On the Theory of
COMBINATORIAL INDEPENDENCE

by

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A.B., University of Michigan
1954

Submitted in Partial Fulfillment
of the Requirements for the
Degree of Doctor of
Philosophy
at the
Massachusetts Institute of
Technology
June, 1964

Signature of Author Signature redacted
Department of Mathematics, 1 May 1964

Certified by Signature redacted
Thesis Supervisor

Accepted by Signature redacted
Chairman, Departmental Committee on
Graduate Students

ABSTRACT

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A theory of combinatorial independence on finite modular lattices is based upon the notion of a differential. This theory is shown to be strong enough to include the classical theory of abstract linear dependence of H. Whitney, as well as the theory of geometric lattices of Birkhoff and Dilworth. Several applications are made.

The main result, obtained as an application of the theory, is extension to arbitrary Whitney independence systems of certain polynomials first defined by W.T. Tutte for the special case of linear graphs. A simplified and order-independent computation of these polynomials is provided, and their characteristic algebraic properties are determined. The chromatic polynomials of G.D. Birkhoff, as well as the classical zeta and Möbius functions of the independence system and of its dual system arise from this polynomial by simple substitutions of variables. The existence of at least 2^n non-isomorphic independence systems on the Boolean algebra of an n-element set is established.

Thesis supervisor: Gian-Carlo Rota

Title: Associate Professor of Mathematics

ON THE THEORY OF COMBINATORIAL INDEPENDENCE

DEDICATION

This work is dedicated to

John Fitzgerald Kennedy.

He leaves us hope.

ACKNOWLEDGEMENTS

The completion of any mathematical work is a fruition engendered by the interaction of many minds, and made possible by a multitude of generous and yielding natures. Footnotes mark my first order contacts with mathematicians via their works, as published in the literature. Second order contacts, through personal acquaintance and a meeting of minds, cannot properly be footnoted.

The teaching of Robert L. Young at the Taft School opened to me the first door of mathematics. This experience was of sufficient depth to remain strong for seven years, during which no mathematical nurture was taken from other sources.

Professor Kenneth Hoffman of the Massachusetts Institute of Technology was first to lead me to expand my experience of mathematics. The personal interest he took in introducing me to mathematics, the tutoring during one long hot Cambridge summer, remained my mainstay throughout my time at the Institute. These

beginnings made it possible to benefit from the teachings of Professors Kenkichi Iwasawa, Willard Ambrose, Norman Levinson, Arthur Mattuck, Franklin Peterson, Reese Prosser, and Professor Emeritus Dirk Struik.

Meanwhile, I was constructing, through my association with Professor Harold Reiche of M.I.T. and Professors Eric Schroeder and Archibald MacLeish of Harvard, the philosophical foundations for a life and work, without which any dedication to mathematics would have been questionable. If there is any unexplored subject matter proper to mathematics, it is the more recondite unities of which these men speak.

Professor Gian-Carlo Rota has seen me through the difficult and fulfilling years devoted to finding a field of inquiry, a question, and a formulation of the answers. A multitude of suggestions arose from his fertile imagination and from his wide-spread and harmonious relations with the mathematical community. His teaching of an undergraduate course in combinatorial analysis, the product of a year's research into what was to him a new field, brought to me the exciting prospect of working in this field. His development of an incidence algebra for partially ordered systems, and

his detailed analysis of Möbius functions now make possible fresh approaches to the tantalizing unsolved problems of elementary combinatorial analysis. The research presented in this thesis could not have begun, had it not been for Professor Rota's introduction of me to the writings of Whitney, Birkhoff, MacLane, Dilworth, and Tutte, and his insistence that the proper study of independence lies in the consideration of the incidence algebra of those lattices induced by closure relations.

I had the good fortune to attend the lectures on graph theory given last summer by Professor W.T. Tutte of Waterloo, at the Canadian Mathematical Congress, Saskatoon, Saskatchewan. Professor Tutte's theory of the "dichromate", together with my desire for an order-independent definition of this polynomial, led to the theorems recounted in chapter four of this paper.

The companionship of my dear wife Elizabeth makes possible the completion of tasks which once could only have been begun. Her interest in this research was unflagging. She worked out examples on all levels of the theory of differentials, both as raw material for extensions of the theory, and as flesh for those results already achieved. Our conversations, as the work proceeded, have lent to the writing a degree of continuity that would otherwise have been lacking. The typescript is a product of her labors.

Professor Harold L. Stubbs, Chairman of the Department of Mathematics at Northeastern University, arranged a leave of absence for me during the first half of this academic year, and assigned a reduced teaching schedule for one term of the present semester, that there might be adequate time at my disposal to complete this work.

To those individuals mentioned above, and to others who have taken an interest in this research, I wish to express my deep feeling of gratitude. To my contemporaries at the Institute, Robert Mosher, David Park, Robert Briney, Dairmuid O'Mathuna, Stuart Newburger, Nicholas Metas and Norton Starr, I give my thanks for filling my days with friendship, and for setting a pace on the road to new mathematics.

INTRODUCTION

One of the most important and characteristic developments in combinatorial analysis in recent years is the renewed attention being given to the generalization of the theory of graphs to a more abstract structure, a structure which has found disparate applications. The original idea goes back to a beautiful paper[1] of Hassler Whitney in 1935. Unfortunately, this paper received little attention during its first twenty years in print, ourside of a small group of mathematicians working mainly in lattice theory.

Whitney's paper set forth equivalent axiomatizations for independence systems on a finite set, phrased alternatively in terms of rank, independence, bases, and circuits. We call these general structures Whitney systems, though they have been variously called matroids, frames, systems of linear independence, closures with the exchange property, exchange structures, matroid lattices, Birkhoff lattices, and geometric lattices.

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[1]H. Whitney, "The Abstract Properties of Linear Dependence", Amer. J., 57 (1935), pp. 507-533.

During the twenty years following the publication of Whitney's paper, two schools of thought developed concerning independence systems. A first direction was established by Garrett Birkhoff, in a paper published that same year.[1] He observed that the closed subspaces of a Whitney system form a lattice. He exhibited the characteristic covering and rank properties of these lattices, and introduced the terms semimodular and geometric to describe these new lattice structures. Saunders MacLane[2] worked out the notions of basis and dimension in the latticial setting, and compared a variety of possible statements of the Steinitz exchange property. R.P. Dilworth[3] further pursued this research, emphasizing the correspondence (which fails to be 1-1) between Whitney systems and semimodular lattices.

The second school of thought sought for extensions of Whitney's theory of independence to infinite sets. A typical paper of this school, with an adequate bibliography to earlier papers on infinite extension, is that

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[1] G. Birkhoff, "Abstract Linear Dependence and Lattices", Amer. J., 57 (1935), pp. 800-804.

[2] S. MacLane, "A Lattice Formulation for Transcendence Degrees and p-Bases", Duke J., 4 (1938) pp. 455-468.

[3] R.P. Dilworth, "Dependence Relations in a Semimodular Lattice", Duke J., 11 (1944), pp. 575-587.

of Jürgen Schmidt[1]. His description of the exchange properties of derived sets reinforced our decision to employ the term differential for the type of function to be defined in Chapter I. R. Rado[2] completed an extensive analysis of such infinite systems.

Only in the last decade has renewed interest attached to Whitney's pioneering work. Though Whitney systems are the natural domain for most theorems of graph theory, their abstract quality has deterred many investigators and even more readers. There is still reason to complain of the relative unpopularity of the general theory, even among avid readers of graph theory.

Advances in the past decade have been the work of one mathematician, W.T. Tutte, of Waterloo. With an assist from the chain-group techniques of algebraic topology, he has succeeded in showing the precise manner in which the theory of graphs and of planar graphs is embedded in the general theory of independence. In a

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[1] J. Schmidt, "Mehrstufige Austauschstrukturen", Zeitsch. f. math. Logik und Grundlagen d. Math., bd 2 (1956).

[2] R. Rado, "Axiomatic Treatment of Rank in Infinite Sets", Can. J., 1 (1949), pp. 337-343.

remarkable series of papers[1] he makes contact with the theory of Kuratowski subgraphs, and provides purely combinatorial analogues of Kuratowski's results.

A substantial increase in interest in Whitney systems manifests itself at the present time. George Minty[2], at the University of Michigan, is generalizing the Bott-Duffin circuit theory to arbitrary Whitney systems, with an eye toward work with error-correcting codes. Alfred Lehman and Jack Edmonds at the Bureau of Standards are actively working on Whitney systems, and applications to circuit and switching theory. Gian-Carlo Rota[3] is continuing his researches into the nature of Möbius functions of partially ordered systems, researches which have already produced a proof that the Möbius function of a geometric lattice alternates in sign on the levels of the lattice, and have produced fundamental relations concerning Galois connections between partially ordered systems. Such activity portends the fruitful development of independence

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[1]W.T. Tutte, "A Class of Abelian Groups", Can J. 8 (1956), pp. 13-28; "A Homotopy Theorem for Matroids, I and II", Trans. A.M.S. 88 (1958) pp. 144-174; "Matroids and Graphs", Trans. A.M.S. 90 (1959), pp. 527-552.

[2]G.J. Minty, "Monotone Networks", Proc. Roy. Soc. London, Ser. A 257 (1960), pp. 194-212.

[3]G.-C. Rota, "On the Foundations of Combinatorial Theory", Zeitschr. fur Wahrshein., 2 (1964), pp. 340-360.

theory in the near future.

The contribution we make to this theory is two-fold: we construct a generalization of, and a natural setting for, Whitney systems, and we produce several new results concerning chromatic polynomials.

It is our aim in writing this paper to provide an appropriate lattice-axiomatic foundation for a theory of independence. It proves expedient to represent independence systems as certain functions on a modular lattice, rather than as lattices themselves. We are able in this way to define a unique dual structure in purely latticial terms, a task which is impossible to complete within the framework of Birkhoff, MacLane and Dilworth.

We show that independence systems are determined by the finite-difference analogues of exact differentials defined on modular lattices. Our development centers on a duplication of the real variable theorem concerning independence of path for integrals of differentials satisfying a local exactness condition. Hassler Whitney's program for proof of the usual theorems on linear independence and of theorems on graphs is carried out for exact differentials on complemented modular lattices.

After a consideration of the Tutte polynomial[1] it was our feeling that there should be some simultaneous computation of the Mobius functions and chromatic polynomials of both a Whitney system and its dual via a single enumerative process applied to all subsets of the underlying set. We determine such an enumeration in the final chapter. The two-variable generating function thus defined provides chromatic polynomials for Whitney systems and their duals, and is shown to reduce by simple substitution of variables to the Tutte polynomial, which has heretofore been defined only for graphs.

The conceptual simplicity of this new formulation makes possible certain advances in the enumeration of exact differentials on a Boolean algebra. We prove the existence of at least 2^n non-isomorphic exact differentials on the Boolean algebra of all subsets of any n-element set. The corresponding problem for graphs is one of the intriguing unsolved problems of elementary combinatorics. Our theory, together with Tutte's embedding of the theory of graphs in the general scheme may be the direct approach to this problem.

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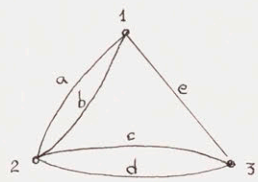
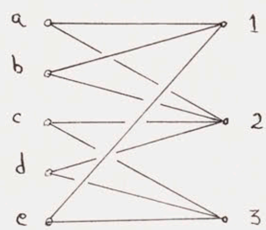
[1] W.T. Tutte, "A Contribution to the Theory of Chromatic Polynomials", Can. J., 6 (1954) pp. 80-91.

It is our hope that the reader will find established a useful conceptual basis for further research in the abstract theory of combinatorial independence. Before turning to the theory proper, and with the purpose of further orienting the reader to our theory of differentials let us consider the interaction of the various theories of graphs due to G.D. Birkhoff, Whitney, Tutte, and those newly introduced in this paper, as they are all applied to a single illustrative example.

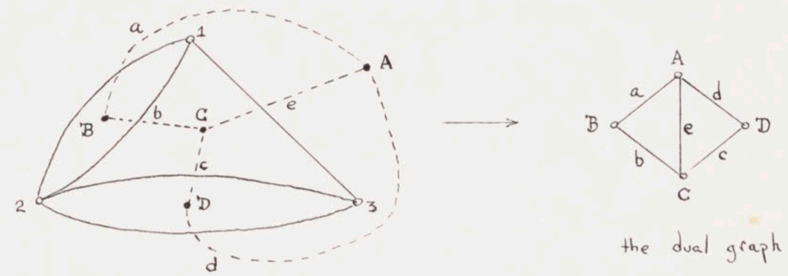
Consider the linear Graph G:



formed of 5 edges and 3 vertices, dividing the plane into 4 regions. This graph arises from the edge-vertex incidence relation



The dual graph G^* may be constructed by placing a vertex in each of the four regions of the plane, and then drawing an edge across every edge of the graph G, connecting the new vertices in regions separated by edges of G. The letter designation of corresponding edges is preserved.

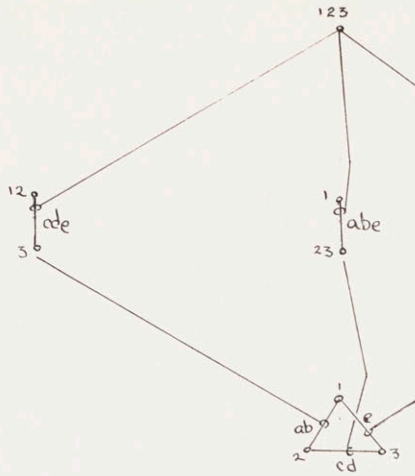


A coloring of a graph is an assignment of colors to vertices such that no two adjacent vertices have the same color. The chromatic polynomial of the graph G is just that of a triangle, as the multiple edges act just as single edges relative to admitting colorings. Given λ colors, use any of the λ colors to color vertex 1, and of the λ-1 different colors to color the adjacent vertex 2, and any of the λ-2 colors different from both of these to color vertex 3.

$$\lambda(\lambda-1)(\lambda-2) = \lambda^3 - 3\lambda^2 + 2\lambda$$

We may color the dual graph in two different ways: vertices B and D may be the same color or different. With B and D the same color, there are $\lambda(\lambda-1)(\lambda-2)$ colorings in λ colors. With B and D different colors, there are $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$ colorings in λ colors. Adding: $\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) = \lambda(\lambda-1)(\lambda-2)^2$ $= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda$.

These polynomials may be obtained by lattice-theoretic means. We construct for the graph G, the lattice of contractions of G along subsets of its edges, as is shown in the lattice diagrams on the next page. Loops are neglected; multiple edges are drawn single for convenience.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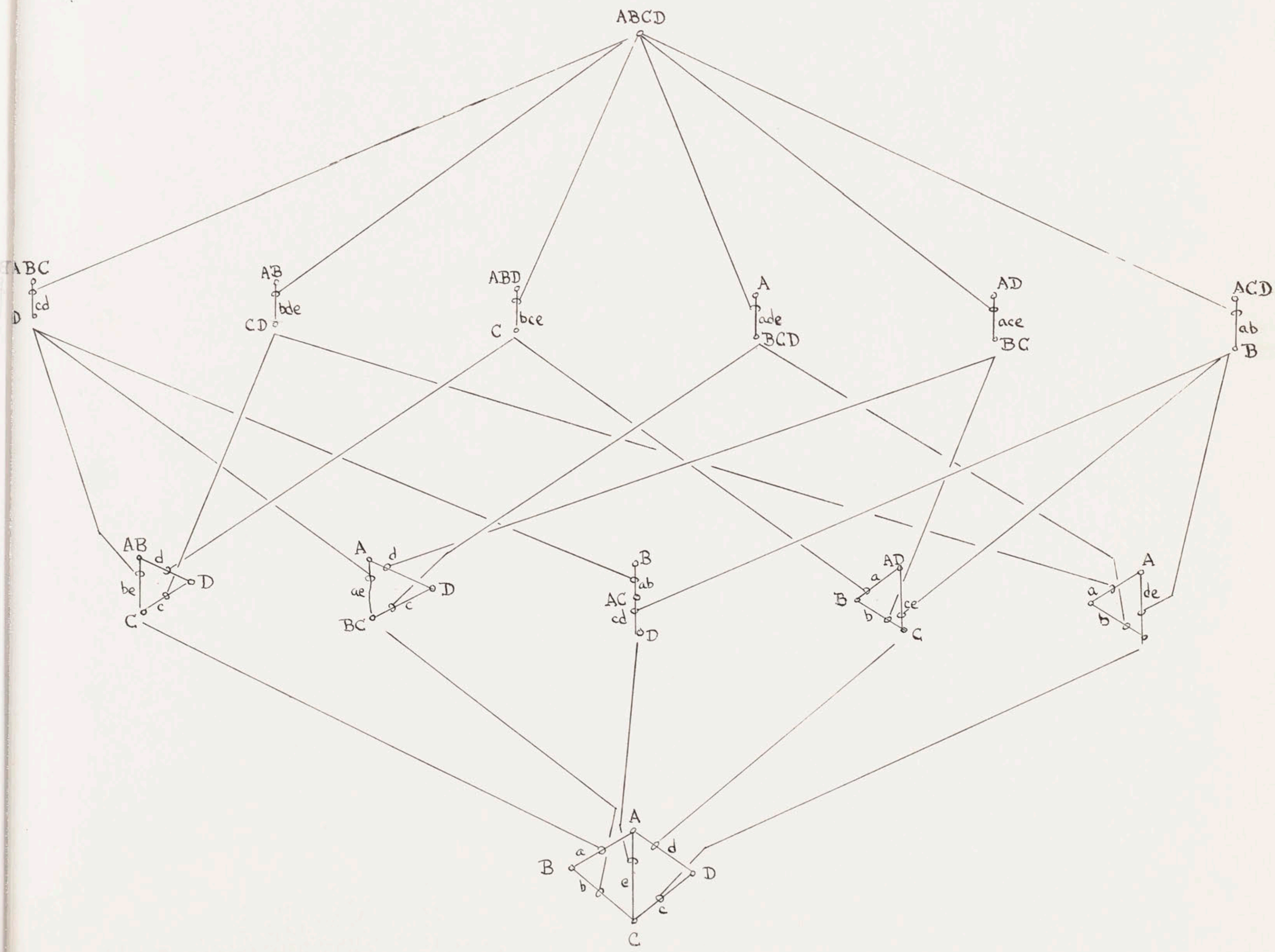
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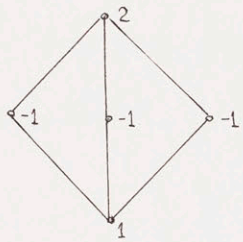
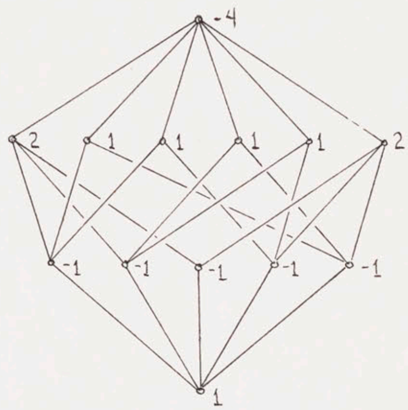
To compute the chromatic polynomials from these lattices, first calculate the Möbius function values $\mu(0,x)$ for each element x of the lattices. Recursively defined, $\mu(0,x)$ has value 1 at zero, $\mu(0,0) = 1$, and the sum over any lattice interval $[0,x]$ is zero. The values are indicated on the lattices as





The sums of the values of the Möbius function on the various lattice levels are the coefficients of the chromatic polynomials of the graph and of the dual graph.

W.T. Tutte obtained these polynomials via an arbitrary ordering of the edges (eg: alphabetically, as we have written them) and the following considerations concerning bases for the graph. A base (or basis) for the graph G is a connected set of edges passing through every vertex, but containing no circuits. If an edge e is added to a base B, the set B + e contains a unique circuit. If the edge e is the highest edge in this circuit, e is said to be externally active with respect to the base B. The complement of the Base B is a base for the dual graph. If an edge e of the base B is deleted from B, and added to the complement, a circuit is formed inthe dual graph G*. If the edge e is the highest edge in this circuit, e is said to be internally active with respect to the base B. Tutte counted the internally active and externally active edges for each base:

Bases	Internally Active	Externally Active
ac		bde
ad		be
ae	e	b
bc		de
bd		e
be	e	
ce	e	d
de	de	

Tabulating the number of bases with activities (i,j), we obtain the array:

$$\begin{array}{|lll} 1 & & \\ 2 & & \\ 1 & 2 & \\ \cdot & 1 & 1 \\ \hline \end{array}$$

This serves as the array of coefficients for what we call the Tutte polynomial:

$$\xi^2 + \xi + 2\xi\eta + \eta + 2\eta^2 + \eta^3$$

Substituting $(1-\lambda,0)$ for (ξ,η), we have $(1-\lambda)^2 + (1-\lambda) = \lambda^2 - 3\lambda + 2$, which, when multiplied by λ, is the chromatic polynomial of the graph G. Substituting $(0,1-\lambda)$ for (ξ,η), we have

$$(1-\lambda)^3 + 2(1-\lambda)^2 + (1-\lambda) = -[\lambda^3 - 5\lambda^2 + 8\lambda - 4];$$

which, when multiplied by $-\lambda$, is the chromatic polynomial of the dual graph.

Our theory of differentials offers an alternative construction of both the geometric lattices and of the Tutte polynomial. On the lattice of subsets of the five-element edge set (shown on the next page) we mark double or color red those covering lines ("steps") along which the rank of the corresponding edge subsets increase. The rank of a subset is the number of vertices which the edges collectively contain, less the number of connected components.

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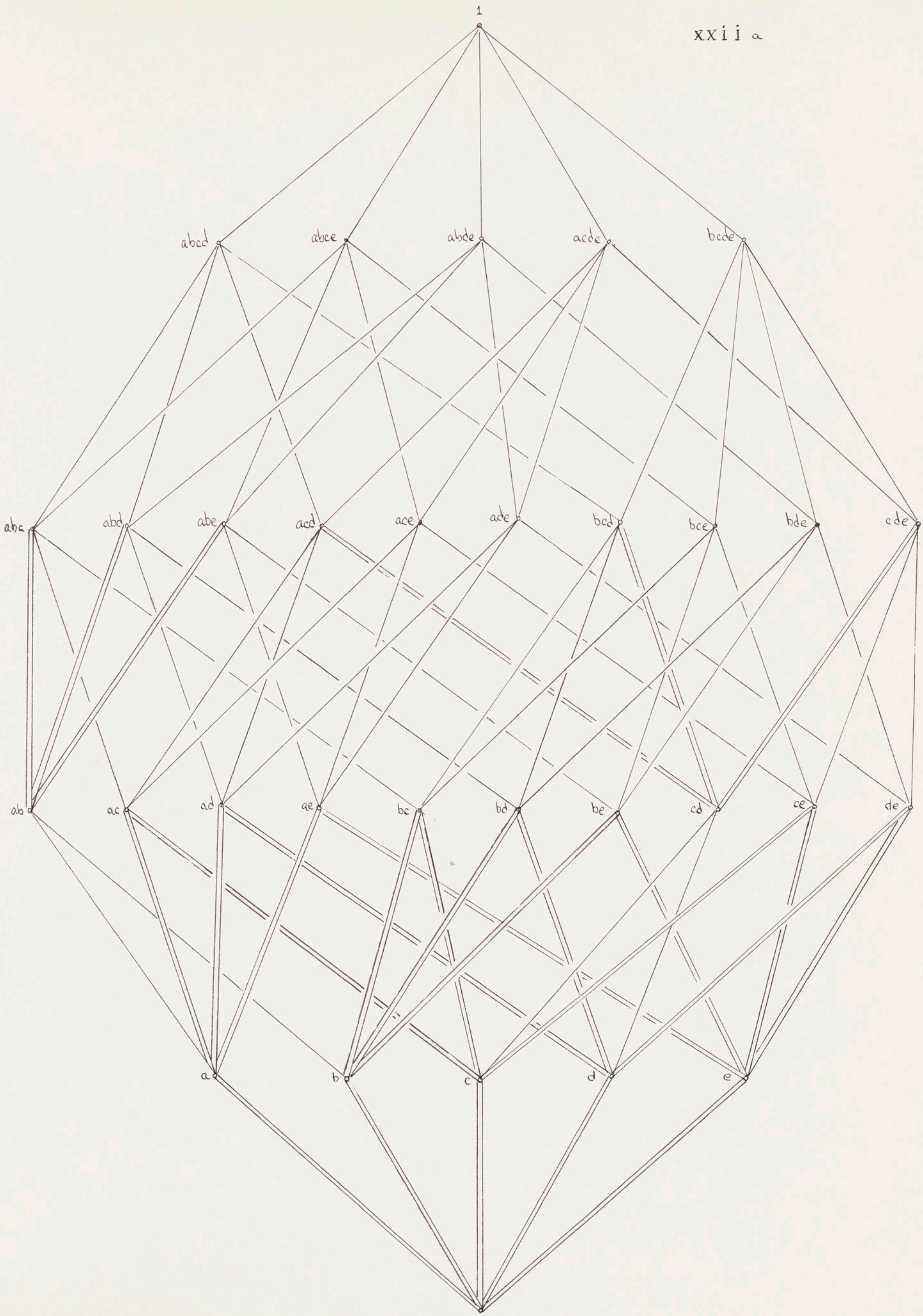
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This done, the lattice elements up from which all steps are double (red) constitute the elements of the geometric lattice of the graph G; the lattice elements down from which all steps are singly-drawn (black) constitute the inverted form of the geometric lattice of the dual graph G*. Indeed, inversion of the Boolean algebra, with interchange of double for single markings of steps, is the differential of the dual graph, ie: the dual differential.

Contractions or removal of edges of G form graphs, the differentials of which are merely the restrictions of the differential of G to various lattice subintervals. Restriction to the lattice interval [e,1] corresponds to contraction of the edge e; restriction to the interval [0,1-e] corresponds to elimination of the edge e.

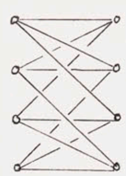
The number of doubly-drawn (red) edges in a path through the Boolean algebra is dependent only upon the end points of the path. If we enumerate all edge subsets with respect to the double grading (the number of red steps from x to 1, then the number of black steps from 0 to x), the resulting array serves as coefficient matrix for a function of two variables.

$$\begin{array}{|lll}1 & & \\ 5 & & \\ 10 & 2 & \\ 8 & 5 & 1 \\ \hline\end{array} \qquad \xi^2 + 5\xi + 8 + \eta(2\xi + 10) + 5\eta^2 + \eta^3$$

This polynomial we refer to as the rank generating function. Substitution of $(\xi-1,\eta-1)$ for (ξ,η) produces the Tutte polynomial:

$$(\xi-1)^2 + 5(\xi-1) + 8 + (\eta-1)(2\xi-2+10) + 5(\eta-1)^2 + (\eta-1)^3$$
$$= \xi^2 - 2\xi + 1 + 5\xi - 5 + 8 + 2\xi\eta - 2\xi + 8\eta - 8 + 5\eta^2$$
$$- 10\eta + 5 + \eta^3 - 3\eta^2 + 3\eta - 1$$
$$= \xi^2 + \xi + 2\xi\eta + \eta + 2\eta^2 + \eta^3.$$

These computations, which could previously be carried out only for planar graphs, can now be performed for any Whitney system. There are differentials dual to those for non-planar graphs; there are differentials, such as that of the incidence relation



for which neither the differential nor its dual are given by graphs.

TABLE OF CONTENTS

Chapter I

§1 Introduction

A variety of terms from the theory of partially ordered systems in general and of lattices[1] in particular will be employed in this paper. The definitions of these terms are collected in section two, together with the statements of important properties of modular lattices, and of the fundamental theorem of Galois connections.

The concept of a differential on a modular lattice is defined in section three. The defining properties are then seen to restrict the local behavior of differentials to one of five basic configurations. An equivalent characterization of differentials is given in terms of these local graphs, in that they permit a unique extension to a differential. The further elimination of the least symmetric of the allowable local graphs provides the concept of exactness of differentials, and marks off the principle objects under investigation

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[1]for more detail, see G. Birkhoff, Lattice Theory, (A.M.S. Colloquium Pub., 1948).

in this paper. Exact differentials are a natural generalization of Whitney's independence systems.

Section three concludes with a discussion of fixed points of differentials. We establish the existence of non-trivial fixed points for any non-zero differential. We investigate an equivalent characterization of differentials in terms of fixed points.

In section four we discuss a variety of different algebraic operations which may be performed on differentials. We produce new differentials from old by any one of the following operations: restriction of the domain lattice to a proper subinterval, taking supremum over a set of differentials, inversion of the domain lattice, a duality operator defined in terms of fixed points, and multiplication of lattices. We prove two important inequalities relating the values of a differential to those of its dual, and prove that the product of exact differentials is exact. The concept of a prime differential arises naturally from a consideration of products.

§2 Lattice Preliminaries

A <u>partial ordering</u> ($\leq$) is a reflexive anticommutative and transitive relation. A set P is a <u>partially</u>

ordered system if a partial ordering is defined thereon. A partially ordered system L is a lattice if, for every pair (x,y) of elements of L there is a unique element $x \wedge y$ of L such that $z \leq x \wedge y \leftrightarrow z \leq x$ and $z \leq y$, and a unique element $x \vee y$ of L such that $x \vee y \leq z \leftrightarrow x \leq z$ and $y \leq z$. These elements are called the inf and sup of x and y, respectively. A least element of L is called 0, and a greatest element, 1.

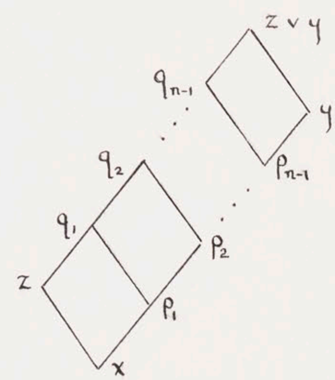
An element y is said to cover an element x if $x < y$, and $x < z \leq y \rightarrow z = y$. A pair (x,y) in which y covers x is termed a step. An increasing sequence of elements $x = p_0 < p_1 < \ldots < p_n = y$ is said to be a path from x to y if p_j covers p_{j-1} for $j = 1, \ldots, n$. If x covers 0, x is an atom; if it is covered by 1, x is a coatom. The notations of closed $[x,y]$ and open (x,y) intervals of L are used.

Certain categories of lattices figure prominently in the theory of independence. A lattice is finite if every path has finite length. A complete lattice (there is a unique supremum for every lattice subset, and, consequently, a unique infimum.) is essential to the definition of differential. A modular lattice ($\forall x,y \in L$, $x \vee y$ covers $x \leftrightarrow y$ covers $x \wedge y$) possesses just enough symmetry to permit the establishment of a duality relation on differentials. It is the finite, complete, modular lattice which serves as substrate for our theory, in contrast to the customary use of the Boolean algebra, or lattice of subsets of a finite set, with ordering given by inclusion.

The exact differentials will distinguish within the modular lattice L a pair of subsets which form semimodular lattices $((\forall x,y \in L)\ x$ covers $x \wedge y \rightarrow x \vee y$ covers $y)$. If for every $x \leq y \leq z$ in L there is an element y' such that $y \vee y' = z$ and $y \wedge y' = x$, and, in particular, if L is a Boolean algebra, the distinguished subsets are geometric lattices (y covers $x \leftrightarrow$ there is an atom e of L such that $x < x \vee e = y$).

The notation $\tilde{L}$ will denote the inverse lattice formed from L by inverting the partial ordering and interchanging the roles of $\wedge$ and $\vee$.

We shall make frequent use of the characteristic property of modular lattices, that paths may be "projected" up or down without variation in length, ie: if p: $x = p_0 < p_1 < \ldots < p_n = y$ is a path from x to y, and if z covers x with $z \nleq y$, then $q_i = z \vee p_i$ defines a path



q: $z = q_0 < \ldots < q_n = z \vee y$ of length equal to that of p. If, on the other hand, p is as above, and y covers z with $x \nleq z$, $q_i = z \wedge p_i$ defines a path q: $z \wedge x = q_0 < \ldots < q_n = z$ of length equal to that of p.

Finally, use of the basic theorem[1] on Galois connections will shorten several of our proofs. A Galois connection is a pair of anti-isotone functions σ and τ, σ mapping a partially ordered system P into a partially ordered system Q, and τ mapping Q into P, such that τ composed with σ is isotone on P, and σ composed with τ is isotone on Q. The basic theorem states that τ composed with σ is a closure operator on the partially ordered system P, as is σ composed with τ on the system Q. Furthermore, the elements of P or of Q which are closed with respect to these operators are exactly those elements in the ranges of τ and σ, respectively. The partially ordered subsystems of closed elements of P and of Q are isomorphic.

§3 Differentials

Throughout this section, L will denote a complete and finite modular lattice. A function R defined for

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[1] Oystein Ore, <u>Theory of Graphs</u>, (1962) AMS, pp. 183-5.

all pairs x,y of elements of L is a <u>differential</u> if and only if

a) R takes values 0 or 1, and $R(x,y) = 0$ unless $x < y$.

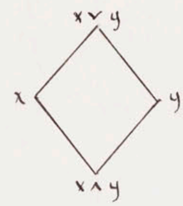
b) R is monotone with respect to intervals, ie: $w \leq x \leq y \leq z \rightarrow R(x,y) \leq R(w,z)$.

c) R is subadditive, ie: $x \leq y \leq z \rightarrow R(x,z) \leq R(x,y) + R(y,z)$.

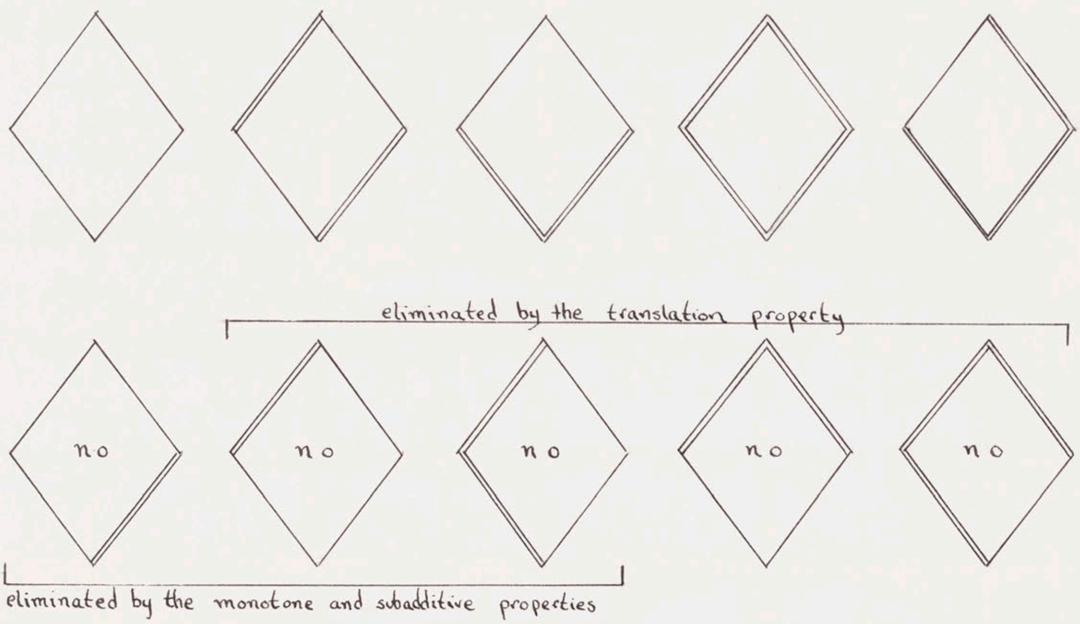
d) Translation property: If $x \vee y$ covers x, $R(x,x \vee y) \leq R(x \wedge y,y)$.

The translation property is the foundation of our theory. It expresses the monotone nature of a differential with respect to upward parallel translation of intervals of length 1.

These conditions can be expressed pictorially if we consider what they require concerning the values of R on pairs drawn from any 4 element subset $\{x,y,x \wedge y,x \vee y\}$ where x and y cover $x \wedge y$ (and consequently $x \vee y$ covers x and y).

As a standard procedure, to be used in all examples, let us color red those steps (ie: lattice intervals of length one) on which R has value 1, leaving black those on which $R = 0$. Of the different possibilities for coloring the edges of the figure 

, only five are allowable in a differential.

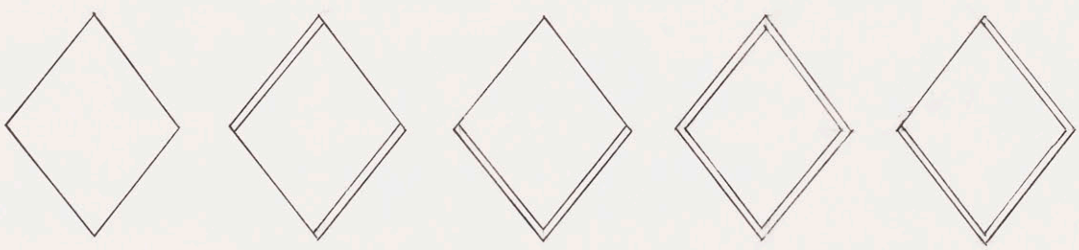


If R has value 1 on any one of the four steps, the monotone property implies $R(x \wedge y, x \vee y) = 1$. In this event, the subadditive property implies there must be at least one step in each of the two paths from $x \wedge y$ to $x \vee y$ on which R has value 1. As this is not the case in the sixth, seventh, and eighth figures, these may not occur in a differential.

On the other hand, the last four figures involve contradictions of the translation property. This leaves the first five figures as the only possibilities for configurations within a differential.

These are the basic building blocks of differentials; within the calculus of finite differences they correspond to the values of second partial derivatives in function theory. Since we shall show that differentials are completely determined by their local characteristics, we shall introduce the term <u>local graph</u> for any such configuration, and shall denote the allowable local graphs by special names, the significance of which will become clear by the end of this chapter.

Allowable local graphs:



zero mixed prime one inexact

Each type of local graph has a characteristic effect on the global properties of the differential.

The zero local graphs appear in the higher lattice regions, the one-type local graphs in the lower regions, by virtue of the translation property. Mixed local graphs predominate in factorable[1] differentials; prime local graphs indicate the existence of non-trivial factors, and are associated with the phenomenon of exchange[2], studied by Steinitz and MacLane.

Note that local graphs which are zero, mixed, prime, or one possess a degree of symmetry lacking in the inexact case. It might be expected that differentials, none of whose local graphs are inexact, have global symmetry properties which reflect their symmetric local character. This is indeed the case.

Definition: A differential R on a finite complete modular lattice L is exact if and only if no local graph of R is inexact.

The properties of exact differentials are set forth in Chapter 3. Examples of exact and inexact differentials are given in Appendix A.

We shall complete this section by proving three propositions which serve to exhibit the essentially

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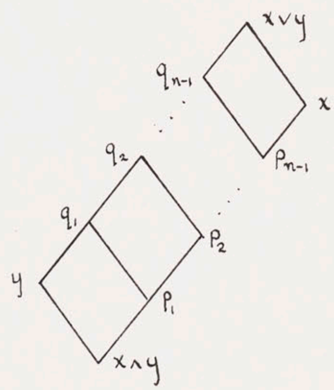
[1]vide §4e, p.22, for products of differentials.

[2]vide Chapter two, §3, p.41.

local character of differentials, and which provide an equivalent characterization of differentials in terms of sets of fixed points on a lattice.

Proposition. Local Character of Differentials: The condition that all local graphs be allowable suffices to characterize differentials. If R is a function defined on all steps of the lattice L, every local graph of which is allowable, R has a unique extension to a differential on L.

Proof: If R is a 0-1 function on the steps of L, satisfying the condition that it have only allowable local graphs, we may define $R(x,y) = 1$ if and only if $x < y$ and $R = 1$ on some step of some path from x to y. If this defines a differential, it must be the unique differential extending R, since all differentials are monotone. First we show R has the translation property. If $x \vee y$ covers x, and $R(x, x\vee y) = 1$, choose a path p: $x \wedge y = p_0 < p_1 < \ldots < p_n = x$, and let $q_i = p_i \vee y$. Since L is modular, q is a path from y to $x \vee y$, $y = q_0 < q_1 < \ldots < q_n = x \vee y$ and q_i covers p_i, $i = 1,\ldots,n$. For each i, $\{p_{i-1}, p_i, q_{i-1}, q_i\}$ compose



a local graph. Let i be the least index for which $R(p_i,q_i) = 1$. Then $R(p_{i-1},q_{i-1}) = 0$, and the local graph on $\{p_{i-1},p_i,q_{i-1},q_i\}$ is not allowable. Thus $1 = R(p_0,q_0) = R(x\wedge y,y)$. Now we prove R to be subadditive. Say $x < y < z$, and $R(x,y) = R(y,z) = 0$. If there is a step $[u,w]$ between x and z for which $R = 1$, we investigate the two projections $[u\vee y,w\vee y]$ and $[u\wedge y,w\wedge y]$. Since L is modular, exactly one of these is a step, the other being a single element. If $[u\wedge y,w\wedge y]$ is a step, $R(u\wedge y,w\wedge y) = 1$ by the translation property, contradicting $R(x,y) = 0$. If $[u\vee y,w\vee y]$ is a step, we choose a path p from u to $u\vee y$, and let $q_i = p_i\vee w$, forming a parallel path q from w to $w\vee y$. Since $R(y,z) = 0$, $R(u\vee y,w\vee u) = 0$. Let i be the least index such that $R(p_i,q_i) = 0$. $i \geq 1$ since $R(p_0,q_0) = R(u,w) = 1$. Then $R(p_{i-1},q_{i-1}) = 1$ and the local graph on $\{p_{i-1},q_{i-1},p_i,q_i\}$ must be prime or inexact. In either case, $R(p_{i-1},p_i) = 1$, so $R(p_{i-1}\wedge y,p_i\wedge y) = 1$ by the translation property, where $[p_{i-1}\wedge y,p_i\wedge y]$ is a step between x and y. Contradiction.

The following proposition resembles a fixed point theorem, and may be considered one if we regard a differential R as defining an upward flow along any paths on which R takes the constant value 0.

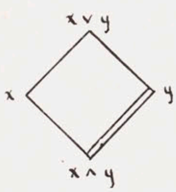
Proposition. Existence of Fixed Points: If a differential R on a finite complete modular lattice L is not everywhere zero, there exists an element z of L, $z \neq 1$, such that $R(z,w) = 1$ for all w covering z. The same is true for any interval [a,b] of L, if we substitute the conditions $z \neq b$, R not everywhere zero on [a,b], and $R(z,w) = 1$ for all $w < b$, w covering z.

Proof: Choose a path p from a to b which is maximal with respect to the number k of initial steps along which $R(p_{i-1},p_i) = 0$, $i = 1,\ldots,k$. If the path p has length n, and $k < n$, then $w \leq b$ and w covers p_k implies $R(p_k,w) = 1$ for otherwise the path p can be replaced above p_k by a path to b via w, which will have k+1 initial steps for which $R = 0$. If $k = n$, we have found a path from a to b along which $R = 0$. By a simple extension, by induction, of the subadditivity property, $R(a,b) = 0$, and R must be zero everywhere on the interval, by the monotone property.

Proposition. Characterization of Differentials in Terms of Sets of Fixed Points: There is an invertible correspondence between differentials R on a complete finite modular lattice L and sets K of lattice elements which are closed with respect to the lattice operation inf, ie: sets K such that $E \subset K \rightarrow \inf E \in K$.

Proof: Given such a set K, define $R(z,w) = 1$ whenever w covers z, and $z \in K$. Then define $R(x,y)$ for any pair x,y where y covers x by the translation property, $R(x,y) = 1$ if and only if there exists $z \in K$ such that $x \leq z$ but $y \vee z$ covers z. Assume $x \vee y$ covers x, and $R(x,x \vee y) = 1$. Then there exists $z \in K$ such that $x \leq z$ but $x \vee y \vee z$ covers z. $x \vee y \vee z = y \vee (x \vee z) = y \vee z$ and y covers $x \wedge y$ implies $R(x \wedge y,y) = 1$, so R, as a function now defined on all steps of L, has the translation property.

Refer now to the types of local graphs pictured on page 7. Since R, as so far defined, has the translation property, it will only fail to extend uniquely to a differential if there is some local graph resembling



. Taking x and y to be situated as in this figure, there must be some element $z \in K$ such that $x \wedge y \leq z$ but $y \vee z$ covers z. Say $x \leq z$. Then $(x \vee y) \vee z = y \vee z$ covers z, and $R(x,x \vee y) = 1$. If $x \not\leq z$, $x \vee z$ covers z, so $R(x \wedge y,x) = 1$. In neither event is the local graph the one we must exclude, so every local graph is allowable. By our previous result, R now extends uniquely to a differential.

With R thus defined for all pairs x,y of elements of L, we note that the set of elements z such that w covers z implies $R(z,w) = 1$ is exactly the set K with which we started. If any element z has this property, and z is not in the set K, then for each element w covering z we can find an element $x(w)$ in K such that $z \leq x(w)$ but $w \vee x(w)$ covers $x(w)$, ie: $w \nleq x(w)$. Then $\inf_w x(w)$, the infimum being taken of a set of elements indexed by the elements w covering z, is an element of K, because each $x(w)$ is in K. $z \leq x(w)$ $(\forall w)$ implies $z \leq \inf_w x(w)$, but $w \nleq x(w)$ $(\forall w)$ implies $w \nleq \inf_w x(w)$ $(\forall w)$. Thus $z = \inf_w x(w)$, and z is an element of K.

For any differential R, the set of elements z such that w covers z implies $R(z,w) = 1$ is a set closed under the lattice operation inf. Let E be a subset of the set K of elements z having this property, and let $u = \inf E$. Then w covers u implies $w \nleq u$, and $w \nleq x$ for some x in E. For this choice of x, $w \vee x$ covers x, so $R(x, w \vee x) = 1$, and $R(u,w) = R(w \wedge x, w) = 1$ by the translation property. Thus $R(u,w) = 1$ for all w covering $u = \inf E$, and the set K is closed with respect to inf.

It remains to be shown merely that, starting from a set K of lattice elements which is closed with respect to inf, if we define a function R to have value 1 on all steps [z,w] where z is an element of K, and extend it to a differential in any way other than that employed at the outset of this proof, we obtain elements z not in K for which w covers z implies R(z,w) = 1.

Since every differential has the translation property, it follows that R(x,y) must equal 1 whenever y covers x, and $x \leq z$ but $y \vee z$ covers z, for some z in K. If y covers x, but these conditions hold for x and y with respect to no z in K, $x \leq z$ must imply $y \vee z = z$, and $y \leq z$, ie: x and y are beneath the same elements of K. Let z be the least element of K which lies above x and y (ie: the meet of all such elements of K). The fixed point property applies to the interval [x,z]. If R(x,y) = 1, there is some element $w < z$, $w \in [x,z]$, such that $u \in [x,z]$ and u covers w implies R(w,u) = 1. But if any other element s covers w, $s \nleq z$, then $s \vee z$ covers z, and $R(z, s \vee z) = 1$. By the translation property, R(w,s) = 1, so R(w,s) = 1 for all elements s covering w, with $x \leq w < z$, in contradiction to our choice of z.

§4 Operations on Differentials

a) Restriction

The simplest yet most important operation on differentials is that of <u>restriction</u> of the domain of the differential to pairs of lattice elements lying within some fixed lattice interval:

$$R|_{[x,y]}(u,v) = R(u,v) \text{ for all } u,v \in [x,y].$$

Proposition. Properties of Restricted Differentials: If R is a differential on a lattice L, and if x and y are elements of L, with $x \leq y$, $R|_{[x,y]}$ is a differential on the sublattice $[x,y]$. If R is exact, $R|_{[x,y]}$ is exact.

Proof: The monotone, subadditive, and translation properties all hold with respect to any elements of L in the interval. If $R|_{[x,y]}$ is not exact, there is some inexact local graph within the interval. A local graph for $R|_{[x,y]}$ is also a local graph for R, so R is not exact.

Restrictions are a central feature of the study of differentials, arising in the theory of graphs, in that they correspond to the operations of contraction and elimination of edges.[1]

b) Supremum

If R_1 and R_2 are two differentials on a finite complete modular lattice L, then

$$[R_1 \vee R_2](x,y) = R_1(x,y) \vee R_2(x,y)$$

defines a function $R_1 \vee R_2$ on pairs of elements in L. $R_1 \vee R_2$, the supremum of R_1 and R_2, is a differential, since it is monotone, and its restriction to steps $[x,y]$ has only allowable local graphs. For a proof, consult the following table of suprema for local graphs:

local graph of R_2

local graph of R_1	$\vee$	zero	mixed	prime	one	inexact
	zero	zero	mixed	prime	one	inexact
	mixed		mixed or one	inexact	one	one or inexact
	prime			prime	one	inexact
	one				one	one
	inexact					one or inexact

[1] infra, Chapter II, §3, p. 43.

If R_1 and R_2 are exact, it does not necessarily follow that the supremum $R_1 \vee R_2$ is exact, since the supremum of a mixed local graph with a prime local graph may be inexact.

The infimum of a pair of differentials is not necessarily a differential, because the infimum of a mixed local graph with a prime local graph is not an allowable local graph.

c) Inverse

There are operations on differentials which yield differentials on other lattices. If R is an exact differential on a finite complete modular lattice L, then the inverse of R, written R', is defined on the inverted lattice $\tilde{L}$. For any element x in L, let $\tilde{x}$ represent the image of x under this inversion. Then if $\tilde{x}$ covers $\tilde{y}$ in $\tilde{L}$, define R' by

$$R'(\tilde{y},\tilde{x}) = 1 - R(x,y)$$

and extend to a differential by the monotone property.

That this defines a differential on $\tilde{L}$ is clear from the change which occurs in local graphs when R values 0 and 1 are interchanged, and the lattice is inverted: those of types zero and one are converted into one another, while local graphs which are mixed or prime retain these descriptions.

Proposition. Properties of Inverse Differentials: If R is an exact differential, R' is exact and R" = R. If R is not exact, the function $S(\tilde{y},\tilde{x}) = 1 - R(x,y)$ for all steps $[\tilde{y},\tilde{x})$ of $\tilde{L}$ cannot be extended to a differential.

Proof: No local graph of R is converted to an inexact local graph under this inversion. R" = R because the operations of subtraction from 1 and lattice inversion are their own inverses, and the extension to a differential from a function on steps, all local graphs of which are allowable, is unique. If R is not exact, there is some inexact local graph in L. Interchange of R-values 0 and 1, followed by lattice inversion, changes an inexact local graph into one which is not allowable in a differential.

d) Dual

Another operation, related, as we shall see, to inversion, also produces a differential on $\tilde{L}$ from one on L. Assuming R is a differential on a complete finite modular lattice L, we determine the set H of lattice elements z for which z covers w implies

$R(w,z) = 0$. The set H is closed with respect to the lattice operation sup, since if E is a subset of H, and if $x = \sup E$, x covers w implies $x \nleq w$, and $z \nleq w$ for some element $z \in E$. Because x covers w, z covers $z \wedge w$, so $R(z \wedge w, z) = 0$, and $R(w,x) = 0$ by the translation property. This being true for all w covered by x, $x = \sup E$ must be in the set H.

The image $\tilde{H}$ of this set H under lattice inversion is closed under the lattice operation inf, and determines a unique differential in accordance with our characterization in terms of fixed points.[1] This differential we call the dual of R, and employ the notation R*.

The main theorem on duality, that $R^* = R'$ if R is exact, will be proven in the equivalence theorem of Chapter three. For any differential R, we still have a relationship between the dual R* and the function S defined, for all elements $\tilde{x}$ covering $\tilde{y}$ in $\tilde{L}$, by $S(\tilde{y},\tilde{x}) = 1 - R(x,y)$, ie: the function used to define the inverse differential R'.[2]

Proposition. Properties of Dual Differentials: If R is a differential on a finite complete modular lattice L, then for all elements $\tilde{x}$ and $\tilde{y}$ of $\tilde{L}$, with

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[1]supra, p. 12.

[2]See examples of differentials and duals, Appendix A.

x covering y,

$$R^*(\tilde{x},\tilde{y}) \leq 1 - R(y,x).$$

Furthermore, for any pair x,y of elements of L,

$$R(x,y) \leq R^{**}(x,y).$$

Proof: To establish the first statement, we must prove that $R(y,x) = 1$ implies $R^*(\tilde{x},\tilde{y}) = 0$ whenever x covers y. Assume x covers y, and $R(y,x) = 1$. Let H be the set of those elements z of L such that z covers w implies $R(w,z) = 0$. Then for no element z of the set H does z lie beneath x but not beneath y. If this were the case, $z \vee y = x$ would cover y, so z would cover $y \wedge z$. This would mean $R(y \wedge z, z) = 0$, so $R(y,x) = 0$ by the translation property, a contradiction. Therefore, carrying all these elements over into their images in $\tilde{L}$, and letting $\tilde{H}$ denote the image of the set H, we find $\tilde{y}$ covers $\tilde{x}$, yet for no element $\tilde{z}$ of $\tilde{H}$ is it the case that $\tilde{x} \leq \tilde{z}$ but $\tilde{y} \not\leq \tilde{z}$ (ie: $\tilde{y} \vee \tilde{z}$ covers $\tilde{z}$). Because of the manner in which we define R^* from $\tilde{H}$, $R^*(\tilde{x},\tilde{y}) = 0$.

The second statement follows from a comparison of the set K, of elements z of L such that w covers z implies $R(z,w) = 1$, with the subset $\tilde{H}$ of L in terms of which the differential R^{**} is defined. This latter set $\tilde{H}$ is the inverse image of the set H of $\tilde{L}$ composed of those elements $\tilde{z}$ for which $\tilde{z}$ covers $\tilde{w}$ implies $R^*(\tilde{w},\tilde{z}) = 0$. Say z is an element of K, and w is any element covering z in L. Then $R(z,w) = 1$, so $R^*(\tilde{w},\tilde{z}) \leq 1 - R(z,w) = 0$, by that portion of the proposition we have already proven. This being true for every element $\tilde{w}$ covered by $\tilde{z}$, $\tilde{z}$ must be an element of the set H, so z is an element of $\tilde{H}$. We have proven the set K to be a subset of the set $\tilde{H}$. Because of the manner in which the differentials R and R** are associated with the sets K and $\tilde{H}$, respectively, ie: for y covering x, $R(x,y) = 1$ if and only if there is an element z of K such that $x \leq z$ but $y \nleq z$, with a similar statement relating R** and $\tilde{H}$, both R and R** then being extended to differentials by the monotone property, we have $R(x,y) \leq R^{**}(x,y)$ for all pairs x,y of elements of L.

e) Products

Let R_1 be a differential on a finite complete modular lattice L_1 and R_2 be a differential on another such lattice L_2. The product $R_1 \times R_2$ of these differentials is defined on the product lattice $L_1 \times L_2$, a lattice which is also finite, complete, and modular[1], according

[1]These three properties follow easily from the observation that (x_1,x_2) covers (y_1,y_2) in $L_1 \times L_2$ if and only if ($x_1=y_1$ and x_2 covers y_2) or (x_1 covers y_1 and $x_2=y_2$).

to the formula:

$[R_1 \times R_2]((x_1,x_2),(y_1,y_2)) = \sup\{R_1(x_1,y_1),R_2(x_2,y_2)\}$
if $(x_1,x_2) \leq (y_1,y_2)$, and is equal to zero otherwise.

We regard an element x of a lattice L, on which a differential R is defined, as a factor of R if x = (1,0) in a product representation of L, for which R is the product differential.

Proposition. Properties of Product Differentials:

i) If $R_1 \times R_2$ is defined as above, $R_1 \times R_2$ is a differential on the lattice $L_1 \times L_2$.

ii) If R is a differential on a lattice L, then x is a factor of R if and only if, for all elements y,z of L,

$$R(y,z) = \sup\{R(y \wedge x, z \wedge x), R(y \vee x, z \vee x)\}.$$

iii) If R is a differential on a lattice L and x is a factor of R, then R is exact if and only if both restricted differentials $R|_{[0,x]}$ and $R|_{[x,1]}$ are exact.

Proof:

i) The value $[R_1 \times R_2]((x_1,x_2),(y_1,y_2))$ is 0 or 1, and has value 0 unless $(x_1,x_2) \leq (y_1,y_2)$, ie: unless $x_1 \leq y_1$ in L_1 and $x_2 \leq y_2$ in L_2. $R_1 \times R_2$ is monotone with respect to intervals:

say $(w_1,w_2) \leq (x_1,x_2) \leq (y_1,y_2) \leq (z_1,z_2)$ in $L_1 \times L_2$. Then $[R_1 \times R_2]((x_1,x_2),(y_1,y_2)) = \sup_{i=1,2} R_i(x_i,y_i) \leq \sup_{i=1,2} R_i(w_i,z_i)$ $= [R_1 \times R_2]((w_1,w_2),(z_1,z_2))$. $R_1 \times R_2$ is subadditive: say $(x_1,x_2) \leq (y_1,y_2) \leq (z_1,z_2)$ in $L_1 \times L_2$, and $1 = [R_1 \times R_2]$ $((x_1,x_2),(z_1,z_2)) = \sup_{i=1,2} R_i(x_i,z_i)$. Then for i = 1 or i = 2, $R_i(y_i,z_i) = 1$. By the subadditive property of R_i for this value of i, either $R_i(x_i,y_i) = 1$ or $R_i(y_i,z_i)$ $= 1$. Thus either $\sup_{i=1,2} R_i(x_i,y_i) = 1$ or $\sup_{i=1,2} R_i(y_i,z_i) = 1$, the former being $[R_1 \times R_2]((x_1,x_2),(y_1,y_2))$, the latter being $[R_1 \times R_2]((y_1,y_2),(z_1,z_2))$. Finally, $R_1 \times R_2$ has the translation property: If $(x_1,x_2) \vee (y_1,y_2)$ covers (x_1,x_2), we must show $[R_1 \times R_2]((x_1,x_2),(x_1,x_2) \vee (y_1,y_2)) < [R_1 \times R_2]$ $((x_1,x_2) \wedge (y_1,y_2),(y_1,y_2))$. The proof is simplified by the observation that $(x_1,x_2) \vee (y_1,y_2)$ covers (x_1,x_2) implies either ($x_1 \vee y_1$ covers x_1 and $x_2 \vee y_2 = x_2$) or ($x_1 \vee y_1 = x_1$ and $x_2 \vee y_2$ covers x_2). Let us assume the former is the case; our proof will apply equally in the latter case. Let us assume $R((x_1,x_2),(x_1 \vee y_1, x_2 \vee y_2)) = 1$. Since $x_2 \vee y_2 = x_2$, $R_2(x_2, x_2 \vee y_2) = 0$. The supremum of this with $R_1(x_1, x_1 \vee y_1)$ being 1, the latter must also have value 1. By the translation property of R_1,

$$R_1(x_1 \wedge y_1) = 1, \text{ so } 1 = \sup_{i=1,2} R_i(x_i \wedge y_i, y_i)$$

$$= [R_1 \times R_2]((x_1,x_2) \wedge (y_1,y_2),(y_1,y_2)).$$

ii) Given a representation of a complete lattice L as the product $L_1 \times L_2$ of complete lattices L_1 and L_2, let π be the projection of L into $L_1 \times L_2$, ie: $\pi(x) = (\pi_1(x), \pi_2(x))$ for all elements x of L. π is then invertible on L_1, in that the lattice L_1 is isomorphic to the sublattice of L composed of all elements x such that, for some fixed element u in L_2, $\pi_2(x) = u$. This sublattice is the interval $[\pi^{-1}(0,u), \pi^{-1}(1,u)]$ of L.

If $\pi(x) = (1,0)$ for a product representation of L in which R is the product differential, we may embed the lattice L_1 as the interval $[0,x]$ of L, and the lattice L_2 as the interval $[x,1]$. Let the functions p_1 and p_2 be the embeddings of L_1 and L_2 in L. Then $p_1(\pi_1(z)) = z \wedge x$ and $p_2(\pi_2(z)) = z \vee x$. Since $R(y \wedge x, z \wedge x) = \sup\{R_1(\pi_1(y), \pi_1(z)), R_2(0,0)\} = R_1(\pi_1(y), \pi_1(z))$ and since $R(y \vee x, z \vee x) = \sup\{R_1(1,1), R_2(\pi_2(y), \pi_2(z))\} =$ $R_2(\pi_2(y), \pi_2(z))$, we conclude that :

$$R(y,z) = \sup_{i=1,2} R_i(\pi_i(y), \pi_i(z)) = \sup\{R(y \wedge x, z \wedge x), R(y \vee x, z \vee x)\}.$$

Conversely, if, for some element x in the lattice L, this formula holds for all pairs of elements y,z in L, the differential R is the product of the restricted differential $R|_{[0,x]}$ on the sublattice [0,x] with the restricted differential $R|_{[x,1]}$ on the sublattice [x,1] of L.

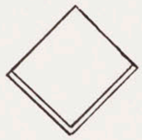
iii) Let us assume the differential R on the lattice L is exact, and an element x in L is a factor. As we proved in the second section of this proof, the lattice L_1 is the image of an R-preserving isomorphism, the domain of which is the interval [0,x] of L, and the lattice L_2 is similarly the image of the interval [x,1] of L. If R_1 or R_2 were to have an inexact local graph, R would have an inexact local graph in the interval [0,x] or [x,1], respectively.

Conversely, assume an element x in the lattice L is a factor of the differential R on L, and both restricted differentials $R|_{[0,x]}$ and $R|_{[x,1]}$ are exact. Consider the various possibilities for formation of local graphs in L, the elements of which are taken to be y,z $y \wedge z$, and $y \vee z$.

A step in L, such as that from $y \wedge z$ to z, results from either a step in L_1 and equality in L_2, or else from equality in L_1 and a step in L_2. Whichever is the case for the step $[y \wedge z, z]$, the same is true for the step $[y, y \vee z]$; the same can be said for the other parallel pair of steps. 2 cases: a) the steps $[y \wedge z, y]$ and $[y \wedge z, z]$ are associated with steps of the same lattice, whether this be L_1 or L_2, and b) the steps $[y \wedge z, y]$ and $[y \wedge z, z]$ arise from steps of different lattices, one from a step in L_1, equality in L_2, the other from equality in L_1, and a step in L_2.

In the first case, the local graph on the elements $y, z, y \wedge z$, and $y \vee z$ has the same R values as does the non-trivial projection of this local graph into L_1 or L_2, as we established in the second section of this proof. In the second case, the two steps in each parallel pair project onto the same step of L_1 or L_2, and thus have the same R value. Local graphs thus formed must be zero, mixed, or one. This completes the proof.

We define a differential R on a finite complete modular lattice L to be <u>prime</u> if there exist no factors of R in L other than 0 or 1, ie: if there is no non-trivial expression for R as a product differential.

The local graph  , which we called "prime",

is indeed a prime differential on the four element lattice pictured. Also prime are both differentials definable on the 2 element lattice, ie:  and  .

Other examples, defined on the Boolean algebras of a three- or of a four-element set, are given in appendix B.

Chapter II

§1 Introduction

Rather than proceed to the principal equivalence and lattice structure theorems for exact differentials, let us pause to consider a number of examples, related characterizations, and fields of application.

In a long section following this introduction, we establish a three-way correspondence between differentials, relations, and closure operators.

A third section relates Whitney's independence systems to exact differentials on Boolean algebras, and extends the concept of a Whitney independence system to a structure definable on any finite complete modular lattice. Graph relations are discussed, and the notion of dual graph is set in differential terms.

In section four we examine atom differentials, which arise from the interaction of set union and lattice supremum on the set of atoms of a general lattice, and which permit us to translate lattice properties into the language of differentials.

A few descriptive paragraphs concerning applications of the theory of differentials to geometry, algebra, logic, and probability bring this chapter to a close.

§2 Relations and Closures

Any relation between two finite sets gives rise to two differentials. The first, or <u>set differential</u> of the relation, will be shown to be associated with the relation by a correspondence which is onto the set of all differentials on complete finite modular lattices, and which is one-one up to the choice of a lattice which will accept the embedding of a certain set as its set of join-irreducible elements. The second, or <u>partition differential</u> of the relation, is of interest in the theory of graphs. (See Appendix A for examples).

A relation β from a finite set X to a finite set Y is an arbitrary subset of the cartesian product $X\times Y$. If d is an element of X, and e an element of Y, we write $d\beta e$ whenever d is related to e, ie: whenever the pair (d,e) is in the subset β of the product $X\times Y$. If d is an element of the set X, let $\beta(d)$ be the subset of Y composed of those elements of Y related to d. If x is any subset of the set X, let $\check{\beta}(x)$ be the subset of

$\mathcal{Y}$ composed of all elements for which there is some element d in the subset x to which they are related, ie: $\check{\beta}(x) = \sup_{d \in x} \beta(d)$, the supremum being taken in the Boolean algebra of subsets of the set $\mathcal{Y}$.

If β is a relation between finite sets $\mathcal{X}$ and $\mathcal{Y}$, we define the <u>set differential</u> R of β, for all pairs x,z of subsets in the Boolean algebra of subsets of the set $\mathcal{X}$, by

$$R(x,z) = 0 \text{ unless } x < z \text{ and } \check{\beta}(x) < \check{\beta}(z),$$
$$\text{in which case } R(x,z) = 1.$$

Proposition: The set differential of a relation is a differential on the Boolean algebra of subsets of the domain of the relation.

Proof: Let X and Y be two finite sets, β a relation from X to Y, and R the set differential of β. The function $\check{\beta}$ from the Boolean algebra of subsets of X to the Boolean algebra of subsets of Y is isotone, ie: $x < z$ implies $\check{\beta}(x) \leq \check{\beta}(z)$. R is monotone, because if $x_1 \leq x_2 \leq x_3 \leq x_4$ are four subsets of $\mathcal{X}$, each contained in the next, then $R(x_2,x_3) = 1$ implies $\beta(x_1) \leq \beta(x_2) < \beta(x_3) \leq \beta(x_4)$, so $\beta(x_1) < \beta(x_4)$, and $R(x_1,x_4) = 1$. R is subadditive, because if $x_1 \leq x_2 \leq x_3$ are three subsets of X, each contained in the next, and if $R(x_1,x_2) = 1$, meaning $\beta(x_1) < \beta(x_3)$, then the set $\check{\beta}(x_2)$, which is contained between $\check{\beta}(x_1)$ and $\check{\beta}(x_3)$, cannot be equal to both of them. Thus $R(x_1,x_2) = 1$ or $R(x_2,x_3) = 1$. Finally, R has the translation property, because if the subset $z \vee x$ covers the subset z, x must contain exactly one element e of X which is not in z, ie: $x = x \wedge z + e$. $R(x \wedge z, x) = 0$ if and only if the element e is related only to elements of Y which are already in the set $\check{\beta}(x \wedge z)$. But if this is the case, $x \wedge z \leq z$ implies $\check{\beta}(x \wedge z) \leq \check{\beta}(z)$, so e is related only to elements of Y which are already in the set $\check{\beta}(z)$. Because $z \vee x = z + e$, $R(z, z \vee x) = 0$.

Proposition. Characterization of Fixed Points of Set Differentials: Under this correspondence of a relation to its set differential, a subset of the domain of the relation is a fixed point of the differential if and only if its complement is the union of sets of the form $\beta^{-1}(e)$ where e is any element of the range of the relation.

Proof: In the notation of the previous proof, the statement $R(x,z) = 1$ for all z covering x is equivalent to the statement that, for any element d not in the subset x, there is an element e of Y such that d is related to e, but no element of x is related to e, ie: for every element d in the complement of x there is an element e of Y such that $\beta^{-1}(e)$ is disjoint from x. This is equivalent to the statement that the complement of x is a union of sets of the form $\beta^{-1}(e)$, where e is an element of the range set Y.

The correspondence from relations to their set differentials has produced a family of differentials, all of which are defined on Boolean algebras. We have seen that fixed points of the set differential R of a relation β are complements of sets expressible as unions of inverse images $\beta^{-1}(e)$ for elements e of the range of the relation β. We may thus use the complete lattice of arbitrary unions of such inverse images as an isomorphic copy of the inverse of the lattice L/R of fixed points of the set differential. In this lattice of unions of sets $\beta^{-1}(e)$, $e \in Y$, an element is join irreducible if and only if it is not expressible as the union of two proper subsets, both of which are elements of the lattice. Since all lattice elements are expressible as unions of elements of the form $\beta^{-1}(e)$, for $e \in Y$, only elements of the form $\beta^{-1}(e)$ may be join irreducible. Not all such elements are join irreducible, however, if some set $\beta^{-1}(e)$ may be expressed as the union

$\beta^{-1}(e_1) \vee \beta^{-1}(e_2) \vee \ldots \vee \beta^{-1}(e_n)$ for some set of one or more elements e_i distinct from e in Y.

Now, returning to the lattice L/R of fixed points of the set differential R of the relation β: $X \rightarrow Y$, we see the complements in L of meet-irreducible elements of L/R are all subsets of the form $\beta^{-1}(e)$ for some element e of Y. This yields an inversion of our correspondence from relations to differentials:

Proposition. Construction of Relations from Differentials: Given an arbitrary differential on a Boolean algebra L of subsets of a finite set X, and letting the set Y be composed of all subsets forming meet-irreducible elements of the lattice L/R of fixed points of R, the differential R is the set differential of the relation β from X to Y defined for all elements d in X and e in Y by

$$d \; \beta \; e \text{ if and only if } d \not\leq e.$$

Proof: Let R_1 be the set differential of the relation β thus defined. The meet irreducible elements of the lattice L/R_1 are, as elements of L, complements of sets of the form $\beta^{-1}(e)$, for some element e of Y. Being thus composed of elements of X not related to an element e of Y, the meet irreducible elements of L/R_1 are thus equal to the sets chosen as the elements of the set Y, namely, the meet irreducible elements of the lattice L/R. Equality of sets of meet irreducible fixed points implies equality of sets of fixed points, and thus $R = R_1$.

We need not, however, restrict ourselves to differentials on finite Boolean algebras in the establishment of this correspondence. Beginning with a differential R on any finite complete modular lattice L, a modification above construction produces a relation from the set X of join irreducible elements of the lattice L to the set Y of meet irreducible elements of the lattice L/R. To reproduce the differential from this relation, we would need an extended notion of set differential as an equivalence class of differentials R_α on various lattices L_α, with their associated lattices L_α/R_α of fixed points, each of which accepts the embedding of the set Y as the set of meet irreducible elements of L_α/R_α.

We turn now to the second type of differential associated with a relation between two finite sets X and Y. Let β be such a relation. With each element d of the set X, associate the partition $\pi(d)$ of the set Y into sections, one section being all elements of Y related to d, and all other sections containing one element each. Then associate with each subset x of the set X, an element $\pi(x)$ in the lattice of all partitions of the set Y, defined by $\pi(x) = \sup_{d \in x} \pi(d)$, the supremum being taken in the partition lattice, wherein the operation inf is common refinement of partitions.

The mapping π: $x \to \pi(x)$ is isotone from the Boolean algebra of subsets of the set to the partition lattice of the set Y. We define the partition differential of the relation β for all pairs of subsets x,z of X by

$$R(x,z) = 0 \text{ unless } x < z \text{ and } \pi(x) < \pi(z),$$
$$\text{in which case } R(x,z) = 1.$$

The proof that the partition differential is a differential is analogous to that given for the set differential, and will be omitted.

Proposition. Comparison of Set and Partition Differentials: If a relation β between two finite sets X and Y has set differential R and partition differential R_1, and if every element of X is related to at least two elements of Y (ie: if the empty set in the Boolean algebra of subsets of the set X is a fixed point of the partition differential R_1), then $R \leq R_1$.

Proof: Say x is a subset of X, d an element of X not in x, and $R(x,x+d) = 1$. Then there must be some element e of the range Y such that $d\beta e$, but $a\beta e$ for no element a of x. Thus e is alone in its section of the partition $\pi(x)$. By our restrictive assumption, d must be related to more than one element of Y, so e is not alone in its section of the partition $\pi(x+d)$. Thus $\pi(x) < \pi(x+d)$, and $R_1(x,x+d) = 1$.

In the next section we shall point out the importance of partition differentials in their relationship to Whitney independence systems in general, and to graphs in particular. Let us turn instead to the discussion of closure operators. We now complete our three-way correspondence, already constructed from relations to differentials, by providing a further link from differentials to closures.

A closure operator on a lattice L is any function $C\ell$, from L into itself, which satisfies the following two conditions, for all elements x,y of L.

i) $x \leq C\ell(x)$

ii) $x \leq C\ell(y)$ implies $C\ell(x) \leq C\ell(y)$.

We have already encountered closure operators in our discussion of set and partition differentials, as in the proof that a set differential is a differential[1]. Also, the mapping $R \to R^{**}$, from a differential to the dual of its dual, is a closure operator on the lattice of all differentials definable on a fixed lattice.

Oystein Ore[2] establishes the equivalence between closure operators and complete intersection rings. Since differentials are characterized by their fixed points, which form such a ring, the correspondence is immediate.

To define a closure operator, given a differential R on a lattice, let

$$C\ell(x) = \sup \{z;\ R(x,z) = 0\}.$$

Conversely, given a closure, $C\ell$, define a differential R by

$$R(x,z) = 0 \text{ unless } x < z \text{ and } C\ell(x) < C\ell(z)$$
$$\text{in which case } R(x,z) = 1,$$

just as we did for the set differential of a relation.

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[1] supra, p.31.

[2] O. Ore, Theory of Graphs, p. 177.

The part played by Galois connections in the establishment of closure operators on lattices is thoroughly discussed by Ore[1]. We will content ourselves with pointing these out as they appear, as in the mapping of a differential to its dual.

§3 Whitney Systems and Graphs

Hassler Whitney begins his theory of independence[2] with the description of a family of increasing integer-valued functions defined on the elements of a Boolean algebra. These functions, which we call Whitney rank functions, are intimately related to exact differentials. The proof that the Whitney rank functions and exact differentials coexist is, however, independent of the special properties of Boolean algebras. Indeed, exact differentials provide an immediate generalization of Whitney's theory to independence systems defined on a finite complete modular lattice.

Whitney defines a rank function on a finite Boolean algebra as follows: a function r on the set of subsets

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[1] ibid. p.183.

[2] H. Whitney, "On the Abstract Properties of Linear Dependence", Amer. J. 57(1935) p.510.

of a finite set X is a rank function if, for all subsets x and y of X, and elements e_1, e_2 of X,

i) $r(\phi) = 0$

ii) $r(x+e) = r(x) + k$, $k = 0$ or 1

iii) $r(x+e_1) = r(x+e_2) = r(x)$ implies $r(x+e_1+e_2) = r(x)$.

Rephrasing these conditions in lattice terminology, we obtain, for all elements x,y of a finite Boolean algebra L,

i) $r(0) = 0$

ii) if y covers x in L, $r(y) - r(x) = 0$ or 1

iii) if x and y cover $x \wedge y$, then $r(x \wedge y) = r(x) = r(y)$ implies $r(x \wedge y) = r(x \vee y)$.

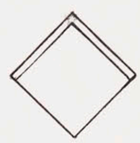
Already it is clear how to establish a connection to the theory of differentials, and in a way which avoids terminology peculiar to Boolean algebras. We shall see shortly that a somewhat simpler statement is equivalent to Whitney's third condition; let us define a _Whitney rank function_ on a finite complete modular lattice L as a function r, defined on the elements of the lattice L, with values in the non-negative integers, and satisfying

i) r has initial value 0, ie: $r(0) = 0$

ii) unit increase condition: the value of r does not decrease, and increases at most by 1 on any step, ie: y covers x in L implies $r(y) - r(x)$ is 0 or 1.

iii) translation property: If y covers x in L, and $x \leq z$, then $r(x) = r(y)$ implies $r(z) = r(z \vee y)$.

Theorem. Integration of Exact Differentials: A function R defined on all pairs of elements in a finite complete modular lattice L is an exact differential if and only if R is the first difference of a Whitney rank function r on the lattice L, ie: $R(x,y) = r(y) - r(x)$ whenever y covers x in L.

Proof: Let r be a Whitney rank function on the lattice L. The simple statement that r increases by at most one on any step, implies $R(x,y) = r(y) - r(x)$ has value 0 or 1, and that there must be the same number of steps for which $R = 1$ on each side of a local graph. The only local graph[1] satisfying this condition, yet not allowable in an exact differential is  .

[1]vide: p.7, supra.

This local graph is excluded by Whitney's third condition, but it is equally well excluded by our translation property. Our theorem[1] on the local character of differentials provides a unique extension of such a function R to an exact differential.

Conversely, if R is an exact differential on the finite, complete modular lattice L, we define an integer valued function r for each element x of the lattice by

$$r(x) = \sum_{i=1}^{n} R(p_{i-1},p_i) \text{ for any path } p \text{ from } 0 \text{ to } x.$$

Induction on the rank $\lambda(x)$ of the element x in the modular lattice L provides a proof of independence of path; the proof is of such a nature that it is more suitably incorporated in the equivalence theorem for exact differentials[2]. If we assume independence of path, r is then well-defined as a function on the elements of the lattice L. The sum r has initial value $0 = R(0,0)$, and increases by 0 or $1 = R(x,y)$ on any step $[x,y]$. To establish the translation property, we notice that if y covers x in L, and $x \leq z$, either $y \leq z$, ie: $z \vee y = z$, or else $z \vee y$ covers z. If $y \leq z$,

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[1]supra, p.10.

[2]infra, p.58. The hypotheses are identical.

$r(z) = r(z \vee y)$. If $z \vee y$ covers z, and if $r(x) = r(y)$, ie: $R(x,y) = 0$, $R(z,z \vee y) = 0$ by the translation property of the differential R, so $r(z) = r(z \vee y)$. Thus the sum r has the translation property of a Whitney rank function.

We shall see in the next chapter how the various structures of Whitney's theory of independence carry over into the realm of exact differentials. As Whitney did for his independence systems, we shall be able to prove for exact differentials several of the standard theorems concerning circuits and trees in graphs.

A graph is a relation. We have seen how to define a set differential and a partition differential of a relation. On the other hand, a graph defines a Whitney system, which in turn defines an exact differential. The connection between the various differentials thus definable from a graph is given by the following proposition. Following W.T. Tutte[1], we understand a graph to be a relation β from a finite set X to a finite set Y in which every element of X is related to at most two elements of Y.

Proposition. The Exact Differentials of a Graph:

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[1] W.T. Tutte, "A Contribution to the Theory of Chromatic Polynomials", Can. J. 6(1954) p.80.

If a Whitney rank function is defined by a graph, the exact differential defined by the rank function is the partition differential of the graph relation. The dual graph has as differential the dual differential.

Proof: The Whitney rank function for a graph G has value k on a set x of edges of G if and only if the number of vertices of the graph G, less k, is equal to the number of arc-connected components of the subgraph composed of all the vertices of G, and the edges of G which are in the subset x. Thus the rank increases on a step $[x,x+e]$, and $R(x,x+e) = r(x+e) - r(x)$, whenever the ends of e are not arc-connected by the subgraph with edges in the set x. Arc connection induces the partition $\pi(x)$[1] on the vertex set, ie: on the range of the graph relation. The rank thus increases on a step $[x,x+e]$ whenever the image of the edge e under the graph relation is not confined to a single section of the partition $\pi(x)$, ie: whenever $\pi(x) < \pi(x+e)$. Thus the exact differential of the Whitney rank function is the partition differential of the graph relation.

The rank function of the dual graph G^* increases on a step $[x,x+e]$ whenever the edge e is in some circuit

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[1]supra, p.36.

of the subgraph of the graph G composed of edges in the complement of the set x, ie: whenever e is not an isthmus in the subgraph of edges in the complementary set x'. The mapping of a set to its complement maps the Boolean algebra L of edge subsets isomorphically onto the inverted Boolean algebra $\tilde{L}$. This isomorphism carries the rank function of the dual graph into a Whitney rank function on the lattice $\tilde{L}$. Since this rank function increases on a step [x',x'-e] whenever e is in some circuit of x', and since circuits are subsets z of the Boolean algebra L such that $\tilde{z}$ is a coatom of the lattice of fixed points of the dual differential[1], the Whitney rank function increases whenever x' is beneath some fixed point of R* which is not above x'-e. This is the local characterization of the dual differential R* in terms of fixed points, so the exact differential of the Whitney rank function of the dual graph is R*, the dual differential.

Corollary: The partition differential of a relation β from a finite set X to a finite set Y is exact whenever β(e) is composed of at most two elements of Y, for every element e of the set X.

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[1]vide: infra,p.86.

§4 Atom and Coatom Differentials

Let us start with an arbitrary finite and complete lattice Q. We construct the Boolean algebra L of subsets of the set C of coatoms of the lattice Q, and define a pair of mappings

$$(\forall x \in Q)\ \beta(x) = C \cap [x,1]$$
$$(\forall y \in L)\ \alpha(y) = \inf y$$

wherein an element of the lattice Q is associated by the map β with the set of coatoms above it, and a subset of coatoms is associated by the map α with its infimum in the lattice Q. This forms a Galois connection[1] $\Gamma:\ Q \underset{\beta}{\overset{\alpha}{\leftrightarrows}} L$. An element x is closed with respect to the closure $\alpha \circ \beta$ on the lattice Q, if and only if it is the infimum of some set of coatoms. A set y of coatoms is closed with respect to the closure $\beta \circ \alpha$ on the lattice L if and only if y is the set of coatoms above some element of the lattice Q. The lattices of closed elements of Q and L are isomorphic, by the main theorem on Galois connections; this lattice is the sublattice of Q composed of all elements expressible as a meet of coatoms.

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[1] G.C. Rota, "On the Foundations of Combinatorial Theory", Zeitsch. fur Wahrshein. [2] 1964, pp.340-360, §5.

As we saw at the close of section two[1], a closure defines a differential. Under this correspondence, applied to the closure $\beta\circ\alpha$ on the Boolean algebra L of subsets of the set C of coatoms, the subsets containing exactly one coatom are invariably fixed points of the differential induced by the closure $\beta\circ\alpha$. Conversely, if we start with a differential R on a finite Boolean algebra L, in which every atom of L is a fixed point of the differential R, the above definition of maps α and β defines a Galois connection between the Boolean algebra L and the lattice L/R of fixed points of R in L. The map α from the Boolean algebra L to the lattice L/R is onto. Thus we have the following statement to relate the study of coatom-meet-expressible sublattices to the study of a certain class of differentials.

Proposition. Characterization of Atom Differentials: The isomorphy classes of complete finite lattices in which every element is a meet of coatoms are in one-one correspondence with the differential-preserving-isomorphism classes of Boolean algebras with differentials in which every atom is a fixed point.

The same may be said, via lattice inversion, about

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[1]supra, p.38.

complete finite lattices in which every element is a join of atoms. Let us call such a differential the <u>atom</u> or <u>coatom</u> <u>differential</u> of the lattice Q, when we construct it on the Boolean algebra of atoms or of coatoms, respectively.

We shall see in chapter three[1] that the lattice of fixed points of an exact differential is always semimodular. As a partial converse to this theorem, we have the following statement.

Proposition. Relating Semimodularity to Exactness of Atom Differentials: If R is an atom differential of a finite complete semimodular lattice Q, R is an exact differential.

Proof: Let α be the map from elements x of the Boolean algebra of subsets of the set A of atoms of the complete finite semimodular lattice Q, defined by $\alpha(x) = \sup x$, the supremum being taken in Q. Assume some local graph on subsets $x, y, x\wedge y, x\vee y$ (x and y covering $x\wedge y$, as usual) of the lattice L is inexact. For instance, say $R(x\wedge y, x) = R(x\wedge y, y) = R(x, x\vee y) = 1$, and $R(y, x\vee y) = 0$. Since subsets x and y each contain exactly one atom

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[1] infra, p.81.

not in $x \wedge y$, since $R(x \wedge y, x) = R(x \wedge y, y) = 1$, and since Q is semimodular, $\alpha(x)$ and $\alpha(y)$ both cover $\alpha(x \wedge y)$ in Q. They may, of course, be the same lattice element, except for the fact that $R(x, x \vee y) = 1$, which excludes the possibility that the one atom in the set $y - x$ is also beneath the element $\alpha(x)$ of Q. Since $\alpha(x)$ and $\alpha(y)$ are not equal, yet both cover $\alpha(x \wedge y)$, and since the supremum $\alpha(x) \vee \alpha(y)$ is also expressible as a join of atoms, $\alpha(x) \vee \alpha(y) = \alpha(x \vee y)$, and this element covers both $\alpha(x)$ and $\alpha(y)$. Thus $R(x, x \vee y) = R(y, x \vee y) = 1$, in contradiction to our assumption of inexactness.

This procedure exemplifies the conversion of lattice structure properties to differential language. It is conceivable that the enumerative work in chapter four of this paper may find application in problems of lattice structure. For detail on these methods, the reader should consult the work of R.P. Dilworth[1].

§5 Various Applications

a) Geometry

Our definition of a set differential and partition

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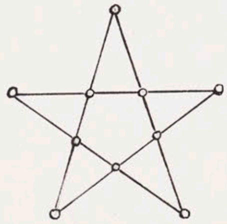
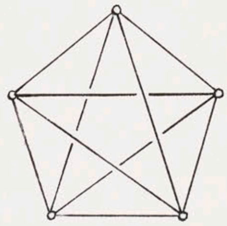
[1] R.P. Dilworth, "Dependence Relations in a Semimodular Lattice", Duke J. 11(1944) pp.575-587.

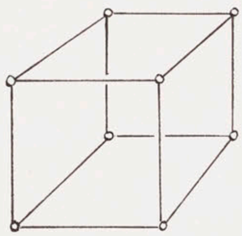
differential for a general relation between two finite sets suggests a new lattice-theoretic approach to geometric problems. Finite geometric configurations are characterized by incidence relations, but these need not be the edge-vertex incidence relations which are the subject of graph theory. Furthermore, the possibility of defining set and partition differentials for the converse relation gives rise to an interesting operator both on geometric configurations and on differentials. Let us consider two examples of non-graphic incidence relations, for which the converses are also incidence relations.

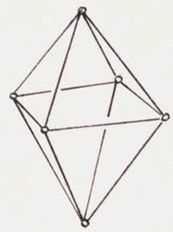
As a first stage of generalization beyond graph theory, consider the edge-vertex incidence relation of the five-pointed star, in which each of five edges has four vertices, and each of ten vertices has two edges. If the converse relation is interpreted as an edge-vertex incidence relation, the resulting ten-edge, five vertex figure is the complete five-graph.





A second stage of generalization introduces more possibilities. Consider the face-vertex incidence relation of the cube. There are six faces, each with four vertices, and eight vertices, each with three faces. If the converse relation is also interpreted as a face-vertex incidence relation, the resulting six-vertex eight-face configuration is the octahedron.





The differentials of a relation and of its converse are defined on different Boolean algebras, but the lattices of fixed points are of the same order of magnitude, and are likely to offer interesting comparisons.

b) Algebra

Substructures of algebraic structures are generally defined as subsets which are closed under certain algebraic operations. Any such definition gives rise to a differential on the Boolean algebra of all subsets of the underlying set:

for any subset x of the underlying set X and for any element e not in x,

$R(x,x+e) = 0$ if e is in the substructure generated by elements of X in x, and $R(x,x+e) = 1$ otherwise.

The lattice of fixed points of these differentials are the lattices of all substructures of that structure, partially ordered by containment.

Typical examples of lattices of substructures include the lattice of subgroups of a group, the lattice of ideals of a ring, and the lattice of subfields of a field. Semimodularity of such a lattice would follow from exactness of the differential[1], but exactness is an uncommon phenomenon in algebra. For example, the differential for the additive group of integers modulo four contains the inexact local graph $\phi,\{0\},\{2\},\{0,2\}$. The differential of the Vier group, restricted to the Boolean subalgebra of sets containing the zero element, is exact, and has a modular subgroup lattice.

The question of exactness is related to the existence of inverses. To prove exactness, we must show that if an element e_2 is not an algebraic combination of elements in a subset x, but is a combination of elements in x together with e_1, then the relation expressing

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[1]infra, p.81.

this, which may involve e_1 with some coefficient other than 1, can be solved for e_1 in terms of elements of x together with e_2. This can be done in Boolean rings, and in fields, so these structures have semimodular substructure lattices.

c) Logic

A differential R is defined on the Boolean algebra of all subsets of a finite set X of statements by

> for any subset x of statements in X, and any statement e not in x, $R(x,x+e) = 0$ if the conjunction of all statements in x implies the statement e.

If the statements compose a set of axioms, and if the differential R has the constant value 1, then the axioms are independent.

d) Continuous Analogues, and an Application to Probability

One continuous analogue of the theory of differentials on a finite Boolean algebra is responsible for our use of the term exact. Let us designate the truth value of the statement "the subset x contains the element e" by some number between 0 and 1, rather than by the discrete values 0 and 1. The subset x may now vary continuously over the unit cube in a space whose dimension n is the number of atoms of the lattice. In this event, the rôle of a Whitney rank function is played by any function increasing on this product space all of whose partial derivatives are bounded by 1. Our theorem, in the following chapter, that exact differentials have sums independent of choice of path, is the discrete analogue of the usual theorem on integration of exact differential in Euclidean n-space.

There is, however, an intermediate level of generalization for our theory of differentials. Consider a finite set X of statements, each with a probability of occurance. On the Boolean algebra of all subsets of this set, we may assign a probability measure m, equal on each subset x to the probability that every statement in the subset x is true, and all statements not in x are false. The sum of the measure m over the lattice is 1.

The measure m bears little resemblance to a differential. Formation of the probability distribution function p brings us closer. For every subset x of the statement set X let p(x) be the probability that all statements in the subset x are true. Then

$$p(x) = \sum_{x \leq y} m(y)$$

The function p(x) is monotone decreasing, and has value 1 on the empty set. For computational simplicity, let us assume $p(1) \neq 0$. Defined for any pair of subsets x,y of X for which $x \leq y$, the conditional probability $p(y/x) = p(x \vee y)/p(x) = p(y)/p(x)$ is a monotone decreasing function of intervals. Let us therefore define a function R(x,y) by

$$R(x,y) = 0 \text{ unless } x \leq y, \text{ in which case}$$
$$R(x,y) = 1 - p(y)/p(x).$$

Proposition. Probabilistic Differentials: The function R, defined as above for all pairs x,y of subsets of the statement set X, satisfies the monotone, subadditive and translation properties of a differential.

Proof: Assume $w \leq x \leq y \leq z$. Then $p(z/w) \leq p(y/x)$ implies $R(x,y) \leq R(w,z)$, so R is monotone. If $x \leq y \leq z$, the product $(p(y)-p(x))(p(y)-p(z))$ is not positive because y is intermediate between x and z. Thus, $p(x)p(y) + p(y)p(z) \geq p(x)p(z) + p^2(y)$ and $1 - p(z)/p(x) \leq 1 - p(z)/p(y) + 1 - p(y)/p(x)$, ie: $R(x,z) \leq R(x,y) + R(y,z)$, the subadditive property. Finally, since the lattice operations $\vee$ and $\wedge$ correspond to the statement subset operations union and intersection, $R(x \wedge y, y) - R(x, x \vee y) = p(x \vee y)/p(x) - p(y)/p(x \wedge y) = (p(x \vee y)p(x \wedge y) - p(x)p(y))/p(x)p(x \wedge y)$ is positive, the numerator representing the probability $p(x-y)p(y-x)$.

Chapter III

§1 Introduction

We begin by gathering together in an equivalence theorem the various conditions which characterize exactness of differentials on a finite complete modular lattice. The proof of this theorem comprises section two of the present chapter.

In section three we explore the analogy between differentials on a lattice and differentials in a space of n real variables. We define higher order partial differentiation of Whitney rank functions in such a way that the Taylor expansion at zero in the lattice is derived from a Möbius inversion formula.

In the fourth section we examine the influence of exactness on the structure of a lattice of fixed points, noting that a stronger connection prevails if the domain of the differential is a complemented lattice.

In section five we carry out the program of Hassler Whitney by generalizing to differentials on a finite complete and complemented modular lattice all the well known theorems from the classical theory of linear independence. The matching and duality properties of bases, circuits and bonds are translated to laticial terminology.

We bring Chapter III to a close with a discussion of factorization of differentials into primes, and show such factorization to be unique.

§2 Characterization of Exact Differentials

The property of exactness, like its counterpart in the theory of functions of several real variables, appears in a variety of forms, and induces strong symmetry and duality properties on differentials and on structures derived from those differentials. We set forth in the following theorem several equivalent formulations of this property. Knowledge of this manifold equivalence is our principal tool for the further development of the theory of exact differentials.

Theorem. Equivalent Characterizations for Exact Differentials: The following statements, all of which concern a differential R defined on a complete finite modular lattice L, are equivalent:

a) Exactness: R is an exact differential, ie: no local graph of R is inexact.

b) Local symmetry: If x, y, $x \wedge y$ and $x \vee y$ form a local graph of L,

$$R(x \wedge y, x) + R(x, x \vee y) = R(x \wedge y, y) + R(y, x \vee y).$$

c) Independence of path: For any pair of elements $x < y$ in the lattice L, the sum $r_p(x,y) = \sum_{i=1}^{n} R(p_{i-1}, p_i)$ of the R values along any path p from x to y is independent of the choice of path.

d) Integrability: There exists a Whitney rank function r on the lattice L (ie: an integer valued function with initial value zero, the unit increase and translation properties) such that $R(x,y) = r(y) - r(x)$ whenever y covers x in L.

e) Fixed point covering property: If an element y covers an element x in the lattice L, then the image in the lattice L/R of $C\ell(y)$, the meet in L of all fixed points above y, at most covers the image in L/R of $C\ell(x)$.

f) Existence of "dual fixed points": If the differential R has value 0 anywhere on an interval $[a,b]$, then there is an element z in the interval, $z \neq a$, such that z covers w and $a \leq w$ imply $R(w,z) = 0$.

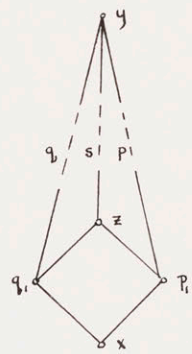
g) Duality: Whenever an element y covers an element x in the lattice L, $R^*(\tilde{y}, \tilde{x}) = 1 - R(x,y)$.

Proof:

a $\rightarrow$ b. Assuming the differential R is exact, its local graphs are then either zero, one, mixed, or prime. Each of these types has the property that the sums of the R values along the two paths from $x \wedge y$ to $x \vee y$ are equal.

b $\rightarrow$ c. We assume that for any local graph on x, y, $x \wedge y$ and $x \vee y$, $R(x \wedge y, x) + R(x, x \vee y) = R(x \wedge y, y) + R(y, x \vee y)$. Given any pair $x < y$ of elements of the lattice L, and any two paths p and q from x to y, let λ be the rank function of the finite modular lattice L, ie: $\lambda(x)$ is the length of any path from 0 to x. We shall establish independence of path by induction on the difference in rank, $\lambda(y) - \lambda(x)$. If the rank difference $\lambda(y) - \lambda(x)$ is one, there is only one path from x to y. If the rank difference is two, and the paths are unequal, the points x, p_1, q_1, and y form a local graph. Equality of the two sums follows from our assumed statement b. Let us assume the R sum $r_p(z,w)$ from z to w is independent of path whenever the rank difference $\lambda(w) - \lambda(z)$ is equal to n-1, and that for the pair $x < y$, the rank difference $\lambda(y) - \lambda(x)$ is equal to n.

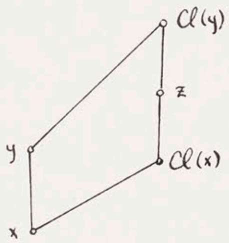
If the elements p_1 and q_1 covering x in the paths p and q, respectively, are equal, the remaining path segments from $p_1 = q_1$ to y via p and via q are of length n-1, and must have the same R-sum. Adding $R(x,p_1)$ to each sum, we have $r_p(x,y) = r_q(x,y)$. If the elements p_1 and q_1 are unequal, the element $z = p_1 \vee q_1$ covers p_1 and q_1. Choose a path s from x to y. The R-sum from x to y along p is equal to the R-sum from x to p_1, to z, then along the path s to y, since these paths agree on the first step, the remaining lengths being equal to n-1. By our assumed formula b, we have $R(x,p_1) + R(p_1,z) = R(x,q_1) + R(q_1,z)$, so we may replace the first two steps of our path from x to y via z by those to q_1, then to z, without altering the value of the R-sum. But this path agrees with the path q from x to y on its first step, the remaining length being n-1, so the path from x to y via q_1, z, and the path s, has the same R-sum as q. Thus the four paths p, via p_1 and s, via q_1 and s, and q have the same R-sum. In particular, $r_p(x,y) = r_q(x,y)$.



We have established independence of path for $\lambda(y) - \lambda(x) = 1$ or 2, and, assuming it for $\lambda(y) - \lambda(x) = n-1$, have proved it for $\lambda(y) - \lambda(x) = n$. By the induction axiom the sum is independent of path for any pair of elements $x < y$ in L.

$c \leftrightarrow d$. We have now proven the sum of an exact differential is independent of path. This is precisely the part missing from the proof of "independence of path if and only if integrable to a Whitney rank function" in Chapter two[1].

$c \rightarrow e$. We assume independence of path for the R-sum along paths between two lattice elements, and prove that if an element y covers an element x, then $C\ell(y)$, the meet in L of all fixed points above y, at most covers $C\ell(x)$ in the lattice L/R. If there is a fixed point z lying between $C\ell(x)$ and $C\ell(y)$, ie: $C\ell(x) < z < C\ell(y)$, we may choose a path q from x via $C\ell(x)$



and z to $C\ell(y)$ and a path p from x via y to $C\ell(y)$. If $R(p_{i-1},p_i)$ were to have value 1 for $i \geq 2$, ie: for any step between y and $C\ell(y)$, our proposition concerning the existence of fixed points implies the existence of an element w in the half-closed interval $[y,C\ell(y))$ such that, for any $u \leq C\ell(y)$, u covering w, $u \nleq C\ell(y)$ implies $u \vee C\ell(y)$ covers $C\ell(y)$, so $R(C\ell(y),u \vee C\ell(y)) = 1$ by the definition

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[1] supra, Chapter II, §3, p.41.

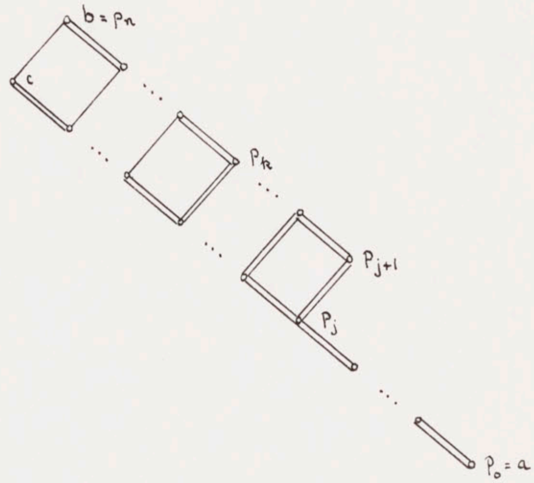
of fixed point, and $R(w,u) = 1$ by the translation property. The element w is a fixed point in the interval $[y, C\ell(y)]$ and lies below $C\ell(y)$, contradicting the definition of $C\ell(y)$ as the least fixed point above y.

The R-sum $r_p(x, C\ell(y))$ is thus no more than 1. The path q, however, passes through two fixed points $C\ell(x)$ and z, before reaching $C\ell(y)$, so along q the R-sum $r_q(x, C\ell(y))$ must be at least 2. This contradicts independence of path.

e $\rightarrow$ f. We assume the contradiction of the existence statement for dual fixed points, ie: we assume there exists an interval $[a, b_1]$ and a step $[c, b_1]$ in the interval, $a \leq c < b \leq b_1$ such that $R(c,b) = 0$, yet for all elements z in the interval $[a, b_1]$, and thus for all elements z in the smaller interval $[a,b]$, there exists an element w, $a \leq w < z$, w covered by z, with $R(w,z) = 1$. Starting from b, and using the existence of downward steps for which R has value 1, we may form a path p: $a = p_0 < p_1 < \ldots < p_n = b$ from a to b along which R has constant value 1. We observed[1] in the proof of the fixed point characterization of differentials that $R(x,y) = 1$ for any element y covering an element x only

[1] supra, p.15.

if there is some fixed point above x but not above y. If $R(x,y) = 1$, the meet $C\ell(x)$ of all fixed points above x cannot lie above y. Applying this condition along the path p from a to b, we see $C\ell(p_{i-1}) < C\ell(p_i)$ for all i, $i = 1,\ldots,n$. Let j be the highest value of the index i such that $p_i \leq c$, and let k be the highest value of the index i such that $k > j$ and $R(p_k \wedge c, p_k) = 1$.



Such a value k exists, and lies between j+1 and n-1, because $R(p_j, p_{j+1}) = 1$ and $R(c,b) = 0$. Then $C\ell(p_k)$ lies properly between $C\ell(p_k \wedge c)$ and $C\ell(p_{k+1}) = C\ell(p_{k+1} \wedge c)$. But $p_{k+1} \wedge c$ covers $p_k \wedge c$, contradicting the fixed point covering property.

f → g. We know in general[1] that, for all pairs of elements x,y of the lattice L with y covering x,

$$R^*(\tilde{y},\tilde{x}) \leq 1 - R(x,y).$$

We fail to have the required equality $R^*(\tilde{y},\tilde{x}) = 1 - R(x,y)$

[1] supra, p.21.

only if $R^*(\tilde{y},\tilde{x}) = R(x,y) = 0$. Assume that such a pair of elements x,y exists, for which y covers x, and $R^*(\tilde{y},\tilde{x}) = R(x,y) = 0$. We prove a contradiction to the "dual fixed point" existence condition. Let w be the join of all elements z less than x, for which $R(s,z) = 0$ for all elements s covered by z. Since $R^*(\tilde{y},\tilde{x}) = 0$, w is also the join of all elements z less than y for which z covers s implies $R(s,z) = 0$. Now $R(x,y) = 0$, so we may apply the dual fixed point condition to the interval $[w,y]$, to produce an element u in the half closed interval $(w,y]$ such that for all elements s greater than or equal to w and covered by u, $R(s,u) = 0$. However, if s is any other element covered by u, then $w \not\leq s$, and $R(s \wedge w,w) = 0$, so $R(s,u) = 0$ by the translation property. Thus u is a "dual fixed point", in contradiction to the definition of the element w.

$g \to a$. We assume the differential R is inexact, and prove a contradiction to the duality condition $R^*(\tilde{v},\tilde{u}) = 1 - R(u,v)$ for some pair of elements u,v with v covering u. Assume the local graph of R on x, y, $x \wedge y$ and $x \vee y$, with x and y covering $x \wedge y$, is inexact, with $R(x,x \vee y) = 0$, the other three R values being 1. Let z be the join of all elements w less than $x \vee y$ for which w covers s implies $R(s,w) = 0$. The element z must lie beneath y, because $R(y,x \vee y) = 1$, the translation property implying $R(y \wedge z, z) = 1$ if $z \not\leq y$. Since $R(x \wedge y, y) = 1$, the same argument proves that the element z lies beneath $x \wedge y$, and hence beneath x. Thus $R^*(\widetilde{x \vee y}, \tilde{x}) = 0$, and $R^*(\widetilde{x \vee y}, \tilde{x}) < 1 - R(x, x \vee y)$. This completes the proof of our equivalence theorem.

We have listed seven properties, any one of which characterizes the property of exactness. These seven properties are phrased in terms of different concepts, and thus will find different uses in the development of a theory of exact differentials.

The local symmetry condition serves as a recursion relation for proofs by induction of global properties of exact differentials. The local symmetry condition, taken with the integrability condition, provides a starting point for an extended analogy with partial differentiation of functions of several variables, and leads to a Taylor theorem for exact differentials. This topic is taken up in the next section.

A variant of the symmetry condition,

$$R(x \wedge y, y) - R(x, x \vee y) = R(x \wedge y, x) - R(y, x \vee y)$$

shows that changes in the value of the differential always occur in pairs. The above equality is either of the form $0 = 0$ or of the form $1 = 1$. The latter occurs whenever $R(x \wedge y, y) = 1$, yet, on raising both points $x \wedge y$ and y by supremum with x, the differential value $R(x, x \vee y)$ is decreased to 0. This form of exactness condition requires a simultaneous decrease in R value as the step $[x \wedge y, x]$ is raised by one to the step $[y, x \vee y]$. This principle is the key to many proofs concerning exact differentials.

The condition for independence of path of differential sums provides an analogue to the usual theorem concerning line integrals of differentials in several real variables.

The fixed point covering property provides a natural link between the algebraic properties of a class of differentials and the structural properties of its fixed point lattices. Theorems of this type are collected in section four of the present chapter.

Perhaps the condition most promising for extensions of the theory of combinatorial independence is the condition of existence of "dual fixed points". Differentials satisfying this condition may well appear in mathematical systems having no obvious structure of independence.

The duality property simplifies the construction of the dual for exact differentials. We may invert the lattice, and interchange 0 and 1 on all steps; the resulting function on steps will extend properly to the dual differential.

§3 A Taylor Theorem for Exact Differentials

The usual test for exactness of a first order differential in n real variables involves a comparison of partial derivatives of coefficient functions. A differential of the first order is of the form

$$\sigma = \sum_{i=1}^{n} M_i(x)dx_i$$

where x is a variable n-dimensional vector $x = (x_1,\ldots,x_n)$, and $M_i(x)$ is one of n real-valued functions on this n-dimensional space. Such a differential is exact if and only if, for every pair i,j of subscripts

$$\partial M_i/\partial x_j = \partial M_j/\partial x_i.$$

Let us restrict the values of the variables x_i to the set {0,1}, and investigate whether the usual notion of exactness resembles the concept we use for differentials on lattices. Under this restriction, the n-dimensional space becomes isomorphic to the Boolean algebra of all subsets of an n element set. Let us define partial differentiation, not by limits, but by finite differences. If M is a function on the Boolean algebra, and x_i is any one of the domain variables, we define the partial derivative of M with respect to x_i as the function whose value of each subset x is given by

$$\frac{\partial M}{\partial x_i}(x) = M(x \vee x_i) - M(x).$$

The partial derivative has value zero whenever the variable x_i is in the set x. If the variable x_i is not in the set x, then the partial derivative $\frac{\partial M}{\partial x_i}(x)$ is a difference of M-values on two subsets, one of which covers the other in the Boolean algebra.

In the differential $\sigma = \sum_{i=1}^{n} M_i(x)dx_i$, each M_i is a function defined on elements of the Boolean algebra. The exactness condition suggested by the theory of real functions of several variables is that, for any pair j,k of indices, between 1 and n, $\partial M_i/\partial x_j = \partial M_j/\partial x_i$, as functions of the subset variable x. Referring to our definition in terms of finite differences, we find the condition becomes:

$$M_i(x \vee x_j) - M_i(x) = M_j(x \vee x_i) - M_j(x).$$

We have only to define, for a differential R on the Boolean algebra,

$$R(x,y) = M_i(x) \text{ where } x_i = y-x;$$

the exactness condition is then the local symmetry condition for exactness of the differential R.

The n-variable analogue of the differential R on a Boolean algebra thus has the form

$$\sigma(x) = \sum_{i=1}^{n} R(x, x \vee x_i)dx_i.$$

Such differentials act on vectors in the n-dimensional "tangent space" as linear transformations. In our theory, the n-space is 2^n, the Boolean algebra, while the vectors are subsets, and the inner product with a subset y is enumeration of elements in the set intersection with y, ie:

$$\sigma(x)\cdot y = \sum_{i=1}^{n} R(x, x \vee x_i)\chi_y(x_i)$$

$$= \lambda(y \wedge \{x_i; R(x, x \vee x_i) = 1\})$$

where λ is rank in the Boolean algebra. If we understand $R(x, x \vee x_i) = 0$ to mean "the element x_i is dependent upon the subset x", then the inner product $\sigma(x)\cdot y$ is the number of elements in y which are independent of the subset x.

If R is an exact differential, it is the first difference of a Whitney rank function r. The value of the differential R on a step $[x, x \vee x_i]$ is given by

$$R(x, x \vee x_i) = r(x \vee x_i) - r(x) = \frac{\partial r}{\partial x_i}(x).$$

It may thus be said that a differential is exact if and only if it is the differential of a function. In our theory, however, all differentials have the translation property, which causes them to resemble closures, and all "integrals" are Whitney rank functions. The extension of the theory of exact differentials to those of functions other than Whitney rank functions will not be discussed in this paper.

We will, however, make use of the notions of gradient and vectorial derivative. Assuming a differential R to be exact, and letting the function r be its Whitney rank function the set

$$\{x_i;\ R(x, x \vee x_i) = 1\}$$

associated with any subset x, is analogous to the gradient of the scalar potential function r. The vectorial derivative of the potential function r, at a subset x, with respect to a subset y, is the number of elements in y which are independent of the subset x. Partial derivatives are thus characteristic functions of inclusion relations.

Let us use the subscript notation for partial differentiation, ie: $\frac{\partial r}{\partial x_i}(x) = r_{x_i}(x)$. Considering a partial derivative of the rank function r with respect to some variable x_i as a function on the Boolean algebra, we may define a second partial derivative with respect to x_i, then x_j, for $x_i \neq x_j$, by

$$r_{x_i x_j} = r_{x_i}(x \vee x_j) - r_{x_i}(x);$$

higher partial derivatives are defined accordingly. Expanding this relation, we find

$$r_{x_i x_j} = r(x \vee x_i \vee x_j) - r(x \vee x_j) - r(x \vee x_i) + r(x)$$

and for a third order derivative, with no two of x_i, x_j, x_k being the same variable,

$$\begin{aligned} r_{x_i x_j x_k} = {} & r(x \vee x_i \vee x_j \vee x_k) - r(x \vee x_i \vee x_j) \\ & - r(x \vee x_i \vee x_k) + r(x \vee x_i) \\ & - r(x \vee x_j \vee x_k) + r(x \vee x_j) \\ & + r(x \vee x_k) - r(x). \end{aligned}$$

These formulae are symmetrical in the subscript variables, so the results are independent of the order of the subscripts, and hence depend only upon the set of variables present as subscripts. We may thus write a set y as subscript, to indicate successive partial differentiation, in any order, with respect to the various elements x_i in the set y, ie:

$$r_y(x) = r_{x_i x_j \ldots x_k}(x) \text{ where } y = x_i, x_j, \ldots, x_k.$$

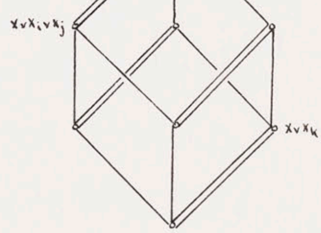
By means of the connection:

$$R(x,z) = r(z) - r(x) \text{ when } z \text{ covers } x,$$

between an exact differential and its Whitney rank function, we may express the partial derivatives r_{x_i}, $r_{x_i x_j}$, $r_{x_i x_j x_k}, \ldots,$ in terms of the values of the differential R, as follows:

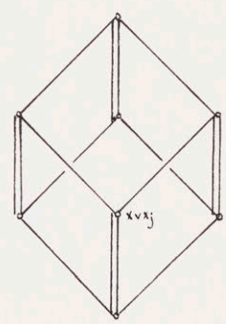
$$\begin{aligned} r_{x_i} &= R(x, x \vee x_i) \\ r_{x_i x_j} &= R(x \vee x_j, x \vee x_i \vee x_j) - R(x, x \vee x_i) \\ r_{x_i x_j x_k} &= R(x \vee x_i \vee x_j, x \vee x_i \vee x_j \vee x_k) \\ &\quad - R(x \vee x_i, x \vee x_i \vee x_k) - R(x \vee x_j, x \vee x_j \vee x_k) \\ &\quad + R(x, x \vee x_k). \end{aligned}$$

These computations are the alternating sums of differential values on all steps parallel to the step connecting the subset $x \vee x_i \vee x_j$, a coatom in the interval $[x, x \vee x_i \vee x_j \vee x_k]$, to the



supremum of the interval. Let us refer to such a step as a coatomic step. The signs of terms alternate in accordance with the difference in rank between the upper end of each parallel step and the supremum of the interval. Since the lattice is a Boolean algebra, these steps are also those parallel to the atomic step $[x, x \vee x_k]$.

It is a curious consequence of exactness of the differential that such alternating sums are the same if another variable appears last in the sequence of differentiations, so that differential values are taken on steps parallel to a different atomic step, eg: $[x, x \vee x_j]$.



A Taylor theorem for Whitney rank functions on a Boolean algebra might well read

$$r(1) = r(0) + \sum_i r_{x_i}(0) + 1/2! \sum_{i \neq j} r_{x_i x_j}(0) + \ldots$$

$$= \sum_{p=0}^{n} 1/p! \sum_{i_1, \ldots, i_p} r_{x_{i_1} x_{i_2} \ldots x_{i_p}}(0)$$

where the interior summation is over all orderings of the n variables, taken p at a time. Using independence of order for partial differentiation and employing the set-subscript notation, we obtain a simpler formula, because there are p! equal terms associated with any subset $y = \{x_{i_1}, \ldots, x_{i_p}\}$. The resulting formula we shall state as a theorem.

Theorem. Taylor's Theorem for Exact Differentials on a Boolean Algebra: If a differential R, on a Boolean algebra L of all subsets of an n-element set, is exact, with Whitney rank function r, and if r_y is the result of successive partial differentiation of the function r with respect to the elements of the subset y, taken in any order, then

$$r(1) = \sum_{y \in L} r_y(0).$$

Proof: The statement $r(x) = \sum_{y \leq x} r_y(0)$ is equivalent to the statement $r_y(0) = \sum_{x \leq y} \mu(x,y) r(x)$ $= \sum_{x \leq y} (-1)^{\lambda(y)-\lambda(x)} r(x)$ by the Möbius inversion formula[1], and the fact that $\mu(x,y) = (-1)^{\lambda(y)-\lambda(x)}$ for $x \leq y$ in a Boolean algebra. The latter statement, $r_y(0)$ $= \sum_{x \leq y} (-1)^{\lambda(y)-\lambda(x)} r(x)$, is the definition of the partial derivative.

If this theorem is to hold for exact differentials on any finite complete modular lattice, we must define

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[1] G.-C. Rota, "On the Foundations of Combinatorial Theory", Zeitschr. fur Wahrshein., 2 (1964), pp. 340-360 §3.

the partial derivative at zero of a Whitney rank function r with respect to the lattice element 1 by

$$r_1(0) = \sum_x \mu(x,1)r(x)$$

and the partial derivative at zero with respect to a general lattice element y by

$$r_y(0) = \sum_x \mu(x,y)r(x).$$

Proposition. Partial Differentiation of Exact Differentials: Let R be an exact differential on a finite complete modular lattice L, and let r be its Whitney rank function. Let c be any coatom of the lattice L, and let e be any atom of L. Then the partial derivatives at zero with respect to the element 1 may be computed as either

$$r_1(0) = \sum_x \mu(x,1)R(x \wedge c,x)$$

or

$$r_1(0) = (-1)^{\lambda(1)} \sum_x \mu(0,x)R(x,x \vee e).$$

Proof: Let us consider the first formula. The terms corresponding to elements x of the lattice which lie beneath the coatom c all vanish, because $R(x \wedge c,x) = R(x,x) = 0$. For elements x which do not lie beneath c, we have $R(x \wedge c,x) = r(x) - r(x \wedge c)$. We must prove

$$r_1(0) = \sum_{x \not\leq c} \mu(x,1)r(x) - \sum_{x \not\leq c} \mu(x,1)r(x \wedge c).$$

This is true if we can show

$$\sum_{x \leq c} \mu(x,1)r(x) = - \sum_{x \not\leq c} \mu(x,1)r(x \wedge c),$$

which, in turn, is established if we show

$-\mu(x,1)$ = the sum of values $\mu(z,1)$ for elements z covering x but not lying beneath some coatom c above x.

This statement will be proven for x = 0 as a lemma, since its validity depends only upon the lower semimodularity of the lattice L. Its lattice-inverted formulation will be of great utility in analysis of fixed point lattices for exact differentials, all of which will be shown to be semimodular.

Lemma. A Recursion Satisfied by Möbius Functions on Semimodular Lattices: If L is a finite semimodular lattice with Möbius function μ, and if e is an arbitrary atom in the lattice L,

$-\mu(0,1)$ = the sum of values $\mu(0,c)$ for all coatoms c not above the atom e.

Inversely, if L is a finite lower semimodular lattice with Möbius function μ, and c is an arbitrary coatom in L,

$-\mu(0,1)$ = the sum of values $\mu(e,1)$ for all atoms e not beneath the coatom c.

Proof: We shall prove the first form of our lemma. The Möbius function is defined as the inverse of the zeta function, ie:

$$\delta_{x,y} = \sum \mu(0,x)\zeta(x,y) = \sum_{x \le y} \mu(0,x)$$

so the function μ is characterized by its property that the sum of $\mu(0,x)$ over any lattice interval $[0,y]$ is zero, unless $y = 0$. Let $c_1,\ldots,c_k$ be an enumeration of the coatoms of the lattice L which lie above the atom e. The sum of values $\mu(0,x)$ for all elements which are less than or equal to at least one of the coatoms c_i, $i = 1,\ldots,k$, may be expressed, by the principle of inclusion-exclusion, as the sum

$$\sum_i \sum_{x \le c_i} \mu(0,x) - \sum_{i<j} \sum_{x \le c_i \wedge c_j} \mu(0,x)$$
$$+ \sum_{i<j<m} \sum_{x \le c_i \wedge c_j \wedge c_m} \mu(0,x) - \ldots$$

where each subscript varies from 1 to k. Since each initial summation is over an entire interval, all sums are zero. Since this sum is zero, and the sum over all elements x in L of $\mu(0,x)$ is also zero, the difference between these, ie: the sum of $\mu(0,x)$ over all elements x which are less than or equal to no coatom above the atom e, is also zero. If an element x is beneath no coatom above the atom e, then $e \not\leq x$, and $x \vee e$ is beneath no coatom, so $x \vee e$ must be the element 1. By the semi-modular assumption for the lattice L, $x \vee e$ covers x, so x is a coatom. Thus the sum of $\mu(0,x)$ over all coatoms x not above the atom e, plus $\mu(0,1)$, is zero, yielding the first form of our lemma. The second form is the inversion of the first, since the Möbius function is the same for a lattice and its inverse, ie:

$$\mu(x,y) = \tilde{\mu}(\tilde{y},\tilde{x}).$$

This completes the proof of our lemma, and establishes the validity of the expression for the partial derivative of a Whitney rank function on a finite complete modular lattice. Partial differentiation was defined in such a way as to make the Taylor theorem hold in the extension from Boolean algebras to finite complete modular lattices. Thus the substance of our general Taylor theorem is embodied in the following three formulae, which are given first in absolute, then in relative terms:

a) Definition of the highest order partial derivative:

$$r_1(\mathbf{o}) = \sum_x \mu(x,1)r(x).$$

b) Characterization of the highest order partial derivative:

$$r_1(0) = \sum_x \mu(x,1)R(x\wedge c,x), \text{ for any coatom } c$$
$$= (-1)^{\lambda(1)}\sum_x \mu(0,x)R(x,x\vee e), \text{ for any atom } e.$$

c) Taylor theorem:

$$r(1) = \sum_x r_x(0).$$

Relative formulae:

a') $r_y(z) = \sum_{z\le x\le y} \mu(x,y)r(x).$

b') $r_y(z) = \sum_{z\le x\le y} \mu(x,y)R(x\wedge c,x)$ for any coatom c of the interval $[z,y]$.

$= (-1)^{\lambda(y)-\lambda(x)} \sum_{z\le x\le y} \mu(z,x)R(x,x\vee e)$ for any atom e of the interval $[z,y]$.

c') $r(y) = \sum_{z\le x\le y} r_x(z).$

§4 Structure of Fixed-Point Lattices

The principal conceptual link between differentials and their lattices of fixed points involves the properties of exactness and semimodularity. MacLane[1] and Dilworth[2]

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[1] S. MacLane, "A Lattice Formulation for Transcendence Degrees and p-Bases", Duke J. 4(1938) pp455-468.

[2] R.P. Dilworth, "Dependence Relations in a Semimodular Lattice", Duke J. 11(1944) pp575-587.

both state theorems of this type; we shall provide proofs which extend the validity of their claims.

Given a differential R on a finite complete modular lattice L, let us denote by h the mapping from elements x of the lattice L to the image in the fixed point lattice L/R of the closure $C\ell(x)$, the meet of all fixed points above x. Let the inverse mapping h^{-1} be the inclusion map from L/R into L. Since the meet of fixed points is also a fixed point, the inverse mapping h^{-1} is a meet homomorphism. The join of elements x and y in the fixed-point lattice yields the image under h of the meet of all fixed points above both $h^{-1}(x)$ and $h^{-1}(y)$, ie: the element $h(C\ell(h^{-1}(x) \vee h^{-1}(y)))$ in L/R.

Proposition. Exactness and Semimodularity: If a differential R on a finite complete modular lattice L is exact, the fixed point lattice L/R is semimodular.

Proof: We must establish the semimodular property for L/R, namely

> If, for a pair of elements x,y of L/R,
> y covers $x \wedge y$, then $x \vee y$ covers x.

Let x and y be a pair of elements in the lattice L/R, for which y covers $x \wedge y$. $h^{-1}(x) \wedge h^{-1}(y) = h^{-1}(x \wedge y) < h^{-1}(y)$ in the lattice L, so we may choose an element z of L covering $h^{-1}(x) \wedge h^{-1}(y)$, and lying beneath $h^{-1}(y)$. Since $h^{-1}(x) \wedge h^{-1}(y)$ is a fixed point, $C\ell(z)$ satisfies the ordering $h^{-1}(x) \wedge h^{-1}(y) < z \leq C\ell(z) \leq h^{-1}(y)$. Thus $x \wedge y < h(C\ell(z)) \leq y$, so $C\ell(z)$ is the point $h^{-1}(y)$. By appealing to both the modularity of the lattice L and the fixed point covering property of exact differentials, we show that the element $x \vee y$ covers the element x in L/R. Since the element z covers $h^{-1}(x) \wedge h^{-1}(y)$ and is less than $h^{-1}(y)$, z is not less than $h^{-1}(x)$, and consequently $z \vee h^{-1}(x)$ covers $h^{-1}(x)$, by the modularity of the lattice L. On one hand, $C\ell(z \vee h^{-1}(x)) \geq C\ell(z) = h^{-1}(y)$ and $C\ell(z \vee h^{-1}(x)) \geq C\ell(h^{-1}(x)) = h^{-1}(x)$ so the closure $C\ell(z \vee h^{-1}(x))$ must be greater than or equal to $h^{-1}(x) \vee h^{-1}(y)$, and hence greater than $C\ell(h^{-1}(x) \vee h^{-1}(y)) = h^{-1}(x \vee y)$. On the other hand, the fixed point covering property of exact differentials implies $h(z \vee h^{-1}(x)) = h(C\ell(z \vee h^{-1}(x)))$ $= h(z \vee h^{-1}(x))$ at most covers x in L/R. Since the element $h(z \vee h^{-1}(x))$ at the same time lies above $x \vee y$ and covers x, $x \vee y$ must cover x. This completes the proof of semimodularity.

A partial converse to this proposition was proven in Chapter II[1], under the assumption that all atoms of the

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[1]supra, Chapter II, §4, p48.

lattice L are fixed points. In that event, the differential R is exact if and only if the fixed-point lattice is semimodular.

Exactness of a differential does not, however, imply that the fixed point lattice is geometric. For a simple counterexample, consider the lattice L of integers $\{0,1,2\}$ ordered $0 < 1 < 2$, and the differential $R(0,1) = R(1,2) = 1$. Since all elements are fixed-points, the fixed point lattice L/R is isomorphic to L, which is semimodular, but not geometric.

Proposition. Exactness and Geometric Lattices: If a differential R on a finite complete and complemented modular lattice is exact, the fixed point lattice L/R is geometric.

Proof: The previous proposition shows the lattice L/R is semimodular. It remains to prove that every element of L/R is a join of atoms, ie: if an element y covers an element x in L/R, there is an atom e of L/R such that $x \vee e = y$. Again letting h and h^{-1} represent the canonical mappings between the lattices L and L/R, and using the fact that in the complemented modular lattice L every element is a join of atoms, we may choose an atom e of L which is less than $h^{-1}(y)$ but not less than $h^{-1}(x)$. The closure $C\ell(e)$ is also not less than $h^{-1}(x)$, but less than $h^{-1}(y)$, since the latter is a fixed point. By the fixed-point covering property of exact differentials, $h(e) = h(C\ell(e))$, which cannot be zero in L/R, is an atom of L/R, lying beneath y but not beneath x.

Whenever the domain of the differential is a complemented modular lattice, as in the above proposition, we also know the lattice structure of L/R relative to coatoms. If the image in L/R of any fixed point is meet-irreducible, the image is covered by the element 1 of L/R.

Proposition. Meet of Coatoms Property for a Fixed-Point Lattice: Let L be a complete finite complemented modular lattice, and let R be an exact differential on L. Then every element of the fixed point lattice L/R is a meet of coatoms of L/R.

Proof: Assume x is a fixed point of R in L, the image of which is meet irreducible in L/R. There is a unique fixed point y in L whose image in L/R covers the image of x. If the element z is any complement of the element y in the interval [x,1], let u be any element of L covering x, with $x < u \leq z$. By the fixed point covering property of exact differentials, the image of $C\ell(u)$ in L/R covers the image of x, so $C\ell(u)$ must be the fixed point y. $u \leq C\ell(u)$ implies $u \leq y$, contradicting the complementary property of the element z, that $z \wedge y = x$. Thus the complement of the fixed point y in the interval [x,1] must be x, and y must be the element 1 of L, the image of which in L/R is the element 1 of that lattice.

§5 Graph-like Properties of Exact Differentials

In the manner of Hassler Whitney, let us now see which of the properties to be expected of a differential of a graph relation actually are true for all exact differentials.

The transition to this more general context is accomplished by the following conventions. We replace the statement "the edge e is dependent upon the subset x of the edge set X" by "the join-irreducible element e of the lattice L and the element x of L have the property that $R(x,x \vee e) = 0$". Bonds and circuits are in some sense the "complements" of meet irreducible elements of the fixed-point lattices L/R and $\tilde{L}/R^*$, respectively.

Perhaps the most striking property of exact differentials is that derived from the statement "if an edge e is dependent upon a subset x, but upon no smaller subset, then x is a circuit."

Proposition. Characterization of Meet-Irreducible Fixed-Points: Let R be an exact differential on a finite complete modular lattice L. If an element y of L covers an element x, and the step $[x,y]$ is maximal among parallel steps with respect to the property $R(x,y) = 1$, then the element x, necessarily a fixed point of the differential R, is meet irreducible in the lattice L/R.

Proof: The precise statement of the maximal condition is as follows: y covers x, $R(x,y) = 1$, and for all elements z with $x < z$, $y \not\leq z$, $R(z,z\vee y) = 0$.

Assume an element y covers an element x in L, $R(x,y) = 1$, and for all elements z properly above x but not above y, $R(z,z\vee y) = 0$. Such an element x must be a fixed point, because if an element $w \neq y$ covers x, with $R(x,w) = 0$, then $R(y,w\vee y) = 0$ by the translation property, while our assumption implies $R(w,w\vee y) = 0$. Such a local graph is not allowable in a differential.

If we show that all elements covering the element x have the same closure, it will follow from the fixed point covering property of exact differentials that the image of x in L/R is covered by exactly one element of L/R, and hence is meet-irreducible.

Let z be any element of L covering the element x. If $z \neq y$, $z \vee y$ covers both z and y, by the modularity of L. The local graph on $\{x,z,y,z\vee y\}$ must be prime, since R is exact, $R(x,z) = R(x,y) = 1$, and $R(z,z\vee y) = 0$ by our maximality assumption concerning x. Thus $C\ell(z) = C\ell(y)$ is true for all z covering x. This completes the proof.

The lattice inverse of this proposition is equally true; we state it separately because it is helpful in discussions of duality.

Corollary. Meet Irreducible Fixed Points of the Dual: Let R be an exact differential on a finite complete modular lattice L. If an element y covers an element x, and the step [x,y] is minimal among parallel steps with respect to the property $R(x,y) = 0$, then the element has as image under lattice inversion an element whose image in the lattice $\tilde{L}/R^*$ is meet irreducible. (Such an element y necessarily has the property, for all elements w of L, that

$$y \text{ covers } w \text{ implies } R(w,y) = 0.)$$

The classical theorems for linear independence in finite-dimensional vector spaces are available for exact differentials on a complemented modular lattice, but fail for such simple modular lattices as linear orderings.

An element x of a finite complete modular lattice L is <u>independent</u>, relative to an exact differential R defined on the lattice L, if the value of the Whitney rank function r at x is equal to the rank λ of the element x in the lattice L. For any element x, the inequality $r(x) \le \lambda(x)$ applies, since r(x) is the sum of R-values on any path from 0 to x, so we may define a <u>dependent</u> element as an element x for which $r(x) < \lambda(x)$.

Proposition. Differential Character of Independent Elements: Let L be a finite complete complemented modular lattice, and R be an exact differential defined on the lattice L. An element x in L is independent if and only if, for any element z of L

$$x \text{ covers } z \text{ implies } R(z,x) = 1.$$

Proof: If the element x is independent, let z be any element covered by x. Choose a path p from 0 to x via z. Since $\lambda(x) = r(x) = \sum_{i=1}^{n} R(p_{i-1},p_i)$,

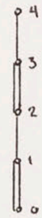
$1 = R(p_{n-1}, p_n) = R(z,x)$. Conversely, $r(x) < \lambda(x)$. Choose a path p from 0 to x. For some subscript k, $1 \leq k \leq n$, $R(p_{k-1}, p_k)$ must have value zero. The element x covers any complement z of the element p_k in the interval $[p_{k-1}, x]$, and $R(z,x) = 0$ by the translation property.

A dual concept and proposition are also available. Define an element x in a finite complete modular lattice L to be a _spanning_ element (or an element which _spans_), relative to an exact differential defined on L, if and only if the value of the Whitney rank function r at x is equal to its value at the element 1 in L.

Corollary. Differential Character of Spanning Elements: Let L be a finite complete complemented modular lattice, and R be an exact differential defined on the lattice L. An element x in L is a spanning element if and only if, for any element z of L

$$z \text{ covers } x \text{ implies } R(x,z) = 0.$$

A base (or basis) for an exact differential on a finite complete modular lattice may be defined alternatively as a maximal independent element, a minimal dependent element, or as an independent element which spans. No two of these concepts are equivalent, as may be seen in the example illustrated in the accompanying diagram. None of the elements is a base, while 1 is the only maximal independent element and 3 is the only minimal spanning element.



Let us define a base as an independent element which spans, and prove that all the suggested definitions are equivalent on a complemented modular lattice.

Proposition. Differential Character of Bases: Let L be a finite complete complemented modular lattice, and let R be an exact differential defined on the lattice L. Then an element x of L is a base if and only if

a) x is a maximal independent element,

b) x is a minimal spanning element,

c) For any element z of L,

$$x \text{ covers } z \text{ implies } R(z,x) = 1, \text{ and}$$
$$z \text{ covers } x \text{ implies } R(x,z) = 0, \text{ or}$$

d) x is minimal with respect to the property: x at least covers $x \wedge z$ for every fixed point z, the image of which is a coatom in the lattice L/R.

Proof: The equivalence, with the definition of a base, of statements a,b, and c follows from arguments analogous to that given for the proposition on the differential character of independent elements. We omit the proof.

Statement d is the lattice counterpart of the property of bases for a graph, that they are minimal matchings of the family of edge sets constituting bonds (or that the base complements are minimal matchings of the family of circuits).

Assume an element x in the lattice L is a base, and an element z in L is a fixed point whose image in the lattice L/R is a coatom. Then the Whitney rank function r of R has value $r(z) = r(1) - 1$, by the proposition on existence of fixed points[1]. Since $r(x) = r(1)$, x cannot satisfy $x \leq z$, so x at least covers $x \wedge z$.

Conversely, assume an element x is minimal with respect to the property that x at least covers $x \wedge z$ for any fixed point z, the image of which in the lattice L/R is a coatom. If y is any element covering x, $R(x,y) = 1$ would imply the existence of a fixed point in the half-closed interval $[x,1)$. A maximal fixed point w in the interval $[x,1)$ necessarily has a coatom as image in the lattice L/R. But $x \wedge w = x$, contradicting

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[1]supra, Chapter I, §3, p12.

our condition on the element x. On the other hand, if y is any element of L covered by x, the minimality condition implies the existence of a fixed point above y but not above x, so $R(y,x) = 1$. Thus the element x is a base.

All the classical theorems on linear independence become properties of bases for exact differentials on finite complete complemented modular lattices. For instance, the theorem "a set of non-zero vectors $\{\alpha_1,\ldots,\alpha_m\}$ in a vector space is linearly dependent if and only if some one of the vectors is a linear combination of the preceding ones" becomes the differential statement "if an element x of the lattice L is dependent, then along any path p from 0 to x, $R(p_{i-1},p_i) = 0$ for some value of the subscript i".

Proposition. Theorems on Linear Independence, Rephrased for Differentials: Let L be a finite complete complemented modular lattice and let R be an exact differential on the lattice L, with Whitney rank function r.

i) If an element y in L is independent, there exists a base x for R in L such that $y \le x$.

ii) If an element x, with rank $\lambda(x)$ in the lattice L, is a base for R, then $\lambda(x) = r(1)$.

iii) Any lattice element of rank λ greater than $r(1)$ is dependent; any element of rank λ less than $r(1)$ does not span.

iv) For a lattice element x of rank $\lambda(x) = r(1)$ to be a base for R, it is sufficient that it span or be independent.

Proof:

i) Let y be an independent element. If y does not span, there is some element z covering y such that $R(y,z) = 1$. If u is any element, other than y, which is covered by z, the elements $y,z,u,u \wedge y$ form a local grpah. Since y is independent, $R(u \wedge y,y) = 1$, so the local graph is of type one, and $R(u,z) = 1$. Thus the element z covering y is also independent. This possibility allows for a recursive definition of a path upward from the element x, all elements of which are independent. This path terminates only in an independent element which spans, ie: in a base for R above x.

ii) If an element x is a base for the differential R in the lattice L, we may choose a path p from 0 to 1 via x. Given any step $[p_{k-1},p_k]$ in the path p above the base x, we may choose an element z covering x such that $z \not\leq p_{k-1}$ but $z \leq p_k$, because the lattice L is complemented. If the differential R were to have value 1 on such a step $[p_{k-1},p_k]$ above x, the translation property would imply $R(x,z) = 1$, contradicting the properties of x as a base. An argument of the same type implies that $R(p_{k-1},p_k) = 1$ for all steps $[p_{k-1},p_k[$ in the path p below the base x. Thus the R-sum $r_p(0,1)$, which must have value $r(1)$, counts the number of steps in the path p below the base x. Thus number is the rank $\lambda(x)$ of the element x in the lattice L.

iii) If, for an element x, $\lambda(x) > r(1)$, it follows that $\lambda(x) > r(x)$, so x is dependent. Since $r \leq \lambda$ for all elements of L, the condition $\lambda(x) < r(1)$ implies $r(x) < r(1)$, so x does not span.

iv) Assume an element x is independent, and $\lambda(x) = r(1)$. Then $\lambda(x) = r(x)$, so $r(x) = r(1)$, and x spans. On the other hand, if the element x spans, and $\lambda(x) = r(1)$, then $r(x) = r(1)$, so $r(x) = \lambda(x)$, and x is independent.

To these properties of bases we may add the exchange property used by Whitney[1] in constructing equivalent axiomatizations for independence systems.

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[1]H. Whitney, "On the Abstract Properties of Linear Dependence", Amer. J. 57(1935) pp. 509-533.

Proposition. Exchange Properties of Bases: Let L be a finite complete complemented modular lattice, and let R be an exact differential on L. If x and y are bases for R in L, and if w is any element covered by x, there is a base z covering w and lying beneath the element $w \vee y$.

Proof: Choose a path p from w to $y \vee w$. Since $y \vee w$ lies above the base y, $r(y \vee w) = r(1)$. Since $r(w) = r(1) - 1$, there is some step $[p_{i-1}, p_i]$ in the path p for which $R(p_{i-1}, p_i) = 1$. Let z be the complement of p_{i-1} in the interval $[w, p_i]$. Then z covers w, and $R(w,z) = 1$ by the translation property. $\lambda(z) = r(z) = r(1)$, so z is a base.

The corresponding exchange property for coatomic (in L/R) fixed points is derived from that given by Whitney for circuits.

Proposition. Exchange Property of Coatomic Fixed Points: Let R be an exact differential on a finite complete complemented modular lattice L. If, for some pair of coatomic (in L/R) fixed points δ_1 and δ_2 there are elements x and y in L covering the element $\delta_1 \wedge \delta_2$, such that x is beneath δ_1 but not beneath δ_2, and y is beneath neither δ_1 nor δ_2, then there is a coatomic fixed point δ_3 which is above the element y but not above the element x.

Proof: Assume δ_1 and δ_2 are coatomic (in L/R) fixed points of R in L and that two elements x and y cover $\delta_1 \wedge \delta_2$, with x beneath δ_1 but not beneath δ_2, and y beneath neither δ_1 nor δ_2. Consider the local graph formed on the elements $\delta_1 \wedge \delta_2, x, y$, and $x \vee y$. The two lower steps of this local graph have R value 1, since $\delta_1 \wedge \delta_2$ is a fixed point. By the translation property, and the fact that y is not beneath the fixed point δ_1, $R(x, x \vee y) = R(\delta_1, \delta_1 \vee y) = 1$. The differential R being exact, the local graph is of type one. Thus R has value 1 on the step $[y, x \vee y]$. Let z be an element of L in the interval $[y, 1]$, maximal with respect to the property that $R(z, z \vee x) = 1$. By our characterization of meet irreducible fixed points[1], the element z is a coatomic (in L/R) fixed point of R lying above y but not above x.

To complete our catalogue of graph-like properties of exact differentials, let us turn to the graph theorem that if a single edge is added to a base, there is a unique circuit in the enlarged edge set. This principle,

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[1] supra, p. 86.

introduced by Whitney[1], has been used by W.T. Tutte throughout his work on chromatic polynomials[2] and the homotopy theory for Whitney systems[3].

Proposition. Fundamental Systems[4] of Coatomic Fixed Points: Let L be a complete finite complemented modular lattice, and let R be an exact differential on L.

i) If an element x is a base for R in L, and x covers an element z, there is a unique coatomic (in L/R) fixed point above z in the lattice L.

ii) If an element x is a base for R in L, and an element y covers x, there is a unique coatomic (in $\tilde{L}/R^*$) dual fixed point beneath y in the lattice L.

iii) The Duality Principle for Exact Differentials: Assume an element x is a base for R in L, and an element y has the property that $x \vee y$ covers both x and y. Let δ be the unique coatomic (in L/R) fixed point above $x \wedge y$, and γ the unique coatomic (in $\tilde{L}/R^*$) dual fixed

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[1] H. Whitney, op. cit.

[2] W.T. Tutte, "A Contribution to the Theory of Chromatic Polynomials", Can. J. 6(1954) pp. 80-91.

[3] W.T. Tutte, "A Homotopy Theorem for Matroids, I and II", Trans. A.M.S., 88(1958) pp. 144-174.

[4] cf: This term is due to Whitney, op cit., p. 517.

point beneath $x \vee y$. The following statements are equivalent:

a) y is a base

b) $y \wedge \delta \leq x$

c) $x \leq y \vee \gamma$.

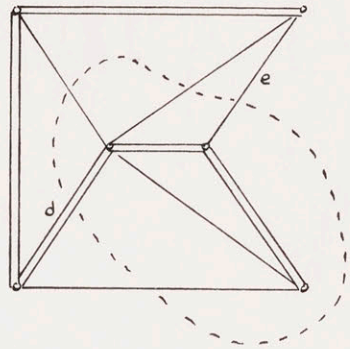
Proof:

i) We have proven the meet of coatoms property for fixed-point lattices of exact differentials on finite complete complemented modular lattices. If an element x is a base for the differential R on L, and x covers an element z, then $R(z,x) = 1$. By the existence proposition for fixed points, there is a fixed point u in the half-closed interval $[z,1)$. Choose a path from the image of u to 1 in the lattice L/R. The penultimate element in this path is a coatom, and is the image of a fixed point w above the element z in L. If there are two such fixed points w_1 and w_2 above z in L, and both have coatomic images in L/R, they are not comparable in the lattice L, so their infimum $w_1 \wedge w_2$ is a lower fixed point also above z in L. Choose a path q from 0 to z, then to $w_1 \wedge w_2$, then to w_1, and finally to 1. The differential sum $r_q(0,1)$ is $r(1) - 1$ from 0 to z, and is at least 2 from z via $w_1 \wedge w_2$ and w_1 to 1, contradicting the independence of path for R-sums of exact differentials. Thus the coatomic fixed point above z is unique.

ii) This proposition is the dual of statement i), and is proven by applying statement i) to the dual differential R^*, which is also exact. Note that if an element x is a base for R, the image $\tilde{x}$ of x in the inverted lattice $\tilde{L}$ is a base for R^*.

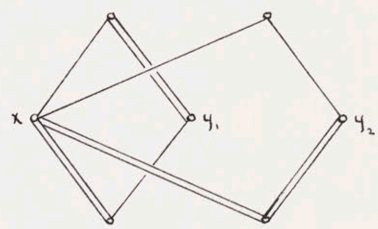
iii) We prove that (a) y is a base if and only if (b) $y \wedge \delta \leq x$. The equivalence (a) $\leftrightarrow$ (c) is the dual of the equivalence we prove, and thus must also hold. If the element y is a base, then $R(x \wedge y, y) = 1$. The half closed interval $[x \wedge y, \delta)$ can contain no fixed point, since δ is the only coatomic fixed point above $x \wedge y$ and every fixed point is a meet of coatomic fixed points. By the proposition concerning existence of fixed points, R must have constant value 0 on the interval $[x \wedge y, \delta]$, so the element y cannot lie beneath δ. Thus $y \wedge \delta = x \wedge y \leq x$. Conversely, if y is an element such that $x \vee y$ covers both x and y, δ is the unique coatomic fixed point above $x \wedge y$, and $y \wedge \delta \leq x$, we know $y \wedge \delta = x \wedge y$ is covered by y, so $y \vee \delta$ covers δ. Since δ is a fixed point, $R(\delta, y \vee \delta) = 1$. By the translation property, $R(x \wedge y, y) = 1$, so $\lambda(y) = r(y) = 1 + r(x \wedge y) = r(1)$, and y is a base.

The duality principle for exact differentials is of critical importance in our enumerative work comprising Chapter IV. Its usefulness arises from the fact that it relates the fixed points coatomic in L/R to the dual fixed points coatomic in $\tilde{L}/R^*$, via the set of bases for the differential R. The duality principle, applied to the graph relation of a planar graph, is illustrated in the accompanying diagram. A base for the graph



is marked in red. If the edge e is added to the base, the enlarged set contains a unique circuit, which involves the edge d. The duality principle implies that if the edge d is removed from the base, the complement of the resulting edge set contains a unique bond (or dual circuit), which involves the edge e. This is marked with a dotted line.

One additional observation should be made concerning the duality principle. Assume, as in the statement of the proposition, that an element x in the finite complete complemented modular lattice L is a base for the exact differential R, and that an element y has the property that $x \vee y$ covers both x and y. Since x is a base,



$R(x \wedge y, x) = 1$ and $R(x, x \vee y) = 0$, the local graph on the elements $x, y, x \wedge y$ and $x \vee y$ must be mixed or prime. The element y is a base if and only if this local graph is prime.

§6 Factorization into Primes

Before proceeding to the enumerative work of Chapter IV, let us prove a unique decomposition property of differentials.

Proposition. Unique Decomposition into Prime Factors: Let L be a finite complete modular lattice and let R be a differential defined on the lattice L. If $R = R_1 \times \ldots \times R_p$ and $R = R_{1'} \times \ldots \times R_{q'}$ are two factorizations of R into prime factors, there is a 1-1 correspondence between the two sets of factors under which corresponding factors are isomorphic.

Proof: If the two decompositions are essentially different, then some y of R on L, corresponding to a differential $R_{i'}$, has a non-trivial intersection with a factor x of R on L, corresponding to a differential R_j, ie: $0 < x \wedge y < x$. Given any step $[u,w]$ in the interval $[0,x]$, we find $u \wedge y < w \wedge y$ if and only if $u \wedge (x \wedge y) < w \wedge (x \wedge y)$ and $u \vee y < w \vee y$ if and only if $u \vee (x \wedge y) < w \vee (x \wedge y)$.

Since y is a factor, $R(u\wedge y,w\wedge y) = R(u\wedge(x\wedge y),w\wedge(x\wedge y))$ and $R(u\vee y,w\vee y) = R(u\vee(x\wedge y),w\vee(x\wedge y))$ so $R(u,w) = \sup\{R(u\wedge(x\wedge y),w\wedge(x\wedge y)),R(u\vee(x\wedge y),w\vee(x\wedge y))\}$ and $x\wedge y$ is a factor of the differential R. This contradicts the assumption that x is a prime factor, so the decomposition must be unique up to a 1-1 correspondence in which corresponding terms differ at most by a differential-preserving isomorphism.

This proposition, although it was suggested by Saunders MacLane's paper[1] on factorization of graphs, and although it applies to more general structures than graphs, does not exhaust the possibilities of generalizing his results, which concern separation across connected subgraphs.

By an inductive use of our observation in Chapter I[2], the prime factors of an exact differential are all exact.

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[1] S. MacLane, "Some Unique Separation Theorems for Graphs", Amer. J. 57(1935), pp. 805-820.

[2] supra, Chapter I, §4e, p. 23.

Chapter IV

§1 Introduction

The content of this chapter is enumerative in nature. By defining a rank generating function for exact differentials, we bring to bear on the theory of exact differentials the techniques of enumerative combinatorial analysis. Emphasis will be laid on those special numerical results available for exact differentials defined on Boolean algebras.

In section two we define the rank generating function of an exact differential as a certain polynomial in two variables, arising from a simultaneous rank grading in the fixed point lattice of the differential and in the fixed point lattice of its dual differential. We prove a fundamental relation which helps to characterize the structure of the domain of the differential.

A discussion of the algebraic properties of the rank generating function comprises section three. We define the relative rank generating function is such a way that, at any numerical evaluation of its two variables, it becomes a matrix element of the incidence algebra of the domain lattice. We prove that the existence of factors of the differential implies factorization of the rank generating function. We exhibit a recursion for the relative rank generating function, in terms of the domain lattice, then calculate the inverse of the relative rank generating function as a function element in the incidence algebra of the domain lattice.

The principal theorem on rank generating functions appears in section four. We prepare the way with a sequence of lemmas which serve to match subsets in a Boolean algebra to bases for an exact differential defined on that Boolean algebra. The substitution theorem establishes the fact that translation of the domain of the rank generating function produces a variety of generalizations of well-known lattice polynomials.

In section five we show that the Möbius function of the fixed point lattices of a differential and of its dual are values of the rank generating function.

An application of our theory to the enumeration of graphs concludes this paper.

§2 Rank Generating Functions

The principal tool of enumerative combinatorial analysis is the generating function of a graded set, a concept introduced by Laplace[1]. A grading of a set S is a function g from the set S into the natural numbers {0,1,2,...} such that any given integer is the image of at most a finite number of elements of S. Such a grading defines a counting sequence ν from the natural numbers into the natural numbers, defined in terms of the grading by

$\nu(j)$ is the number of elements of grade j in the set S.

The (ordinary) generating function γ of the grading g on the set S is defined by

$$\gamma(\xi) = \sum_{i=1}^{\infty} \nu(i)\xi^{i}.$$

Many variants of this concept are available. Generally, they involve either multiple gradings, or the use of functions other than powers of the variable in the definition of the generating function. We shall deal

[1] Laplace, "Théorie Analytique des Probabilites", Courcier, Paris, 1812; 3rd ed. 1820.

with ordinary generating functions of a bi-grading.

We have shown that an exact differential on a finite complete modular lattice is associated with a unique Whitney rank function on the lattice. The same may be said for the dual of such a differential. The Whitney rank functions of an exact differential and of its dual provide a natural bi-grading of the elements of the lattice.

Beginning with the bigrading defined by the two Whitney rank functions, let us define the rank generating function. Assume R is an exact differential on a finite complete modular lattice L, with Whitney rank function r on L. For every element x in the lattice L, define gradings g_1 and g_2 by

$$g_1(x) = r(1) - r(x)$$
$$g_2(x) = r^*(\tilde{0}) - r^*(\tilde{x})$$

where $\tilde{x}$ represents the image of the element x in the inverted lattice $\tilde{L}$. Then define a bigrading g on the lattice L by setting

$$g(x) = (g_1(x), g_2(x))$$

for every element x in L.

The double counting sequence of the bigrading g will be denoted by ρ, and is defined for all pairs of natural numbers $i = 0,1,\dots$; $j = 0,1,\dots$ by

ρ_{ij} = the number of elements x in L with $g(x) = (i,j)$.

Finally, the <u>rank generating function</u> ρ of the exact differential R on the finite complete modular lattice L is given by

$$\rho(\xi,\eta) = \sum_{i,j} \rho_{ij}\xi^i\eta^j.$$

The rank generating function is a polynomial in two variables, because the lattice L is assumed finite. We now set forth in detail a number of simple properties of this polynomial, and of the bigrading which gives rise to it.

Proposition. Rank in the Fixed-Point Lattices: Let R be an exact differential on a finite complete modular lattice L, with Whitney rank function r. Let r^* be the Whitney rank function of the dual differential R^*. Then the rank in the lattice L/R of the image of a fixed point x in L is $r(x)$, and the rank in $\tilde{L}/R^*$ of the image of a fixed point $\tilde{x}$ of R^* on $\tilde{L}$ is $r^*(\tilde{x})$.

More generally, for any element x in L, $r(x)$ = rank in L/R of the image of $C\ell(x)$, and $r^*(\tilde{x})$ = rank in $\tilde{L}/R^*$ of the image of $C\ell^*(\tilde{x})$.

Proof: The more general statement follows from the first form, because the presence of a step on which R has value 1 in a path from x to $C\ell(x)$ implies the existence of an intermediate fixed point, so $R(x,C\ell(x)) = 0$, and $r(x) = r(C\ell(x))$.

The first form of our statement follows from the fact that if, for two fixed points x,y in L, the image of y in L/R covers the image of x, then $r(x) + 1 = r(y)$. This covering property is proven as follows. Assume the image in L/R of a fixed point y covers the image of a fixed point x, and let z be any element of L covering x. Since x is a fixed point, $R(x,z) = 1$. Since there are no fixed points in the half-closed interval $[z,y)$, $R(z,y) = 0$, and $r_p(x,y) = R(x,z) = 1$ for any path p from x to y via the element z. Thus $r(x) + 1 = r(y)$.

To prove the rank property, we consider an arbitrary fixed point x in L, and form a path q in L/R from the zero of L/R to the image of x. The elements of this path correspond to an increasing sequence of elements in L, a sequence which may be extended to a path p from

0 to x in L. Since the images of the fixed points in the path p form a path in L/R, the R-sum $r_p(0,x) = r(x)$ is equal to the rank of the image of the fixed point x in the lattice L/R. Applying this result to the dual differential R*, we find also that $r^*(x)$ is the rank of the image of $\tilde{x}$ in $\tilde{L}/R^*$, whenever x is a dual fixed point of R on L.

If we make use of the duality[1] property of exact differentials, we shall be able to simplify the computation of the rank generating function. Just as we may characterize the grading g_1 by

$g_1(x)$ = the number of steps on which R = 1 in any path from x to 1,

so also we may characterize the grading g_2 by

$g_2(x)$ = the number of steps on which R = 0 in any path from 0 to x.

Proposition. Grading Duality: Let $g = (g_1, g_2)$ by the bigrading of a finite complete modular lattice L with respect to the Whitney rank functions r and r*

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[1]supra, Chapter III, §2, p. 59.

of an exact differential R and of its dual R*. If λ is rank in the lattice L, then $g_2(x) = \lambda(x) - r(x)$ for any element x in L.

Proof: The duality property of exact differentials states that $R^*(\tilde{y},\tilde{x}) = 1 - R(x,y)$ whenever an element y covers an element x in the lattice L. The value of the grading g_2 at x is the difference $r^*(\tilde{0}) - r^*(\tilde{x})$, or the R^* sum from $\tilde{x}$ to $\tilde{0}$ in the lattice $\tilde{L}$. This equals the sum of 1 - R from 0 to the element x in the lattice L, which is $\lambda(x) - r(x)$.

Generating functions may always be expressed either as a sum over the range of the grading, or as a sum over the graded set. Thus

$$\rho(\xi,\eta) = \sum_{i,j} \rho_{ij}\xi^i\eta^j = \sum_{x\in L} \xi^{g_1(x)}\eta^{g_2(x)}.$$

The latter form is often more convenient in theoretical work.

It is to be expected that the structure of the lattice serving as domain of an exact differential will have some influence on the algebraic properties of the rank generating function. That all elements x of the same rank in the lattice L have the same difference $g_2(x) - g_1(x)$ is an immediate consequence of the grading duality property. Following Laplace, we state this as a property of the rank generating function, rather than as a property of its coefficients.

Proposition. The Fundamental Domain Relation for a Rank Generating Function: Let ρ be the rank generating function of an exact differential R on a finite complete modular lattice L. Let λ be rank in the lattice L, and let r be the Whitney rank function of the exact differential R. Then

$$t^{r(1)}\rho(1/t,t) = \sum_{x\in L} t^{\lambda(x)}.$$

Proof: The bi-grading g is given, for all elements x in L, by

$$g_1(x) = r(1) - r(x)$$
$$g_2(x) = \lambda(x) - r(x),$$

so, for all elements x,

$$g_2(x) - g_1(x) = \lambda(x) - r(1).$$

Evaluating the rank generating function at $\xi = 1/t$, $\eta = t$, and multiplying by $t^{r(1)}$, we have

$$\begin{aligned} t^{r(1)}\rho(1/t,t) &= t^{r(1)} \sum_{x\in L} (1/t)^{r(1)-r(x)} t^{\lambda(x)-r(x)} \\ &= t^{r(1)} \sum_{x\in L} t^{\lambda(x)-r(1)} = \sum_{x\in L} t^{\lambda(x)}. \end{aligned}$$

In the particular case in which the lattice L is the Boolean algebra of all subsets of an n-element set, we have

$$\rho(1/t,t) = t^{-r(1)}(t+1)^n,$$

because there are $\binom{n}{k}$ elements of rank $\lambda = k$ in L, for $k = 0,1,\ldots,n$. A considerable amount of information is available from the fundamental domain relation, because it consists of $\lambda(1)$ independent relations on the set of $(r(1)+1)(\lambda(1)-r(1)+1)$ coefficients of the rank generating function. In particular, there is always a monomial $\xi^{r(1)}$ with coefficient 1, which corresponds to the element 0 of the lattice, and a monomial $\eta^{\lambda(1)-r(1)}$, also with coefficient 1, corresponding to the element 1 of the lattice.

§3 Algebraic Properties of the Rank Generating Function

a) The Relative Rank Generating Function

The rank generating function of an exact differential is also, in some sense, a function on pairs of elements of the lattice. We observed in Chapter I[1]

[1]supra, Chapter I, §4a, p. 16.

that the restriction of an exact differential to any lattice interval is itself an exact differential. Beginning, then, with an exact differential R on a finite complete modular lattice L, we have, for every pair of elements x,y of L with $x \leq y$, a rank generating function which we denote

$$\rho(x,y;\xi,\eta),$$

associated with the restricted differential $R|_{[x,y]}$ on the sublattice [x,y].

If we further define $\rho(x,y;\xi,\eta)$ to be the zero function unless $x \leq y$, we obtain the relative rank generating function $\rho(x,y;\xi,\eta)$, defined for all elements x and y in L, and for all real numbers ξ and η.

It will be convenient for future claculations to set down in detail the enumerations which yield the coefficients $\rho_{ij}(x,y)$ of monomials $\xi^i\eta^j$ in the relative rank generating function $\rho(x,y;\xi,\eta)$:

$\rho_{ij}(x,y)$ = the number of lattice elements z,
with $x \leq z \leq y$, for which
$i = r(y) - r(z)$ and
$j = (\lambda(z) - \lambda(x)) - (r(z) - r(x))$.

These conditions may also be written in terms of rank in the fixed point lattices of the restricted differential and its dual. The index i represents the rank difference between the element 1 in the fixed point lattice $[x,y] \,/\, R|_{[x,y]}$ and the image in this lattice of the closures of elements in the lattice interval $[x,y]$. The index j represents the rank difference in the lattice $[\tilde{y},\tilde{x}] \,/\, R^*|_{[\tilde{y},\tilde{x}]}$ between the image of $\tilde{0}$ and the image of the *closures of elements in the interval $[\tilde{y},\tilde{x}]$ of the lattice $\tilde{L}$.

b) Factorization

There exists a connection between multiplication of differentials and multiplication of their rank generating functions.

Proposition. A Sufficient Condition for Factorization of Rank Generating Functions: Assume an element x in a finite complete modular lattice L is a factor of an exact differential R on L. Then the rank generating function of the exact differential R on L is the product of the evaluations of the relative rank generating function at the pairs (0,x) and (x,1) of the lattice elements, ie:

$$\rho(\xi,\eta) = \rho(0,x;\xi,\eta)\rho(x,1;\xi,\eta).$$

Proof: Since the element x is a factor of the differential R on the lattice L, we know the values of R are given by

$$R(y,z) = \sup\{R(y\wedge x, z\wedge x), R(y\vee x, z\vee x)\}$$

for all elements y and z in the lattice L. If p is a path from 0 to an element y in L, then for every step $[p_{i-1}, p_i]$ we have either $p_{i-1}\wedge x < p_i \wedge x$ or $p_{i-1}\vee x < p_i \vee x$, but not both. Thus the rank λ in L has the property

$$\lambda(y) = (\lambda(y\wedge x) - \lambda(0)) + (\lambda(y\vee x) - \lambda(x))$$

and the Whitney rank function r of the exact differential R satisfies

$$r(y) = (r(y\wedge x) - r(0)) + (r(y\vee x) - r(x)).$$

The rank generating function ρ of R may be written

$$\rho(\xi,\eta) = \sum_{y\in L} \xi^{r(1)-r(y)}\eta^{\lambda(y)-r(y)}.$$

We express these exponents of ξ and η as follows:

$$r(1)-r(y) = [r(x)-r(y\wedge x)] + [r(1)-r(y\vee x)]$$
$$\lambda(y)-r(y) = [\lambda(y\wedge x)-r(y\wedge x)] + [\lambda(y\vee x)-\lambda(x)-r(y\vee x)+r(x)],$$

wherein the first of the two terms in each sum depends only on the projection $y \to y\wedge x$ of the element y into the interval [0,x], and the remaining term in each sum depends only on the projection $y \to y \vee x$ of the element y into the interval $[x,1]$. Summing instead over all pairs of elements y_1, y_2, with y_1 in $[0,x]$ and y_2 in $[x,1]$, we have

$$\begin{aligned} \rho(\xi,\eta) &= \sum_{y_1 \in [0,x]} \xi^{r(x)-r(y_1)} \eta^{\lambda(y_1)-r(y_1)} \\ &\cdot \sum_{y_2 \in [x,1]} \xi^{r(1)-r(y_2)} \eta^{\lambda(y_2)-\lambda(x)-r(y_2)+r(x)} \\ &= \rho(0,x;\xi,\eta) \cdot \rho(x,1;\xi,\eta). \end{aligned}$$

c) Recursion

We have extablished the property which rank generating functions possess relative to the multiplicative structure of differentials. If an exact differential is defined on the Boolean algebra of all subsets of an n-element set, the relative rank generating function of the differential also satisfies a recursion relation relative to the "additive" structure of the lattice.

Proposition. The Recursion Formula: Given an exact differential R with rank generating function ρ on the Boolean algebra L of all subsets of the n-element set X, and given any element e of the set X, let e' denote the complementary subset $X - e$, then

$$\rho(\xi,\eta) = \eta^{1-R(0,e)}\rho(0,e';\xi,\eta) + \xi^{R(e',1)}\rho(e,1;\xi,\eta).$$

Proof: We consider three cases:

a) $R(0,e) = 0$, in which case $R(e',1) = 0$

b) $R(e',1) = 1$, in which case $R(0,e) = 1$

c) $R(0,e) = 1$ and $R(e',1) = 0$.

Case a): Assume an element y covers an element x in the interval $[0,e']$. Consider the local graph on $x,y,x\vee e,y\vee e$. Since $R(0,e) = 0$, the translation property implies $R(x,x\vee e) = R(y,y\vee e) = 0$. The local graph must be either zero or mixed, so $R(x,y) = R(x\vee e,y\vee e)$, and the mapping $x \to x\vee e$ from the interval $[0,e']$ to the interval $[e,1]$ is a differential-preserving isomorphism. $\rho(0,e';\xi,\eta) = \rho(e,1;\xi,\eta)$, and every subset in the interval $[e,1]$ has grading one higher in the second component than the corresponding subset in the interval $[0,e']$. Thus

$$\begin{aligned}\rho(\xi,\eta) &= (1+\eta)\rho(e,1;\xi,\eta)\\ &\ \eta\cdot\rho(0,e';\xi,\eta) + \rho(e,1;\xi,\eta).\end{aligned}$$

Note that the factor $1+\eta$ in the above expression is $\rho(0,e;\xi,\eta)$. The first form of the above expression is the product formula derived from the product $R = R|_{[0,e]}\times R|_{[e,1]}$, the element e being a factor of the differential R on the Boolean algebra L.

case b): If $R(e',1) = 1$, the element e is again a factor of the differential R on the Boolean algebra L, so

$$\begin{aligned}\rho(\xi,\eta) &= \rho(0,e;\xi,\eta)\rho(e,1;\xi,\eta) \\ &= (1+\xi)\rho(e,1;\xi,\eta) \\ &= \rho(0,e';\xi,\eta) + \xi\rho(e,1;\xi,\eta).\end{aligned}$$

case c): If $R(0,e) = 1$ and $R(e',1) = 0$, we must prove

$$\rho(\xi,\eta) = \rho(0,e';\xi,\eta) + \rho(e,1;\xi,\eta).$$

If y is any subset containing e, we choose a path p from 0 to 1 via e and y, and show the bi-grading of the subset y is the same as that with respect to the restricted differential $R|_{[e,1]}$. The sum of R values along the path p from y to 1 is the same with respect to R and $R|_{[e,1]}$, as is the sum of values of the function $1 - R$ on that portion of the path p from 0 to y in one instance and from e to y in the other. These are required bi-gradings.

On the other hand, if y is a subset not containing e, we choose a path q from 0 to 1 via y and the subset e', and show the bi-grading of the subset y is the same as that with respect to the restricted differential

$R|_{[0,e']}$, by a similar argument.

Thus, separating the expression for $\rho(\xi,\eta)$ into a sum over subsets containing the element e and a sum over subsets not containing e, we have:

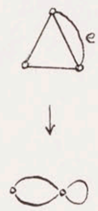
$$\begin{aligned}\rho(\xi,\eta) &= \sum_{y\in L} \xi^{g_1(y)}\eta^{g_2(y)} \\ &= \sum_{y\in[0,e']} \xi^{g_1(y)}\eta^{g_2(y)} + \sum_{y\in[e,1]} \xi^{g_1(y)}\eta^{g_2(y)} \\ &= \rho(0,e';\xi,\eta) + \rho(e,1;\xi,\eta)\end{aligned}$$

since we have shown that the gradings coincide.

To obtain a well-known example of this recursion formula, let the differential R be the partition differential of a graph relation. We know such a differential to be exact; the lattice on which it is defined is the Boolean algebra of all subsets of the set of edges. The restricted differential $R|_{[0,e']}$ is the partition differential of the graph formed by removing the edge e; the restricted differential $R|_{[e,1]}$ is the partition differential of the graph formed by contracting the edge e to a single vertex, and contracting all other edges connecting the same pair of vertices to loops. For example:

elimination

contraction



That we have established a general recursion formula which applies simultaneously to the Tutte polynomial[1] and to the chromatic polynomials[2] of a graph and its dual will be clear once we derive these various polynomials from the rank generating function[3].

Any recursion formula, valid for exact differentials on a complemented modular lattice, must take into account the possibility that an atom may have several complements in the lattice, all of which are necessarily coatoms.

Proposition. General Recursion Formula: Given an exact differential R with Whitney rank function r and rank generating function ρ on a finite complete complemented modular lattice L, the following formula holds for any atom e of L, the complements of which are

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[1]W.T. Tutte, "A Contribution to the Theory of Chromatic Polynomials", Can. J. 6(1954) p. 85.

[2]G.D. Birkhoff, "A Determinant Formula for the Number of Ways of Coloring a Map", Annals, 14(1912) p. 42-46.

[3]infra, sections 4 and 5 of the present chapter.

enumerated $e'_1, \ldots, e'_k$ in some order:

$$\rho(\xi,\eta) = \eta^{1-R(0,e)}\rho(e,1;\xi,\eta)$$
$$+ \sum_i \xi^{R(e'i,1)}\rho(0,e'_i;\xi,\eta)$$
$$- \sum_{i<j} \xi^{r(1)-r(e'_i\ e'_j)}\rho(0,e'_i\ e'_j;\xi,\eta)$$
$$+ \ldots$$
$$+ (-1)^{k-1}\xi^{r(1)-r(e'_1\ \ldots\ e'_k)}\rho(0,e'_1\ \ldots e'_k;\xi,\eta)$$

Proof: The proof used for case c) of the recursion formula for rank generating functions of differentials on a Boolean algebra applies in this more general situation. The first component of the bi-grading is the same for elements above e as it is on the interval [e,1]. The second component differs by 1 if $R(0,e) = 0$. Thus arises the first term in the recursion formula. The remaining terms arise by application of the classical inclusion-exclusion principle to sums over lattice elements lying beneath complements e'_i comprising various subsets of the set $\{e'_1,\ldots,e'_k\}$ of all complements of the element e of L[1].

d) Matrix Inversion

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[1]vide infra, Appendix A, example

The function $\rho(x,y;\xi,\eta)$, regarded as a function of x and y alone, for any fixed ξ and η, may be thought of as an upper triangular square matrix with numerical entries, indexed in both dimensions by the elements of the lattice, arranged in some increasing order. These matrices all have the property that the entry in the row corresponding to the element x in L and in the column corresponding to the element y in L is zero unless $x \leq y$ in L. Such matrices form an algebra under addition, scalar multiplication, and matrix multiplication, an algebra called the incidence algebra[1] of the lattice L.

The principal objects of interest in the incidence algebra are the identity, the zeta function, and its inverse, the Möbius function. These are usually defined as follows:

identity δ: $\delta(x,y) = 0$ if $x \neq y$

$= 1$ if $x = y$.

zeta ζ: $\zeta(x,y) = 0$ if $x \nless y$

$= 1$ if $x < y$.

Möbius μ: the inverse of ζ, thus characterized by the relation $\sum_{y \in [x,z]} \mu(x,y) = \delta(x,z)$

[1] G.-C. Rota, "On the Foundations of Combinatorial Theory", Zeitschr. fur Wahrshein., 2 (1964), pp. 340-360 §3.

for any choice of interval [x,z].

We shall employ a polynomial generalization of the zeta function, but shall use the same name. Thus we define the zeta function of an exact differential R with Whitney rank function r on a finite complete modular lattice L by

$$\zeta(\xi,\eta) = \xi^{r(1)}\eta^{\lambda(1)-r(1)}.$$

As a function of pairs of lattice elements, the relative zeta function may then be computed, for $x \le y$, as

$$\zeta(x,y;\xi,\eta) = \xi^{r(y)-r(x)}\eta^{\lambda(y)-r(y)-\lambda(x)+r(x)}.$$

It should be noted that $\zeta(x,y;1,1)$ is the usual zeta function, a numerical matrix element of the incidence algebra. The relationship between the relative zeta function of a differential and the rank generating function of that same differential is exhibited in the following statement.

Proposition. The Rank Generating Function Derived from the Zeta Function: Given an exact differential R on a finite complete modular lattice L, its rank generating function ρ and its zeta function ζ, we have

$$\rho(x,y;\xi,\eta) = \sum_{z\in[x,y]} \zeta(x,z;1,\eta)\zeta(z,y;\xi,1),$$

ie: the rank generating function is the product, in the incidence algebra, of two partial evaluations of the relative zeta function.

Proof: The number of elements z in the lattice interval $[x,y]$ for which $\zeta(x,z;1,\eta) = \eta^j$ and for which $\zeta(z,y;\xi,1) = \xi^i$ is equal to the coefficient ρ_{ij}.

We shall now develop an inversion formula for the relative rank generating function, the inverse being with respect to the operation of matrix multiplication in the incidence algebra of the domain of the differential, and valid whenever that domain is a Boolean algebra.

Theorem. The Multiplicative Inverse of a Rank Generating Function: If an exact differential R on the Boolean algebra L of all subsets of an n-element set has rank generating function ρ and zeta function ζ, then for all pairs x,y of subsets in L and for all pairs ξ,η of non-zero real numbers,

$$\rho^{-1}(x,y;\xi,\eta) = \zeta(x,y;-\xi,-\eta)\rho(x,y;1/\xi,1/\eta).$$

Proof: We begin by analysing the numerical product $\zeta(x,y;\xi,\eta)\rho(x,y;1/\xi,1/\eta)$. The coefficient of $\xi^i\eta^j$ in the expansion of this product is equal to the number of elements z in the interval $[x,y]$ for which

$$i = r(y) - r(x) - [r(y) - r(z)]$$
$$= r(z) - r(x)$$

and for which

$$j = \lambda(y) - r(y) - \lambda(x) + r(x)$$
$$-[\lambda(z) - r(z) - \lambda(x) + r(x)]$$
$$= \lambda(y) - r(y) - \lambda(z) + r(z)$$

ie: the coefficient of $\xi^i\eta^j$ in the expansion of the product $\zeta(x,y;\xi,\eta)\rho(x,y;1/\xi,1/\eta)$ is the number of elements z in the lattice interval [x,y] such that any path from x to y via z has R value 1 on i steps beneath z and R value 0 on j steps above z. This means that the product $\zeta(x,y;\xi,\eta)\rho(x,y;1/\xi,1/\eta)$ may be expressed, in much the same manner as the rank generating function, as a matrix product of partial evaluations of the relative zeta function:

$$\zeta(x,y;\xi,\eta)\rho(x,y;1/\xi,1/\eta)$$
$$= \sum_{z \in [x,y]} \zeta(x,z;\xi,1)\zeta(z,y;1,\eta).$$

No use has yet been made of the assumption that the domain of the differential is a Boolean algebra. The above formulation holds for any finite complete modular lattice.

We may now proceed to prove that the matrix product

$$\sum_{z\in[x,y]} \zeta(x,z;-\xi,-\eta)\rho(x,z;1/\xi,1/\eta)\rho(z,y;\xi,\eta)$$

is equal to the identity function $\delta(x,y)$. Factor the first two factors and the last factor into their respective formulations as matrix products of partial evaluations of the zeta function; this introduces a triple summation

$$\sum_{z\in[x,y]} \ \sum_{w_1\in[x,z]} \ \sum_{w_2\in[z,y]}$$

of the product of four terms, signed in accordance with the degrees of terms in $\zeta(x,z;-\xi,-\eta)$, namely:

$$(-1)^{\lambda(z)-\lambda(x)}\zeta(x,w_1;\xi,1)\zeta(w_1,z;1,\eta)\zeta(z,w_2;1,\eta)\zeta(w_2,y;\xi,1).$$

If we interchange the order of summation, summing first with respect to subsets z in the interval $[w_1,w_2]$, all terms in the product are constant except

$$\begin{aligned}&(-1)^{\lambda(z)}\zeta(w_1,z;1,\eta)\zeta(z,w_2;1,\eta)\\ &= (-1)^{\lambda(z)}\eta^{[\lambda(z)-r(z)-\lambda(w_1)-r(w_1)]+[\lambda(w_2)-r(w_2)-\lambda(z)+r(z)]}\\ &= (-1)^{\lambda(z)}\eta^{\lambda(w_2)-r(w_2)-\lambda(w_1)+r(w_1)}.\end{aligned}$$

Since this power of η is independent of z, it remains to sum $(-1)^{\lambda(z)}$ over the interval $[w_1,w_2]$. This summation yields $(-1)^{\lambda(w_1)}\delta(w_1,w_2)$, because the lattice is Boolean, the number of elements of each rank is a binomial coefficient, and the alternating sum of binomial coefficients is zero unless the sum is over an interval of length zero.

We now ignore all terms in which $w_1 \neq w_2$ and set $w = w_1 = w_2$ in the rest. There remains to be performed a single summation, for all subsets w in the interval [x,y], of the products

$$\begin{aligned}&(-1)^{\lambda(w)-\lambda(x)}\zeta(x,w;\xi,1)\zeta(w,y;\xi,1)\\&= (-1)^{\lambda(w)-\lambda(x)}\xi^{r(w)-r(x)}\xi^{r(y)-r(w)}\\&= \xi^{r(y)-r(x)}(-1)^{\lambda(w)-\lambda(x)}.\end{aligned}$$

Once more, the summation of $(-1)^{\lambda(w)-\lambda(x)}$ over the interval [x,y] yields $\delta(x,y)$, which is zero except when $x = y$. When $x = y$, $\xi^{r(y)-r(x)}$ is also 1, so the entire matrix product yields $\delta(x,y)$, and the proof of the inversion formula is complete.

The coefficient array for the inverse of the rank generating function of an exact differential on a finite Boolean algebra may be obtained by rotation (180°) of the coefficient array of the rank generating function, then applying the appropriate sign to the array as a whole. For example, the rank generating function $\rho(\xi, \eta)$ for the partition differential of the graph 

is $\eta^2 + \eta(4 + 3\xi + \xi^2) + 3 + 3\xi + \xi^2$ with coefficient

array $\begin{matrix} 1 & & \\ 4 & 3 & 1 \\ 3 & 3 & 1 \end{matrix}$. The inverse of the relative form,

$\rho(x,y;\xi,\eta)$, of this function has, at the subset pair

$(x,y) = (0,1)$, the coefficient array $\begin{matrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ & & 1 \end{matrix}$.

This inversion process may be traced through two stages. Substitution of $1/\xi$, $1/\eta$ for ξ and η reflects the coefficient array through the origin. Multiplication by $\zeta(\xi,\eta)$ translates the array up and to the right by exactly its own dimensions. The composite effect is equivalent to a rotation by 180°, preserving the outlines of the array.

We now turn from this compilation of algebraic properties to establish the relationship of the rank generating function to other well-known polynomials.

§4 Associated Lattice Polynomials

Embodied in a sequence of lemmas below is the proof of the principal theorem concerning rank generating functions on a Boolean algebra. Our objective is to establish a 1-1 correspondence between bases for the differential and intervals in the Boolean algebra, such that the collection of intervals is a 1-1 covering of the Boolean algebra. Matchings of this type correspond to substitutions in the rank generating function.

We shall first establish the matching process, then state the theorem concerning substitution in the rank generating function. Let R be an exact differential with Whitney rank function r on a Boolean algebra L of all subsets of an n-element set X. Since the Boolean algebra is complemented, every subset which is a fixed point of R on L is the intersection, in L, of coatomic (in L/R) fixed points, and every dual fixed point is a join in L of coatomic (in $\tilde{L}/R^*$) dual fixed points. We shall refer to the set C of coatomic (in L/R) fixed points of R in L, and to the set C^* of coatomic (in $\tilde{L}/R^*$) dual fixed points of R in L.

Place the elements of the set X in some linear order ω. The statement $\omega(d) < \omega(e)$ will mean the element d is lower in the ordering ω than is the element e. Relative to this ordering of the set X, we define a complex of four operators on the Boolean algebra.

The first operator ℓ is designed to produce in one operation the result of sequentially deleting from a subset y the highest element of y which is in any dual fixed point contained in the set y, repeating this operation until the resulting subset is independent. The definition: for any subset y in the Boolean algebra L,

> $\ell(y)$ is a subset of y, and an element e is in $y - \ell(y)$ if and only if there exists a dual fixed point in C* contained in y, in which the element e is the highest element in the ordering ω.

The mate to the operator ℓ is the operator u, which is designed to produce in one operation the result of sequentially adding to a subset y the highest element not in some coatomic fixed point which contains the subset y, repeating this operation until the resulting subset spans. The definition: for any subset y in L,

> y is a subset of u(y), and an element e is in $u(y) - y$ if and only if there exists a fixed point in the set C which contains the subset y, and in the complement of which e is the highest element in the ordering ω.

Note that, under the anti-isomorphism carrying the Boolean algebra L into the inverted Boolean algebra $\tilde{L}$, the operator u defined for the differential R becomes the operator ℓ defined for the differential R*. The ordering ω is left the same in both instances, and non-containment in fixed points is the dual of containment in dual fixed points. This observation will make it unnecessary to provide separate proofs for what may be seen to be the duals of statements already proven.

This first pair of operators map subsets into sets which resemble bases in one or more respects; we shall prove that the image subsets are independent sets and spanning sets, respectively. The other two operators also form a pair, but work in a direction opposed to that of the first pair: the images of subsets are less like bases than are the subsets themselves.

The operator $^-$ is defined to be the local opposite of the operator u. Given any subset y in the Boolean algebra, we define:

> y^- is a subset of y, and an element e is in
> $y - y^-$ if and only if the element e is in the
> set $u(y-e) - (y-e)$.

To characterize the operator $^-$ without reference to the operator u, we say an element e is in the set $y - y^-$ if and only if there exists a coatomic fixed point, in the set C, which contains y - e, and in the complement of which e is the highest element in the ordering ω.

Similarly, the operator $^+$ is defined to be the local opposite of the operator ℓ. Given any subset y is the Boolean algebra L, we define:

> y is a subset of y^+, and an element e is in $y^+ - y$ if and only if the element e is in the set $(y+e) - \ell(y+e)$.

Thus an element e is in the difference set $y^+ - y$ if and only if there exists a dual coatom, in the set C*, which is contained in the set y+e, and in which the element e is highest in the ordering ω.

We shall prove that the operators $^+$ and $^-$ are closure operators, the differentials of which are greater than or equal to the differentials R and R*, respectively. The operators $^+$ and $^-$ induce a bi-grading which we define, in terms of the Boolean algebra rank λ, for any subset y in the Boolean algebra, by

$$\iota(y) = \lambda(y) - \lambda(y^-)$$

$$\varepsilon(y) = \lambda(y^+) - \lambda(y).$$

The grading $\iota(y)$, the number of elements deleted from the subset y by the operator $^-$, we refer to as the internal activity[1] of the subset y. The grading $\varepsilon(y)$, the number of elements added to the subset y by the operator $^{+}$, we call the external activity of the subset y.

We now prove a succession of lemmas leading to the proof of the main substitution theorem. Rather than repeat standard assumptions in the statement of each lemma, let us agree that in each lemma the differential R is exact, that it is defined on a Boolean algebra L of all subsets of a finite set with n elements, and that it has Whitney rank function r inducing the usual bi-grading $g = (g_1, g_2)$ on subsets in L. Further, we agree that λ is rank in the Boolean algebra L, and that the operators $\ell, u, {}^{+}, {}^{-}$ and the bi-grading (ι, ε) are defined as above.

Lemma. Ranges of the Operators ℓ and u: The operator ℓ maps onto the set of independent subsets in the Boolean algebra L; the operator u maps onto the set of spanning subsets of L.

Proof: Let y be any subset in L, and $u(y)$ its image under the operator u. If an element z covers $u(y)$, and $R(u(y),z) = 1$, there is a fixed point of R in the

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[1] cf. W.T. Tutte, "A Contribution to the Theory of Chromatic Polynomials", Can. J., 6, (1954) p. 85.

half closed interval $[u(y),1)$, and thus there is a fixed point w of R, above u(y), whose image in L/R is a coatom. But u(y) must contain the highest element not in this coatomic fixed point w, contradicting the statement $u(y) \leq w$. By the characterization of spanning elements in terms of differentials, u(y) spans. Conversely, if y spans, $u(y) = y$, so the map u is onto.

The corresponding statement concerning the range of the operator ℓ may be proven by applying the above result to the dual differential R*, and employing our observation that the roles of the operators u and ℓ are interchanged by duality.

Lemma. Successive Operations Have the Same Effect as u and ℓ: Given a subset x in the Boolean algebra L, and any subset y containing x but contained in u(x), it must be true that $u(y) = u(x)$. Dually, for the operator ℓ, $\ell(x) \leq y \leq x$ implies $\ell(y) = \ell(x)$.

Proof: Let x and y be subsets in the Boolean algebra L such that $x \leq y \leq u(x)$. If an element e is in the set u(y), but e is not in y, there exists a coatomic fixed point δ in the set C, with δ containing y and with the element e being highest in the order ω among elements in the complement of δ. Then $x \leq y \leq \delta$ implies $e \in u(x)$, so u(y) is a subset containing y and contained by u(x).

The remainder of the proof that $x \leq y \leq u(x)$ implies $u(y) = u(x)$ is somewhat more involved. We need the recursive relationship, valid for any subset x and any pair of elements e_1 and e_2 not in the subset x, that if the elements e_1 and e_2 are in the set u(x), then the element e_2 is in the set $u(x \vee e_1)$. To prove this relationship, let e_1 and e_2 be any two distinct elements in the set u(x) - x. There exist coatomic fixed points δ_1 and δ_2 containing x, such that e_1 is the highest element, with respect to the ordering ω, not in the set δ_1, and e_2 is the highest element not in the set δ_2. In particular, $\delta_1 \neq \delta_2$. If e_1 is an element of δ_2, then δ_2 is a coatomic fixed point containing $x \vee e_1$. Since e_2 is the highest element not in δ_2, e_2 is in the set $u(x \vee e_1)$. If, on the other hand, the element e_1 is not in the set δ_2, then e_2 is higher than e_1 in the ordering ω. If the element e_2 were not in the fixed point δ_1, the contrary ordering, e_1 higher than e_2, would apply. Thus the element e_1 is in neither δ_1 nor δ_2, and the element e_2 is in δ_1 but not in δ_2, with e_2 higher than e_1 in the ordering ω. Applying the exchange property of coatomic fixed points[1], there exists

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[1] supra, Chapter III, 5, p. 95.

a coatomic fixed point δ_3 containing the subset $(\delta_1 \wedge \delta_2) \vee e_1$, and not containing the element e_2. Since elements not in δ_3 are either not in δ_1 or not in δ_2, they are lower in the ordering ω than either e_1 or e_2, and thus lower than e_2. Consequently, the element e_2 is in the set $u(x \vee e_1)$, and our recursive relationship is proven.

We now complete the proof that $x \leq y \leq u(x)$ implies $u(y) = u(x)$. Assume some element e is in the set $u(x)$ but not in the set $u(y)$. List the elements $e_1, e_2, \ldots, e_k$ of the set $y - x$ in some order. Since the elements e and $e_1, \ldots, e_k$ are in the set $u(x)$, the elements e and $e_2, \ldots, e_k$ are in the set $u(x \vee e_1)$, the elements e and $e_3, \ldots, e_k$ are in the set $u(x \vee e_1 \vee e_2), \ldots,$ and the element e is in the set $u(x \vee e_1 \ldots \vee e_k) = u(y)$. This completes the proof of our lemma.

Lemma. The Rank and Bi-grading of Images Under the Operators ℓ and u: The increase in rank between a subset y and its image $u(y)$ under the operator u, is equal to the value $g_1(y)$ of the first component of the bigrading $g(y) = (g_1(y), g_2(y))$. The value of the second component of the grading is unchanged: $g_2(y) = g_2(u(y))$. The operator ℓ accounts for a decrease in rank, between a subset y and its image $\ell(y)$, equal to the value $g_2(y)$ of the second component of the bigrading $g(y)$. The value of the first component of the grading is unchanged: $g_1(y) = g_1(\ell(y))$.

Proof: Choose a path p from the subset y to its image $u(y)$. For any step $[p_{i-1},p_i]$ of the path p, let e_i be the single element in the difference set $p_i - p_{i-1}$. By our previous lemma, e_i is an element of $u(p_{i-1})$, so $R(p_{i-1},p_i) = 1$. Thus the first component grading difference is equal to the difference in Boolean algebra rank, and the second component grading is unchanged. The corresponding statement for the operator ℓ follows by application of the foregoing result to the dual differential R^*.

Lemma. The Operators $^+$ and $^-$: The operator $^+$ is a closure operator on the Boolean algebra L. The operator $^-$ is a closure operator on the inverted Boolean algebra $\tilde{L}$.

Proof: We shall prove for the operator $^-$ that, for any subsets x and y in the Boolean algebra,

i) $x^- \leq x$

ii) $x \leq y$ implies $x^- \leq y^-$

iii) $x^{--} = x^-$.

The image x^- is defined as a subset of the set x, so property i holds. Assume a subset x is contained in a subset y, and that an element e is in the set x^- but not in the set y^-. By the definition of the operator $^-$, there is a coatomic (in L/R) fixed point δ containing the subset y - e, in the complement of which e is the highest element in the ordering ω. Since $x \leq y$, the fixed point δ also contains the set x - e, so e is not an element of the image set x^-, contradicting our assumption. Thus $x^- \leq y^-$, proving property ii.

Now assume, in contradiction to property iii, that, for some element e in an image set x^- of some set x in L, there is a coatomic fixed point δ containing the set x^- - e, and the element e is the highest in the ordering ω among those elements not in the fixed point δ. Under these assumptions, we shall establish the existence of a sequence of coatomic fixed points having successively larger intersection with the set $x - x^-$, all containing the set x^- - e, and all having the element e as the highest element in their complement. The extablishment of this sequence implies the contradictory situation in which some coatomic fixed point terminates the process by containing x - e, which would mean the element e must be in the set $x - x^-$.

Let δ_1 be any coatomic fixed point containing the set x^- - e, for which the element e is highest among elements in the complement of δ_1. Let e_1 be any element which is in the complement of δ_1 and also in the difference set $x - x^-$. Since e_1 is in the difference set $x - x^-$, there exists a coatomic fixed point δ_2 containing $x - e_1$, such that e_1 is the highest among elements not in δ_2. Since e_1 is also in the complement of δ_1, e is higher than e_1 in the ordering. By the exchange property of coatomic fixed points, there exists a coatomic fixed point δ_3 containing $\delta_1 \wedge \delta_2$ and the element e_1 but not the element e. Such an element δ_3 contains whatever intersection δ_2 had with the difference set $x - x^-$, plus the element e_1. Any element not in δ_3 is either not in δ_1 or not in δ_2; in either case it is lower in the ordering ω than is the element e. Thus there is established a sequence of coatomic fixed points with successively smaller intersection with the set $x - x^-$, and leading to the contradiction outlined above.

The corresponding statement for the operator $^+$ follows by application of the above result to the dual exact differential R*.

Lemma. Mutual Containment Relations: For any pair of subsets x and y of the Boolean algebra, containment of y between x^- and x is equivalent to containment of x between y and u(y). Similarly, containment of y between x and x^+ is equivalent to containment of x between $\ell(y)$ and y.

Proof: A subset y is contained between a subset x and its image x^- if and only if, for every element e in the difference set x - y there is a coatomic fixed point δ containing the subset x - e, for which e is the highest element in the complement of δ. This is true if and only if the subset x is contained between the subset y and its image u(y). The corresponding relation between the operators $^+$ and ℓ is the dual of this relation.

Lemma. Partial Matching Property: A subset x is the image of a subset y under the operator u if and only if x is a spanning subset and y is contained between x and its image x^-. Dually, a subset x is the image of a subset y under the operator ℓ if and only if x is independent and y is contained between x and its image x^+.

Proof: Our lemma concerning the ranges of the operators ℓ and u states that any image subset x = u(y) is a spanning set. Since x is between y and u(y), we apply the previous lemma to imply that y lies between x and its image x^-. Conversely, if a subset x spans,

$u(x) = x$. If, moreover, a subset y is contained between x and its image x^-, the subset x must be contained between y and its image $u(y)$. By the lemma concerning successive operations for u and ℓ, $u(y) = u(x) = x$. The dual property follows by application of this result to the dual differential R*.

We are now in a position to prove the fundamental matching property, upon which all our subsequent enumerative work is based. Using the operator ℓ, then the operator u, we map every element onto a base. The inverse image of a base is invariably a lattice interval. The collection of intervals associated with bases jointly cover the entire Boolean algebra without overlapping.

Lemma. The Fundamental Base-Interval Matching Property: A subset x is the image of a subset y under the composite operator ℓ, then u, if and only if the subset x is a base, and the subset y is contained between the image subsets x^- and x^+. The composition of the operators u and ℓ in the opposite order results in the same operator: $u(\ell(y)) = \ell(u(y))$ for all subsets y.

Proof: Assume a subset x is the image of a subset y under the composite operator, $x = u(\ell(y))$. By the lemma concerning the ranges of u and ℓ, the subset x spans, and the subset $\ell(y)$ is independent. By the lemma on bi-grading of images of u and ℓ, $g_2(u(\ell(y))) = g_2(\ell(y)) = 0$, so $x = u(\ell(y))$ is also independent, and must be a base. We now show that the subset y is contained between x^- and x^+. Because $\ell(y) \leq x = u(\ell(y))$, the mutual containement relation implies $x^- \leq \ell(y) \leq x$, so $x^- \leq \ell(y) \leq x \wedge y$. Choose a path p from $\ell(y)$ to $x \wedge y$. Since $\ell(y) \leq x \wedge y \leq y$, the differential R must have value 0 on every step of the path p. Since $x^- \leq \ell(y) \leq x \wedge y \leq x$, the differential R must have value 1 on every step of the path p. Thus the path must be of length zero, and $\ell(y) = x \wedge y$. Let e be any element in the difference set $y - x = y - \ell(y)$, and let z be the subset $\ell(y) \vee e = (x \vee e) \wedge y$. By our lemma comparing successive operations with u and ℓ, we know e is not an element of $\ell(z)$. There exists a coatomic (in $\tilde{L}/R^*$) dual fixed point contained in z, and thus in $x \vee e$, in which e is the highest element in the ordering ω. Thus e is an element of the difference set $x^+ - x$, establishing the fact that the subset y is contained between x^- and x^+.

Conversely, we assume a subset y is contained between the images x^- and x^+ of a base x for the differential. The proof that $x = u(\ell(y))$ rests on the duality principle for exact differentials[1]. Let e be any element in the difference set $y - x$. Since e is thus an element in the set $x^+ - x$, there exists a coatomic (in $\tilde{L}/R^*$) dual fixed point γ contained in $x \vee e$, such that e is the highest element in γ. We show that γ is actually contained in the set $(y \wedge x) \vee e$. Let e_1 be any element other than e in the dual fixed point γ. We apply the duality principle for exact differentials to the dual fixed point γ contained in $x \vee e$ and the unique fixed point δ containing $x - e_1$. Let y be the subset $(x - e_1) \vee e$. The element e_1 is in the dual fixed point γ if and only if the base x is contained in the set $y \vee \gamma$, if and only if the set $y \wedge \delta$ is contained in the base x, if and only if the element e is not in the coatomic fixed point δ. Since e is the highest element of γ in the ordering ω, no element e_1 in γ can be in the difference set $x - x^-$. Thus the dual fixed point γ is a subset of $(x \wedge y) \vee e$, which in turn is contained in the subset y. Therefore the element e is in the difference set $y - \ell(y)$. This being true for all elements e in the difference set $y - x$, we know $\ell(y)$ is a subset of $x \wedge y$. Since $x \wedge y$ is contained between y

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[1] supra, Chapter III, §5, p. 97.

and $\ell(y)$, the lemma concerning successive operations implies $\ell(x \wedge y) = \ell(y)$. But since $x \wedge y$ is a subset of a base, $x \wedge y$ is independent, and $\ell(x \wedge y) = x \wedge y$, and $\ell(y) = x \wedge y$. Now apply the operator u to the subset $\ell(y)$. Since $\ell(y)$ is contained between the base x and its image x^-, x is contained between $\ell(y)$ and its image $u(\ell(y))$. Thus $u(x) = u(\ell(y))$. But the base x is a spanning set, so $u(x) = x$, and $x = u(\ell(y))$.

This completes the proof of the base-interval matching property

$$x = u(\ell(y)) \text{ if and only if}$$
$$x \text{ is a base, and } x^- \leq y \leq x^+.$$

The equality of the two composite operators

$$u(\ell(y)) = \ell(u(y)) \text{ for all subsets } y$$

is a consequence of the symmetry of the base-interval matching condition with respect to lattice inversion and replacement of the differential R by its dual R*.

We conclude the presentation of preparatory material with separate statements of two observations made in the course of the previous proof.

Lemma. The Composite Operators: If a base subset x, with images x^- and x^+, is the image under the composite operator ℓ, then u, of a subset y in L, $x = u(\ell(y))$, then the image $\ell(y)$ is the intersection $x \wedge y$, and the image $u(y)$ is the union $x \vee y$.

Proof: Under these assumptions, we proved $\ell(y) = x \wedge y$. The corresponding statement for the operator u is the dual of the statement for the operator ℓ.

Lemma. Internal and External Activity: If an independent subset x is the image of a subset y under the operator ℓ, then the internal activity of y is equal to the internal activity of x. If a spanning subset x is the image of a subset y under the operator u, then the external activity of y is equal to the external activity of x.

Proof: Assume an independent subset x is the image $\ell(y)$ of a subset y. Since the operator $^-$ is a closure operator on the inverted lattice $\tilde{L}$, the inequality $x \leq y$ implies $x^- \leq y^-$, so the difference set $y - y^-$ is a subset of the difference set $x - x^-$. As in the proof of the fundamental base-interval matching property, we use the duality principle for exact differentials to prove these two difference sets are equal: $x - x^- = y - y^-$. Since the numbers of elements in these sets are the internal activities $\iota(x)$ and $\iota(y)$, if follows that $\iota(x) = \iota(y)$.

Given any element e in the difference set $x - x^-$, there exists a coatomic fixed point δ containing the set $x - e$, such that e is the highest element not in δ. The condition $y \geq x = \ell(y)$ implies $x^+ \geq y \geq x$, by the lemma on mutual containment relations. Let e_1 be any element in the set $x^+ - x$. There exists a coatomic (in $\tilde{L}/R^*$) dual fixed point γ contained in the set $x \vee e_1$, such that e_1 is the highest element in γ. Since the set x is independent, and is a subset of the set $\delta \vee e$, which spans, we may select a base w containing x and contained in $\delta \vee e$. The element e_1 is not in the base w, since the set $x \vee e_1$ is dependent. Thus the fixed point δ is the unique coatomic fixed point containing the set $w - e$, and the dual fixed point γ is the unique coatomic dual fixed point contained in the set $w \vee e_1$, by the proposition on fundamental systems of coatomic fixed points[1]. The duality principle for exact differentials then applies, making equivalent the statements $e \in \gamma$ and $e_1 \notin \delta$. Since e_1 is the highest element in γ and e is the highest element not in δ, these statements cannot both be true.

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[1]supra, Chapter III, §5, p. 97

Hence both are false. Since $e_1 \in \delta$ for all elements e_1 in the difference set $x^+ - x$, the coatomic fixed point δ contains the sets $x^+ - e$ and $y - e$, so e is an element of the difference set $y - y^-$. This completes the proof.

The completion of the preceding sequence of lemmas makes available an immediate proof of the substitution theorem for rank generating functions. We recall that the rank generating function of an exact differential R enumerates all subsets in the Boolean algebra relative to the rank bi-grading $g = (g_1, g_2)$. The substitution theorem states that simple substitutions of variables transform the rank generating function into generating functions enumerating independent sets, spanning sets, or bases, relative to a bi-grading made up partly from the rank bigrading g and partly from the bi-grading (ι, ε), with respect to internal and external activity.

Theorem. The Substitution Theorem for Exact Differentials on a Boolean Algebra: Assume an exact differential R with rank generating function ρ is defined on a Boolean algebra L of all subsets of a finite set. If we define double counting sequences α_{ij}, m_{ij}, and m^*_{ij} by

α_{ij} is the number of bases of internal activity i and external activity j,

m_{ij} is the number of independent subsets with first-component rank grading i and external activity j,

m^*_{ij} is the number of spanning subsets with internal activity i and second-component rank grading j,

and if we let α, m, and m* be the associated two-variable generating functions of these double sequences, then the formulae

$$\rho(\xi,\eta) = m(\xi,\eta+1) = m^*(\xi+1,\eta) = \alpha(\xi+1,\eta+1)$$

hold as identities in ξ and η.

Proof: Expanding the generating functions $m(\xi,\eta+1)$, $m^*(\xi+1,\eta)$ and $\alpha(\xi+1,\eta+1)$ by the binomial formula, we see that the statement of the substitution theorem is equivalent to the following equations among the coefficients of these four polynomials:

$$\rho_{ij} = \sum_{k,\ell} \binom{k}{i}\binom{\ell}{j}\alpha_{k\ell}$$

$$m_{ij} = \sum_{k} \binom{k}{i}\alpha_{kj}$$

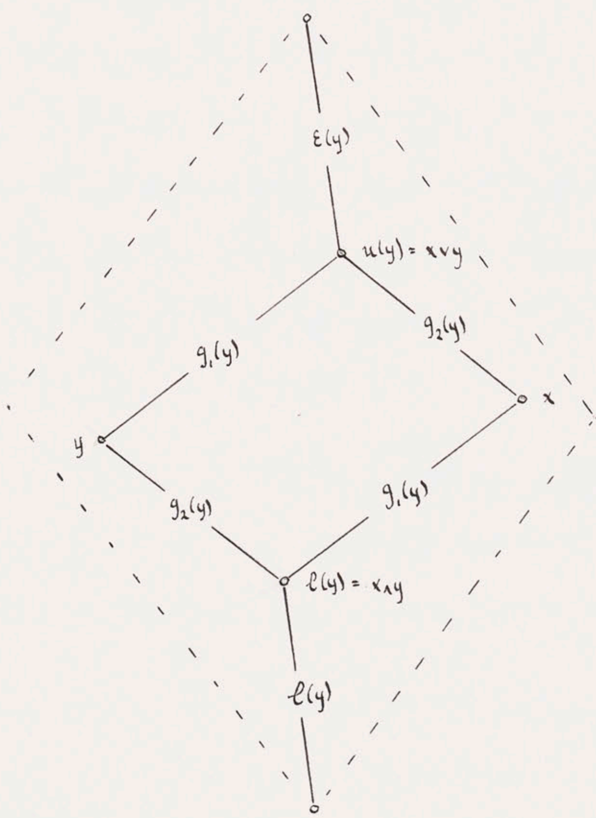
$$m^*_{ij} = \sum \binom{\ell}{j}\alpha_{i\ell}$$

It will be sufficient to prove that $m_{ij} = \sum_k \binom{k}{i} \alpha_{kj}$, and then that $\rho_{ij} = \sum_\ell \binom{\ell}{j} m_{i\ell}$. The symmetry of the gradings and of the definitions of independent and span with respect to lattice inversion and duality then provides a proof for the remainder of the theorem.

With any base x of internal activity k and external activity j we associate the various independent sets y such that $u(y) = x$. All such sets y lie in the interval $[x^-, x]$, and conversely. By the final lemma, the external activities of y and x are equal; by the lemma on rank bigrading under the operators ℓ and u, the difference in Boolean algebra rank, $\lambda(x) - \lambda(y)$, is equal to the first component rank grading. An independent set with first component rank grading i and external activity j must be one of the $\binom{k}{i}$ subsets with i fewer elements than a base with internal activity k and external activity agreeing with that of the independent set. The total number of independent sets enumerated by m_{ij} is obtained by summing over all bases, so $m_{ij} = \sum_k \binom{k}{i} \alpha_{kj}$.

Finally, a subset with rank bi-grading $g_1 = i$, $g_2 = j$, is mapped by the operator ℓ onto an independent set with the same first component rank grading, and with a number of elements j less than the number in the original subset. Since such subsets lie in intervals of the form $[x,x^+]$ above independent sets x, a subset with rank bigrading i,j must be one of the $\binom{\ell}{j}$ sets containing j elements in excess of an independent set of external activity ℓ. Summing over all such independent sets, we have the formula $\rho_{ij} = \sum_{\ell}\binom{\ell}{j}m_{i\ell}$. This completes the proof of the substitution theorem.

The following diagram of the Boolean algebra rank differences involved in the above argument may serve for further clarification:



The substitution formula, $\rho(\xi,\eta) = \alpha(\xi+1,\eta+1)$, offers an alternative in every enumerative problem concerning exact differentials on a Boolean algebra. On one hand, we may perform the grading $g = (g_1,g_2)$ for every lattice element. This operation is conceptually simple, but may involve extensive computations if the underlying set X is large. On the other hand, we may assign an ordering ω to the elements of the set X, form the images of all bases x under the operators $^+$ and $^-$ and measure the rank difference $\iota(x)$ between the base x and its image x^-, and the rank difference $\varepsilon(x)$ between x and its image x^+. While we achieve a considerable decrease in the size of the set of objects to be graded, we encounter a more complex grading process. But the size of the set to be graded is an issue only in practical problems. In theoretical work, the size of the set to be graded is of no consequence, while the conceptual simplicity and order-independence of the rank grading is all-important.

We close this section with an example, then proceed to a closer inspection of the associated lattice polynomials m,m^*, and α, in section five.

Consider the exact differential on a set of four elements in which all zero and one-element sets are independent, all two-element sets are bases, and all three- and four-element sets span. The differential R has value 1 on all steps below rank 2, and value 0 on all steps above rank 2. The coefficient array for

the rank generating function ρ is then
$$\begin{array}{r|ccc} 2 & 1 & & \\ 1 & 4 & & \\ q_2 = 0 & 6 & 4 & 1 \\ \hline q_1 = & 0 & 1 & 2 \end{array},$$

so the coefficient array for the polynomial α is

$$\begin{array}{r|ccc} 2 & 1 & & \\ 1 & 2 & & \\ \varepsilon = 0 & & 2 & 1 \\ \hline \iota = & 0 & 1 & 2 \end{array}.$$
Since the sum of the entries in the α coefficient array is equal to the (0,0) entry in the ρ coefficient array, the numerical simplification in passing from the rank generating function to the Tutte polynomial is evident. However, all bases are structurally identical; the differences between their internal and external activities are introduced by the choice of an ordering.

The working our of examples is aided by the observation that each base with internal activity k and external activity j contributes $\binom{k}{i}\binom{\ell}{j}$ to the coefficient of $\xi^i\eta^j$ in the rank generating function. The total contribution of one such base to the coefficient array of the rank generating function is a rectangular array formed of products of binomial coefficients. For example, one base of internal activity 3, external activity 2

$$\text{contributes} \quad \begin{matrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 6 & 2 \\ 1 & 3 & 3 & 1 \end{matrix} \quad \text{to the coefficient array of } \rho.$$

§5 Enumeration on the Fixed-Point Lattice

The Möbius functions of the fixed-point lattices both of an exact differential on a Boolean algebra and of its dual differential may be obtained by evaluation of the relative rank generating function at a pair of integers. An immediate proof of this fact is provided by G.-C. Rota's theorem[1] comparing the Möbius functions of two lattices joined by a Galois connection. Since we must distinguish between fixed points in the Boolean algebra and their images in the fixed point lattice,

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[1]G.-C. Rota, "On the Foundations of Combinatorial Theory", Zeitschr. fur Wahrshein., 2 (1964), pp. 340-360, §4.

let us use script letters $\mathcal{x},\mathcal{y}$, etc., for the latter.

Proposition. Mobius Functions of Fixed-Point Lattices: If R is an exact differential with rank generating function ρ on a Boolean algebra L of all subsets of a finite set, then, for all elements $\mathcal{x},\mathcal{y}$ in the fixed point lattice L/R, $\mathcal{x}$ and $\mathcal{y}$ being the images of fixed points x and y of R in the Boolean algebra, the value $\mu(\mathcal{x},\mathcal{y})$ of the Mobius function of the lattice L/R is given by

$$\mu(\mathcal{x},\mathcal{y}) = (-1)^{r(y)-r(x)}\rho(x,y;0,-1).$$

Proof: We define a Galois connection from the interval [x,y] in the Boolean algebra L to the interval $[\tilde{\mathcal{y}},\tilde{\mathcal{x}}]$ in the inverted fixed-point lattice $\widetilde{L/R}$. The mapping from L to $\widetilde{L/R}$ is the composite of closure $C\ell$ in L, image of the resulting fixed point in L/R, and inversion of the lattice L/R. The mapping from $\widetilde{L/R}$ to L is lattice inversion followed by embedding. Both maps are anti-isotone. The composite map from L to $\widetilde{L/R}$ and back to L is equal to the closure operator $C\ell$ in L. The composite map from $\widetilde{L/R}$ to L and back to $\widetilde{L/R}$ is the identity. The pair of maps thus constitutes a Galois connection.

G.-C. Rota's theorem is proven under the additional assumptions that $\tilde{x}$ is the only element in the interval $[\tilde{y},\tilde{x}]$ which is mapped to x in the interval [x,y], and that the element x in the interval [x,y] is mapped to $\tilde{x}$ in L/R. These assumptions are valid, since the element x is a fixed point of R in L.

The conclusion, in this application of Rota's theorem, is that the Möbius function of the lattice $\widetilde{L/R}$, evaluated at the pair of elements x,y, is the sum of all $\mu_L(x,z)$ values of the Möbius function μ_L of the Boolean algebra L at pairs x,z of subsets in L, where z is any subset not equal to x and mapped to the element $\tilde{y}$ in $\widetilde{L/R}$. Such subsets z in L are the spanning subsets for y with respect to the restricted differential $R|_{[x,y]}$, and are thus characterized by the value zero for the first component rank grading $g_1(z) = r(y) - r(z) = 0$. The Möbius function μ_L for the Boolean algebra has value $\mu_L(x,z) = (-1)^{\lambda(z)-\lambda(x)}$. The Möbius functions on the lattice L/R and on its inverse $\widetilde{L/R}$ are equal. Thus,

$$\begin{aligned}\mu(x,y) &= \sum_{\substack{z\in[x,y]\\ g_1(z)=0}} (-1)^{\lambda(z)-\lambda(x)}\\ &= (-1)^{r(y)-r(x)} \sum_{z\in[x,y]} (0)^{g_1(z)}(-1)^{\lambda(z)-r(z)}\\ &= (-1)^{r(y)-r(x)}\rho(x,y;0,-1).\end{aligned}$$

Corollary. Mobius Functions of Fixed-Point Lattices of Dual Differentials: If R is an exact differential with rank generating function ρ on a Boolean algebra L of all subsets of a finite set, then, for all elements $x \wedge y$ in the fixed point lattice $\tilde{L}/R^*$, x and y being the images of dual fixed points x and y in the Boolean algebra, the value $\mu(x,y)$ of the Mobius function of the lattice $\tilde{L}/R^*$ is given by

$$\mu(x,y) = (-1)^{\lambda(y)-r(y)-\lambda(x)+r(x)}\rho(x,y;-1,0).$$

Proof: A Galois connection is established between the inverted Boolean algebra $\tilde{L}$ and the inverted fixed point lattice of the dual differential, $\widetilde{\tilde{L}/R^*}$. We obtain an expression for the Mobius function value $\mu(x,y)$ on the lattice $\tilde{L}/R^*$ as the sum of all values $\mu_L(z,x) = (-1)^{\lambda(x)-\lambda(z)}$ of the Mobius function μ_L on the Boolean algebra, where x is the dual fixed point corresponding to the element x in $\tilde{L}/R^*$, and z is any subset in the interval [y,x] of L for which there is no dual fixed point in the half-closed interval (y,z]. Such subsets z are those mapped to y in the Galois connection, and are characterized by the value zero for the second component rank grading $g_2(z) = \lambda(z)-r(z)-\lambda(y)-r(y) = 0$. Thus

$$\mu(x,y) = \sum_{\substack{z\in[y,x] \\ g_2(z)=0}} (-1)^{\lambda(x)-\lambda(z)}$$

$$= (-1)^{\lambda(x)-r(x)-\lambda(y)+r(y)} \sum_{z\in[y,x]} (-1)^{r(x)-r(z)}{}_{(0)}g_2(z)$$

$$= (-1)^{\lambda(x)-r(x)-\lambda(y)+r(y)} \rho(y,x;-1,0).$$

The exponents of (-1) in the expressions for the Möbius functions on the fixed point lattices L/R and $\tilde{L}/R^*$ are equal to the g_1 and g_2 rank grading differences, respectively, between the corresponding subsets in L. We proved in section two[1] that these rank grading differences are equal to the differences in lattice rank in the lattices L/R and $\tilde{L}/R^*$, respectively. If we had reason to believe that the values $\rho(x,y;-1,0)$ and $\rho(x,y;0,-1)$ would always be positive, we would have an alternative proof of Rota's theorem[2] that the Möbius functions of geometric lattices alternate in sign. Our main substitution theorem, makes this proof possible. We know

$$\rho(x,y;-1,0) = m(x,y;0,0) = m_{0,0}(x,y), \text{ and}$$
$$\rho(x,y;0,-1) = m^*(x,y;0,0) = m^*_{0,0}(x,y)$$

which enumerate spanning subsets of zero external activity and independent subsets of zero internal activity,

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[1]supra, p. 107.

[2]G.-C. Rota, op. cit., Theorem 4, §7.

respectively, all statements being relative to the Boolean algebra interval [x,y]. Being enumerants of non-empty classes of subsets, the values $\rho(x,y;-1,0)$ and $\rho(x,y;0,-1)$ are invariably positive.

§6 Classification and Enumeration of Exact Differentials

Having laid general foundations for an enumerative theory of exact differentials, we are now in a position to make substantial progress on the problem of enumerating graphs. This problem, the enumeration of the isomorphy classes of graphs with n edges, has remained unsolved, despite the application of Polya's Theorem[1].

The sequence of values of an exact differential on the steps of a path from 0 to 1 in the domain lattice may be thought of as forming a word in a language employing two letters, the letter 0 coming before the letter 1 in the alphabet. Of the words thus associated with a given exact differential the work coming first in alphabetical order is an isomorphy invariant of the differential, and serves as an index for a classification

[1] J. Riordan, "An Introduction to Combinatorial Analysis", Wiley, New York, 1958, pp. 143-147.

system.

We use the term least path to indicate a path, from 0 to 1 in the domain lattice of an exact differential, if the word formed of the differential values along this path is alphabetically the first among all such words associated with the same differential. The word associated with a least path of an exact differential we shall call the first word of the differential.

Given any n-letter word formed of the letters 0 and 1, we shall prove the existence of an exact differential having that word as first word. This establishes the existence of at least 2^n isomorphically inequivalent exact differentials defined on the Boolean algebra of subsets of an n-element set.

The existence proof proceeds by induction, and involves some facts of independent interest concerning extensions of exact differentials. First, however, we shall establish the basic properties of least paths, and indicate their relationship with the rank generating function.

The ordering of rational numbers on the interval [0,1] provides a clear picture of the ordering we have placed on words. Given a path p from 0 to 1 in the

Boolean algebra L of all subsets of an n-element set, and given an exact differential R on L, map the path p to the rational number

$$\sum_{i=1}^{n} R(p_{i-1},p_i)2^{-i}.$$

The lowest rational number in the range of this map has binary decimal expansion equal to the first word of the differential R. Any path mapped to this rational number is a least path.

Proposition. Least Paths and Fixed Points: Let R be an exact differential on a Boolean algebra of all subsets of an n-element set, and let p be a least path from 0 to 1 in L. If $R(p_{i-1},p_i) = 1$ for any step $[p_{i-1},p_i]$ of the path p, the subset p_{i-1} is a fixed point of the differential R. If $R(p_{i-1},p_i) = 0$ for any step $[p_{i-1},p_i]$ of the path p, the subset p_i is a dual fixed point of the differential R.

Proof: Assume $R(p_{i-1},p_i) = 1$ on a step $[p_{i-1},p_i]$ in a least path p for the differential R. If there were a subset x covering p_{i-1} such that $R(p_{i-1},x) = 0$, the path via p to p_{i-1}, thence via x to 1, would be alphabetically prior to the path p. Thus the subset p_{i-1} is a fixed point of R on L. Now assume $R(p_{j-1},p_j) = 0$

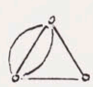
for a step $[p_{j-1},p_j]$ in a least path p for the differential. If there is a subset x covered by p_j, such that $R(x,p_j) = 1$, let e be the element in the difference set $p_j - x$, and define subsets q_i by $q_i = p_i - e$, for $i \leq j$. For some value k of the subscript, $q_{k-1} = q_k = p_k - e = p_{k-1}$, because e is an element in the subset p_j, and the path p begins at the empty set. For subscript values i between k and j, the subsets q_i form a path from p_{k-1} to x. By the translation property, the differential R has value 1 on all steps $[q_i,p_i]$, $i = k,\ldots,j$. By the exactness of the differential, $R(q_{i-1},q_i) = R(p_{i-1},p_i)$, $i = k+1,\ldots,j$. Compare the words associated with the path p and with this alternate path, q, agreeing with p except on the path segment $[p_{k-1},p_j]$. The word for the path q is formed by deleting the letter 1 in the k^{th} position, shifting the $k+1^{st}$ through j^{th} letters forward on space, and replacing the letter 1 in the j^{th} position. There is at least one letter 0 between the $k+1^{st}$ and j^{th} letters in the word for the path p. The first letter 0 between the $k+1^{st}$ and j^{th} letters is moved forward by one position when the word for q is formed, so the latter word is alphabetically prior to the word for the path p. This contradiction implies that the differential R has value

0 on all steps $[x,p_i]$ for subsets x covered by p_i, so the subset p_i is a dual fixed point for R on L.

Consider for a moment the coefficient array ρ_{ij} for the rank generating function ρ of an exact differential R with Whitney rank function r on a Boolean algebra L of all subsets of an n-element set. If a subset y covers a subset x, and the subset x has grading $g(x) = (g_1(x),g_2(x)) = (i,j)$, the grading $g(y)$ is either $(i,j+1)$ or $(i-1,j)$, if the differential R has value 0 or 1, respectively, on the step $[x,y]$.

If a subset x with grading $g(x) = (i,j)$ is on a least path for the differential R, and if y is the subset covering x in this least path, then $R(x,y) = 0$ if and only if there exists a subset with grading $(i,j+1)$. Thus the subsets in a least path are enumerated by the extreme entries in the coefficient array for the rank generating function.

For example, the partition differential for the

five-edged graph,  , has coefficient array

$$\begin{array}{lll} 1 & & \\ 5 & 1 & \\ 9 & 3 & \\ 7 & 5 & 1 \end{array}$$

for the rank generating function ρ.

The least path p, the intervals $[p_i, 1]$ of which correspond to successive contractions of the 3 edges in the tribond, then the two single edges, has word 10010, the first word of this differential. This first word corresponds to the outline of the coefficient array,

$$\begin{array}{ccc} 1 & \cdot & \cdot \\ 5 & 1 & \cdot \\ 9 & 3 & \cdot \\ 7 & 5 & 1 \end{array}$$

starting from the empty subset 0

in position (r(1),0), and ending with the full subset 1 in position (0,n-r(1)). Further considerations of this nature will follow our proof of the enumeration theorem.

In order to establish the inductive step in our proof that any word is a first word for some differential, we need to know precisely the nature of extensions of exact differentials to enlarged domain lattices. All proper extensions may be given in terms of the lattice of fixed points of the original differential. For this purpose we introduce the concept of a modular cut of a lattice.

Definition: A <u>modular</u> <u>cut</u> of a lattice Q is a bipartition of the elements of Q (one section of which may be empty), such that each bipartition section is complete with respect to betweenness, and the section containing the element 1 is complete under the formation of meets, $x \wedge y$, where $x \wedge y$ is covered by one of the elements x or y.

The notion of modular cut thus combines the properties of a Dedekind cut with an essential covering condition.

Lemma. Extensions Produce Modular Cuts: Let R be an exact differential on a Boolean algebra L of all subsets of an n-element set. Let e be any element of that finite set and let $\{H_0, H_1\}$ be the bipartition, of the subalgebra of subsets not containing the element e, defined by

$$H_0 = \{x \in [0,e']; R(x,x \vee e) = 1\}$$
$$H_1 = \{x \in [0,e']; R(x,x \vee e) = 0\}.$$

Then $\{H_0, H_1\}$ is generated by a modular cut of the fixed point lattice L/R, in the sense that the section H_0 consists of subsets contained in some fixed point in the lower half of the modular cut.

Proof: Any maximal element of the set H_0 is a fixed point (even coatomic), as we established in our characterization of meet irreducible fixed points[1]. The translation property of differentials provides completeness of the bipartition sections with respect to betweenness. For example, if $x \leq y \leq z$ with x and z in H_1, we know $R(x, x \vee e) = 0$ and $x \leq y$, so $R(y, y \vee e) = 0$, and y is also in the section H_1.

It remains to prove that the bipartition of the fixed point lattice L/R induced by the bipartition $\{H_0, H_1\}$ on the image of L/R under inclusion in L, is a modular cut. Assume fixed points x and y in the section H_1 have an intersection z in H_0, and that the image of x covers the image of z in the fixed point lattice L/R. Choose a path p in the Boolean algebra L from z to x. By our proof of the fixed point covering property of exact differentials[2], $R(p_{i-1}, p_i)$ has value 1 only for $i = 1$. Since the fixed point x is in the section H_1, $R(x, x \vee e) = 0$, so the R-sum from z to $x \vee e$ is 1. Lift the path p to a path $p \vee e$ from $z \vee e$ to $x \vee e$. Since $R(z, z \vee e) = 1$, R is zero along the entire path $p \vee e$. Since the fixed point z is also the meet in the Boolean algebra L of the fixed points x and y, the path p may

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[1] supra, Chapter III, §5, p. 86.

[2] supra, Chapter III, §2, p. 59.

be lifted to a path $p \vee y$ from y to $x \vee y$, and lifted again to a path $(p \vee y) \vee e$ from $y \vee e$ to $(x \vee y) \vee e$. Since y is a fixed point of the restricted differential $R|_{[0,e']}$, and since y is an element of H_1 the differential R has value zero on the steps $[w, w \vee e]$ and $[y, y \vee e]$ for any subset w covering y and not containing the element e. The fact that $1 = R(y,w) = R(y \vee e, w \vee e)$ implies $y \vee e$ is a fixed point of the differential R. Thus $R(p_0 \vee y \vee e, p_1 \vee y \vee e) = 1$, contradicting the translation property, because $R(p_0 \vee e, p_1 \vee e) = 0$. Thus the bipartition $\{H_0, H_1\}$ is determined by a modular cut of the fixed point lattice L/R.

The converse of this lemma is also true, as we now prove.

Lemma. Modular Cuts Produce Extensions: Let R_0 be an exact differential on a Boolean algebra L_0 of all subsets of an (n-1)-element set X and let $\{H_0, H_1\}$ be any bipartition of the lattice L_0 generated by a modular cut of the fixed point lattice L_0/R_0. Then the following three statements define an exact differential R on the Boolean algebra L of all subsets of the n-element set $X+e$, for $e \notin X$: for any pair x,y of elements of L_0, with y covering x,

i) $R(x,y) = R_0(x,y)$

ii) $R(x, x \vee e) = 1$ if $x \in H_0$

$= 0$ if $x \in H_1$

iii) $R(x \vee e, y \vee e) = 0$ if $R_0(x,y) = 0$

or if $R(x, x \vee e) > R(y, y \vee e)$

and $R(x \vee e, y \vee e) = 1$ if $R_0(x,y) = 1$

and $R(x, x \vee e) = R(y, y \vee e)$.

Proof: Fixed points of R_0 on L_0 are fixed points of R on L if and only if they are in the section H_0. Fixed points of R which contain the element e must, on deletion of e, become fixed points of R_0, by the translation property. To ascertain which subsets of the form $x \vee e$, where x is a fixed point of R_0, are fixed points of R, we observe that the R value of 1 on a step $[x,w]$, for $w \in L_0$, decreases to 0 on the step $[x \vee e, w \vee e]$ if and only if x is in the partition section H_0, while w is in the section H_1. This decrease thus occurs if and only if the image of the fixed point x is covered in L_0/R_0 by the image of $C\ell(w)$, with x in the partition section H_0, and $C\ell(w)$ in the section H_1.

In summary, the fixed points of R are those of R_0 in H_0, plus those subsets $x \vee e$ covering fixed points x of R_0, where x is either in H_1, or is an element of H_0

not covered in L/R by any fixed point in H_1.

To establish that R, as defined by the three properties listed in the statement of the lemma, is a differential, we show that the family of fixed points of R is closed under intersection, and then quote the characterization of differentials in terms of fixed points, as given in Chapter I[1]. Let x and y be two fixed points of R on the enlarged Boolean algebra L. If the appended element e is not an element of $x \wedge y$, e is not in one of the two fixed points. Say $e \notin x$. Then $x \wedge y = x \wedge (y-e)$ is a meet of fixed points of R_0, and is thus a fixed point of the differential R_0. Since x is a fixed point of R, and $e \notin x$, x is in the bipartition section H_0. Thus $x \wedge y$ is also in H_0, so $x \wedge y$ is a fixed point of the differential R on the enlarged Boolean algebra L. If, on the other hand, the element e is in both fixed points x and y, e is in $x \wedge y$. If $(x \wedge y) - e$ is a fixed point of R_0 in the partition section H_1, $x \wedge y$ is a fixed point of R. If $(x \wedge y) - e$ is in the partition section H_0, we must show it is covered in L_0/R_0 by no fixed point of R_0 in the section H_1. If a fixed point z of R_0 in the section H_1 covers the

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[1]supra, Chapter 1, §3, p. 12.

fixed point $(x \wedge y) - e$ in the lattice L_0/R_0, neither $z \wedge x$ not $z \wedge y$ may be $x \wedge y$, since $\{H_0, H_1\}$ is generated by a modular cut of the lattice L_0/R_0. Since z covers $x \wedge y$, we know $z \wedge x = z \wedge y = z$, contradicting the statement $x \wedge y < z$.

Thus the set of fixed points of R is closed under intersection, and gives rise to a differential, as shown in Chapter I. That the differential is exact follows from the manner of extension. If a local graph is entirely within L_0, it may not be inexact. If a local graph has any subset in the extended portion $L - L_0$, either a pair of parallel sides corresponds to addition of the element e, or all four subsets contain the element e. In the first instance, the local graph consists of subsets x,y in L_0, together with subsets $x \vee e$, $y \vee e$ in the extension. Condition iii) guarantees that R values on the steps $[x,y]$, $[x \vee e, y \vee e]$, differ if and only if the R values $[x, x \vee e]$, $[y, y \vee e]$ differ. Such local graphs may not be inexact. Finally, assume the subsets $x, y, x \vee y, x \wedge y$ all contain the element e, and that the local graph on these four subsets is inexact, with $R(x, x \vee y) = 0$ and the other three R values 1. The downward projection of this local graph into the Boolean algebra L_0 has three R values 1 by the translation property. By exactness of R_0, this projected local graph is of type 1. Since no change occurs in the R values on three parallel sets of steps, $R((x\wedge y)-e,x-e) = R(x\wedge y,x)$, etc., we know that on the four steps $[(x\wedge y)-e,x\wedge y],[x-e,x],[y-e,y],[(x\vee y)-e,x\vee y]$ the values of the differential R must be equal. This contradicts the assumption that $R(x,x\vee y) = 0$, while $R(x-e,(x\vee y)-e) = 1$.

Theorem. Existence of Exact Differentials With a Given First Word: Let W be any n-letter word, ie: any sequence of length n consisting of zeros and ones. There exists an exact differential R on a Boolean algebra L of all subsets of an n-element set, such that the word W is the first word of the differential R on L.

Proof: The theorem is obvious for n = 1. Assume we are given a word $W = w_1,\ldots,w_n$ of length n, and that for any word of length n - 1 there exists an exact differential defined on a Boolean algebra of all subsets of an (n-1)-element set, such that the word of length n-1 is the first word. If $w_n = 1$, find an exact differential R_0 on the sublattice $[0,1-e]$, for which $w_1,\ldots,w_{n-1}$ is first word. Define $R(x,x\vee e) = 1$ for all subsets x in the sublattice $[0,1-e]$, and define

$R(x \vee e, y \vee e) = R(x,y)$ for all pairs x,y of elements in the sublattice $[0,1-e]$. This defines an exact differential on the Boolean algebra of all subsets of an n element step. That W is the first word for R follows from the fact that the word for a path employing a step parallel to $[0,e]$ in any position but the last may be obtained from the corresponding word for the projection of this path into the sublattice $[0,1-e]$ by insertion of the letter 1 at some point, moving all later letters back one space. This results in a word at least as late in the alphabetical ordering.

If the final letter w_n in the word W is 0, the proof is a bit more intricate. Form an exact differential R_0 on the Boolean algebra interval $[0,1-e]$, having $w_1,\ldots,w_{n-1}$ as first word. On the fixed point lattice $[0,1-e]/R_0$, let the upper section of a modular cut contain only the fixed point $1-e$ of R_0. Construct an extended exact differential R on the Boolean algebra L of all subsets of the n-element set in accordance with the above lemma. The coatomic fixed points of the resulting differential R are the coatoms of R_o on the interval $[0,1-e]$, together with the elements of the form $x \vee e$, where x is a fixed point of L_0/R_0 of rank $r(1)-2$.

If a subset x in the interval $[0,1-e]$ is not a spanning set, it is contained in some coatomic fixed point of R_0, so is in the bipartition section H_0, with $R(x,x \vee e) = 1$. In any path from 0 to $x \vee e$ for a non-spanning set x, the projection of this path into the lattice interval $[0,1-e]$, then via the step $[x,x \vee e]$, is a lesser path. Let p be a least path for the extended differential R, and one which passes through a minimum number of subsets containing the element e. By the argument just given, if the subset p_k is the first subset containing the element e occurring in the path p, the subset p_{k-1} must be either a spanning subset in the interval $[0,1-e]$, or else a maximal non-spanning subset: a coatomic fixed point of the differential R_0. If the subset p_{k-1} is a spanning subset for R_0, all steps above p_{k-1} in the path p have R value zero, so the path following p to p_{k-1}, then via 1-e to 1 gives rise to the same word, and involves the element e in fewer subsets. On the other hand, if p_{k-1} is a coatomic fixed point of R_0, the step $[p_{k-1},p_k]$ is the final step of p for which R has value zero. Any path along p to p_{k-1}, then via 1-e to 1 gives rise to the same word, and involves the element e in fewer subsets. Thus the path p passes through 1-e, and is a least path. The restriction of p to the interval [0,1-e] is a least path for R_0, so must give rise to the word $w_1,\ldots,w_{n-1}$. Since w_n is 0, the word for the least path p of the differential R is the word W.

Corollary. A Lower Bound for the Number of Exact Differentials: There are at least 2^n non-isomorphic exact differentials on a Boolean algebra of all subsets of an n-element set.

Proof: There are 2^n different words of length n, and each may be first word for some exact differential.

The construction of an exact differential with a given first word may be carried out methodically. This has been done for all words of length 2,3,4 and 5 in Appendix B. On a Haase diagram of the Boolean algebra of all subsets of an n-element set, we color red those steps on which the differential R is to have value 1, leaving black those steps on which R is to have value 0. Choose any path to be the least path, and color it to conform to the given first word. Then color all steps according to the requirements of translation and exactness, together with the requirement that the given path be least. Most helpful is our proof that the lower ends of intervals on which R = 1 in a least path are fixed points, and that the upper ends of intervals on which R = 0 are dual fixed points.

When all the implications of the first word are exhausted, an exact differential may not be fully determined. An exact differential can then be defined in more than one way with the given first word; the number of such ways we shall term the multiplicity of the word. Thus,

> the <u>multiplicity</u> $\theta(W)$ of a word W of length n is the number of isomorphically inequivalent exact differentials with first word W, definable on the Boolean algebra of all subsets of an n-element set.

We have proven that all words have multiplicity θ at least equal to one. All words of one, two, or three letters have multiplicity equal to one. A single four letter word, 1010, has multiplicity two.

An application of the fundamental domain relation[1] to the array of coefficients α_{ij} for the Tutte polynomial, $\alpha(\xi,\eta) = \rho(\xi-1,\eta-1)$, bears on this question of

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[1]supra, Chapter IV, §2, p. 111.

multiplicity. As a formula concerning coefficients, the fundamental domain relation may be written

$$\text{for all } k,\ \binom{n}{k} = \sum_i \rho_{i,k+i-r(1)}.$$

Substitution of Tutte polynomial coefficients for rank generating function coefficients, we have

$$\text{for all } k,\ \binom{n}{k} = \sum_i \sum_{s,t} \binom{s}{i}\binom{t}{k+i-r(1)}\alpha_{st}$$

$$= \sum_{s,t} \Big(\sum_i \binom{s}{i}\binom{t}{k+i-r(1)}\Big)\alpha_{st}$$

This may be simplified, using the fact that the product $\binom{s}{i}\binom{t}{k+i-r(1)}$ enumerates $(s+k-r(1))$-element subsets chosen from the union of an s-element set with a t-element set, in which i elements are chosen from the s-element set. The sum over i of these products must be the binomial coefficient $\binom{s+t}{s+k-r(1)}$. Thus

$$\text{for all } k,\ \binom{n}{k} = \sum_{s,t} \binom{s+t}{s+k-r(1)}\alpha_{st}.$$

Substituting $q = r(1) - s$ and $p = n - s - t$, we obtain an equivalent expression in which the coefficients α_{st} provide a recursion relation for the binomial coefficients.

$$\binom{n}{k} = \sum_{p,q} \alpha_{r(1)-q,n-r(1)+q-p} \binom{n-p}{k-q}.$$

One solution for a set of α coefficients satisfying the fundamental domain relation is

$$\alpha_{ij} = 0 \text{ unless } i = r(1),\ j = n-r(1),$$
$$\text{in which case } \alpha_{r(1),n-r(1)} = 1.$$

Given any solution for α coefficients satisfying this relation, another solution is obtained by reducing any positive coefficient α_{ij}, for $1 \leq i \leq r(1)$, $1 \leq j \leq n-r(1)$, by one, and simultaneously increasing by one the adjacent coefficients $\alpha_{i-1,j}$ and $\alpha_{i,j-1}$.

We may now form the possible coefficient arrays for the Tutte polynomial of an exact differential with a given first word. Say the word has n letters, k of which are 1. Place a single 1 in the (k,n-k) position, and continue with the transfer of units from positions (i,j) to positions (i-1,j) and (i,j-1), until the array first falls withing the outline prescribed by the first word. This array, and the arrays resulting from further transfers which do not affect the outline, constitute the possible coefficient arrays for the Tutte polynomial of an exact differential with that first word.

For example, the word 1010 gives rise to two possible arrays.

$$\begin{array}{ccc} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \to \begin{array}{ccc} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{array} \to \begin{array}{ccc} 1 & \cdot & \cdot \\ \cdot & 1 & 1 \\ \cdot & \cdot & \cdot \end{array} \to \begin{array}{ccc} 1 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \cdot & \cdot & 1 \end{array} \to \begin{array}{ccc} 1 & \cdot & \cdot \\ 1 & 1 & \cdot \\ \cdot & 1 & 1 \end{array}$$

The last two arrays satisfy both the fundamental domain relation and the first word condition. Of these two, the former yields rank generating function coefficients

$$\begin{array}{lll} 1 & & \\ 4 & 2 & \\ 4 & 4 & 1 \end{array}$$

, and arises from the rank grading for the partition differential for the graph  . The latter of these two arrays yields rank generating function coefficients

$$\begin{array}{lll} 1 & & \\ 4 & 1 & \\ 5 & 4 & 1 \end{array}$$

, which arise from the rank grading for the partition differential of the graph  .

Further examples will be found in Appendix B, wherein all coefficient arrays are computed for exact differentials on Boolean algebras of ranks two through five.

Appendix A

As examples of differentials, and to illustrate the independence from one another of the properties

i) $R^{**} = R$ ("R is closed")

ii) R^* is exact ("R is pre-exact")

iii) The fixed point lattice L/R is semimodular

iv) Every element in L/R is a join of atoms.

We exhibit four differentials, possessed of properties i, ii, iii, and iv according to the following table: (1 = yes, 0 = no)

	a	b	c	d
$R^{**} = R$	0	1	0	1
R^* is exact	1	0	1	0
L/R semimodular	0	1	1	0
L/R join of atoms	1	0	0	1

Examples e and f exhibit the difference between set differentials and partition differentials.

Example g shows a product of differentials.

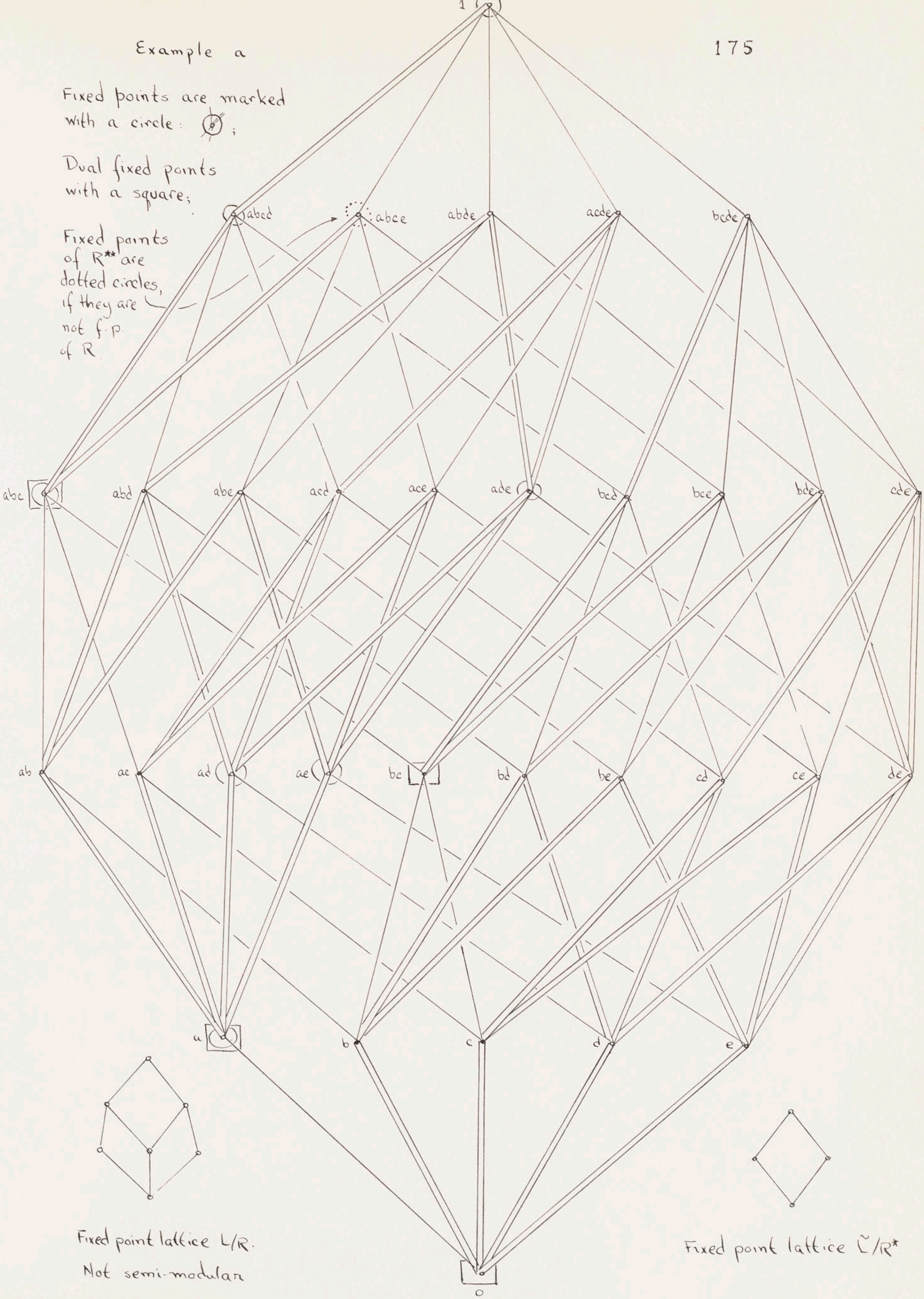
Example h is a differential on a modular but not Boolean lattice, to illustrate the theory of page 121.

Example a

Fixed points are marked with a circle: ;

Dual fixed points with a square;

Fixed points of R^{**} are dotted circles, if they are not f.p. of R



Fixed point lattice L/R.

Not semi-modular

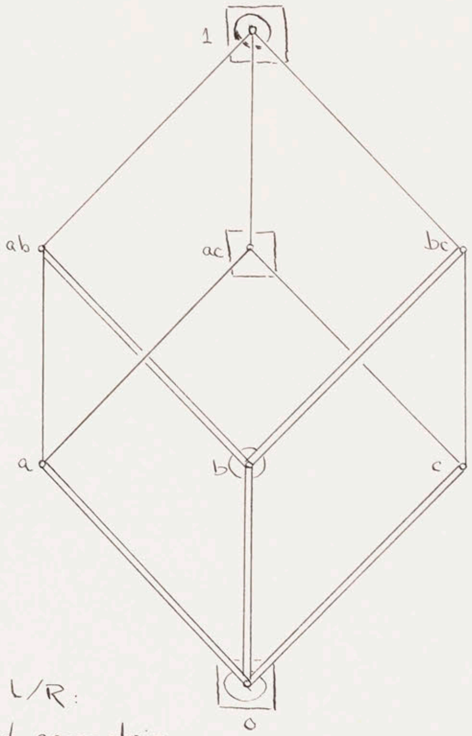
Fixed point lattice $\tilde{L}/R^*$

Notice that the fixed point lattice of R^{**} is semimodular: 

Example b

$R^{**} = R$

R^{*} is inexact

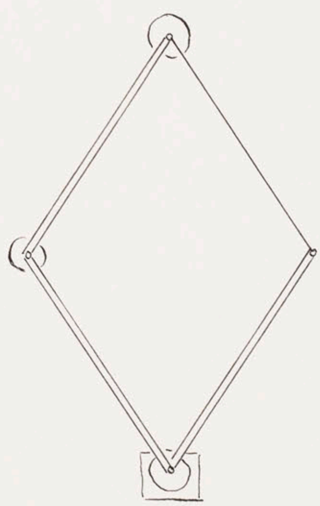


Fixed point lattice L/R:
semimodular, but not geometric

Example c

$R^{**} = 1 \neq R$

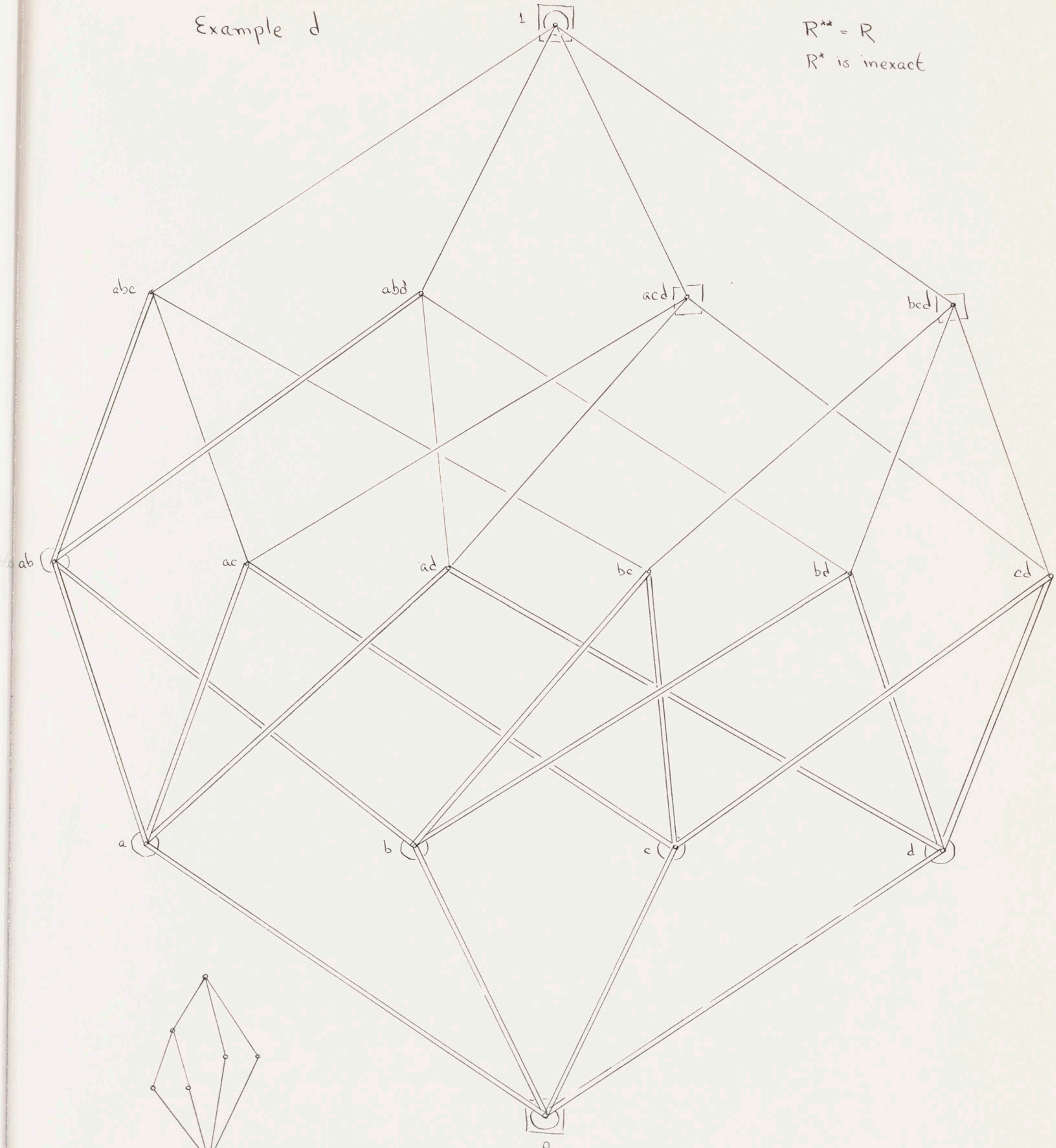
$R^{*} = 0$ is exact



Example d

$R^{**} = R$

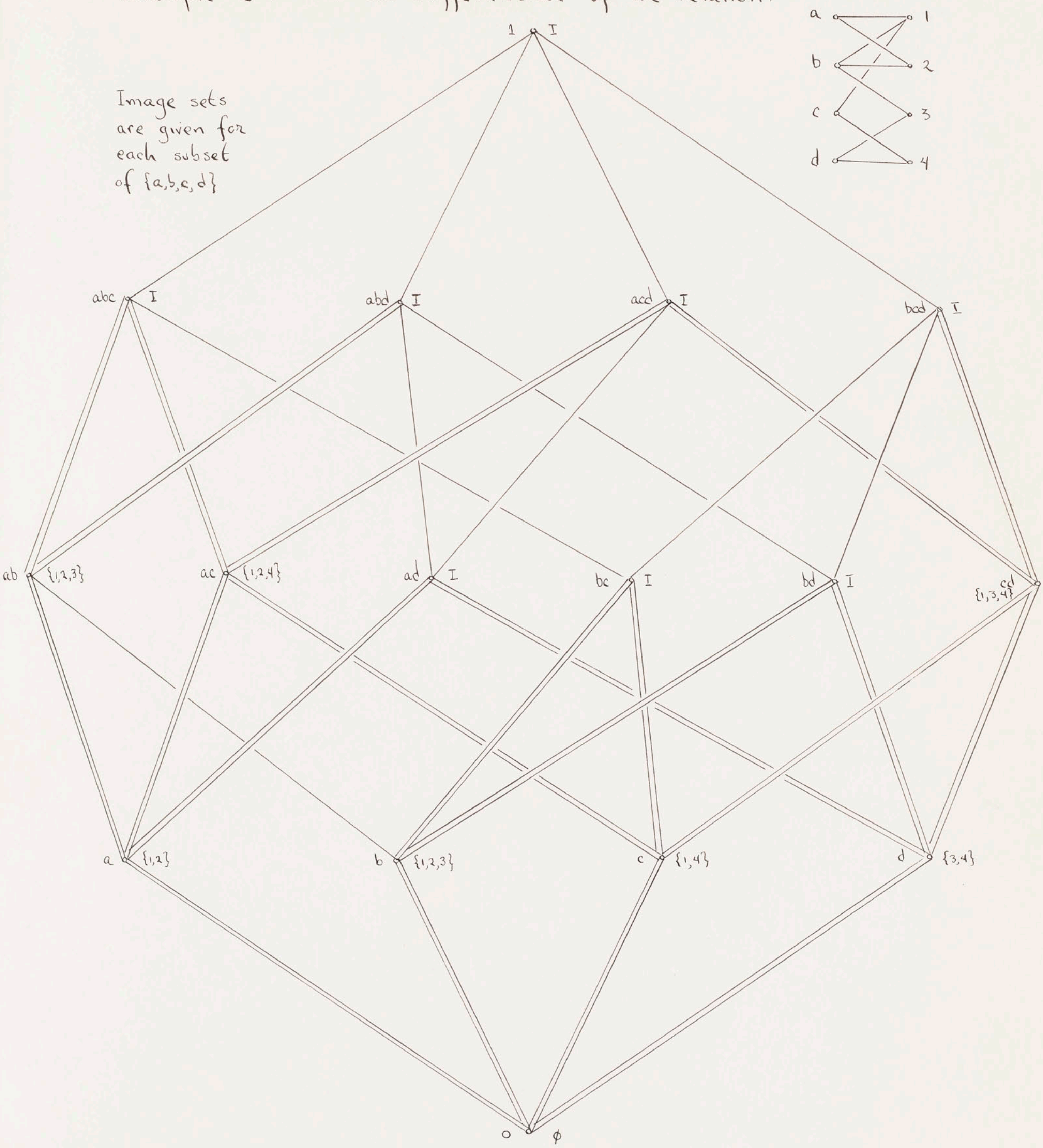
R^{*} is inexact



Fixed point lattice L/R;
not semimodular. — Every element of L/R is, however, a join of atoms.

Example e: The set differential of the relation:

Image sets are given for each subset of {a,b,c,d}

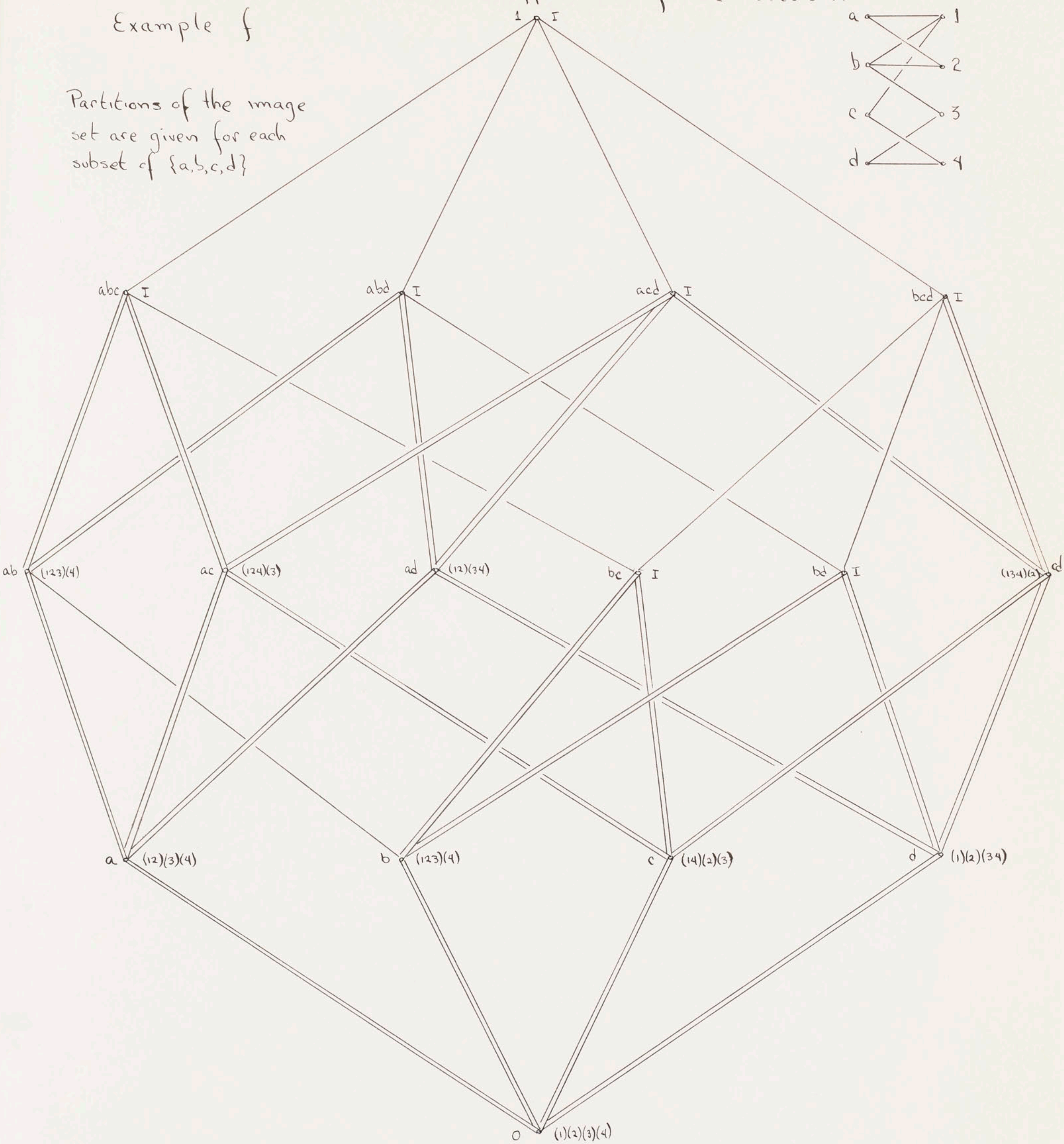


Compare with the partition differential, p. 179

Partition differential of the relation:

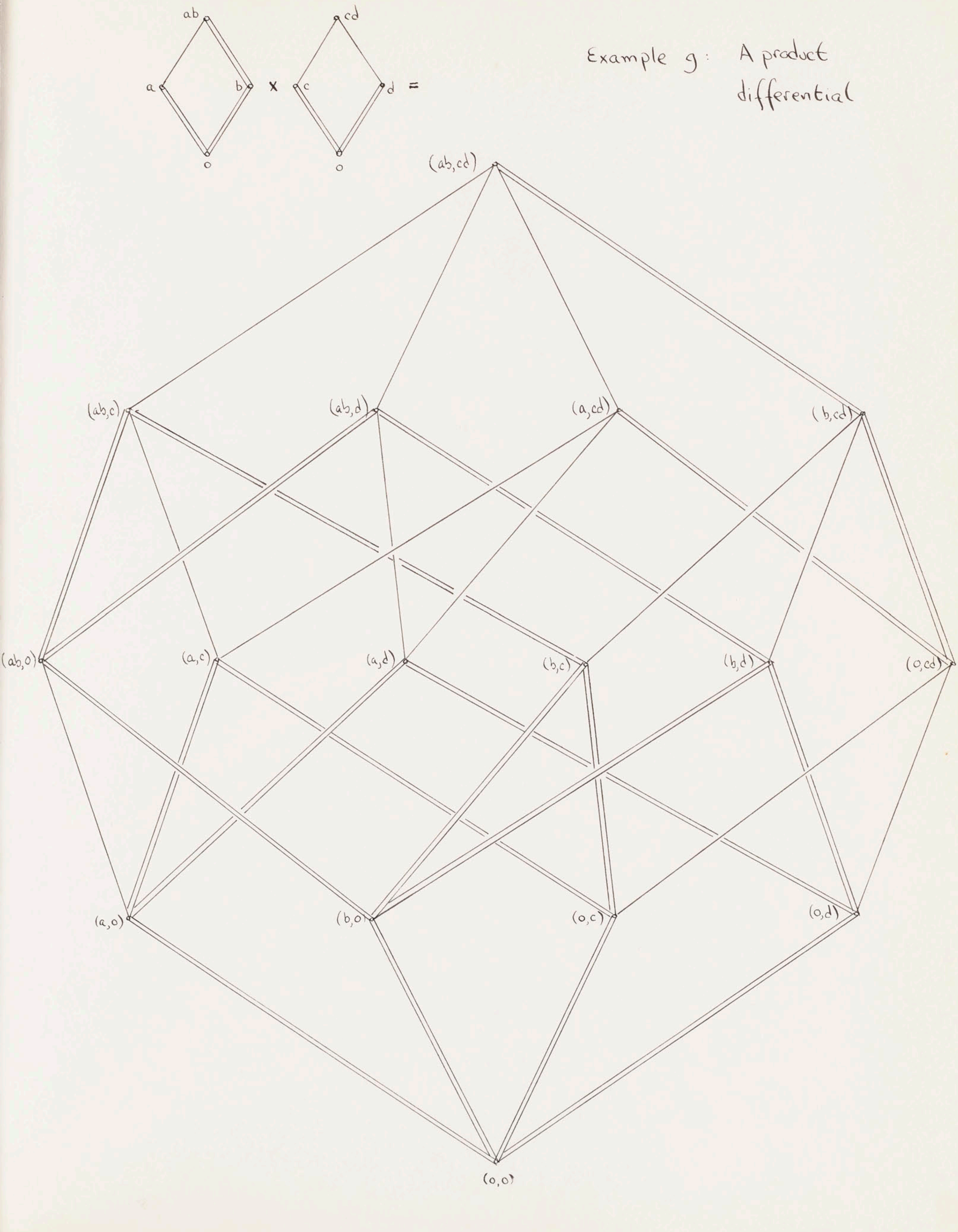
Example f

Partitions of the image set are given for each subset of {a,b,c,d}



Compare with the set differential of the same relation, as drawn on page 178.

Example 9: A product differential



Example h: see p 121, above.

	0	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	1
0	1	11	11	11	11	$\begin{smallmatrix}1&\\3&1\end{smallmatrix}$	121	121	121	$\begin{smallmatrix}1&1&\\3&4&1\end{smallmatrix}$
a_1		1				$\begin{smallmatrix}1\\1\end{smallmatrix}$	11			$\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$
a_2			1			$\begin{smallmatrix}1\\1\end{smallmatrix}$		11		$\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$
a_3				1		$\begin{smallmatrix}1\\1\end{smallmatrix}$			11	$\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$
a_4					1		11	11	11	$\begin{smallmatrix}1&\\3&1\end{smallmatrix}$
b_1						1				11
b_2							1			$\begin{smallmatrix}1\\1\end{smallmatrix}$
b_3								1		$\begin{smallmatrix}1\\1\end{smallmatrix}$
b_4									1	$\begin{smallmatrix}1\\1\end{smallmatrix}$
1										1

The coefficient arrays

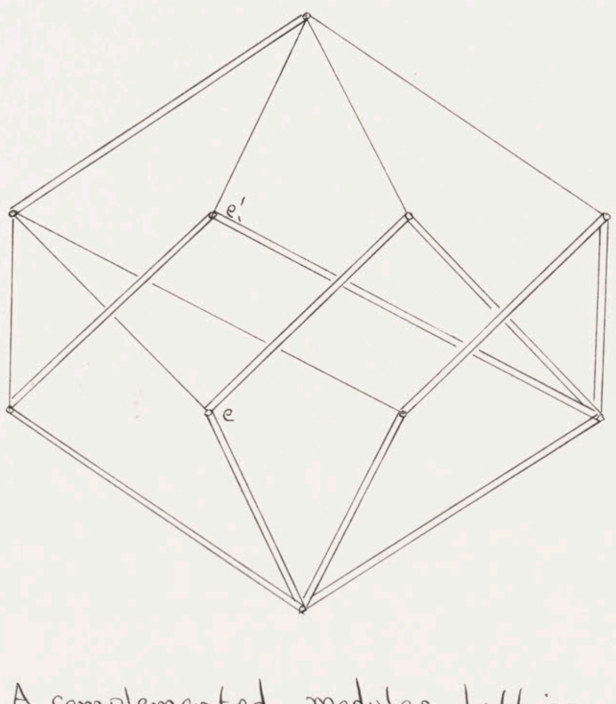
$\rho_{ij}(x,y)$

for all pairs

(x,y)

in the differential given below

[a_i refer to elements of rank 1, b_i to those of rank 2.]



A complemented modular lattice

(the element e has complements e_1', e_2')

To compute $\rho_{ij}(0,1) = \begin{smallmatrix}1&1&\\3&4&1\end{smallmatrix}$ according to the formula on p.121,

<u>add</u>

$\rho_{ij}(e,1) = \begin{smallmatrix}1&1\\1&1\end{smallmatrix}$

$\rho_{ij}(0,e_1') = 1\ 2\ 1$

$\rho_{ij}(0,e_2') = 1\ 2\ 1$

and <u>subtract</u>

the right-translate by one unit of $\rho_{ij}(0, e_1' \wedge e_2')$

$= 1\ 1$

ie: subtract

$0\ 1\ 1$

Total: $\begin{smallmatrix}1&1&\\3&4&1\end{smallmatrix}$

$\rho(0,1;\xi,\eta) = \eta(\xi+1) + \xi^2 + 4\xi + 3$

Appendix B

There follows a compilation of exact differentials whose domain lattices are the Boolean algebras of all subsets of a 1,2,3,4, or 5-element set. These are listed in the order of their first words.

If an exact differential is the partition differential of a graph relation, the appropriate graph is listed. If no such graph exists, a higher order geometric relation is indicated, as is the case for the word 1100.

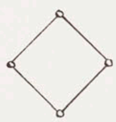
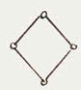
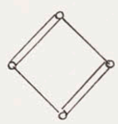
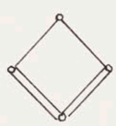
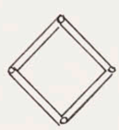
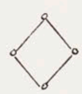
Beside the first word and indication of the graph relation, there are listed the coefficient arrays of the rank generating function and of the Tutte polynomial. The fixed point lattices of the differential and of its dual are then drawn or described.

For exact differentials defined on the smaller Boolean algebras, the differentials themselves are drawn, with double (or red) lines indicating steps on which the differential has value 1. The method for drawing all exact differentials with a given first word is indicated at the end of this appendix.

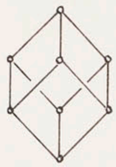
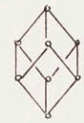
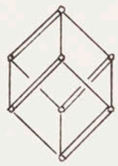
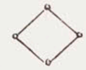
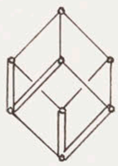
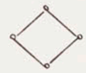
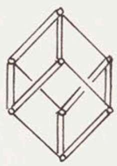
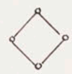
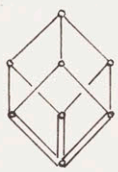
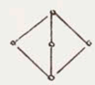
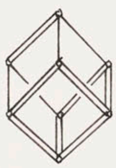
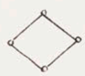
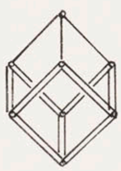
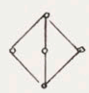
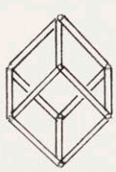
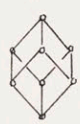
Exact Differentials on the Boolean Algebra of a One-element Set

First Word	Differential	Graph	ρ	α	L/R	$\tilde{L}/\tilde{R}$	factorization
0			1 1	1 ·	∘		prime
1			1 1	· 1		∘	prime

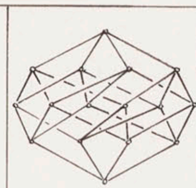
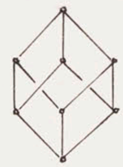
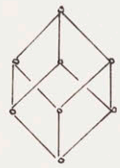
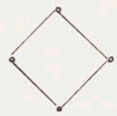
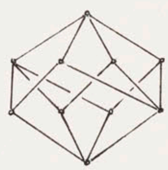
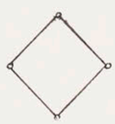
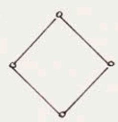
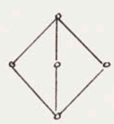
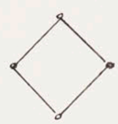
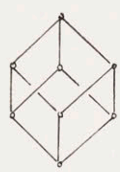
Exact Differentials on the Boolean Algebra of a Two-element Set

00			1 2 1	1 · ·	∘		0 × 0
01			1 1 1 1	· 1 · ·			0 × 1
10			1 2 1	1 · 1			prime
11			1 2 1	· · 1		∘	1 × 1

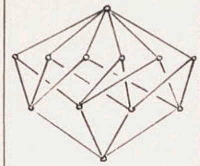
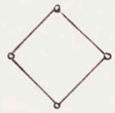
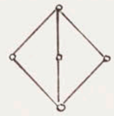
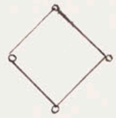
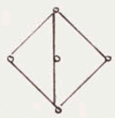
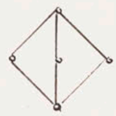
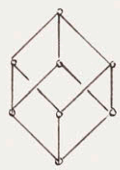
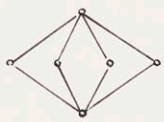
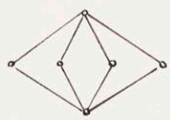
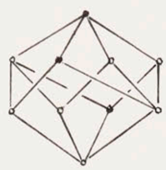
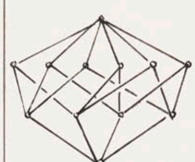
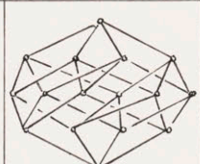
Exact Differentials on the Boolean Algebra of a Three-element Set

First Word	Differential	Graph	ρ	α	L/R	$\tilde{L}/R^*$	factorization
000			1 3 3 1	1 · · ·	∘		0×0×0
001			1 1 2 2 1 1	· 1 · · · ·			0×0×1
010			1 3 1 2 1	1 · 1 · ·			0 × 10
011			1 2 1 1 2 1	· · 1 · · ·			0×1×1
100			1 3 3 1	1 1 · 1			prime
101			1 1 2 3 1	· 1 · · 1			10 × 1
110			1 3 3 1	1 · 1 1			prime
111			1 3 3 1	· · · 1		∘	1×1×1

Exact Differentials on the Boolean Algebra of a Four-element Set

First Word	Graph	ρ	α	L/R	$\tilde{L}/R^*$	factorization
0000		1 4 6 4 1	1	∘		0×0×0×0
0001		1 1 3 3 3 3 1 1	. 1			0×0×0×1
0010		1 4 1 5 2 2 1	1 . 1			0×0×10
0011		1 2 1 2 4 2 1 2 1	. . 1			0×0×1×1
0100		1 4 6 1 3 1	1 1 . 1 . .			0×100
0101		1 1 3 4 1 2 3 1	. 1 . . 1 . . .			0×10×1
0110		1 4 3 1 3 3 1	1 . 1 1 . . .			0×110
0111		1 3 3 1 1 3 3 1	. . . 1			0×1×1×1

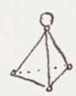
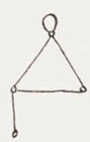
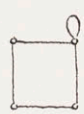
... Four - element Set (continued)

First Word	Graph	ρ	α	L/R	$\tilde{L}/R^*$	factorization
1000		1 4 6 4 1	1 1 1 · 1			prime
1001		1 1 3 3 3 4 1	· 1 · 1 · · 1			100 × 1
1010		1 4 2 4 4 1	1 · 2 · · 1			10 × 10
		1 4 1 5 4 1	1 1 1 · 1 1			prime
1011		1 2 1 2 5 4 1	· · 1 · · · 1			10 × 1 × 1
1100	(non-graphic)  face - edge incidence of tetrahedron	1 4 6 4 1	1 2 2 1			prime
1101		1 1 3 6 4 1	· 1 · · 1 1			110 × 1
1110		1 4 6 4 1	1 · 1 1 1			prime
1111		1 4 6 4 1	· · · · 1			1 × 1 × 1 × 1

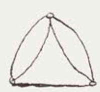
Exact Differentials on the Boolean Algebra of a Five-element Set

First Word	Graph	ρ	α	factorization
00000		1 5 10 10 5 1	1 · · · · ·	0×0×0×0×0
00001		1 1 4 4 6 6 4 4 1 1	· 1 · · · · · · · ·	0×0×0×0×1
00010		1 5 1 9 3 7 3 2 1	1 · 1 · · · · · ·	0×0×0×10
00011		1 2 1 3 6 3 3 6 3 1 2 1	· · 1 · · · · · · · · ·	0×0×0×1×1
00100		1 5 10 1 9 2 3 1	1 1 · 1 · · · ·	0×0×100
00101		1 1 4 5 1 5 7 2 2 3 1	· 1 · · 1 · · · · · ·	0×0×10×1
00110		1 5 3 1 7 6 2 3 3 1	1 · 1 1 · · · · · ·	0×0×110
00111		1 3 3 1 2 6 6 2 1 3 3 1	· · · 1 · · · · · · · ·	0×0×1×1×1

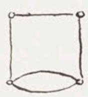
... of a Five-element Set (continued)

First Word	Graph	ρ	α	factorization
01000		1 5 10 10 1 4 1	1 1 1 · 1 · ·	0 × 1000
01001		1 1 4 4 6 7 1 3 4 1	· 1 · 1 · · 1 · · ·	0 × 100 × 1
01010		1 5 2 8 6 1 4 4 1	1 · 2 · · 1 · · ·	0 × 10 × 10
		1 5 1 9 5 1 5 4 1	1 1 1 · 1 1 · · ·	0 × 1010
01011		1 2 1 3 7 5 1 2 5 4 1	· · 1 · · · 1 · · · ·	0 × 10 × 1 × 1
01100	(non-graphic)  Loop added to face-vertex incidence	1 5 10 4 1 6 4 1	1 2 · 2 1 · · ·	0 × 1100
01101		1 1 4 7 4 1 3 6 4 1	· 1 · · 1 1 · · · ·	0 × 110 × 1
01110		1 5 6 4 1 4 6 4 1	1 · 1 1 1 · · · ·	0 × 1110

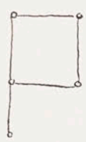
··· of a Five-element Set (continued)

First Word	Graph	ρ	α	factorization
01111		1 4 6 4 1 1 4 6 4 1	· · · · 1 · · · · ·	0×1×1×1×1
10000		1 5 10 10 5 1	1 1 1 1 · 1	prime
10001		1 1 4 4 6 6 4 5 1	· 1 · 1 · 1 · · 1	1000 × 1
10010		1 5 1 9 4 6 5 1	1 1 1 · 2 · · 1	100 × 10
		1 5 1 9 3 7 5 1	1 1 1 1 1 · 1 1	prime
10011		1 2 1 3 6 3 3 7 5 1	· · 1 · · 1 · · · 1	100×1×1
10100		1 5 10 2 8 5 1	1 2 1 2 1 1	prime
	(non-graphic) face-vertex incidence of a tetrahedron with one face duplicated.	1 5 10 1 9 5 1	1 2 2 1 2 1	prime

... of a Five-element Set (continued)

First Word	Graph	ρ	α	factorization
10101		1 1 4 6 2 4 8 5 1	· 1 · · 2 · · · 1	10 × 10 × 1
		1 1 4 5 1 5 9 5 1	· 1 · 1 1 · · 1 1	1010 × 1
10110		1 5 4 1 6 9 5 1	1 · 2 1 · · 1 1	10 × 110
		1 5 3 1 7 9 5 1	1 1 1 1 · 1 1 1	prime
10111		1 3 3 1 2 7 9 5 1	· · · 1 · · · · 1	10 × 1 × 1 × 1
11000	(non-graphic)	1 5 10 10 5 1	1 2 3 · 3 1	prime
11001	(non-graphic) face-vertex incidence of the tetrahedron plus a free edge	1 1 4 4 6 10 5 1	· 1 · 2 · · 2 1	1100 × 1

... of a Five-element Set (continued)

First Word	Graph	ρ	α	factorization
11010		1 5 2 8 10 5 1	1 1 2 · 1 2 1	prime
	(non-graphic)	1 5 1 9 10 5 1	1 2 1 · 2 2 1	prime
11011		1 2 1 3 9 10 5 1	· · 1 · · · 1 1	110 x 1 x 1
11100	(non-graphic)	1 5 10 10 5 1	1 3 · 3 2 1	prime
11101		1 1 4 10 10 5 1	· 1 · · 1 1 1	1110 x 1
11110		1 5 10 10 5 1	1 · 1 1 1 1	prime
11111		1 5 10 10 5 1	· · · · · 1	1x1x1x1x1

The method for drawing all exact differentials with a given first word is now indicated by an example. The differential values of the first word are assigned to steps in some convenient path, here taken to be 0,a,ab,abc,1. Then all implications are worked out, and the appropriate differential values indicated, according to the following principles:

i) The lower end of a step on which R = 1 in the least path is a fixed point, so all steps of which it is the lower end have differential value R = 1.

i') The upper end of a step on which R = 0 in the least path is a dual fixed point, so all steps of which it is the upper end have differential value R = 0.

ii) The translation property: Any step below and parallel to a step on which R = 1 also has R = 1; any step above and parallel to a step on which R = 0 also has R = 0.

iii) Exactness, in the form of the statement of independence of path for differential sums.

The method:

First word: 1010

Least path: R(0,a) = 1 R(a,ab) = 0 R(ab,abc) = 1
R(abc,1) = 0.

FIxed points: R(0,b) = R(0,c) = R(0,d) = 1

$R(b,ab) = 0$

$R(ab,abd) = 1$

$R(abd,1) = R(acd,1) = R(bcd,1) = 0.$

Translation: $R(a,ac) = R(a,ad) = R(b,bc) = R(b,bd) = 1$

$R(ac,abc) = R(ac,acd) = R(ad,abd) =$

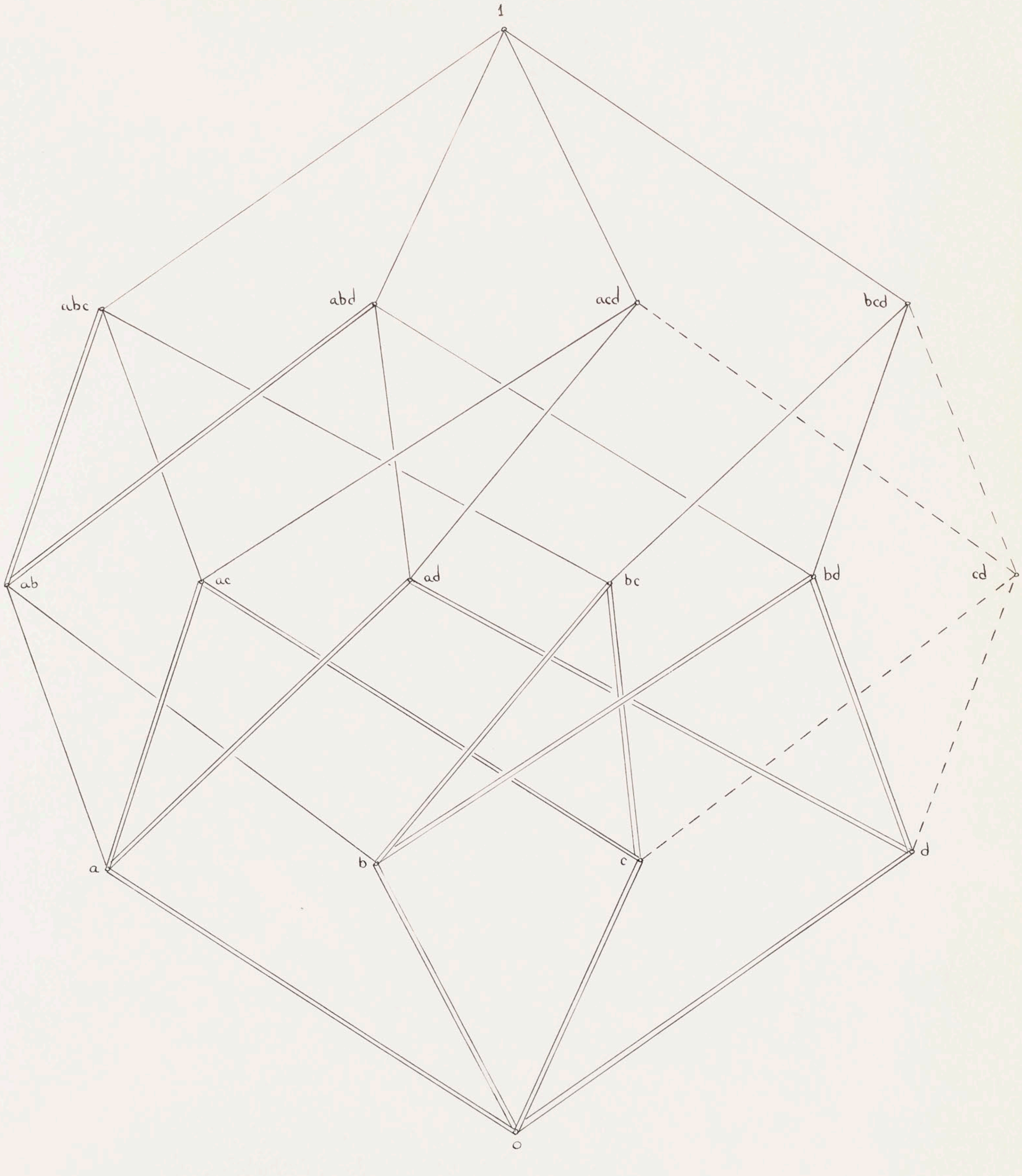
$R(ad,acd) = 0.$

Exactness: $R(c,ac) = R(c,bc) = R(d,ad) = R(d,bd) = 1$

$R(ac,acd) = R(ad,acd) = R(bc,bcd) = R(bd,bcd)$

$= 0.$

This leaves four steps with indeterminate differential values, namely [c,cd], [d,cd], [cd,acd], [cd,bcd]. By exactness, we know R(c,cd) = R(d,cd) and 1 - R9c,cd) = R(cd,acd) = R(cd,bcd). There are thus two possible exact differentials, as indicated following the word 1010 in the foregoing tables.


1
abc
abd
acd
bcd
ab
ac
ad
bc
bd
cd
a
b
c
d
o

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Biographical Note

The author was born on 12 August, 1932, in Detroit, Michigan, and received his early schooling in that city. The high school years were spent at the Taft School, Watertown, Connecticut. He was graduated second in his class in June 1950.

He did his undergraduate work at the University of Michigan, Ann Arbor, Michigan, pursuing a major curriculim in Physics. He was elected to the scholastic honorary societies Phi Eta Sigma, Phi Kappa Phi, and Phi Beta Kappa. The award of his diploma in June of 1954 was with distinction in Physics.

Upon graduation, the author began active duty as Ensign in the U.S. Naval Reserve, aboard the tanker U.S.S. Marias with the U.S. Sixth Fleet in the Mediterranean Sea.

Discharged in 1956, he recommenced his study of Physics as a special graduate student at the Massachusetts Institute of Technology. He was admitted to the graduate school as a student of Mathematics in January 1957, and was appointed to a part-time teaching assistantship at the end of that year. He was promoted to a part-time instructorship in 1961, and served in that capacity until June 1962.

The author was married in 1962 and began teaching at Northeastern University, Boston, in September of that year. He was appointed to an Assistant Professorship in the spring of 1963. The research for this thesis was accomplished during the year 1963, and during the spring of 1964.