to the parametrization problem, which avoids all the difficulties mentioned above, and is as follows. Once one has selected a set of integers $\kappa = (k_1, k_2, \ldots, k_m)$ such that the columns of $P(A,B,\kappa)$ are clearly independent, then one can use the canonical form for this set κ given in Theorem 2.4. This parametrization will then be well-posed for all systems (A,B) such that the columns of $P(A,B,\kappa)$ are reasonably independent. (i.e. all systems in a large neighborhood of the nominal values for the system). A particular system could have a finite number of different realizations using this method if several different sets κ_i make $\det P(A,B,\kappa_i) \neq 0$. However this should not be a practical problem since canonical forms will be used because there is no obvious physical interpretation for the states, so that the particular realization is not important. Also if one were comparing the identified parameters of a system in two situations one can artificially assign the indices for the second situation to be the same as those in the first, so that the parameter values can be compared directly. The example 2.4 given earlier in this section would be realized as follows.

Example 2.4 (continued)

$$G(s) = \begin{bmatrix} s^{-1} & 0 \\ \varepsilon s^{-2} & s^{-2} \end{bmatrix}$$

$$\begin{bmatrix} CB & CAB \\ CAB & CA^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 1 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 1 & 0 & 0 \end{bmatrix}$$

Clearly the columns 1, 2 and 4 of the Hankel matrix are independent so we will set $k_1 = 1$, $k_2 = 2$. Therefore the parametrization will be,

$$A = \begin{bmatrix} x & 0 & x \\ x & 0 & x \\ x & 1 & x \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} x & x & x \\ x & x & x \end{bmatrix}$$

where x denotes a free parameter.

In fact G(s) is realized by

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varepsilon & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is clearly well-posed for all ε. ■

The family of parametrizations suggested here is thus a series of parametrizations, with each one representing "almost all" systems, and a particular one is chosen by finding the most appropriate set of integers $K = (k_1,\ldots,k_m)$ such that $\det P(A,B,K) \neq 0$.

The assumption that K is known is only a slightly greater assumption than knowing the order, in that in order to determine the order of a system one has to essentially find a set K.

2.5 A New Realization Algorithm

Here we use the ideas presented in the previous two sections to derive a realization algorithm, (i.e. to find (A,B,C) satisfying $CA^kB = H_k$ for some given H_k, k = 0, 1, 2,..). It has the pleasing property that it produces A and B matrices in the special form given in Theorem 2.4 and uses no more computational effort than other methods (e.g. Ho and Kalman (1967)). We first give a preliminary lemma proved in Luenberger (1967).

Lemma 2.6: If (A,B) is controllable the following algorithm generates n independent vectors, P(A,B,S(n)).

Let $S(r) = (s_1(r), s_2(r), \ldots, s_m(r))$

where $s_j(r) \geq 0$ for all j = 1, 2, ... m

and $\sum_{j=1}^{m} s_j(r) \leq n$.

Let P(A,B,S(r)) be as defined in Section 2.3.

Algorithm

1) Set r = 0, $s_j(0) = 0$, j = 1,....m.

2) pick any j $\in$ {1, 2,... m}, say $\hat{j}$, such that

$A^{s_{\hat{j}}(r)} b_{\hat{j}}$ is independent of P(A,B,S(r))

$$\text{Set } s_j(r+1) = \begin{cases} s_j(r) + 1 & \text{if } j = \hat{j} \\ s_j(r) & \text{if } j \neq \hat{j} \end{cases}$$

3) increase the index r by one.

4) if r < n then return to 2), otherwise stop. ■

The above lemma produces a basis for the matrix $W = [B, AB, \ldots A^{n-1}B]$ of a particular type, (i.e. if $A^k b_j$ is in the basis then so are $A^\ell b_j$ for $\ell < k$), and the integers $s_j(n)$ are such that $A^{s_j(n)-1} b_j$ is in the basis and $A^{s_j(n)} b_j$ is not in the basis.

The following realization algorithm produces such a basis for the columns of the Hankel matrix (see Brockett (1970)) in the same way as Lemma 2.6.

Theorem 2.7: Given that the rank of the infinite Hankel matrix

$$\mathcal{H} = \begin{bmatrix} H_0 & H_1 & H_2 \ldots \\ H_1 & H_2 & \\ H_2 & & \\ \cdot & & \\ \cdot & & \end{bmatrix}$$

is less than or equal to N, the triple (A,B,C) as given by the following algorithm is a realization of $\mathcal{H}$, i.e. $CA^kB = H_k$ for $k = 0, 1, 2, \ldots$

Firstly define h_ℓ as the ℓ^{th} column of the finite Hankel matrix

$$H^N = \begin{bmatrix} H_0 & H_1 \ldots H_{N-1} \\ H_1 & \\ \cdot & \\ \cdot & \\ \cdot & \\ H_{N-1} \ldots H_{2N-2} \end{bmatrix}$$

Algorithm

Step 1: initialization $r = 1$

$s_j(1) = 0 \qquad j = 1,2,..m.$

$f_j(1) = h_j \qquad j = 1,2,..m.$

Step 2:

a) Choose $\hat{\ell}$ such that $\hat{\ell} = ms_j(r) + \hat{j}$ for some $\hat{j} \in \{1,2,\ldots,m\}$ and

$$||f_{\hat{\ell}}(r)||^2 \geq ||f_{\ell}(r)||^2, \ \forall\ \ell > \hat{\ell} \text{ and } \ell \in \{ms_j(r) + j\}_{j=1}^{m}$$

$$> ||f_{\ell}(r)||^2, \ \forall\ \ell < \hat{\ell} \text{ and } \ell \in \{ms_j(r) + j\}_{j=1}^{m}$$

b) Set $e_{\hat{\ell}} = f_{\hat{\ell}}(r)$, $\quad \ell_{r+1} = \hat{\ell}$.

c) For $\ell = ms_j(r) + j$, $\quad j = 1,2,\ldots,\hat{j} - 1, \hat{j} + 1,..m.$

set

$$\gamma_{\ell,\hat{\ell}} = \frac{< h_{\ell}, e_{\hat{\ell}} >}{< e_{\hat{\ell}}, e_{\hat{\ell}} >}$$

$$f_{\ell}(r+1) = f_{\ell}(r) - \gamma_{\ell,\hat{\ell}} e_{\hat{\ell}}$$

d) For $q = 1,\ldots,r + 1$ set

$$\gamma_{\hat{\ell}+m,\ell_q} = \frac{< h_{\hat{\ell}+m}, e_{\ell_q} >}{< e_{\ell_q}, e_{\ell_q} >}$$

set $\quad f_{\hat{\ell}+m}(r+1) = h_{\hat{\ell}+m} - \sum_{q=1}^{r+1} \gamma_{\hat{\ell}+m,\ell_q} e_{\ell_q}$

e) Set

$$s_j(r+1) = \begin{cases} s_j(r) + 1 & j = \hat{j} \\ s_j(r) & j \neq \hat{j}. \end{cases}$$

Step 3: Increase the index r by one.

Step 4: If $||f_{\ell}(r)||^2 = 0$ for all $\ell = ms_j(r) + j$, $j = 1,2,\ldots m$, then go to Step 5, otherwise return to Step 2.

Step 5: Set $n = r - 1$, $k_j = s_j(r) - 1 \quad j = 1,\ldots,m.$

Set $\begin{bmatrix} \beta_{i1} \\ \beta_{i2} \\ \\ \\ \beta_{in} \end{bmatrix} = \begin{bmatrix} 1 & \gamma_{\ell_2\ell_1} & \gamma_{\ell_3\ell_1} & \cdots & \gamma_{\ell_n\ell_1} \\ 0 & 1 & \gamma_{\ell_3\ell_2} & \cdots & \cdot \\ & & & & \cdot \\ 0 & & 1 & & \cdot \\ \cdot & & & & \gamma_{\ell_n\ell_{n-1}} \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_{\ell_i\ell_1} \\ \gamma_{\ell_i\ell_2} \\ \cdot \\ \cdot \\ \cdot \\ \gamma_{\ell_i\ell_n} \end{bmatrix}$

Then for $i = 1,2,\ldots m;\quad j = 1,2,\ldots,m;\quad k = 0,1,\ldots k_j - 1;$

Set $\boxed{\alpha_{ijk} = \beta_{iq} \text{ where } q \text{ is such that } km + j = \ell_q}$

Set (A,B) in the form of Theorem 2.4 given the above $\{k_i\}$ and α_{ijk}.

Step 6: Let $C = [c_1\ c_2\ \ldots\ c_n] \qquad c_i \in {}^{p}$

Set $\boxed{\begin{array}{l} c_{\sigma_j+k} = [I_p \vdots 0_{p,p(N-1)}]\ h_{(k-1)m+j+1} \\ \text{for } j = 0,1,..m-1;\ k = 1,2,\ldots k_{j+1} \end{array}}$

Remarks:

1) The above algorithm produces a sequence of independent columns of the Hankel matrix, e_{ℓ_q}, $q = 1,\ldots,r+1$, which will be of the form $M A^k b_j$, where

$$M = \begin{bmatrix} C \\ C A \\ \vdots \\ C A^{N-1} \end{bmatrix}$$

according to the rule given in Lemma 2.6. Since the realization will have dimension n, (which equals the rank of $\mathcal{H}$), the rank of M is n. Therefore any linear dependence or independence of the columns of H^N will be exactly the same as for the columns of $W = [B, AB...A^{N-1}B]$.

The set of independent columns of H^N is represented as an orthogonal set by a Gram-Schmidt type procedure, and the vectors eligible to join this basis at the r^{th} step (i.e. $M A^{s_j(r)} b_j$, $j = 1,...,m$) are represented in terms of this basis and a component orthogonal to it, and hence if the orthogonal component is non zero it is independent. The orthogonalization procedure also produces γ_{ij} which give the dependence of the vectors $M A^{s_j(r)} b_j$ on the basis, then the α_{ijk} required in the A (and perhaps B) matrices can be found by inverting an upper triangular matrix, which is computationally very easy.

Once the basis is found the C matrix follows immediately since $P(A,B,K) = I_n$. A formal proof of the algorithm is not given but it is clear from an understanding of Theorem 2.4.

2) The rule for selecting the new vector to enter the basis is to take that vector, of those eligible, with the greatest component orthogonal to the basis. This rule is chosen because it ensures that the basis has a determinant far from zero. Further if the data is noisy the chosen basis will remain independent for comparatively large variations in the parameters.

3) In the stochastic case such a method would be well-suited for finding a parametrization and approximate values for the parameters, to be used subsequently in a more efficient identification method, (e.g. maximum likelihood). The selection procedure for the basis works best if the inputs are of similar magnitude and the outputs are observed with similar accuracy. Ideally one might want to choose a basis, κ_i, with the least probability of becoming ill-posed. For example if $\eta(\kappa_i)$ is the largest probability such that in the $\eta(\kappa_i)$ confidence region $\det P(A,B,\kappa_i) \neq 0$, then one might choose the set κ_i with the greatest $\eta(\kappa_i)$.

CHAPTER 3

PARAMETER IDENTIFIABILITY FROM INPUT/OUTPUT OBSERVATIONS

3.1 Introduction

In this chapter we consider the identification of systems described by linear differential or difference equations;

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

or

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = Cx(k) + Du(k)$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, $A \in R^{nxn}$, $B \in R^{nxm}$, $C \in R^{pxn}$, $D \in R^{pxm}$. Also define $N = n(m + m + p) + mp$, the total number of elements of the matrices.

The problem is to identify these system matrices from input/output observations. As explained in Chapter 2 there is no unique solution to this identification problem because there are infinitely many equivalent realizations of a particular input/output response. In Chapter 2 it was shown how canonical forms can be used to overcome the non-uniqueness problem. In the present chapter it is assumed that the system equations are derived from physical knowledge of the system. That is, the elements of the A,B,C, and D matrices are either,

1) zero,
2) known physical constants,
or 3) known functions of some unknown parameters.

Thus if the unknown parameters are denoted $\alpha \in \Omega \subset R^q$, then the matrices may be written as $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, and $D(\alpha)$ where $A:\Omega \to R^{nxn}$;

$B:\Omega \to R^{nxm}$; $C:\Omega \to R^{pxn}$; and $D:\Omega \to R^{pxm}$. That is the system matrices are parametrized by the unknown parameters, α.

In practice it is very often the case that such equations can be postulated with relatively few unknown parameters and this is a very useful way of incorporating one's a priori knowledge, (e.g. in aerospace problems). The identification problem is then to find estimates of the unknown parameters based on the observed data.

When such a model can be formulated it has two main advantages over using canonical forms (as given in Chapter 2). Firstly the parameters being identified have a physical interpretation and secondly for multiple input/multiple output systems, the canonical forms have the disadvantage that a set of integers (e.g. the Kronecker invariants) must be determined before the real valued parameters can be identified.

A natural question that arises in the context of such identification problems is whether or not the unknown parameters, α, can be identified from observations of the system. This is the so-called identifiability problem and will be the subject of this chapter. Firstly we will give some simple examples to illustrate the main concepts.

Example 3.1

Consider the two parametrizations of a single input/single output, second order system.

$$1)\quad A(\alpha) = \begin{bmatrix} -1 & \alpha_1 \\ 0 & -2 \end{bmatrix}, \quad B(\alpha) = \begin{bmatrix} 0 \\ \alpha_2 \end{bmatrix}, \quad C(\alpha) = [1 \quad 0]$$

This will have transfer function, $G(s) = \dfrac{\alpha_1\alpha_2}{(s+1)(s+2)}$

Here only the product $\alpha_1\alpha_2$ can be identified and neither α_1 nor α_2 can be identified individually. (This system would thus be said to be not locally or globally identifiable).

$$2) \quad A(\alpha) = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix}, \quad B(\alpha) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C(\alpha) = [1 \quad 0]$$

This will have transfer function, $G(s) = \dfrac{1}{(s-\alpha_1)(s-\alpha_2)}$

In this case if $\alpha_1 \neq \alpha_2$ then α_1 and α_2 can be uniquely identified in a neighborhood of their nominal values. However if α_1 and α_2 have their values exchanged then the transfer function is not altered, and therefore the parametrization would be called not globally identifiable. If however α_1 and α_2 are restricted to be in the set

$$\{(\alpha_1,\alpha_2) \in \mathbb{R}^2 | \alpha_1 \geq \alpha_2\},$$

then α_1 and α_2 are globally identifiable. ■

Example 3.2

Consider the system with n = m = p = 1 and a,b,c arbitrary real numbers, then the transfer function $G(s) = \dfrac{cb}{s-a}$ and clearly (a,b,c) is not identifiable. However if cb ≠ 0 then a can be identified independently from b and c. This parametrization will be called partially identifiable in a, independent of b and c. ■

The above examples illustrate that the identifiability of a particular parametrization is not obvious and has several aspects to it.

In the following section the solutions to most of these identifiability questions will be given.

3.2 Local Identifiability

Identifiability of parameter means roughly that parameter estimates can be determined which are asymptotically exact. Identifiability will thus depend on the data available, and in this chapter we will assume that we could obtain asymptotically consistent estimates of the transfer function and nothing else, which is assured by the following assumptions;

A1) Both the input and output are observed, perhaps with observation noise.
A2) The input is independent of the observations and is persistently exciting (that is the input excites all the system modes, see Åström and Bohlin (1966)).
A3) The observation noise statistics are such that the system transfer function, or Markov parameters, can be identified asymptotically.
A4) The system was either started an arbitrarily long time before identification was started, or that the initial condition was zero.

The assumptions imply that if noise is present on all the observations then the system must be stable, since otherwise A2 implies that some outputs would tend to infinity with increasing time. The correct way to identify unstable systems in the presence of noise is to insert a known stabilizing feedback system and identify the resulting composite system, from which the open-loop system could be deduced.

Assumption A4 is included so that no more than the transfer function can be identified. For reachable systems there is no difference between the cases with the initial condition zero and non-zero but unknown, since in the latter case the initial condition can be replaced by an equivalent input. However for unreachable systems the

initial condition response gives additional information that is not available from an input/output test, but such information will only be finite if the observation noise covariance is positive definite.

Under the above assumptions the following definition for local identifiability of a system parametrization, as given in Section 3.1, is natural if one has nominal values for the unknown parameters, (e.g. wind tunnel tests on an airframe).

Definition 3.1

Let $(A,B,C,D)(\alpha) : \Omega \subset R^{q} \to R^{N}$ $(N = n(n+m+p) + mp)$, be a parametrization of the system matrices (A,B,C,D) of a linear dynamical system. This parametrization is said to be locally identifiable (from the transfer function) at $\alpha = \hat{\alpha} \in \Omega$ if there exists an $\varepsilon > 0$ such that

(i) $||\alpha - \hat{\alpha}|| < \varepsilon, ||\beta - \hat{\alpha}|| < \varepsilon, \alpha, \beta \in \Omega,$

and (ii) $C(\alpha)A^{k}(\alpha)B(\alpha) = C(\beta)A^{k}(\beta)B(\beta), \quad k = 0,1,2..$

imply $\alpha = \beta$. ■

In other words, in a neighborhood of $\hat{\alpha}$, there are no two systems with distinct parameters, which have the same transfer function. This definition is similar to the definition of "non-degeneracy" as given by Kalman (1966). Definition 3.1 is equivalent to requiring that the map from the parameters, α, into the Markov parameters is locally one-to-one. A standard result on injective maps is given by the following lemma.

Lemma 3.1 (Rank Theorem, see for example Narasimham (1968) page 18).

Let Ω be an open set in R^{n} and $f : \Omega \to R^{m}$ be a C^{k} map. Suppose that rank $\frac{\partial f(x)}{\partial x} = r$ for all x in Ω. Then there exist open neighborhoods

U of a and V of b = f(a), cubes Q,Q' in R^n, R^m respectively and C^k diffeomorphisms $u : Q \to U$ and $u' : V \to Q'$, such that if $\phi = u' \circ f \circ u$ then ϕ has the form

$$\phi(x_1,x_2,\ldots,x_n) = (x_1,x_2,..x_r, 0,0,\ldots 0)$$

(Note: a cube in R^n is a set of the form $\{x \mid |x_j - a_j| < r_j\}$, and $\circ$ denotes composition). ■

An immediate consequence of the Rank Theorem is;

Corollary 3.2

Let Ω be an open set in R^n and $f : \Omega \to R^m$ be a C^k map with $k \geq 1$. Then if $\frac{\partial f(x)}{\partial x}$ has constant rank r in a neighborhood of $\hat{x}$, f is locally injective if and only if r = n. ■

We can now obtain an identifiability condition as follows.

Theorem 3.3

Let $(A,B,C,D)(\alpha) : \Omega \subset R^q \to R^N$ (with Ω an open set in R^q) be a C' parametrization of the system matrices (A,B,C,D). Then if rank $\frac{\partial G(\alpha)}{\partial \alpha}$ = r (see below) for all α in some neighborhood of $\hat{\alpha}$, then the parametrization is locally identifiable at $\hat{\alpha}$, if and only if r = q.

In here $G : \Omega \to R^{(2n+1)mp}$ is given by,

$$G'(\alpha) = [D'(\alpha),(C(\alpha)B(\alpha))',(C(\alpha)A(\alpha)B(\alpha))',\ldots,(C(\alpha)A^{2n-1}(\alpha)B(\alpha))']'$$

Further the Jacobian of $\bar{\bar{G}}$ can be written as,

$$\frac{\partial \bar{\bar{G}}(\alpha)}{\partial \alpha} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & I_p \otimes I_m \\ 0 & C \otimes I_m & I_p \otimes B' & 0 \\ C \otimes B' & CA \otimes I_m & I_p \otimes B'A' & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{r=1}^{k} CA^{k-r} \otimes B'A'^{r-1} & CA^k \otimes I_m & I_p \otimes B'A'^k & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{r=1}^{2n-1} CA^{2n-1-r} \otimes B'A'^{r-1} & CA^{2n-1} \otimes I_m & I_p \otimes B'A'^{2n-1} & 0 \end{array}\right] . M(\alpha)$$

where the dependence of (A,B,C,D) on α is understood. And

$$M(\alpha) = \left[\begin{array}{c} \frac{\partial \bar{\bar{A}}(\alpha)}{\partial \alpha} \\ \frac{\partial \bar{\bar{B}}(\alpha)}{\partial \alpha} \\ \frac{\partial \bar{\bar{C}}(\alpha)}{\partial \alpha} \\ \frac{\partial \bar{\bar{D}}(\alpha)}{\partial \alpha} \end{array}\right]$$

where if X is an nxm matrix given by $X' = [x_1, x_2, \ldots x_n]$, with $x_i \in R^m$, then $\bar{\bar{X}}$ is the nmx1 vector given by $\bar{\bar{X}}' = [x_1', x_2', \ldots, x_n']$. Also $\otimes$ denotes Kronecker product (see Appendix I).

Proof: If a system has order less than or equal to n, the set $(D, CB, CAB, \ldots, CA^{2n-1})$ is sufficient to determine all subsequent Markov parameters. Thus G is locally injective if and only if the function from α into all the Markov parameters is locally injective. Therefore the result follows immediately from Corollary 3.2, and it only remains to show that the Jacobian of $\bar{\bar{G}}$ is as given above.

$$\lim_{h \to 0} \left\{ \frac{1}{h} \left[(C + h\Delta C)(A + h\Delta A)^k (B + h\Delta B) - CA^k B \right] \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{1}{h} \left[h\Delta CA^k B + hCA^k \Delta B + h \sum_{r=1}^{k} CA^{k-r} \Delta A A^{r-1} B + O(h^2) \right] \right\}$$

$$= \Delta CA^k B + CA^k \Delta B + \sum_{r=1}^{k} CA^{k-r} \Delta A A^{r-1} B$$

$$= (\Delta C \quad C) \begin{pmatrix} A & 0 \\ \Delta A & A \end{pmatrix}^k \begin{pmatrix} B \\ \Delta B \end{pmatrix}$$

and the expression given for $\frac{\partial \bar{\bar{G}}}{\partial \alpha}$ is obtained by ordering the elements of ΔA, ΔB, ΔC, and ΔD as given for $M(\alpha)$. ■

The expression above reduces to the evaluation of a q x q determinant. It is however unnecessarily complex if we know that a system is of minimal order, in which case we know that all equivalent systems are related by a similarity transformation as explained in Chapter 2.

The following theorem gives conditions for the local identifiability of minimal systems.

Theorem 3.4

Let $(A,B,C,D)(\alpha) : \Omega \subset R^q \to R^N$ (with Ω an open subset of R^q) be a C^1 (i.e. continuously differentiable on Ω) parametrization of the system matrices (A,B,C,D) and suppose $(A,B,C,D)(\hat{\alpha})$ is minimal. Then

1) $(A,B,C,D)(\alpha)$ is locally identifiable at $\alpha = \hat{\alpha}$ if and only if $F : GL(n) \times \Omega \to R^N$ is locally injective at $T = I$ and $\alpha = \hat{\alpha}$, where

$$F(T,\alpha) = (TA(\alpha)T^{-1}, TB(\alpha), C(\alpha)T^{-1}, D(\alpha)).$$

2) if rank $\frac{\partial F(I,\alpha)}{\partial(T,\alpha)} = r$ for all α in some neighborhood of $\hat{\alpha}$, then $(A,B,C,D)(\alpha)$ is locally identifiable at $\alpha = \hat{\alpha}$ if and only if $r = n^2 + q$, or equivalently $\det[X'(\hat{\alpha})X(\hat{\alpha})] \neq 0$, where

$$X(\alpha) = \left[\frac{\partial \tilde{F}}{\partial T}(I,\alpha) \;;\; \frac{\partial \tilde{F}}{\partial \alpha}(I,\alpha)\right]$$

$$= \left[\begin{array}{c|c} \begin{array}{c} I_n \times A'(\alpha) - A(\alpha) \times I_n \\ I_n \times B'(\alpha) \\ -C(\alpha) \times I_n \\ 0_{mp,n^2} \end{array} & M(\alpha) \end{array}\right]$$

$\tilde{F}$ is a reordering of F given by

$$\tilde{F}(T,\alpha) = \left[\begin{array}{c} \underline{\underline{TA(\alpha)T^{-1}}} \\ \underline{\underline{TB(\alpha)}} \\ \underline{\underline{C(\alpha)T^{-1}}} \\ \underline{\underline{D(\alpha)}} \end{array}\right]$$

$M(\alpha)$ and $\underline{\underline{\quad}}$ notation are as defined in Theorem 3.3.

Proof

1) **Necessity**

If F is not locally injective then for all $\varepsilon > 0$ there exist $(T_\varepsilon,\alpha_\varepsilon),(S_\varepsilon,\beta_\varepsilon) \in N_\varepsilon(I,\hat{\alpha})$ such that $F(T_\varepsilon,\alpha_\varepsilon) = F(S_\varepsilon,\beta_\varepsilon)$ and therefore

$$S_\varepsilon^{-1}T_\varepsilon A(\alpha_\varepsilon)T_\varepsilon^{-1}S_\varepsilon = A(\beta_\varepsilon)$$
$$S_\varepsilon^{-1}T_\varepsilon B(\alpha_\varepsilon) = B(\beta_\varepsilon)$$
$$C(\alpha_\varepsilon)T_\varepsilon^{-1}S_\varepsilon = C(\beta_\varepsilon)$$
$$D(\alpha_\varepsilon) = D(\beta_\varepsilon)$$

Therefore there are equivalent systems in an arbitrarily small neighborhood of $\hat{\alpha}$, and the parametrization is not locally identifiable. (Note that the fact that GL(n) is an open subset of R^{nxn} is used)

Sufficiency

First note that since $(A,B,C,D)(\hat{\alpha})$ is minimal there exists a neighborhood $W \subset \Omega$ of $\hat{\alpha}$ such that $(A,B,C,D)(\alpha)$ is minimal for all $\alpha \in W$. (Since minimal systems form an open set in parameter space and the parametrization is assumed to be continuous). Therefore when restricted to W all equivalent systems are related by a similarity transformation. Therefore the parametrization is locally identifiable if F is injective when restricted to GL(n) x V, where $V \subset W$ is any open set containing $\hat{\alpha}$.

In order to prove the result we will prove the contrapositive. Assume therefore that the parametrization is not locally identifiable, then for all $\varepsilon > 0$ there exist $T_\varepsilon, S_\varepsilon \in GL(n)$, $\alpha_\varepsilon \neq \beta_\varepsilon \in N_\varepsilon(\hat{\alpha}) \subset W$ such that $F(T_\varepsilon,\alpha_\varepsilon) = F(S_\varepsilon,\beta_\varepsilon)$. Therefore we have that

$$S_\varepsilon^{-1}T_\varepsilon = W(\beta_\varepsilon)W'(\alpha_\varepsilon)[W(\alpha_\varepsilon)W'(\alpha_\varepsilon)]^{-1}$$

where $W(\alpha) = [B(\alpha), A(\alpha)B(\alpha), \ldots, A^{n-1}(\alpha)B(\alpha)]$

$S_\varepsilon^{-1}T_\varepsilon$ is therefore a continuous function of $(\alpha_\varepsilon,\beta_\varepsilon)$ since $W(\alpha)$ has full rank for all $\alpha \in W$ by the reachability assumption. Therefore $||S_\varepsilon^{-1}T_\varepsilon - I||$ can be made arbitrarily small by taking ε sufficiently small and $F(S_\varepsilon^{-1}T_\varepsilon,\alpha_\varepsilon) = F(I,\beta_\varepsilon)$. Hence there does not exist a neighborhood of $(I,\hat{\alpha})$ in which F is injective, and thus F is not locally injective.

2) To prove this we use part (1) above, and Corollary 3.2. First we will compute the Jacobian of F or equivalently $\tilde{F}$, which is given by,

$$\lim_{h \to 0} \frac{1}{h} \left(F(T + h\delta T, \alpha) - F(T,\alpha)\right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left((T + h\delta T)A(\alpha)(T + h\delta T)^{-1} - TA(\alpha)T^{-1},\right.$$

$$\left.(T + h\delta T)B(\alpha) - B(\alpha), C(\alpha)(T + h\delta T)^{-1} - C(\alpha), 0\right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(h\delta TA(\alpha)T^{-1} - hTA(\alpha)T^{-1}\delta TT^{-1} + O(h^2),\right.$$

$$\left. + h\delta TB(\alpha), - hC(\alpha)T^{-1}\delta TT^{-1} + O(h^2), O\right)$$

$$= (\delta TA(\alpha)T^{-1} - TA(\alpha)T^{-1}\delta TT^{-1}, \delta TB(\alpha), - C(\alpha)T^{-1}\delta TT^{-1}, O)$$

Therefore using notation of Appendix I,

$$\frac{\partial \tilde{F}}{\partial T}(T,\alpha) = \begin{bmatrix} I \otimes T'^{-1}A'(\alpha) - TA(\alpha)T^{-1} \otimes T^{-1'} \\ I \otimes B'(\alpha) \\ - C(\alpha)T^{-1} \otimes T^{-1'} \\ 0 \end{bmatrix}$$

Similarly

$$\frac{\partial \tilde{F}}{\partial \alpha}(T,\alpha) = \left[\begin{array}{c|c|c|c} T \otimes T^{-1'} & 0 & 0 & 0 \\ 0 & T \otimes I & 0 & 0 \\ 0 & 0 & I \otimes T^{-1}, & 0 \\ 0 & 0 & 0 & I \otimes I \end{array}\right] M(\alpha)$$

and thus

$$\left[\frac{\partial \tilde{F}}{\partial T}(T,\alpha), \frac{\partial \tilde{F}}{\partial \alpha}(T,\alpha)\right] =$$

$$\left[\begin{array}{c|c|c|c} T \otimes T^{-1'} & 0 & 0 & 0 \\ 0 & T \otimes I & 0 & 0 \\ \hline 0 & 0 & I \otimes T^{-1}, & 0 \\ \hline 0 & 0 & 0 & I \otimes I \end{array}\right] X(\alpha) \left[\begin{array}{c|c} T^{-1} \otimes I & 0 \\ \hline 0 & I_q \end{array}\right]$$

Therefore since $T \in GL(n)$ the rank of the Jacobian of F at (T,α) is equal to the rank of $X(\alpha)$, hence the assumption that rank $X(\alpha) = r$ for all $\alpha \in N_\varepsilon(\hat{\alpha})$ implies that rank $\frac{\partial \tilde{F}(T,\alpha)}{\partial (T,\alpha)} = r$ for all (T,α) in some neighborhood of $(I,\hat{\alpha})$. Therefore the assumptions of Corollary 3.2 are valid and the result follows immediately. ■

Remarks

1) If rank $X(\hat{\alpha}) = n^2 + q$ then rank $X(\alpha) = n^2 + q$ in some neighborhood of $\hat{\alpha}$, and hence the condition for identifiability is simply $\det[X'(\hat{\alpha})X(\hat{\alpha})] \neq 0$. Requiring that the rank $X(\alpha)$ is constant in a neighborhood of $\hat{\alpha}$ is specified so that unidentifiable parametrizations can be found from the test. Those systems which have rank $X(\alpha) < q$ at $\alpha = \hat{\alpha}$ but not in a neighborhood of $\hat{\alpha}$ may or may not be locally identifiable, however it can be said that the sensitivity of the input/output response to certain small changes in the parameter values is zero. This situation is analogous to trying to estimate α from noisy observations of α^3, in a neighborhood of $\alpha = 0$.

2) Theorem 3.4 gives a comparatively simple test for the local identifiability of a parametrization, with the unknown parameter entering in a straightforward manner. It is significantly simpler than the methods based on the information matrix (see Section 3.6), and more elegant than the condition of Theorem 3.3. The computational comparison between the tests of Theorems 3.3 and 3.4 is not clear, in that although Theorem 3.3 reduces to evaluating a q x q determinant, whereas the condition of Theorem 3.4 involves a determinant of dimension

$n^2 + q$ (can be reduced to n^2), the precomputing required in Theorem 3.3 is considerable. The test of Theorem 3.4 allows some unknown parameters to be left as free parameters and the determinant evaluated as a function of them, so that regions of local identifiability can be deduced. However in Theorem 3.3 such calculations could be exceedingly tedious.

3) If a parametrization is locally identifiable, this ensures that any well-conceived algorithm which minimizes some cost function over the parameters will be well-posed and have a unique solution in some neighborhood of the nominal values. Further if a parametrization is locally identifiable for all values of $\alpha \in \Omega$ then an algorithm will always be well-posed but may converge to one of several solutions depending on the initial parameter estimates and the actual data received. This is the problem of global identifiability which will be discussed in the next section. First we will give some examples to illustrate the local identifiability theorems.

Example 3.1 (continued)

(i) For the parametrization of example 3.1 (i),

$$X(\alpha) = \left[\begin{array}{cc|cc|cc} 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ \alpha_1 & -1 & 0 & \alpha_1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ \hline 0 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last 3 columns are clearly dependent for all $(\alpha_1,\alpha_2) \in R^2$, and hence the parametrization is not locally identifiable for any α_1 and α_2.

(ii) For the parametrization of Example 3.1 (ii)

$$X(\alpha) = \left[\begin{array}{cc|cc|cc} 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & \alpha_2-\alpha_1 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & \alpha_1-\alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

And $\det[X'(\alpha)X(\alpha)] = (\alpha_1 - \alpha_2)^2$ and so it is locally identifiable if $\hat{\alpha}_1 \neq \hat{\alpha}_2$, however the region of local identifiability is small when $\hat{\alpha}_1 \approx \hat{\alpha}_2$. The fact that it is not locally identifiable at $\hat{\alpha}_1 = \hat{\alpha}_2$ is true but cannot be deduced from the theorems.

For comparison using Theorem 3.3 for this parametrization gives

$$\frac{\partial \bar{G}}{\partial \alpha}(\alpha) = \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & \alpha_1 & 1 \\ 1 & \alpha_1+\alpha_2 & 0 & 1 & \alpha_1^2 & \alpha_1+\alpha_2 \\ 2\alpha_1+\alpha_2 & \alpha_1^2+\alpha_1\alpha_2+\alpha_2^2 & 1 & \alpha_1+2\alpha_2 & \alpha_1^3 & \alpha_1^2+\alpha_1\alpha_2+\alpha_2^2 \end{array}\right]$$

$$\begin{bmatrix} 0 & 1 \\ 1 & \alpha_2 \\ \alpha_1+\alpha_2 & \alpha_2^2 \\ \alpha_1^2+\alpha_1\alpha_1+\alpha_2^2 & \alpha_2^3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 2\alpha_1+\alpha_2 & 2\alpha_2+\alpha_1 \end{bmatrix}$$

and det $\left[\left(\frac{\partial \bar{\bar{G}}}{\partial \alpha}\right)' \left(\frac{\partial \bar{\bar{G}}}{\partial \alpha}\right)\right] = (\alpha_1-\alpha_2)^2$ ■

Example 3.3

A simple minded extension of the standard controllable form for single input systems is given in the following proposition, and it is shown that except when all the indices are equal the parametrization is not locally identifiable anywhere.

Proposition 3.5

Consider the set $(k_1,k_2,\ldots,k_n)$ summing to n, then the parametrization of the system matrices (A,B,C) given below is never locally identifiable anywhere unless the k_i are all equal in which case it is locally identifiable for all $\alpha \in R^{n(m+p)}$.

C is completely free.

A and B are block matrices

$A = (A_{ij})_{i,j=1,..m}$ $\quad B = (B_{ij})_{i,j=1,..,m}$.

$$
\underset{(k_i \times k_j)}{A_{ij}} = \begin{cases} \left[\begin{array}{c|c} 0_{k_i-1,1} & I_{k_i-1} \\ \hline x & x\ x \ldots\ x \end{array}\right] & i = j \\ \left[\begin{array}{c} 0_{k_i-1,k_j} \\ \hline x\ \ x \ \ldots\ldots\ x \ \ x \end{array}\right] & i \neq j \end{cases}
$$

$$
\underset{(k_i \times 1)}{B_{ij}} = \begin{cases} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} & i = j \\ 0_{k_i,1} & i \neq j \end{cases}
$$

where x's are free parameters.

Proof

The local identifiability condition of Theorem 3.4 is equivalent to,

$$
(\star) \qquad \left.\begin{aligned} Q\,A(\alpha) - A(\alpha)Q &= \Delta A \\ Q\,B(\alpha) &= \Delta B \\ -C\,Q(\alpha) &= \Delta C \end{aligned}\right\} \Longrightarrow Q = 0
$$

where $(\Delta A, \Delta B, \Delta C)$ represent admissible local variations in (A,B,C). (i.e. if the implication holds at some $\hat{\alpha}$ then local identifiability results, and if the implication does not hold for all α in some neighborhood of $\hat{\alpha}$ then it is not locally identifiable).

Now because of the structure of $(\Delta A, \Delta B, \Delta C)$ $(\star)$ is equivalent to

$$(**) \qquad \left.\begin{matrix} E(Q\,A(\alpha) - A(\alpha)Q) = 0 \\ Q\,B = 0 \end{matrix}\right\} \Rightarrow Q = 0$$

where

$$E = \begin{bmatrix} I_{k_1-1} & O_{k_1-1,1} & & & \\ & & I_{k_2-1} & O_{k_2-1,1} & \\ & & & & \ddots \\ & & & & I_{k_m-1} \quad O_{k_m-1,1} \end{bmatrix}$$

Let $$Q = \begin{bmatrix} Q_{11} & Q_{12} \cdots Q_{1m} \\ Q_{21} & \\ \vdots & \\ Q_{m1} \cdots\cdots Q_{mm} \end{bmatrix}$$

with Q_{ij} a $(k_i \times k_j)$ matrix.

Then $Q\,B = 0 \Rightarrow Q_{ij}\,B_{ij} = 0$, or the ℓ,k_j th element of Q_{ij}, $q^{ij}_{\ell,k_j} = 0$ for $\ell = 1,2,..,k_i$.

The i-j th block of (**) gives

$$\left[I_{k_i-1},\ O_{k_i-1,1}\right]\left\{\sum_{k=1}^{m} Q_{ik}A_{kj}(\alpha) - \sum_{k=1}^{m} A_{ik}(\alpha)Q_{kj}\right\} = 0$$

And since $q^{ij}_{\ell,kj} = 0$ this becomes

$$\left[I_{k_i-1},\ O_{k_i-1,1}\right]\left\{Q_{ij}A_{jj}(\alpha) - A_{ii}(\alpha)Q_{ij}\right\} = 0$$

that is for i,j = 1,...,m,

$$\begin{bmatrix} 0 & q_{11} & q_{12} \cdots q_{1,k_j-1} \\ 0 & q_{21} & \\ \vdots & \vdots & \\ 0 & q_{k_i-1,1} \cdots & q_{k_i-1,k_j-1} \end{bmatrix}_{ij} = \begin{bmatrix} q_{21} & q_{22} \cdots q_{2,k_j-1} & 0 \\ q_{31} & & 0 \\ \vdots & & \vdots \\ q_{k_i,1} & q_{k_i,k_j-1} & 0 \end{bmatrix}_{ij}$$

Therefore there are $(k_i-1)k_j$ equations in $(k_j-1)k_i$ unknowns, and if $k_i < k_j$ there are more unknowns than linear equations and a non-zero solution exists. If $k_i \geq k_j$ there is a unique solution for Q_{ij} (i.e. zero) and this can be verified recursively, ($q_{\ell 1} = 0 \Rightarrow q_{\ell+1,2} = 0 \Rightarrow$ $\ldots \Rightarrow \ell = 2,3,..,k_i$ and $q_{\ell,k_j-1} = 0 \Rightarrow q_{\ell-1,k_j-2} = 0 \ldots \ell = 1,2,..,k_i-1)$.

Therefore the parametrization is not locally identifiable for any α if $k_i \neq k_j$ for some i and j, and it is locally identifiable for all α if $k_i = k_j$ for all i and j. (In fact it is globally identifiable in this case by Theorem 3.6 , see next section). ∎

Some authors imply that the above parametrization is useful (e.g. Jordan and Shridar (1973)) but the above proposition shows that it is rarely identifiable, and hence there will not be a unique representation of a particular system response. The correct extension of the standard controllable form is given by Theorem 2.5. ∎

Example 3.4

We now give an example of a parametrization which is locally identifiable for all $\alpha \in R$ but is not globally identifiable.

$$A(\alpha) = \begin{bmatrix} -1 & 0 \\ 1+2\alpha & -5 \end{bmatrix}, \qquad B(\alpha) = \begin{bmatrix} 2+\alpha \\ 1 \end{bmatrix}, \qquad C(\alpha) = [1-\alpha, \ \alpha]$$

which has transfer function,

$$= \frac{s(2-\alpha^2) + (2\alpha^3 - 2\alpha + 10)}{(s+1)(s+5)}$$

and Markov parameters,

$$G(\alpha) = \begin{bmatrix} 2 - \alpha^2 \\ -2 - 2\alpha + 6\alpha^2 + 2\alpha^3 \\ 2 + 12\alpha - 31\alpha^2 - 12\alpha^3 \\ -2 - 62\alpha + 156\alpha^2 + 62\alpha^3 \end{bmatrix}$$

Therefore as in Theorem 3.3

$$\frac{\partial G(\alpha)}{\partial \alpha} = \begin{bmatrix} -2\alpha \\ -2 + 12\alpha + 6\alpha^2 \\ 12 - 62\alpha - 36\alpha^2 \\ -62 + 312\alpha + 186\alpha^2 \end{bmatrix}$$

which is clearly of full rank for all $\alpha \in R$, and hence by Theorem 3.3 the parametrization is locally identifiable for all $\alpha \in R$. However the systems with $\alpha = 1$ and $\alpha = -1$ have the same transfer function $\frac{s+10}{(s+1)(s+5)}$ and therefore the parametrization is not globally identifiable.

The variation of the transfer function with α is shown below in Figure 3.1.

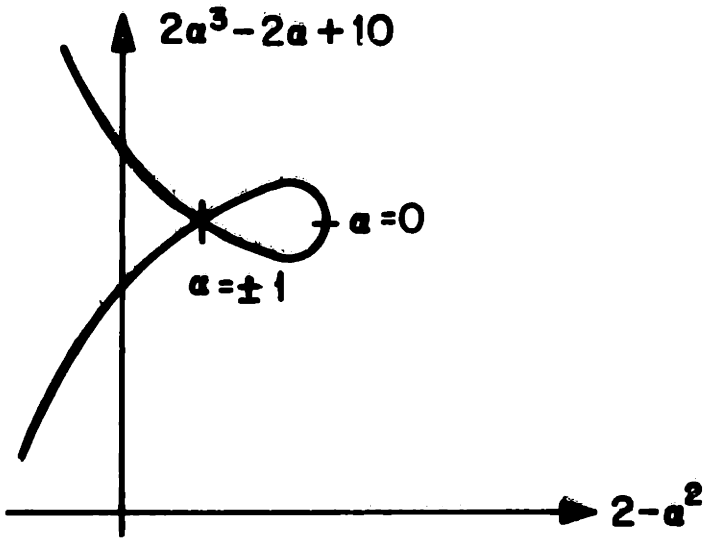

Figure 3.1

3.3 Global Identifiability

As remarked in section 3.2 from a practical point of view local identifiability has the disadvantages that a nominal value for α is required, and that given local identifiability the extent of the region of identifiability is not easily found. Hence the concept of global identifiability is now defined.

Definition 3.2

Let $(A,B,C,D)(\alpha) : \Omega \subset R^q \to R^N$ be a parametrization of the system matrices (A,B,C,D). This parametrization is said to be globally identifiable (from the transfer function) if, for all $\alpha, \beta \in \Omega$,

(i) $D(\alpha) = D(\beta)$

(ii) $C(\alpha)A^k(\alpha)B(\alpha) = C(\alpha)A^k(\alpha)C(\alpha), \quad k = 0,1,2,\ldots$

and (iii) $(A,B,C,D)(\alpha)$ is minimal

imply $\alpha = \beta$. ∎

Condition (iii) in the above definition could be deleted, but then the definition would be very restrictive since most useful parametrizations admit multiple representations of non-minimal systems.

The following proposition gives a sufficient condition for global identifiability when the parametrization is affine. (i.e. $f : R^m \to R^n$ is affine if $f(x) = c + \ell(x)$ where ℓ is linear and c is a constant). Affine parametrizations occur frequently in practice, for example all the standard canonical forms are affine.

Theorem 3.6

An affine parametrization $(A,B,C,D)(\alpha) : \Omega \subset R^q \to R^N$, is globally identifiable if for all $\alpha, \beta \in \Omega$,

$$R(Z(\alpha,\beta)) \cap R(Z(\beta,\alpha)) \cap R(M) = \{0\}$$

and this is implied by $\det[Y'(\alpha,\beta)Y(\alpha,\beta)] \neq 0$ for all $\alpha, \beta \in \Omega$. In here

$$Z(\alpha,\beta) = \begin{bmatrix} I \otimes A'(\alpha) - A(\beta) \otimes I \\ I \otimes B'(\alpha) \\ - C(\beta) \otimes I \\ 0 \end{bmatrix}$$

M is as defined in Theorem 3.3.

$$Y(\alpha,\beta) = \begin{bmatrix} Z(\alpha,\beta) & 0 & M \\ 0 & Z(\beta,\alpha) & M \end{bmatrix}$$

Proof

Since we are only concerned with minimal systems, global identifiability is implied if the following equations have a unique solution for all $\alpha, \beta \in \Omega$,

$$\left.\begin{aligned} T\,A(\alpha) &= A(\beta)T \\ T\,B(\alpha) &= B(\beta) \\ C(\alpha) &= C(\beta)T \\ D(\alpha) &= D(\beta) \end{aligned}\right\} \qquad (*)$$

Let $Q_1 = T - I$ and $Q_2 = I - T^{-1}$ then (*) is equivalent to,

$$Z(\alpha,\beta)\,\bar{\bar{Q}}_1 = M(\beta-\alpha)$$

or $$Z(\beta,\alpha)\,\bar{\bar{Q}}_2 = M(\beta-\alpha)$$

Which will have a unique solution if $R(Z(\alpha,\beta)) \cap R(z(\beta,\alpha)) \cap R(M) = \{0\}$. Finally the condition on $Y(\alpha,\beta)$ is immediate. ■

Remarks:

1) The condition is not necessary since $(\bar{\bar{Q}}_1', \bar{\bar{Q}}_2', \alpha-\beta) \in N(Y(\hat{\alpha},\hat{\beta}))$ does not imply that $(Q_1+I)^{-1} = I - Q_2$ or that $\alpha - \beta = \hat{\alpha} - \hat{\beta}$ which

are required for a system not to be globally identifiable.

2) A somewhat more restrictive sufficient condition for global identifiability is that $R(z(\alpha,\beta)) \cap R(M) = \{0\}$ for all $\alpha, \beta \in \Omega$. We remark that this condition is in fact satisfied by the canonical forms given in Theorems 2.4 and 2.5 and the former case is proven in the following proposition.

Proposition 3.7

Let the set of integers $(s_0, s_1, .., s_m)$ satisfy

$$0 = s_0 < s_1 < s_2 \ldots < s_{m-1} < s_m = n.$$

Then the following parametrization satisfies $\det[(Z(\alpha,\beta),M)'(Z(\alpha,\beta),M)] \neq 0$ for all $\alpha, \beta \in$ $^{n(m+p)}$.

$$A = [a_1\ a_2\ \ldots\ a_n], \qquad B = [b_1, b_2, \ldots b_m]$$

$$a_i = \begin{cases} \text{free if } i = s_j \quad j = 1,\ldots m. \\ e_{i+1} \qquad \text{otherwise} \end{cases}$$

$$b_i = e_{s_i+1}$$

$$C = \text{free}$$

where $e_i = i^{th}$ unit vector.

Proof

$\det[(Z(\alpha,\beta),M)'(Z(\alpha,\beta),M)] \neq 0 \quad \forall\ \alpha,\beta$ is implied if

$$\left.\begin{aligned} Q\,A(\alpha) - A(\beta)Q &= \Delta A \\ Q\,B(\alpha) &= \Delta B \\ -\,C(\beta)Q &= \Delta C \end{aligned}\right\} \Rightarrow Q = 0$$

where $(\Delta A, \Delta B, \Delta C)$ are admissible variations in (A,B,C).

Set $Q = [q_1\ q_2\ \dots\ q_n]$

$QA(\alpha) = [p_1\ p_2\ \dots\ p_n]$

$A(\beta)Q = [r_1\ r_2\ \dots\ r_n]$.

Then
$$p_i = \begin{cases} Qa(\alpha)_{s_j} & \text{if } i = s_j, \quad j = 1,\dots,m. \\ q_{i+1} & \text{otherwise} \end{cases}$$

$$r_i = A(\beta)q_i$$

(*) $$Q\,B = [q_1\ q_{s_1+1}\ \dots\ q_{s_m+1}] = \delta B \doteq 0$$

$Q\,A(\alpha) - A(\beta)Q = \delta A$ implies

$p_i = r_i$ for $i \neq s_j$ $j = 1,\dots,m$.

Set $i = s_j + 1$ then

$$q_{s_j+2} = A(\beta)\ q_{s_j+1}$$

but $q_{s_j+1} = 0$ by (*) and hence $q_{s_j+2} = 0$ and hence recursively until $q_{s_{j+1}} = 0$. Hence $Q = 0$ and the result is proven. ∎

Example 3.5

The following is an example of a globally identifiable parametrization which fails the conditon, $R(Z(\alpha,\beta)) \cap K(M) = \{0\}$ for all α,β.

$$A(\alpha) = \begin{bmatrix} \alpha_1 & 1 + 2\alpha_2 \\ 0 & \alpha_1+\alpha_2 \end{bmatrix}, \quad B(\alpha) \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C(\alpha) = [1 \quad 0].$$

Now, the transfer function $= \dfrac{1 + 2\alpha_2}{s^2 - (2\alpha_1+\alpha_2)s + \alpha_1(\alpha_1+\alpha_2)}$

and the parametrization is globally identifiable, since for (α_1,α_2) such that the system is minimal, the three coefficients of the transfer function can be identified, and α_2 can be determined from the numerator and α_1 can then be determined from the coefficients of s.

Now in this parametrization $\det[(Z(\alpha,\beta),M)'(Z(\alpha,\beta),M)] = 0$ if and only if

$$\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix} + \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}$$

in which case $A(\beta) = \begin{pmatrix}1+\alpha_1+\alpha_2 & -(1+2\alpha_2)\\ 0 & \alpha_2\end{pmatrix}$

and setting $Q = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ gives

$$Q\,A(\alpha) - A(\beta)Q = (1+2\alpha_2)I_2$$

$$Q\,B(\alpha) = 0$$

$$-C(\beta)Q = 0$$

which are all admissible variations in A,B and C. Thus we have constructed a non-zero element in $R(Z(\alpha,\beta))$ and $R(M)$.

If the better condition of Theorem 3.6 is used instead of the more restrictive one above, no non-zero solution exists verifying that the parametrization is globally identifiable. It is also noted that the parametrization satisfies the sufficient condition for local identifiability given in Theorem 3.4 for all parameter values. ■

Remarks on Global Identifiability

Finding conditions for global identifiability has been the major time consumer of this research and is perhaps the least productive in its results. It is the purpose of this section to outline the mathematical problems in determining global identifiability.

Definition 3.2 of global identifiability removes non-minimal systems from consideration. (see remark following the definition). Therefore define U to be the largest subset of Ω such that $(A(\alpha), B(\alpha), C(\alpha))$ is minimal for all $\alpha \in U$. Then global identifiability is exactly equivalent to any of the following three conditions.

1) (Markov parameters) G restricted to U is injective (see Theorem 3.3 for definition of G).

2) (Transfer function) H restricted to U is injective, where $H(\alpha)$ is the set of coefficients of the transfer function, i.e. the coefficients of $\det(Is - A(\alpha))$ and the coefficients of $\det(Is - A(\alpha)) \times [C(\alpha)(Is - A(\alpha))^{-1}B(\alpha) + D(\alpha)]$.

3) (Similarity transformation) F restricted to $GL(n) \times U$ is injective (see Theorem 3.4 for definition of F).

The restriction of these functions to U makes any analysis intractable in all but the simplest cases, since U is not easily described and most useful results in global analysis require that the function's domain is well-behaved.

Assuming that the above conditions are indeed intractable in practice, sufficient conditions can be obtained if the functions F,G, or H are injective without the restriction to U. However such modified conditions may be too strong. G will never be injective if multiple

representations of non-minimal systems are possible. H will not be injective in many cases; for example in any parametrization with the C matrix completely free and more than n free parameters in the A and B matrices combined, H cannot be injective since when C is zero the only non-zero coefficients in the transfer function are the n coefficients of $\det(Is - A(\alpha))$ and hence since we assumed greater than n free parameters in A and B, there will be infinitely many values of α with the same image under H. Requiring H to be injective would thus seem to be overly restrictive. The function F will not be injective for a globally identifiable parametrization, only if two non-minimal systems are related by a similarity transformation. This is in fact much less restrictive than with the two other functions G and H but there exist examples where F is not injective but the parametrization is globally identifiable, as in the following example.

Example 3.6

$$A(\alpha) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B(\alpha) \begin{pmatrix} \alpha_1 \\ 1 \end{pmatrix}, \quad C(\alpha) = (\alpha_2 \quad 0)$$

$$(\alpha_1, \alpha_2) \in R^2$$

$$F(T,\alpha) = (TA(\alpha)T^{-1}, TB(\alpha), C(\alpha)T^{-1}, D(\alpha))$$

and $$F\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -t \\ 0 \end{pmatrix}\right) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, (0 \ , \ 0), 0\right)$$

for all $t \neq 0$ so that F is not injective at $\alpha_2 = 0$. Notice that $\alpha_2 = 0$ corresponds to the unobservable systems so does not affect global identifiability, which is ensured if we look at the transfer

function = $\frac{\alpha_2(1 + \alpha_1 s)}{s^2}$, and assume $\alpha_2 \neq 0$. For this system the functions G and H are also not injective at $\alpha_2 = 0$. ∎

We can thus conclude that a reasonable sufficient condition for global identifiability is that $F : GL(n) \times \Omega \to R^N$ is injective. Such a question is in general exceedingly difficult to answer without some additional assumptions. The general mathematical problem of determining whether a map from a subset of one Euclidean space into another is injective is non-trivial. Clearly a necessary condition is that it is locally injective everywhere, however this is not sufficient. If the image space is of higher dimension than the domain the functions which are locally injective everywhere but not globally injective are very easy to construct. (see for example Example 3.4), and general results are very restrictive.

If the domain and image spaces are of the same dimension then better results are available. For example Palais' Theorem (see Palais (1959), Wu and Desoer (1972), Ortega and Rheinbolt (1970)) which states that : If f is a C^k map ($k \geq 1$) from R^n into R^n, then f is a C^k diffeomorphism if and only if, (i) $\det \frac{\partial f(x)}{\partial x} \neq 0 \quad \forall x \in R^n$ and (ii) $\lim_{||x|| \to \infty} ||f(x)|| = \infty$. (Note, a C^k diffeomorphism is by definition a bijective C^k map whose inverse is also C^k). This is a very strong and also surprising result in that the conditions are both necessary and sufficient. However we are only interested in maps being

injective and do not require them to be surjective. Palais' Theorem could be applied to single input/single output systems with $2n$ degrees of freedom, to show that H is bijective.

General conditions for a map just to be injective tend to be very restrictive and hard to verify. For example in the related control area of the global observability of nonlinear dynamical systems it is required to have the map from the initial condition into the output sequence injective, and this has been considered in some detail by Fitts(1970), who did not find any sufficient conditions for maps to be injective, that were not too restrictive for our present purposes.

The global identifiability problem is complicated considerably by the domain of F being $GL(n) \times \Omega$. However the condition that $\det \left[\frac{\partial F}{\partial (T,\alpha)} (T,\alpha)\right] \neq 0$ for all T and α can be replaced by $\det \left[\frac{\partial F}{\partial (T,\alpha)} (I,\alpha)\right] \neq 0$ for all α. (see proof of Theorem 3.4). Also if we assume that $(A,B,C,D)(\alpha)$ is affine then F is a highly structured function. We now make a conjecture for which no proof or counter examples are known to us.

(Open) Conjecture

Let $(A,B,C,D)(\alpha) : R^{nm+np+mp} \to R^{N}$ be an affine parametrization, then it is globally identifiable if $\det \left[\frac{\partial F}{\partial (T,\alpha)} (I,\alpha)\right] \neq 0$ for all $\alpha \in R^{nm+np+mp}$.

If the dimension of α is allowed to be less than (nm+np+mp) then counter examples to the conjecture can be found (see Example 3.4). Further if the affine restriction is not made counter examples can be found (see below).

Example 3.7

Consider the parametrization,

$A(\alpha) = \alpha_1$, $B(\alpha) = (\alpha_2, 1 - \alpha_2^2)$, $C(\alpha) = \alpha_3$, $D(\alpha) = (0, 0)$ then

$$\det\left[\frac{\partial F(I,\alpha)}{\partial(T,\alpha)}\right] = \det\begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha_2 & 0 & 1 & 0 \\ 1-\alpha_2^2 & 0 & -2\alpha_2 & 0 \\ -\alpha_3 & 0 & 0 & 1 \end{bmatrix}$$

$$= 1 + \alpha_2^2 \geq 1 \quad \forall\, \alpha \in R^3$$

However if $\alpha_2 \neq 0$ then

$$F\left(-\frac{1}{\alpha_2^2}, (\alpha_1, \alpha_2, \alpha_3)\right) = \left(\alpha_1, \left(-\frac{1}{\alpha_2}, 1 - \frac{1}{\alpha_2^2}, -\frac{\alpha_3}{\alpha_2^2}\right)\right)$$

$$= F\left(1, (\alpha_1, -\alpha_2^{-1}, -\alpha_3\alpha_2^{-2})\right)$$

So that F is not injective (even if restricted to minimal systems) and so the parametrization is not globally identifiable. ■

In conclusion it would seem from the above discussion that the sufficient condition for global identifiability given in Theorem 3.6 is a good condition, but that if the number of degrees of freedom is (nm+np+mp) then the better condition of the Open Conjecture may be true.

3.4 Partial Identifiability

An interesting question that arises in some practical applications is: given a parametrization $(A,B,C,D)(\alpha,\beta) : \Omega_1 \times \Omega_2 \to R^N$. can α be identified independently from β? The implication here is that we are only interested in α and are not concerned if β is not identified uniquely. For example β could represent the feedback gains and α some open loop parameters. This motivates the following definition.

Definition 3.3

A parametrization $(A,B,C,D)(\alpha,\beta) : \Omega_1 \times \Omega_2 \subset R^{q_1} \times R^{q_2} \to R^N$ is said to be locally partically identifiable in α at $\alpha = \hat{\alpha}$ and $\beta = \hat{\beta}$ if there exists an $\varepsilon > 0$ such that

(i) $||\alpha_i - \hat{\alpha}|| < \varepsilon$, $||\beta_i - \hat{\beta}|| < \varepsilon$, $i = 1,2$.

(ii) $D(\alpha_1,\beta_1) = D(\alpha_2,\beta_2)$

and (iii) $C(\alpha_1,\beta_1)A^k(\alpha_1,\beta_1)B(\alpha_1,\beta_1) = C(\alpha_2,\beta_2)A^k(\alpha_2,\beta_2)B(\alpha_2,\beta_2)$

$k = 0,1,2,...$

imply $\alpha_1 = \alpha_2$. ■

The following Theorem gives conditions for local partial identifiability.

Theorem 3.8

Let $(A,B,C,D)(\alpha,\beta) : \Omega_1 \times \Omega_2 \to R^N$ (with Ω_i open subsets of R^{q_i} $i = 1,2$) be a C^1 parametrization of the system matrices (A,B,C,D), and assume that $(A,B,C,D)(\hat{\alpha},\hat{\beta})$ is minimal. Suppose

1) rank $[Z((\alpha,\beta),(\alpha,\beta)); M_\beta(\alpha,\beta)] = r_2$ for all (α,β) in some neighborhood of $(\hat{\alpha},\hat{\beta})$.

2) rank $[Z((\alpha,\beta),(\alpha,\beta)), M_\beta(\alpha,\beta), M_\alpha(\alpha,\beta)] = r_1 + r_2$ for all (α,β) in some neighborhood of $(\hat{\alpha},\hat{\beta})$.

Then $(A,B,C,D)(\alpha,\beta)$ is locally partially identifiable in α at $(\hat{\alpha},\hat{\beta})$ if and only if $r_1 = q_1$.

In here $Z((\alpha,\beta), (\alpha,\beta))$ is as defined in Theorem 3.6 and $M_\alpha(\alpha,\beta)$ and $M_\beta(\alpha,\beta)$ are derivatives of the parametrization with respect to α and β respectively (see Theorem 3.3). ■

The proof relies on the following lemma.

Lemma 3.9

Let Ω_1 and Ω_2 be open sets in R^{m_1} and R^{m_2} and $f: \Omega_1 \times \Omega_2 \to R^n$ be a C^k map with $k \geq 1$, thus f maps (x,y) into f(x,y), with $x \in \Omega_1$ and $y \in \Omega_2$. Also assume

1) rank $\frac{\partial f(x,y)}{\partial y} = r_1$ $\forall (x,y)$ in some neighborhood of $(\hat{x},\hat{y})$.

2) rank $\frac{\partial f}{\partial x \partial y}(x,y) = r_1 + r_2$ $\forall$ (x,y) in some neighborhood of $(\hat{x},\hat{y})$.

Then, there exists a neighborhood of $(\hat{x},\hat{y})$, say W, such that

$$f(x_1,y_1) = f(x_2,y_2) \text{ and } (x_1,y_1),(x_2,y_2) \in W$$

imply $x_1 = x_2$, if and only if $r_1 = m_1$.

Proof of Lemma 3.9

We will now use the rank theorem (Lemma 3.1) to find $f^{-1}(f(\tilde{x},\tilde{y}))$ for any $(\tilde{x},\tilde{y})$ in some neighborhood of $(\hat{x},\hat{y})$.

From Lemma 3.1 there exist neighborhoods U of $\hat{x}$ and V of $\hat{y}$ such that

$$f(x,y) = u \circ \phi \circ u'$$

for all $(x,y) \in U \times V$, where u and u' are C^k diffeomorphisms and

$$\phi(x_1,x_2,\ldots,x_{m_1+m_2}) = (x_1,x_2\ldots x_{r_1+r_2}, 0 \ldots 0)$$

and further

$$f(\tilde{x},y) = v_{\tilde{x}} \circ \psi \circ v'_{\tilde{x}} \text{ for all } y \in V \text{ and any fixed } \tilde{x} \in U.$$

where $v_{\tilde{x}}$ and $v'_{\tilde{x}}$ are C^k diffeomorphisms and

$$\psi(x_1,x_2,..x_{m_2}) = (x_1,x_2,\ldots x_{r_2}, 0 \ldots 0)$$

Therefore

$$S_1(\tilde{x},\tilde{y}) = f^{-1}(f(\tilde{x},\tilde{y})) \cap (U \times V)$$

$$= (U \times V) \cap u^{-1}(u'_1(f(\tilde{x},\tilde{y})),\ldots,u'_{r_1+r_2}(f(\tilde{x},\tilde{y})),z_1,\ldots,z_k)$$

where $z_i \in R$ for $i = 1,2,\ldots k$, and $k = m_1 + m_2 - r_1 - r_2$.

also $f^{-1}(f(\tilde{x},\tilde{y})) \supset S_2(\tilde{x},\tilde{y}) = \{(\tilde{x},y) \mid f(\tilde{x},y) = f(\tilde{x},\tilde{y}),\ y \in V\}$

$$= (U \times V) \cap v_{\tilde{x}}^{-1}(v'_{\tilde{x},1}(f(\tilde{x},\tilde{y})),\ldots,v'_{\tilde{x},r_2}(f(\tilde{x},\tilde{y})),z_1,\ldots z_\ell)$$

where $z_i \in R$, $i = 1,2,\ldots\ell$, and $\ell = m_2 - r_2$.

Hence for all $(\tilde{x},\tilde{y}) \in U \times V$,

$S_1(\tilde{x},\tilde{y})$ is homeomorphic to a neighborhood in R^k

$S_2(\tilde{x},\tilde{y})$ is homeomorphic to a neighborhood in R^ℓ

and $S_2(\tilde{x},\tilde{y}) \subset S_2(\tilde{x},\tilde{y})$ and therefore $S_1(\tilde{x},\tilde{y}) = S_2(\tilde{x},\tilde{y})$

if and only if $r_1 = m_1$.

Clearly if $S_1(\tilde{x},\tilde{y}) = S_2(\tilde{x},\tilde{y})$ the implication that $x_2 = x_1$ in the theorem statement holds and otherwise the implication is false since there

will always be $x_2 \neq x_1$ such that $f(x_1,y_1) = f(x_2,y_2)$ no matter how small a neighborhood of $(\hat{x},\hat{y})$ is taken. ■

Proof of Theorem 3.8

Define $F(T,\alpha,\beta) = (TA(\alpha,\beta)T^{-1}, TB(\alpha,\beta), C(\alpha,\beta)T^{-1}, D(\alpha,\beta))$ then the result holds if and only if there exist neighborhoods U of $\hat{\alpha}$ and V of $\hat{\beta}$ such that

$$F(T,\alpha,\beta) = F(\bar{T},\bar{\alpha},\bar{\beta})$$

and $\alpha,\bar{\alpha} \in U$ and $\beta,\bar{\beta} \in V$ and $T,\bar{T} \in GL(n)$ imply $\alpha = \bar{\alpha}$.

Now by an exactly analagous argument to that of Theorem 3.4 we can restrict T to be in a neighborhood of T = I. Therefore Lemma 3.9 applies and the result follows immediately. ■

An application of this result is given in Corollary 3.10.

Example 3.2 (continued)

n = m = p = 1 and a,b,c are free parameters. Now referring to Theorem 3.8, let α = a and β = (b,c) then conditions 1) and 2) become

1) rank $\begin{bmatrix} 0 & 0 & 0 \\ b & 1 & 0 \\ -c & 0 & 1 \end{bmatrix} = 2 = r_2$ for all $(a,b,c) \in R^3$

2) rank $\begin{bmatrix} 0 & 0 & 0 & 1 \\ b & 1 & 0 & 0 \\ -c & 0 & 1 & 0 \end{bmatrix} = 3 = r_1 + r_2$ for all $(a,b,c) \in R^3$

Thus $r_1 = 1 = q_1$ = dimension of α, and local partial identifiability of a results if $(\hat{a},\hat{b},\hat{c})$ is minimal. ■

3.5 Identifiability in the Presence of Feedback

As pointed out by Åström and Eykhoff (1971) identification in the presence of feedback can cause significant problems. Consider Åström's example given in Figure 3.2.

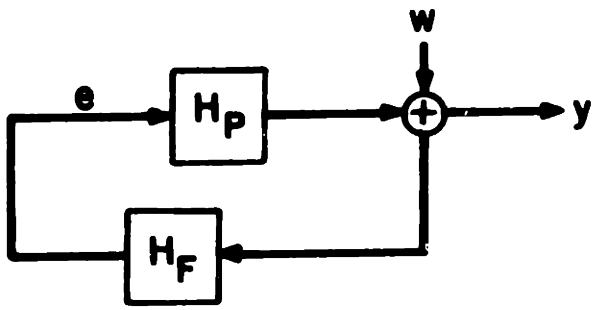

Figure 3.2

In this example a simple-minded identification algorithm would be to ignore H_F, observe e and y and assume that

$$H_p \approx \frac{y}{e}$$

but $\frac{y}{e} = (H_F)^{-1}$ and therefore such estimates would be completely false. The feedback can enter quite subtly as for example an aircraft pilot's response to external disturbances. The correct way to model a system with feedback is to write down the state space equations in open-loop form with the feedback matrix modifying the system matrices. Then the identifiability questions can be asked and answered as in the previous sections of this chapter.

For the general problem when the system and feedback matrices are all parametrized by some unknown parameters, and it is desired to identify some of the parameters and not others then the conditions of Theorem 3.8 would have to be used.

Two particular situations have been worked out by way of example in the following Corollary (a direct consequence of Theorems 3.4 and 3.8)

Corollary 3.10

Consider the linear feedback system,

$$\frac{dx(t)}{dt} = A(\alpha)x(t) + B(\alpha)u(t)$$

$$y(t) = C(\alpha)x(t), \qquad u(t) = -Fx(t) + v(t)$$

with $F \in R^{m \times n}$ where $(A,B,C)(\alpha) : R^{q} \to R^{n(n+m+p)}$ is a C^{r} parametrization of (A,B,C). Assume

1) rank $\begin{bmatrix} W(\alpha,F); & B(\alpha) \otimes I \\ & 0 \end{bmatrix} = r_2$ in a neighborhood of $\hat{\alpha}$. And

2) rank $\begin{bmatrix} W(\alpha,F); & B(\alpha) \otimes I \\ & 0 \end{bmatrix} ; M(\alpha) = r_1 + r_2$ in a neighborhood of $\hat{\alpha}$.

where $W(\alpha,F) = \begin{bmatrix} I \otimes (A(\alpha) - B(\alpha)F)' - (A(\alpha) - B(\alpha)F) \otimes I \\ I \otimes B'(\alpha) \\ - C(\alpha) \otimes I \end{bmatrix}$

Then the parameters α and F are locally identifiable if and only if $r_1 + r_2 = nm + q$.

Further the parameters α are locally partially identifiable if and only if $r_1 = q$. ■

When considering systems under feedback the analysis of invariants has been an active area of research. (see Popov (1972), Morse (1972), Wang and Davison (1972), Wolovich and Falb (1969)). Therefore if one had a completely unknown system and wanted to identify as much as possible in the presence of an unknown state feedback matrix it would be natural to use a canonical form under the transformation $(A,B,C) \rightarrow (T(A - BF)T^{-1}, TB, CT^{-1})$ with $F \in R^{m \times n}$ and $T \in GL(n)$. However such an analysis seems more suited to the design rather than the identification problem, since it is unlikely that only the invariants under feedback are required to be identified.

3.6 Comparison with the Information Matrix and Sensitivity Analysis

Previous work on identifiability has been of two types. Firstly there has been work on deterministic single input systems with the (A,B) matrices in standard controllable form and the question answered is what inputs will enable the unknown parameters in the (A,C) matrices to be identified. (see Stanley and Yue (1970), Fisher (1965), Lee (1964)).

Secondly there has been work in the statistics and stochastic control literature on the type of observation noise statistics, control inputs and system parametrizations that enable the system matrices to be identified asymptotically. This work is generally based on the so-called (Fisher) Information Matrix, and needs knowledge such as the conditional probability density of the present observations given all previous observations and the parameters. (see Rothenburg (1971), Tse (1973), Mehra (1972), Aström and Bohlin (1966)). The information matrix is a quite general approach, and indeed also gives approximations of the covariance of the parameter estimates, however for the problem that has been considered in this chapter it gives computationally difficult tests.

The present work is complementary to the above work in that it assumes the inputs and observations are sufficient to identify the transfer function and then determines the identifiability of the system parametrization. The equivalence of the two approaches will now be shown for a particular situation.

Example 3.8

Consider the linear discrete time dynamical system

$$x(k+1) = Ax(k) + Bu(k), \qquad x(0) = 0$$

$$z(k) = Cx(k) + w(k)$$

where $w(k)$ is a Gaussian white noise sequence with $E(w(k)) = 0$ and $E(w(k)w(j)) = R\,\delta_{jk}$. (with $R = R' > 0$).

The evolution equation for s sample points (in time), can be written as

$$\begin{bmatrix} z(1) \\ z(2) \\ \vdots \\ z(s) \end{bmatrix} = \begin{bmatrix} H_0 & 0 & \cdots & 0 \\ H_1 & H_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ H_{s-1} & & H_1 & H_0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(s-1) \end{bmatrix} + \begin{bmatrix} w(1) \\ w(2) \\ \vdots \\ w(s) \end{bmatrix}$$

where $H_k = C\,A^k B$

Now if (A,B,C) are parametrized as (A,B,C)(α) the equation can be written as,

$$\tilde{z}_s = \tilde{h}_s(\alpha) + \tilde{w}_s$$

with the obvious interpretation of the symbols.

For such a system Schweppe (1973) shows that the information matrix at $\alpha = \hat{\alpha}$ is,

$$M(\hat{\alpha}) = \left(\frac{\partial \tilde{h}_s}{\partial \alpha}(\hat{\alpha})\right)' \quad \tilde{R}_s^{-1} \left(\frac{\partial \tilde{h}_s}{\partial \alpha}(\hat{\alpha})\right)$$

where $\tilde{R}_s = I_s \otimes R$

Now let $\tilde{h}_s(\alpha) = \begin{bmatrix} h_c(\alpha) \\ h_1(\alpha) \\ \vdots \\ h_{s-1}(\alpha) \end{bmatrix}$

where $h_k(\alpha) = \sum_{j=0}^{k} H_{k-j}(\alpha)\, u(j)$ and $H_k(\alpha) = C(\alpha)A^k(\alpha)B(\alpha)$.

Then $\dfrac{\partial h_k(\alpha)}{\partial \alpha} = \sum_{j=0}^{k} \dfrac{\partial H_{k-j}(\alpha) u(j)}{\partial \alpha_i}$

and $\dfrac{\partial h_k(\alpha)}{\partial \alpha} = \sum_{j=0}^{k} (I_p \otimes u'(j)) \dfrac{\partial \bar{\bar{H}}_{k-j}}{\partial \alpha}$ (where $\bar{\bar{H}}_k$ is H_k listed as a vector by rows). Hence

$$\frac{\partial \tilde{h}(\alpha)}{\partial \alpha} = \begin{bmatrix} U_0 & 0 & \cdots & 0 \\ U_1 & U_0 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ U_{s-1} & & U_1 & U_0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial \bar{\bar{H}}_0(\alpha)}{\partial \alpha} \\ \dfrac{\partial \bar{\bar{H}}_1(\alpha)}{\partial \alpha} \\ \vdots \\ \dfrac{\partial \bar{\bar{H}}_{s-1}(\alpha)}{\partial \alpha} \end{bmatrix} = U\, K(\alpha) \quad \text{say.}$$

where $U_k = I_p \otimes u'(k)$ and hence

$$M_s(\alpha) = K'(\alpha) U' \tilde{R}^{-1} U\, K(\alpha)$$

Now $M_s^{-1}(\alpha)$ gives a lower bound on the covariance of any <u>unbiased</u> estimator of α. (Cramér-Rao lower bound.) Therefore to asymptotically identify α exactly with an unbiased estimator we need that all the values of $M_s(\alpha)$ tend to infinity as $s \to \infty$. This is a condition on both the parametrization and the input sequence. It is clearly necessary that $K(\hat{\alpha})$ must be of full rank for the information to tend to infinity as $s \to \infty$, in which case inputs will exist to ensure this is so. (This is similar to the persistant excitation required by Åström and Bohlin (1966)). The condition that the rank $K(\hat{\alpha}) = q$ is an identical sufficient

condition for local identifiability as that given in Theorem 3.3, however this condition is only necessary if rank $K(\alpha)$ is constant in a neighborhood of $\hat{\alpha}$. Indeed it is not necessary for identifiability that the Cramér-Rao lower bound tend to zero as $s \to \infty$, because biased estimators can sometimes improve on the Cramér-Rao lower bound. For example the maximum likelihood estimator (see Box and Jenkins (1970)) is in general biased for any fixed sample length, s, but as $s \to \infty$ the estimates tend to the true values. For maximum likelihood estimators the Cramér-Rao lower bound will only be tight when $M_s(\hat{\alpha}) > 0$ and when $s \to \infty$. If $M(\alpha)$ is singular for $\alpha = \hat{\alpha}$ but not in a neighborhood of $\hat{\alpha}$, then Cramér-Rao lower bound does not give meaningful results, since in this case no linearization is valid near $\hat{\alpha}$.

In a very similar manner to the above analysis for the noise free case, the sensitivity of the outputs with respect to the parameters given the inputs, can be produced. (see for example Kokotović and Rutman (1965)). Identifiability will then result if the sensitivity of the outputs is of full rank. Such a result would depend on the inputs, but given that the input sequence is satisfactory, a condition equivalent to that of Theorem 3.3 will be obtained for the identifiability of a parametrization. One can also consider the sensitivity of the transfer function or Markov parameters with respect to the unknown system parameters, and then this would be equivalent to Theorem 3.3.

CHAPTER 4

IDENTIFIABILITY FROM OUTPUT CORRELATION

4.1 Introduction

In this chapter we consider the identifiability of linear system parametrizations when the system is driven by white noise and only the output is observed. This is in general a significantly more difficult problem than when input observations are also made, and is referred to variously as the spectral factorization problem and the inverse problem of covariance generation. Before the identifiability problem can be approached characterizations of indistinguishable systems in these situations are required. Sections 4.2 and 4.3 give the appropriate background material for the continuous time and discrete time situations respectively. Then in Section 4.4 the identifiability problem is considered.

4.2 Continuous Time Systems

In this section we consider the system,

$$\frac{dx(t)}{dt} = A\,x(t) + B\,u(t)$$

$$y(t) = C\,x(t) + D\,u(t)$$

with $x(\cdot) \in R^{n}$, $u(\cdot) \in R^{m}$, $y(\cdot) \in R^{p}$ and the following assumptions.

A1. The input u(t) is not observed directly but is assumed to be a white noise process normalized such that $E(u(t)u(\tau)) = I\delta(t-\tau)$.

A2. The matrix A is asymptotically stable (i.e. the eigen values of A are strictly in the left half plane).

A3. The system has reached steady state when the observations begin (i.e. the output process y(t) is a stationary random process).

A4. The system to be identified is globally minimal, i.e. the dimension of the state is less than or equal to that of any other system with the same output spectral density when driven by white noise. (Anderson (1969)).

Under these assumptions the most information that may be obtained from output observations is the output spectral density, $\Phi(s) = G(s)G'(-s)$, where $G(s) = C(Is - A)^{-1} B + D$.

The identification problem is thus, given observations of $\Phi(s)$ find a system $G(s)$ such that $\Phi(s) = G(s)G'(-s)$. This is the so-called spectral factorization problem. It has been extensively studied, and a general solution in the frequency domain has been given by Youla (1961). A general time domain treatment of this problem has been given by Anderson (1969). Since we are primarily concerned with state space representations the results of Anderson are most useful for our purposes, and are restated here for easy reference.

Let $Z(s)$ be a positive real matrix of rational functions such that

$$\Phi(s) = Z(s) + Z'(-s) \qquad \text{(sum decomposition)}$$

$Z(s)$ is in fact the Laplace transform of the correlation function $R_{yy}(\tau)$ for $\tau \geq 0$.

Now let (A,G,C,J) be a minimal realization of $Z(s)$. Then we have the following result.

Lemma 4.1 (Anderson (1969))

Consider the matrix equation

$$(ME)\qquad \begin{bmatrix} AP + PA' & PC' - G \\ CP - G' & -J - J' \end{bmatrix} = - \begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B' & D' \end{bmatrix}$$

in the unknown matrices P(nxn), B(nxm), and D(pxm). Then every globally minimal solution, G(s) to $\Phi(s) = G(s)G'(-s)$ has a state space realization (A,B,C,D) with B and D satisfying (ME) together with some $P = P' > 0$.

Conversely if B,D and $P = P' > 0$ satisfy (ME), $G(s) = C(Is - A)^{-1} B + D$ is a globally minimal solution to $\Phi(s) = G(s)G'(-s)$. ■

Lemma 4.1 essentially characterizes all equivalent state space solutions to the spectral factorization problem, and is used in the following corollary.

Corollary 4.2

If (A_1,B_1,C_1,D_1) and (A_2,B_2,C_2,D_2) are globally minimal systems then

$$G_1(s)G_1'(-s) = G_2(s)G_2'(-s)$$

(where $G_i(s) = C_i(Is - A_i)^{-1}B_i + D_i$, $i = 1,2$.)

if and only if there exists $T \in GL(n)$ and $Q = Q'$ such that

$$\left.\begin{aligned} A_1 &= T A_2 T^{-1} \\ C_1 &= C_2 T^{-1} \\ QA_1' + A_1Q &= - B_1B_1' + TB_2B_2'T' \\ QC_1' &= - B_1D_1' + TB_2D_2' \\ D_1D_1' &= D_2D_2' \end{aligned}\right\}$$

Further if D_1D_1' is nonsingular the above is equivalent to there being a similarity transformation between the Kalman filters of the two systems.

Proof

We know from global minimality and Lemma 4.1 that if

$$\phi(s) = Z(s) + Z'(-s)$$

with Z(s) positive real, then a minimal realization of Z(s) is given by (A_1,G,C_1,J) where

$$G = P_1C_1' + B_1D_1'$$

and $J + J' = D_1D_1'$, and where $P = P' > 0$ satisfies

$$P_1A_1' + A_1P_1 = - B_1B_1'.$$

Also Lemma 4.1 implies that there exists a unique similarity transformation $T \in GL(n)$ between (A_2,C_2) and $A_1,C_1)$, i.e.

$$A_1 = T A_2T^{-1}, \qquad C_1 = C_2T^{-1}$$

Therefore $(A_2,T^{-1}G, C_2,J)$ is also a minimal realization of Z(s) and there exist $P_2 = P_2 > 0$ such that

$$\begin{cases} P_2A_2' + A_2P_2 = - B_2B_2' \\ T^{-1}G = P_2C_2' + B_2D_2' \\ J + J' = D_2D_2' \end{cases}$$

Simple manipulation of the above equations gives that

$$(P_1-TP_2T')A_1' + A_1(P_1-TP_2T') = -B_1B_1' + TB_2B_2'T'$$

$$G = TP_2T'C_1' + TB_2D_2 = P_1C_1' + B_1D_1'$$

Hence, setting $Q = P_1 - TP_2T'$ the 'only if' statement follows. The if statement can be verified by direct substitution.

The equivalence of the condition with the equivalence of the system Kalman filters is proved as follows. (That two systems with equivalent Kalman filter's are indistinguishable is essentially shown in Geesay and Kailath (1969)).

The Kalman filter is realized by

$$\frac{d}{dt}\hat{x}(t) = A\,\hat{x}(t) - K\,\nu(t)$$

$$y(t) = C\,\hat{x}(t) + \nu(t)$$

where $\nu(t) = C\,\hat{x}(t) - y(t)$

$$K = (\Pi C' + BD')(DD')^{-1}$$

and

$$\Pi A' + A\Pi + BB' - (\Pi C' + BD')(DD')^{-1}(\Pi C' + BD')' = 0$$

Now defining $\Sigma = P - \Pi$, and using Lemma 4.1 we get

$$(*)\quad \Sigma A' + A\Sigma - (G - \Sigma C')(J + J')^{-1}(G - \Sigma C')' = 0$$

for which there is a unique minimal solution for Σ.

Thus two Kalman filters are equivalent if there exists $T \in GL(n)$ such that

$$A_1 = TA_2T^{-1}, \quad C_1 = C_2T^{-1}, \quad K_1 = TK_2$$

and the covariance of ν_1 = covariance of ν_2 i.e.

$$D_1D_1' = D_2D_2'$$

Hence the only equation to prove is that $K_1 = TK_2$.

Consider (*) for system (1), then substituting for A_1 and C_1 and setting $\tilde{\Sigma} = T^{-1}\Sigma\, T^{-1}{}'$, gives

$$\tilde{\Sigma}\, A_2' + A_2\, \tilde{\Sigma} - (T\, G - \tilde{\Sigma}\, C_2')(J + J')^{-1}(T\, G - \tilde{\Sigma}\, C_2') = 0$$

but (*) for system (2) gives

$$\Sigma_2\, A_2' + A_2\, \Sigma_2 - (T\, G - \Sigma_2\, C_2')(J + J')^{-1}(T\, G - \Sigma_2\, C_2') = 0$$

Hence since there is a unique minimal solution

$$\Sigma_2 = \tilde{\Sigma} = T^{-1}\, \Sigma_1\, T^{-1}{}',$$

Thus

$$\begin{aligned} K_1 &= (\Pi_1\, C_1' + B_1 D_1')(D_1 D_1')^{-1} \\ &= (P_1 C_1' + B_1 D_1' - \Sigma_1 C_1')(D_1 D_1')^{-1} \\ &= (G - T\, \Sigma_2\, C_2')(D_2 D_2')^{-1} \\ &= T(T^{-1}\, G - \Sigma_2\, C_2')(D_2 D_2')^{-1} \\ &= T(P_2 C_2' + B_2 D_2' - \Sigma_2 C_2')(D_2 D_2')^{-1} \\ &= T\, K_2 \end{aligned}$$

as desired.

Conversely that the equivalence of the Kalman filters implies the existence of $Q = Q'$ can be established analagously. ■

This relationship between the solutions to the spectral factorization problem will be used in Section 4.4 where we discuss the identifiability problem stated earlier.

4.3 Discrete Time Systems

In this section analagous results to those of Section 4.2 are derived for the discrete-time case. The results presented are probably equivalent to other work discrete time spectral factorization (e.g. Mehra (1970 and 1971) and Motyka and Cadzow (1967)) but the particular form of the results does not seem to have appeared in the literature.

Consider the discrete-time linear dynamical system:

$$x(k+1) = A\,x(k) + B\,w(k)$$

$$y(k) = C\,x(k) + D\,w(k)$$

where $w(\cdot)$ is white Gaussian noise with $E\{w(k)w'(j)\} = I\,\delta_{kj}$. Now assume that the output spectral density, $\Phi(z)$ (= the z-transform of $E\{y(k)y'(k-i)\}$ is known then the (discrete-time) spectral factorization problem is to find an asymptotically stable (i.e. $\lambda_i(A) < 1$) transfer function $G(z)$ such that

$$\Phi(z) = G(z)G'(z^{-1})$$

Any state space realization of $G(z)$ will then give possible values for the parameters (A,B,C,D).

Since $\Phi(z)$ is a spectral density matrix we can assume without loss of generality that:

A1. $\Phi(z) = \Phi'(z^{-1})$

A2. $\Phi(e^{j\theta})$ is Hermitian nonnegative definite for $-\pi \leq \theta < \pi$.

Further we will assume that,

A3. $\Phi(z)$ is analytic for $z = e^{j\theta}$ with $-\pi \leq \theta < \pi$, i.e. $\Phi(z)$ has no poles on the unit circle.

From the partial fraction expansion of $\Phi(z)$ one can decompose $\Phi(z)$ as

$$\Phi(z) = Z(z) + Z'(z^{-1})$$

where the poles of $Z(z)$ are strictly inside the unit disc. $Z(z)$ is essentially the one-sided z-transform of $E\{y(k)y'(k-i)\}$. We are thus looking for a factorization of the form:

$$\text{(SF)} \qquad \Phi(z) = Z(z) + Z'(z^{-1}) = G(z)G'(z^{-1})$$

Assume that $Z(z)$ has a minimal realization (A,G,C,J), then the following Lemma characterizes all solutions to (SF).

Lemma 4.3

Consider the matrix equation

$$\text{(ME)}' \qquad \begin{bmatrix} APA' - P, & APC' - G \\ CPA' - G', & -J-J' + CPC' \end{bmatrix} = -\begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B' & D' \end{bmatrix}$$

in the unknown matrices $P(n\times n)$, $B(n\times m)$, and $D(p\times m)$. Then every globally minimal solution, $G(z)$ to $\Phi(z) = G(z)G'(z^{-1})$ has a state space realization (A,B,C,D) with B and D satisfying (ME)' together with some $P = P' > 0$.

Conversely if B,D and $P = P' > 0$ satisfy (ME)', $G(z) = C(Iz - A)^{-1} B + D$ is a globally minimal solution to $\Phi(z) = G(z)G'(z^{-1})$.

Proof

We will make the transformation $s = \frac{z-1}{z+1}$ and reduce the problem to the continuous time case. First we note an observation about this transformation which is easily verified by direct substitution.

Fact 4.4

If (A,B,C,D) is a minimal realization of G(z) (with $\lambda_i(A) < 1$) and $W(s) = G\left(\frac{1+s}{1-s}\right)$ then the McMillan degrees of W(s) and G(z) are equal and further

$$(-(I-A)(I+A)^{-1},\ \sqrt{2}\,(I+A)^{-1}B,\ \sqrt{2}\,C(I+A)^{-1},\ D - C(I+A)^{-1}B)$$

is a minimal realization of W(s).

Conversely if (F,G,H,J) is a realization of W(s) and $G(z) = W\left(\frac{z+1}{z-1}\right)$ then

$$(\,(I-F)^{-1}(I+F),\ \sqrt{2}\,(I-F)^{-1}G,\ \sqrt{2}\,H(I-F)^{-1},\ J + H(I-F)^{-1}G)$$

is a minimal realization of G(z).

Now define

$$W(s) = G\left(\frac{1+s}{1-s}\right)$$

$$T(s) = Z\left(\frac{1+s}{1-s}\right)$$

$$Y(s) = \Phi\left(\frac{1+s}{1-s}\right)$$

then

$$Y(s) = T(s) + T'(-s) = W(s)W'(s)$$

and T(s) is positive real.

Since G(z) is globally minimal as a solution to $G(z)G'(z^{-1}) = \Phi(z)$, W(s) is globally minimal as a solution to $W(s)W'(-s) = Y(s)$. Using Fact 4.4 T(s) has a minimal realization

$$(-(I-A)(I+A)^{-1},\ \sqrt{2}\,(I+A)^{-1}G,\ \sqrt{2}\,C(I+A)^{-1},\ J - C(I+A)^{-1}G)$$

and using Lemma 4.1 W(s) will have a realization of the form

$$(-(I-A)(I+A)^{-1},\ \sqrt{2}\ (I+A)^{-1}B,\ \sqrt{2}\ C(I+A)^{-1},\ D - C(I+A)^{-1}B)$$

where B and D satisfy the following equations for some $P = P' > 0$.

$$\begin{cases} -(I-A)(I+A)^{-1}P - P(I+A')^{-1}(I-A') = -2(I+A)^{-1}B\,B'(I+A')^{-1} \\ P(I+A')^{-1}C'\sqrt{2} - \sqrt{2}\ (I+A)^{-1}G = -\sqrt{2}\ (I+A)^{-1}B(D' - B'(I+A')^{-1}C') \\ -J + C(I+A)^{-1}G - J' + G'(I+A')^{-1}C' = -(D-C(I+A)^{-1}B)(D-C(I+A)^{-1}B)' \end{cases}$$

Now since $G(z) = W\left(\frac{z+1}{z-1}\right)$ and using Fact 4.4, G(z) will have a realization (A,B,C,D) satisfying the above equations. Straight forward manipulation then gives the result.

The converse is easily established by direct substitution. ■

Lemma 4.3 shows that if the sum decomposition can be identified then the spectral factor satisfies a relatively simple matrix equation.

The following corollary uses Lemma 4.3 to derive a relationship between the solutions to the spectral factorization problem.

Corollary 4.5

If (A_1,B_1,C_1,D_1) and (A_2,B_2,C_2,D_2) are globally minimal discrete time systems then

$$G_1(z)G_1(z^{-1}) = G_2(z)G_2(z^{-1})$$

(where $G_i(z) = C_i(Is-A_i)^{-1}B_i + D_i$, $i = 1,2.$) if and only if there exists $T \in GL(n)$ and $Q = Q'$ such that

$$\left.\begin{aligned} A_1 &= TA_2T^{-1} \\ C_1 &= C_2T^{-1} \\ A_1QA_1' - Q &= -B_1B_1' + TB_2B_2'T' \\ A_1QC_1' &= TB_2D_2' - B_1D_1' \\ C_1QC_1' &= D_2D_2' - D_1D_1' \end{aligned}\right\}$$

Further if $D_1 D_1'$ is nonsingular the above conditions are equivalent to there being a similarity transformation between the Kalman filter of the two systems.

Proof

The proof is analagous to that of Corollary 4.2. The equivalence of the Kalman filters is also shown by Tse and Weinert (1973) by different techniques. ■

Comments on the Correlation Identification Technique due to Mehra (1971).

To illustrate how the previous results can be applied, the correlation technique of Mehra (1971) is now considered by way of example.

This algorithm estimates the system parameters from estimates of output correlation function,

$$C_i = E\{y(k)y'(k-i)\}$$

$$\simeq \hat{C}_i^N = \frac{1}{(N-i)} \sum_{k=i}^{N} y(k)y'(k-i)$$

Now

$$\Phi(z) = \sum_{1}^{\infty} C_i z^{-i} + \sum_{1}^{\infty} C_i' z^{i} + C_0$$

$$= Z(z) + Z'(z^{-1}) \qquad \text{(see Lemma 4.3)}$$

$$= C(Iz-A)^{-1} B + J + G'(Iz^{-1} - A')^{-1} C' + J'$$

$$= \sum_{1}^{\infty} CA^{i-1} G z^{-i} + \sum_{1}^{\infty} G'A'^{i-1} C' z^{i} + J + J'$$

(A,G,C,J) can be a realization of the sum decomposition, $Z(z)$, of $\Phi(z)$

by Lemma 4.3. Further the $\hat{C}_i$ will now be estimates of the Markov parameters of $Z(z)$. Hence the matrices A and C can be estimated using a standard realization algorithm (e.g. Ho and Kalman (1967)). Finally, the B and D matrices will be the solution to the algebraic equations given in Lemma 4.3.

In order to obtain a unique solution for (A,C),(A,C) could be put in a canonical form such as those of Chapter 2, but still B and D will not be uniquely determined even if a minimum phase assumption is made (unless m=p=1). This problem is pointed out by Mehra (1971) where he suggests identifying the Kalman filter instead, whose transfer function will indeed be identifiable by Corollary 4.5. The identifiability problem will now be discussed in more detail in the following section.

4.4 Identifiability from Output Observation

The question considered in this section is when a parametrization is identifiable from output observation alone.

Definition 4.2

Let $(A,B,C,D)(\alpha) : \Omega \subset R^{q} \to R^{n(n+m+p)+mp}$ be a parametrization of the system matrices (A,B,C,D). This parametrization is said to be **locally identifiable from its output spectral density** at $\alpha = \hat{\alpha} \in \Omega$ if there exists an $\varepsilon > 0$ such that

(i) $||\alpha - \hat{\alpha}|| < \varepsilon, \quad ||\beta - \hat{\alpha}|| < \varepsilon, \ \alpha,\beta \in \Omega.$

and (ii a) (continuous time),
$G(s,\alpha)G'(-s,\alpha) = G(s,\beta)G'(-s,\beta)$ for all $s \in \mathbb{C}$.

(ii b) (discrete time)
$G(z,\alpha)G'(z^{-1},\alpha) = G(z,\beta)G'(z^{-1},\beta)$ for all $s \in \mathbb{C}$.

imply $\alpha = \beta$

(where $G(s,\alpha) = C(\alpha)(Is-A(\alpha))^{-1} B(\alpha) + D(\alpha)$. ■

A condition for local identifiability in this sense can be obtained via the characterizations of all globally minimal solutions to the spectral factorization problem given in Corollaries 4.2 and 4.5. Thus local identifiability from the output spectral density is implied if the following equations have a unique solution $\alpha = \beta$, $T = I$, $P = 0$ for all $\alpha,\beta \in N_{\varepsilon}(\hat{\alpha})$.

1. Continuous Time

$$Q = Q', \; A(\alpha) = TA(\beta)T^{-1}, \; C(\alpha) = C(\beta)T^{-1}$$

$$A(\alpha)Q + QA'(\alpha) = - B(\alpha)B'(\alpha) + T\,B(\beta)B'(\beta)T'$$

$$Q\,C'(\alpha) = - B(\alpha)D'(\alpha) + T\,B(\beta)D'(\beta)$$

$$D(\alpha)D'(\alpha) = D(\beta)D'(\beta)$$

2. Discrete Time

$$Q = Q', \; A(\alpha) = T\,A(\beta)T^{-1}, \; C(\alpha) = C(\beta)T^{-1}$$

$$A(\alpha)Q\,A'(\alpha) - Q = - B(\alpha)B'(\alpha) + T\,B(\beta)B'(\beta)T'$$

$$A(\alpha)Q\,C'(\alpha) = - B(\alpha)D'(\alpha) + T\,B(\beta)D'(\beta)$$

$$C(\alpha)Q\,C'(\alpha) = - D(\alpha)D'(\alpha) + D(\beta)D'(\beta)$$

The following theorem can be proved in an analagous manner to Theorem 3.4.

Theorem 4.6

Set $(A,B,C,D)(\alpha) : \Omega \subset R^{q} \to R^{n(n+m+p)+mp}$ (with Ω an open set in R^{q}) be a C^1 parametrization of the system matrices (A,B,C,D) of continuous time system satisfying (A1) - (A4) of Section 4.2. Then this parametrization is locally identifiable from its output spectral density at $\hat{\alpha} \in \Omega$, if the following linear equations in $(\delta\beta,\delta B,\delta D,\delta T,\delta Q)$,

have a unique solution (i.e. zero).

(i) $\delta Q = \delta Q'$

(ii) $(\hat{A}\delta Q + \delta B\hat{B}' - \delta T\hat{B}\hat{B}') + (\hat{A}\delta Q + \delta B\hat{B}' - \delta T\hat{B}\hat{B}')' = 0$

(iii) $\delta Q\hat{C}' = - \delta B\hat{D}' - \hat{B}\delta D' + \delta T\hat{B}\hat{D}'$

(iv) $\delta D\hat{D}' + \hat{D}\delta D' = 0$

(v)
$$\begin{bmatrix} \delta T\hat{A} - \hat{A}\delta T \\ \hline \delta B \\ \hline - \hat{C}\delta T \\ \hline \delta D \end{bmatrix} = M(\hat{\alpha})\delta\beta$$

where $M(\alpha)$ is defined in Theorem 3.3, and $(\hat{A},\hat{B},\hat{C},\hat{D}) = (A,B,C,D)(\hat{\alpha})$.

The analagous equations for discrete time systems are

(i)' $\delta Q = \delta Q'$

(ii)' $\hat{A}\delta Q\hat{A}' - \delta Q = - \delta B\hat{B}' - \hat{B}\delta B' + \delta T\hat{B}\hat{B}' + \hat{B}\hat{B}'\delta T'$

(iii)' $\hat{A}\delta Q\hat{C}' = - \delta B\hat{D}' - \hat{B}\delta D' + \delta T\hat{B}\hat{D}'$

(iv)' $\hat{C}\delta Q\hat{C}' = -\delta D D' - D\delta D'$

(v)' as (v) above. ■

The above condition is equivalent to a nonzero determinant condition of dimension $[\frac{n}{2}(3n + 2m + 1) + pm]$.

Notice that although the theorem uses implicity the matrices P,G,J of Lemmas 4.1 and 4.3 only the nominal values of the system matrices $(\hat{A},\hat{B},\hat{C},\hat{D})$ are required.

In general fewer parameters can be identified than when input observations are allowed. In fact the number of identifiable parameters

is bounded by $[2np + \frac{p(p+1)}{2}]$, which if $m = p$ is $\frac{p(p-1)}{2}$ less than the $[2np + mp]$ identifiable parameter when input observations are permitted.

CHAPTER 5

GEOMETRICAL PROPERTIES OF MINIMAL SYSTEMS

5.1 Introduction[†]

In this chapter we examine some geometrical properties of minimal linear systems which are of interest in identification and also in their own right.

As mentioned in Chapters 2 and 3 many useful parametrizations admit multiple representations of nonminimal systems. This implies that in such cases if the system being identified with such a parametrization is not minimal then the identification problem no longer has a unique solution, and many minimization algorithms will become ill-posed. Now in on-line algorithms where new estimates of the unknown parameters are made after each new data point, a cost function is essentially minimized at each point in time and there is no reason to suppose that after relatively few data points the estimates will represent minimal systems. Whether estimates become nonminimal or nearly nonminimal depends on the nature of the set of nonminimal systems in the parameter space.

In this chapter the following problem is considered, "given a parametrization of a linear system which may represent both minimal and nonminimal systems does the set of nonminimal systems separate the minimal systems into unconnected regions?" For single input/single output systems the natural parametrization of the standard controllable

[†] I would like to acknowledge that many of the original ideas for this chapter are due to Professor R.W. Brockett of Harvard University.

(or observable) form is considered and indeed the minimal systems do not form a connected subset of the parameter space. However for multi-input/multi-output systems there is no "natural" parametrization and the problem is more complex, but in general it would seem that the minimal systems form a connected subset of the parameter space, and this is proven for certain examples.

5.2 Single Input/Single Output Systems

The following unpublished result of Brockett (private communication) shows that for single input/single output systems the set of minimal systems does not form a connected subset of the parameter space. The proof that the space is disconnected is the same as that of Brockett, but the proof that each region is connected is new.

Theorem 5.1 (Brockett)

Given the rational function

$$g(z) = \frac{\beta_{n-1}s^{n-1} + \ldots + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_0}$$

Then the parameter space, R^{2n}, is divided into (n+1) connected regions in which there are no pole/zero cancellations. Each such region is characterized by the Cauchy index of g(z) (or the signature of the corresponding Hankel matrix) and these disconnected regions are separated by rational functions of lower order.

Note: Cauchy index of g(z) = $I_{-\infty}^{\infty}(g(z))$

= (number of times g(z) changes from $-\infty$ to $+\infty$) - (number of times g(z) changes from $+\infty$ to $-\infty$) as z goes from $-\infty$ to ∞ on the real line.

If S = S' then the signature of S, $\sigma(S)$= (number of positive eigen values of S) - (number of negative eigen values of S).

Proof

1) The regions are separated

Let $(A(\alpha),b,c'(\beta))$ be the standard observable realization of $g(z)$ and set

$$s_i(\alpha,\beta) = c'(\beta)A^i(\alpha)b$$

Define

$$S_r = \begin{bmatrix} s_o & s_1 & \cdots & s_{r-1} \\ s_1 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ s_{r-1} & \cdot & \cdots & s_{2r-2} \end{bmatrix}$$

There are no pole/zero cancellations if and only if

$$\det S_n(\alpha,\beta) \neq 0.$$

Now it is shown in Gantmacher (1959) that the Cauchy index of $g(z)$ = signature of S_n and that if two symmetric matrices have different signatures then every continuous path in the space of symmetric matrices that connects them passes through a singular matrix. Therefore it is not possible to continuously connect two rational functions with different Cauchy indices without passing through a pole/zero cancellation. (This is clear since if the signature changes, the eigen value must change sign and if the path is continuous it must pass through zero (since the matrix is symmetric) when the matrix is singular.)

2) Each region is connected

To prove connectedness we must exhibit a continuous path connecting any two rational functions with the same Cauchy index and degree.

Firstly we show that it is sufficient to find a path in the space s_i, $i = 1,\ldots,2n-1$. (Lemma 5.2). Then it is shown that any Hankel matrix with a particular signature can be continuously deformed into a standard form without its determinant becoming zero. (Lemma 5.3). ■

Lemma 5.2

For any continuous function $s_i(t) : [0 , 1] \to R^1$ $i = 0,\ldots,2n-1$ and such that $\det S_n(t) \neq 0$, there exist continuous functions $\alpha_i(t) : [0 , 1] \to R^1$ and $\beta_i(t) : [0 , 1] \to R^1$ $i = 1,2,\ldots,n$ such that for each t

$$\frac{\beta_n(t)^{n-1} + \ldots + \beta_0(t)}{z^n + \alpha_{n-1}(t)z^{n-1} + \ldots + \alpha_0(t)} = \frac{s_0(t)}{z} + \frac{s_1(t)}{z^2} + \ldots + \frac{s_{2n-1}(t)}{z^{2n}} + \sum_{i=0}^{\infty} \frac{s_{2n+i}(t)}{z^{2n+i+1}}$$

where $s_{2n+i}(t) = - \sum_{g=1}^{n} \alpha_{n-g}(t) s_{2n+i-g}(t)$ $\quad i = 0,1,\ldots$ and $S_n(t)$ is defined in Theorem 5.1.

Proof (See Gantmacher Vol. II, page 207)

If $\det S_n(t) \neq 0$ then

$$\begin{bmatrix} \alpha_0(t) \\ \alpha_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n-1}(t) \end{bmatrix} = - S_n^{-1}(t) \begin{bmatrix} s_n(t) \\ s_{n+1}(t) \\ \cdot \\ \cdot \\ \cdot \\ s_{2n-1}(t) \end{bmatrix}$$

$$\begin{bmatrix} \beta_{n-1}(t) \\ \beta_{n-2}(t) \\ \vdots \\ \beta_0(t) \end{bmatrix} = \begin{bmatrix} s_0(t) & 0 & \cdots & 0 \\ s_1(t) & s_0(t) & 0 & \cdots \\ \vdots & & \ddots & \\ s_{n-1}(t) & \cdots & s_1(t) & s_0(t) \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_{n-1}(t) \\ \vdots \\ \alpha_0(t) \end{bmatrix}$$

and the result follows immediately. ■

Lemma 5.3

Any Hankel matrix S_n of a particular signature and rank,n, can be continuously deformed into a standard form without reducing its rank.

Proof (outline)

The proof defines some standard forms for S_n, one associated with each possible signature, and then perturbs an arbitrarily S_n of a particular signature and R rank n into the standard form without allowing rank $S_n < n$, as follows.

1) Perturb S_n very slightly so that $D_r \neq 0$ for r = 1,2,..,n. (see proof for definition of D_r).

2) Perturb S_n so that $|D_r| = 1$ r = 1,2,..,n.

3) Perturb S_n such that the order of the + 1's and -1's in the sequence $(1,D_1,D_2,..,D_n)$ is in a standard form.

Jacobi's Theorem which determines the signature from the number of sign changes in the sequence $(1,D_1,...D_n)$ is used.

Proof

Let $$S_r = \begin{bmatrix} s_0 & s_1 & \cdots & s_{r-1} \\ s_1 & & & \\ \vdots & & & \\ s_{r-1} & \cdots & & s_{2r-2} \end{bmatrix}, \qquad D_r = \det S_r$$

Then there are (n+1) possible signatures for S_n namely, n,n-2,...,-n+2,-n.

Consider now the following standard forms, $S^{(n-2\nu)}$ $\nu = 0,\ldots,n$ given by,

$$s_{2j+1} = 0 \qquad j = 0,1,\ldots,n-2$$

and s_{2j} $j = 0,1,..n-1$ is implicitly defined by,

$$D_j = (-1)^j \qquad j = 0,1,\ldots\nu$$

and

$$D_j = (-1)^\nu \qquad j = \nu + 1,\ldots n.$$

This does indeed define s_{2j} since using the formula for the determinant of a partitioned matrix we get,

$$D_{r+1} = D_r \left(s_{2r} - (s_r \ldots s_{2r-1}) S_r^{-1} \begin{pmatrix} s_r \\ \cdot \\ \cdot \\ \cdot \\ s_{2r-1} \end{pmatrix} \right)$$

therefore $s_{2r} = \frac{D_{r+1}}{D_r} + (s_r \ldots s_{2r-1}) S_r^{-1} \begin{pmatrix} s_r \\ \cdot \\ \cdot \\ \cdot \\ s_{2r-1} \end{pmatrix}$

and since $s_{2r-1} = 0$, s_{2r} is fixed by the above formula as a function of D_1, D_2, .. D_{r+1}.

We will now show that there exist continuous deformations of $(s_0,\ldots,s_{2n-2})$ such that D_i are as given above and $s_{2j+1} = 0$, $j = 0,1;;n-2$.

Firstly consider $(1,D_1,D_2,..,D_n)$, we know that $D_n \neq 0$ by assumption. If any $D_i = 0$ $i = 1,..,n-1$ then vary $(s_0,s,\ldots s_{2n-2})$ continuously so that the new values $(s_0^*,s_1^*,\ldots s_{2n-2}^*)$ satisfy $D_r^* \neq 0$ $r = 0,\ldots,n$,

and in the process of variation no non-zero D_i becomes zero. Such a variation is always possible since(to quote from Gantmacher page 354 Vol I), in the space of parameters $(s_0, s_1, \ldots s_{2n-2})$ an equation of the form $D_i = 0$ determines a certain algebraic hypersurface. If a point lies in some such hypersurface, then it can always be approximated arbitrarily closely by points not in these hypersurfaces. Then since the rank does not change its signature does not change, and hence the signature is given by Jacobi's Theorem (Gantmacher Vol. I, p. 303),

$$\sigma(S_n) = n - 2V(1, D_1, D_2, \ldots D_n)$$

Let $\nu = V(1, D_1, \ldots, D_n)$ = number of variations of sign in the sequence $1, D_1, \ldots, D_n$. Now since $D_r^* \neq 0$ for $r = 0, 1, \ldots n$, we can make the following perturbation.

1) $\tilde{D}_r(t) = (1-t)D_r + t \operatorname{sgn} D_r^* \qquad r = 0, \ldots n$

2) $\tilde{s}_{2r+1}(t) = (1-t)\, s^*_{2r+1} \qquad r = 0, 1, \ldots, n-1.$

3) $$\tilde{s}_{2r}(t) = \frac{\tilde{D}_{r+1}}{\tilde{D}_r}(t) + (\tilde{s}_r(t) \ldots \tilde{s}_{2r-1}(t))\tilde{S}_r^{-1}(t) \begin{pmatrix} \tilde{s}_r(t) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{s}_{2r-1}(t) \end{pmatrix}$$

for $r = 1, \ldots, n-1$ and $\tilde{s}_0(t) = \tilde{D}_1(t)$.

The third equation is well-posed since $\tilde{s}_{2r}(t)$ only depends on $\tilde{s}_i(t)$ for $i < 2r$ and $\tilde{D}_i(t)$ for $i = 1, \ldots, r+1$ and $\tilde{D}_r(t) \neq 0$ for all t. Also equation 3) is consistent with 1).

Let $\tilde{D}_r(1) = \hat{D}_r \quad r = 1,2,\ldots n$ and $\tilde{s}_r(1) = \hat{s}_r \quad r = 0,..,2n-2.$

Thus $|\hat{D}_r| = 1 \quad r = 1,\ldots n$ and $\hat{s}_{2r+1} = 0 \quad r = 0,1,\ldots.n-2.$

This is in the standard form except that the signs of $\hat{D}_r$ are not necessarily in the desired order, and a perturbation to change the signs is now given. Consider the following partition of S_{r+2}

$$S_{r+2} = \left[\begin{array}{ccc|cc} s_0 & s_1 \cdots & s_{r-1} & s_r & s_{r+1} \\ s_1 & & \vdots & & \\ \vdots & & & & \\ s_{r-1} & & s_{2r-2} & s_{2r-1} & s_{2r} \\ \hline s_r & & s_{2r-1} & s_{2r} & s_{2r+1} \\ s_{r+1} & & s_{2r} & s_{2r+1} & s_{2r+2} \end{array}\right]$$

Using the formula of the determinant of a partitioned matrix assuming $D_r \neq 0$ it is easy to verify that

$$D_{r+2} = D_{r+1}\left[s_{2r+2} - (s_{r+1}\cdots s_{2r})\, S_r^{-1} \begin{pmatrix} s_{r+1} \\ \vdots \\ s_{2r} \end{pmatrix}\right]$$

$$- D_r\left[s_{2r+1} - (s_r \ldots s_{2r-1})S_r^{-1} \begin{pmatrix} s_{r+1} \\ \vdots \\ s_{2r} \end{pmatrix}\right]^2$$

and

$$D_{r+1} = D_r \left[s_{2r} - (s_r \ldots s_{2r-1}) S_r^{-1} \begin{pmatrix} s_r \\ \cdot \\ \cdot \\ \cdot \\ s_{2r-1} \end{pmatrix} \right]$$

Now assume sgn $\hat{D}_r$ = - sgn $\hat{D}_{r+2}$ and $|\hat{D}_r| = |\hat{D}_{r+2}| = 1$

Let $s_0 \ldots s_{2r-1}$ be fixed, this implies D_r is fixed then s_{2r} can be continuously varied so that $\hat{D}_{r+1}$ becomes - $\hat{D}_{r+1}$, and as D_{r+1} varies we can vary s_{2r+2} and s_{2r+1} continuously so that D_{r+2} remains fixed. (Since sgn D_{r+2} = - sgn D_r and $D_r \neq 0$) Note that if sgn D_r = sgn D_{r+2}, sgn D_{r+1} cannot change without making $D_{r+2} = 0$ at some point.

Using the above continuous deformation, if the sequence $1, D_1, D_2, \ldots D_n$ contains the triple 1,1,-1 it can be continuously deformed to 1,-1,-1 or a sequence -1,-1,1 can be changed to -1,1,1. That is the variation in sign can be moved one place to the left.

Therefore in summary we have continuously deformed S_n to $\hat{S}_n$ such that $|\hat{D}_i| = 1$ for i = 1,..n and then we can "concentrate" all the changes in sign in the sequence $(1, D_1, \ldots D_n)$ at the left and hence have the standard form given initially. ■

Implications of this result are given in Section 5.4.

5.3 Multi-input/Multi-output Systems

For multi-input/multi-output systems the questions one would like to answer are "when is the set of minimal systems a connected subset of parameter space?" and "when is the set of nonminimal systems of codimension 1?" The codimension of a p-dimensional hypersurface in R^n is n-p. Hence a surface with codimension greater than 1 cannot separate any points, whereas a surface of codimension 1 can form a barrier.

One method of approach is to consider the codimension of the nonminimal systems as follows. Suppose $(A(\alpha),B(\alpha),C(\alpha))$ is an affine parametrization of the system matrices then the set of nonminimal systems is given by,

$$N = \{ \alpha \mid \text{rank } [B(\alpha), A(\alpha)B(\alpha),\ldots,A^{n-1}(\alpha)B(\alpha)] < n \}$$

$$\cup \{ \alpha \mid \text{rank } [C'(\alpha),A'(\alpha)C'(\alpha),\ldots,A'^{n-1}(\alpha)C'(\alpha)] < n\}$$

That is the class of nonminimal systems is the union of the set of uncontrollable and unobservable systems. Referring back to Chapter 2 let

$$P(A,B,\kappa) = [b_1,Ab_1,\ldots,A^{k_1-1}b_1,\ldots,b_m,Ab_m,\ldots A^{k_m-1}b_m]$$

where $\kappa = (k_1,k_2,\ldots,k_m)$ and $\sum_{i=1}^{m} k_i = n$ with $k_i \geq 0$.

Therefore the pair $(A(\alpha),B(\alpha))$ is uncontrollable if and only if $\det (A(\alpha),B(\alpha),\kappa) = 0$ for all k. The solution of each equation such as this will be an algebraic variety of codimension ≥ 1. Now if two such surfaces can be found which are independent then their intersection will be of codimension ≥ 2 and hence the set of uncontrollable systems

has codimension ≥ 2 . For two such independent surfaces to exist it is clearly necessary that $n > 1$ and $(A(\alpha),B(\alpha))$ is not controllable for some α. This is a completely general approach but verifying that two such surfaces are independent is not easy for any arbitrary parametrization. The following theorem considers some particular parametrizations by explicitly constructing paths connecting arbitrary minimal systems.

Theorem 5.2

Systems that are controllable and observable form a connected subset of the parameter space for the following parametrizations of the (A,B,C) matrices.

(i) A,B, and C arbitrary matrices with $m > 1$, $p > 1$.

(ii) C arbitrary and (A,B) given by any affine parametrization $(A(\alpha),B(\alpha))$ $(\alpha \in R^q)$ such that,

(a) $(A(\alpha),B(\alpha))$ is controllable for all $\alpha \in R^q$.

(b) $A(\alpha)$ is such that there exists $\hat{C} \in R^{pxn}$ such that $(A(\alpha),\hat{C})$ is observable for all $\alpha \in R^q$, and

(c) there exists $\tilde{c} \in R^{1xn}$ and $\hat{\alpha} \in R^q$ such that $(A(\hat{\alpha}),\tilde{c})$ is observable.

[For example all the controllable canonical parametrizations given in Chapter 2 with $p \geq m$ satisfy (a),(b) and (c) above].

Proof

(ii) We will construct a perturbation of an arbitrary $(A(\alpha),B(\alpha),C)$ that is minimal to a standard form namely $(A(0),B(0),\hat{C})$, as follows.

(1) perturb C and α very slightly to C^1 and α^1 so that $(A(\alpha^1),c_1^1)$ is observable. This is possible by the same

arguments as in the proof of Theorem 5.1 and by assumption (c).

(2) perturb $c_2,\ldots,c_p$ to $\hat{c}_2,\ldots,\hat{c}_p$.

(3) perturb c_2 to c_2^2 and α to α^2 arbitrarily close to $\hat{c}_2$ and α^1 such that $(A(\alpha^2),c_2^2)$ is observable. Make c_2^2 sufficiently close to $\hat{c}_2$ so that $\left(A(\alpha^2),\begin{bmatrix}\hat{c}_1\\ c_2\\ \hat{c}_3\\ \vdots\\ \hat{c}_p\end{bmatrix}\right)$ is observable for $c_2 = \lambda c_2^2 + (1-\lambda)\hat{t}_2$ and $\lambda \in [0\ ,\ 1]$.

(4) perturb c_1 to $\hat{c}_1$.

(5) perturb c_2 to $\hat{c}_2$ by $c_2 = \lambda\ \hat{c}_2 + (1-\lambda)c_2^2$, $\lambda \in [0\ ,\ 1]$.

(6) perturb α to zero.

It is clear that during this perturbation observability is preserved.

(i) We will construct a continuous perturbation of any (A,B,C) which takes (A,B,C) while preserving minimality to a particular parametrization that satisfies the assumption of (ii). Assume that $p \geq m$ (if not consider (A',C',B') in exactly the same way) we will perturb an arbitrary system to the canonical form of Theorem 2.4 for some $\kappa = (k_1,k_2,..,k_m)$. We know there exists $\tilde{\kappa}$ such that $\det(P(A,B,\tilde{\kappa})) \neq 0$ by controllability assumption.

(1) Perturb (A,B) very slightly to (A^1, B^1) so that $\det(P(A,B,\hat{\kappa})) \neq 0$ for some $\hat{\kappa} \neq \kappa$.

(2) By Theorem 2.4 there exists a similarity transformation, T, between (A^1,B^1) and (A^2,B^2) where (A^2,B^2) are in the canonical form associated with $\hat{K}$. If det T > 0 then I and T can be continuously connected in GL(n), so perturb (A^1,B^1,C) to (A^2,B^2,C^2) continuously by a sequence of similarity transformations in GL(n). If det T < 0 then let $S = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}$ and perturb continuously in the same way to $(S^{-1}A^2S,\ S^{-1}B^2,C\ S)$, which will be in a canonical parametrization similar to Theorem 2.4.

(3) Perturb (A,B,C) in the canonical parametrization avoiding unobservable systems as in (ii) above until $\det(P(A,B,K)) > 0$.

(4) Take similarity transformation as in (2) (which necessarily has positive determinant) to obtain (A,B,C) in the canonical parametrization for K.

5.4 Implications of Theorems 5.1 and 5.2

Theorem 5.1 has implications in the following situations for single input/single output systems.

1) Any on line algorithm where minimality of the estimates is required if the algorithm is to be well-posed may have problems of the type mentioned in Section 5.1. Namely if the initial data implies that the system is in the wrong region of the parameter space then the only way that the successive estimates can tend to the correct solution is for the parameter estimates to pass through a surface of norminimal systems, when the algorithm will become ill-posed. A simulation example of this type is given in Section 5.5.

2) Consider the following adaptive stochastic regulator. The output is passed through a Kalman filter to estimate the state and the input is obtained by the solution of the infinite-time Ricatti equation. Further the gains in the Kalman filter and the solution of the Ricatti equation are based on the present best estimates of the system parameters (sometimes this is referred to as open-loop feedback). Now if stabilizability is lost then the Ricatti equation's solution becomes infinite and if detectability is lost the Kalman filter gains become infinite. Such an algorithm will thus become ill-posed when detectability or stabilizability are lost, moreover the surfaces of undetectable (unstabilizable) systems has local codimension 1 for the standard controllable (observable) form, and thus such surfaces are likely to be encountered as the algorithm progresses if the system is unstable and the initial data estimates the system to be on the wrong side of a non-minimal surface of codimension 1.

Theorem 5.2 implies that problems such as those that are outlined above are unlikely to occur for multi-input/multi-output systems because the set of nonminimal systems forms a surface of codimension greater than 1 and if it is encountered one can reasonably blame this on "bad luck" rather than on almost inevitable consequence of bad initial data as with the single input/single output case. However for particular parametrizations difficulties may occur, for example if the Hankel matrix is symmetric for all α.

5.5 Simulation Results

We now present the results of a computer simulation of an identification algorithm which exhibits difficulty due to the phenomena mentioned in the preceeding sections of this chapter. The algorithm chosen for this simulation is the output correlation method due to Mehra (1971) that has been discussed in Chapter 4, for single input/single output systems. In order to illustrate the difficulties it was only necessary to estimate the A matrix when (A,C) is in standard observable form.

The system simulated was,

$$\begin{pmatrix} x_1(k+k) \\ x_2(k+2) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha_0 & \alpha_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + d\, v(k)$$

where u and v are independent Gaussian white noise sequences with unit covariance. The steady state was reached before any observations were used.

Estimates of the output correlation function, $C_j = E\{y(k)y(k+j)\}$ are given by

$$\hat{C}_j^N = \frac{1}{N} \sum_{k=1}^{N} y(k)y(k+j)$$

Then the coefficients in the A matrix are estimated by,

$$\begin{bmatrix} \hat{\alpha}_0^N \\ \hat{\alpha}_1^N \end{bmatrix} = \begin{bmatrix} \hat{C}_1^N & \hat{C}_2^N \\ \hat{C}_2^N & \hat{C}_3^N \end{bmatrix}^{-1} \begin{bmatrix} \hat{C}_3^N \\ \hat{C}_4^N \end{bmatrix}$$

Now the indicated inverse will only exist if the corresponding system is second order. However the inverse will not necessarily exist for the estimated correlation coefficients, indeed if for the true correlation coefficients the above determinant is negative and the initial estimates of the determinant are positive then as better estimates are made the determinant must pass through zero and the parameter estimates in this region will become arbitrarily large. This behaviour was indeed manifested in several examples one of which is now given.

True parameter values $\alpha_0 = -0.24$, $\alpha_1 = 1.0$ $d = 0.1$. With these parameter values the pulse transfer function ,

$g(z) = \frac{1}{(z-0.4)(z-0.6)}$. Typical sample paths are given in Figures 5.1, 5.2 and, 5.3 which shows the difficulties encountered. Further if the matrix inverse is evaluated by some recursive scheme then large numerical errors may accumulate if the determinant becomes very small.

If however the alternate method suggested by Mehra (1971) given by,

$$\begin{bmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_1 \end{bmatrix} = (\hat{H}'\hat{H})^{-1} \hat{H}' \begin{bmatrix} \hat{c}_3 \\ \hat{c}_4 \\ \cdot \\ \cdot \\ \cdot \\ \hat{c}_{4+k} \end{bmatrix}$$

where

$$\hat{H} = \begin{bmatrix} \hat{c}_1 & \hat{c}_2 \\ \hat{c}_2 & \hat{c}_3 \\ \cdot & \\ \cdot & \\ \cdot & \\ \hat{c}_{2+k} & \hat{c}_{3+k} \end{bmatrix} \quad \text{for some } k \geq 1$$

does not have these difficulties because the surface

$\{A \in R^{mxn}, m > n \mid \det(A'A) = 0\}$ has codimension ≥ 2 and hence cannot separate any regions of the parameter space.

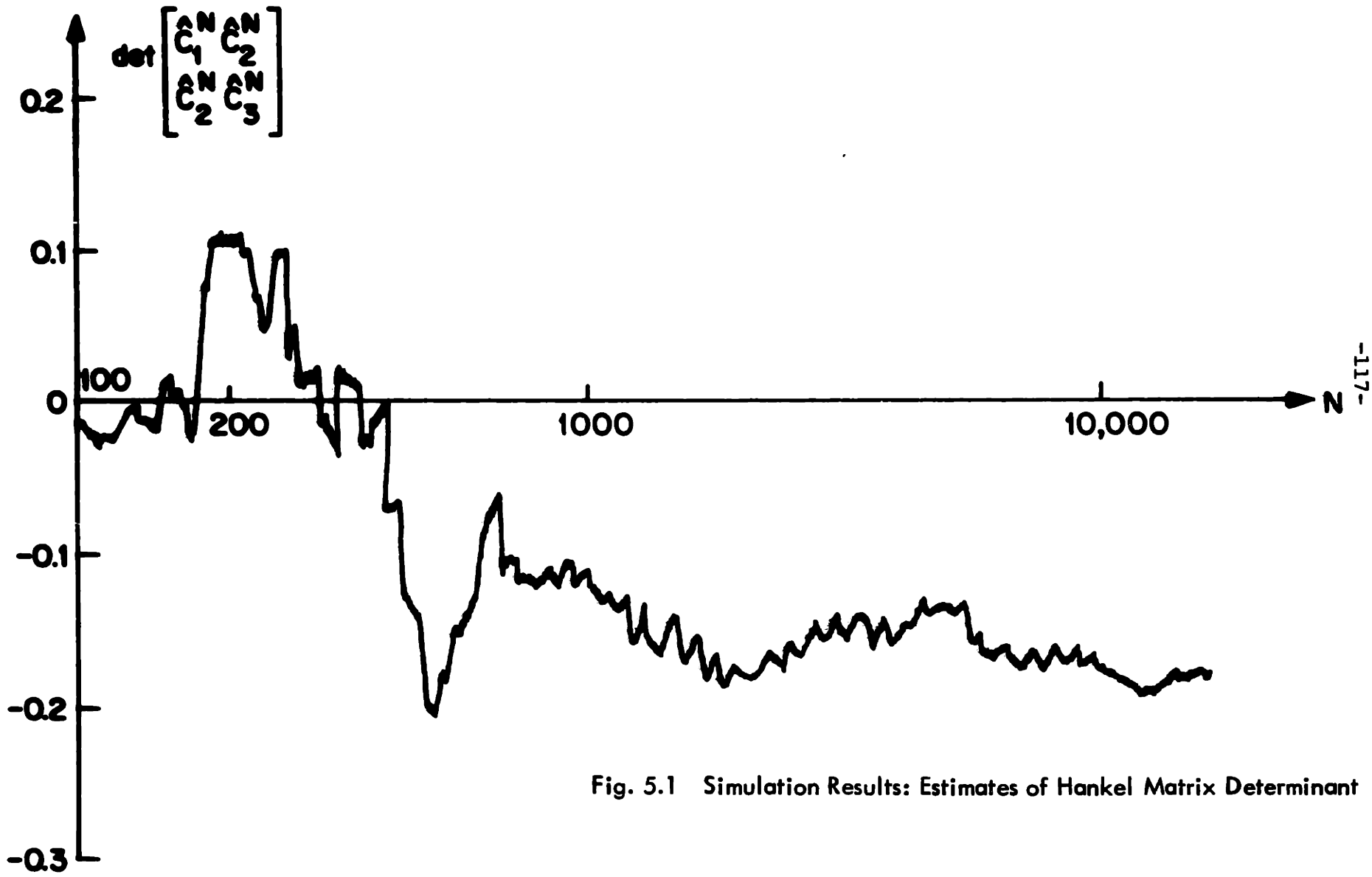

Fig. 5.1 Simulation Results: Estimates of Hankel Matrix Determinant

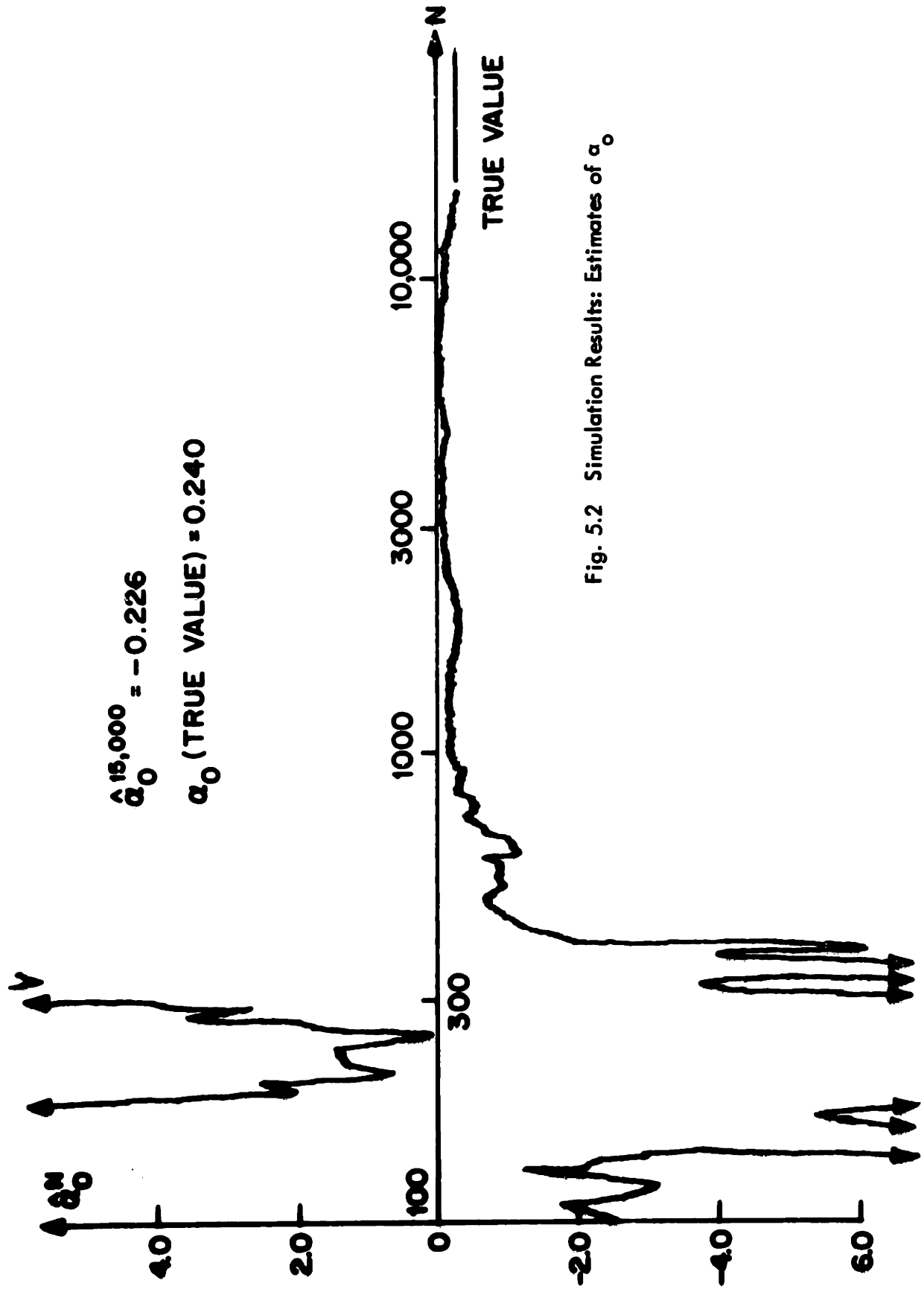

Fig. 5.2 Simulation Results: Estimates of α_0

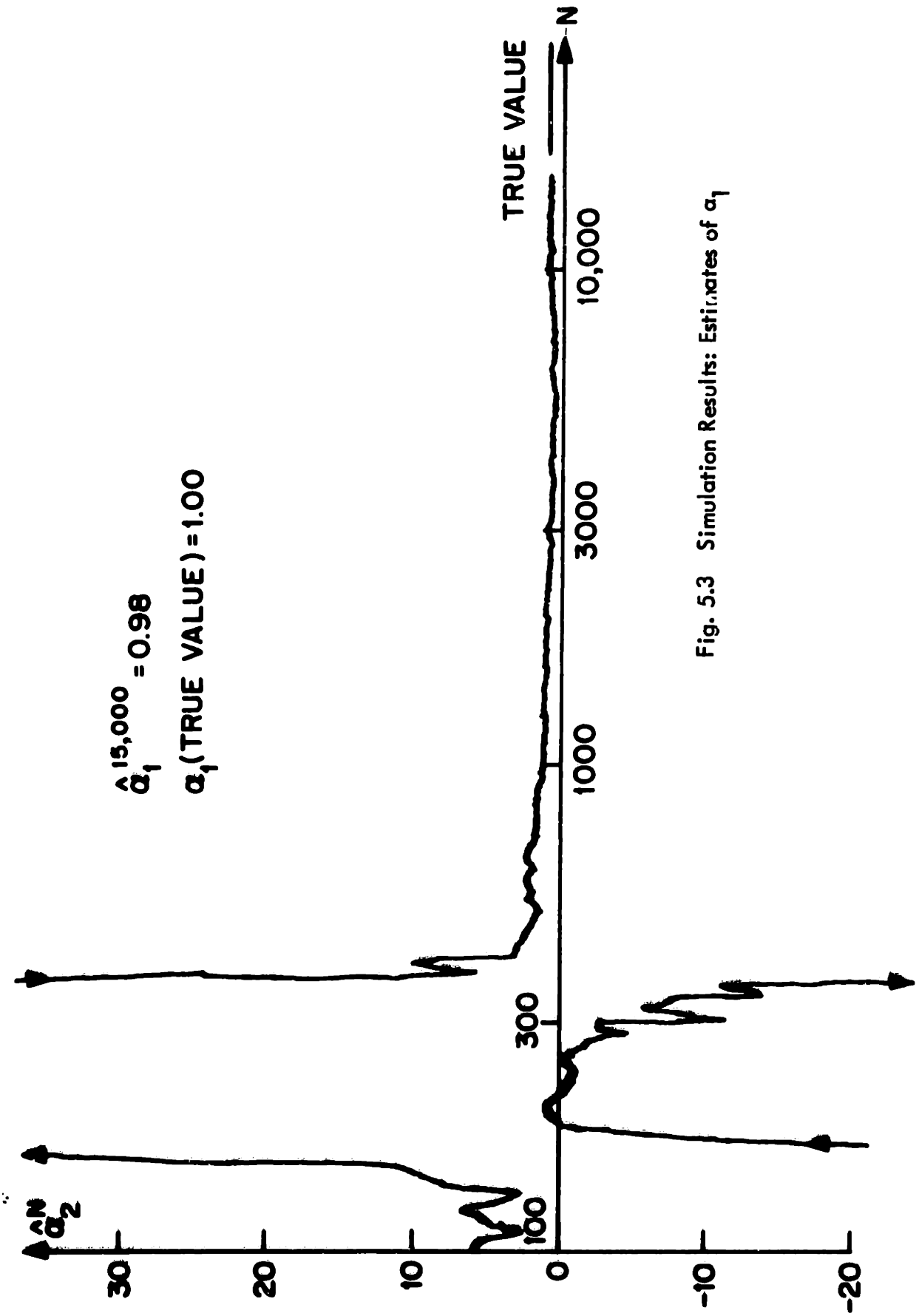

Fig. 5.3 Simulation Results: Estimates of α_1

5.5 Other Geometrical Properties

If one could parametrize linear systems so that all the systems of a given order were represented but none of a lower order were represented, then all the problems outlined previously would be avoided. For the single input/single output case one would require (n+1) parametrizations one for each of the regions of Theorem 5.1. Referring to the proof of Lemma 5.3 one can see that each region is characterized by the number of changes in sign of the principal minors of the Hankel matrix. Two regions are easily parametrized, that is those with a positive definite Hankel matrix and a negative definite Hankel matrix, in which cases all the principal minors are always nonzero. However the other regions correspond to some collection of minors which may be positive, negative or zero. This parametrization problem seems very difficult and has not been solved except when $m = 2$. One observation which perhaps illustrates why this parametrization is difficult is that when $n \geq 3$ it is possible for two complex conjugate poles to be cancelled by two zeros and such a cancellation occurs on nonminimal surface of codimension 2. This means that some regions in parameter space will have as their boundaries nonminimal surfaces of codimension 1 and be "punctured" by surfaces of codimension 2.

The parametrization problem for multi-input or multi-output minimal systems is even more complex.

CHAPTER 6

CONCLUSIONS

It is hoped that this thesis has pointed out the importance in identification of some structural properties of linear systems, particularly the parametrization of linear systems. We would like to conclude by stating the practical implication of the results in this research by way of some specific suggestions concerning identification.

1) If it is not required to have a physical interpretation of the state space realization of a particular system, then standard linear system parametrizations are appropriate. The discussion of Chapter 2 suggests that for multivariable systems true canonical forms (e.g. Popov's) are not desirable because of numerical difficulties near boundary points. An alternate family of globally identifiable parametrizations is then given that are well-suited for identification.

2) If a natural parametrization of the system matrices is given by physical considerations, then before any identification is attempted it is recommended that at least the local identifiability of the parametrization is checked at nominal values of the unknown parameters. Then if the parametrization is found not to be identifiable it is straightforward from Theorem 3.4 to see which parameters need to be fixed at their nominal values in order to make the remainder identifiable.

3) If feedback is present around a system then it is suggested that the system equations are rewritten as a linear system without feedback, but with the feedback matrix parametrizing the A matrix. Then the local or partial identifiability results can be used to determine whether

CHAPTER 6

CONCLUSIONS

It is hoped that this thesis has pointed out the importance in identification of some structural properties of linear systems, particularly the parametrization of linear systems. We would like to conclude by stating the practical implication of the results in this research by way of some specific suggestions concerning identification.

1) If it is not required to have a physical interpretation of the state space realization of a particular system, then standard linear system parametrizations are appropriate. The discussion of Chapter 2 suggests that for multivariable systems true canonical forms (e.g. Popov's) are not desirable because of numerical difficulties near boundary points. An alternate family of globally identifiable parametrizations is then given that are well-suited for identification.

2) If a natural parametrization of the system matrices is given by physical considerations, then before any identification is attempted it is recommended that at least the local identifiability of the parametrization is checked at nominal values of the unknown parameters. Then if the parametrization is found not to be identifiable it is straightforward from Theorem 3.4 to see which parameters need to be fixed at their nominal values in order to make the remainder identifiable.

3) If feedback is present around a system then it is suggested that the system equations are rewritten as a linear system without feedback, but with the feedback matrix parametrizing the A matrix. Then the local or partial identifiability results can be used to determine whether

the unknown parameters in the open-loop system can be identified, as well as or independently from the feedback system.

4) If a system is driven by an unobserved white noise process then special care should be exercised in choosing the parametrization and the results of Chapter 4 used.

5) If minimality of successive system estimates is required for an on-line algorithm to be well-posed, then (at least for single input/single output systems and for certain multivariable parametrizations), on-line identification should only proceed from initially good parameter estimates since the set of non-minimal systems forms a surface of co-dimension one in the parameter space. Further it is recommended that algorithms should not require the successive system estimates to be minimal so that these problems will not occur.

A general procedure for determining identifiability that can be applied to other situations is as follows. Firstly characterize by a set of equations all systems indistinguishable from one another given the observations, then determine whether the set of equations has a unique solution when the systems are restricted to be in some parametrization. A local result can then be obtained by linearizing the equation.

Open problems that have originated from this research are:

1) Finding good sufficient conditions for the global identifiability of an arbitrary parametrization.

2) Finding globally identifiable parametrizations.

3) Determining whether a parametrization will be troubled by local minima. This will depend both on the cost function being minimized and the parametrization.

4) Parametrizing linear systems driven by unobserved white noise, in a similar way to the results of Chapter 2 for input/output systems.

5) Parametrizing minimal linear systems, i.e. finding families of globally identifiable parametrizations that do not admit multiple representations of nonminimal systems but represent every minimal system.

6) Further study of the mathematical structure of the class of linear systems modulo equivalence, to give greater insight into the nature of the object one is trying to identify.

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APPENDIX I

KRONECKER PRODUCTS

In Chapter 3 it is often required to rewrite linear equations in an unknown matrix as an equation in an unknown vector so that rank conditions can be written explicitly. In this section the necessary background is given (see also Halmos (1958) and Pease (1965)).

Consider the matrix equation

$$(*) \qquad A\,X\,B = C$$

in the unknown matrix X(nxm), with A(pxn), B(mxr) and C(pxr) known matrices. This is a linear equation in X and thus if X is rewritten as a vector by some lexicographical ordering then (*) can be written as a vector equation. Two natural orderings of the elements of X come to mind, firstly to list X row by row and secondly column by column. The first ordering has been chosen here arbitrarily.

Let $X' = [x_1\ x_2\ \ldots\ x_n]$ with $x_i \in R^m$ and define $\bar{\bar{X}} \in R^{nm}$ as the vector,

$$\bar{\bar{X}}' = [x_1'\ x_2'\ \ldots\ x_n']$$

Also let $C' = [c_1\ c_2\ \ldots\ c_p]$ with $c_i \in R^r$ and $\bar{\bar{C}}' = [c_1'\ c_2'\ \ldots\ c_p']$. Now equation (*) gives for i = 1,...,p

$$c_i' = \sum_{j=1}^{n} a_{ij}\, x_j'\, B$$

$$= \bar{\bar{X}}' \begin{bmatrix} a_{i1}B \\ a_{i2}B \\ \vdots \\ a_{i}B \end{bmatrix}$$

and hence

$$\begin{bmatrix} a_{11}B' & a_{12}B' & \cdots & a_{1m}B' \\ a_{21}B' & a_{22}B' & \cdots & a_{2n}B' \\ \vdots & & & \\ a_{p1}B' & a_{p2}B' & \cdots & a_{pn}B' \end{bmatrix} \bar{\bar{X}} = \bar{\bar{C}}$$

Now the Kronecker product of two matrices A and B, denoted A x B, is defined as,

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & & & \\ a_{p1}B & a_{p2}B & \cdots & a_{pm}B \end{bmatrix}$$

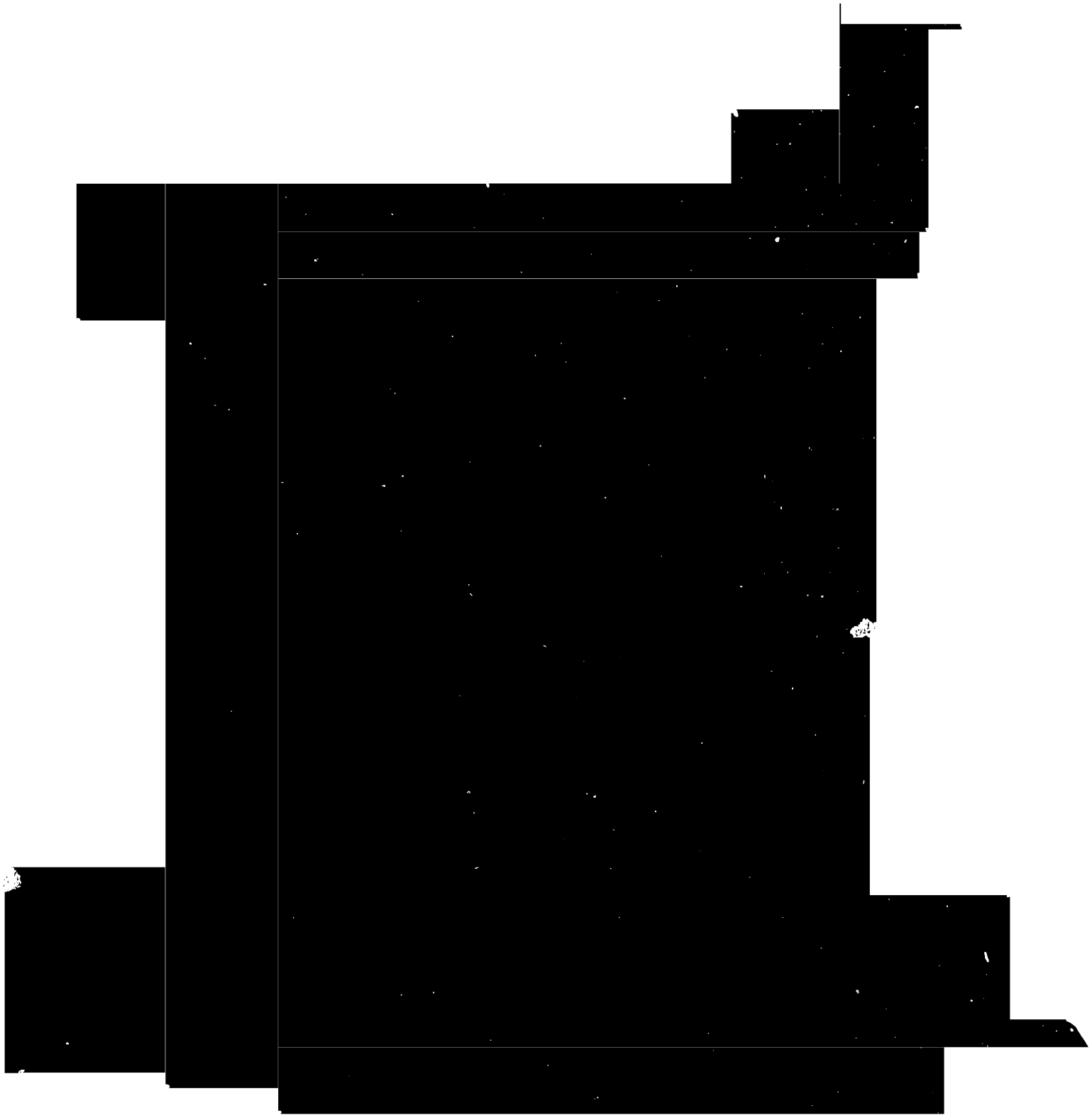
Therefore equation (*) can be written concisely as the vector equation.

$$\boxed{(A \otimes B')\,\bar{\bar{X}} = \bar{\bar{C}}}$$

If the lexicographical ordering by columns is used then letting

$X = [x_1\ x_2\ \ldots\ x_m]$ and $\bar{\bar{X}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ equation (*) is easily shown to be

$$(B' \otimes A)\,\bar{\bar{X}} = \bar{\bar{C}}$$



BIOGRAPHIC NOTE

Keith Glover was born in Bromley, Kent, England on April 23, 1946. He attended the Grammar School, Dartford, Kent, from September 1957 to June 1964. In September 1964 he entered the Imperial College of Science and Technology, London University, graduating in August 1967, when he received the degree B.Sc.(Eng.) in Electrical Engineering with first class honours. While an undergraduate he obtained a black belt in judo and was captain of London University Judo Team for 1966-67.

In September 1967 Mr. Glover joined the Marconi Company, Chelmsford, Essex, England, where he entered a graduate industrial training program for six months, and then worked as a development engineer in the Pulse Code Modulation group for eighteen months.

Mr. Glover has been a full-time graduate student in the Department of Electrical Engineering at M.I.T. since September 1969. He has been supported by a Kennedy Memorial Fellowship from September 1969 to August 1971, by a teaching assistantship from September 1971 to May 1972 and by a research assistantship in the Electronic Systems Laboratory from June 1972 to the present time. He was awarded the degrees of Master of Science and Electrical Engineer in June 1971.

During summers Mr. Glover has worked in an electronic company (1964), a telephone company (1965), a television company (1966), on the environmental control of commercial glasshouses (1967), and for an electronics company (1970).

Mr. Glover is married to the former Jean Elizabeth Priestley.