THE PROPAGATION OF WATER WAVES OVER SEDIMENT POCKETS

by

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ABSTRACT

Wave diffraction due to a rectangular domain of finite depth and width containing two fluids of constant but different densities connected to two channels of constant but different depths is considered. The implied scattering problem is modeled subject to the usual assumptions of linearized water-wave theory. The analysis is restricted to monochromatic plane progressive surface waves normally incident on the inhomogeneous domain. This results in a linear two-dimensional boundary-value problem for the velocity-potential. The scattering problem is formulated in terms of complementary variational integrals of Schwinger's type; symmetry relations between the complex amplitudes of the scattered potential are developed; and numerical calculations for the complex reflection and transmission coefficients are presented for a range of the physical variables. The analysis shows that sediment pockets can exhibit resonant behavior due to the combined effects of density stratification and pocket geometry. This offers a plausible explanation for offshore pipeline migration during storms.

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1.0 INTRODUCTION

The physical oceanographer and the ocean engineer share a common interest in the effects of bottom topography upon the propagation of water waves. While the oceanographer might consider the scattering properties of a sea mount, the ocean engineer will consider the scattering of plane progressive waves by a submerged petroleum storage tank. After making similar geometric idealizations, the mathematical analyses are virtually equivalent. The essential difference is one of scale where the oceanographer might include Coriolis effects while the engineer might include real fluid effects. Recently, oceanographers have developed mathematical descriptions which include the effect of density stratification within the oceanic water column. Ocean engineers have shown little interest in density stratification presumably because the loads implied by internal gravity waves have little influence on the design of offshore facilities. Yet, there is reason to believe that at least one form of density stratification, namely, the strong density contrast exhibited by sediment pockets, should be examined for its effect on the design of offshore facilities. Here, again, the difference between oceanographer and ocean engineer is one of scale.

The geological evolution of a sediment pocket requires the presence of fine-grained muds and depressions which allow these muds to localize. The obvious example is a submerged beach zone near a river delta. In this case, inactive erosion cuts caused by the distributory system of the delta or deformations in the beach plain act to localize transported sediments. Many of the world's more prolific offshore petroleum reservoirs are located beneath such an environment. Off the Gulf Coast of the United States near the Mississippi Delta, the bottom topography exhibits elongated troughs of from a few hundred to several thousand feet in width partially filled with up to thirty feet of unconsolidated sediments whose specific gravity ranges from 1.3 to 1.6. During major storms when low frequency components dominate the sea spectrum, offshore pipelines crossing such areas are known to experience significant migration which, in some cases, has led to pipeline rupture (Krieg, 1966). In this thesis, we investigate the excitation of "internal gravity waves" at the water-sediment interface by propagating surface waves. Our results quantify the resonant behavior which can develop within the sediment pocket due to the combined effects of stratification and sediment pocket geometry. Thus we establish a reasonable mechanism capable of transmitting large unsteady loads to any structure within the sediment pocket. We do not pretend that this is the only physical process which influences pipeline migration, only that this is a rather clear example where the consequences of density stratification are of engineering importance.

The propagation of water waves in fluids of variable depth has been studied for three distinct physical idealizations--finite obstacles, finite step changes in depth, and beach problems where the depth tends to zero. While qualitative properties of water-wave scattering in homogeneous fluids, such as symmetry relations and bounds for special geometries were established in a theoretical treatment by Kriesel (1949), explicit calculations have been performed only in limiting cases. For homogeneous fluids, the known solutions range from the exact solution for the scattering properties of a thin vertical barrier in waters of infinite depth by Ursell (1947) to the extensive numerical investigations of the scattering properties of rectangular obstacles in waters of finite depth by Mei and Black (1969). For inhomogeneous fluids, the known solutions range from studies of internal wave propagation in fluids of weakly variable depth by Keller and Mow (1969) to investigations of the diffraction of internal waves by a semi-infinite barrier by Manton et al. (1970). In spite of the differences in problem statement, all of these investigations demonstrate the common difficulty of solving the implied integral equations once the problem has been formulated. This has led many authors to investigate analytical approximation techniques.

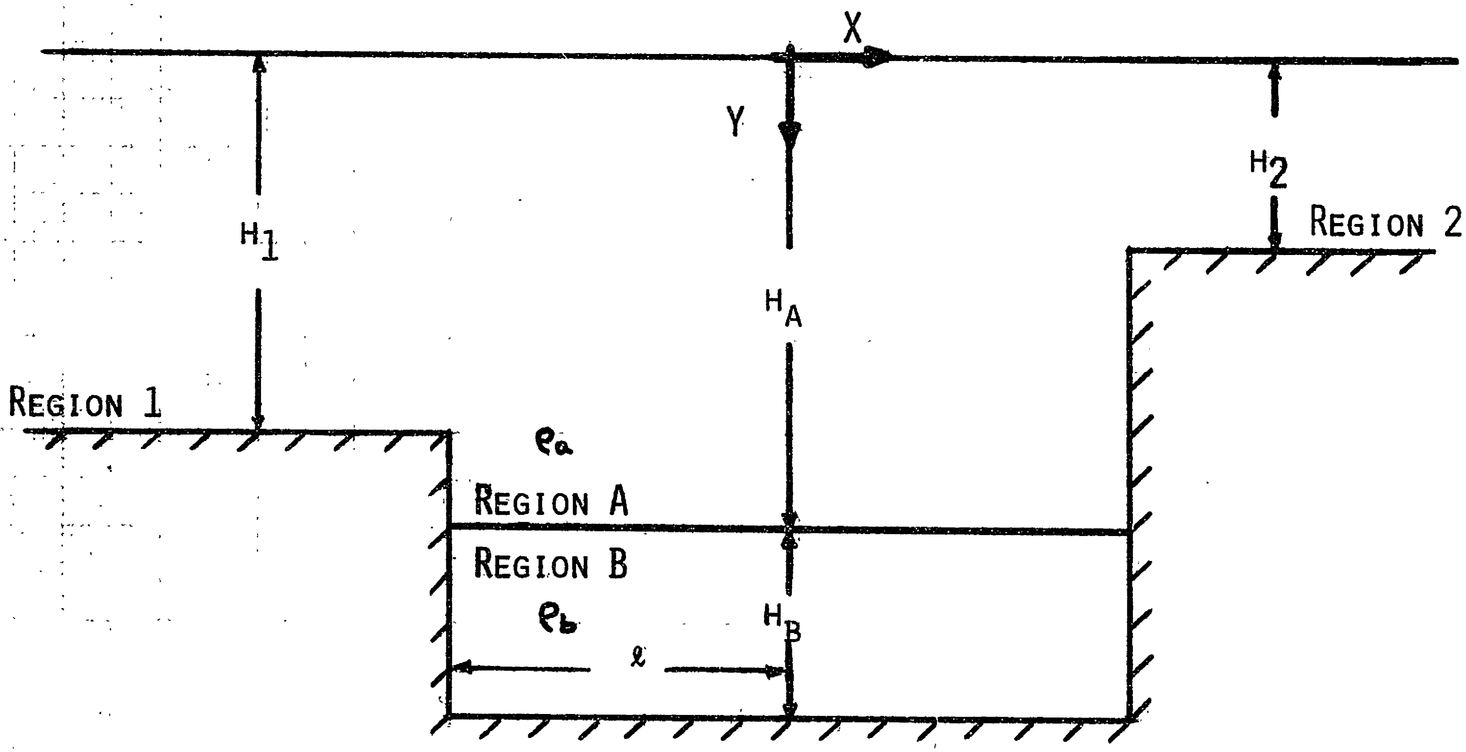
Bartholomuesz (1958) applied a long wave approximation in his detailed study of step-shelf scattering. Newman (1965a) was able to demonstrate the oscillatory nature of scattering for long symmetrical obstacles by coupling back-to-back step shelves. Mei (1967) constructed approximate solutions for bottom obstacles assuming small depth variations. Wunsch (1969) employed a similar argument in his analysis of beach problems. Miles (1967) has shown that Schwinger's variational method may be used to construct approximate solutions for finite step changes in depth in a manner analogous to electromagnetic wave guide problems. Each of these approximate solutions demonstrates important qualitative features of water-wave scattering. Each, in turn, has its limitations. Long wave approximations and special geometric configurations often do not yield results valid for the broad ranges of physical dimension or physical environment which the engineer must analyze. While Schwinger's method is slightly more general, it can be difficult to apply to nonrectangular geometries. Finally, the investigator must make a compromise between the geometry which he would like to study and the geometry which is amenable to solution.

In this thesis, we have elected to employ Schwinger's variational method to construct approximate solutions. A rectangular geometric idealization is physically plausible and allows the consideration of asymmetric as well as symmetric geometries. The accuracy for step shelf problems has been established by Miles (1967) by comparing his results to Newman's (1965b) more accurate numerical solution. The method has been previously applied to inhomogeneous fluid systems by Kelly (1969) in his study of the transmission of deep-sea internal waves into shallower waters. In addition, Miles (1971) has shown that Schwinger's method can be used to form complementary variational principles which imply upper and lower bound approximations to the true solution for the scattering properties of a circular dock in waters of finite depth. While Miles' bounding argument appears to be useful for only a special class of problems, we have been able to show that the complementary variational principles developed for this asymmetric, inhomogeneous problem yield roughly identical estimates of the far field even when extremely crude approximations to the near field are assumed. However, without the bounding argument, we are faced with the possibility that our estimates are invalid for some combinations of the physical variables.

The idealized model discussed in this thesis is shown in Figure 1. The implied scattering problem will be modeled subject to the usual assumptions of linearized water-wave theory. The two-fluid system is assumed to be ideal, the motion irrotational, and the wave amplitudes small compared to both the wave length and the water depth. The analysis is restricted to monochromatic plane progressive waves normally incident on the sediment pocket. This results in a linear two-dimensional boundary-value problem for the velocity potential. The scattering problem is formulated in terms of complementary variational integrals of Schwinger's type, symmetry relations between the complex amplitudes of the scattered potential are developed, and numerical calculations for the complex reflection and transmission coefficients are presented for a range of the physical variables.

2.0 THE BOUNDARY-VALUE PROBLEM

Let (x,y) be Cartesian coordinates with the plane y=0 being the undisturbed free surface and y positive downwards. The plane x = 0 bisects a rectangular domain of depth, d and half-width, ℓ, in which are contained two fluids with constant, but different densities ρ_a for $0 < y < h_a$ and ρ_b for $h_a < y < d$. This rectangular domain is connected to two channels of con-



THE PHYSICAL IDEALIZATION - FIGURE 1

stant, but different depths, h_1 and h_2. Thus, as shown in Figure 1, the fluid domain is defined by four regions:

Region 1	$0 < y < h_1$,	$-\infty < x < -\ell$
Region A	$0 < y < h_a$,	$\|x\| < \ell$
Region B	$h_a < y < d$,	$\|x\| < \ell$
Region 2	$0 < y < h_2$,	$\ell < x < +\infty$

Assuming plane progressive waves incident on the inhomogeneous domain from either or both infinities and linearized theory throughout the fluid domain, the velocity vector will be expressed by

$$\underline{V} = \mathrm{Re}\{e^{-i\omega t}\,\nabla\Phi(x,y)\}$$

where $\Phi(x,y)$ is the velocity-potential scalar. This potential function must satisfy Laplace's Equation throughout the fluid domain, subject to the boundary conditions on the free surface, at the interface between the two fluids, and along the rigid bottom as follows (cf. Wehausen and Laitone, 1960):

$$K\phi + \phi_y = 0 \quad \text{on} \quad y = 0 \;,\; |x| < \infty \tag{2.1}$$

$$\rho_a(K\phi_a + \phi_{ay}) = \rho_b(K\phi_b + \phi_{by}) \quad \text{on} \quad y = h_a \;,\; |x| < \ell \tag{2.2}$$

$$\phi_{1y} = 0 \quad \text{on} \quad y = h_1 \;,\; -\infty < x < -\ell \tag{2.3}$$

$$\phi_{ay} = \phi_{by} \quad \text{on} \quad y = h_a \;,\; |x| < \ell \tag{2.4}$$

$$\phi_{by} = 0 \quad \text{on} \quad y = d \;,\; |x| < \ell \tag{2.5}$$

$$\phi_{2y} = 0 \quad \text{on} \quad y = h_2 \;,\; \ell < x < +\infty \tag{2.6}$$

$$\phi_{ax} = 0 \quad \text{on} \quad h_1 < y < h_a \;,\; x = -\ell \tag{2.7}$$

$$\phi_{ax} = 0 \quad \text{on} \quad h_2 < y < h_a \;,\; x = +\ell \tag{2.8}$$

$$\phi_{bx} = 0 \quad \text{on} \quad h_a < y < d \;,\; |x| = \ell \tag{2.9}$$

where $K=\omega^2/g$ and the subscripts (1,2,a,b) denote the respective regions. In addition to these boundary conditions, we must require that both the horizontal velocity and the potential be continuous at the boundary between Region 1 and Region A as well as at the boundary between Region A and Region 2. The need for these additional constraints results from the mathematical convenience of solving for the potential in each of the four regions prior to requiring flow continuity throughout the fluid domain. These "matching" conditions are

$$\left.\begin{aligned} \phi_{ax} &= \phi_{mx} \\ \phi_a &= \phi_m \end{aligned}\right\} \text{ on } x = (-1)^m \ell,\ 0<y<h_m \qquad \begin{aligned}(2.10)\\(2.11)\end{aligned}$$

where the subscript, m, denotes either the Region 1 or the Region 2 when m assumes the notational value 1 or 2 respectively. Finally, as plane progressive waves are to be allowed at both infinities, we choose a radiation condition such that $|x|\to\infty$ $\Phi_m(x,y)$ is of the form of two plane progressive waves propagating in opposite directions.

As is common in water-wave problems, one assumes a separable solution for the potential where the x dependence must be composed of the complete set of functions (sinh kx, cosh kx) while the y dependence must be composed of the complete set (sinh ky, cosh ky). Applying the boundary conditions to the separable solution to Laplace's Equation, the form of the potential valid for $|x|<\ell$ is

$$\phi_r(x,y) = \sum_{n=0}^{\infty} \left(C_n e^{P_n x} + D_n e^{-P_n x}\right) Y_{rn}(y) \qquad (2.12)$$

$f_m(y)$. Second, the potentials at $x = \pm\ell$ will be assumed to be known functions, $g_m(y)$ over the interval $(0,d)$. The Fourier coefficients of Φ_r and Φ_m will be expressed in terms of $g_m(y)$. Then, the horizontal velocities, Φ_{mx} and Φ_{ax} will be matched over the interval $(0,h_m)$ subject to the boundary conditions on the rigid walls, (2.7) through (2.9). This will yield a set of simultaneous integral equations in $g_m(y)$. If exact solution of either set of integral equations was expedient, the aforementioned steps would correspond to redundant, yet equally valid, means of obtaining a unique solution to the boundary-value problem. However, as noted earlier, exact solutions to the integral equations common to water-wave problems are not tractable in general. It is from this seemingly redundant pair that Miles (1971) is able to construct complementary variational principles via Schwinger's method. We shall see that the first step, a velocity formulation, will yield for a special class of problems an upper bound approximation to the true solution to the scattering problem. We shall see that the second step, a potential formulation, will yield for a special class of problems a lower bound approximation to the true solution to the scattering problem. Even though we are unable to show that this bounding argument is valid for the problem at hand, we shall demonstrate that the complementary features of the two approaches are helpful in interpreting the physics of scattering and provide a practical vehicle for performing numerical calculations.

We shall begin with the velocity formation and show its development in some detail. We proceed as described by specifying that

$$\Phi_{mx} = f_m(y) \quad \text{on} \quad x = (-1)^m \ell \ , \quad 0 < y < h_m \tag{3.1}$$

$$\Phi_{rx} = f_m(y) \quad \text{on} \quad x = (-1)^m \ell \ , \quad 0 < y < d \tag{3.2}$$

Combining (3.1) and the boundary conditions (2.7) through (2.9), we form the Fourier coefficients of Φ_{mx}:

$$k_{mn}\left[(\delta_{1m} + \delta_{2m}\delta_{on}) A_{mn} - (\delta_{2m} + \delta_{1m}\delta_{on}) B_{mn}\right] = \rho_a \int_m f_m(y) Y_{mn}(y)\, dy \tag{3.3}$$

Combining (3.2) and the boundary conditions (2.7) through (2.9), we form the Fourier coefficients of Φ_{rx}:

$$p_n (C_n e^{-p_n \ell} - D_n e^{p_n \ell}) = \sum_r \rho_r \int_r f_1(y) Y_{rn}(y)\, dy \rightarrow \rho_a \int_0^{h_1} f_1(y) Y_{an}(y)\, dy \tag{3.4a}$$

$$p_n (C_n e^{p_n \ell} - D_n e^{-p_n \ell}) = \sum_r \rho_r \int_r f_2(y) Y_{rn}(y)\, dy \rightarrow \rho_a \int_0^{h_2} f_2(y) Y_{an}(y)\, dy \tag{3.4b}$$

As might be expected, (3.4a) and (3.4b) are simultaneous in $f_m(y)$, which allows us to express the behavior of the inhomogeneous domain entirely in terms of the velocities, $f_m(y)$, without loss of information. Finally, matching the potentials at $x = \pm\ell$, we obtain the simultaneous integral equations

$$\sum_{n=0}^{\infty}\left[(\delta_{1m}+\delta_{2m}\delta_{0n})A_{mn}+(\delta_{2m}+\delta_{1m}\delta_{0n})B_{mn}\right]Y_{mn}(y)=$$

$$\rho a\sum_{n=0}^{\infty}\left[\sum_{s=1}^{2}\kappa_{smn}\int_{S} f_s(\eta)Y_{an}(\eta)\,d\eta\right]\frac{Y_{an}(y)}{p_n} \quad (3.5)$$

where

$$\kappa_{smn}=\left[\delta_{2s}(\delta_{1m}+\delta_{2m}\cosh 2p_n\ell)-\delta_{1s}(\delta_{2m}+\delta_{1m}\cosh 2p_n\ell\right]\operatorname{csch} 2p_n\ell$$

The coefficient κ_{smn} is trigonometric for the imaginary values (p_o, p_1) and otherwise hyperbolic. The resulting expressions for the potential for all x and y in terms of $f_m(y)$ and the incident potentials from infinity by virtue of (3.1) are:

$$\Phi_r(x,y)=\rho a\sum_{n=0}^{\infty}(p_n\sinh 2p_n\ell)^{-1}\left[\cosh p_n(x+\ell)\int_0^{h_2} f_2(\eta)Y_{an}(\eta)\,d\eta - \right.$$

$$\left. -\cosh p_n(x-\ell)\int_0^{h_1} f_1(\eta)Y_{an}(\eta)\,d\eta\right]Y_{rn}(y) \quad (3.6)$$

$$\Phi_m(x,y)=\left[A_{mo}e^{k_{mo}[x-(-1)^m\ell]}+B_{mo}e^{-k_{mo}[x-(-1)^m\ell]}\right]Y_{mo}(y)+$$

$$+(-1)^m\rho a\sum_{n=1}^{\infty}\left[\int_0^{h_m} f_m(\eta)Y_{mn}(\eta)\,d\eta\right]k_{mn}^{-1}Y_{mn}(y)\,e^{-k_{mn}|x-(-1)^m\ell|}$$

(3.7)

Now we have formed two coupled integral equations, (2.3), in terms of $f_m(y)$ and the unknown scattered potential amplitudes (B_{10}, A_{20}). To decouple (3.5), we recognize that the horizontal velocity, $f_m(y)$, is the linear sum of the normalized velocity contributions associated with the incident potentials from $\pm\infty$. In other words, we are free to invoke the linear independence of $(A_{m0}+B_{m0})$ and $(A_{m0}-B_{m0})$ implied by (3.5) and (3.1). We shall choose to expand $f_m(y)$ in terms of $(A_{m0}+B_{m0})$ because this leads to a more conventional formulation and allows us to retain the scattered potential amplitudes explicitly which will aid in physical interpretation. Formally introducing the expansion for $f_m(y)$,

$$f_m(y) = \sum_{s=1}^{2} (A_{s0} + B_{s0})\, u_{ms}(y) \tag{3.8}$$

into (3.1) we obtain the linear scattering equations

$$k_{m0}(A_{m0} - B_{m0}) = \sum_{s=1}^{2} (A_{s0} + B_{s0})\, S_{ms} \tag{3.9}$$

where S_{ms} denotes the scattering element in terms of the normalized velocity contribution, $u_{ms}(y)$.

$$S_{ms} = \rho_a \int_m u_{ms}(y) Y_{m0}(y)\, dy \tag{3.10}$$

S_{ms} are the scattering elements which form the scattering matrix, $\underline{S}$. If a unique solution to the linear inhomogeneous scattering equations (3.9) exists, the determinate of $\underline{S}$ must not vanish by virtue of Cramer's rule. For the moment, we shall assume that

$\underset{\sim}{S}$ is non-singular.

Expanding, in turn, (3.5) with the substitution of (3.8) and invoking the linear independence of $(A_{m0}+B_{m0})$, we obtain the following set of four decoupled integral equations which are entirely independent of the parameter set (A_{m0},B_{m0}).

$$Y_{m0}(y) = \rho_a \sum_{n=0}^{\infty} \Bigg[\sum_{s=1}^{2} \frac{K_{smn}}{P_n} \int_s u_{ms}(\eta) Y_{an}(\eta)\, d\eta\, Y_{an}(y) + \\ + \frac{(\delta_{1m}-\delta_{2m})(1-\delta_{0n})}{k_{mn}} \int_m u_{mm}(\eta) Y_{mn}(\eta)\, d\eta\, Y_{mn}(y) \Bigg] \quad (3.12)$$

$$0 = \rho_a \sum_{n=0}^{\infty} \Bigg[\sum_{s=1}^{2} \frac{K_{smn}}{P_n} \int_s u_{sm}(\eta) Y_{an}(\eta)\, d\eta\, Y_{an}(y) + \\ + \frac{(\delta_{1m}-\delta_{2m})(1-\delta_{0n})}{k_{mn}} \int_m u_{sm}(\eta) Y_{mn}(\eta)\, d\eta\, Y_{mn}(y) \Bigg] \quad (3.13)$$

Multiplying (3.12) and (3.13) by $\rho_a u_{ms}(y)$ and then integrating over the region of validity $(0,h_m)$, a set of four simultaneous equations for S_{ms} in terms of the normalized velocities, $u_{ms}(y)$ are recovered.

For m=s

$$S_{mm} = \frac{\left[\int_m u_{mm}(y)\, Y_{m0}(y)\, dy \right]^2}{\sum_{n=0}^{\infty} \left\{ \frac{-K_{mm}}{P_n} \left[\int_s u_{sm}(y) Y_{an}(y)\, dy \right]^2 + \frac{(\delta_{1m}-\delta_{2m})(1-\delta_{0n})}{k_{mn}} \left[\int_m u_{mm}(y) Y_{mn}(y)\, dy \right]^2 \right.}$$

(3.14)

For $m \neq s$

$$S_{ms} = \frac{\int_{s} u_{sm}(\eta) Y_{so}(\eta)\, d\eta \int_{m} u_{ms}(y) Y_{mo}(y)\, dy}{\sum_{n=0}^{\infty} \frac{\kappa_{smn}}{p_n} \iint_{sm} \left[u_{ss}(\eta) u_{mm}(y) - u_{sm}(\eta) u_{ms}(y) \right] Y_{an}(\eta) Y_{an}(y)\, d\eta\, dy} \tag{3.15}$$

Recalling that κ_{smn} is simply the negative of κ_{msn}, a reciprocity relation similar to that derived by Miles (1967) is obtained.

$$S_{12} = -S_{21} \tag{3.16}$$

At the same time, we may conclude that S_{ms} is pure real since the decoupled integral equations (3.12) and (3.13) are pure real, and hence the true solution, $u_{ms}(y)$ must also be real. Application of these general properties will be deferred until we consider the relationships between the scattered potentials and the complementary variational principles.

Having constructed the scattering element forms for (3.12) and (3.13), we may apply the tests which determine the validity of Schwinger's variational procedure for our problem. To apply the method, we must establish two properties:

- the scattering elements, S_{ms}, must be stationary with respect to first order variations of the true solution to the integral equations (3.12) and (3.13).
- the scattering elements, S_{ms}, must be invariant to a scale transformation of $u_{ms}(y)$.

The variational property is concerned with approximation to the true y-dependence of the horizontal velocity at $x = \pm \ell$. To verify this property for our problem, substitute the approximation

$$u^*_{ms}(y) = u_{ms}(y) + \varepsilon g_{ms}(y)$$

into (3.14) and (3.15). After some algebra, one recovers the approximation to the scattering element,

$$S^*_{ms} = S_{ms} + O(\varepsilon^2)$$

which demonstrates the accuracy to $O(\varepsilon^2)$. The invariance property is concerned with the magnitude of the chosen approximation, $u^*_{ms}(y)$ with respect to the true solution. This is verified for our problem by substituting $Cu_{ms}(y)$ into (3.14) and (3.15). Again after some algebra, S^*_{ms} is shown to be invariant and implies that the net energy flux through $x=\pm\ell$ is zero. Thus, the applicability of Schwinger's approximation method is established for the velocity formulation.

The potential formulation proceeds analogously. We assume that the potential on $|x|=\ell$ is a known function

$$\phi_r(y) = g_m(y) \quad \text{on} \quad x = (-1)^m \ell \,, \quad 0 < y < d$$
$$\phi_m(y) = g_m(y) \quad \text{on} \quad x = (-1)^m \ell \,, \quad 0 < y < h_m \tag{3.17}$$

Combining (3.17) with the matching conditions, (2.10) and (2.11), and applying the boundary conditions, (2.7) through (2.9), the resulting set of integral equations simultaneous in $q_m(y)$ is:

for $0 < y < h_m$

$$\sum_{n=0}^{\infty} k_{mn} \left[(\delta_{1m} + \delta_{2m}\delta_{0n}) A_{mn} - (\delta_{2m} + \delta_{1m}\delta_{0n}) B_{mn} \right] Y_{mn}(y) =$$
$$\sum_{n=0}^{\infty} p_n \left[\sum_{s=1}^{2} \kappa_{smn} \sum_r \rho_r \int_r g_s(\eta) Y_{rn}(\eta)\, d\eta \right] Y_{an}(y) \tag{3.18a}$$

for $h_m < y < h_a$

$$0 = \sum_{n=0}^{\infty} p_n \left[\sum_{s=1}^{2} \kappa_{smn} \sum_{r} \rho_r \int_{r} g_s(\eta) Y_{rn}(\eta)\, d\eta \right] Y_{an}(y) \tag{3.18b}$$

for $h_a < y < d$

$$0 = \sum_{n=0}^{\infty} p_n \left[\sum_{s=1}^{2} \kappa_{smn} \sum_{r} \rho_r \int_{r} g_s(\eta) Y_{rn}(\eta)\, d\eta \right] Y_{bn}(y) \tag{3.18c}$$

where by virtue of the orthogonality of $Y_{mn}(y)$ and the radiation condition

$$(\delta_{1m} + \delta_{2m}\delta_{on}) A_{mn} + (\delta_{2m} + \delta_{1m}\delta_{on}) B_{mn} = \rho_a \int_{m} g_m(y) Y_{mn}(y)\, dy \tag{3.19}$$

The complication of the internal sum over r=(a,b) as defined by (2.14) is due to the stratification within $|x|<\ell$. However, aside from the cumbersome sum over r, the kernels of (3.5) and (3.18) differ only through the inversion of the wave modes, p_n and k_{mn}. To form the scattering elements associated with (3.18), we invoke the linear independence implied by (3.18) and (3.19). Again, we are free to expand $g_m(y)$ in terms of any linear combinations of (A_{10}, B_{20}). We choose to expand $g_m(y)$ in a manner complementary to the velocity formulation,

$$g_m(y) = \sum_{s=1}^{2} k_{so} [A_{so} - B_{so}]\, v_{ms}(y) \tag{3.20}$$

so that the linear scattering equations

$$[A_{mo} + B_{mo}] = \sum_{s=1}^{2} k_{so} [A_{so} - B_{so}]\, S'_{ms} \tag{3.21}$$

may be formed from (3.19) upon the substitution of (3.20) where S'_{ms} denotes the scattering element in terms of the normalized potential $v_{ms}(y)$:

$$S'_{ms} = \rho_a \int_m v_{ms}(y) Y_{m0}(y)\, dy \tag{3.22}$$

By comparing (3.20) and (3.9), we are able to draw the conclusion that the scattering matrix S'_{ms} derived from the potential formulation is simply the inverse of the scattering matrix S_{ms} derived from the velocity formulation. Since S_{ms} is assumed to be non-singular, its inverse may be constructed. This shows that a solution to the scattering problem may be derived from either the velocity formulation or the potential formulation. But, of practical importance, we have established a convenient vehicle for comparing approximate solutions to our problem.

To derive the inverse scattering matrix $S^{-1}_{ms}(y)$, (3.20) is substituted into (3.18) and the linear independence of $k_{m0}(A_{m0}-B_{m0})$ is invoked to decouple the integral equations. The decoupled integral equations are independent of the parameter set (A_{m0}, B_{m0}).

$$Y_{m0}(y) = \sum_{n=0}^{\infty}\Big[p_n \sum_{s=1}^{2} \kappa_{smn} \sum_{r} \rho_r \int_r v_{sm}(\eta) Y_{rn}(\eta)\, d\eta\, Y_{an}(y) + (\delta_{2m}-\delta_{1m})(1-\delta_{0n}) k_{mn} \cdots$$

$$\cdots \int_0^{h_m} v_{mm}(\eta) Y_{mn}(\eta)\, d\eta\, Y_{mn}(y)\Big] \tag{3.23a}$$

$$0 = \sum_{n=0}^{\infty}\Big[p_n \sum_{s=1}^{2} \kappa_{smn} \sum_{r} \rho_r \int_r v_{ms}(\eta) Y_{rn}(\eta)\, d\eta\, Y_{an}(y) + (\delta_{2m}-\delta_{1m})(1-\delta_{0n}) \cdots$$

$$\cdots k_{mn} \int_0^{h_m} v_{ms}(\eta) Y_{mn}(\eta)\, d\eta\, Y_{mn}(y)\Big] \tag{3.23b}$$

$$0 = \sum_{n=0}^{\infty} p_n \sum_{s=1}^{2} \kappa_{smn} \sum_{r} \rho_r \int_r v_{sm}(\eta) Y_{rn}(\eta)\, d\eta \, Y_{a,bn}(y) \tag{3.24}$$

$$0 = \sum_{n=0}^{\infty} p_n \sum_{s=1}^{2} \kappa_{smn} \sum_{r} \rho_r \int_r v_{ms}(\eta) Y_{rn}(\eta)\, d\eta \, Y_{a,bn}(y) \tag{3.25}$$

This resulting set of uncoupled integral equations is multiplied by $\rho_r v_{ms}(y)$, integrated over the region of validity, and summed. Thus, the inverse scattering elements are constructed as follows:

for m=s

$$S^{-1}_{mm} = \frac{\rho_a^2 \left[\int_m v_{mm}(y) Y_{m0}(y)\, dy \right]^2}{\sum_{n=0}^{\infty} p_n \left\{ \sum_{s=1}^{2} \kappa_{ssn} \left[\sum_r \rho_r \int_r v_{sm}(y) Y_{rn}(y)\, dy \right]^2 + (\delta_{1m} - \delta_{2m})(1-\delta_{0n}) k_{mn} \rho_a^2 \left[\int_m v_{mm}(y) Y_{mn}(y)\, dy \right]^2 \right.} \tag{3.26}$$

for m≠s

$$S^{-1}_{ms} = \frac{\rho_a^2 \int_m v_{ms}(y) Y_{m0}(y)\, dy \int_s v_{sm}(\eta) Y_{s0}(\eta)\, d\eta}{\sum_{n=0}^{\infty} p_n \kappa_{smn} \left[\sum_r \sum_{r'} \left\{ \rho_r \int_r v_{sm}(y) Y_{rn}(y)\, dy \, \rho_{r'} \int_{r'} v_{ms}(\eta) Y_{r'n}(\eta)\, d\eta - \rho_r \int_r v_{mm} Y_{rn} \, \rho_{r'} \int_{r'} v_{ss} Y_{r'n} \right] \right.} \tag{3.27}$$

As in the velocity formulation, we can conclude that (3.26) and (3.27) are stationary with respect to first order variations of $v_{ms}(y)$ about the true solution to (3.18), are invariant to scale transformations of $v_{ms}(y)$, yield the reciprocity relation

$$S^{-1}_{12} = -S^{-1}_{21}$$

and confirm that $(S^{-1}_{ms}, v_{ms}(y))$ are real. Thus, the applicability of Schwinger's variational procedure for constructing approximate solutions from the potential formulation is verified.

4.0 SYMMETRY RELATIONS BETWEEN THE SCATTERED POTENTIALS

To establish the theoretical symmetry relations, actual solution of the implied scattering problem is not necessary. Therefore, we may make completely general statements from the properties of either S_{ms} or S_{ms}^{-1}. We choose to use the velocity formulation. With the definition of S_{ms}, the reflection and transmission coefficients may be constructed by solving the simultaneous linear scattering equations (3.9), for the unknown scattered amplitudes (B_{10}, A_{20}) in terms of the incident potentials (A_{10}, B_{20}), the propagating wave modes k_{m0}, and the scattering elements S_{ms}. We obtain

$$B_{10} = \frac{A_{10}\left[(k_{20}-S_{22})(k_{10}-S_{11})-S_{12}S_{21}\right] - 2B_{20}k_{20}S_{12}}{(k_{10}+S_{11})(k_{20}-S_{22}) + S_{12}S_{21}} \tag{4.1}$$

$$A_{20} = \frac{2A_{10}k_{10}S_{21} + B_{20}\left[(k_{10}+S_{11})(k_{20}+S_{22})-S_{12}S_{21}\right]}{(k_{10}+S_{11})(k_{20}-S_{22}) + S_{12}S_{21}} \tag{4.2}$$

Defining the complex reflection and transmission coefficients from (4.1) and (4.2) such that

$$R^- = \frac{B_{10}e^{-2k_{10}\ell}}{A_{10}}, \quad T^- = \frac{A_{20}e^{-(k_{10}+k_{20})\ell}}{A_{10}}, \quad B_{20} \equiv 0. \tag{4.3}$$

$$T^+ = \frac{B_{10}e^{-(k_{10}+k_{20})\ell}}{B_{20}}, \quad R^+ = \frac{A_{20}e^{-2k_{20}\ell}}{B_{20}}, \quad A_{10} \equiv 0. \tag{4.4}$$

we can develop symmetry relations. Recalling the properties of S_{ms} following the discussion of (3.15), we make use of the reciprocity relation for $m \neq s$ and the proof that S_{ms} is real. We recover the results

$$|R^+| = |R^-| \tag{4.5}$$

$$|R^+|^2 + \frac{|k_{20}|}{|k_{10}|}|T^+|^2 = |R^-|^2 + \frac{|k_{10}|}{|k_{20}|}|T^-|^2 = 1 \tag{4.6}$$

$$|k_{10}||T^+| = |k_{20}||T^-| \tag{4.7}$$

$$\arg(T^+) = \arg(T^-) = \arg(T) \tag{4.8}$$

$$\arg(R^+) + \arg(R^-) = 2\arg(T) + \pi \tag{4.9}$$

For example, to demonstrate the relation between the transmission phase shifts (4.8), we divide T^+ by T^- with the incident amplitudes (A_{10}, B_{20}) assumed to be of the same magnitude and phase, invoke the reciprocity relation between (S_{21}, S_{12}), and find that the result is pure real. Therefore, the transmission phase shifts must be the same:

$$\frac{T^+}{T^-} = -\frac{k_{20} S_{12}}{k_{10} S_{21}} = \frac{k_{20}}{k_{10}} = \frac{iK_2}{iK_1} = \frac{K_2}{K_1}$$

These relations between the amplitudes of the scattered potential are identical to those previously derived in studies of the scattering properties of rigid geometries in homogeneous fluid domains. Except for slight notational differences, the symmetry relations (4.5) through (4.7) are shown in Kriesel (1949) and Wehausen and Laitone (1960) while (4.8) and (4.9) are given by Newman (1965a, 1965b). As the far-field behavior of the scattered potential obeys the same symmetry relations exhibited by a rigid body, we are able to conclude that the inhomogeneous domain could be replaced by an equivalent rigid body peculiar to each frequency parameter, K. At the same time, since both S_{ms} and $u_{ms}(y)$ are real, we are able to conclude that the phase of the horizontal velocity $f_m(y)$ must be independent of y. This phase constancy has been proven for an open-bottomed circular cylinder in waters of finite depth and for a circular dock in waters of finite depth by Garrett (1970,1971).

The convenient derivation of the scattering symmetry relations from a single formulation is of more than pedagogic value. We may conclude that any domain whose Green's function is real will obey the scattering properties shown by (4.5) through (4.9).

5.0 RELATIONS BETWEEN THE COMPLEMENTARY VARIATIONAL PRINCIPLES

Miles (1971) was able to prove that the complementary variational integrals peculiar to the scattering problem posed by a circular dock in waters of finite depth could be used to calculate upper and lower bound approximations to the true scattering coefficient. To prove this, Miles observed that the kernels of the integral equations formed via the velocity approach and the kernels of the integral equations formed via the potential approach were real, positive, and symmetric. This meant that the associated variational integrals were positive definite. Upon applying Bessel's inequality, Miles was able to conclude that any assumed trial function describing either the velocity or the potential over his matching boundary would yield an upper bound estimate to the true scattering coefficient. Using our notation, we express this as

$$\frac{S^{*}_{ms}}{S_{ms}} = 1 + \varepsilon_v^2 \qquad (5.1)$$

where ε_v is the unprescribed error associated with the assumed description of the velocity on the matching boundary and

$$\frac{S^{*\,-1}_{ms}}{S^{-1}_{ms}} = 1 + \varepsilon_p^2 \qquad (5.2)$$

where ε_p is the unprescribed error associated with the assumed description of the potential on the matching boundary. Inverting the scattering element particular to the potential formulation so that it may be compared directly to the scattering element particular to the velocity formulation, we find that (5.2) is

$$\frac{S^{*}_{ms}}{S_{ms}} = \frac{1}{1 + \varepsilon_p^2} \qquad (5.3)$$

Comparing (5.1) and (5.3), we see that the complementary variational forms yield upper and lower bound estimates to the true scattering element. Admittedly, the magnitudes of ε_v and ε_p are unknown. However, the practical value of the bounding argument may be established numerically. In Miles (1971), detailed calculations show that the upper and /er bounds differ by at most a few percent.

To apply this bounding argument to our problem, we must refer to the kernels of the integral equations peculiar to the velocity formulation (3.12, 3.13) and to the kernels of the integral equations peculiar to the potential formulation (3.23, 3.24, 3.25). By inspection, we observe that the kernels are real and symmetric, but indefinite. The kernels are indefinite because they both contain the coefficient κ_{smn} which represents the coupling between the two matching boundaries. For n=0,1, κ_{smn} is a trigonometric function in $p_n \ell$. In particular, when $p_n \ell$ is an integer multiple of π, κ_{smn} is singular. Therefore, the variational integrals associated with both formulations are indefinite. Therefore, the sign of S^*_{ms} or its inverse cannot be established in general. This appears to preclude the bounding argument for our problem since approximate forms of the potential or the velocity will not approach the true solution to the scattering matrix monotonically. In Miles (1971), the floating dock pierces the free surface. This eliminates the imaginary eigenvalue in the domain of the obstacle and, as a result, his κ_{smn} is always positive. In our problem, the domain of the obstacle contains two imaginary eigenvalues corresponding to a propagating mode, due to the depth of the basin, and to another propagating mode, due to the stratification within the basin. Even without the complication of stratification, the propagating surface mode would remain and the associated variational forms would be indefinite. Therefore, we conclude that the bounding argument is applicable only when the domain of the obstacle contains no imaginary eigenvalues. This forces us to abandon the bounding argument.

Having lost the attractive feature of a bounding argument, we shall investigate the implications of the singular behavior when $p_n \ell$ for n=0,1 is an integer multiple of π. To do this, we must consider the validity of our assumption that $\underset{\sim}{S}$ is non-singular. Suppose that $u^*_{ms}(y)$ and $u_{ms}(y)$ are two distinct solutions of (3.12, 3.13). Multiplying the integral equations through by $u^*_{ms}(y)$, integrating over the region of validity $(0,h_m)$, invoking the symmetry of the kernels, and our hypotheses that $u^*_{ms}(y)$ is a solution to the integral equations, we find that $\underline{S}^*$ is equivalent to $\underline{S}$. Therefore, $\underline{S}$ is unique and non-singular except when the integer multiple of π occurs. This proves that the two complementary approaches are equivalent. When $\underline{S}$ is singular, $\underline{S}^{-1}$ is undefined. This means that a solution via the velocity approach would not agree with a solution via the potential approach. For clarity, we shall show the above procedure for the scattering element S_{11}. The integral equations are:

$$Y_{10}(y) = -\int_0^{h_1} u_{11}(\eta)\, G_1(\eta,y)\, d\eta + \int_0^{h_2} u_{21}(\eta)\, G(\eta,y)\, dy$$

$$0 = -\int_0^{h_1} u_{11}(\eta)\, G(\eta,y)\, d\eta + \int_0^{h_2} u_{21}(\eta)\, G_2(\eta,y)\, dy$$

where G, G_1, and G_2 represent a compact notation for the kernels of (3.12) and (3.13). Multiplying by $u_{11}(y)$ and $u_{21}(y)$, the scattering elements are:

$$S_{11} = \int_0^{h_1} u_{11}(y) Y_{10}(y)\, dy = -\int_0^{h_1}\!\!\int_0^{h_1} u_{11}(y)\, u^*_{11}(\eta)\, G_1(\eta,y)\, d\eta\, dy + \int_0^{h_1}\!\!\int_0^{h_2} u_{11}(y)\, u^*_{21}(\eta)\, G(\eta,y)\, d\eta\, dy$$

$$S^*_{11} = \int_0^{h_1} u^*_{11}(y) Y_{10}(y)\, dy = -\int_0^{h_1}\!\!\int_0^{h_1} u^*_{11}(y)\, u_{11}(\eta)\, G_1(\eta,y)\, d\eta\, dy + \int_0^{h_2}\!\!\int_0^{h_1} u^*_{11}(y)\, u_{21}(\eta)\, G(\eta,y)\, d\eta\, dy$$

$$0 = -\int_0^{h_1}\!\!\int_0^{h_2} u_{21}(y)\, u^*_{11}(\eta)\, G(\eta,y)\, d\eta\, dy + \int_0^{h_2}\!\!\int_0^{h_2} u^*_{21}(\eta)\, u_{21}(y)\, G_2(\eta,y)\, d\eta\, dy$$

Combining these after substituting for the cross terms, we establish:

$$S_{11}^{*} = S_{11}$$

$$\therefore \quad S_{ms}^{*} = S_{ms}$$

Having established the uniqueness of $\underline{S}$, we shall consider if $u_{ms}(y)$ is the unique solution to (3.12, 3.13). Assuming that two distinct solutions exist, we consider the integrals

$$\left\{ \begin{aligned} \Delta &= S_{11}\left\{u_{11}^{*}-u_{11}, u_{21}^{*}-u_{21}\right\} = -\iint (u_{11}^{*}-u_{11})(u_{11}^{*}-u_{11}) G_1 + \\ &\qquad\qquad \iint (u_{11}^{*}-u_{11})(u_{21}^{*}-u_{21}) G \\ 0 &= -\iint (u_{21}^{*}-u_{21})(u_{11}^{*}-u_{11}) G + \iint (u_{21}^{*}-u_{21})(u_{21}^{*}-u_{21}) G_2 \end{aligned} \right\}$$

$$\Delta = S_{11}^{*} - 2\left[\iint u_{11}u_{11}^{*} G_1 - \iint u_{21}u_{21}^{*} G_2\right] + S_{11}$$

$$\Delta = S_{11}^{*} - S_{11}$$

based on the difference between the two solutions. If the kernels of $\underline{S}$ were definite, Δ would always be greater than or equal to zero. Δ, by virtue of (5.4), would vanish if and only if $u_{ms}^{*}(y) = u_{ms}(y)$. This would prove the uniqueness of $u_{ms}(y)$. Even if the kernels of $\underline{S}$ were semi-definite, Δ would obey the same properties except at an infinite set of discrete values. This would allow a bounding argument. As the kernels in our problem are indefinite, the sign of Δ cannot be established in general. Therefore, we cannot prove the uniqueness of our solution or apply the associated bounding argument.

Considering the application of complementary variational principles to water-wave scattering problems in general, we conclude that it is sufficient for the Green's function to be real, symmetric, and indefinite to prove the uniqueness of the scattering matrix. This covers a broad range of scattering problems. To apply the bounding argument and the proof of uniqueness, it is sufficient for the Green's function to be real, symmetric, and semi-definite. Examples for which this complete argument appear possible include obstacles of infinitesimal width or a single step change in depth for stratified as well as homogeneous fluid domains and arbitrary obstacles piercing the free surface in homogeneous domains. Physically, these are cases where there is no interference between the waves at the ends of the obstacle.

6.0 NUMERICAL APPLICATION OF THE COMPLEMENTARY VARIATIONAL PRINCIPLES

The numerical value of the complementary forms is useful even though we have been forced to abandon the bounding argument. Since either formulation is theoretically identical to the other, we may compare the numerical results yielded by the complementary formulations. If there is a significant difference between the theoretically identical scattering matrices, the variational integrals are sensitive to the assumed trial functions describing the velocity and the potential at $x=\pm\ell$. If the complementary estimates are in tolerable agreement, the variational integrals are insensitive to the assumed trial functions. Here we must draw the distinction between mathematical description and physical description. Without the bounding argument, we are forced to admit that our estimates are possibly invalid for at least some combination of the physical variables. Until more detailed experimental or numerical analyses are available for comparison, a more definitive statement is untenable. In particular, solutions near the resonant modes of the basin require careful study.

Having verified the applicability of Schwinger's variational argument to our problem, we shall assume that the horizontal velocity and the potential at $x=\pm\ell$ may be described by simple functions. The variational argument assures us that an $O(\varepsilon)$ error in our assumed trial function contributes no more than an $O(\varepsilon^2)$ error to our estimate of the scattering matrix. We shall choose the trial functions $u^*_{ms}(y)$ and $v^*_{ms}(y)$ with the orthogonality statements (2.19) and (2.14) in mind. These are:

$$u^*_{ms}(y) = C_{ms} Y_{m0}(y) \quad \text{on} \quad x = (-1)^m \ell\ ,\ 0<y<h_m \tag{6.1}$$

$$\begin{aligned} v^*_{ms}(y) &= D_{ms} Y_{a0}(y) \quad \text{on} \quad x = (-1)^m \ell\ ,\ 0<y<h_a \\ &= D_{ms} Y_{b0}(y) \quad \text{on} \quad x = (-1)^m \ell\ ,\ h_a<y<d \end{aligned} \tag{6.2}$$

Physically, we are assuming that the velocity distribution on the matching boundary (6.1) may be adequately described by the velocity contribution associated with a constant multiple of the far-field potential propagating modes, k_{m0}. At high frequencies, we expect the body to have little effect on the incident potential so, for at least part of the range of interest, this trial function is physically plausible. This approximation is often referred to as the Borne approximation in the literature of acoustics and electromagnetics. Miles (1967) used a similar trial function in this study of step-shelf scattering and showed good numerical agreement with the more accurate numerical results obtained by Newman (1965b). In (6.2), we are assuming that the potential at the matching boundary may be adequately described by the potential contribution associated with the propagating mode related to the depth of the basin, p_0. While there are no precedents for this choice, it allows us to consider the effects of the stratification separately from the effects of depth. At high frequencies, we will expect that these two solutions are in good agreement. At low frequencies, the difference between them will be due solely to the presence of stratification. We shall clarify this momentarily. Referring to the actual forms of the assumed trial functions (2.20) and (2.15, 2.16), we remark that these are quite different physical descriptions. The numerical test must be the final judge.

Formally introducing (6.1) and (6.2) into the variational integrals, we find that both (3.14, 3.15) and (3.26, 3.27) consist of a common integral form. For the velocity formulation, this is:

$$\int_0^{h_m} Y_{m0}(y) Y_{an}(y)\, dy = \left(\Lambda^h_{m0} \Lambda^s_n\right)^{-1/2} \frac{p_n}{p_n^2 - k_{m0}^2} \left[\frac{p_n \sin p_n h_m + K \cos p_n h_m}{p_n \sin p_n h_a + K \cos p_n h_a}\right] \tag{6.3}$$

For the potential formulation, the corresponding form is:

$$\int_0^{h_m} Y_{mn}(y) Y_{a0}(y)\, dy = \left(\Lambda^h_{mn} \Lambda^s_0\right)^{-1/2} \frac{p_0}{p_0^2 - k_{mn}^2} \left[\frac{p_0 \sin p_0 h_m + K \cos p_0 h_m}{p_0 \sin p_0 h_a + K \cos p_0 h_a}\right] \tag{6.4}$$

In the special case where $p_n = k_{mn}$, both (6.3) and (6.4) reduce to

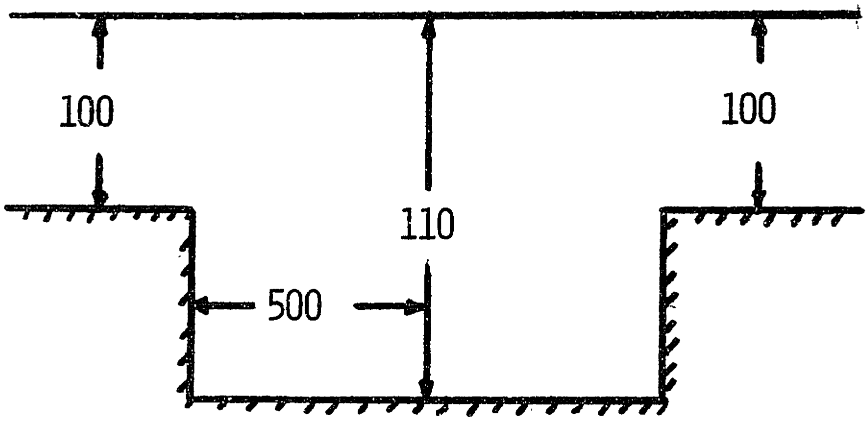
$$\int_0^{h_m} Y_{m0}(y)Y_{a0}(y)\,dy = (\Lambda^h_{m0}\Lambda^s_0)^{-1/2} \frac{\sin p_0 h_m}{2} \left[\frac{1 - h_m K^{-1}(p_0^2+K^2)}{p_0 \sin p_0 h_a + K \cos p_0 h_a} \right]$$

We can see that the largest contribution occurs when both p_n and k_{mn} are imaginary as the difference of their squares appears in the denominator of (6.3, 6.4). However, when $k_{mn}=p_n$, the integral remains non-singular even though the contribution can be quite large. This means that the only singular terms in $\underline{S}$ are associated with the normal modes on the basin, $p_n \ell = n\pi$ for n=0,1. The physical difference implied by our choice is confirmed by comparing these two integrals. In the velocity case (6.1) for n=0,1, we include both the imaginary wave modes of the basin. This will possibly give two large contributions. In the potential form (6.2) for n=0, we include only one imaginary wave mode. Therefore, any differences between the results are solely attributable to the presence of the propagating mode due to stratification within the basin.

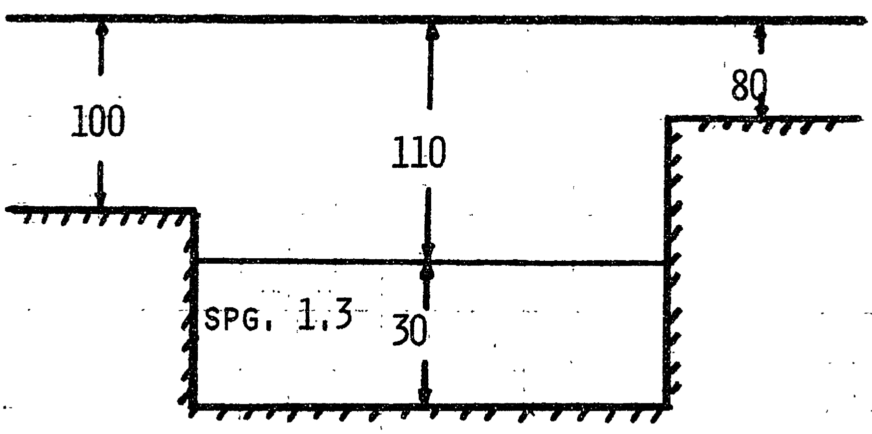
In actual calculations for the scattering elements, the integrals (6.3, 6.4) appear as products in an infinite series over all the possible wave modes. A truncated series of some twenty-five terms has proven to be more than adequate numerically. While the variational integrals are simple to evaluate, the solution of the exact eigenvalue relations (2.13, 2.18) for the roots (p_n, k_{mn}) is troublesome. It is difficult to insure that one is not missing roots during an iterative solution technique. We have found that either Halley's method or Mueller's method, Traub (1962) offer a reasonable compromise between the order of convergence and the likelihood of missing roots. We caution that neither iterative method is foolproof.

7.0 RESULTS AND CONCLUSIONS

In our problem, the number of physical variables is prohibitive with respect to a definitive treatment of sensitivity to each parameter. We shall restrict ourselves to an analysis of two separate problems. The first of these considers a set of variables of interest to the ocean engineer. This particular set emphasizes the small geometrical contrast and large density contrast peculiar to a sediment pocket. The second set of variables addresses the large geometric contrast and minute density contrast of interest to the oceanographer. Physically, the oceanographer's problem might correspond to a fjord. These and variations which we have chosen to investigate are shown in Figures 2 and 3 respectively. By comparing these extrema, the reader should be able to draw his own conclusions concerning the utility of the approximate solution method developed in this thesis. We have chosen to plot only the reflection coefficient magnitude as it is the poorest numerically. The "velocity" solution is shown in a solid line while the "potential" solution is shown in a dashed line. The reflection coefficient magnitudes are shown as a function of the nondimensional frequency parameter, Kh_1. The value for $Kh_1=0$ was established by taking the long-wave limit to our integral forms. For the symmetric case, $|R|$ tends to zero. In other words, the basin disappears as the incident wave length tends to infinity. For the asymmetric case, $|R|$ tends to the

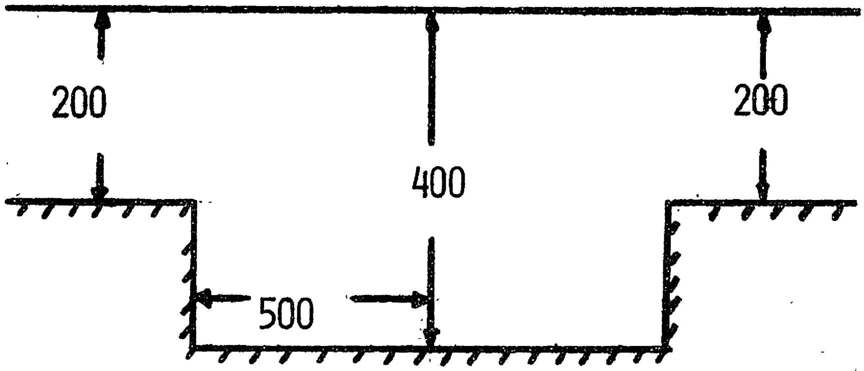


THE SYMMETRIC HOMOGENOUS CASE

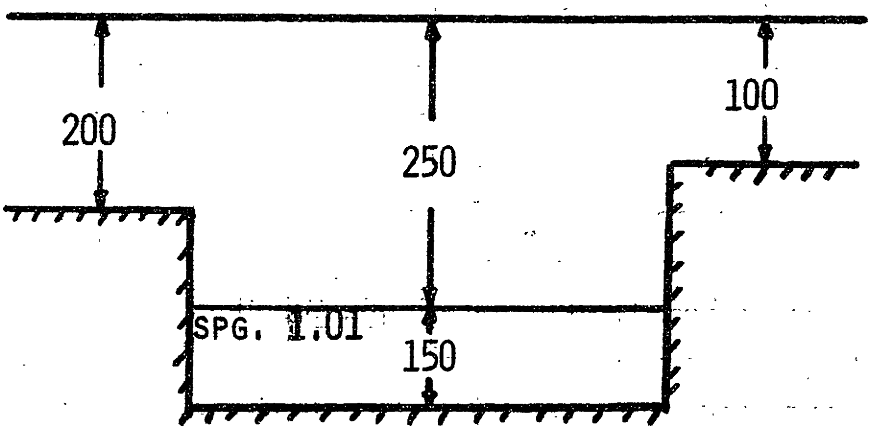


THE ASYMMETRIC INHOMOGENOUS CASE

THE SEDIMENT POCKET VARIABLES - FIGURE 2



THE SYMMETRIC HOMOGENOUS CASE



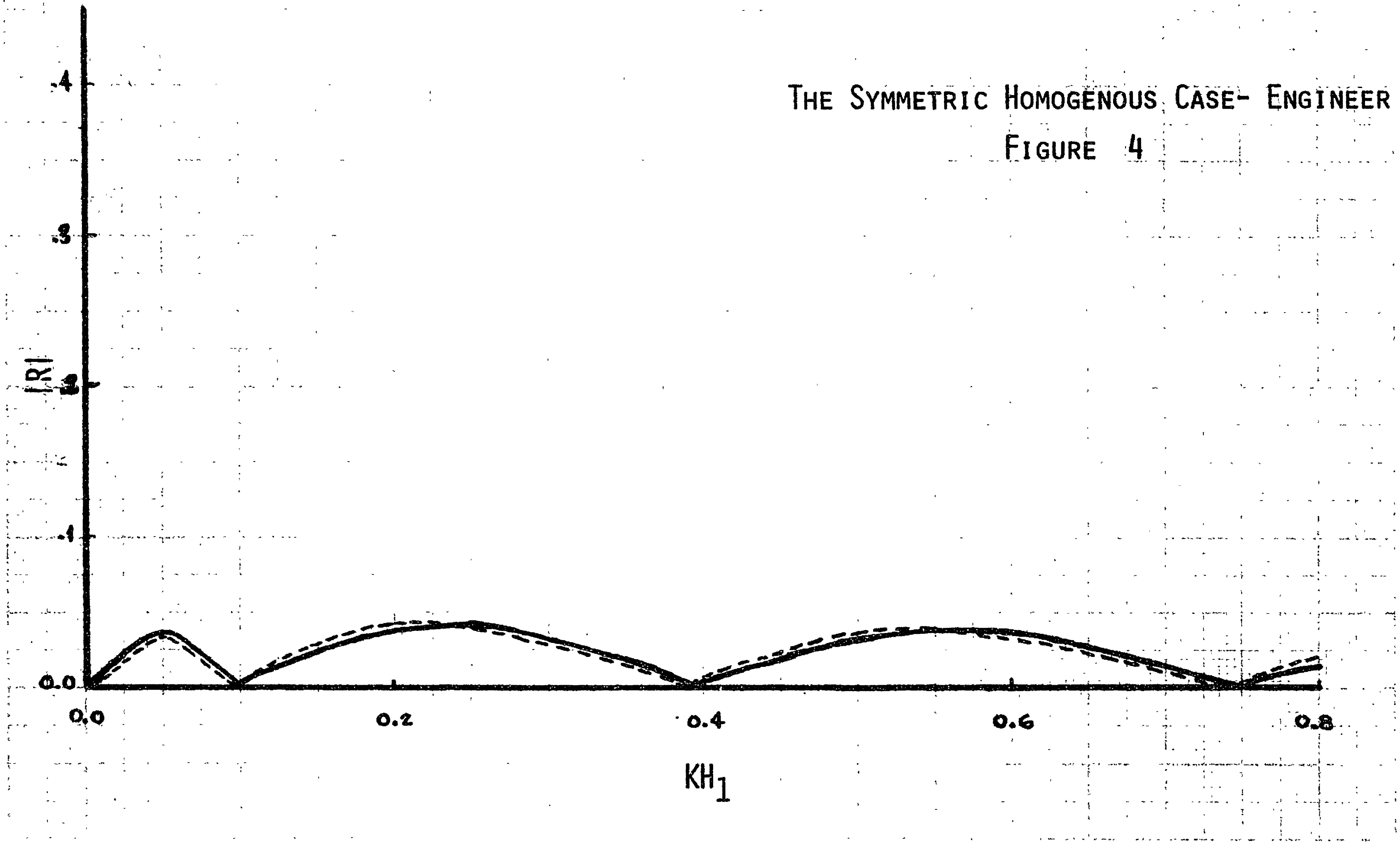
THE ASYMMETRIC INHOMOGENOUS CASE

THE FJORD VARIABLES - FIGURE 3

reflection coefficient of a step shelf of height, h_1-h_2. This is given by Lamb (1932).

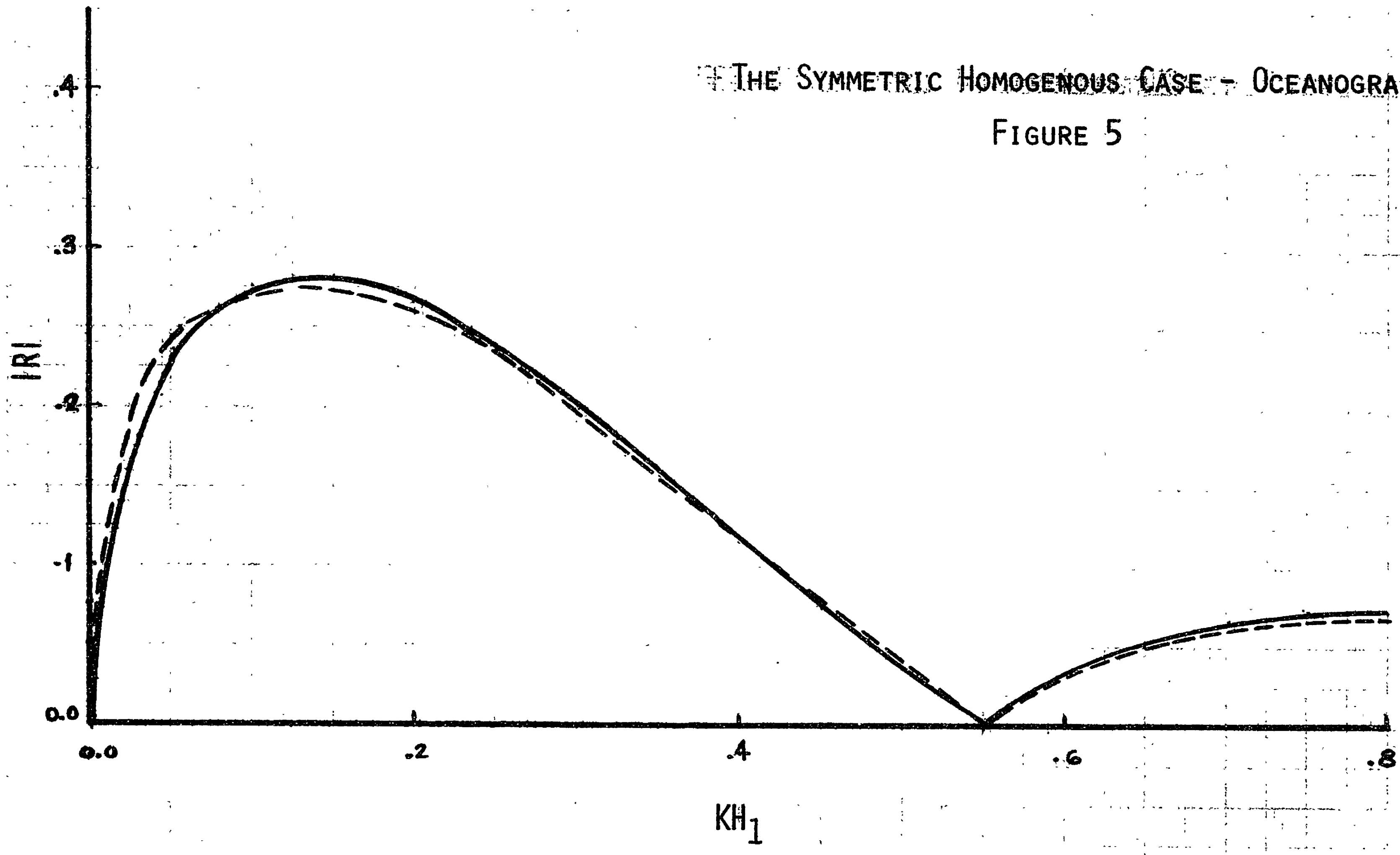
$$|R| = \frac{1-(h_2/h_1)^{1/2}}{1+(h_2/h_1)^{1/2}} \quad ; \; h_1 > h_2$$

We shall comment on the simplest configuration first--a symmetric basin in a homogeneous domain. Figures 4 and 5 show the results for the engineer's problem and the oceanographer's problem respectively. The most striking feature of these two graphs is the oscillatory behavior of $|R|$. The sharp zeroes correspond to incident frequencies at which the basin is transparent to the incident potential. These occur when $p_o \ell = n\pi$ and correspond physically to the resonant modes of the basin. This same type of oscillatory behavior has been treated analytically by Newman (1965a) and numerically by Mei and Black (1969) for a symmetric rectangular obstacle resting on the bottom. As in our problem, the sharp zeroes are connected with the excitation of standing waves over the obstacle. The only difference between the oceanographer's problem and the ocean engineer's problem is the periodicity of $|R|$ which increases as the depth of the basin increases. In Figures 6 and 7 we add the complication of asymmetry in the geometric idealizations. Here the most striking feature is the elimination of the sharp zeroes associated with the symmetric basin. In the terminology of electrical engineering, the fluctuating AC behavior of the symmetric basin is imposed on the steadily decreasing DC behavior



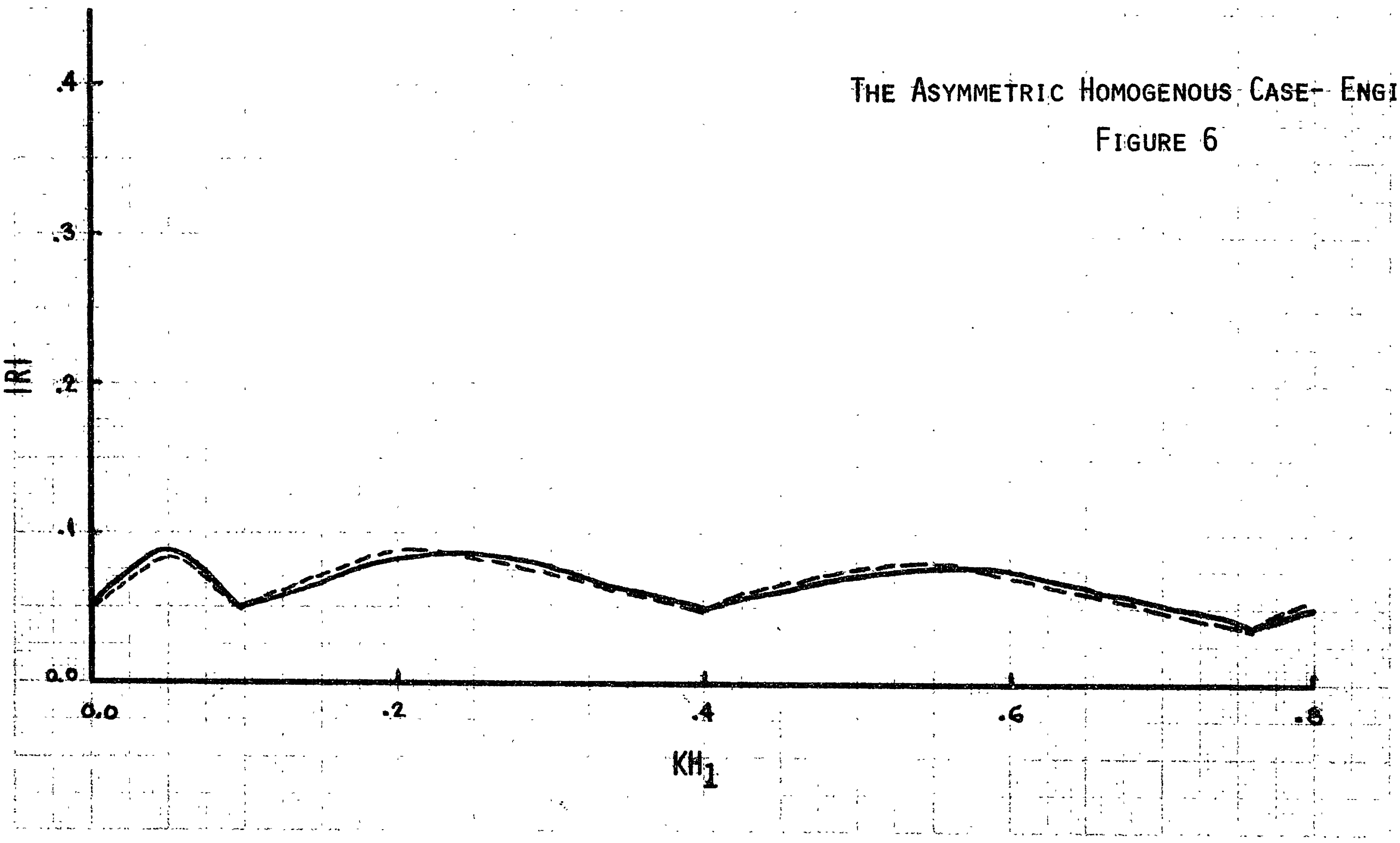
THE SYMMETRIC HOMOGENOUS CASE- ENGINEER

FIGURE 4



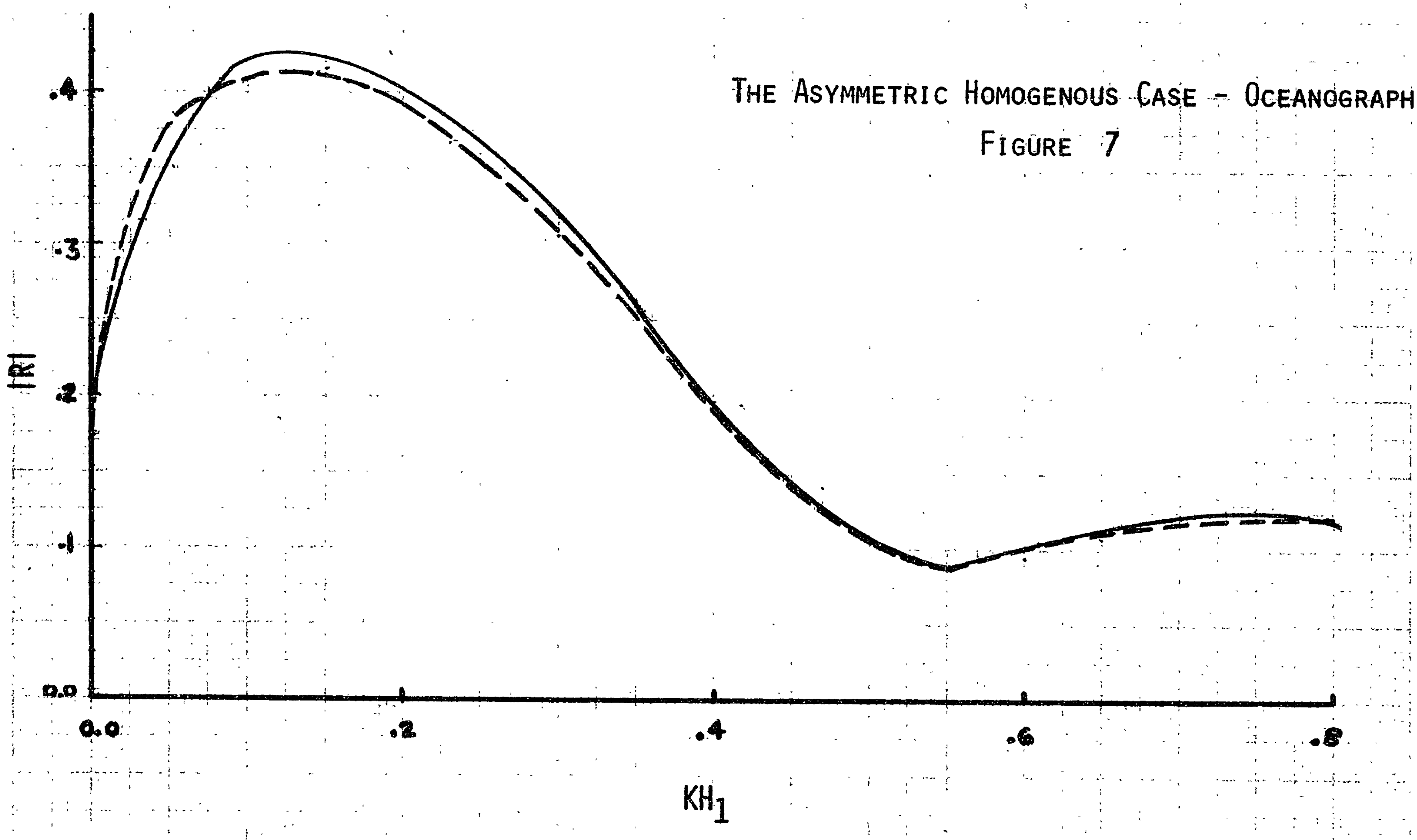
THE SYMMETRIC HOMOGENOUS CASE - OCEANOGRAPHER

FIGURE 5



THE ASYMMETRIC HOMOGENOUS CASE- ENGINEER

FIGURE 6

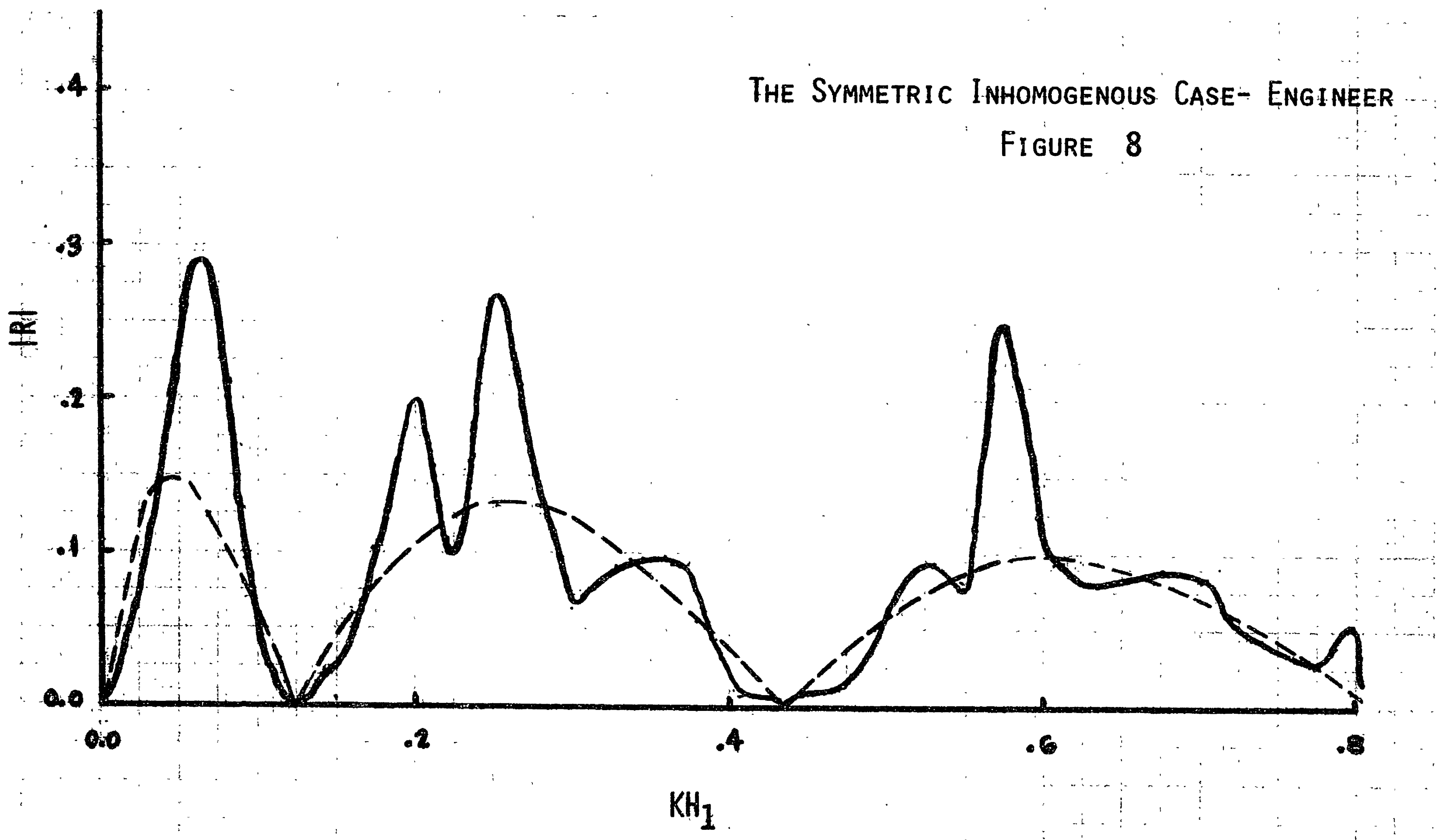


THE ASYMMETRIC HOMOGENOUS CASE - OCEANOGRAPHER

FIGURE 7

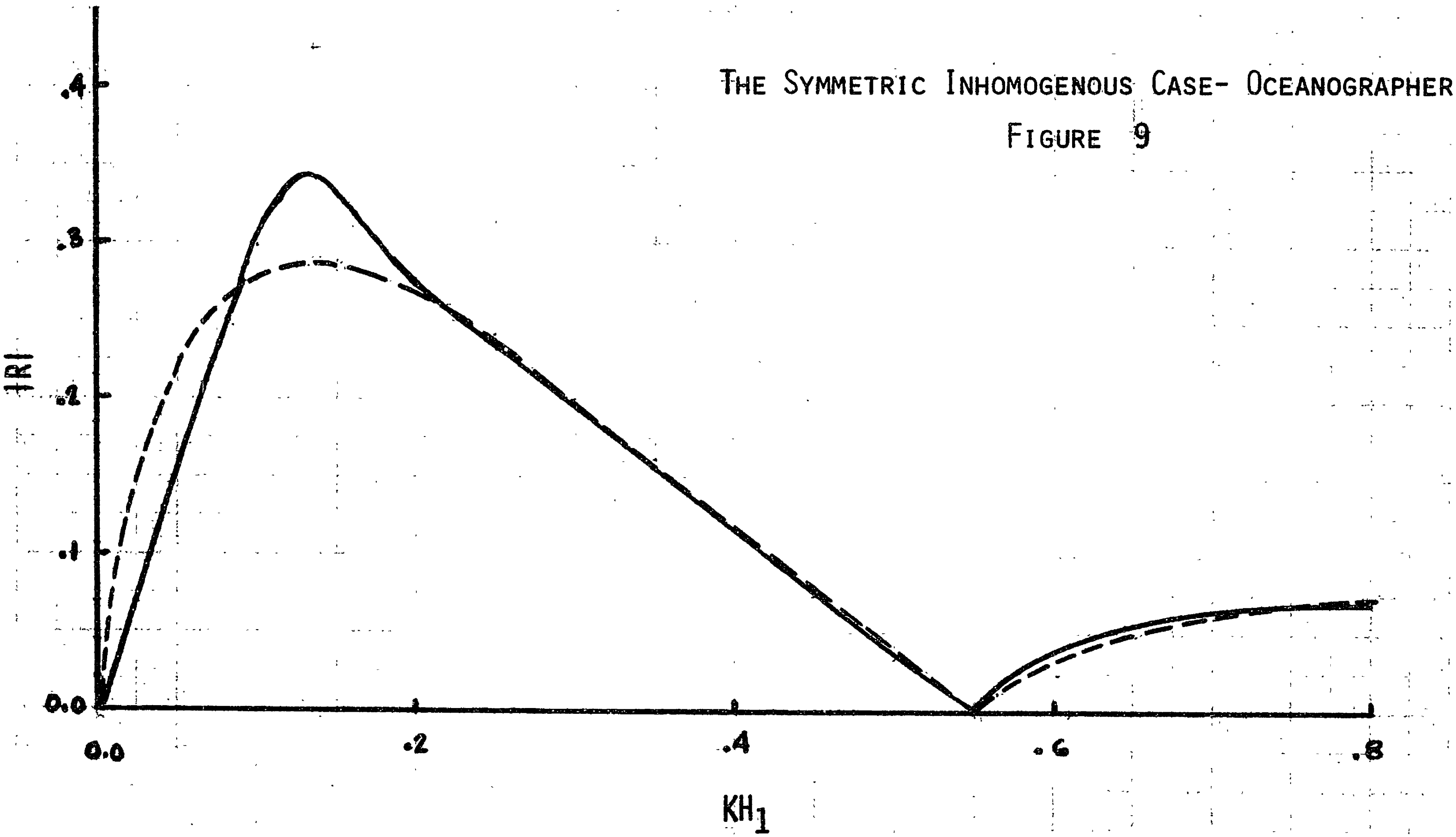
of a finite step shelf. At low frequencies, $K\ell$ small, the step shelf dominates. At high frequencies, $K\ell$ large, the symmetric obstacle dominates. This suggests that the solution to our linear boundary-value problem should behave approximately like the sum of the two problems taken separately--a symmetric obstacle and a geometrical contrast of the step shelf.

Rather than repeat a similar development for the variations of geometry for the inhomogeneous complication, we simply state that the same behavior is observed. We shall focus our attention on the symmetric, inhomogeneous problem. Figures 8 and 9 show the results for engineer and oceanographer respectively. Considering the first graph, we note that the sediment pocket is assumed to have a depth of thirty feet and a mean specific gravity of 1.3. Recalling the discussion of the physics implied in our choice of trial function, the "velocity" solution should differ only because of the influence of the stratification within the basin. The low frequency periodicity is due to the excitation of standing waves of frequency $p_0\ell=n\pi$. Thus, the behavior due to stratification is physically important only for low frequency excitation. We expect this since, at high frequencies, the incident potential will die off exponentially with depth. These standing waves will be highly oscillatory and provide a plausible forcing mechanism for pipeline migration. The important thing to remember is that this resonant behavior can occur at physically realistic values of $K\ell$. For this particular case, the incident wave length is on the order of 1200 feet. This wave length is not uncommon during



THE SYMMETRIC INHOMOGENOUS CASE- ENGINEER

FIGURE 8

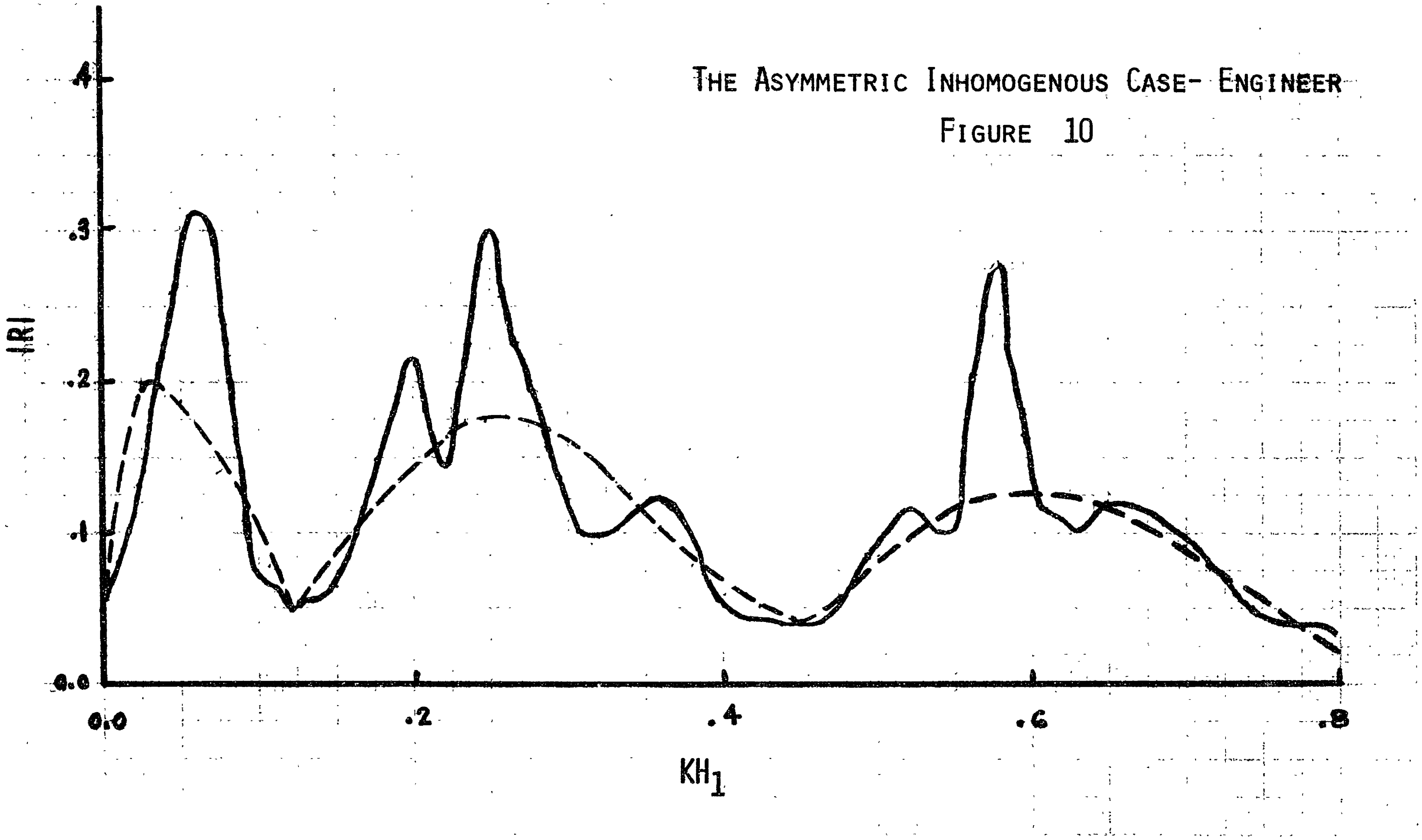


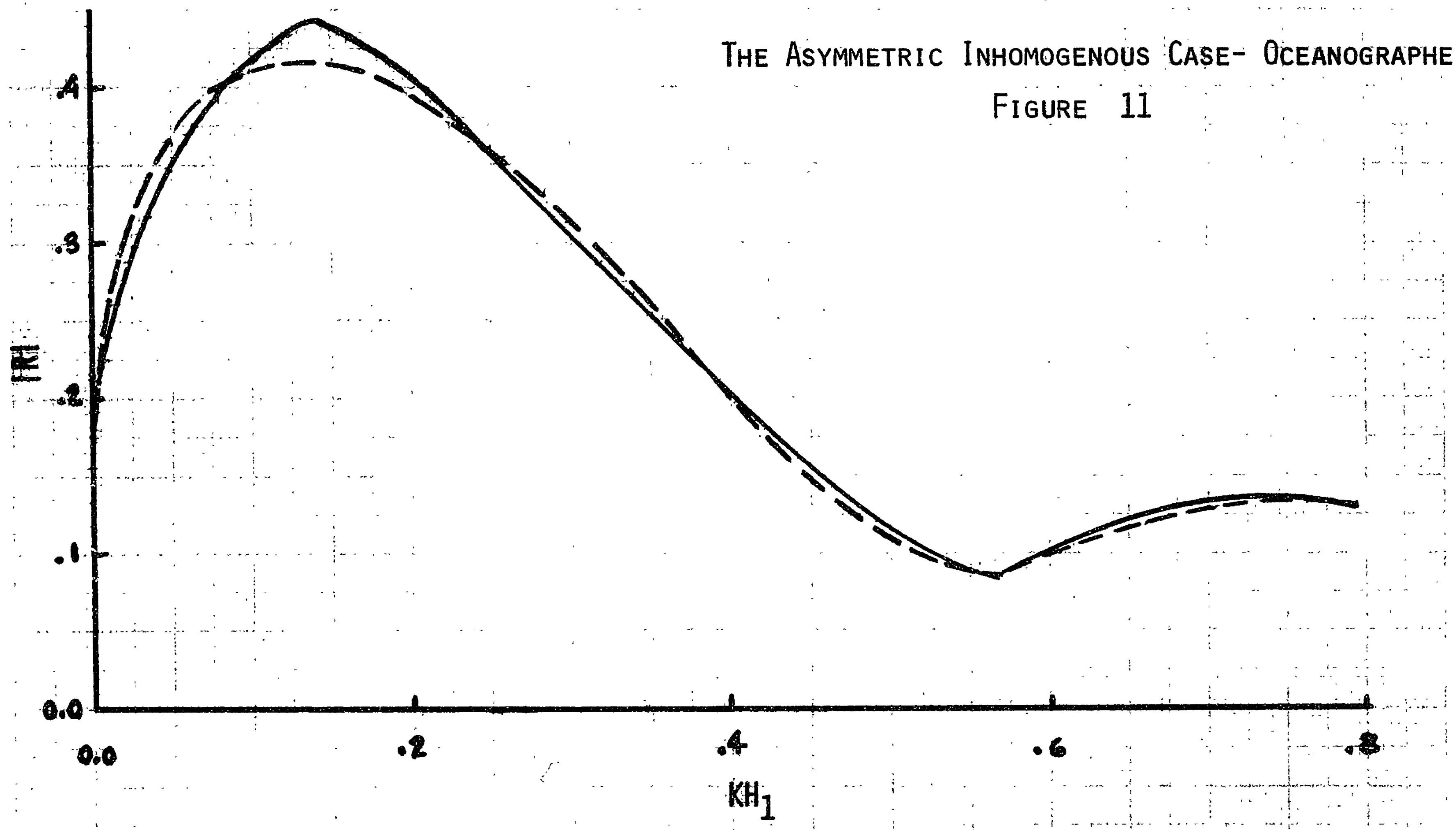
THE SYMMETRIC INHOMOGENOUS CASE- OCEANOGRAPHER

FIGURE 9

major storms. This confirms our hypothesis that pipeline migration is at least partially attributable to resonant behavior in sediment pockets due to storm wave excitation. For the oceanographer's problem, we consider a depth of 150 feet of dense water and a density contrast of 1.03. We readily see that there is virtually no effect on the propagating surface mode. This has been theorized by Phillips (1966) and proven experimentally by Wunsch (1969). It is possible to excite resonant behavior at extremely low frequencies on the tidal range. This has been theorized by Rattray (1960). Figures 10 and 11 show the asymmetric cases, but require no further discussion.

In concluding, we shall comment on the mathematical and on the practical aspects of this thesis. Dealing with the engineering applications first, we have established a plausible mechanism for initiating pipeline migration. We have noticed that sediment pockets can be significant reflectors. This could still be true in a more realistic geometrical model--even though we expect a smoothing due to viscosity and the continuous spectrum of eigenvalues which nature provides. This might be useful in optimizing offshore platform locations and, in a true extreme, developing a novel breakwater. This reflection characteristic could explain why shrimp fishermen in the Gulf of Mexico anchor over silty areas during unexpected storms. From the mathematical viewpoint, we remark that the complementary solution technique has led to an informative analysis--numerically, physically and analytically. We have shown under what conditions complementary variational integrals of Schwinger's


THE ASYMMETRIC INHOMOGENOUS CASE- ENGINEER
FIGURE 10
|R|
.4
.3
.2
.1
0.0
0.0
.2
.4
.6
.8
KH1



THE ASYMMETRIC INHOMOGENOUS CASE- OCEANOGRAPHER

FIGURE 11

type imply upper and lower bound approximate solutions. We have shown that the necessary and sufficient conditions for establishing a bounding argument are closely coupled to the proof of uniqueness. We have demonstrated that the physics of the problem must be satisfied in the choice of trial functions for variational arguments. Finally, we must emphasize that the close numerical agreement between our complementary formulations may be misleading. Without the bounding argument, experimental or numerical studies are necessary to definitively verify our solution.

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BIOGRAPHICAL NOTE

Joseph Baker Lassiter III was born in El Dorado, Arkansas, on October 29, 1947. He received the Bachelor of Science degree as recommended by the Department of Mechanical Engineering at M.I.T. and the Master of Science degree in Ocean Engineering at M.I.T. in September 1970. Subsequently, he was appointed as Instructor in the Department of Ocean Engineering while pursuing his doctoral studies. Mr. Lassiter has been employed by E. G. Frankel, Inc. (1970), Litton Industries (1968-1970), Descon Engineers (1967), Brown & Root, Inc. (1966), and Texas A & M University (1963-65). He is or has been a consultant to the Federal Power Commission, Pratt & Whitney, United Aircraft Corp., Descon Engineers, Woods Hole Oceanographic Institution, E. G. Frankel, Inc., and the National Academy for Engineering. His publications include "The Economics of Arctic Oil Transportation," Schiff und Hofen, 1970, with J. W. Devanney III, "Some Chemical Aspects Affecting the Utilization of the Red Sea Heavy Metal Deposits," Proceedings of the American Chemical Society, 1970, and The Evolution and Utilization of Marine Mineral Resources, M.I.T. Sea Grant Publication, 1972, with H. S. Lahman. Mr. Lassiter is a member of Sigma Xi and the American Society for Oceanography. He serves on technical panels of the Society of Naval Architects and Marine Engineers and of the Marine Technology Society. He is married to the former Patricia Allison Mayfield and has no children.