Spectral Fukaya categories for Liouville manifolds

by

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ABSTRACT

This thesis constructs stable homotopy types underlying symplectic Floer homology, realizing a program proposed by Cohen, Jones and Segal twenty-five years ago. We work in the setting of Liouville manifolds with a stable symplectic trivialization of their tangent bundles, where we prove that the moduli spaces of Floer trajectories are smooth and stably framed. We then develop a basic TQFT formalism, in the stable homotopy category, for producing operations on these Floer homotopy types from families of punctured Riemann surfaces. As a byproduct, we can generalize many familiar algebraic constructions in traditional Floer homology over the integers to Floer homotopy theory: among them symplectic cohomology, wrapped Floer cohomology, and the Donaldson-Fukaya category.

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1. Floer homotopy

1.1. **Overview.** In 1994, Cohen-Jones-Segal [3] proposed that underlying the various incarnations of Floer homology, in the right topological and geometric circumstances, there should be a stable homotopy type, whose integer or $\mathbb{Z}/2$-valued classical homology recovers the original Floer groups. Since then, Floer theory has burgeoned into an enormous field in its own right, with Floer homology groups in various settings often coming with sophisticated algebraic structures that are amenable to computation. On the other hand, there have been major technical advances in the foundations of stable homotopy theory. In particular there are now several models for a symmetric monoidal model category of spectra, opening up the possibility of defining many of the algebraic structures now familiar to symplectic geometers— chief amongst those, the Fukaya category— over the sphere spectrum.

The purpose of this thesis is to provide some of the first technical steps towards realizing this vision. We must necessarily work in a very restrictive setting:

(i) We restrict our attention to *Liouville manifolds and their exact Lagrangian submanifolds*, to avoid any issues with bubbling or non-transversality in the theory of pseudoholomorphic curves.
(ii) We also impose a strong topological condition on the Liouville manifolds we study, namely that their *tangent bundles admit a stable symplectic trivialization*, in order to guarantee that the moduli spaces of pseudoholomorphic curves admit the requisite orientation structures.
(iii) Our construction will furnish only algebraic structures on Floer *homotopy types*, rather than on a point-set level spectrum: this means that while we can produce spectral analogues of things like the pair of pants product on Hamiltonian Floer cohomology, versions of chain-level structures such as the Fukaya A_∞-product are beyond the scope of this work.

Within these parameters, the main outcome of this thesis is a sort of "open-closed topological quantum field theory" with values in the stable homotopy category arising from the Floer theory of a Liouville manifold equipped with a stable symplectic framing of its tangent bundle.

There are three main components to this theory. First, we produce Floer homotopy types associated to a non-degenerate Hamiltonian on a Liouville manifold or a pair of transversely intersecting exact Lagrangians, by filling in the analytic and topological details of Cohen-Jones-Segal's proposal. Second, we produce from a family of Riemann surfaces, with marked input and output points and some additional decorations from the target Liouville manifold, multi-linear operations on these Floer homotopy types. Third, we extend this to families of possibly nodal Riemann surfaces, and show that conversely "gluing" two families of Riemann surfaces into a nodal family yields compositions of the corresponding operations.

The bulk of this thesis is devoted to establishing the major technical ingredients for this theory to work. Broadly speaking, the first half of this thesis, comprising chapters 2, 3 and 4, covers the algebraic and homotopy theoretic questions: how does one take the geometric information from the moduli spaces of pseudoholomorphic curves and turn it into homotopy types and operations relating them. The perspective here is mostly abstract: we develop an extension of Cohen-Jones-Segal's language of flow categories to talk about "multilinear" maps between them, and explain how to produce "geometric realizations" of such maps given appropriate orientation data.

The second half of the thesis, comprising chapters 5, 6, 7 and 8, covers the analytic and geometric questions: how do we guarantee that in our setting, the moduli spaces carry the structures that are needed as inputs in the first half of the thesis. Here, the main steps are to prove that the moduli spaces in exact Floer theory can be endowed with smooth structures with corners, and to show that under hypotheses on the tangent bundle of the Liouville manifold in question, the virtual Fredholm index bundle of the linearized Floer operators over these moduli spaces can be trivialized.

Before we dive into these technical considerations, let us extrapolate on the geometric outcomes, beginning from the perspective of topological quantum field theory.

1.2. **A Floer-theoretic TQFT in spectra.** Let us fix, for the entirety of this thesis, a target geometry: take a Liouville manifold M, equipped with what we call a *background framing*:

- a stable symplectic trivialization Φ_M of the tangent bundle TM.

Whenever we consider Lagrangian submanifolds of M, we require them to be exact, and either compact or cylindrical at infinity. We moreover require them to carry an additional topological structure, called a *framed brane structure*:

- a null-homotopy of the stable Gauss map
$$L \longrightarrow U/O$$
classifying the Lagrangian sub-bundle TL of the stably trivialized bundle $TM|_L$.

We also need a suitable class of Hamiltonian functions. We will take those $H : M \to \mathbb{R}$ that are *linear at infinity*, meaning in the natural coordinate r on the conical part of M given by the Liouville flow we have
$$H = \tau e^r \qquad \text{outside a compact subset}$$
where $\tau \in \mathbb{R}_{>0}$ is some real number that is *not* the period of a Reeb orbit of the contact boundary at infinity of M; write $\mathcal{H}^\tau$ for this class of functions. We say that such H is *of slope* τ.

For an S^1-parameter family H_t of elements of $\mathcal{H}^\tau$ for a fixed τ satisfying a non-degeneracy condition, the 1-periodic orbits of the corresponding Hamiltonian flow, together with the moduli spaces of solutions to Floer's equation for cylinders for an auxiliary regular choice J_t

of a family of almost complex structures, define an object called a *flow category*. This flow category is equipped with some additional topological data from the background framing, which we call a *spectral system*– the definitions will be covered in Chapter 2. The pair of choices (H_t, J_t) will be called a Hamiltonian Floer datum. The outcome of the geometric realization construction for a flow category with a spectral system is then a spectrum, which gives:

Theorem 1.1. *In the presence of a background framing, associated to a Hamiltonian H_t and a regular almost complex structure J_t (a Hamiltonian Floer datum) there is a spectrum*

$$HF(M, H_t; \mathbb{S}) \tag{1.1}$$

which we call the Hamiltonian Floer homotopy type, and will usually think of as the Floer homology with coefficients in the sphere spectrum.

Similarly, given two cylindrical Lagrangians L_0, L_1 equipped with framed brane structures, we choose a $\tau > 0$ so that there are no Reeb chords of length τ connecting the corresponding Legendrians in the contact boundary at infinity, and then take a $[0,1]$-parameter family H_t of Hamiltonians in $\mathcal{H}^\tau$ so that the time 1 flow of L_0 intersects L_1 transversely. These transverse intersection points, together with the moduli spaces of Floer's equation for strips, define a flow category, and the framed brane structures induce a spectral system. Again, the pair of choices (H_t, J_t) is called a Lagrangian Floer datum. Thus we have

Theorem 1.2. *Given framed brane structures on Lagrangians L_0, L_1, which are compact or cylindrical at infinity, and a Floer datum (H_t, J_t), there is a spectrum*

$$HF(L_0, L_1, H_t; \mathbb{S}) \tag{1.2}$$

which we call the Lagrangian Floer homotopy type, regarding it as the Floer homology with coefficients in $\mathbb{S}$.

Now, suppose that we have some family

$$\hat{\mathcal{S}} \longrightarrow \mathcal{R}$$

of Riemann surfaces with boundary, and with marked points Σ, over a closed smooth manifold $\mathcal{R}$. The marked points have two different attributes: they can be incoming or outgoing; and they can lie on the boundary (and be referred to as "open"), or lie in the interior (and be referred to as "closed"); and are thus partitioned

$$\Sigma = \Sigma_o^- \sqcup \Sigma_c^- \sqcup \Sigma_o^+ \sqcup \Sigma_c^+$$

into the outgoing open and closed, and incoming open and closed points respectively. We always assume that there is exactly one outgoing marked point, which we call ζ^- (but there can be any number of incoming points, possibly zero). The closed points will moreover come with fixed "asymptotic markers", that is a distinguished real tangent direction. For technical reasons related to how we produce "spectral systems" on flow categories, we make the following restriction on such families of surfaces:

- the fibres $\hat{\mathcal{S}}_r$ of the family are all genus zero, and are thus either spheres or discs.

Now, take a (locally constant in r) labelling of the boundary components of $\hat{\mathcal{S}}_r \backslash \Sigma$ by exact Lagrangian submanifolds with framed brane structures. At each open marked point ζ, we have a pair of associated Lagrangians with branes $L_{\zeta,0}$ and $L_{\zeta,1}$. Then, equip each marked point ζ with a Hamiltonian Floer datum (H_ζ, J_ζ) in the case ζ is closed, and a Lagrangian Floer datum for $(L_{\zeta,0}, L_{\zeta,1})$ in the case that ζ is open. Let us write τ_ζ for the slope of the Hamiltonian at ζ. In order to define operations, we require that these slopes satisfy the inequality

$$\tau_{\zeta^-} \geq \sum_{\zeta^+ \in \Sigma^+} \tau_{\zeta^+}. \tag{1.3}$$

We also make an additional choice of a "perturbation datum" on the family $\mathcal{S}$, the details of which are spelt out in chapter 5, compatible with the earlier choices. The outcome of our analysis of the moduli space of pseudoholomorphic maps from $\mathcal{S}$ to M in chapters 6,7,8 and the general algebraic construction in chapter 3 is then the following:

Theorem 1.3. *For such a family of genus zero surfaces over $\mathcal{R}$ equipped with Lagrangian labels, Floer data and perturbation data, we have a "coproduct" operation on the Floer homotopy types: in the case that ζ^- is closed, this is a map in the stable homotopy category*

$$HF(M, H_{\zeta^-}; \mathbb{S}) \wedge \mathcal{R}_+ \longrightarrow \bigwedge_{\zeta \in \Sigma_c^+} HF(M, H_\zeta; \mathbb{S}) \ \wedge \bigwedge_{\zeta \in \Sigma_o^+} HF(L_{\zeta,0}, L_{\zeta,1}, H_\zeta; \mathbb{S}). \tag{1.4}$$

In the case that ζ^- is open, this is likewise a map of stable homotopy types

$$HF(L_{\zeta^-,0}, L_{\zeta^-,1}, H_{\zeta^-}; \mathbb{S}) \wedge \mathcal{R}_+ \longrightarrow \bigwedge_{\zeta \in \Sigma_c^+} HF(M, H_\zeta; \mathbb{S}) \wedge \bigwedge_{\zeta \in \Sigma_o^+} HF(L_{\zeta,0}, L_{\zeta,1}, H_\zeta; \mathbb{S}). \tag{1.5}$$

Finally, consider a family of possibly nodal genus zero Riemann surfaces with marked points, Lagrangian labels, Floer data and so forth,

$$\hat{\mathcal{S}} \longrightarrow \mathcal{R}$$

over a compact manifold with boundary $\mathcal{R}$. We require the fibres to be smooth over the interior of $\mathcal{R}$, and so that each boundary component $\mathcal{R}^\beta$ has a decomposition $\mathcal{R}^\beta = \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$, with the family decomposing along $\mathcal{R}^\beta$ as the gluing of two families

$$\hat{\mathcal{S}}_1^\beta \longrightarrow \mathcal{R}_1^\beta, \qquad \hat{\mathcal{S}}_2^\beta \longrightarrow \mathcal{R}_2^\beta$$

at the outgoing point of $\hat{\mathcal{S}}_2^\beta$ and a chosen incoming point ζ^β of $\hat{\mathcal{S}}_1^\beta$. We then have:

Theorem 1.4. *Writing $HF(\zeta)$ for the Floer homotopy type associated to the Floer data (Hamiltonian or Lagrangian) at the marked point ζ, we have a coproduct operation on Floer homotopy types*

$$HF(\zeta^-) \wedge \mathcal{R}_+ \longrightarrow \bigwedge_{\zeta \in \Sigma^+} HF(\zeta).$$

Moreover, at each boundary component of $\mathcal{R}$, the operations defined by the families over $\mathcal{R}_1^\beta$ and $\mathcal{R}_2^\beta$ fit into a commutative diagram in the stable homotopy category

$$\begin{array}{ccc} HF(\zeta^-) \wedge (\mathcal{R}_1^\beta)_+ \wedge (\mathcal{R}_2^\beta)_+ & \longrightarrow & HF(\zeta^-) \wedge \mathcal{R}_+ \\ \downarrow & & \downarrow \\ \left(\bigwedge_{\zeta \in (\Sigma_1^\beta)^+ \setminus \{\zeta^\beta\}} HF(\zeta) \right) \wedge HF(\zeta^\beta) \wedge (\mathcal{R}_2^\beta)_+ & \longrightarrow & \bigwedge_{\zeta \in \Sigma^+} HF(\zeta^+). \end{array} \tag{1.6}$$

Should one wish to work with product structures, rather than coproducts, recall for each spectrum X we can take its Spanier-Whitehead dual $DX = F(X, \mathbb{S})$. Thus, in both the Hamiltonian and Lagrangian case, we can define a *Floer cohomotopy type*, distinguished by a decorative upper $*$, by

$$HF^*(M, H_t; \mathbb{S}) = D(HF(M, H_t; \mathbb{S})), \qquad HF^*(L_0, L_1, H_t; \mathbb{S}) = D(HF(L_0, L_1, H_t; \mathbb{S})). \tag{1.7}$$

The cohomological operations induced by a family of surfaces then take the form

$$\mathcal{R}_+ \wedge \bigwedge_{\zeta \in \Sigma^+} HF^*(\zeta) \longrightarrow HF^*(\zeta^-); \tag{1.8}$$

these satisfy the analogous dual commutative diagram to (1.6). It is more convenient technically to produce the homological operations rather than the cohomological ones, however, and thus we have chosen to highlight them in this overview.

1.3. **Examples of such operations.** Before we move on to the main technical body of the thesis to establish the above constructions, let us first record a couple of examples, by no means at all exhaustive, where the formalism directly extends a structure well-known in traditional symplectic Floer homology to the spectral world.

Continuation maps. Given a pair of Floer data (H_t, J_t) and (H_t', J_t') of slopes τ, τ' respectively, either Hamiltonian or Lagrangian for a pair of Lagrangians (L_0, L_1) with brane structures, as long as $\tau \geq \tau'$ we can take perturbation data on a fixed cylinder (in the Hamiltonian case) or strip (in the Lagrangian case) interpolating from (H_t, J_t) to (H_t', J_t'). This thus defines a map on the homotopy types

$$HF(M, H_t; \mathbb{S}) \longrightarrow HF(M, H_t'; \mathbb{S}) \tag{1.9}$$

in the Hamiltonian case, or

$$HF(L_0, L_1, H_t; \mathbb{S}) \longrightarrow HF(L_0, L_1, H_t'; \mathbb{S}) \tag{1.10}$$

in the Lagrangian case. Given a triple of Floer data $(H_t, J_t), (H_t', J_t'), (H_t'', J_t'')$ with slopes $\tau \geq \tau' \geq \tau''$, we can build a family of surfaces with perturbation data over the interval $[0, 1]$ that then produces a homotopy between a composition of continuation maps

$$HF(H_t; \mathbb{S}) \longrightarrow HF(H_t'; \mathbb{S}) \longrightarrow HF(H_t'', \mathbb{S})$$

and a continuation map $HF(H_t;\mathbb{S}) \longrightarrow HF(H''_t;\mathbb{S})$.

Pair of pants product. Choose Hamiltonians H_0, H_1, H_2 of slopes τ_0, τ_1, τ_2 with $\tau_0 \geq \tau_1 + \tau_2$. Then, choose perturbation data over the pair of pants with one outgoing end (with Floer data having Hamiltonian H_0) and two incoming ends (Floer data having Hamiltonians H_1, H_2). We then have a coproduct

$$HF(M, H_0;\mathbb{S}) \longrightarrow HF(M, H_1;\mathbb{S}) \wedge HF(M, H_2;\mathbb{S}). \tag{1.11}$$

In the event that $H_1 = H_2$, this coproduct is co-commutative, by considering a family which swaps the two incoming marked points. These coproducts are moreover homotopy co-associative for quadruples of Hamiltonians H_0, H_1, H_2, H_3 of slopes $\tau_0, \tau_1, \tau_2, \tau_3$ with $\tau_0 \geq \tau_1 + \tau_2 + \tau_3$.

The Batalin-Vilkovisky operator. Take a Hamiltonian Floer datum (H, J), and consider the S^1-family of cylinders where the asymptotic marker on one end rotates, with the same Floer data on either end. This induces a Batalin-Vilkovisky operator on Hamiltonian Floer homotopy

$$HF(M, H;\mathbb{S}) \wedge (S^1)_+ \longrightarrow HF(M, H;\mathbb{S}). \tag{1.12}$$

Triangle product of Lagrangians. If L_0, L_1, L_2 are three mutually transversely intersecting Lagrangians, we can take the zero Hamiltonian for our Floer data. By considering a disc with three boundary punctures with these labels, we obtain coproduct maps

$$HF(L_0, L_2;\mathbb{S}) \longrightarrow HF(L_0, L_1;\mathbb{S}) \wedge HF(L_1, L_2;\mathbb{S}). \tag{1.13}$$

By considering degenerations of a disc with four boundary punctures, we moreover see these coproducts are co-associative for quadruples of mutually transversely intersecting L_0, L_1, L_2, L_3. Of course, one can dualize to cohomotopy to obtain the perhaps more traditional product structures.

Now note that for any pair of *compact* Lagrangians, observe we can always choose a Floer datum with slope zero: in other words, a compactly supported Hamiltonian perturbation. In particular, using the triangle product we have:

Definition 1.5. *The spectral compact Donaldson-Fukaya category $\mathscr{F}(M;\mathbb{S})$ of a Liouville domain M with a background framing is the category enriched over the stable homotopy category with objects the compact exact Lagrangians L of M equipped with brane framings, and morphism spectra $HF^*(L_0, L_1;\mathbb{S})$ for some choice of regular Floer data with a compactly supported Hamiltonian. Composition is given by the triangle product.*

Symplectic cohomology. Take a system of Hamiltonians H_n on M, of slopes $\tau_n \longrightarrow \infty$. In particular, we have an inverse system of maps of spectra

$$\ldots \longrightarrow HF(M, H_{n+1};\mathbb{S}) \longrightarrow HF(M, H_n;\mathbb{S}) \longrightarrow HF(M, H_{n-1};\mathbb{S}) \longrightarrow \ldots$$

and thus a dual directed system on the cohomotopy types HF^*.

Definition 1.6. *The symplectic cohomology of M with coefficients in the sphere spectrum is the direct limit*

$$SH^*(M;\mathbb{S}) = \varinjlim HF^*(M, H_n; \mathbb{S}). \tag{1.14}$$

The pair of pants products endow $SH^*(M;\mathbb{S})$ with the structure of a commutative and associative algebra in the stable homotopy category (with its usual symmetric monoidal structure from the smash product).

This product is moreover unital: the sphere with one outgoing boundary puncture defines a co-unit for the pair of pants co-product, which dualizes to a unit for the product on SH^*.

Wrapped cohomology. Fix an exact Lagrangian with brane framing L, cylindrical at infinity. Again by taking a system of Hamiltonians H_n on M of slopes $\tau_n \to \infty$, we can define its wrapped Floer cohomology over the sphere spectrum

$$HW^*(L;\mathbb{S}) = \varinjlim HF^*(L, L, H_n; \mathbb{S}). \tag{1.15}$$

The triangle product endows $HW^*(L;\mathbb{S})$ with the structure of an associative algebra object in the stable homotopy category. It is also unital, by taking the disc with one outgoing boundary puncture.

By considering the disc with one incoming closed marked point, and one outgoing boundary point, we obtain the *closed-open map*

$$\mathcal{CO} : SH^*(M,\mathbb{S}) \longrightarrow HW^*(L;\mathbb{S}). \tag{1.16}$$

Moreover, by gluing into the incoming closed point a sphere with no incoming points, we see that this map sends the unit of $SH^*(M,\mathbb{S})$ to the unit of $HW^*(L;\mathbb{S})$. In particular, we obtain a simple non-vanishing principle for spectral symplectic cohomology:

Proposition 1.7. *If there exists an exact Lagrangian with a brane framing such that $HW^*(L,\mathbb{S})$ is not null, then $SH^*(M,\mathbb{S})$ is not null.*

The proof follows immediately from the ring structures on either side: a unital associative ring object in the homotopy category of spectral is null if and only if its unit map is. We indeed expect that the closed-open map $\mathcal{CO}$ is a map of homotopy ring spectra; however the types of degenerations of Riemann surfaces required to prove this are beyond the scope of this thesis.

1.4. **Remarks on the category of spectra.** While our results are phrased in terms of the stable homotopy category of spectra, the actual construction of the stable homotopy types requires some use of point-set models for spectra with a symmetric monoidal smash product. Thankfully, these exist; in this thesis let us officially declare that we are using the orthogonal spectra of Mandell-May-Schwede-Shipley [8].

However, we will at no point use a point-set level model for the smash product of spectra in anything other than a totally formal fashion. All the intermediate spectra used to build the Floer homotopy types and maps between them will be Thom spectra of vector bundles over spaces, or perhaps their Spanier-Whitehead duals. All the commutative diagrams of spectra we will use will arise directly as commutative diagrams of spaces, possibly with isomorphisms of pull-back vector bundles.

In this light, the only "serious" properties we need to know about the category of orthogonal spectra is that its smash product is symmetric monoidal, that it admits an internal Hom, and that the suspension spectrum functor from the category of pointed spaces is symmetric monoidal.

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2. Flow categories and the Cohen-Jones-Segal construction

2.1. **Morse theory.** Let us begin by discussing the motivating example. Suppose M^n is an n-dimensional closed smooth manifold, and $f : M \to \mathbb{R}$ is a Morse function. Denote the set of critical points of f by $\mathrm{Crit}(f)$. We will also choose a Riemannian metric on M, so that one may speak of the gradient flow of f.

Broadly speaking, there are two ways to use the function f and its gradient flow to study the topology of M. Classically, the descending discs of each critical point give M the structure of a cell complex, yielding a cellular filtration

$$M_0 \subset M_1 \subset \ldots \subset M_n = M \tag{2.1}$$

on M; topological invariants of M such as its integer cohomology can then be computed from this structure.

In Witten's reinterpretation of Morse theory on the other hand, a cochain complex computing the cohomology is directly constructed: the free abelian group generated by $\mathrm{Crit}(f)$ (and graded by the Morse index) is endowed with a differential which counts the number of points in the dimension zero *moduli spaces of flow lines* between two critical points $x_-, x_+ \in \mathrm{Crit}(f)$:

$$\mathcal{M}(x_-, x_+) = \left\{ \gamma : \mathbb{R} \to M \text{ such that } \frac{d}{ds}\gamma(s) = -\nabla f; \quad \lim_{s\to\pm\infty} \gamma(s) = x_\pm \right\} /\mathbb{R} \tag{2.2}$$

where the quotient is by the translation action reparametrizing the domain of γ. In order to define homology and cohomology over coefficient rings outside characteristic two, some additional data is required, namely orientations on $\mathcal{M}(x_-, x_+)$.

The natural question, addressed by Cohen-Jones-Segal [3], is what more topological information about M other than its singular homology and cohomology can be extracted from $\mathrm{Crit}(f)$ and the moduli spaces $\mathcal{M}(x_-, x_+)$. Cohen-Jones-Segal's answer was that by using some additional *framing data*, the *stable* homotopy type $\Sigma^\infty_+ M$ of M can be reconstructed, as well as the filtration (2.1) in the stable homotopy category.

Let us explain what we mean by framing data. Writing $\mu : \mathrm{Crit}(f) \to \mathbb{Z}$ for the Morse index, at each critical point $x \in \mathrm{Crit}(f)$, there is a $\mu(x)$-dimensional vector space $V(x)$ (the tangent space at x of its descending disc). For a generic Riemannian metric on M, the moduli space of flow lines $\mathcal{M}(x_-, x_+)$ is a smooth manifold of dimension $\mu(x_-) - \mu(x_+) - 1$. Moreover, the corresponding moduli space of parametrized flow lines (non-canonically diffeomorphic to $\mathcal{M}(x_-, x_+) \times \mathbb{R}$) can be thought of as the transverse intersection of the descending disc of x_- and the ascending disc of x_+; this yields an isomorphism of vector bundles over $\mathcal{M}$:

$$\mathbb{R} \oplus T\mathcal{M} \oplus V(x_+) \cong V(x_-) \tag{2.3}$$

(where $V(x_\pm)$ and $\mathbb{R}$ respectively are the trivial bundles with this fibre). In particular, the manifolds $\mathcal{M}$ are stably framed. In the case that $\mathcal{M}$ is a closed manifold (for instance, when x_-, x_+ are the only critical points of f), a stable homotopy type can be directly extracted by means of the Pontrjagin-Thom construction. Explicitly, choose an embedding

$$\mathcal{M} \hookrightarrow \mathbb{R}^N$$

for $N \gg 0$; taking the one-point compactifications and desuspending yields a map of spectra

$$\mathbb{S} \longrightarrow \mathcal{M}^{-T\mathcal{M}} \tag{2.4}$$

from the sphere spectrum to the Thom spectrum of the virtual normal bundle to $\mathcal{M}$. Moreover, (2.3) yields a map of spectra

$$S^{V(x_-)} \wedge \mathcal{M}^{-(\mathbb{R} \oplus T\mathcal{M})} \longrightarrow S^{V(x_+)}$$

where $S^{V(x_\pm)}$ is the Thom spectrum of a trivial vector bundle $V(x_\pm)$ over a point (of course, non-canonically isomorphic to a shift of the sphere spectrum). Composing with a shift of the Pontrjagin-Thom map (2.4) we have a map of spectra

$$\Sigma^{-1} S^{V(x_-)} \longrightarrow S^{V(x_+)};$$

whose cofiber defines a stable homotopy type invariant of $\mathcal{M}$ and its framing.

While in general $\mathcal{M}(x_-, x_+)$ is not compact, it admits a natural compactification by adding in "broken flow lines":

$$\bar{\mathcal{M}}(x_-, x_+) = \mathcal{M}(x_-, x_+) \sqcup \bigsqcup_{\substack{k \geq 1 \\ x_1, \ldots, x_k \in \mathrm{Crit}(f)}} \mathcal{M}(x_-, x_1) \times \ldots \times \mathcal{M}(x_k, x_+)$$

with the appropriate topology of Gromov convergence. It is, in fact, a smooth manifold with corners [12], with natural smooth charts given by the "gluing maps" of Morse flow lines. Its codimension one boundary faces are the images of closed embeddings

$$\bar{\mathcal{M}}(x_-, x) \times \bar{\mathcal{M}}(x, x_+) \hookrightarrow \bar{\mathcal{M}}(x_-, x_+)$$

for critical points x with $\mu(x_-) > \mu(x) > \mu(x_+)$; the combinatorial structure of its deeper codimension faces is determined by the poset of chains $x_-, x_1, \dots, x_k, x_+$ of critical points with descending Morse index. This gives the collection of manifolds $\{\bar{\mathcal{M}}(x_-, x_+)\}_{x_-,x_+}$ the structure of what Cohen-Jones-Segal call a *flow category*. In the rest of this section, we will give explain Cohen-Jones-Segal's method of extracting homotopy types out of a flow category with additional framing data.

2.2. **Abstract flow categories.** In order to state the definition of a flow category, it is useful to be more precise about the sort of manifolds with corners we are working with. A manifold with corners X is covered by smooth local charts on $U \subset X$ of the form

$$\phi : U \to \mathbb{R}^n_{\geq 0}.$$

For such a chart ϕ and a point $x \in U \subset X$, the number of coordinates that are zero in $\phi(x)$ is independent of the choice of chart, and is called the codimension $c(x)$ of x. A *boundary face* of X is the closure of a connected component of $c^{-1}(1)$; likewise a *codimension* ℓ *facet* of X is the closure of a connected component of $c^{-1}(\ell)$. Without extra combinatorial structure, however, these notions are in general poorly behaved; a boundary face is not guaranteed to be a manifold with corners in this set-up. One such choice of extra structure is the notion of a $< k >$-manifold, which we recall from [7].

Definition 2.1. *A* $< k >$*-manifold* X *is a smooth manifold with corners, together with an ordered tuple* $\partial_1 X, \dots, \partial_k X$ *of disjoint of unions of boundary faces of* X*, so that:*

(i) each $x \in X$ *is contained in exactly* $c(x)$ *boundary faces of* X*;*
(ii) the union of $\partial_a X$ *covers the whole boundary of* X*;*
(iii) for $a \neq b$*, the intersection* $\partial_a X \cap \partial_b X$ *is a boundary face of each of* $\partial_a X$ *and* $\partial_b X$*.*

This definition ensures, for instance, that each codimension ℓ facet of X is a $< k - \ell >$-manifold. It is also well behaved under products; the product of a $< k_1 >$-manifold and a $< k_2 >$-manifold is a $< k_1 + k_2 >$-manifold.

The most obvious example of a $< k >$-manifold is the Euclidean space $\mathbb{R}^k_{\geq 0}$; the distinguished faces $\partial_a \mathbb{R}^k_{\geq 0}$ are the subspaces where the a-th coordinate vanishes. The compactified moduli spaces of flow lines $\bar{\mathcal{M}}(x_-, x_+)$ are an example of a $< \mu(x_-) - \mu(x_+) - 1 >$-manifold; in this example $\partial_a \bar{\mathcal{M}}$ is the moduli space of broken flow lines where one of the breaks is at a critical point of Morse index $\mu(x_+) + a$.

Ultimately, when we will construct operations between homotopy types defined by the Cohen-Jones-Segal construction, more combinatorial structure than just that of a $< k >$-manifold will be required. There it will be useful to notice that a $< k >$-manifold can be described by a cubical diagram of topological spaces

$$\underline{2}^k = \{0, 1\}^k \longrightarrow \mathrm{Top}$$

by setting $X_T = \cap_{a \in T} \partial_a X$ for a non-empty subset $T \subset \{1, 2, \dots, k\}$, and $X_\emptyset = X$.

We can now give the definition of a flow category. Of course we have the example of Morse theory in mind, but we wish to state something more general so that we can apply it later to Floer theory.

Definition 2.2. *The data of a flow category $\mathcal{M}$ consists of:*

(i) *a set of objects* $\mathrm{Ob}(\mathcal{M})$*, which we will sometimes call the* generators*, equipped with an integer-valued function* $\mu : \mathrm{Ob}(\mathcal{M}) \to \mathbb{Z}$*, which we will often call the* index*;*
(ii) *for each* $x_0, x_1 \in \mathrm{Ob}(\mathcal{M})$ *with* $\mu(x_0) > \mu(x_1)$*, a (possibly empty) smooth compact manifold with corners* $\mathcal{M}(x_0, x_1)$ *of dimension* $\mu(x_0) - \mu(x_1) - 1$*;*
(iii) *for each* $x_0, x_1, x_2 \in \mathrm{Ob}(\mathcal{M})$ *with* $\mu(x_0) > \mu(x_1) > \mu(x_2)$*, a composition map*

$$G : \mathcal{M}(x_0, x_1) \times \mathcal{M}(x_1, x_2) \longrightarrow \mathcal{M}(x_0, x_2)$$

which is a diffeomorphism onto a boundary face of $\mathcal{M}(x_0, x_2)$*.*

The composition maps must be associative. Moreover for x_0, x_2 *and* $1 \le a \le \mu(x_0) - \mu(x_2) - 1$ *we require that the images under composition* $G(\mathcal{M}(x_0, x_1) \times \mathcal{M}(x_1, x_2))$ *for varying* x_1 *with* $\mu(x_1) = \mu(x_2) + a$ *are disjoint, and if we write*

$$\partial_a \mathcal{M}(x_0, x_2) = \bigsqcup_{x_1 : \mu(x_1) = \mu(x_2) + a} G(\mathcal{M}(x_0, x_1) \times \mathcal{M}(x_1, x_2))$$

we require that $\partial_1 \mathcal{M}(x_0, x_2), \ldots, \partial_{\mu(x_0) - \mu(x_2) - 1} \mathcal{M}(x_0, x_2)$ *endows* $\mathcal{M}(x_0, x_2)$ *with the structure of a* $< \mu(x_0) - \mu(x_2) - 1 >$*-manifold.*

We will usually further require that $\mathrm{Ob}(\mathcal{M})$ is finite, in which case we say that the flow category is *finite*. The structure of a flow category makes $\mathrm{Ob}(\mathcal{M})$ into a poset, by declaring $x_0 > x_1$ whenever $\mu(x_0) > \mu(x_1)$ and $\mathcal{M}(x_0, x_1)$ is non-empty.

A metric on a flow category is a Riemannian metric on each manifold $\mathcal{M}(x_0, x_1)$ such that whenever $x_0 > x_1 > x_2$ the product metric on $\mathcal{M}(x_0, x_1) \times \mathcal{M}(x_1, x_2)$ coincides with the pull-back of the metric on $\mathcal{M}(x_0, x_2)$ by the composition map. A metric induces an isomorphism of vector bundles over $\mathcal{M}(x_0, x_1) \times \mathcal{M}(x_1, x_2)$

$$T\mathcal{M}(x_0, x_1) \oplus \mathbb{R} \oplus T\mathcal{M}(x_1, x_2) \cong G^* T\mathcal{M}(x_0, x_2) \tag{2.5}$$

where the trivial $\mathbb{R}$ factor is the inward pointing normal direction. Write $\nu_{x_1} \in G^* T\mathcal{M}(x_0, x_1)$ for the image of the canonical generator of $\mathbb{R}$ under this map. We then define, for each $x_0 > x_1$, the *index bundle* of $\mathcal{M}(x_0, x_1)$ to be the vector bundle

$$I(x_0, x_1) = \mathbb{R} \oplus T\mathcal{M}(x_0, x_1)$$

and write $\tau_{x_0, x_1} \in I(x_0, x_1)$ for the generator of the trivial $\mathbb{R}$ factor. The isomorphisms (2.5) then yield isomorphisms of vector bundles over $\mathcal{M}(x_0, x_1) \times \mathcal{M}(x_1, x_2)$

$$\phi_{x_0, x_1, x_2} : I(x_0, x_1) \oplus I(x_1, x_2) \cong G^* I(x_0, x_2) \tag{2.6}$$

defined as the inverse of the map

$$\mathbb{R}\tau_{x_0, x_2} \oplus \underbrace{T\mathcal{M}(x_0, x_1) \oplus \mathbb{R}\nu_{x_1} \oplus T\mathcal{M}(x_1, x_2)}_{G^* T\mathcal{M}(x_0, x_2)} \longrightarrow \mathbb{R}\tau_{x_0, x_1} \oplus T\mathcal{M}(x_0, x_1) \oplus \mathbb{R}\tau_{x_1, x_2} \oplus T\mathcal{M}(x_1, x_2)$$

that is the identity on the $T\mathcal{M}$ components, and sends

$$\tau_{x_0,x_2} \mapsto \tau_{x_0,x_1} + \tau_{x_1,x_2}; \qquad \nu_{x_1} \mapsto \tau_{x_0,x_1} - \tau_{x_1,x_2}.$$

It is not difficult to check from this definition:

Proposition 2.3. *The isomorphisms ϕ_{x_0,x_1,x_2} of (2.6) are associative for quadruples of generators $x_0 > x_1 > x_2 > x_3$, meaning we have a commutative diagram of isomorphisms of vector bundles over $\mathcal{M}(x_0,x_1) \times \mathcal{M}(x_1,x_2) \times \mathcal{M}(x_2,x_3)$:*

$$\begin{array}{ccc} I(x_0,x_1) \oplus I(x_1,x_2) \oplus I(x_2,x_3) & \xrightarrow{\mathrm{id}\oplus\phi} & I(x_0,x_1) \oplus G^* I(x_1,x_3) \\ \downarrow{\scriptstyle \phi\oplus\mathrm{id}} & & \downarrow{\scriptstyle \phi} \\ G^* I(x_0,x_2) \oplus I(x_2,x_3) & \xrightarrow{\phi} & G^* I(x_0,x_3). \end{array}$$

These isomorphisms then allow us to define a *stable framing* of a flow category:

Definition 2.4. *A stable framing of a flow category $\mathcal{M}$ with metric is the data of a (virtual) vector space $V(x)$ for each $x \in \mathrm{Ob}(\mathcal{M})$, and isomorphisms of vector bundles over $\mathcal{M}(x_0,x_1)$*

$$I(x_0,x_1) \oplus V(x_1) \cong V(x_0) \tag{2.7}$$

whenever $x_0 > x_1$, which over the boundary faces $\mathcal{M}(x_0,x_1) \times \mathcal{M}(x_1,x_2)$ are compatible with the isomorphisms of (2.6).

From an isomorphism (2.7) over $\mathcal{M}(x_0,x_1)$, one obtains a map *on Thom spectra*

$$\mathcal{M}^{-I(x_0,x_1)} \wedge S^{V(x_0)} \longrightarrow S^{V(x_1)} \tag{2.8}$$

where for a (virtual) vector space V, S^V is the Thom spectrum of V thought of as a bundle over a point, which is equivalent to a suspension of the sphere spectrum by $\dim(V)$.

These maps on Thom spectra also satisfy some (strict) associativity relations, which can be handled by defining the *Atiyah dual* flow category:

Definition 2.5. *For a flow category $\mathcal{M}$, the* Atiyah dual flow category *$\mathcal{M}^{-I}$ is the category enriched over spectra with the same objects $\mathrm{Ob}(\mathcal{M})$ and with morphism spectra*

$$\mathcal{M}^{-I}(x_0,x_1) = \begin{cases} \mathcal{M}(x_0,x_1)^{-I(x_0,x_1)} & \text{if } x_0 > x_1; \\ \mathbb{S} & \text{if } x_0 = x_1; \\ * & \text{otherwise.} \end{cases}$$

Composition is given by the composition maps in $\mathcal{M}$ together with the isomorphisms (2.6) for a choice of Riemannian metric.

This allows us to deal with slightly more general objects than the stable framings of Definition 2.4. Writing Sp for the symmetric monoidal category of orthogonal spectra, we have:

Definition 2.6. *A* spectral system *on a flow category* $\mathcal{M}$ *is a spectrally enriched functor*

$$\mathfrak{o} : \mathcal{M}^{-I} \longrightarrow \mathrm{Sp}.$$

Explicitly, this is the data of an orthogonal spectrum $\mathfrak{o}(x)$ *for each* $x \in \mathrm{Ob}(\mathcal{M})$*, together with maps of spectra for every* $x_0 > x_1$

$$\mathfrak{o}(x_0) \wedge \mathcal{M}(x_0, x_1)^{-I(x_0,x_1)} \longrightarrow \mathfrak{o}(x_1) \tag{2.9}$$

so that for $x_0 > x_1 > x_2$ *we have commutative diagrams*

$$\begin{array}{ccc} \mathfrak{o}(x_0) \wedge \mathcal{M}(x_0, x_1)^{-I(x_0,x_1)} \wedge \mathcal{M}(x_1, x_2)^{-I(x_1,x_2)} & \longrightarrow & \mathfrak{o}(x_1) \wedge \mathcal{M}(x_1, x_2)^{-I(x_1,x_2)} \\ \downarrow & & \downarrow \\ \mathfrak{o}(x_0) \wedge \mathcal{M}(x_0, x_2)^{-I(x_0,x_2)} & \longrightarrow & \mathfrak{o}(x_2). \end{array}$$

The goal of this chapter is to explain how, from an abstract flow category $\mathcal{M}$ equipped with a spectral system $\mathfrak{o}$, one can define "level n realization spectra" $|\mathfrak{o}|_n$ for each $n \in \mathbb{Z}$. Roughly speaking, each $|\mathfrak{o}|_n$ will be a filtered spectrum

$$|\mathfrak{o}|_n = \varinjlim \left(|\mathfrak{o}|_n^n \longrightarrow |\mathfrak{o}|_n^{n+1} \longrightarrow |\mathfrak{o}|_n^{n+2} \longrightarrow \dots \right)$$

whose sub-quotients are identified with the spectra

$$\bigvee_{x:\mu(x)=a} \mathfrak{o}(x) \tag{2.10}$$

for each integer $a \geq n$. We will extract this structure out of a collection of maps between (shifts of) the spectra (2.10), obtained from a version of the Pontrjagin-Thom construction on the flow category $\mathcal{M}$. Part of the topological difficulty is that $\mathcal{M}$ is not a closed smooth manifold, but rather a manifold with corners, and so the Pontrjagin-Thom construction must be appropriately modified.

The approach of Cohen-Jones-Segal in [3] is to embed each $\mathcal{M}(x_0, x_1)$ with $\mu(x_0) - \mu(x_1) = w$ into a standard "$< w - 1 >$-manifold Euclidean space"

$$\mathcal{M}(x_0, x_1) \longrightarrow \mathbb{R}^{w-1}_{\geq 0} \times \mathbb{R}^N$$

for some $N \geq 0$, so that each boundary face $\mathcal{M}(x_0, x') \times \mathcal{M}(x', x_1)$ of $\mathcal{M}(x_0, x_1)$ is sent to the boundary face of $\mathbb{R}^{w-1}_{\geq 0} \times \mathbb{R}^N$ where the $\mu(x_0) - \mu(x')$-coordinate vanishes. If we further suppose that $\mathcal{M}$ is orthogonal to each of these boundary faces of $\mathbb{R}^{w-1}_{\geq 0} \times \mathbb{R}^N$ (in this case the embedding is called "neat"), then $\mathcal{M}(x_0, x_1)$ has a well-defined normal bundle ν that extends over all the boundary strata. We can then define a Pontrjagin-Thom collapse map from the one-point compactification of $\mathbb{R}^{w-1}_{\geq 0} \times \mathbb{R}^N$ to the Thom space of ν

$$\Sigma^N (\mathbb{R}^{w-1}_{\geq 0})^+ \cong \left(\mathbb{R}^{w-1}_{\geq 0} \times \mathbb{R}^N \right) \longrightarrow \mathcal{M}^\nu$$

and then, desuspending, we have a map of spectra

$$\Sigma^{\infty - w} (\mathbb{R}^{w-1}_{\geq 0})^+ \longrightarrow \mathcal{M}^{-I}.$$

Thus, from a spectral system $\mathfrak{o}$ over $\mathcal{M}$, for each pair of integers $a_0 > a_1$ with $w = a_0 - a_1$ we get a map of spectra

$$\Sigma^{\infty-w}(\mathbb{R}^{w-1}_{\geq 0}) \wedge \left(\bigvee_{\mu(x)=a_0} \mathfrak{o}(x) \right) \longrightarrow \bigvee_{\mu(x)=a_1} \mathfrak{o}(x).$$

These maps satisfy certain coherence properties, and from these the realization spectra $|\mathfrak{o}|_n$ are defined from a purely algebraic procedure. However, this is not quite the route we will take in this thesis, where we will want to define maps between realization spectra from "maps of flow categories" in the next chapter. Our approach is much closer to that of [1], and uses Pontrjagin-Thom theory in a significantly more flexible way (without requiring the sorts of highly structured "neat embeddings" used by Cohen-Jones-Segal), through a general framework of Atiyah duality on manifolds with boundary.

2.3. **Weighted chains.** Before describing the construction for general flow categories $\mathcal{M}$, let us detail some of the underlying combinatorics. At first sight the notation may seem a serious overkill for the particular task of this chapter (to define realizations of spectral systems on flow categories), however it will be essential for the discussion in the next chapter (to define "multi-linear" maps between such realizations).

Definition 2.7. *A* weighted unstable chain T *is an oriented, linear graph on vertices* $V(T) = v_1, v_2, \dots, v_\ell$ *read outgoing to incoming, with two semi-infinite edges (an "outgoing" edge attached to* v_1*, which we call the* root*; and an "incoming" edge attached to* v_ℓ*, which we call the* leaf*), and a* weighting *function*

$$w : V(T) \longrightarrow \mathbb{Z}$$

so that $w(v) \geq 1$ *for each* $v \in V(T)$*. We refer to* $w(T) = \sum_{v \in V(T)} w(v)$ *as the* total weight *of* T*.*

We will always write $E(T)$ for the set of *interior* (meaning, non-semi-infinite) edges of T. Given a subset $E_0 \subset E(T)$, we can form another weighted unstable chain by a procedure we call *contraction*:

- one-by-one delete the edges $e \in E_0$, and identify the two vertices $v_-(e), v_+(e)$ on either side of e, forming a new vertex whose weight is the sum $w(v_-) + w(v_+)$.

The resulting chain, which we call T/E_0, is independent of the order in which we chose to contract the edges. The edges of $E(T/E_0)$ can be naturally thought of as a subset of $E(T)$; indeed

$$E(T) \cong E(T/E_0) \sqcup E_0.$$

Definition 2.8. *For* $w \geq 1$*, the poset* $\mathcal{C}(w)$ *has as elements the (isomorphism classes of) weighted unstable chains of total weight* w*. The partial order is defined by declaring* $T_2 \leq T_1$ *if* T_1 *is obtained from* T_2 *by contracting some subset of its edges.*

This poset clearly has a maximal element, the weighted unstable chain with a single vertex of weight w. Indeed, we can fully determine the structure of $\mathcal{C}(w)$. Fix some integers $a > b$ such that $a - b = w$. Then, there is an isomorphism of posets between $\mathcal{C}(w)$ and the power set $\underline{2}^{\{a+1,\ldots,b-1\}}$. This isomorphism takes a subset $a_1 > a_2 > \ldots > a_{\ell-1}$ of $\{a+1,\ldots,b-1\}$ to the weighted chain on vertices $v_1,\ldots,v_\ell$ of respective weights $a-a_1, a_1-a_2, \ldots, a_{\ell-1}-b$.

Remark 2.9. *There is a slightly different perspective on this identification of posets, which will become useful later. Consider the subset*

$$\mathcal{E}(w) \subset \mathcal{C}(w)$$

of weighted unstable chains with precisely one interior edge; we call such chains elementary*. Then, for an arbitrary $T \in \mathcal{C}(w)$, we can associate to each edge $e \in E(T)$ an elementary chain by collapsing every edge of T other than e. This determines an injective function*

$$E(T) \longrightarrow \mathcal{E}(w)$$

from which we can regard $E(T)$ as a subset of $\mathcal{E}(w)$. This correspondence then in fact defines an isomorphism of posets

$$\mathcal{C}(w) \cong \underline{2}^{\mathcal{E}(w)}.$$

Finally, for $w_1, w_2 \geq 1$, there are natural *grafting* operations

$$\# : \mathcal{C}(w_1) \times \mathcal{C}(w_2) \longrightarrow \mathcal{C}(w_1 + w_2) \tag{2.11}$$

which take $T_1 \in \mathcal{C}(w_1)$ and $T_2 \in \mathcal{C}(w_2)$ to the chain $T_1 \# T_2$ obtained by gluing in the incoming edge of T_1 to the outgoing edge of T_2 to form an interior edge of the new chain, and leaving the vertices and weights intact. For a triple w_1, w_2, w_3 of positive integers, the grafting maps satisfy an obvious associativity relation.

2.4. **The spaces $\mathcal{J}$ and the spectra $\mathscr{J}$.** We now define some auxiliary spaces and spectra, which will be the basis for this version of Cohen-Jones-Segal's construction.

Definition 2.10. *A* metrization *of an unstable weighted chain T is an assignment of a length $\ell_e \in (0, \infty]$ for each $e \in E(T)$. We can also formally allow some of the edges to acquire length $\ell_e = 0$, in which case we identify this metrized weighted chain with its contraction T/E_0. The set of isomorphism classes $\mathcal{J}(w)$ of all metrized unstable weighted chains of total weight w is then a topological space*

$$\mathcal{J}(w) \cong [0, \infty]^{w-1}.$$

For a fixed (*unmetrized*) weighted chain $T \in \mathcal{C}(w)$, we can likewise consider the topological space $\mathcal{J}_T$ of all metrized weighted chains (T', ℓ) with $T' \leq T$, and so that the edges $e \in E(T) \subset E(T')$ has maximal length $\ell_e = \infty$. It is not difficult to check that

$$\mathcal{J}_T = \prod_{v \in V(T)} \mathcal{J}(w(v)) \cong [0, \infty]^{w - |V(T)|}.$$

There are natural inclusions $\mathcal{J}_{T_2} \hookrightarrow \mathcal{J}_{T_1}$ whenever $T_2 \le T_1$. Even better, there are natural inclusions

$$\mathcal{J}_{T_2} \times (0,\infty]^{E(T_2)\setminus E(T_1)} \hookrightarrow \mathcal{J}_{T_1} \tag{2.12}$$

which we refer to as *the collar* of $\mathcal{J}_{T_2}$ in $\mathcal{J}_{T_1}$, and its image as *the collar neighbourhood*. These collars are moreover associative, in that whenever $T_3 \le T_2 \le T_1$ with edges $E_i = E(T_i)$, the composition of the collars

$$\mathcal{J}_{T_3} \times (0,\infty]^{E_3\setminus E_1} = \left(\mathcal{J}_{T_3} \times (0,\infty]^{E_3\setminus E_2}\right) \times (0,\infty]^{E_2\setminus E_1} \longrightarrow \mathcal{J}_{T_2} \times (0,\infty]^{E_2\setminus E_1} \longrightarrow \mathcal{J}_{T_1}$$

is exactly the collar of $\mathcal{J}_{T_3}$ in $\mathcal{J}_{T_1}$.

In addition to the collars, let us also single out the embeddings

$$\iota : \mathcal{J}_{T_2} = \mathcal{J}_{T_2} \times \{1\}^{E(T_2)\setminus E(T_1)} \hookrightarrow \mathcal{J}_{T_1} \tag{2.13}$$

formed by declaring the edges of $E(T_2)\setminus E(T_1)$ have length exactly one. These embeddings are similarly associative, and moreover we can identify the normal bundle to $\iota(\mathcal{J}_{T_2}) \subset \mathcal{J}_{T_1}$ with $\mathbb{R}^{E(T_2)\setminus E(T_1)}$.

In particular, $\mathcal{J}(w)$ becomes a space stratified by the poset $\mathcal{C}(w)$, with closed strata $\mathcal{J}_T$. Of course, under the identification $\mathcal{J}(w) = [0,\infty]^{w-1}$, the strata are exactly the subsets where some of the coordinates are ∞.

We will write $\partial\mathcal{J}(w)$ and $\partial\mathcal{J}_T$ for the union of the proper strata, and in each case refer to it as *the boundary*. We warn the reader that this is not the boundary of $[0,\infty]^{w-1}$ in the usual sense.

Given metrized chains (T_1,ℓ_1) and (T_2,ℓ_2), the grafting $T_1\#T_2$ acquires a metrization, by setting the length of the new, glued edge to be ∞. In this way we have homeomorphisms

$$\mathcal{J}_{T_1} \times \mathcal{J}_{T_2} \cong \mathcal{J}_{T_1\#T_2}.$$

The images of all these maps lie in the boundaries of $\mathcal{J}_T$; and moreover for any $T \in \mathcal{C}(w)$, the images also cover the boundary of $\partial\mathcal{J}_T$.

Now, we equip each $\mathcal{J}_T$ with a trivial real vector bundle

$$\nu_T = \mathbb{R}^{V(T)} \times \mathcal{J}_T \tag{2.14}$$

with basis indexed by the vertices of T.

Now, suppose that we have chains $T_2 \le T_1$; the resulting contraction produces a surjection of sets $p : V(T_2) \to V(T_1)$ and an inclusion $E(T_1) \subset E(T_2)$, and in particular a short exact sequence of vector spaces

$$0 \longrightarrow \mathbb{R}^{V(T_1)} \longrightarrow \mathbb{R}^{V(T_2)} \longrightarrow \mathbb{R}^{E(T_2)\setminus E(T_1)} \longrightarrow 0 \tag{2.15}$$

as follows: the first map sends

$$(s_v)_{v\in V(T_1)} \mapsto (s_{p(v)})_{v\in V(T_2)};$$

the second map sends

$$(s_v)_{v\in V(T_2)} \mapsto (s_{v_+(e)} - s_{v_-(e)})_{e\in E_2\setminus E_1}$$

where $v_-(e), v_+(e)$ are the incoming and outgoing vertices of an edge e respectively.

These short exact sequences are moreover compatible with eachother, meaning that whenever $T_3 \leq T_2 \leq T_1$, writing $V_i = V(T_i)$ and $E_i = V(T_i)$, we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{R}^{V_1} & \longrightarrow & \mathbb{R}^{V_2} & \longrightarrow & \mathbb{R}^{E_2 \setminus E_1} \\ \| & & \downarrow & & \downarrow \\ \mathbb{R}^{V_1} & \longrightarrow & \mathbb{R}^{V_3} & \longrightarrow & \mathbb{R}^{E_3 \setminus E_1} \\ & & \downarrow & & \downarrow \\ & & \mathbb{R}^{E_3 \setminus E_2} & = & \mathbb{R}^{E_3 \setminus E_2} \end{array} \tag{2.16}$$

where the right column arises from the canonical splitting $\mathbb{R}^{E_3 \setminus E_1} = \mathbb{R}^{E_2 \setminus E_1} \oplus \mathbb{R}^{E_3 \setminus E_2}$.

Now recall that if $\xi \to X$ is a vector bundle on a space X, the Thom space X^ξ is the based space formed by the quotient $D_X(\xi)/S_X(\xi)$, where $D_X(\xi)$ and $S_X(\xi)$ are the disk and sphere bundles respectively. Then if $Y \subset X$ is a closed subset so that there is a neighbourhood U of Y homeomorphic to the total space of a vector bundle $N_Y \to Y$, then there is a Pontrjagin-Thom collapse map $X \to Y^{N_Y}$, identifying all of $X \setminus U$ to the basepoint. Moreover, if X itself also carries a vector bundle ξ, there is an open neighbourhood V of $Y \subset X \subset D_X(\xi)$, homeomorphic to the total space of the vector bundle $\xi|_Y \oplus N_Y$; we then have a collapse map $X^\xi \to Y^{\xi|_Y \oplus N_Y}$.

In particular, applying this to $\iota(\mathcal{J}_{T_2}) \subset \mathcal{J}_{T_1}$, we have maps of Thom spaces

$$q'_{T_2 \leq T_1} : \mathcal{J}_{T_1}^{\nu_{T_1}} \longrightarrow \mathcal{J}_{T_2}^{\nu_{T_2}} \tag{2.17}$$

whenever $T_2 \leq T_1$. The commutativity of (2.15) ensures that these maps are associative: whenever $T_3 \leq T_2 \leq T_1$ we have $q'_{T_3 \leq T_2} \circ q'_{T_2 \leq T_1} = q'_{T_3 \leq T_1}$. In order to simplify the notation, we will when there is no ambiguity write $\nu = \nu_T$ for the vector bundle over $\mathcal{J}_T$.

Moreover, $q'_{T_2 \leq T_1}$ carries $(\partial \mathcal{J}_{T_1})^\nu$ into $(\partial \mathcal{J}_{T_2})^\nu$, and so we have maps on the quotient spaces

$$q_{T_2 \leq T_1} = \mathcal{J}_{T_1}^\nu / \partial \mathcal{J}_{T_1}^\nu \longrightarrow \mathcal{J}_{T_2}^\nu / \partial \mathcal{J}_{T_2}^\nu \tag{2.18}$$

which likewise satisfy $q_{T_3 \leq T_2} \circ q_{T_2 \leq T_1} = q_{T_3 \leq T_1}$.

Moreover, under the identification $\mathcal{J}_{T_1 \# T_2} = \mathcal{J}_{T_1} \times \mathcal{J}_{T_2}$ when two chains T_1, T_2 are grafted, we can further identify

$$\mathcal{J}_{T_1 \# T_2}^\nu / \partial \mathcal{J}_{T_1 \# T_2}^\nu = \left(\mathcal{J}_{T_1}^\nu / \partial \mathcal{J}_{T_1}^\nu \right) \wedge \left(\mathcal{J}_{T_2}^\nu / \partial \mathcal{J}_{T_2}^\nu \right). \tag{2.19}$$

Combining (2.18) and (2.19), we see that whenever w_1, w_2 are positive integers we have maps of based spaces

$$\mathcal{J}(w_1 + w_2)^\nu / \partial \mathcal{J}(w_1 + w_2)^\nu \longrightarrow (\mathcal{J}(w_1)^\nu) / \partial \mathcal{J}(w_1)^\nu) \wedge (\mathcal{J}(w_2)^\nu / \partial \mathcal{J}(w_2)^\nu). \tag{2.20}$$

To pass to spectra, we take the Spanier-Whitehead duals of all these quotients:

Definition 2.11. *For a positive integer $w \geq 1$, the spectrum $\mathscr{J}(w)$ is defined to be*

$$\mathscr{J}(w) = D(\Sigma^\infty \mathcal{J}(w)^\nu / \partial \mathcal{J}(w)^\nu) = F(\Sigma^\infty \mathcal{J}(w)^\nu / \partial \mathcal{J}(w)^\nu, \mathbb{S})$$

where $D(-)$ is the Spanier-Whitehead dual, and $F(-,-)$ denotes the function spectrum. Similarly, for $T \in \mathcal{C}(w)$, define the spectrum

$$\mathscr{J}_T = D(\Sigma^\infty \mathcal{J}_T^\nu / \partial \mathcal{J}_T^\nu, \mathbb{S}) = F(\Sigma^\infty \mathcal{J}_T^\nu / \partial \mathcal{J}_T^\nu, \mathbb{S}).$$

Dualizing (2.20), we have natural maps of spectra

$$\mathscr{J}(w_1) \wedge \mathscr{J}(w_2) \longrightarrow \mathscr{J}(w_1 + w_2) \tag{2.21}$$

whenever w_1, w_2 are positive integers. Moreover, these maps are strictly associative for triples w_1, w_2, w_3, since they arise from dualizing strictly associative diagrams of based spaces. In particular, we have a category enriched over spectra:

Definition 2.12. *The spectral category $\mathscr{J}$ has for objects the integers $\mathbb{Z}$, and morphisms*

$$\mathscr{J}(a_0, a_1) = \begin{cases} * & \text{if } a_0 < a_1; \\ \mathbb{S} & \text{if } a_0 = a_1; \\ \mathscr{J}(a_0 - a_1) & \text{if } a_0 > a_1 \end{cases}$$

with composition maps

$$\mathscr{J}(a_0, a_1) \wedge \mathscr{J}(a_1, a_2) \longrightarrow \mathscr{J}(a_0, a_2)$$

determined by (2.21).

Note that our convention in this thesis is for compositions of morphisms to be read left to right.

Definition 2.13. *An* $\mathscr{J}$-spectrum *is a functor from $\mathscr{J}$ to spectra*

$$X : \mathscr{J} \longrightarrow \mathrm{Sp}.$$

Explicitly, this is an assignment of a spectrum $X(a)$ for each $a \in \mathbb{Z}$, together with maps of spectra

$$X(a_0) \wedge \mathscr{J}(a_0, a_1) \longrightarrow X(a_1)$$

such that the composite $X(a) \cong X(a) \wedge \mathbb{S} \to X(a) \wedge \mathscr{J}(a,a) \to X(a)$ is the identity, and so that the diagrams

$$\begin{array}{ccc} X(a_0) \wedge \mathscr{J}(a_0, a_1) \wedge \mathscr{J}(a_1, a_2) & \longrightarrow & X(a_1) \wedge \mathscr{J}(a_1, a_2) \\ \downarrow & & \downarrow \\ X(a_0) \wedge \mathscr{J}(a_0, a_2) & \longrightarrow & X(a_2) \end{array}$$

strictly commute. We will often equivalently refer to X as a right $\mathscr{J}$-module.

One can think of an $\mathscr{J}$-spectrum as a sort of "homotopy coherent chain complex" in spectra. Indeed, one can directly compute the homotopy types

$$\mathscr{J}(a_0, a_1) \simeq \begin{cases} \mathbb{S}^{-1} & \text{if } a_0 = a_1 + 1; \\ \mathbb{S} & \text{if } a_0 = a_1; \\ * & \text{otherwise.} \end{cases} \tag{2.22}$$

Accordingly, the data of an $\mathscr{J}$-spectrum X is that of a spectrum $X(a)$ for each $a \in \mathbb{Z}$, together with "differentials"

$$\delta : X(a+1) \longrightarrow \Sigma X(a)$$

and a coherent system of nullhomotopies for $\delta \circ \delta$.

2.5. **$\mathscr{J}$-spectra from flow categories.** At the end of this chapter, we will explain how to produce the realization spectra $|X|_n$ of a $\mathscr{J}$-spectrum X. But before we do this, let us explain how to obtain an $\mathscr{J}$-spectrum from a flow category $\mathcal{M}$ equipped with a spectral system $\mathfrak{o}$ in the same of Definition 2.6.

Fix, for the rest of this chapter, a flow category $\mathcal{M}$. We also fix a smooth structure on $(0, \infty]$, say by a choice of gluing profile as in Section 6.5. Using this, we also fix a system of collars for the boundary strata of $\mathcal{M}$:

Definition 2.14. *A* coherent system of collars *for $\mathcal{M}$ is a choice, for every $x_0 > x_1 > \ldots > x_\ell$, of a smooth map*

$$C : \mathcal{M}(x_0, x_1) \times \ldots \times \mathcal{M}(x_{\ell-1}, x_\ell) \times (0, \infty]^{\ell-1} \longrightarrow \mathcal{M}(x_0, x_\ell)$$

which is a diffeomorphism onto a neighbourhood of the boundary stratum $\mathcal{M}(x_0, x_1) \times \ldots \times \mathcal{M}(x_{\ell-1}, x_\ell)$ of $\mathcal{M}(x_0, x_\ell)$, and are coherent in that for any choice of $0 \leq i < j \leq \ell$ the diagram

$$\begin{array}{ccc} \mathcal{M}(x_0,x_1) \times \ldots \underbrace{\mathcal{M}(x_i,x_{i+1}) \times \ldots \times \mathcal{M}(x_{j-1},x_j)} \times \mathcal{M}(x_{\ell-1},x_\ell) \times (0,\infty]^{\ell-1} & & \\ \downarrow & \searrow & \\ \mathcal{M}(x_0,x_1) \times \ldots \mathcal{M}(x_i,x_j) \ldots \mathcal{M}(x_{\ell-1},x_\ell) \times (0,\infty]^{\ell-j+i} & \longrightarrow & \mathcal{M}(x_0,x_\ell) \end{array}$$

commutes. We moreover require that the images of the collars associated to $\{x_0, x_1, \ldots, x_{\ell-1}, x_\ell\}$ and $\{x_0, x_1', \ldots, x_{\ell-1}', x_\ell\}$ be disjoint unless the subset $\{x_0, x_1, x_1', \ldots, x_{\ell-1}, x_{\ell-1}', x_\ell\}$ of $\mathrm{Ob}(\mathcal{M})$ *is totally ordered, in which case the intersection of the two collar neighbourhoods is required to be precisely the collar neighbourhood associated to this combined subset.*

Given a coherent system of collars for $\mathcal{M}$, we can take the corresponding closed embeddings for $x_0 > x_1 > \ldots > x_\ell$

$$\iota : \mathcal{M}(x_0, x_1) \times \ldots \times \mathcal{M}(x_{\ell-1}, x_\ell) = \mathcal{M}(x_0, x_1) \times \ldots \times \mathcal{M}(x_{\ell-1}, x_\ell) \times \{1\}^{\ell-1} \xrightarrow{C} \mathcal{M}(x_0, x_\ell). \tag{2.23}$$

These embeddings satisfy the appropriate associativity property, and moreover the normal bundle of $\iota(\mathcal{M}(x_0, x_1) \times \ldots \times \mathcal{M}(x_{\ell-1}, x_\ell))$ is exactly $\mathbb{R}^{\ell-1}$.

In particular, equip each $\mathcal{M}(x_0,x_1)$ with a trivial rank one vector bundle

$$\nu = \mathbb{R} \times \mathcal{M}(x_0,x_1)$$

and consider the quotient of Thom spaces

$$\mathcal{M}(x_0,x_1)^\nu/\partial\mathcal{M}(x_0,x_1)^\nu.$$

For short, we will usually write this space as $\mathcal{M}^\nu/\partial\mathcal{M}^\nu(x_0,x_1)$. For any $x_0 > x_1 > \ldots > x_\ell$, the Pontrjagin-Thom collapse map associated to the closed embedding (2.23) yields a map of spaces

$$q_{x_0,\ldots,x_\ell} : \mathcal{M}^\nu/\partial\mathcal{M}^\nu(x_0,x_\ell) \longrightarrow \prod_{i=1}^{\ell} \mathcal{M}^\nu/\partial\mathcal{M}^\nu(x_{i-1},x_i) \tag{2.24}$$

similarly to the construction of the maps (2.18). These maps are moreover strictly associative, in that whenever $0 \le i < j \le \ell$ we have

$$q_{x_i,\ldots,x_j} \circ q_{x_0,\ldots,x_i,x_j,\ldots,x_\ell} = q_{x_0,\ldots,x_\ell}.$$

Remark 2.15. *There is another, slightly more geometric perspective on these spaces and maps, in the event that $\mathcal{M}$ is a flow category from Morse or Floer theory. Over the interior of $\mathcal{M}(x_0,x_1)$, the total space of the vector bundle ν can be thought of as the space $W(x_0,x_1)$ of* parametrized *Morse or Floer trajectories from x_0 to x_1. This space is non-compact, however its one-point compactification is precisely the quotient space $\mathcal{M}(x_0,x_1)^\nu/\partial\mathcal{M}(x_0,x_1)^\nu$. Then, given $x_0 > x_1 > x_2$, assuming the existence of "global" gluing maps we would have an open embedding*

$$W(x_0,x_1) \times W(x_1,x_2) \longrightarrow W(x_0,x_2)$$

and the map q_{x_0,x_1,x_2} could be constructed as the associated Pontrjagin-Thom collapse map.

A coherent system of collars also induces a unique continuous map on each $\mathcal{M}(x_0,x_1)$

$$\pi : \mathcal{M}(x_0,x_1) \longrightarrow \mathcal{J}(\mu(x_0)-\mu(x_1)) \tag{2.25}$$

with the properties:

(i) it sends complement $\mathring{\mathcal{M}}(x_0,x_1)$ of all the collar neighbourhoods to $\{0\}^{\mu(x_0)-\mu(x_1)-1} \subset \mathcal{J}(\mu(x_0)-\mu(x_1))$;

(ii) the collars on $\mathcal{M}$ are compatible with those on $\mathcal{J}$, meaning we have commutative diagrams for each $x_0 > \ldots > x_\ell$

$$\begin{array}{ccc} \prod_{i=1}^{\ell} \mathcal{M}(x_{i-1},x_i) \times (0,\infty]^{\ell-1} & \xrightarrow{C} & \mathcal{M}(x_0,x_\ell) \\ \downarrow & & \downarrow \\ \prod_{i=1}^{\ell} \mathcal{J}(\mu(x_{i-1})-\mu(x_i)) \times (0,\infty]^{\ell-1} & \longrightarrow & \mathcal{J}(\mu(x_0)-\mu(x_\ell)). \end{array}$$

where the bottom arrow is a collar map (2.12) for $\mathcal{J}$.

Observe that these maps preserve the boundaries: $\partial\mathcal{M}$ is sent to $\partial\mathcal{J}$. Moreover, they automatically preserve the closed embeddings ι, for both $\mathcal{M}$ and for $\mathcal{J}$ as in (2.13): we have commutative diagrams

$$\begin{array}{ccc}
\prod_{i=1}^{\ell} \mathcal{M}(x_{i-1}, x_i) & \xrightarrow{\quad\iota\quad} & \mathcal{M}(x_0, x_\ell) \\
\downarrow & & \downarrow \\
\prod_{i=1}^{\ell} \mathcal{J}(\mu(x_{i-1}) - \mu(x_i)) & \xrightarrow{\iota} & \mathcal{J}(\mu(x_0) - \mu(x_\ell)).
\end{array}$$

In particular, we have for each x_0, x_1 a continuous map we will also call π

$$\pi : \mathcal{M}^\nu/\partial\mathcal{M}^\nu(x_0, x_1) \longrightarrow \mathcal{J}^\nu/\partial\mathcal{J}^\nu(\mu(x_0) - \mu(x_1))$$

which is compatible with the product maps (2.18) and (2.24) in that whenever $x_0 > \ldots > x_\ell$ the square

$$\begin{array}{ccc}
\mathcal{M}^\nu/\partial\mathcal{M}^\nu(x_0, x_\ell) & \longrightarrow & \prod_{i=1}^{\ell} \mathcal{M}^\nu/\partial\mathcal{M}^\nu(x_{i-1}, x_i) \\
\downarrow & & \downarrow \\
\mathcal{J}^\nu/\partial\mathcal{J}^\nu(\mu(x)_0 - \mu(x_\ell) & \longrightarrow & \prod_{i=1}^{\ell} \mathcal{J}^\nu/\partial\mathcal{J}^\nu(\mu(x)_{i-1} - \mu(x_i))
\end{array}$$

commutes.

We now take the Spanier-Whitehead dual of π to obtain maps of spectra

$$\pi^* : \mathscr{J}(\mu(x_0) - \mu(x_1)) \longrightarrow F\left(\mathcal{M}^\nu/\partial\mathcal{M}^\nu(x_0, x_1), \mathbb{S}\right). \tag{2.26}$$

and these maps are moreover compatible with the composition structures.

Now, recall that over each $\mathcal{M}(x_0, x_1)$, we have an "index bundle" $I(x_0, x_1) = \mathbb{R} \oplus T\mathcal{M}(x_0, x_1)$, which we used to define the Atiyah dual flow category $\mathcal{M}^{-I}$ in the previous section. In order to relate $F\left(\mathcal{M}^\nu/\partial\mathcal{M}^\nu, \mathbb{S}\right)$ to $\mathcal{M}^{-I}$, and thus to our notions of framings and spectral systems, we appeal to Atiyah duality for manifolds with boundary [2]. In the incarnation we will use, if $(X, \partial X)$ is a compact smooth manifold with boundary and E is a vector bundle over X, then there is a pairing

$$X^{-TX-E} \wedge X^E/\partial X^E \longrightarrow \mathbb{S}$$

such that the adjoint map is a weak equivalence

$$X^{-TX-E} \xrightarrow{\sim} F(X^E/\partial X^E, \mathbb{S}). \tag{2.27}$$

The Atiyah duality equivalence is moreover functorial for closed embeddings: if $(Y, \partial Y) \subset (X, \partial X)$ is a closed submanifold with normal bundle N, the vector bundle $(TX \oplus E)|_Y$

splits as $TY \oplus E \oplus N$. There is then a commutative diagram of spectra

$$\begin{array}{ccc} Y^{-TY-E-N} & \longrightarrow & F(Y^{E\oplus N}/\partial Y^{E\oplus N}, \mathbb{S}) \\ \downarrow & & \downarrow \\ X^{-TX-E} & \longrightarrow & F(X^E/\partial X^E, \mathbb{S}) \end{array}$$

where the top arrow is the Atiyah duality equivalence for Y, the left arrow is the induced map on Thom spectra of the embedding $Y \to X$, and the right map is the dual of the Pontrjagin-Thom collapse map $X^E/\partial X^E \longrightarrow Y^{E\oplus N}/Y^{E\oplus N}$.

Atiyah duality for the index bundles I over the flow category $\mathcal{M}$ then yields maps of Thom spectra

$$\mathcal{M}^{-I}(x_0, x_1) \xrightarrow{\sim} F(\mathcal{M}^{\nu}/\partial\mathcal{M}^{\nu}(x_0, x_1), \mathbb{S}) \tag{2.28}$$

which are weak equivalences. Moreover, the functoriality of Atiyah duality ensures that these equivalences are compatible with the multiplication structures on either side: whenever $x_0 > x_1 > x_2$ we have commutative diagrams

$$\begin{array}{ccc} \mathcal{M}^{-I}(x_0, x_1) \wedge \mathcal{M}^{-I}(x_1, x_2) & \longrightarrow & \mathcal{M}^{-I}(x_0, x_2) \\ \downarrow & & \downarrow \\ F(\mathcal{M}^{\nu}/\partial\mathcal{M}^{\nu}(x_0, x_1), \mathbb{S}) \wedge F(\mathcal{M}^{\nu}/\partial\mathcal{M}^{\nu}(x_1, x_2), \mathbb{S}) & \longrightarrow & F(\mathcal{M}^{\nu}/\partial\mathcal{M}^{\nu}(x_0, x_2), \mathbb{S}). \end{array}$$

In particular, suppose that $\mathfrak{o} : \mathcal{M}^{-I} \longrightarrow \mathrm{Sp}$ is a spectral system on $\mathcal{M}$ in the sense of Definition 2.6. Inverting the Atiyah duality map and composing with π^*, we then obtain an $\mathscr{J}$-spectrum

$$X : \mathscr{J} \longrightarrow \mathrm{Sp}$$

which on objects is given by

$$X(a) = \bigvee_{x:\mu(x)=a} \mathfrak{o}(x).$$

2.6. **$\mathscr{J}$-complexes and their realizations.** We now explain Cohen-Jones-Segal's realization procedure. Fix an integer m. For any integer a such that $a \geq m$, consider the spectrum

$$\mathscr{V}_m(a) := \Sigma\mathscr{J}(a, m-1) \simeq \begin{cases} \mathbb{S} & \text{if } m = a; \\ * & \text{if } a > m. \end{cases}$$

This spectrum is a left $\mathscr{J}$-module, in that there are appropriately associative structure maps

$$\lambda_{\mathscr{V}} : \mathscr{J}(a_0, a_1) \wedge \mathscr{V}_m(a_1) \longrightarrow \mathscr{V}_m(a_0) \tag{2.29}$$

whenever $a_0 > a_1 \geq m$.

We can give a more direct construction of these spectra and their structure maps using the combinatorics of weighted graphs.

Definition 2.16. *A* weighted flower-stem *is an oriented, linear graph on vertices* $V(T) = v_1, \ldots, v_\ell$ *read outgoing to incoming, with just one semi-infinite edge attached to* v_1*, and a weighting function* $w : V(T) \to \mathbb{Z}$*. We require that* $w(v_i) \geq 1$ *for* $i = 1, \ldots, \ell - 1$ *and that* $w(v_\ell) \geq 0$*. We will often refer to* v_ℓ *as the "flower" vertex.*

Again, the set of weighted flower-stems of a fixed total weight forms a poset under construction. Of course, by taking a flower stem with vertices $v_1, \ldots, v_\ell$, attaching a semi-infinite edge to v_ℓ and raising its weight by 1, there is an isomorphism between the poset of flower-stems of total weight w and the poset of weighted unstable chains of weight $w+1$. Given a weighted unstable chain T_1 and a weighted flower-stem T_2, we can also form their grafting $T_1 \# T_2$.

Definition 2.17. *A metrization of a weighted flower-stem* T *is an assignment of a length* $\ell \in (0, \infty]$ *for each interior edge of* T*. We formally identify those flower-stems with edges of length zero with their contractions. In particular, the set of isomorphism classes* $\mathcal{V}(w)$ *of metrized weighted flower-stems of total weight* w *is a topological space*

$$\mathcal{V}(w) = [0, \infty]^w.$$

This is naturally a stratified space, with strata indexed by the poset of weighted flower-stems. We will again refer to the union of its proper strata as the boundary $\partial \mathcal{V}(w)$; this is exactly the space where one of the lengths is ∞.

Given metrizations on a weighted unstable chain T_1 and a flower-stem T_2, the choice of a length for the new grafted edge of $T_1 \# T_2$ determines a "collar neighbourhood map"

$$\mathcal{J}(w_1) \times \mathcal{V}(w_2) \times (0, \infty] \longrightarrow \mathcal{V}(w_1 + w_2).$$

By setting this grafted length to be 1, we then have a canonical closed embedding

$$\iota : \mathcal{J}(w_1) \times \mathcal{V}(w_2) \hookrightarrow \mathcal{V}(w_1 + w_2).$$

Taking Pontrjagin-Thom collapse maps associated to these embeddings ι, we then have maps of pointed spaces

$$\mathcal{V}(w_1 + w_2)/\partial \mathcal{V}(w_1 + w_2) \longrightarrow \mathcal{J}(w_1)^\nu / \partial \mathcal{J}(w_1)^\nu \wedge \mathcal{V}(w_2)/\partial \mathcal{V}(w_2).$$

We can then define the spectrum

$$\mathscr{V}(w) = F(\mathcal{V}(w)/\partial \mathcal{V}(w), \mathbb{S})$$

which comes equipped with multiplication maps

$$\lambda : \mathscr{J}(w_1) \wedge \mathscr{V}(w_2) \longrightarrow \mathscr{V}(w_1 + w_2)$$

which satisfy the appropriate assocativity diagram for the composite maps $\mathscr{J}(w_1) \wedge \mathscr{J}(w_2) \wedge \mathscr{V}(w_3) \to \mathscr{V}(w_1 + w_2 + w_3)$. In particular, we can then construct the left $\mathscr{J}$-module $\mathscr{V}_m$ as just

$$\mathscr{V}_m(a) = \begin{cases} \mathscr{V}(a - m) & \text{if } a \geq m; \\ * & \text{otherwise} \end{cases}.$$

Observe that, whenever $a \geq m_0 \geq m_1$, there are natural maps compatible with the left $\mathscr{J}$-module structure

$$\mathscr{V}_{m_1}(a) \longrightarrow \mathscr{V}_{m_0}(a) \tag{2.30}$$

defined by taking the Spanier-Whitehead dual of the natural closed embedding of pairs

$$(\mathcal{V}(a-m_0), \partial\mathcal{V}(a-m_0)) \hookrightarrow (\mathcal{V}(a-m_1), \partial\mathcal{V}(a-m_1))$$

that takes a metrized flower-stem of total weight $a - m_0$ to the flower-stem of total weight $a - m_1$ which exactly the same lengths and weights except for the weight of the flower vertex, which is increased by $m_0 - m_1$. These maps of spectra are also clearly compatible under taking triples $m_0 \geq m_1 \geq m_2$ of integers, in that the composite map

$$\mathscr{V}_{m_2}(a) \longrightarrow \mathscr{V}_{m_1}(a) \longrightarrow \mathscr{V}_{m_0}(a)$$

is exactly the original map $\mathscr{V}_{m_2}(a) \to \mathscr{V}_{m_0}(a)$.

We can now define the Cohen-Jones-Segal realizations of a $\mathscr{J}$-spectrum X.

Definition 2.18. *For an integer $m \in \mathbb{Z}$, the m-realization $|X|_m$ of a $\mathscr{J}$-spectrum X is the derived tensor product of left and right $\mathscr{J}$-modules*

$$|X|_m = X \otimes_{\mathscr{J}} \mathscr{V}_m. \tag{2.31}$$

More explicitly, this is given by the bar construction: it is the geometric realization of the simplicial spectrum whose ℓ-simplices are given by the spectrum

$$\bigvee_{a_0 \geq a_1 \geq \ldots \geq a_\ell} X(a_0) \wedge \mathscr{J}(a_0, a_1) \wedge \ldots \wedge \mathscr{J}(a_{\ell-1}, a_\ell) \wedge \mathscr{V}_m(a_\ell).$$

Moreover, for $m_0 > m_1$, the maps $\{\mathscr{V}_{m_1}(a) \to \mathscr{V}_{m_0}(a)\}_{a \geq m_0}$ of (2.30) assemble to induce maps of spectra well-defined up to homotopy

$$|X|_{m_1} \longrightarrow |X|_{m_0} \tag{2.32}$$

which are moreover compatible up to homotopy for triples $m_0 > m_1 > m_2$. In particular, we have an inverse system of spectra $\{|X|_m\}_{m \in \mathbb{Z}}$.

We say that a $\mathscr{J}$-spectrum X is *finite type* if $X(a) \simeq *$ for all but finitely many a. In particular, the $\mathscr{J}$-spectrum associated to a finite flow category with a spectral system is finite type. For a $\mathscr{J}$ spectrum of finite type, the maps

$$|X|_{m-1} \to |X|_m$$

are equivalences for $m \ll 0$. In particular, this inverse system of spectra can be thought of as a single homotopy type $|X|$ equipped with a decreasing filtration. In the context of a finite flow category $\mathcal{M}$ with a spectral system $\mathfrak{o}$, the *geometric realization* (without qualification to the level m) is the corresponding homotopy type $|\mathfrak{o}|$.

3. Multi-linear operations on flow categories

One of the goals of this thesis is to describe how Floer homotopy types come naturally equipped with operations arising from parameter spaces of families of Riemann surfaces. In this section, we develop a formalism to do this, in the case that the parameter spaces are closed manifolds. In the subsequent section, we will deal with parameter spaces with boundary, which will allow us to deduce algebraic relations (up to homotopy) between the corresponding operations.

The operations we will define in this thesis will all have one input, and multiple outputs; thus they model structures such as co-algebras. This reflects the fact that the Floer homotopy type of, for instance, the Morse flow category on a closed manifold should come equipped with a co-algebra structure, dual to the cup product on the stable cohomotopy of the manifold. Of course, to recover Floer cohomology (with product, rather than co-products) one can take the Spanier-Whitehead duals of the geometric realizations of the flow categories. This choice to work homologically rather than cohomologically may seem somewhat unnatural, but will turn out to be technically convenient.

For this whole section, fix a compact smooth manifold $\mathcal{R}$, and a (possibly empty) finite set Σ^+. The manifold $\mathcal{R}$ is to be thought of as the *parameter space* for a family of operations with inputs Σ^+. We will fix non-negative integers

$$k = |\Sigma| \quad \text{the } \textit{arity}; \qquad d = \dim \mathcal{R} \quad \text{the } \textit{dimension}.$$

For this section, we will almost always identify $\Sigma \cong \{1, 2, \ldots, k\}$.

Later in this section, we will give the definition of a $\mathcal{R}$-parametrized map

$$\mathcal{F} : \mathcal{M}_0 \longrightarrow (\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k)$$

from a flow category $\mathcal{M}_0$ to a tuple of flow categories $\mathcal{M}_1, \ldots, \mathcal{M}_k$. Roughly speaking, this will consist of a smooth manifold with corners

$$\mathcal{F}(x_0; x_1, \ldots, x_k)$$

of dimension $\mu(x_0) - \sum_{i=1}^k \mu(x_i) + d$ for each choice of $x_0 \in \mathrm{Ob}(\mathcal{M}_0), \ldots, x_k \in \mathrm{Ob}(\mathcal{M}_k)$, which comes equipped with a smooth "projection" map

$$\pi : \mathcal{F}(x_0; x_1, \ldots, x_k) \longrightarrow \mathcal{R}.$$

It will also come with associative composition maps, which are inclusions of boundary faces,

$$\mathcal{F}(x_0; x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_k) \times \mathcal{M}_i(x_i', x_i) \hookrightarrow \mathcal{F}(x_0; x_1, \ldots, x_k)$$

for each $i = 1, \ldots, k$ and $x_i' > x_i$, as well as

$$\mathcal{M}_0(x_0, x_0') \times \mathcal{F}(x_0'; x_1, \ldots, x_k) \hookrightarrow \mathcal{F}(x_0; x_1, \ldots, x_k)$$

for each $x_0 > x_0'$, which commute with the projection maps π.

The combinatorial structure of the boundary facets of $\mathcal{F}$ is more complicated than just that of a $< w >$-manifold for some w; in particular the facets are indexed by a poset that is not quite the cubical posets $\underline{2}^w$. In order to give a more formal definition of $\mathcal{F}$, and then study its Pontrjagin-Thom theory, let us talk about the combinatorics of trees.

3.1. **Weighted trees.** Throughout this thesis, a *k-tree* for $k \geq 0$ will always mean a graph with no cycles, with $k+1$ semi-infinite edges, that are attached to only one vertex, one of which is singled out as the *root*, and the rest are referred to as *leaves.* The leaves are customarily labelled $1, 2, \ldots, k$; an isomorphism of trees is understood to preserve the labels.

For a tree T, we will always denote the set of vertices by $V(T)$, and the set of *interior edges* (non-semi-infinite) as $E(T)$. The set of leaves will be denoted $E^{+\infty}(T)$ (we could also denote the set of roots by $E^{-\infty}(T)$, although this is a singleton). Under the orientation of T, for an edge $e \in E(T)$ we write $v_-(e)$ for the incoming vertex adjacent to e, and $v_+(e)$ for the outgoing vertex; for a leaf $e \in E^{+\infty}(T)$, only have an incoming vertex $v_-(e)$. Likewise, for a vertex $v \in V(T)$, we have a set $E^+(v)$ of outgoing edges (each of which satisfies $v_-(e) = v$), and one incoming edge $e_-(v)$.

A connected subset of interior edges $E_0 \subset E(T)$ of a tree T determines a new tree $T_{sub}(E_0)$, the *subtree*, with vertices given by those adjacent to E_0 and interior edges exactly those of E_0, and semi-infinite edges given by those edges of T which are adjacent to, but not included in, E_0. We can then form *the contraction* T/E_0, which is a tree formed by deleting all the edges of E_0 and identifying as one all the vertices adjacent to E_0. The orientation on T induces orientations on each of $T_{sub}(E_0)$ and T/E_0; the leaves and root of T/E_0 are identified with those of T, but not for $T_{sub}(E_0)$.

Definition 3.1. *A* simple k-tree *is a k-tree with a distinguished vertex, which we call the* stable vertex*, of valence* $k+1$*, such that all other vertices have valence exactly two. These other vertices are called* unstable*.*

The terminology is because eventually the vertices of a tree T will correspond to component curves of a “broken” Riemann surface which is the domain for a family of pseudoholomorphic maps. The “stable” vertex corresponds to a component curve that is stable, while the “unstable” vertices correspond to infinite strips or cylinders on which one takes Floer's equation; these curves are unstable since they carry an action by $\mathbb{R}$-translation.

The interior edges of a simple k-tree are on one of the k stems (between one of the leaves and the stable vertex), in which case we call them *stem* edges, or between the stable vertex and the root, in which case we call them *trunk* edges.

Definition 3.2. *For non-negative integers k, d, a* weighted simple k-tree of type d *is a simple k-tree together with an integer weight $w(v)$ at each vertex. We moreover require that*

$$w(v) \geq \begin{cases} 1 & \text{if } v \text{ is an unstable vertex;} \\ -d & \text{if } v \text{ is the unique stable vertex.} \end{cases}$$

The total weight is the sum of all the weights. We will write $\mathcal{T}_{k,d}(w)$ for the set of isomorphism classes of such trees of a fixed total weight $w \geq -d$.

Any contraction of a simple k-tree is also a simple k-tree; moreover if the original tree was weighted of type d, the contraction also acquires a weighting by summing the weights of identified vertices, and this weighting satisfies the above condition. In particular, $\mathcal{T}_{k,d}(w)$ is a poset, by declaring $T_2 \leq T_1$ if T_1 can be obtained by contracting some collection of interior edges of T_2.

Example 3.3. *$w = -d$ is the minimum weight for $\mathcal{T}_{k,d}(w)$ to be non-empty; $\mathcal{T}_{k,d}(-d)$ consists of a single simple k-tree with no unstable vertices.*

For $w = -d+1$, the set $\mathcal{T}_{k,d}(w)$ has $k+2$ elements: the tree with a single stable vertex of weight $-d+1$ and no unstable vertices, and the $k+1$ trees each with a single unstable vertex of weight $w(v) = 1$ adjacent to respectively the root semi-infinite edge or one of the k leaves. The poset structure is that the unique tree with a single vertex is maximal, and all the other trees are incomparable.

We say that a weighted simple tree with a single interior edge is elementary. Consider the subset $\mathcal{E}_{k,d}(w) \subset \mathcal{T}_{k,d}(w)$ of the elementary trees. For any arbitrary $T \in \mathcal{T}_{k,d}(w)$, each interior edge $e \in E(T)$ determines an element of $\mathcal{E}_{k,d}(w)$ by collapsing all other interior edges; this determines an injection

$$E(T) \hookrightarrow \mathcal{E}_{k,d}(w).$$

The following is then elementary, and broadly characterizes the posets $\mathcal{T}_{k,d}(w)$:

Proposition 3.4. *The assignment above of a subset $E(T) \subset \mathcal{E}_{k,d}(w)$ for each $T \in \mathcal{T}_{k,d}(w)$ determines an order-preserving injective map of posets*

$$\mathcal{T}_{k,d}(w) \longrightarrow \underline{2}^{\mathcal{E}_{k,d}(w)}.$$

Thinking of $\mathcal{T}_{k,d}(w)$ as thus as sub-poset of the power set $\underline{2}^{\mathcal{E}_{k,d}(w)}$, this sub-poset satisfies:

(i) each singleton subset $S = \{e\} \subset \mathcal{E}_{k,d}(w)$ is in $\mathcal{T}_{k,d}(w)$;
(ii) if $T \in \mathcal{T}_{k,d}(w)$ and $S \in \underline{2}^{\mathcal{E}_{k,d}(w)}$ such that $T \leq S$, then $S \in \mathcal{T}_{k,d}(w)$.

We will call such sub-posets of power sets $\mathcal{T} \subset 2^{\mathcal{E}}$ satisfying the conditions (i), (ii) above *quasi-cubical posets*. While this will be a useful perspective in some of the topological constructions we are about to make, working more directly with isomorphism classes of trees has many advantages.

Trees come with natural *grafting* operations. Suppose that $T_1 \in \mathcal{C}(w_1)$ is a weighted chain, and $T_2 \in \mathcal{T}_{k,d}(w_2)$ is a weighted simple tree. One can similarly glue the incoming edge of T_1 to the outgoing edge of T_2 to form a new tree $T_1 \#_0 T_2$; this process gives an order-preserving map

$$\#_0 : \mathcal{C}(w_1) \times \mathcal{T}_{k,d}(w_2) \longrightarrow \mathcal{T}_{k,d}(w_1 + w_2).$$

Similarly for $T_1 \in \mathcal{T}_{k,d}(w_1)$, $T_2 \in \mathcal{C}(w_2)$ and $\ell = 1, 2, \ldots, k$, one can glue the incoming edge of T_1 labelled by ℓ to the outgoing edge of T_2 to form $T_1 \#^{\ell} T_2$, giving maps

$$\#_\ell : \mathcal{T}_{k,d}(w_1) \times \mathcal{C}(w_2) \longrightarrow \mathcal{T}_{k,d}(w_1 + w_2).$$

These *grafting* maps $\#, \#_0$ and $\#_\ell$ for $\ell = 1, \ldots, k$ are appropriately associative; we will not explicitly spell out the associativity relations. These grafting maps are the combinatorial underpinnings for the structure maps of "multi-linear" maps between flow categories, which we will now define.

3.2. **$\mathcal{R}$-maps of flow categories.** Suppose that $\mathcal{T}$ is a subset of $\underline{2}^{\mathcal{E}}$ for some finite set $\mathcal{E}$, which satisfies the conditions of Proposition 3.4:

(i) $\mathcal{T}$ contains each singleton $\{e\} \in \underline{2}^{\mathcal{E}}$;
(ii) whenever $T \in \mathcal{T}$ and $S \in \underline{2}^{\mathcal{E}}$ are such that $T \leq S$, then $S \in \mathcal{T}$ as well.

The set $\mathcal{E}$ can be recovered just from $\mathcal{T}$ and its poset structure: $\mathcal{T}$ has a maximal element T_0, and $\mathcal{E}$ is in bijection with the non-maximal elements $T \in \mathcal{T}$ such that there are no T' with $T < T' < T_0$.

Definition 3.5. *For such a poset $\mathcal{T} \subset \underline{2}^{\mathcal{E}}$, a* $\mathcal{T}$-manifold with corners X *is an $< \mathcal{E} >$-manifold so that for distinct $e_1, \ldots, e_\ell \in \mathcal{E}$ we have*

$$\partial_{e_1} X \cap \ldots \cap \partial_{e_\ell} X = \emptyset$$

unless $\{e_1, \ldots, e_\ell\} \in \mathcal{T}$.

In particular, for each $T \in \mathcal{T}$ the subspaces

$$\partial_T X = \cap_{e \in T} \partial_e X$$

are disjoint unions of codimension $|T|$ facets of X, and $\partial_{T_1} X \subset \partial_{T_2} X$ if and only if $T_1 \leq T_2$.

Now let us return to flow categories. Recall we have a fixed compact smooth manifold $\mathcal{R}$ of dimension d (a parameter space of operations), and a non-negative integer k (the arity of the operations). For each $w \geq -d$, we have a poset $\mathcal{T}_{k,d}(w)$ of weighted trees, naturally a sub-poset of $\underline{2}^{\mathcal{E}_{k,d}(w)}$.

Definition 3.6. *Suppose we have a flow category $\mathcal{M}_0$, and a k-tuple of flow categories $(\mathcal{M}_1, \ldots, \mathcal{M}_k)$. An $\mathcal{R}$-parametrized map*

$$\mathcal{F} : \mathcal{M}_0 \longrightarrow (\mathcal{M}_1, \ldots, \mathcal{M}_k)$$

is the data of, for each tuple $(x_i \in \mathrm{Ob}(\mathcal{M}_i))_{i=0,\ldots,k}$, a $\mathcal{T}_{k,d}(w)$-manifold with corners together with a smooth map to $\mathcal{R}$

$$\pi : \mathcal{F}(x_0; x_1, \ldots, x_k) \longrightarrow \mathcal{R}$$

for $w = \mu(x_0) - \sum_{i \geq 1} \mu(x_i)$, which is of dimension $w + d$.

This comes with composition maps: for each $i = 1, \ldots, k$ and $x_i' \in \mathrm{Ob}(\mathcal{M}_i)$ there is a map

$$G : \mathcal{F}(x_0; x_1, \ldots, x_i', \ldots, x_k) \times \mathcal{M}_i(x_i', x_i) \longrightarrow \mathcal{F}(x_0; x_1, \ldots, x_k)$$

whose image lies in the distinguished subset $\partial_e \mathcal{F}$ for $e \in \mathcal{E}_{k,d}(w)$ being the tree with a single unstable vertex adjacent to the i-th outgoing leaf of weight $\mu(x_i') - \mu(x_i)$. There is also a composition map

$$G : \mathcal{M}_0(x_0, x_0') \times \mathcal{F}(x_0'; x_1, \ldots, x_k) \longrightarrow \mathcal{F}(x_0; x_1, \ldots, x_k)$$

whose image lies in the distinguished subset $\partial_e\mathcal{F}$ for $e \in \mathcal{E}_{k,d}(w)$ being the tree with a single unstable vertex adjacent to the root of weight $\mu(x_0') - \mu(x_0)$. These structures satisfy the following properties:

(i) the composition maps are associative with each-other as well as the composition maps on each flow category $\mathcal{M}_0, \ldots, \mathcal{M}_k$;
(ii) the composition maps commute with the projections π to $\mathcal{R}$;
(iii) the composition maps are each diffeomorphisms onto a boundary face of $\mathcal{F}$;
(iv) for $e \in \mathcal{E}_{k,d}(w)$, then the images of all the composition maps contained in $\partial_e\mathcal{F}$ are disjoint, and moreover these images cover $\partial_e\mathcal{F}$.

As in this case of individual flow categories, a $\mathcal{R}$-parametrized map will carry certain canonical (virtual) vector bundles over each space $\mathcal{F}(x_0; \ldots, x_k)$. Recall that for $x_i' > x_i$ in $\mathrm{Ob}(\mathcal{M}_i)$, we defined the "index bundle" to be

$$I(x_i', x_i) = \mathbb{R}\tau_{x_i', x_i} \oplus T\mathcal{M}(x_i', x_i).$$

Now over $\mathcal{F}(x_0; \ldots, x_k)$, we define the index bundle to be the *formal difference*

$$I(x_0; \ldots, x_k) := T\mathcal{F}(x_0; \ldots, x_k) - \pi^* T\mathcal{R}.$$

After placing a Riemannian metric on $\mathcal{F}$, over each boundary face we have decompositions of the form

$$T\mathcal{M}(x_0, x_0') \oplus \mathbb{R}\nu_{x_0'} \oplus T\mathcal{F}(x_0'; \ldots, x_k) \cong G^* T\mathcal{F}(x_0; \ldots, x_k)$$

or alternatively

$$T\mathcal{F}(x_0; \ldots, x_i', \ldots, x_k) \oplus \mathbb{R}\nu_{x_i'} \oplus T\mathcal{M}(x_i', x_i) \cong G^* T\mathcal{F}(x_0; \ldots, x_i, \ldots, x_k).$$

according to the placement of the single unstable vertex on the corresponding elementary tree indexing this face. In particular, in the former case, we have isomorphisms

$$I(x_0, x_0') \oplus I(x_0'; \ldots, x_k) \cong G^* I(x_0; \ldots, x_k)$$

which send $\nu_{x_0'} \mapsto \tau_{x_0 x_0'} \in I(x_0, x_0')$; and in the latter case we have isomorphisms

$$I(x_0; \ldots, x_i', \ldots, x_k) \oplus I(x_i', x_i) \cong G^* I(x_0; \ldots, x_i, \ldots, x_k).$$

It is not difficult to verify that these isomorphisms are moreover compatible with the decomposition isomorphisms (2.6) for the index bundles over the flow categories $\mathcal{M}_i$.

In particular, we can take the Thom spectra:

Definition 3.7. *Given a $\mathcal{R}$-map of flow categories $\mathcal{F} : \mathcal{M}_0 \to (\mathcal{M}_1, \ldots, \mathcal{M}_k)$, the Atiyah dual map is the collection of spectra*

$$\mathcal{F}^{-I}(x_0; x_1, \ldots, x_k) := \begin{cases} \mathcal{F}(x_0; x_1, \ldots, x_k)^{-I(x_0; x_1, \ldots, x_k)} & \text{if } \mu(x_0) - \Sigma_{i \geq 1}\mu(x_i) \geq -d; \\ * & \text{otherwise.} \end{cases}$$

This comes with composition maps

$$\mathcal{M}_0^{-I}(x_0, x_0') \wedge \mathcal{F}^{-I}(x_0'; \ldots, x_k) \longrightarrow \mathcal{F}^{-I}(x_0; \ldots, x_k)$$

as well as for each $i = 1, \dots, k$

$$\mathcal{F}^{-I}(x_0; \dots, x_i', \dots, x_k) \wedge \mathcal{M}^{-I}(x_i', x_i) \longrightarrow \mathcal{F}^{-I}(x_0, \dots, x_k)$$

which are compatible with the composition maps on each Atiyah dual flow category $\mathcal{M}_i^{-I}$*.*

Remark 3.8. *A useful perspective on this structure is that of a bimodule of spectral categories: consider the spectral category*

$$\wedge_{i=1}^{k} \mathcal{M}_i^{-I}$$

whose objects are $\mathrm{Ob}(\mathcal{M}_1) \times \ldots \times \mathrm{Ob}(\mathcal{M}_k)$ *and whose morphism spectrum from* $(x_1', \dots, x_k')$ *to* $(x_1, \dots, x_k)$ *is given by*

$$\mathcal{M}_1^{-I}(x_1', x_1) \wedge \ldots \wedge \mathcal{M}_k^{-I}(x_k', x_k).$$

Then, $\mathcal{F}^{-I}$ *is naturally a* $(\mathcal{M}_0^{-I}, \wedge_{i=1}^{k} \mathcal{M}_i^{-I})$*-bimodule.*

Now, suppose that each flow category $\mathcal{M}_i$ was also equipped with a spectral system $\mathfrak{o}_i$.

Definition 3.9. *An* extension of the spectral systems $\mathfrak{o}_i$ over an $\mathcal{R}$-map $\mathcal{F} : \mathcal{M}_0 \to (\mathcal{M}_1, \dots, \mathcal{M}_k)$ *is a collection of maps of spectra for each* $x_0 \in \mathcal{M}_0, \dots, x_k \in \mathcal{M}_k$

$$\mathfrak{o}_0(x_0) \wedge \mathcal{F}^{-I}(x_0; x_1, \dots, x_k) \longrightarrow \mathfrak{o}_1(x_1) \wedge \ldots \wedge \mathfrak{o}_k(x_k)$$

so that we have commutative diagrams for each x_0'

$$\begin{array}{ccc} \mathfrak{o}_0(x_0) \wedge \mathcal{M}_0^{-I}(x_0, x_0') \wedge \mathcal{F}^{-I}(x_0'; \dots x_k) & \longrightarrow & \mathfrak{o}_0(x_0) \wedge \mathcal{F}^{-I}(x_0; \dots x_k) \\ \downarrow & & \downarrow \\ \mathfrak{o}_0(x_0') \wedge \mathcal{F}^{-I}(x_0'; \dots, x_k) & \longrightarrow & \mathfrak{o}_1(x_1) \wedge \ldots \wedge \mathfrak{o}_k(x_k) \end{array} \tag{3.1}$$

as well as for $i = 1, \dots, k$ *and* x_i'

$$\begin{array}{ccc} \mathfrak{o}_0(x_0) \wedge \mathcal{F}^{-I}(x_0; \dots, x_i', \dots, x_k) \wedge \mathcal{M}_i^{-I}(x_i', x_i) & \longrightarrow & \mathfrak{o}_0(x_0) \wedge \mathcal{F}^{-I}(x_0; \dots, x_k) \\ \downarrow & & \downarrow \\ \mathfrak{o}_1(x_1) \wedge \ldots \wedge \mathfrak{o}_i(x_i') \wedge \ldots \wedge \mathfrak{o}_k(x_k) \wedge \mathcal{M}_i^{-I}(x_i', x_i) & \longrightarrow & \mathfrak{o}_1(x_1) \wedge \ldots \wedge \mathfrak{o}_k(x_k). \end{array} \tag{3.2}$$

The basic problem we will solve in the rest of this section is to explain how a $\mathcal{R}$-map $\mathcal{F} : \mathcal{M}_0 \to \mathcal{M}_1, \dots, \mathcal{M}_k)$ together with an extension of a collection of spectra systems $\mathfrak{o}_i$ for each $\mathcal{M}_i$ gives rise to a map on the Cohen-Jones-Segal realizations. Explicitly, if $|\mathfrak{o}_i|_{m_i}$ is the level m_i realization for $i = 0, \dots, k$ and $m_0 \leq m_1 + \ldots + m_k$ we wish to construct a map of spectra

$$|\mathfrak{o}_0|_{m_0} \wedge \mathcal{R}_+ \longrightarrow |\mathfrak{o}_1|_{m_1} \wedge \ldots \wedge |\mathfrak{o}_k|_{m_k}.$$

We will not quite achieve the above; instead we will obtain a well-defined map on the homotopy category of spectra, as we will now explain.

3.3. **The spaces $\mathcal{R}(w)$ and the spectra $\mathscr{R}(w)$.** We now define some auxiliary spaces and spectra, which will be used to link the Cohen-Jones-Segal construction for flow categories to the earlier notion of an $\mathcal{R}$-map.

Definition 3.10. *A* metrization and $\mathcal{R}$-marking *of a weighted simple k-tree T is the assignment of:*

- *to each interior edge e of T, a length $\ell_e \in (0, \infty]$;*
- *to the stable vertex, a point $r \in \mathcal{R}$.*

We can formally allow some of the edges to acquire length zero, in which case we identify the metrized $\mathcal{R}$-marked weighted tree with its contraction. The topological space

$$\mathcal{R}(w)$$

is then the space of isomorphism classes of metrized $\mathcal{R}$-marked weighted trees of total weight w.

Observe $\mathcal{R}(w)$ is always just the product of $\mathcal{R}$ with the corresponding space in the case that $\mathcal{R}$ is a point; however in the sequel (when we model the composition of $\mathcal{R}$-maps) this will no longer be the case.

Example 3.11. *In the case where $k = 0$ and $\mathcal{R}$ is a point, weighted simple k-trees are precisely the weighted "flower-stems" defined earlier, and $\mathcal{R}(w) = \mathcal{V}(w)$.*

As with the spaces $\mathcal{J}(w)$, the space $\mathcal{R}(w)$ is stratified by the poset $\mathcal{T}_{k,d}(w)$: for $T \in \mathcal{T}_{k,d}(w)$, the open T-stratum is those metrized trees which, after contracting every edge of non-infinite length, have underlying weighted tree T. Alternatively, the closed stratum, which we denote $\mathcal{R}_T$, is the subspace of metrized $\mathcal{R}$-marked trees (T', ℓ) with $T' \leq T$ and so that the edges $e \in E(T) \subset E(T')$ have maximal length $\ell_e = \infty$. As before, we have

$$\mathcal{R}_T = \mathcal{R}(w(v^{st})) \times \prod_{\substack{v \in V(T) \\ \text{unstable}}} \mathcal{J}(w(v))$$

where v^{st} is the stable vertex of T. Again, we declare that the "boundary" $\partial\mathcal{R}(w)$ is the union of the proper strata, or equivalently the subspace of those metrized trees with at least one interior edge of infinite length.

Metrized weighted chains can be grafted onto metrized $\mathcal{R}$-marked weighted trees. If $(T_1, \ell_1) \in \mathcal{J}(w_1)$ is a metrized weighted chain and $(T_2, \ell_2) \in \mathcal{J}(w_2)$ is a metrized weighted tree, we declare the grafting $T_1 \#_0 T_2$ to have infinite length. Likewise for the graftings $T_2 \#_i T_1$ for $i = 1, 2, \ldots, k$. In particular we see that

$$\mathcal{R}_{T_1 \#_0 T_2} = \mathcal{J}_{T_1} \times \mathcal{R}_{T_2}$$

and likewise for the graftings $\#_i$ for $i = 1, 2, \ldots, k$.

The natural inclusions $\mathcal{R}_{T_2} \hookrightarrow \mathcal{R}_{T_1}$ for $T_2 \leq T_1$ extend, as before, to collar neighbourhood maps

$$\mathcal{R}_{T_2} \times (0, \infty]^{E(T_2) \setminus E(T_1)} \hookrightarrow \mathcal{R}_{T_1}$$

which are moreover associative for triples $T_3 \le T_2 \le T_1$. We again single out the embeddings

$$\iota : \mathcal{R}_{T_2} = \mathcal{R}_{T_2} \times \{1\}^{E(T_2)\setminus E(T_1)} \hookrightarrow \mathcal{R}_{T_1}.$$

Now, write $V^{un}(T)$ for the set of unstable vertices of a weighted simple tree T. Observe that whenever $T_2 \le T_1$, there are again short exact sequences of trivial vector spaces

$$0 \longrightarrow \mathbb{R}^{V^{un}(T_1)} \longrightarrow \mathbb{R}^{V^{un}(T_2)} \longrightarrow \mathbb{R}^{E(T_2)\setminus E(T_1)} \longrightarrow 0 \tag{3.3}$$

given as follows. For the first map, write $p : V(T_2) \to V(T_1)$ for the associated map on vertices, and note that $p^{-1}(V^{un}(T_1)) \subset V^{un}(T_2)$. Then the first map sends a tuple

$$(s_v)_{v\in V^{un}(T_1)} \mapsto (s_{p(v)})_{v\in V^{un}(T_2)}$$

where $s_{p(v)}$ is understood to be zero if $p(v)$ is the stable vertex. The second map sends

$$(s_v)_{v\in V^{un}(T_2)} \mapsto (s_{v_+(e)} - s_{v_-(e)})_{e\in E_2\setminus E_1}$$

where $v_-(e), v_+(e)$ are the incoming and outgoing vertices of an edge e respectively; again s_v is understood to be zero if v is the stable vertex.

We then consider the *virtual* vector bundle ν over $\mathcal{R}_T$ given as the formal difference

$$\nu = \mathbb{R}^{V^{un}(T)} - T\mathcal{R}$$

where $T\mathcal{R}$ is the tangent bundle of $\mathcal{R}$, pulled back to $\mathcal{R}_T$ by the marking. Whenever $T_2 \le T_1$, since the normal bundle to $\iota(\mathcal{R}_{T_2}) \subset \mathcal{R}_{T_1}$ is canonically identified with $\mathbb{R}^{E(T_2)\setminus E(T_1)}$, associated to the closed embedding ι we have a Pontrjagin-Thom collapse map of Thom spectra

$$\mathcal{R}^{\nu}_{T_1} \longrightarrow \mathcal{R}^{\nu}_{T_2}.$$

Remark 3.12. *Strictly speaking, we should fix once and for all a complementary vector bundle F for $T\mathcal{R}$ and an isomorphism $T\mathcal{R} \oplus F \cong \mathbb{R}^m$, and instead of using the Thom spectrum $\mathcal{R}^{\nu}_T$ we should use the Thom space $\mathcal{R}_T^{\mathbb{R}^{V^{un}(T)}\oplus F}$ in this whole discussion so as to work on the space level, rather than the spectrum level; and only desuspend at the very end.*

Since ι preserves the boundaries of $\mathcal{R}_{T_2}$ and $\mathcal{R}_{T_1}$, we have a corresponding map on the cofibers (which we will continue to write as quotients)

$$q_{T_1T_2} : \mathcal{R}^{\nu}_{T_1}/\partial\mathcal{R}^{\nu}_{T_1} \longrightarrow \mathcal{R}^{\nu}_{T_2}/\partial\mathcal{R}^{\nu}_{T_2}. \tag{3.4}$$

Moreover, these maps are coherent, in that whenever $T_3 \le T_2 \le T_1$, we have $q_{T_1T_3} = q_{T_2T_3} \circ q_{T_1T_2}$. We will for short write $\mathcal{R}^{\nu}/\partial\mathcal{R}^{\nu}(w)$ for $\mathcal{R}^{\nu}(w)/\partial\mathcal{R}^{\nu}(w)$, for each $w \ge -d$.

Now, by taking in the above T_1 to be a weighted simple tree with no unstable vertices of total weight $w_1 + w_2$, and T_2 to be an elementary tree with an unstable vertex attached to the root of weight w_1, and the stable vertex of weight w_2, we have maps

$$q_0 : \mathcal{R}^{\nu}/\partial\mathcal{R}^{\nu}(w_1 + w_2) \longrightarrow \mathcal{J}^{\nu}/\partial\mathcal{J}^{\nu}(w_1) \wedge \mathcal{R}^{\nu}/\partial\mathcal{R}^{\nu}(w_2).$$

Likewise, by taking an elementary tree with an unstable vertex attached to the i-th leaf, we have

$$q_i : \mathcal{R}^{\nu}/\partial\mathcal{R}^{\nu}(w_1 + w_2) \longrightarrow \mathcal{R}^{\nu}/\partial\mathcal{R}^{\nu}(w_1) \wedge \mathcal{J}^{\nu}/\partial\mathcal{J}^{\nu}(w_2).$$

The coherence conditions for $q_{T_1T_2}$ ensure that these maps are co-associative in the obvious fashion.

We then take the Spanier-Whitehead duals to give the main definition of this section.

Definition 3.13. *For $w \geq -d$, the spectrum $\mathscr{R}(w)$ is defined to be the function spectrum*

$$\mathscr{R}(w) = F(\mathcal{R}^\nu/\partial\mathcal{R}^\nu(w), \mathbb{S}).$$

This comes with canonical composition maps

$$q_0^* : \mathscr{J}(w_1) \wedge \mathscr{R}(w_2) \longrightarrow \mathscr{R}(w_1 + w_2)$$

as well as for each $i = 1, \dots, k$

$$q_i^* : \mathscr{R}(w_1) \wedge \mathscr{J}(w_2) \longrightarrow \mathscr{R}(w_1 + w_2)$$

which are appropriately strictly associative.

In particular, for $a_0, a_1, \dots, a_k \in \mathbb{Z}$ if we set

$$\mathscr{R}(a_0; a_1, \dots, a_k) = \begin{cases} \mathscr{R}(a_0 - \Sigma_{i\geq 1} a_i) & \text{for } a_0 - \Sigma_{i\geq 1} a_i \geq -d; \\ * & \text{otherwise} \end{cases}$$

then $\mathscr{R}$ is a $(\mathscr{J}, \mathscr{J}^{\wedge k})$-bimodule, where $\mathscr{J}^{\wedge k}$ is the spectral category with objects $\mathbb{Z}^k$ and morphism spectra from (a_i') to (a_i) given by $\wedge_{i=1}^k \mathscr{J}(a_i', a_i)$.

Example 3.14. *For $w = -d$ the minimum weight, the boundary $\partial\mathcal{R}(-d)$ is empty, and $\mathscr{R}(-d)$ is the Spanier-Whitehead dual of $\mathcal{R}^{-T\mathcal{R}}$, which by Atiyah duality is weak equivalent to $\Sigma^\infty \mathcal{R}_+$.*

Now, suppose that

$$\mathcal{F} : \mathcal{M}_0 \longrightarrow (\mathcal{M}_1, \dots, \mathcal{M}_k)$$

is an $\mathcal{R}$-map of flow categories. Recall that by choosing a coherent system of collars as in Definition 2.14 on each $\mathcal{M}_i$, we obtained maps $\mathcal{M}_i(x_i, x_i') \to \mathcal{J}(\mu(x_i) - \mu(x_i'))$ and thus maps of spectra

$$\mathscr{J}(\mu(x_i) - \mu(x_i')) \longrightarrow F(\mathcal{M}_i^\nu/\partial\mathcal{M}_i^\nu(x_i, x_i'), \mathbb{S});$$

by further applying Atiyah duality to $\mathcal{M}_i^\nu$ we have a map of a spectra to the Atiyah dual flow category

$$\mathscr{J}(\mu(x_i) - \mu(x_i')) \longrightarrow \mathcal{M}^{-I}(x_i, x_i').$$

The same process can be applied to the $\mathcal{R}$-map of flow categories $\mathcal{F}$. First, choose a coherent system of collar neighbourhoods compatible with those on $\mathcal{M}_i$ (the notion is totally analogous to that of Definition 2.14); then take the induced maps

$$\pi : \mathcal{F}(x_0; x_1, \dots, x_k) \longrightarrow \mathcal{R}(\mu(x_0) - \Sigma_{i\geq 1}\mu(x_i))$$

which lifts the existing projection map $\mathcal{F} \to \mathcal{R}$. Taking the Thom spaces of the bundles ν, the quotients by the boundaries and then Spanier-Whitehead duals, we have maps of spectra

$$\pi^* : \mathscr{R}(\mu(x_0) - \Sigma_{i\geq 1}\mu(x_i)) \longrightarrow F(\mathcal{F}^\nu/\partial\mathcal{F}^\nu(x_0; x_1, \dots, x_k), \mathbb{S}) \tag{3.5}$$

which are moreover compatible with the composition operations. Composing again with Atiyah duality for the virtual vector bundle $\nu = -T\mathcal{R}$ over $\mathcal{F}$, we have a map to the "Atiyah dual map" of Definition 3.7

$$\mathscr{R}(\mu(x_0) - \Sigma_{i\geq 1}\mu(x_i)) \longrightarrow \mathcal{F}^{-I}(x_0; x_1, \ldots, x_k). \tag{3.6}$$

Suppose further that each $\mathcal{M}_i$ comes equipped with a spectral system $\mathfrak{o}_i$, and that there is an extension of spectral systems over $\mathcal{F}^{-I}$. We then have, for each $x_0, x_1, \ldots, x_k$ maps of spectra

$$\mathfrak{o}_0(x_0) \wedge \mathscr{R}(\mu(x_0); \mu(x_1), \ldots, \mu(x_k)) \longrightarrow \mathfrak{o}_1(x_1) \wedge \ldots \wedge \mathfrak{o}_k(x_k)$$

which satisfy associativity relations analogous to those of (3.1) and (3.2).

Recall that the data of a spectral system $\mathfrak{o}_i$ and its corresponding maps $\mathfrak{o}_i \wedge \mathscr{J} \to \mathfrak{o}_i$ is packaged into a $\mathscr{J}$-spectrum X_i, where $X_i(a) = \vee_{\mu(x)=a}\mathfrak{o}_i(a)$. From the above, we obtain maps of spectra for each tuple of integers $a_0, a_1, \ldots, a_k$

$$X_0(a_0) \wedge \mathscr{R}(a_0; a_1, \ldots, a_k) \longrightarrow X_1(a_1) \wedge \ldots \wedge X_k(a_k).$$

Let us now deal which such maps in the abstract, and explain how these induce coproduct structures on the realization spectra of $X_0, \ldots, X_k$.

3.4. $\mathscr{R}$-coproducts of $\mathscr{J}$-spectra.

. Suppose we have $k+1$ fixed $\mathscr{J}$-spectra $X_0, X_1, \ldots, X_k$.

Definition 3.15. *An $\mathscr{R}$-coproduct from X_0 to $(X_1, \ldots, X_k)$ a collection of maps of spectra for every tuple of integers $(a_0, \ldots, a_k)$*

$$X_0(a_0) \wedge \mathscr{R}(a_0; a_1 \ldots, a_k) \longrightarrow X_1(a_1) \wedge \ldots \wedge X_k(a_k)$$

so that the following two types of diagram commute:

$$\begin{array}{ccc} X_0(a_0) \wedge \mathscr{J}(a_0, a_0') \wedge \mathscr{R}(a_0'; a_1, \ldots, a_k) & \longrightarrow & X(a_0') \wedge \mathscr{R}(a_0'; a_1, \ldots, a_k) \\ \downarrow & & \downarrow \\ X_0(a_0) \wedge \mathscr{R}(a_0; a_1, \ldots, a_k) & \longrightarrow & X_1(a_1) \wedge \ldots \wedge X_k(a_k) \end{array}$$

where the top horizontal arrow is the structure map of X_0, the left vertical map the structure map of $\mathscr{R}$; and for each $i = 1, \ldots k$

$$\begin{array}{ccc} X_0(a_0) \wedge \mathscr{R}(a_0; \ldots a_i', \ldots, a_k) \wedge \mathscr{J}(a_i', a_i) & \longrightarrow & X_1(a_1) \wedge \ldots X_i(a_i') \ldots \wedge X_k(a_k) \wedge \mathscr{J}(a_i', a_i) \\ \downarrow & & \downarrow \\ X_0(a_0) \wedge \mathscr{R}(a_0; a_1, \ldots, a_k) & \longrightarrow & X_1(a_1) \wedge \ldots \wedge X_k(a_k) \end{array}$$

where the left vertical arrow is the structure map of $\mathscr{R}$, and the right vertical arrow the structure map of X_i.

In particular, recalling that $\mathscr{R}$ is a $(\mathscr{J}, \mathscr{J}^{\wedge k})$ bimodule, for any $\mathscr{J}$-complex X_0 the derived tensor product

$$X_0 \otimes_{\mathscr{J}} \mathscr{R}$$

has the structure of a right $\mathscr{J}^{\wedge k}$-module. An $\mathscr{R}$-coproduct from X_0 to $(X_1, \ldots, X_k)$ then induces a map of right $\mathscr{J}^{\wedge k}$-modules

$$X_0 \otimes_{\mathscr{J}} \mathscr{R} \longrightarrow X_1 \wedge \ldots \wedge X_k \tag{3.7}$$

where $X_1 \wedge \ldots \wedge X_k$ is the right $\mathscr{J}^{\wedge k}$-module defined by $(X_1 \wedge \ldots \wedge X_k)(a_1, \ldots, a_k) = X_1(a_1) \wedge \ldots \wedge X_k(a_k)$ with the obvious structure maps.

Now, fix integers $m_1, \ldots, m_k$, and recall the left $\mathscr{J}$-modules $\mathscr{V}_{m_i}$ of the previous section; these were used previously to define the geometric realizations

$$|X_i|_{m_i} = X_i \otimes_{\mathscr{J}} \mathscr{V}_{m_i}.$$

There is also a left $\mathscr{J}^{\wedge k}$ module $\mathscr{V}_{m_1} \wedge \ldots \wedge \mathscr{V}_{m_k}$, and moreover there is an equivalence of spectra

$$\left(\wedge_{i=1}^{k} X_i\right) \otimes_{\mathscr{J}^{\wedge k}} \left(\wedge_{i=1}^{k} \mathscr{V}_{m_i}\right) \simeq (X_1 \otimes_{\mathscr{J}} \mathscr{V}_{m_1}) \wedge \ldots \wedge (X_k \otimes_{\mathscr{J}} \mathscr{V}_{m_k}).$$

We can take the derived $\mathscr{J}^{\wedge k}$-tensor product of (3.7) with the left $\mathscr{J}^{\wedge k}$ module $\mathscr{V}_{m_1} \wedge \ldots \wedge \mathscr{V}_{m_k}$ to obtain a map of spectra

$$X_0 \otimes_{\mathscr{J}} \mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} \left(\wedge_{i=1}^{k} \mathscr{V}_{m_i}\right) \longrightarrow (X_1 \otimes_{\mathscr{J}} \mathscr{V}_{m_1}) \wedge \ldots \wedge (X_k \otimes_{\mathscr{J}} \mathscr{V}_{m_k}).$$

The main result of this section, which underlies the whole theory, is a computation of the homotopy type of the left $\mathscr{J}$-module

$$\mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} (\mathscr{V}_{m_1} \wedge \ldots \wedge \mathscr{V}_{m_k}).$$

Proposition 3.16. *There is a map of left $\mathscr{J}$-modules*

$$\mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} (\mathscr{V}_{m_1} \wedge \ldots \wedge \mathscr{V}_{m_k}) \longrightarrow \mathscr{V}_{m_1+\ldots+m_k-d} \wedge \mathcal{R}_+ \tag{3.8}$$

which, on the level of each integer $a \in \mathbb{Z}$, is a weak homotopy equivalence. In particular, for any $\mathscr{J}$-spectrum X, there is a natural weak equivalence of spectra

$$X \otimes_{\mathscr{J}} \mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} (\mathscr{V}_{m_1} \wedge \ldots \wedge \mathscr{V}_{m_k}). \longrightarrow |X|_{m_1+\ldots+m_k-d} \wedge \mathcal{R}_+.$$

In particular, in the homotopy category of spectra there is a natural map

$$|X|_{m_1+\ldots+m_k-d} \wedge \mathcal{R}_+ \longrightarrow |X_1|_{m_1} \wedge \ldots \wedge |X_k|_{m_k}$$

which is the desired objective of this section.

The rest of this section concerns the proof of Proposition 3.16. First, observe that

$$\left(\mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} (\mathscr{V}_{m_1} \wedge \ldots \wedge \mathscr{V}_{m_k})\right)(a) \cong \left(\mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} (\mathscr{V}_0 \wedge \ldots \wedge \mathscr{V}_0)\right)(a - \Sigma m_i);$$

we can thus assume that each $m_1 = \ldots = m_k = 0$. We will accordingly write $\mathscr{V}$ for $\mathscr{V}_0$. In order to produce the required map

$$\mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} \mathscr{V}^{\wedge k} \longrightarrow \mathscr{V}_{-d} \wedge \mathcal{R}_+$$

we will pass through an intermediate $\mathscr{J}$-module, called $\mathscr{R}\mathscr{V}$, which will be defined as a by now familiar Spanier-Whitehead dual of a certain space of metrized graphs, and directly compute the homotopy type of this spectrum.

Recall that $\mathscr{V}(w)$ was constructed as

$$\mathscr{V}(w) = F(\mathcal{V}(w)/\partial\mathcal{V}(w), \mathbb{S})$$

where $\mathcal{V}(w)$ is the space of metrized weighted flower-stems of total weight w. We thus introduce an analogous notion of weighted trees with flowers.

Definition 3.17. *A* weighted simple k-tree with flowers T *is an oriented tree graph with precisely one semi-infinite edge, directed outgoing, referred to as the root; one distinguished vertex of valence $k+1$, which we call the stable vertex, and k distinguished vertices of valence exactly one (with their one edge being outgoing), distinct from the stable vertex, which we call the flowers. The flowers are considered labelled 1 through k. All other vertices are valence two, and are referred to as unstable.*

The weighting $w : V(T) \to \mathbb{Z}$ again has $w(v) \geq -d$ for the stable vertex, $w(v) \geq 1$ for the unstable vertices, and now also must satisfy $w(v) \geq 0$ for the flowers v.

The set of isomorphism classes of weighted simple k-trees with flowers of a fixed weight w is a poset under contraction, which we denote $\mathcal{TF}_{k,d}(w)$.

Observe that, by adding 1 to the weight of each flower and attaching a semi-infinite edge, we can identify $\mathcal{TF}_{k,d}(w)$ with the sub-poset of $\mathcal{T}_{k,d}(w+k)$ consisting of those weighted simple trees where there is an unstable vertex adjacent to each of the k leaf semi-infinite edges.

Definition 3.18. *A* metrization and $\mathcal{R}$-marking *of a weighted tree with flowers T is the assignment of a point $r \in \mathcal{R}$, together with a length $\ell_e \in (0, \infty]$ to each interior edge, which is required to satisfy:*

- *for each $i = 1, 2, \ldots, k$, at least one of the edges that lies between the i-th flower and the stable vertex has length $\ell = \infty$.*

We formally allow some of the edges to acquire length zero, and identify these with their contractions. The topological space

$$\mathcal{R}\mathcal{V}(w)$$

is then the space of isomorphism classes of metrized $\mathcal{R}$-marked trees with flowers of total weight w.

By grafting metrized weighted chains onto the root and declaring the length of the grafted edge to be ∞, we have maps of spaces

$$\mathcal{J}(w_1) \times \mathcal{RV}(w_2) \longrightarrow \mathcal{RV}(w_1 + w_2)$$

for each $w_1 \geq 1, w_2 \geq 0$. These maps are moreover associative when considering composites $\mathcal{J}(w_1) \times \mathcal{J}(w_2) \times \mathcal{RV}(w_3) \to \mathcal{RV}(w_1 + w_2 + w_3)$. The union of their images in $\mathcal{RV}(w)$ is declared to be the *boundary*, and written $\partial\mathcal{RV}(w) \subset \mathcal{RV}(w)$. Of course, this is precisely the subspace of metrized trees with flowers where at least one edge between the stable vertex and the root has length ∞.

Take the virtual vector bundle $\nu = -T\mathcal{R}$ over $\mathcal{RV}(w)$. As before, by taking closed embeddings

$$\iota : \mathcal{J}(w_1) \times \mathcal{RV}(w_2) \hookrightarrow \mathcal{RV}(w_1 + w_2)$$

which assign the length of the grafted edge to be 1, we have Pontrjagin-Thom collapse maps

$$\mathcal{RV}^\nu/\partial\mathcal{RV}^\nu(w_1 + w_2) \longrightarrow \mathcal{J}^\nu/\partial\mathcal{J}^\nu(w_1) \wedge \mathcal{RV}^\nu/\partial\mathcal{RV}^\nu(w_2)$$

which satisfy the appropriate co-associativity property. Taking Spanier-Whitehead duals, we have a function spectrum

$$\mathscr{RV}(w) = F(\mathcal{RV}^\nu/\partial\mathcal{RV}^\nu, \mathbb{S}) \tag{3.9}$$

which comes with appropriately associative structure maps

$$\mathscr{J}(w_1) \wedge \mathscr{RV}(w_2) \longrightarrow \mathscr{RV}(w_1 + w_2).$$

By setting $\mathscr{RV}(w) = *$ for $w < -d$, this then endows $\mathscr{RV}$ with the structure of a left $\mathscr{J}$-module.

Now, observe that there is a natural closed embedding of spaces for each $w \geq 0$

$$\mathcal{V}(w) \times \mathcal{R} \longrightarrow \mathcal{RV}(w - d) \tag{3.10}$$

which takes a metrized weighted flower-stem T, and produces a metrized marked tree with flowers by doing the following:

(i) attaching k vertices of valence exactly 1 to the flower, declaring these to be the new flowers, and the old flower to be the stable vertex;
(ii) reducing the weight of the old flower (and new stable vertex) by d;
(iii) assigning the k new edges length ∞;
(iv) assigning to the stable vertex the mark given by the $\mathcal{R}$-coordinate;
(v) leaving all the other lengths unchanged,

Note that when $w = 0$, the map (3.10) is precisely the identity map $\mathcal{R} \to \mathcal{R}$. Moreover, this map sends $\partial\mathcal{V}(w) \times \mathcal{R}$ to the boundary of $\mathcal{RV}(w - d)$, and so induces a map

$$\mathcal{V}/\partial\mathcal{V}(w) \wedge \mathcal{R}^{-T\mathcal{R}} \longrightarrow \mathcal{RV}^\nu/\partial\mathcal{RV}^\nu(w - d)$$

which is appropriately co-associative for the $\mathcal{J}^\nu/\partial\mathcal{J}^\nu$-comodule structures. In particular, taking function spectra, we have a map

$$\mathscr{RV}(w - d) \longrightarrow \mathscr{V}(w) \wedge F(\mathcal{R}^{-T\mathcal{R}}, \mathbb{S}) \tag{3.11}$$

which moreover as w ranges produces a map of left $\mathscr{J}$-modules.

We claim that (3.11) is a weak equivalence for every $w \geq 0$. For $w = 0$, this follows from the fact that the only possible tree with flowers of weight $-d$ is the one with no unstable vertices, and the minimal possible weight $-d$ on the stable vertex and weight 0 on each flower. For $w > 0$, this amounts to the claim that the spectrum $\mathscr{R}\mathscr{V}(w-d)$ is null.

It suffices to prove that, for $w > -d$, the inclusion of the boundary $\partial\mathcal{R}\mathcal{V}(w) \hookrightarrow \mathcal{R}\mathcal{V}(w)$ is a homotopy equivalence. It is sufficient to prove this in the case that $\mathcal{R} = \{\mathrm{pt}\}$ is a point, since the general space $\mathcal{R}\mathcal{V}(w)$ is the product of that for a point and the original space $\mathcal{R}$.

We claim that, in this case where $\mathcal{R}$ is a point and $w > -d$, both spaces $\mathcal{R}\mathcal{V}(w)$ and $\partial\mathcal{R}\mathcal{V}(w)$ are in fact contractible. We first prove the contractibility of $\mathcal{R}\mathcal{V}(w)$. To see this, observe that $\mathcal{R}\mathcal{V}(w)$ is homotopy equivalent to the *classifying space* $|N(\mathcal{T}\mathcal{F}_{k,d}(w))|$ of the poset $\mathcal{T}\mathcal{F}_{k,d}(w)$ of weighted trees with flowers (recall that the classifying space of a poset is the geometric realization of its simplicial nerve).

For an inclusion of posets $\mathcal{A} \subset \mathcal{B}$, a *deformation retraction* is an order-preserving map $r : \mathcal{B} \to \mathcal{A}$ which restricts to the identity on $\mathcal{A}$ and such that either:

(i) for each $b \in \mathcal{B}$, we have $r(b) \leq b$, in which case we say r retracts *downwards*;
(ii) for each $b \in \mathcal{B}$, we have $b \leq r(b)$, in which case we say r retracts *upwards*.

For any such retraction, the induced map $|N(r)| : |N(\mathcal{B})| \to |N(\mathcal{A})|$ on the classifying spaces is a homotopy equivalence. Thus, in order to show that $|N(\mathcal{T}\mathcal{F}_{k,d}(w))|$ is contractible, we produce a sequence of deformation retractions to poset with just one element.

First, write $\mathcal{T}\mathcal{F}^{(1)}_{k,d}(w) \subset \mathcal{T}\mathcal{F}_{k,d}(w)$ for the weighted trees with flowers such that the weight of each flower is exactly zero. Then, take the retraction

$$\mathcal{T}\mathcal{F}_{k,d}(w) \to \mathcal{T}\mathcal{F}^{(1)}_{k,d}(w)$$

which, for a tree with flowers T, takes each flower v and either:

(i) leaves it untouched if its weight $w(v)$ is exactly zero;
(ii) turns it into an unstable vertex if its weight is $w(v) \geq 1$, and attaches to it a new flower of weight exactly zero.

Now, observe that in fact $\mathcal{T}\mathcal{F}^{(1)}_{k,d}(w)$ has a maximal element: the tree with flowers where each flower has weight zero, the stable vertex has weight w, and where there are no unstable vertices. In particular, $|N(\mathcal{T}\mathcal{F}^{(1)}_{k,d}(w))|$ is contractible (it admits an upward deformation retraction to the singleton poset). This proves that $\mathcal{R}\mathcal{V}(w)$ is contractible.

We similarly prove that $\partial\mathcal{R}\mathcal{V}(w)$ is contractible. First, observe that $\partial\mathcal{R}\mathcal{V}(w)$ is homotopy equivalent to the classifying space $|N(\mathcal{T}\mathcal{F}^{(2)}_{k,d}(w))|$ of the poset $\mathcal{T}\mathcal{F}_{k,d}(w)$ of trees with flowers with at least one unstable vertex between the root and the stable vertex. To simplify the notation, we suppress k, d, w.

First, define $\mathcal{TF}^{(3)} = \mathcal{TF}^{(1)} \cap \mathcal{TF}^{(2)}$; these are the trees where each flower has weight exactly zero, and there is an unstable vertex of any weight adjacent to the root. The original deformation retraction $\mathcal{TF} \to \mathcal{TF}^{(1)}$ then yields a downwards deformation retraction

$$\mathcal{TF}^{(2)} \longrightarrow \mathcal{TF}^{(3)}.$$

Now, consider the subset $\mathcal{TF}^{(4)}$ of $\mathcal{TF}^{(3)}$ of the weighted trees where each flower is directly adjacent to the stable vertex. There is an upwards deformation retraction

$$\mathcal{TF}^{(3)} \longrightarrow \mathcal{TF}^{(4)}$$

which takes a tree $T \in \mathcal{TF}^{(3)}$ and contracts all the unstable vertices between the stable vertex and each flower to the stable vertex.

Then, consider the subset $\mathcal{TF}^{(5)}$ of $\mathcal{TF}^{(4)}$ where the stable vertex has the minimum possible weight $-d$. There is a downwards deformation retraction

$$\mathcal{TF}^{(4)} \longrightarrow \mathcal{TF}^{(5)}$$

which takes a tree T with stable vertex weight w_0, and inserts a new unstable vertex adjacent to and outgoing from the stable vertex, of weight $w_0 + d$, and changes the weight of the stable vertex to $-d$. Finally, observe that $\mathcal{TF}^{(5)}$ has a maximal element: the tree with flowers with each flower having weight 0, the stable vertex weight $-d$, and exactly one unstable vertex of weight $w + d$ adjacent to the root. Thus, $|N(\mathcal{TF}^{(5)}|$ is contractible, and hence so is $\partial \mathcal{RV}(w)$, completing the proof.

In summary, whenever we have an $\mathscr{R}$-coproduct from X_0 to $(X_1, \ldots, X_k)$ for $\mathscr{J}$-spectra $X_0, \ldots, X_k$, we have maps in the homotopy category of spectra for each $m_1, \ldots, m_k \in \mathbb{Z}$

$$|X_0|_{m_1+\ldots+m_k-d} \wedge \mathcal{R}_+ \longrightarrow |X_1|_{m_1} \wedge \ldots \wedge |X_k|_{m_k}.$$

Moreover, it is not difficult to see that these maps are compatible up to homotopy with the inverse systems $\{|X_i|_{m_i}\}_{m_i \in \mathbb{Z}}$, in that whenever $m_i' < m_i$ we have a homotopy commutative diagram

$$\begin{array}{ccc} |X_0|_{m_1+\ldots+m_i'+\ldots+m_k-d} \wedge \mathcal{R}_+ & \longrightarrow & |X_1|_{m_1} \wedge \ldots |X_i|_{m_i'} \ldots \wedge |X_k|_{m_k} \\ \downarrow & & \downarrow \\ |X_0|_{m_1+\ldots+m_i+\ldots+m_k-d} \wedge \mathcal{R}_+ & \longrightarrow & |X_1|_{m_1} \wedge \ldots |X_i|_{m_i} \ldots \wedge |X_k|_{m_k}. \end{array}$$

4. Composing operations on flow categories

We will now explain how the co-product-like maps of flow categories we have just defined can be composed.

For this whole section, let us fix a compact, smooth manifold $\mathcal{R}$, which now has boundary, together with a finite set Σ. Again we also fix the integers

$$k = |\Sigma| \quad \text{the arity,} \qquad d = \dim \mathcal{R} \quad \text{the dimension.}$$

Index the components of $\partial\mathcal{R}$ by a finite set B, and for each $\beta \in B$ write $\mathcal{R}^\beta$ for the corresponding boundary component. We assume that we have the following structures, for each $\beta \in B$:

(i) a decomposition
$$\mathcal{R}^\beta = \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$$
for compact closed smooth manifolds $\mathcal{R}_1^\beta, \mathcal{R}_2^\beta$, of dimension d_1^β and d_2^β respectively;

(ii) finite sets Σ_1^β and Σ_2^β, of cardinality k_1^β and k_2^β respectively, together with an element $\zeta^\beta \in \Sigma_1^\beta$;

(iii) a bijection of sets

$$\varphi^\beta : \left(\Sigma_1^\beta \backslash \{\zeta^\beta\}\right) \sqcup \Sigma_2^\beta \longrightarrow \Sigma. \tag{4.1}$$

On its own, this data of each $\mathcal{R}^\beta$ is to be thought of as, for each $\beta \in B$, a pair of families operations, parametrized by $\mathcal{R}_1^\beta$ and $\mathcal{R}_2^\beta$, of arity k_1^β and k_2^β, being composed, with the output of the $\mathcal{R}_2^\beta$-operation plugged into the ζ^β-input of the $\mathcal{R}_1^\beta$ operation. Together with the whole manifold with boundary $\mathcal{R}$, this data is a "homotopy" between various different ways of composing Σ inputs.

4.1. **Weighted B-shaped trees.** Before we use this data to define compositions of maps between flow categories, we need to again briefly discuss the combinatorics of trees. Recall that for us, a k-tree has $k+1$ semi-infinite edges, one of which is distinguished as the root, and the others referred to as leaves; the leaves are moreover labelled $1, 2, \ldots, k$. Slightly relaxing this definition, for a finite set Σ, a Σ-tree is a tree with $|\Sigma|+1$ semi-infinite edges, one of which is labelled the root, and the rest of which (the leaves) come with a bijection to Σ.

Definition 4.1. *For $\beta \in B$, the* stable β-shaped Σ-tree *is the unique Σ-tree T^β with precisely two vertices, v_1^β and v_2^β, so that:*

(i) v_1^β has valence $k_1^\beta + 1$, and is adjacent to the root, the single interior edge e connecting v_1^β and v_2^β, and $k_1^\beta - 1$ leaves;

(ii) v_2^β has valence $k_2^\beta + 1$, and is adjacent to the single interior edge e, and k_2^β leaves;

(iii) the incoming edges of v_1^β are labelled by Σ_1^β, with the interior edge having label ζ^β, and the incoming edges of v_2^β are labelled by Σ_2^β. Consequently, the leaves of the tree are labelled by Σ via the bijection (4.1).

We then write $\mathcal{T}^B$ for the set containing the above collection of Σ-trees, together with the unique Σ-tree with a just a single vertex of valence $k+1$. $\mathcal{T}^B$ is endowed with a poset structure, by declaring that simple Σ-tree is maximal (and with no other relations). We will say that each stable β-shaped Σ-tree contracts *to the maximal tree along its unique interior edge.*

As before, we use such trees as the basis for constructing more complicated trees with integer weights.

Definition 4.2. *For $\beta \in B$, a* weighted β-shaped Σ-tree *is a Σ-tree T with a weight function*

$$w : V(T) \longrightarrow \mathbb{Z}$$

and two distinguished vertices, v_1^β and v_2^β, which are called the stable *vertices; all other vertices are called* unstable. *These satisfy:*

- *(i) each unstable vertex v has valence exactly two, and positive weight $w(v) \geq 1$.*
- *(ii) each stable vertex v_i^β for $i = 1, 2$ has valence $k_i^\beta + 1$, and weight $w(v) \geq -d_i^\beta$.*
- *(iii) upon deleting each unstable vertex and identifying its two adjacent edges, and forgetting about the weights, the resulting labelled tree is exactly T^β.*

We refer to T^β as the underlying *stable tree, and to $w(T) = \sum_{v \in V(T)} w(v)$ as the* total weight. *Write $\mathcal{T}^\beta(w)$ for the set of isomorphism classes of weighted β-shaped Σ-trees of fixed total weight w.*

As before, contracting edges and summing the weights of identified vertices gives a partial order on the set $\mathcal{T}^\beta(w)$. The new feature here is that we can also have an interior edge connecting the two stable vertices v_1^β and v_2^β; its contracting such an edge will no longer give a tree of shape β, but rather give a weighted simple Σ-tree as in Definition 3.2.

Definition 4.3. *The poset $\mathcal{T}^B(w)$ is the union*

$$\mathcal{T}^B(w) = \mathcal{T}_{k,d}(w) \cup \bigcup_{\beta \in B} \mathcal{T}^\beta(w) \tag{4.2}$$

of the set of weighted simple trees and the the weighted β-shaped trees of a fixed weight, with the partial order given by $T_2 \leq T_1$ if T_1 is obtained from T_2 by iteratively contracting edges.

Note this poset has a maximal element, the tree with a single valence $k+1$ vertex of weight w.

Example 4.4. *This poset is empty for $w < -d$. For weight $w = -d$, the poset $\mathcal{T}^B(-d)$ has a single element, given by the tree with a single vertex of weight $-d$. For weight $w = -d+1$, the poset $\mathcal{T}^B(-d+1)$ has $|B| + k + 2$ elements, with one of them being maximal and no other relations:*

- *(i) the maximal tree has a single vertex of weight $-d+1$ and valence $k+1$;*
- *(ii) for each $\beta \in B$, there is the β-shaped tree with two vertices v_1^β and v_2^β of weight $-d_1^\beta$ and $-d_2^\beta$ respectively;*
- *(iii) there are k weighted simple trees with a stable vertex of weight $-d$ and a weight 1 unstable vertex adjacent to one of the leaves, as well as the simple tree with a stable vertex of weight $-d$ and a weight 1 unstable vertex adjacent to the root.*

For each $\beta \in B$, within the poset $\mathcal{T}^\beta(w)$ we can single out the *elementary* weighted trees $\mathcal{E}^\beta(w)$ as those with exactly one interior edge (or equivalently, no unstable vertices); there are exactly $w - d$ such trees. We can then similarly write

$$\mathcal{E}^B(w) = \mathcal{E}_{k,d}(w) \cup \bigcup_{\beta \in B} \mathcal{E}^\beta(w) \subset \mathcal{T}^B(w) \tag{4.3}$$

for the elementary weighted trees in $\mathcal{T}^B(w)$.

Again, for a general tree $T \in \mathcal{T}^B(w)$, each choice of interior edge $e \in E(T)$ determines an element of $\mathcal{E}^B(w)$ by contracting every other interior edge of T. This again defines an injection of sets

$$E(T) \hookrightarrow \mathcal{E}^B(w).$$

We then have the following direct analogue of Proposition 3.4:

Proposition 4.5. *The assignment above of the subset $E(T) \subset \mathcal{E}^B(w)$ for each $T \in \mathcal{T}^B(w)$ determines an order-preserving injective map of posets*

$$\mathcal{T}^B(w) \longrightarrow \underline{2}^{\mathcal{E}^B(w)}.$$

Thinking of $\mathcal{T}^B(w)$ thus as a sub-poset of the power set of $\mathcal{E}^B(w)$, this sub-poset satisfies

- *(i) each singleton $S = \{e\} \subset \mathcal{E}^B(w)$ is in $\mathcal{T}^B(w)$;*
- *(ii) if $T \in \mathcal{T}^B(w)$ and $S \in \underline{2}^{\mathcal{E}^B(w)}$ such that $T \leq S$, then $S \in \mathcal{T}^B(w)$.*

Finally, before we talk about $\mathcal{T}^B(w)$-manifolds and maps between flow categories, let us record that the poset $\mathcal{T}^B(w)$ comes with natural grafting operations.

As before, we have grafting operations that attach weighted unstable chains to the root of a tree

$$\#_0 : \mathcal{C}(w_1) \times \mathcal{T}^B(w_2) \longrightarrow \mathcal{T}^B(w_1 + w_2);$$

and for each $\sigma \in \Sigma$ we have a grafting operation that attached a weighted unstable chain to the leaf labelled σ:

$$\#_\sigma : \mathcal{T}^B(w_1) \times \mathcal{C}(w_2) \longrightarrow \mathcal{T}^B(w_1 + w_2).$$

The new element here is we also have grafting operations that combine a weighted simple Σ_1^β-tree T_1 and a weighted simple Σ_2^β-tree T_2 into a β-shaped tree, for each $\beta \in B$, by grafting the root of T_2 onto the leaf of T_1 labelled ζ^β, and relabelling the leaves in accordance to the bijection φ^β of (4.1). The resulting family of operations

$$\#^\beta : \mathcal{T}_{k_1^\beta, d_1^\beta}(w_1) \times \mathcal{T}_{k_2^\beta, d_2^\beta}(w_2) \longrightarrow \mathcal{T}^B(w_1 + w_2) \tag{4.4}$$

are moreover appropriately associative with any composition of the earlier grafting operations $\#$ and $\#_\sigma$. Such grafting operations are the combinatorial underpinnings allowing us to talk about composing maps between flow categories, which we now move onto.

4.2. $(\mathcal{R},\{\mathcal{R}^\beta\})$**-maps of flow categories.** First recall Definition 3.5: for a sub-poset $\mathcal{T}\subset \underline{2}^{\mathcal{E}}$, a $\mathcal{T}$-manifold with corners X is an $<\mathcal{E}>$-manifold so that for distinct $e_1,\dots,e_\ell$ we have $\cap_i \partial_{e_i} X=\emptyset$ unless $\{e_1,\dots,e_\ell\}\in\mathcal{T}$.

Now, recall throughout this section we have a fixed smooth manifold with boundary $\mathcal{R}$, with its boundary components $\mathcal{R}^\beta$ enjoying decompositions $\mathcal{R}^\beta=\mathcal{R}^\beta_1\times\mathcal{R}^\beta_2$. Throughout, we fix an identification $\Sigma\cong\{1,2,\dots,k\}$.

Definition 4.6. *Suppose we have a flow category $\mathcal{M}_0$, a k-tuple of flow categories $(\mathcal{M}_1,\dots,\mathcal{M}_k)$, as well as an additional flow category $\mathcal{M}_\beta$ for each $\beta\in B$. Suppose we also have, for each $\beta\in B$:*

(i) an $\mathcal{R}^\beta_1$-map of flow categories

$$\mathcal{F}^\beta_1:\mathcal{M}_0\longrightarrow(\mathcal{M}_\beta,\mathcal{M}_{\varphi^\beta(\sigma)})_{\sigma\in\Sigma^\beta_1\setminus\{\zeta^\beta\}}$$

(ii) an $\mathcal{R}^\beta_2$-map of flow categories

$$\mathcal{F}^\beta_2:\mathcal{M}_\beta\longrightarrow(\mathcal{M}_{\varphi^\beta(\sigma)})_{\sigma\in\Sigma^\beta_2}.$$

Then, an $(\mathcal{R},\{\mathcal{R}^\beta\})$-map of flow categories extending $\{\mathcal{F}^\beta_1,\mathcal{F}^\beta_2\}_{\beta\in B}$

$$\mathcal{F}:\mathcal{M}_0\longrightarrow(\mathcal{M}_1,\dots,\mathcal{M}_k)$$

is the data of, for each tuple $(x_i\in\mathrm{Ob}(\mathcal{M}_i))_{i=0,\dots,k}$ a $\mathcal{T}^B(w)$-manifold with corners together with a smooth map to $\mathcal{R}$

$$\pi:\mathcal{F}(x_0;x_1,\dots,x_k)\longrightarrow\mathcal{R}$$

for $w=\mu(x_0)-\Sigma_{i\geq1}\mu(x_i)$, which is of dimension $w+d$. We require for each β that the pre-image of π over the boundary component $\mathcal{R}^\beta\subset\partial\mathcal{R}$ consists of those faces of $\mathcal{F}$ indexed by elementary β-shaped trees:

$$\pi^{-1}(\mathcal{R}^\beta)=\bigcup_{e\in\mathcal{E}^\beta(w)}\partial_e\mathcal{F}.$$

These manifolds come with three types of composition maps:

(i) for $i=1,\dots,k$ and $x'_i\in\mathrm{Ob}(\mathcal{M}_i)$, a composition map

$$G_i:\mathcal{F}(x_0;\dots,x'_i,\dots,x_k)\times\mathcal{M}_i(x'_i,x_i)\longrightarrow\mathcal{F}(x_0;x_1,\dots,x_k)$$

whose image lies in $\partial_e\mathcal{F}$ for $e\in\mathcal{E}^B(w)$ the tree with a single unstable vertex adjacent to the i-th leaf of weight $\mu(x'_i)-\mu(x_i)$;

(ii) for $x'_0\in\mathrm{Ob}(\mathcal{M}_0)$, a composition map

$$G_0:\mathcal{M}(x_0,x'_0)\times\mathcal{F}(x'_0;x_1,\dots,x_k)\longrightarrow\mathcal{F}(x_0;x_1,\dots,x_k)$$

whose image lies in $\partial_e\mathcal{F}$ for $e\in\mathcal{E}^B(w)$ the tree with a single unstable vertex adjacent to the i-th leaf of weight $\mu(x_0)-\mu(x'_0)$;

(iii) for $\beta \in B$ and $x_\beta \in \mathrm{Ob}(\mathcal{M}_\beta)$, a composition map

$$G_\beta : \mathcal{F}_1^\beta(x_0; \ldots, x_\beta, \ldots) \times \mathcal{F}_2^\beta(x_\beta; \ldots) \longrightarrow \mathcal{F}(x_0; x_1, \ldots, x_k)$$

whose image lies in $\partial_e \mathcal{F}$ for $e \in \mathcal{E}^B(w)$ the β-shaped tree with exactly two vertices v_1^β and v_2^β of respective weights

$$\mu(x_0) - \mu(x_\beta) - \sum_{\sigma \in \Sigma_1^\beta \setminus \{\zeta\}} \mu(x_{\varphi^\beta(\sigma)}), \qquad \mu(x_\beta) - \sum_{\sigma \in \Sigma_2^\beta} \mu(x_{\varphi^\beta(\sigma)}).$$

These composition maps satisfy the conditions:

(i) the composition maps are associative with each-other, as well as with the composition maps for the individual flow categories $\mathcal{M}$ and the $\mathcal{R}_i^\beta$-maps $\mathcal{F}_i^\beta$;
(ii) the composition maps commute with the projections π to $\mathcal{R}$; in the case of G_β this means there is a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_1^\beta(x_0; \ldots, x_\beta, \ldots) \times \mathcal{F}_2^\beta(x_\beta; \ldots) & \xrightarrow{G_\beta} & \mathcal{F}(x_0; x_1, \ldots, x_k) \\ \downarrow & & \downarrow \\ \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta & \longrightarrow & \mathcal{R}; \end{array}$$

(iii) the composition maps are each diffeomorphisms onto a boundary face of $\mathcal{F}$;
(iv) for $e \in \mathcal{E}^B(w)$, then the images of all the composition maps contained in $\partial_e \mathcal{F}$ are disjoint, and moreover these images cover $\partial_e \mathcal{F}$.

As in earlier cases, an $(\mathcal{R}, \{\mathcal{R}^\beta\})$-map $\mathcal{F}$ carries canonical virtual vector bundles over each space $\mathcal{F}(x_0; x_1, \ldots, x_k)$, namely it has an *index bundle* which is defined to be the formal difference

$$I(x_0; \ldots, x_k) = T\mathcal{F}(x_0; \ldots, x_k) - \pi^* T\mathcal{R}. \tag{4.5}$$

After putting a Riemannian metric on $\mathcal{F}$, over each boundary face with a decomposition of the index bundle. For the boundary faces $\partial_e \mathcal{F}$ for $e \in \mathcal{E}_{k,d}(w) \subset \mathcal{E}^B(w)$ corresponding to the elementary simple trees (so with just one unstable vertex and one stable vertex), these decompositions are just as in the case of $\mathcal{R}$-maps for $\mathcal{R}$ without boundary. The remaining boundary faces are the images of the composition maps

$$G_\beta \left(\mathcal{F}_1^\beta(x_0; \ldots, x_\beta, \ldots) \times \mathcal{F}_2^\beta(x_\beta; \ldots) \right).$$

where we have a decomposition

$$T\mathcal{F}_1^\beta(x_0; \ldots, x_\beta, \ldots) \oplus \mathbb{R}\nu_{x_\beta} \oplus T\mathcal{F}_2^\beta(x_\beta; \ldots) \cong G_\beta^* T\mathcal{F}(x_0; \ldots, x_k)$$

for ν_{x_β} the inward pointing normal vector. However we also have a decomposition of the tangent bundle of the base parameter space $\mathcal{R}$:

$$T\mathcal{R}_1^\beta \oplus \mathbb{R}\nu_\beta \oplus T\mathcal{R}_2^\beta \cong T\mathcal{R}|_{\mathcal{R}^\beta}$$

where ν_β is again the inward pointing normal.

In particular, we have a stable isomorphism of virtual vector bundles over $\mathcal{F}_1^\beta(x_0;\ldots,x_\beta,\ldots)\times\mathcal{F}_2^\beta(x_\beta;\ldots)$:

$$I(x_0;\ldots,x_\beta,\ldots)\oplus I(x_\beta;\ldots)\cong G_\beta^* I(x_0;\ldots,x_k).$$

Again, these isomorphisms are coherent with the decomposition isomorphisms for index bundles of each $\mathcal{F}_i^\beta$, as well as those on the individual flow categories.

Definition 4.7. *Given an $(\mathcal{R},\{\mathcal{R}^\beta\})$-map of flow categories $\mathcal{F}$, the Atiyah dual map is the collection of spectra*

$$\mathcal{F}^{-I}(x_0;\ldots,x_k):=\begin{cases}\mathcal{F}(x_0;\ldots,x_k)^{-I(x_0;\ldots,x_k)} & \text{if } \mu(x_0)-\sum_{i\geq 1}\mu(x_i)\geq -d;\\ * & \text{otherwise.}\end{cases}$$

These come with composition maps of the form

$$\mathcal{M}_0^{-I}(x_0,x_0')\wedge\mathcal{F}^{-I}(x_0;\ldots,x_k)\longrightarrow\mathcal{F}^{-I}(x_0;\ldots,x_k),$$

as well as for $i=1,\ldots,k$

$$\mathcal{F}^{-I}(x_0;\ldots,x_i',\ldots,x_k)\wedge\mathcal{M}_i^{-I}(x_i',x_i)\longrightarrow\mathcal{F}^{-I}(x_0;\ldots,x_k)$$

and also, for $\beta\in B$, composition maps from the Atiyah dual $\mathcal{R}^\beta$-maps:

$$(\mathcal{F}_1^\beta)^{-I}(x_0;\ldots,x_\beta,\ldots)\wedge(\mathcal{F}_2^\beta)^{-I}(x_\beta;\ldots)\longrightarrow\mathcal{F}^{-I}(x_0;\ldots,x_k)$$

which are moreover compatible with the internal composition maps for each Atiyah dual flow category $\mathcal{M}^{-I}$, as well as those for the Atiyah dual maps $(\mathcal{F}_1^\beta)^{-I}$ and $(\mathcal{F}_2^\beta)^{-I}$.

Now, suppose that each flow category $\mathcal{M}_i$ and $\mathcal{M}_\beta$ comes equipped with a spectral system, which we respectively denote $\mathfrak{o}_i$ and $\mathfrak{o}_\beta$, such that for each $\beta\in B$ there is an extension of spectral systems over each $\mathcal{R}_1^\beta$-map $\mathcal{F}_1^\beta$ and each $\mathcal{R}_2^\beta$-map $\mathcal{F}_2^\beta$.

Definition 4.8. *An extension of these spectral systems over an $(\mathcal{R},\{\mathcal{R}^\beta\})$-map $\mathcal{F}$ is a collection of maps of spectra for each $x_0\in\mathcal{M}_0,\ldots,x_k\in\mathcal{M}_k$*

$$\mathfrak{o}_0(x_0)\wedge\mathcal{F}^{-I}(x_0;\ldots,x_k)\longrightarrow\mathfrak{o}_1(x_1)\wedge\ldots\wedge\mathfrak{o}_k(x_k)$$

which satisfy the same commutative diagrams (3.1) *and* (3.2) *as in the case for $\mathcal{R}$ a closed manifold, and additionally a commutative diagram for each $\beta\in B$ and $x_\beta\in\mathrm{Ob}(\mathcal{M}_\beta)$:*
(4.6)

$$\begin{array}{ccc}\mathfrak{o}_0(x_0)\wedge(\mathcal{F}_1^\beta)^{-I}(x_0;\ldots,x_\beta,\ldots)\wedge(\mathcal{F}_2^\beta)^{-I}(x_\beta;\ldots) & \longrightarrow & \mathfrak{o}_0(x_0)\wedge\mathcal{F}^{-I}(x_0;\ldots,x_k)\\ \downarrow & & \downarrow\\ \left(\bigwedge_{\sigma\in\Sigma_1^\beta\setminus\{\zeta^\beta\}}\mathfrak{o}_{\varphi^\beta(\sigma)}(x_{\varphi^\beta(\sigma)})\right)\wedge\mathfrak{o}_\beta(x_\beta)\wedge(\mathcal{F}_2^\beta)^{-I}(x_\beta;\ldots) & \longrightarrow & \mathfrak{o}_1(x_1)\wedge\ldots\wedge\mathfrak{o}_k(x_k).\end{array}$$

In the previous chapter, we explained how to use an extension of spectral systems over a $\mathcal{R}$-map to define a map of geometric realizations. Suppressing the integer levels from the notation for a moment, this means for each $\beta \in B$ we have maps of spectra

$$|\mathfrak{o}_0| \wedge (\mathcal{R}_1^\beta)_+ \longrightarrow \left(\bigwedge_{\sigma \in \Sigma_1^\beta \setminus \{\zeta^\beta\}} |\mathfrak{o}_{\varphi^\beta(\sigma)}| \right) \wedge |\mathfrak{o}_\beta|$$

as well as

$$|\mathfrak{o}_\beta| \wedge (\mathcal{R}_2^\beta)_+ \longrightarrow \bigwedge_{\sigma \in \Sigma_2^\beta} |\mathfrak{o}_{\varphi^\beta(\sigma)}|.$$

The task then, for the rest of this chapter, is to construct a map of spectra

$$|\mathfrak{o}_0| \wedge \mathcal{R}_+ \longrightarrow |\mathfrak{o}_1| \wedge \ldots \wedge |\mathfrak{o}_k|$$

so that as to make the diagram

$$\begin{array}{ccc} |\mathfrak{o}_0| \wedge (\mathcal{R}_1^\beta)_+ \wedge (\mathcal{R}_2^\beta)_+ & \longrightarrow & |\mathfrak{o}_0| \wedge \mathcal{R}_+ \\ \downarrow & & \downarrow \\ \left(\bigwedge_{\sigma \in \Sigma_1^\beta \setminus \{\zeta^\beta\}} |\mathfrak{o}_{\varphi^\beta(\sigma)}| \right) \wedge |\mathfrak{o}_\beta| \wedge (\mathcal{R}_2^\beta)_+ & \longrightarrow & |\mathfrak{o}_1| \wedge \ldots \wedge |\mathfrak{o}_k| \end{array} \tag{4.7}$$

commute up to homotopy.

4.3. **Pontrjagin-Thom theory for $(\mathcal{R}, \{\mathcal{R}^\beta\})$-maps.** We now define spaces $\mathcal{R}(w)$ and spectra $\mathscr{R}(w)$, analogous to those of Definition 3.10 and 3.13 for this case when $\mathcal{R}$ is a manifold with boundary

$$\partial \mathcal{R} = \coprod_{\beta \in B} \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta.$$

Definition 4.9. *For $\beta \in B$, a* metrization and marking *of a weighted β-shaped Σ-tree T is the data of:*

- *for each interior edge e of T, a length $\ell_e \in (0, \infty]$;*
- *for each of the stable vertices v_1^β, v_2^β a point $r_1 \in \mathcal{R}_1^\beta, r_2 \in \mathcal{R}_2^\beta$ respectively.*

We formally allow some of the edges to acquire length zero, in which case we identify the marked weighted tree with its contraction. In the event that all the edges between v_1^β and v_2^β acquire length zero, the contracted tree is a simple Σ-tree, and we give the stable vertex the marking $(r_1, r_2) \in \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta \subset \mathcal{R}$. The topological space

$$\mathcal{R}(w)$$

is then the space of isomorphism classes of metrized $\mathcal{R}$-marked weighted trees of total weight w which are either simple or β-shaped for some $\beta \in B$.

The space $\mathcal{R}(w)$ is stratified by the poset $\mathcal{T}^B(w)$: for $T \in \mathcal{T}^B(w)$ the open T-stratum is those marked metrized trees which after contracting every edge of non-infinite length have underlying weighted tree T. If we write $\mathcal{R}_T$ for the closed T-stratum, similarly to the case when $\mathcal{R}$ is closed, we have if T is β-shaped

$$\mathcal{R}_T = \mathcal{R}(w(v_1^\beta)) \times \mathcal{R}(w(v_2^\beta)) \times \prod_{v \in V^{un}(T)} \mathcal{J}(w(v))$$

and likewise $\mathcal{R}_T = \mathcal{R}(w(v^{st})) \times \prod_{v \in V^{un}(T)} \mathcal{J}(w(v))$ is T is simple. We again say that the boundary $\partial\mathcal{R}(w)$ is the union of the proper strata; this is just the subspace of metrized trees with at least one interior edge of infinite length.

The natural inclusions $\mathcal{R}_{T_2} \hookrightarrow \mathcal{R}_{T_1}$ for $T_2 \leq T_1$ in $\mathcal{T}^B(w)$ extend to collar neighbourhood maps

$$\mathcal{R}_{T_2} \times (0, \infty]^{E(T_2) \setminus E(T_1)} \hookrightarrow \mathcal{R}_{T_1}$$

and we single out the embeddings

$$\iota : \mathcal{R}_{T_2} = \mathcal{R}_{T_2} \times \{1\}^{E(T_2) \setminus E(T_1)} \hookrightarrow \mathcal{R}_{T_1}$$

whose normal bundles are identified with $\mathbb{R}^{E(T_2) \setminus E(T_1)}$.

Consider the virtual vector bundle ν_T over each closed stratum $\mathcal{R}_T$, which in the case that T is simple is as before

$$\nu_T = \mathbb{R}^{V^{un}(T)} - T\mathcal{R}$$

and in the case that T is β-shaped, is defined to be

$$\nu_T = \mathbb{R}^{V^{un}(T)} - T(\mathcal{R}_1^\beta \times \mathcal{R}_2^\beta).$$

We now discuss how these virtual vector bundles fit together under inclusions of the strata and under the distinguished closed embeddings $\iota(\mathcal{R}_{T_2}) \subset \mathcal{R}_{T_1}$ for $T_2 \leq T_1$.

In the case that either both T_2, T_1 are β-shaped, or that they are both simple, there is a short exact sequence of trivial vector spaces

$$0 \longrightarrow \mathbb{R}^{V^{un}(T_1)} \longrightarrow \mathbb{R}^{V^{un}(T_2)} \longrightarrow \mathbb{R}^{E(T_2) \setminus E(T_1)} \longrightarrow 0$$

exactly as in (3.3). We remind the reader that the first map takes (s_v) to $(s_{p(v)})$, and the second map takes (s_v) to $(s_{v_+(e)} - s_{v_-(e)})$, where $p : V(T_2) \to V(T_1)$ is the contraction map, and where the s_v co-ordinate is understood to be zero if v is not unstable. In these cases, by taking the Pontrjagin-Thom collapse map associated to ι, we then have continuous maps

$$q_{T_1 T_2} : \mathcal{R}_{T_1}^\nu / \partial\mathcal{R}_{T_1}^\nu \longrightarrow \mathcal{R}_{T_2}^\nu / \partial\mathcal{R}_{T_2}^\nu.$$

The situation is slightly different when T_2 is β-shaped, and T_1 is simple. Instead of the above short exact sequence of vector spaces, we have an exact sequence

$$0 \longrightarrow \mathbb{R}^{V^{un}(T_1)} \longrightarrow \mathbb{R}^{V^{un}(T_2)} \longrightarrow \mathbb{R}^{E(T_2) \setminus E(T_1)} \longrightarrow \mathbb{R} \longrightarrow 0 \tag{4.8}$$

where the first two maps are as before; the third map $\mathbb{R}^{E(T_2)\setminus E(T_1)} \longrightarrow \mathbb{R}$ takes a tuple $(t_e)_{e\in E(T_2)\setminus E(T_1)}$ and sends it to

$$\sum_{\substack{e\in E(T_2)\setminus E(T_1)\\ e \text{ between } v_1^\beta, v_2^\beta}} t_e.$$

Comparing this to the decomposition

$$T\mathcal{R} = \mathbb{R} \oplus T\mathcal{R}_1^\beta \oplus T\mathcal{R}_2^\beta$$

over $\mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$ we see that we have a short exact sequence of virtual vector bundles over $\mathcal{R}_{T_2}$

$$0 \longrightarrow \nu_{T_2} \longrightarrow \iota^*\nu_{T_1} \longrightarrow \mathbb{R}^{E(T_2)\setminus E(T_1)} \longrightarrow 0$$

and so in particular we also have a Pontrjagin-Thom collapse map

$$q_{T_1T_2} : \mathcal{R}^\nu_{T_1}/\partial\mathcal{R}^\nu_{T_1} \longrightarrow \mathcal{R}^\nu_{T_2}/\partial\mathcal{R}^\nu_{T_2} \tag{4.9}$$

in this case where T_2 is β-shaped and T_1 is simple.

Remark 4.10. *Again, strictly speaking, we should choose once and for all complementary vector bundles F_1^β, F_2^β for $T\mathcal{R}_1^\beta$ and $T\mathcal{R}_2^\beta$, as well as a complementary vector bundle F for $T\mathcal{R}$, with isomorphisms $F \oplus \mathbb{R} \cong F_1^\beta \oplus F_2^\beta$ over $\mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$, to give distinguished space level versions of the spectra $\mathcal{R}^\nu_T/\partial\mathcal{R}^\nu_T$.*

These maps are associative, in that whenever $T_3 \le T_2 \le T_1$, we have $q_{T_2T_3} \circ q_{T_1T_2} = q_{T_1T_3}$. We will again write $\mathcal{R}^\nu/\partial\mathcal{R}^\nu(w)$ for $\mathcal{R}^\nu(w)/\partial\mathcal{R}^\nu(w)$, the version of this space for the maximal tree in $\mathcal{T}^B(w)$ with just a single vertex.

Observe that as before, if we took T_1 to be the maximal tree of weight $w_1 + w_2$, and T_2 to be an elementary, simple tree, we obtain maps

$$q_0 : \mathcal{R}^\nu/\partial\mathcal{R}^\nu(w_1 + w_2) \longrightarrow \mathcal{J}^\nu/\partial\mathcal{J}^\nu(w_1) \wedge \mathcal{R}^\nu/\partial\mathcal{R}^\nu(w_2)$$

as well as for $i = 1, \dots, k$

$$q_i : \mathcal{R}^\nu/\partial\mathcal{R}^\nu(w_1 + w_2) \longrightarrow \mathcal{R}^\nu/\partial\mathcal{R}^\nu(w_1) \wedge \mathcal{J}^\nu/\partial\mathcal{J}^\nu(w_2)$$

corresponding to grafting an unstable chain onto a weighted simple tree. The new feature is that we can also take T_2 to be an elementary β-shaped weighted tree, to obtain maps

$$q_\beta : \mathcal{R}^\nu/\partial\mathcal{R}^\nu(w_1 + w_2) \longrightarrow (\mathcal{R}_1^\beta)^\nu/\partial(\mathcal{R}_1^\beta)^\nu(w_1) \wedge (\mathcal{R}_2^\beta)^\nu/\partial(\mathcal{R}_2^\beta)^\nu(w_2).$$

The coherence conditions then ensure that these maps are appropriately co-associative. Once again, we then dualize these maps, taking function spectra as follows.

Definition 4.11. *For $w \ge -d$, the spectrum $\mathscr{R}(w)$ is the function spectrum*

$$\mathscr{R}(w) = F(\mathcal{R}^\nu/\partial\mathcal{R}^\nu(w), \mathbb{S}).$$

This comes with canonical composition maps

$$q_0^* : \mathscr{J}(w_1) \wedge \mathscr{R}(w_2) \longrightarrow \mathscr{R}(w_1 + w_2)$$

and for each $i = 1, \ldots, k$

$$q_i^* : \mathscr{R}(w_1) \wedge \mathscr{J}(w_2) \longrightarrow \mathscr{R}(w_1 + w_2)$$

as well as, for each $\beta \in B$,

$$q_\beta^* : \mathscr{R}_1^\beta(w_1) \wedge \mathscr{R}_2^\beta(w_2) \longrightarrow \mathscr{R}(w_1 + w_2)$$

where $\mathscr{R}_1^\beta, \mathscr{R}_2^\beta$ *are the spectra of Definition 3.13 for* $\mathcal{R}_1^\beta, \mathcal{R}_2^\beta$. *These composition maps satisfy strict associativity, with respect to each-other and the composition maps for* $\mathscr{J}$, $\mathscr{R}_1^\beta$ *and* $\mathscr{R}_2^\beta$.

If we set, for $a_0, a_1, \ldots, a_k \in \mathbb{Z}$

$$\mathscr{R}(a_0; a_1, \ldots, a_k) = \begin{cases} \mathscr{R}(a_0 - \Sigma_{i \geq 1} a_i) & \text{for } a_0 - \Sigma_{i\geq 1} a_i \geq -d; \\ * & \text{otherwise} \end{cases}$$

then $\mathscr{R}$ is again a $(\mathscr{J}, \mathscr{J}^{\wedge k})$-bimodule. Moreover, it comes equipped with a $(\mathscr{J}, \mathscr{J}^{\wedge k})$-bimodule homomorphism

$$\mathscr{R}_1^\beta \otimes_{\mathscr{J}} \mathscr{R}_2^\beta \longrightarrow \mathscr{R}$$

where the left-$\mathscr{J}$-module structure on $\mathscr{R}_2^\beta$ is tensored with the ζ^β-th right-$\mathscr{J}$-module structure on $\mathscr{R}_1^\beta$.

Example 4.12. *For* $w = -d$ *the minimum weight, the boundary* $\partial\mathcal{R}(-d)$ *is again empty (not to be confused with the boundary of the original manifold* $\mathcal{R}$*!).* $\mathscr{R}(-d)$ *is then the Spanier-Whitehead dual of* $\mathcal{R}^{-T\mathcal{R}}$, *which by Atiyah duality for manifolds with boundary is weak equivalent to* $\Sigma^\infty \mathcal{R}/\partial\mathcal{R}$.

Now, suppose that $\mathcal{F} : \mathcal{M}_0 \to (\mathcal{M}_1, \ldots, \mathcal{M}_k)$ is an $(\mathcal{R}, \{\mathcal{R}^\beta\})$-map of flow categories that extends, for each β, an $\mathcal{R}_1^\beta$-map of flow categories $\mathcal{F}_1^\beta$ and a $\mathcal{R}_2^\beta$-map $\mathcal{F}_2^\beta$. Recall that a choice of coherent collars for $\mathcal{F}_1^\beta$ and $\mathcal{F}_2^\beta$ induced maps of spectra

$$\mathscr{R}_1^\beta \xrightarrow{\pi^*} F((\mathcal{F}_1^\beta)^\nu / \partial(\mathcal{F}_1^\beta)^\nu, \mathbb{S}); \qquad \mathscr{R}_2^\beta \xrightarrow{\pi^*} F((\mathcal{F}_2^\beta)^\nu / \partial(\mathcal{F}_2^\beta)^\nu, \mathbb{S})$$

and thus after applying Atiyah duality we have maps of spectra

$$\mathscr{R}_1^\beta \longrightarrow (\mathcal{F}_1^\beta)^{-I}, \qquad \mathscr{R}_2^\beta \longrightarrow (\mathcal{F}_2^\beta)^{-I}.$$

Similarly, by making a choice of coherent collars on $\mathcal{F}$, compatible with those choices on each flow category $\mathcal{M}$ and on $\mathcal{F}_1^\beta, \mathcal{F}_2^\beta$, we have canonical projection maps for $x_0, x_1, \ldots, x_k$ objects of the respective flow categories

$$\pi : \mathcal{F}(x_0; x_1, \ldots, x_k) \longrightarrow \mathcal{R}(\mu(x_0) - \Sigma\mu(x_i))$$

which sends $\partial\mathcal{F}$ to $\partial\mathcal{R}(\mu(x_0) - \Sigma\mu(x_i))$, and lifts the original projection map $\mathcal{F} \to \mathcal{R}$. We thus also have dual maps on function spectra

$$\pi^* : \mathscr{R}(\mu(x_0) - \Sigma\mu(x_i)) \longrightarrow F(\mathcal{F}^\nu / \partial\mathcal{F}^\nu(x_0; \ldots, x_k), \mathbb{S})$$

and after applying Atiyah duality a map of spectra

$$\mathscr{R}(\mu(x_0) - \Sigma\mu(x_i)) \longrightarrow \mathcal{F}^{-I}(x_0; x_1, \ldots, x_k).$$

Thus, suppose further that each $\mathcal{M}_i$ comes equipped with a spectral system $\mathfrak{o}_i$, and that these admit an extension of spectral systems over each map $\mathcal{F}_1^\beta, \mathcal{F}_2^\beta$ and then over $\mathcal{F}$. We thus not only have maps of spectra

$$\mathfrak{o}_0(x_0) \wedge \mathscr{R}_1^\beta(x_0; \ldots x_\beta \ldots) \longrightarrow \bigwedge_{\sigma \in \Sigma_1^\beta} \mathfrak{o}_{\phi^\beta(\sigma)}(x_{\phi^\beta(\sigma)})$$

$$\mathfrak{o}_\beta(x_\beta) \wedge \mathscr{R}_2^\beta(x_\beta; \ldots) \longrightarrow \bigwedge_{\sigma \in \Sigma_2^\beta} \mathfrak{o}_{\phi^\beta(\sigma)}(x_{\phi_\sigma^\beta})$$

but also a map of spectra

$$\mathfrak{o}_0(x_0) \wedge \mathscr{R}(x_0; x_1, \ldots, x_k) \longrightarrow \mathfrak{o}_1(x_1) \wedge \ldots \wedge \mathfrak{o}_k(x_k)$$

so that the analogue of the diagram (4.6) commutes. Let us now explain how these structures give rise to the desired maps on the geometric realizations of the $\mathscr{J}$-spectra associated to each flow category.

4.4. **$(\mathscr{R}, \{\mathscr{R}^\beta\})$-coproducts of $\mathscr{J}$-spectra.** Suppose we have $k+1$ fixed $\mathscr{J}$-spectra $X_0, X_1, \ldots, X_k$ together with additional $\mathscr{J}$-spectra X_β for each $\beta \in B$. Suppose further that for each $\beta \in B$ we have a $\mathscr{R}_1^\beta$-coproduct from X_0 to $(X_{\phi^\beta(\sigma)})_{\sigma \in \Sigma_\beta^1}$ and a $\mathscr{R}_2^\beta$-coproduct from X_β to $(X_{\phi^\beta(\sigma)})_{\sigma \in \Sigma_\beta^2}$.

Definition 4.13. *An $(\mathscr{R}, \{\mathscr{R}^\beta\})$-coproduct from X_0 to $X_1, \ldots, X_k$ extending these $\mathscr{R}_1^\beta$ and $\mathscr{R}_2^\beta$ coproducts is a collection of maps of spectra for integers $a_0, \ldots, a_k$*

$$X_0(a_0) \wedge \mathscr{R}(a_0; a_1, \ldots, a_k) \longrightarrow X_1(a_1) \wedge \ldots \wedge X_k(a_k)$$

satisfying the same two commutative diagrams as in Definition 3.15, together with the additional collection of diagrams

$$\begin{array}{ccc} X_0(x_0) \wedge \mathscr{R}_1^\beta(x_0; \ldots, x_\beta, \ldots) \wedge \mathscr{R}_2^\beta(x_\beta; \ldots) & \longrightarrow & X_0(x_0) \wedge \mathscr{R}(x_0; \ldots, x_k) \\ \downarrow & & \downarrow \\ \left(\bigwedge_{\sigma \in \Sigma_1^\beta \setminus \{\zeta^\beta\}} X_{\varphi^\beta(\sigma)}(x_{\varphi^\beta(\sigma)}) \right) \wedge X_\beta(x_\beta) \wedge \mathscr{R}_2^\beta(x_\beta; \ldots) & \longrightarrow & X_1(x_1) \wedge \ldots \wedge X_k(x_k). \end{array}$$

where the top arrow is the composition $\mathscr{R}_1^\beta \wedge \mathscr{R}_2^\beta \to \mathscr{R}$, the left arrow is the $\mathscr{R}_1^\beta$ coproduct, the bottom arrow is the $\mathscr{R}_2^\beta$ coproduct, and the right arrow is the $\mathscr{R}$ coproduct.

In the language of bimodules for spectral categories, $\mathscr{R}$ is a $(\mathscr{J}, \mathscr{J}^{\wedge k})$-bimodule, and we then have an induced map of right $\mathscr{J}^{\wedge k}$ modules

$$X_0 \otimes_{\mathscr{J}} \mathscr{R} \longrightarrow X_1 \wedge \ldots \wedge X_k. \tag{4.10}$$

Recall for each β we also have a map of $(\mathscr{J}, \mathscr{J}^{\wedge k})$ bimodules $\mathscr{R}_1^\beta \otimes_{\mathscr{J}} \mathscr{R}_2^\beta \to \mathscr{R}$; we then have a commutative diagram of right $\mathscr{J}^{\wedge k}$-modules

$$\begin{array}{ccc} X_0 \otimes_{\mathscr{J}} \mathscr{R}_1^\beta \otimes_{\mathscr{J}} \mathscr{R}_2^\beta & \longrightarrow & X_0 \otimes_{\mathscr{J}} \mathscr{R} \\ \downarrow & & \downarrow \\ \left(\bigwedge_{\sigma \in \Sigma_1^\beta \setminus \{\zeta^\beta\}} X_{\varphi^\beta(\sigma)} \right) \wedge X_\beta \otimes_{\mathscr{J}} \mathscr{R}_2^\beta & \longrightarrow & X_1 \wedge \ldots \wedge X_k. \end{array} \tag{4.11}$$

Now, fix integers $m_1, \ldots, m_k$; we will now explain how to produce an induced coproduct on the geometric realizations

$$|X_i|_{m_i} = X_i \otimes_{\mathscr{J}} \mathscr{V}_{m_i}.$$

Again, consider the left $\mathscr{J}^{\wedge k}$ module $\mathscr{V}_{m_1} \wedge \ldots \wedge \mathscr{V}_{m_k}$; taking its derived tensor product with (4.10) we have a map

$$X_0 \otimes_{\mathscr{J}} \mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} (\wedge_i \mathscr{V}_{m_i}) \longrightarrow |X_1|_{m_1} \wedge \ldots \wedge |X_k|_{m_k}.$$

The objective for the remainder of this section is to compute the left $\mathscr{J}$-module $\mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} (\wedge_i \mathscr{V}_{m_i})$. As before, it suffices to do this in the case that $m_1 = \ldots = m_k = 0$. The computation will have the following steps:

(i) first, produce an explicit model $\mathscr{R}\mathscr{V}$ for $\mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} \mathscr{V}^{\wedge k}$, using a similar construction as before of metrized "trees with flowers";

(ii) second, produce an explicit model $\partial\mathscr{R}\mathscr{V}$ for $\vee_{\beta \in B} \mathscr{R}_1^\beta \otimes_{\mathscr{J}} \mathscr{R}_2^\beta \otimes_{\mathscr{J}^{\wedge k}} \mathscr{V}^{\wedge k}$, together with a map of left $\mathscr{J}$-modules

$$\partial\mathscr{R}\mathscr{V} \longrightarrow \mathscr{R}\mathscr{V};$$

(iii) third, produce an explicit left $\mathscr{J}$-module which we will call $\mathscr{R}/\partial\mathscr{R}\mathscr{V}$, which is canonically level-wise in a cofiber sequence of spectra

$$\partial\mathscr{R}\mathscr{V}(w) \longrightarrow \mathscr{R}\mathscr{V}(w) \longrightarrow \mathscr{R}/\partial\mathscr{R}\mathscr{V}(w); \tag{4.12}$$

(iv) fourth, produce maps of $\mathscr{J}$-modules which are levelwise weak equivalences

$$\partial\mathscr{R}\mathscr{V} \longrightarrow \mathscr{V}_{-d} \wedge \partial\mathcal{R}_+; \qquad \mathscr{R}/\partial\mathscr{R}\mathscr{V} \longrightarrow \mathscr{V}_{-d} \wedge \mathcal{R}/\partial\mathcal{R}$$

so that the connecting maps of (4.12) fit into a commutative diagram

$$\begin{array}{ccc} \mathscr{R}/\partial\mathscr{R}\mathscr{V}(w) & \longrightarrow & \Sigma\partial\mathscr{R}\mathscr{V}(w) \\ \downarrow & & \downarrow \\ \mathscr{V}_{-d}(w) \wedge \mathcal{R}/\partial\mathcal{R} & \longrightarrow & \mathscr{V}_{-d}(w) \wedge \Sigma\partial\mathcal{R}_+ \end{array}$$

where the bottom arrow is given by the Puppe map $\mathcal{R}/\partial\mathcal{R} \to \Sigma\partial\mathcal{R}_+$.

In particular, from these steps we can deduce that there is a map of left-$\mathscr{J}$-modules, which is a levelwise weak equivalence

$$\mathscr{R}\mathscr{V} \longrightarrow \mathscr{V}_{-d} \wedge \mathcal{R}_+.$$

This moreover fits into a homotopy commutative diagram of left-$\mathscr{J}$-modules for each $\beta \in B$

$$\begin{array}{ccccc}
\mathscr{R}_1^\beta \otimes_{\mathscr{J}} \mathscr{R}_2^\beta \otimes_{\mathscr{J}^{\wedge k}} \mathscr{V}^{\wedge k} & & \longrightarrow & & \mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} \mathscr{V}^{\wedge k} \\
\downarrow & \searrow & & & \downarrow \\
\mathscr{R}_1^\beta \otimes_{\mathscr{J}^{\wedge k_1^\beta}} \mathscr{V}^{\wedge k_1^\beta} \wedge (\mathcal{R}_2^\beta)_+ & & \partial\mathscr{R}\mathscr{V} & \longrightarrow & \mathscr{R}\mathscr{V} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{V}_{-d} \wedge (\mathcal{R}_1^\beta)_+ \wedge (\mathcal{R}_2^\beta)_+ & \longrightarrow & \mathscr{V}_{-d} \wedge \partial\mathcal{R}_+ & \longrightarrow & \mathscr{V}_{-d} \wedge \mathcal{R}_+.
\end{array} \tag{4.13}$$

where the left vertical maps are given by the equivalences

$$\mathscr{R}_1^\beta \otimes_{\mathscr{J}^{\wedge k_1^\beta}} \mathscr{V}^{\wedge k_1^\beta} \longrightarrow \mathscr{V}_{-d_1^\beta} \wedge (\mathcal{R}_1^\beta)_+, \qquad \mathscr{R}_2^\beta \otimes_{\mathscr{J}^{\wedge k_2^\beta}} \mathscr{V}^{\wedge k_2^\beta} \longrightarrow \mathscr{V}_{-d_2^\beta} \wedge (\mathcal{R}_2^\beta)_+$$

of the previous chapter.

In particular, from these steps we will be able to conclude that for any $\mathscr{J}$-spectrum X there is a weak equivalence

$$X \otimes_{\mathscr{J}} \mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} (\wedge_i \mathscr{V}_{m_i}) \xrightarrow{\sim} |X|_{\Sigma m_i - d} \wedge \mathcal{R}_+.$$

fitting into a homotopy commutative diagram for each $\beta \in B$

$$\begin{array}{ccc}
X \otimes_{\mathscr{J}} \mathscr{R}_1^\beta \otimes_{\mathscr{J}} \mathscr{R}_2^\beta \otimes_{\mathscr{J}^{\wedge k}} (\wedge_i \mathscr{V}_{m_i}) & \xrightarrow{\sim} & |X|_{\Sigma m_i - d} \wedge (\mathcal{R}_1^\beta)_+ \wedge (\mathcal{R}_2^\beta)_+ \\
\downarrow & & \downarrow \\
X \otimes_{\mathscr{J}} \mathscr{R} \otimes_{\mathscr{J}^{\wedge k}} (\wedge_i \mathscr{V}_{m_i}) & \xrightarrow{\sim} & |X|_{\Sigma m_i - d} \wedge \mathcal{R}_+.
\end{array} \tag{4.14}$$

Then, in the presence of an $(\mathscr{R}, \{\mathscr{R}^\beta\})$-coproduct from X_0 to $X_1, \ldots, X_k$, combining the commutative diagrams (4.11) and the homotopy commutative (4.14) yields a commutative diagram in the homotopy category of spectra, for each $m_1, m_2, \ldots, m_k$ and β:

$$\begin{array}{ccc}
|X_0|_{\Sigma m_i - d} \wedge (\mathcal{R}_1^\beta)_+ \wedge (\mathcal{R}_2^\beta)_+ & \longrightarrow & |X_0|_{\Sigma m_i - d} \wedge \mathcal{R}_+ \\
\downarrow & & \downarrow \\
\left(\bigwedge_{\sigma \in \Sigma_1^\beta \setminus \{\zeta^\beta\}} |X_{\phi^\beta(\sigma)}|_{m_{\phi^\beta(\sigma)}} \right) \wedge |X_\beta|_{m_\beta - d_2^\beta} \wedge (\mathcal{R}_2^\beta)_+ & \longrightarrow & |X_1|_{m_1} \wedge \ldots \wedge |X_k|_{m_k}.
\end{array} \tag{4.15}$$

where the left and bottom arrows are the coproducts on the realizations induced by $\mathscr{R}_1^\beta$ and $\mathscr{R}_2^\beta$.

The task of the remainder of this chapter is to carry out steps (i)-(iv) above.

Step (i). For $\beta \in B$, a *weighted β-shaped tree with flowers* is an oriented tree graph T with precisely one semi-infinite edge (oriented outgoing, and called the root); k distinguished vertices of valence exactly one, labelled 1 to k, which we call the flowers; and a weighting function $w : V(T) \to \mathbb{Z}$; with two conditions:

- the flower vertices have non-negative weight $w(v) \geq 0$;
- after deleting the flower vertices, the weighted resulting tree with k incoming semi-infinite edges is β-shaped.

Explicitly, such a tree has vertices in three classes: the k flowers, exactly two stable vertices v_1^β and v_2^β of weights at least $-d_1^\beta$ and d_2^β respectively, and any number of unstable vertices which are valence exactly two.

Write $\mathcal{TF}^\beta(w)$ for the set of isomorphism classes of weighted β-shaped trees with flowers of a fixed weight w; this is a poset under contraction. By varying over all $\beta \in B$, and also considering the weighted simple k-trees with flowers, we can also allow an interior edge connecting v_1^β and v_2^β to contract to a single vertex; we thus have a combined poset

$$\mathcal{TF}^B(w) = \mathcal{TF}_{k,d}(w) \sqcup \bigsqcup_{\beta \in B} \mathcal{TF}^\beta(w). \tag{4.16}$$

Again, by adding 1 to the to the weight of each of the flower vertices and attaching semi-infinite edges to them, we can identify $\mathcal{TF}^B(w)$ with the sub-poset of $\mathcal{T}^B(w+k)$ consisting of the weighted trees with an unstable vertex adjacent to each of the incoming semi-infinite edges.

Similarly to before, a *metrization and marking* of a β-shaped tree with flowers is an assignment of points $r_1 \in \mathcal{R}_1^\beta$ and $r_2 \in \mathcal{R}_2^\beta$ to the two stable vertices, and a length $\ell_e \in (0, \infty]$ to each interior edge, which is required to satisfy the condition that at least one of the edges between each flower and the next stable vertex has length $\ell = \infty$.

We can then allow some of these edges to acquire length zero, in which case we identify the metrized $\mathcal{R}$-marked tree with its contraction (and in the case that in this contraction the two stable vertices are identified, the new stable vertex is marked by the image of $(r_1, r_2) \in \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$ in $\mathcal{R}$). We then have a topological space

$$\mathcal{RV}(w) \tag{4.17}$$

of isomorphism classes of metrized marked trees with flowers of total weight w.

As before metrized weighted chains can be grafted onto the root of a tree with flowers, giving maps

$$\mathcal{J}(w_1) \times \mathcal{RV}(w_2) \longrightarrow \mathcal{RV}(w_1 + w_2) \tag{4.18}$$

for $w_1 \geq 1$ and $w_2 \geq -d$. We also have, for each β and $w_1 \geq -d_1^\beta, w_2 \geq -d_2^\beta$, grafting maps

$$\mathcal{R}_1^\beta \mathcal{V}(w_1) \times \mathcal{R}_2^\beta \mathcal{V}(w_2) \longrightarrow \mathcal{R}\mathcal{V}(w_1 + w_2) \tag{4.19}$$

where for $i = 1, 2$, $\mathcal{R}_i^\beta \mathcal{V}$ are the spaces of metrized simple trees with flowers for the closed manifolds $\mathcal{R}_i^\beta$: this grafting map takes a pair of metrized trees with flowers (T_1, T_2), removes the β-th flower f_β from T_1, increases the weight of the vertex of T_2 adjacent to its root by $w(f_\beta)$, and then grafts it onto T_1 in place of the β-th flower.

The union images of all these maps (4.18) and (4.19) form a subspace $\partial\mathcal{R}\mathcal{V}(w) \subset \mathcal{R}\mathcal{V}(w)$, which is the subspace of metrized trees with flowers where there is either at least one interior edge between v_1^β and the root of length ∞, or at least one interior edge between v_2^β and v_1^β of length ∞. Taking the virtual vector bundle $\nu = -T\mathcal{R}$ over $\mathcal{R}\mathcal{V}(w)$, we obtain Pontrjagin-Thom collapse maps

$$\mathcal{R}\mathcal{V}^\nu / \partial\mathcal{R}\mathcal{V}^\nu(w_1 + w_2) \longrightarrow \mathcal{J}^\nu / \partial\mathcal{J}^\nu(w_1) \wedge \mathcal{R}\mathcal{V}^\nu / \partial\mathcal{R}\mathcal{V}^\nu(w_2).$$

We thus take the function spectrum

$$\mathscr{R}\mathscr{V}(w) = F(\mathcal{R}\mathcal{V}^\nu / \partial\mathcal{R}\mathcal{V}^\nu(w), \mathbb{S}) \tag{4.20}$$

which comes with natural maps

$$\mathscr{J}(w_1) \wedge \mathscr{R}\mathscr{V}(w_2) \longrightarrow \mathscr{R}\mathscr{V}(w_1 + w_2)$$

which make it into a left $\mathscr{J}$-module.

Step (ii). For $\beta \in B$, consider those metrized marked β-shaped trees so that at least one of the edges between v_1^β and v_2^β is length ∞. For a fixed total weight w, these form a topological subspace $\mathcal{R}^\beta\mathcal{V}(w) \subset \mathcal{R}\mathcal{V}(w)$, and we write $\partial\mathcal{R}^\beta\mathcal{V}(w)$ for the sub-space of those trees with an additional edge between v_1^β and the root of infinite length. In addition, it comes with a closed embedding

$$\iota : \mathcal{R}^\beta\mathcal{V}(w) \hookrightarrow \mathcal{R}\mathcal{V}(w)$$

defined by choosing a homeomorphism $(0, \infty] \cong (0, 1]$, and using it to rescale the metric. $\mathcal{R}^\beta\mathcal{V}(w)$ space carries a virtual vector bundle $\nu = -T\mathcal{R}^\beta$; by identifying the normal bundle to $\iota(\mathcal{R}^\beta\mathcal{V}(w)) \subset \mathcal{R}\mathcal{V}(w)$ with the trivial rank one summand in the splitting $T\mathcal{R}|_{\mathcal{R}^\beta} = T\mathcal{R}^\beta \oplus \mathbb{R}$ we have a collapse map

$$\mathcal{R}\mathcal{V}^\nu / \partial\mathcal{R}\mathcal{V}^\nu(w) \longrightarrow \mathcal{R}^\beta\mathcal{V}^\nu / \partial\mathcal{R}^\beta\mathcal{V}^\nu(w).$$

In particular, the dual spectrum

$$\mathscr{R}^\beta\mathscr{V} = F(\mathcal{R}^\beta\mathcal{V}^\nu / \partial\mathcal{R}^\beta\mathcal{V}^\nu(w), \mathbb{S}) \tag{4.21}$$

admits a natural map

$$\mathscr{R}^\beta\mathscr{V}(w) \longrightarrow \mathscr{R}\mathscr{V}(w)$$

which is additionally a map of left $\mathscr{J}$-modules. We then write

$$\partial\mathscr{R}\mathscr{V}(w) = \bigvee_{\beta\in B} \mathscr{R}^\beta\mathscr{V}(w). \tag{4.22}$$

Step (iii). For the third left $\mathscr{J}$-module, consider the space of all metrized simple trees with flowers of a fixed weight w together with all those metrized trees of some shape $\beta \in B$ so that all the edges between v_1^β and v_2^β have finite length ≤ 2. Call this space $\mathcal{R}_{int}\mathcal{V}(w)$ to distinguish it from the space $\mathcal{R}\mathcal{V}(w)$ of (4.17). It is naturally a closed subset of $\mathcal{R}\mathcal{V}(w)$, whose complement is a union of tubular neighbourhoods of $\mathcal{R}^\beta\mathcal{V} \subset \mathcal{R}\mathcal{V}$ for each $\beta \in B$. We can define its boundary set

$$\partial\mathcal{R}_{int}\mathcal{V} = \mathcal{R}_{int}\mathcal{V} \cap \partial\mathcal{R}\mathcal{V}$$

which here consists of all those metrized trees with an edge of infinite length along the root, and with all edges between v_1^β and v_2^β having length ≤ 2. In particular we have cofiber sequences of suspension spectra

$$\mathcal{R}_{int}\mathcal{V}^\nu/\partial\mathcal{R}_{int}\mathcal{V}^\nu(w) \longrightarrow \mathcal{R}\mathcal{V}^\nu/\partial\mathcal{R}\mathcal{V}^\nu(w) \longrightarrow \bigvee_{\beta\in B} \mathcal{R}^\beta\mathcal{V}^\nu/\partial\mathcal{R}^\beta\mathcal{V}^\nu(w).$$

If we then define

$$\mathscr{R}/\partial\mathscr{R}\mathscr{V}(w) = F(\mathcal{R}_{int}\mathcal{V}^\nu/\partial\mathcal{R}_{int}\mathcal{V}^\nu(w), \mathbb{S}) \tag{4.23}$$

we have the desired cofiber sequence, which is again compatible with the left $\mathscr{J}$-module structures

$$\partial\mathscr{R}\mathscr{V}(w) \longrightarrow \mathscr{R}\mathscr{V}(w) \longrightarrow \mathscr{R}/\partial\mathscr{R}\mathscr{V}(w).$$

Step (iv). To better understand this cofiber sequence, consider the following "model" case. Suppose X is a manifold with boundary ∂X, with distinguished collar neighbourhood $\partial X \times (0,\infty] \subset X$. Pontrjagin-Thom collapse of X onto $\partial X \times (0,\infty)$ defines the Puppe map of based spaces

$$X/\partial X \longrightarrow \Sigma\partial X_+. \tag{4.24}$$

This map has a different incarnation however in terms of Atiyah duality: $\Sigma^\infty X/\partial X$ is dual to X^{-TX}, and $\Sigma^{\infty+1}\partial X_+$ is dual to $\partial X^{-T\partial X-\mathbb{R}} \simeq \partial X^{-TX}$. The map (4.24) is then dual to the obvious inclusion map

$$\partial X^{-TX} \longrightarrow X^{-TX}.$$

Returning to spaces of metrized and marked trees with flowers, observe that as before we have natural closed embeddings of spaces

$$\mathcal{V}(w) \wedge \mathcal{R}_+ \longrightarrow \mathcal{R}_{int}\mathcal{V}(w+d)$$

defined exactly as for (3.10), which induces a map on the Thom spectra

$$\mathcal{V}^\nu/\partial\mathcal{V}^\nu(w) \wedge \mathcal{R}^{-T\mathcal{R}} \longrightarrow \mathcal{R}_{int}\mathcal{V}^\nu/\partial\mathcal{R}_{int}\mathcal{V}^\nu(w+d); \tag{4.25}$$

taking the duals and appealing to Atiyah duality we have a map of left $\mathscr{J}$-modules

$$\mathscr{R}/\partial\mathscr{R}\mathscr{V} \longrightarrow \mathscr{V}_{-d} \wedge \mathcal{R}/\partial\mathcal{R}. \tag{4.26}$$

Similarly for each β we have a closed embedding of spaces

$$\mathcal{V}(w) \wedge \mathcal{R}^{\beta}_{+} \longrightarrow \mathcal{R}^{\beta}\mathcal{V}(w+d)$$

inducing a map on Thom spectra

$$\mathcal{V}^{\nu}/\partial\mathcal{V}^{\nu}(w) \wedge (\mathcal{R}^{\beta}_{+})^{-T\mathcal{R}^{\beta}} \longrightarrow \mathcal{R}^{\beta}\mathcal{V}^{\nu}/\partial\mathcal{R}^{\beta}\mathcal{V}^{\nu}(w+d); \tag{4.27}$$

and thus a dual map

$$\partial\mathscr{R}^{\beta}\mathscr{V} \longrightarrow \mathscr{V}_{-d} \wedge \mathcal{R}^{\beta}_{+}. \tag{4.28}$$

Before dualizing (4.25) and (4.27), observe that the maps fit into natural commutative squares (coming from commutative squares of maps of spaces with vector bundles)

$$\begin{array}{ccc} \mathcal{V}^{\nu}/\partial\mathcal{V}^{\nu}(w) \wedge \Sigma(\mathcal{R}^{\beta}_{+})^{-T\mathcal{R}^{\beta}} & \longrightarrow & \Sigma\mathcal{R}^{\beta}\mathcal{V}^{\nu}/\partial\mathcal{R}^{\beta}\mathcal{V}^{\nu}(w+d) \\ \downarrow & & \downarrow \\ \mathcal{V}^{\nu}/\partial\mathcal{V}^{\nu}(w) \wedge \mathcal{R}^{-T\mathcal{R}} & \longrightarrow & \mathcal{R}_{int}\mathcal{V}^{\nu}/\partial\mathcal{R}_{int}\mathcal{V}^{\nu}(w+d) \end{array}$$

where the suspensions in the top row come from the addition $\mathbb{R}$ summand of the vector bundle ν. Dualizing and taking the direct sum over all $\beta \in B$, we have a commutative square of spectra

$$\begin{array}{ccc} \mathscr{R}/\partial\mathscr{R}\mathscr{V}(w) & \longrightarrow & \Sigma\partial\mathscr{R}\mathscr{V}(w) \\ \downarrow & & \downarrow \\ \mathscr{V}(w+d) \wedge \mathcal{R}/\partial\mathcal{R} & \longrightarrow & \mathscr{V}(w+d) \wedge \Sigma\partial\mathcal{R}_{+} \end{array} \tag{4.29}$$

where the top arrow is the connecting map of (4.23) and the bottom arrow is the Puppe map for $\partial\mathcal{R} \subset \mathcal{R}$.

Finally, observe that the vertical maps of (4.29) are weak equivalences for all w. This follows by direct calculation for the minimum possible non-trivial weight $w = -d$, where we can describe all possible metrized trees with flowers (in each case, there can be no unstable vertices); and for $w \geq -d$, it follows from the same argument as in Section 3.4: the relevant homotopy type is the smash product of either $\mathcal{R}/\partial\mathcal{R}$ or $\partial\mathcal{R}$ with that for the case $\mathcal{R} = \{\text{pt}\}$, which can be directly computed to be contractible by producing a sequences of upwards and downwards deformation retractions of the combinatorial poset of trees of flowers.

In particular, there is then a map of $\mathscr{J}$-spectra, well-defined only up to homotopy, which is a level-wise weak equivalence

$$\mathscr{R}\mathscr{V} \longrightarrow \mathscr{V}_{-d} \wedge \mathcal{R}_{+}. \tag{4.30}$$

From this we conclude that an $(\mathscr{R}, (\mathscr{R}^{\beta}))$-coproduct from X_0 to $(X_1, \ldots, X_k)$ induces maps in the homotopy category of spectra

$$|X_0|_{m_1+\ldots+m_k-d} \wedge \mathcal{R}_{+} \longrightarrow |X_1|_{m_1} \wedge \ldots |X_k|_{m_k}$$

for each $m_1, \ldots, m_k \in \mathbb{Z}$. Moreover, these maps moreover fit into the natural homotopy commutative diagrams (4.15), meaning that the above $\mathcal{R}_+$-parametrized coproduct can be rightly thought of as a homotopy between between the compositions of $\mathcal{R}_1^\beta$- and $\mathcal{R}_2^\beta$-parametrized coproducts.

5. Pseudoholomorphic curves

5.1. **The target geometry.** Throughout this section, fix a Liouville domain $(M, \omega = d\theta)$. Recall that this is a compact manifold M^{2n} with boundary, together with a one-form θ_M (called the Liouville form), satisfying the two conditions that $\omega = d\theta$ is symplectic and that the Liouville vector field Z, defined by $\iota_Z \omega = \theta$, points strictly outwards along ∂M. In particular, $(\partial M, \theta|_{\partial M})$ is a contact manifold; the completion $\hat{M}$ of M is the open exact symplectic manifold defined by

$$\hat{M} = M \cup_{\partial M} \partial M \times [0, \infty)$$

where the symplectic form on the semi-infinite cylinder $\partial M \times [0, \infty)$ is the symplectization of the contact form on ∂M. Writing r for the coordinate on $[0, \infty)$, on $\partial M \times [0, \infty)$ the Liouville form is given by $\hat{\theta}_M = e^r \theta|_{\partial M}$, and the corresponding Liouville vector field is $\hat{Z} = \partial_r$.

We will also consider certain exact Lagrangian submanifolds, which are of two forms. First, we those that we will call the compact exact Lagrangians, which are those closed Lagrangian submanifolds $L \subset \mathrm{int}(M)$ of the interior of M for which $\theta|_L$ is an exact one-form. Second, we have those that we will say are *cylindrical at infinity*, which are exact Lagrangian submanifolds $L \subset M$ so that $\partial L = L \cap \partial M$ is a Legendrian submanifold of ∂M. In this case the completion $\hat{L} = L \cup_{\partial L} \partial L \times [0, \infty)$ is an exact Lagrangian submanifold of $\hat{M}$.

For the rest of this thesis, we will usually elide the distinction between M and $\hat{M}$, and for Lagrangians between L and $\hat{L}$.

We will want to make additional topological assumptions on M and the exact Lagrangians L which we consider. For this section, it is not strictly necessarily, but will be nevertheless convenient, to assume the existence of *gradings*. To this end, assume that M admits a everywhere non-vanishing complex volume form η, and that each Lagrangian L comes equipped with a grading in the sense of [10]. Ultimately, we will assume far more structures in order to guarantee stable framings: in the final chapter of this thesis, we will work with a stable symplectic trivialization of the tangent bundle of M, and a compatible trivialization of TL.

5.2. **Floer data.** As a class of Hamiltonian functions on M, consider those that are linear at infinity: smooth functions $H \in C^\infty(\hat{M}, \mathbb{R})$ so that on $\partial M \times [0, \infty)$, for $r \gg 0$ we have

$$H = \tau e^r$$

for some non-negative real number $\tau \geq 0$, which we require to not be period of any Reeb orbit of $(\partial M, \theta|_{\partial M})$. We say that such a Hamiltonian is *linear at ∞ of slope τ.*

More generally, we consider S^1-parameter families $\{H_t\}_{t\in S^1}$ of such Hamiltonians for a fixed τ. For such a time-dependent Hamiltonian $\{H_t\}_{t\in S^1}$, its time-dependent Hamiltonian vector field X_{H_t} is defined by

$$\omega(\cdot, X_{H_t}) = dH_t;$$

a 1-periodic Hamiltonian orbit of H_t is a map $x : S^1 \to M$ such that

$$\frac{d}{dt}x(t) = X_{H_t}(x(t))$$

for all $t \in S^1$. We say such a time-dependent Hamiltonian is non-degenerate if all its 1-periodic orbits are non-degenerate, which is a generic condition; in this case there are necessarily finitely many such orbits; in this case we write $\mathcal{P}(H_t)$ for the set of 1-periodic orbits; we will often call these the *Floer generators*. Associated to such a Floer generator is its Maslov index $\mu(x) \in \mathbb{Z}$, which depends on the original choice of complex volume form on M.

An almost complex structure J on M is said to be of contact type at infinity if, for $r \gg 0$, in the decomposition of the tangent space

$$T(\partial M \times [0,\infty)) = \mathbb{R}\partial_r \oplus \mathbb{R}R \oplus \ker(\theta|_{\partial M})$$

the almost complex structure is the standard one on the first two summands, for R the Reeb vector field. We will usually also require that for $r \gg 0$ the almost complex structure is independent of r.

Given a fixed time-dependent Hamiltonian H_t and a time-dependent family $\{J_t\}_{t\in S^1}$ of almost complex structures of contact type, we can write down the Floer equation for maps u from the cylinder $Z = \mathbb{R} \times S^1$ to M:

$$\partial_s u + J_t(\partial_t u - X_t(u)) = 0.$$

Solutions to this equation of finite energy $\int |du|^2 < \infty$ necessarily have as limits

$$u(s,t) \longrightarrow x_\pm(t) \qquad \text{as } s \to \pm\infty$$

for some pair of Floer orbits $x_-, x_+ \in \mathcal{P}(H_t)$. It is a result of Robbin-Salamon [9] that this convergence is moreover exponential in the C^∞-topology: there are positive constants δ and $c_1, c_2, \ldots$ such that for $s \gg 0$ sufficiently large and all k

$$||\partial_s u||_{C^k([s,\infty)\times S^1)}, \quad ||\partial_s u||_{C^k((-\infty,-s]\times S^1)} \leq c_k e^{-\delta s}.$$

For fixed $x_-, x_+ \in \mathcal{P}(H_t)$, we write $W(x_-, x_+)$ of the space of solutions to the Floer equation with these limits, and

$$\mathcal{M}(x_-, x_+) = W(x_-, x_+)/\mathbb{R} \tag{5.1}$$

for space of solutions modulo translation; we will refer to this as the moduli space of (unparametrized) trajectories. We can also speak of the space of *broken* trajectories, which is the union

$$\overline{\mathcal{M}}(x_-, x_+) = \bigsqcup_{x_1, x_2, \ldots, x_{\ell-1} \in \mathcal{P}(H_t)} \mathcal{M}(x_-, x_1) \times \ldots \times \mathcal{M}(x_{\ell-1}, x_+)$$

topologized by Gromov convergence.This space of broken trajectories is compact: this follows from a maximum principle argument to constrain the image of any such curve to a compact set in M, together with Gromov's original convergence argument.

We say that the pair $(H_t, J_t)_{t\in S^1}$ is a *regular Hamiltonian Floer datum* if for all pairs $x_-, x_+ \in \mathcal{P}(H_t)$ the solutions to the Floer equation are transversely cut out: this means that for each $u \in W(x_-, x_+)$ and any order of differentiability $k \geq 1$ the linearized Floer operator

$$D_u : L^2_k(Z, u^*TM) \longrightarrow L^2_{k-1}(Z, \Omega^{0,1}_Z \otimes_J u^*TM) \cong L^2_{k-1}(Z, u^*TM)$$

is surjective. In this situation, the moduli spaces $\mathcal{M}(x_-, x_+)$ are smooth manifolds of dimension $\mu(x_-) - \mu(x_+) - 1$, whose tangent spaces are modelled on quotients of $\ker(D_u)$ by a trivial rank one sub-bundle arising from the translation action.

In the Lagrangian case, fix two Lagrangian submanifolds L_0, L_1, each one of which is either compact or cylindrical at infinity. We then consider time-dependent Hamiltonians $\{H_t\}_{t\in[0,1]}$ parametrized by the interval $[0, 1]$ which are linear at infinity of some fixed slope $\tau \geq 0$, where in the case that both L_0, L_1 are cylindrical at infinity we require that τ is not the length of a Reeb chord in ∂M connecting ∂L_0 to ∂L_1.

A Hamiltonian chord connecting L_0 to L_1 is a map $x : [0, 1] \to M$ with $x(0) \in L_0, x(1) \in L_1$ and $\partial ddtx(t) = X_{H_t}(x(t))$; again we say that H_t is non-degenerate if all of its Hamiltonian chords are; these chords then form a finite set which we will write $\chi(L_0, L_1, H_t)$ and call these again the *Floer generators*. We will often replace L_0 by its time 1 Hamiltonian flow $\phi^1_{H_t} L_0$, in which case $\chi(L_0, L_1)$ is just the set of (necessarily transverse) intersection points $L_0 \cap L_1$. The Reeb chord length condition on the slope translates here into the statement that ∂L_0 and ∂L_1 are disjoint.

Given a family of almost complex structures of contact type $\{J_t\}_{t\in[0,1]}$, the Floer equation from the strip $\mathbb{R} \times [0, 1]$, which by a slight abuse of notation we will typically also write as Z, to the M with boundary conditions on L_0, L_1 is

$$\partial_s u + J_t(\partial_t u - X_t(u)) = 0, \qquad u(\cdot, 0) \in L_0, \qquad u(\cdot, 1) \in L_1.$$

Again, solutions to this equation of finite energy necessarily have limits $x_\pm \in \chi(L_0, L_1, H_t)$, and the convergence is exponential in the C^∞ topology. We will also write $W(x_-, x_+)$ for the space of solutions to this equation with these limits, and $\mathcal{M}(x_-, x_+) = W(x_-, x_+)/\mathbb{R}$ for the moduli space of trajectories modulo translation. Again, we can form the space of broken trajectories $\overline{\mathcal{M}}(x_-, x_+)$ in the same manner, which is compact.

We say that $(H_t, J_t)_{t\in[0,1]}$ is a *regular Lagrangian Floer datum* for (L_0, L_1) if the space of solutions is transversely cut out. Explicitly, for a Floer solution u, at any order of differentiability $k \geq 2$ we will consider the Banach space

$$W_k^2(Z, u^*TM) \subset L_k^2(Z, u^*TM)$$

of L_k^2 sections ξ of u^*TM so that $\xi(\cdot, 0) \in u^*TL_0$ and $\xi(\cdot, 1) \in u^*TL_1$. Then (H_t, J_t) is regular if at all $u \in W(x_-, x_+)$ the linearized Floer operator

$$D_u : W_k^2(Z, u^*TM) \longrightarrow L_{k-1}^2(Z, \Omega_Z^{0,1} \otimes_J u^*TM)$$

is surjective. In this situation, the moduli space $\mathcal{M}(x_-, x_+)$ is again a smooth manifold of dimension $\mu(x_-) - \mu(x_+) - 1$.

In the next chapter, we will show that in fact $\overline{\mathcal{M}}(x_-, x_+)$ is a smooth manifold with corners, in both the Hamiltonian and Lagrangian cases. In particular, associated to a regular Floer datum (H, J) we have a *flow category* with objects given by $\mathcal{P}(H_t)$ in the Hamiltonian case, and $\chi(L_0, L_1, H_t)$ in the Lagrangian case, index map given by the Maslov index, and morphism spaces given by the moduli spaces of pseudoholomorphic strips or cylinders.

5.3. **Maps from surfaces.** Let us briefly review the details of producing a TQFT formalism in symplectic Floer theory; our treatment largely follows that of [11]. Take $\hat{S}$ a compact Riemann surface, possibly with boundary, together with a finite set of points Σ. These points are partitioned in two ways: first, we have a partition

$$\Sigma = \Sigma^- \sqcup \Sigma^+$$

into the *incoming and outgoing* points respectively: in this thesis, Σ^- will always be a singleton set $\{\zeta^-\}$, but we have no restrictions on Σ^+. Second, we have the partition

$$\Sigma = \Sigma_o \sqcup \Sigma_c$$

into respectively those points that are on the boundary $\partial\hat{S}$, which we called *open*, and those which are in the interior, which we will call *closed*. Thus all in all we have a partition

$$\Sigma = \Sigma_o^- \sqcup \Sigma_c^- \sqcup \Sigma_o^+ \sqcup \Sigma_c^+$$

where one of Σ_o^-, Σ_c^- is a singleton and the other is empty. We will also assume that each point $\zeta \in \Sigma_c$ comes with an *asymptotic marker*: a choice of real tangent ray in $T_\zeta\hat{S}$. We will refer to this structure as an *open-closed pointed Riemann surface*.

We will moreover assume that the punctured surface $S = \hat{S}\backslash\Sigma$ comes equipped with *strip-like and cylindrical/tubular ends*. This means, at each $\zeta \in \Sigma$, a proper holomorphic embedding

$$\varepsilon_\zeta : \mathbb{R}^\pm \times [0,1] \longrightarrow S \qquad \text{if } \zeta \in \Sigma_o^\pm;$$
$$\varepsilon_\zeta : \mathbb{R}^\pm \times S^1 \longrightarrow S \qquad \text{if } \zeta \in \Sigma_c^\pm$$

such that $\lim_{s\to\pm\infty} \varepsilon_\zeta(s, \cdot) = \zeta$, and in the open case $\varepsilon_\zeta^{-1}(\partial S) = \mathbb{R}^\pm \times \{0,1\}$, and in the closed case that $\varepsilon_\zeta(\mathbb{R}^\pm \times \{1\})$ is tangent to the asymptotic marker. We moreover require their images to be pairwise disjoint.

We say that S has *Lagrangian labels* if each component $C \subset \partial S$ comes with a choice L_C of exact Lagrangian submanifold (either compact or cylindrical at infinity) of M. In particular, associated to each $\zeta \in \Sigma_o$ is a pair of Lagrangians $L_{\zeta,0}$ and $L_{\zeta,1}$ corresponding to the components on either side. By convention, in the natural (anticlockwise) boundary orientation, for $\zeta \in \Sigma_o^-$ we have the component labelled by $L_{\zeta,1}$ coming before that labelled by $L_{\zeta,0}$, and the reverse for $\zeta \in \Sigma_o^+$.

Now, suppose that at each $\zeta \in \Sigma$, we choose a regular Floer datum (H_ζ, J_ζ): either Hamiltonian or Lagrangian for $(L_{\zeta,0}, L_{\zeta,1})$ depending on whether ζ is closed or open. A perturbation datum for S is then a pair (K, J) where $\{J_z\}_{z\in S}$ is a domain-dependent family of almost complex structures of contact type, and $K \in \Omega^1(S, C^\infty(M,\mathbb{R}))$ is a function valued 1-form, that satisfy:

- over each end of S, we have $\varepsilon_\zeta^* K \longrightarrow H_\zeta dt$ and $\varepsilon_\zeta^* J \longrightarrow J_\zeta$ exponentially fast in any C^k norm as $s \to \pm\infty$.
- at a point $z \in \partial S$, for each $\xi \in T\partial S$ we have $K(\xi)|_{L_z} = 0$, where L_z is the Lagrangian label for the component of ∂S containing z.
- there is some real 1-form $\beta \in \Omega^1(S,\mathbb{R})$, so that near infinity on M, we have $K = e^r \beta$ where r is the coordinate on $\partial M \times [0,\infty)$. The 1-form must moreover satisfy the pointwise bound
$$d\beta \leq 0.$$

In particular, together this means that at each point ζ, if τ_ζ is the slope of the chosen Floer datum, we must have $\varepsilon_\zeta^*\beta = \tau_\zeta dt$. The condition that $d\beta \leq 0$ then ensures by Stokes' theorem that
$$\tau_{\zeta^-} \geq \sum_{\zeta^+ \in \Sigma^+} \tau_{\zeta^+}.$$

A choice of perturbation datum (K, J) then determines a vector-field-valued 1-form $Y \in \Omega^1(S, C^\infty(M, TM))$; the corresponding Floer equation for maps $u : S \to M$ is
$$(Du - Y)^{0,1} = 0, \qquad u(C) \subset L_C \text{ for each component } C \subset \partial S.$$

Solutions of finite energy $\int_S |Du - Y|^2 < \infty$ again have as limits Floer generators: at each $\zeta \in \Sigma$, there exists x_ζ in either $\mathcal{P}(H_\zeta)$ or $\chi(L_{\zeta,0}, L_{\zeta,1}, H_\zeta)$ so that on the corresponding end
$$u(\varepsilon_\zeta(s,t)) \longrightarrow x_\zeta(t) \qquad \text{as } s \to \pm\infty.$$

and this convergence is exponential to all derivatives. For fixed choices $\{x_\zeta\}$ of Floer generators, let us denote the moduli space of all such maps with these asymptotics as

$$\mathcal{F}(\{x_\zeta\}). \tag{5.2}$$

In practice, we will usually identify Σ with the set $\{0, 1, \ldots, k\}$, so that the unique outgoing point ζ^- is identified with 0; in this case we will write this moduli space of solutions as
$$\mathcal{F}(x_0; x_1, \ldots, x_k).$$

At a point $u \in \mathcal{F}(x_0; x_1, \ldots, x_k)$, this moduli space is locally a smooth manifold its linearized operator, which is always Fredholm,
$$D_{S,u} : W_k^2(S, u^*TM) \longrightarrow L_{k-1}^2(S, \Omega_S^{0,1} \otimes_J u^*TM)$$

is moreover surjective, in which case the tangent bundle is modelled on $\ker(D_{S,u})$. Here we are writing $W^2_k(S, u^*TM)$ for those L^2_k sections of u^*TM with Lagrangian boundary conditions given by the Lagrangian labels. If this holds at every point of $\mathcal{F}(x_0; x_1, \dots, x_k)$, we say that the perturbation datum is regular.

The condition $d\beta \leq 0$ on the perturbation datum guarantee that solutions of the equation satisfy a maximum principle, and are a priori constrained to lie in a compact subset of M. In particular, the moduli space $\mathcal{F}(\{x_\zeta\})$ can be compactified by adding in maps from "broken" surfaces.

The data of a broken map from S to M with endpoints $\{x_0 = x_{\zeta^-}, x_1, \dots, x_k\}$ consists of a choice of sequences $x_0 = y^1_0, y^2_0, \dots, y^{\ell_0}_0$ of Floer generators for the Floer datum $(H_{\zeta^-}, J_{\zeta^-})$, together with Floer trajectories $u^j_0 \in \mathcal{M}(y^j_0, y^{j+1}_0)$ for each $j = 1, \dots, \ell_0 - 1$; for each $i = 1, \dots, k$ a sequence $y^{\ell_j}_j, \dots, y^1_j = x_j$ of Floer generators for the datum $(H_{\zeta^+(i)}, J_{\zeta^+(i)})$, together with Floer trajectories $u^j_i \in \mathcal{M}(y^{j+1}_i, y^j_i)$ for each $j = \ell_i - 1, \dots, 1$, as well as an element $u \in \mathcal{F}(y^{\ell_0}_0; y^{\ell_1}_1, \dots, y^{\ell_k}_k)$. The set of all possible broken maps with these endpoints is thus

$$\overline{\mathcal{F}}(x_0; x_1, \dots, x_k) = \bigsqcup_{y^j_i} \left(\prod \mathcal{M}(y^j_0, y^{j+1}_0)\right) \times \mathcal{F}(y^{\ell_0}_0; \dots, y^{\ell_k}_k) \times \dots \times \left(\prod \mathcal{M}(y^{j+1}_k, y^j_k)\right).$$

and with the topology of Gromov convergence is a compact space.

5.4. **Families of surfaces.** More generally, we wish to consider not just a single pointed surface $S = \hat{S} \backslash \Sigma$ equipped with a regular perturbation datum, but a family of such surfaces. Take a smooth manifold $\mathcal{R}$ (for now without boundary), and a smooth fiber bundle

$$p : \mathcal{S} \to \mathcal{R}$$

with fibre S and structure group the oriented diffeomorphisms of $\hat{S}$ that are the identity on Σ and preserve the asymptotic markers at $\zeta \in \Sigma_c$. This can be compactified to a proper fibre bundle

$$\widehat{p} : \widehat{\mathcal{S}} \longrightarrow \mathcal{R}$$

with fibre $\widehat{S}$; each marked point $\zeta \in \Sigma$ yields a section of $\widehat{p}$; we will abuse notation and also refer to the collection of sections as Σ. If $\widehat{\mathcal{S}}$ comes equipped with an almost complex structure $I_{\mathcal{S}}$ on the fibrewise tangent bundle, each fibre is a Riemann surface; we say that $\mathcal{S}$ is a *family of open-closed pointed Riemann surfaces*. In this paper, we will always assume that $\mathcal{R}$ is a smooth manifold, and that p is a smooth fibre bundle.

We will always equip $\mathcal{S} \to \mathcal{R}$ with strip-like and cylindrical ends, meaning proper embeddings $\mathcal{R} \times \mathbb{R}^{\pm} \times [0,1] \to \mathcal{S}$ fibred over $\mathcal{R}$ in the open case, and $\mathcal{R} \times \mathbb{R}^{\pm} \times S^1 \to \mathcal{S}$ in the closed case, which restrict to strip-like/cylindrical ends on each fibre compatible with the asymptotic markers.

It will later be analytically important to choose local smooth trivializations of the family $\mathcal{S} \to \mathcal{R}$, in order to think of this as a family of varying complex structures on a fixed smooth surface. We will always choose these trivializations which *make the ends constant*, meaning that the given trivialization, when restricted to the strip-like and tubular ends, agrees with the natural trivialization given by the parametrizations ε_ζ. In particular, we can think of this as a family of complex structures which vary only away from the ends.

A set of Lagrangian labels for $p : \mathcal{S} \to \mathcal{R}$ is a locally constant choice over the whole family. Assuming that $\mathcal{R}$ is connected, this associates to each $\zeta \in \Sigma_o$ a pair of Lagrangians $L_{\zeta,0}, L_{\zeta,1}$. Now, choose a Floer datum (H_ζ, J_ζ) for each $\zeta \in \Sigma$. A perturbation datum (K, J) for $\mathcal{S}$ is then simply a pair (K_r, J_r) of smoothly varying perturbation data on the fibres $\mathcal{S}_r$ of the family.

For fixed choices of Floer generators x_ζ for $\zeta \in \Sigma$, we then have the moduli space $\mathcal{F}(\{x_\zeta\})$ of maps from the family $\mathcal{S}$ to M, consisting of pairs (r, u) where $r \in \mathcal{R}$ and $u : \mathcal{S}_r \to M$ satisfies the inhomogeneous Cauchy Riemann equation for $\mathcal{S}_r$ with the perturbation data (K_r, J_r).

At $(r, u) \in \mathcal{F}(\{x_\zeta\})$, there are two linearized operators of importance to the theory. On the one hand, there is the linearized operator of the previous section, which is always Fredholm

$$D_{\mathcal{S}_r,u} : W^2_k(\mathcal{S}_r, u^*TM) \longrightarrow L^2_{k-1}(\mathcal{S}_r, \Omega^{0,1}_{\mathcal{S}_r} \otimes_J u^*TM)$$

for some order of differentiability $k > 1$. We will typically refer to $D_{\mathcal{S}_r,u}$ as just the *linearized operator*, and if there is confusion with the next notion, we will sometimes call it the *pointwise* linearized operator. In general, this is not surjective, however its virtual index bundle $\operatorname{ind}(D_{\mathcal{S}_r,u})$ over $\mathcal{F}$ still plays a crucial role in the theory.

Restoring the $\mathcal{R}$-dependence, the full equations for varying $r \in \mathcal{R}$ can then be linearized as

$$D^{\text{ext}}_{\mathcal{S},u} : T\mathcal{R}_r \oplus W^2_k(\mathcal{S}_r, u^*TM) \longrightarrow L^2_{k-1}(\mathcal{S}_r, \Omega^{0,1}_{\mathcal{S}_r} \otimes_J u^*TM) \tag{5.3}$$

where the component on the second factor is exactly the linearized operator from before, while the component on the first factor is the derivative of the equation with respect to changes to the conformal structure $r \in \mathcal{R}$ while fixing the map u. We refer to this as the *extended linearized operator*.

We say that (r, u) is *regular* if $D^{\text{ext}}_{\mathcal{S},u}$ is surjective. After fixing a local smooth trivialization making the ends constant, of the family $\mathcal{S} \to \mathcal{R}$ near r, we can treat the domain surface $\mathcal{S}_r$, with the complex structure (and thus the Cauchy-Riemann term $(Du - Y)^{0,1}$) varying with in $\mathcal{R}$. We can then apply the implicit function theorem to see that the moduli space $\mathcal{F}(\{x_\zeta\})$ is a smooth manifold near (r, u), with tangent space

$$T_{(r,u)}\mathcal{F}(\{x_\zeta\}) = \ker D^{\text{ext}}_{\mathcal{S}_r,u}.$$

and dimension given by $\dim(\mathcal{R}) + \mu(x_0) - \sum_{\zeta \in \Sigma^+} (x_\zeta)$. Observe that the virtual index bundle $\operatorname{ind}(D_{\mathcal{S}_r,u})$ of the *pointwise* linearized operator can then be computed as the formal difference

$$\operatorname{ind}(D_{\mathcal{S}_r,u}) = T_{(r,u)}\mathcal{F}(\{x_\zeta\}) - T_r\mathcal{R}. \tag{5.4}$$

Assuming that $\mathcal{R}$ is compact, the moduli space space $\mathcal{F}(\{x_\zeta\})$ of maps from the family again admits a compactification by broken maps. Again writing $\{x_0 = x_{\zeta^-}, x_1, \ldots, x_k\}$ for $\{x_\zeta\}$, this compactified moduli space is also given by
(5.5)

$$\overline{\mathcal{F}}(x_0; x_1, \ldots, x_k) = \bigsqcup_{y_i^j} \quad \left(\prod \mathcal{M}(y_0^j, y_0^{j+1})\right) \times \mathcal{F}(y_0^{\ell_0}; \ldots, y_k^{\ell_k}) \times \ldots \times \left(\prod \mathcal{M}(y_k^{j+1}, y_k^j)\right)$$

where this time $\mathcal{F}(y_0^{\ell_0}; \ldots, y_k^{\ell_k})$ is the moduli space of maps from the family $\mathcal{S}$ to M.

The strata in of this compactification are indeed indexed by the poset $\mathcal{T}_{k,d}(\mu(x_0) - \sum_{i\geq 1} \mu(x_i))$ of weighted simple k-trees from earlier. Each vertex of the tree corresponds to a component of the domain nodal curve; the "stable" vertex of valence $k+1$ corresponds to the component just given by the family $\mathcal{S}$ with its perturbation datum (K, J), while the remaining "unstable" vertices of valence 2 correspond to components that are just strips or cylinders with translation-invariant perturbation data (given, indeed, by the original choice of Floer data).

In the next chapter, we will endow $\overline{\mathcal{F}}(x_0; x_1, \ldots, x_k)$ with the structure of a smooth manifold with corners. Given this, the moduli spaces of maps from this family of surfaces has precisely the structure of an $\mathcal{R}$-parametrized map of flow categories

$$\overline{\mathcal{F}} : \mathcal{M}_0 \longrightarrow (\mathcal{M}_1, \ldots, \mathcal{M}_k) \tag{5.6}$$

where $\mathcal{M}_i$ is the flow category associated to the Floer data $(H_{\zeta(i)}, J_{\zeta(i)})$.

5.5. **Families of degenerating surfaces.** We now allow our families of surfaces to degenerate and acquire nodes, in such a way to model composition of operations in a Floer-theoretic TQFT.

Let us first take two open-closed pointed Riemann surfaces $S_1 = \hat{S}_1 \backslash \Sigma_1$ and $S_2 = \hat{S}_2 \backslash \Sigma_2$. Suppose we have two families open-closed Riemann surfaces with fibres modelled on S_1 and S_2:

$$p_1 : \mathcal{S}_1 \longrightarrow \mathcal{R}_1, \qquad p_2 : \mathcal{S}_2 \longrightarrow \mathcal{R}_2$$

together with a distinguished choice $\zeta_1^+ \in \Sigma_1^+$, which is of the same type (open or closed) as ζ_2^-. As a topological surface, we can form $S_1 \#_\zeta S_2$ by gluing the positive end $\varepsilon_{\zeta_1^+}$ of S_1 to the negative end $\varepsilon_{\zeta_2^-}$ of S_2. This topological surface has $\Sigma^+ = (\Sigma_1^+ \backslash \{\zeta_1^+\}) \cup \Sigma_2^+$. We then have a glued family of Riemann surfaces

$$p : \mathcal{S}_1 \# \mathcal{S}_2 \longrightarrow \mathcal{R}_1 \times \mathcal{R}_2 \times (0, \infty)$$

where the fibre over (r_1, r_2, L) is the Riemann surface built in the following manner: it has two "body" regions given by $(\mathcal{S}_1)_{r_1} \backslash \mathrm{im}(\varepsilon_{\zeta_1^+})$ and $(\mathcal{S}_2)_{r_2} \backslash \mathrm{im}(\varepsilon_{\zeta_2^-})$ respectively, glued together with a "neck" region $[-L, L] \times [0, 1]$ in the open case and $[-L, L] \times S^1$ in the closed case.

Now, consider a smooth manifold with boundary $\mathcal{R}$, and a finite set $\Sigma = \Sigma^- \sqcup \Sigma^+$ where $|\Sigma^-| = 1$, further partitioned into open and closed points. Write the boundary of $\mathcal{R}$ as a disjoint union of connected components

$$\partial\mathcal{R} = \bigsqcup_{\beta \in B} \mathcal{R}^\beta$$

for some finite set B. Let us also assume that, for each β, we have:

(i) a decomposition $\mathcal{R}^\beta = \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$ into a product of closed smooth manifolds;
(ii) finite sets $(\Sigma_1^+)^\beta$ and $(\Sigma_2^+)^\beta$ for each $\beta \in B$, together with a choice of an element $\zeta^\beta \in (\Sigma_1^+)^\beta$, and a bijection

$$\varphi^\beta : \Big((\Sigma_1^+)^\beta \backslash \{\zeta^\beta\}\Big) \sqcup (\Sigma_2^+)^\beta \longrightarrow \Sigma^+. \tag{5.7}$$

A *degenerating family of open-closed pointed Riemann surfaces* over $\mathcal{R}$ is the following structure.

(i) A family $\mathcal{S} \to \mathrm{int}(\mathcal{R})$ of open-closed pointed Riemann surfaces over the interior of $\mathcal{R}$, with marked points given by Σ.
(ii) For each $\beta \in B$, a pair of families $\mathcal{S}_1^\beta \to \mathcal{R}_1^\beta$ and $\mathcal{S}_2^\beta \to \mathcal{R}_2^\beta$ of open-closed Riemann surfaces, with incoming marked points given by $(\Sigma_1^+)^\beta$ and $(\Sigma_2^+)^\beta$ respectively.
(iii) For each $\beta \in B$, a collar neighbourhood

$$\mathcal{R}_1^\beta \times \mathcal{R}_2^\beta \times (L, \infty]$$

for $\mathcal{R}^\beta$, pairwise disjoint for each $\beta \in B$, together with a fibre-wise biholomorphism between the pull-back of the family $\mathcal{S} \to \mathrm{int}(\mathcal{R})$ to $\mathcal{R}_1^\beta \times \mathcal{R}_2^\beta \times (L, \infty)$ and the glued family $\mathcal{S}_1 \# \mathcal{S}_2$, which preserves the marked points via the bijection φ^β, and preserves the strip-like or cylindrical ends.

For such a degenerating family, a *consistent* choice of Lagrangian labels is a choice of labels on $\mathcal{S} \to \mathrm{int}(\mathcal{R})$ as well as $\mathcal{S}_1^\beta$ and $\mathcal{S}_2^\beta$, so that at the glued end we have $L_{\zeta_1^+,0} = L_{\zeta_2^-,0} = L_{C_0}$ and $L_{\zeta_1^+,1} = L_{\zeta_2^-,1} = L_{C_1}$ where C_0, C_1 are the corresponding boundary components of the glued surface. A choice of consistent Floer data is the choice of Floer data (in either the Hamiltonian or Lagrangian case as is appropriate) at the ends of each $\mathcal{S}$, $\mathcal{S}_1^\beta$ and $\mathcal{S}_2^\beta$, so that at each β the Floer data associated to the glued ends ζ_1^+ and ζ_2^- coincide, and so that the Floer data on $\mathcal{S}$ agrees with those on $\mathcal{S}_1^\beta$ and $\mathcal{S}_2^\beta$ under the bijection φ^β of (5.7)

A choice of perturbation data (K, J) on $\mathcal{S}$ and $(K_1^\beta, J_1^\beta), (K_2^\beta, J_2^\beta)$ on $\mathcal{S}_1^\beta, \mathcal{S}_2^\beta$ is then said to be *consistent* if, in the collar neighbourhood of each boundary component, we have:

(i) on each "body" part of $\mathcal{S}_1^\beta \# \mathcal{S}_2^\beta$, which recall is each $\mathcal{S}_1^\beta, \mathcal{S}_2^\beta$ with the glued end removed, we have

$$K|_{\mathcal{S}^\beta_{i,body}} \longrightarrow K_i^\beta, \qquad J|_{\mathcal{S}^\beta_{i,body}} \longrightarrow J_i^\beta$$

for $i = 1, 2$ as the gluing length $L \to \infty$, where this convergence exponential is in any C^k norm;

(ii) on the "neck" part $[-L, L] \times [0, 1]$ or $[-L, L] \times S^1$ of $\mathcal{S}_1^\beta \# \mathcal{S}_2^\beta$, with corresponding Floer data (H_ζ, J_ζ), we have

$$K|_{neck} \longrightarrow H_\zeta dt, \qquad J|_{neck} \longrightarrow J_\zeta$$

as the gluing lengths $L \to \infty$, where this convergence exponential on any C^k norm on any compact set.

Now, fix an identification of Σ with $\{0, 1, \ldots, k\}$, and pick Floer generators $x_0, x_1, \ldots, x_k$ for some consistent choice of Lagrangian labels and Floer data. Given also a choice of consistent perturbation data, the moduli space

$$\mathcal{F}(x_0; x_1, \ldots, x_k)$$

of maps from surfaces $\mathcal{S}_r \to M$ satisfying the equation and boundary conditions, with asymptotics to $x_0, \ldots, x_k$, admits a compactification by broken maps, in the following fashion. Let us write $w = \mu(x_0) - \sum_{i \geq 1} \mu(x_i)$. Recall the poset

$$\mathcal{T}^B(w) = \mathcal{T}_{k,d}(w) \sqcup \bigsqcup_{\beta \in B} \mathcal{T}^\beta(w)$$

where $\mathcal{T}_{k,d}(w)$ are the total weight w simple k-trees (so just one stable vertex), and $\mathcal{T}^\beta(w)$ are the total weight w β-shaped trees (so, these have two stable vertices v_1^β and v_2^β).

For $T \in \mathcal{T}^B(w)$, each interior edge of T comes with an associated Floer datum: for the edges between the stable vertices and the leaves or root, these are just the Floer data $(H_{\zeta(i)}, J_{\zeta(i)})$; in the case that T is β-shaped, each interior edge between v_1^β and v_2^β has the associated Floer datum for $(H_{\zeta^\beta}, J_{\zeta^\beta})$ for the glued ends of $\mathcal{S}_1^\beta$ and $\mathcal{S}_2^\beta$. We say that a tree $T \in \mathcal{T}^B(w)$ is *decorated by Floer generators* if at each interior edge, we have a Floer generator x_e for the corresponding Floer datum.

In particular, for a decorated tree $(T, \{x_e\})$, we have a moduli space of broken T-shaped maps with asymptotics given by x_e:

$$\mathcal{F}_T(\{x_e\}) = \prod_{\substack{v \in V(T) \\ \text{stable}}} \mathcal{F}(\{x_e\}) \quad \times \prod_{\substack{v \in V(T) \\ \text{unstable}}} \mathcal{M}(x_{e_-(v)}, x_{e_+(v)}) \tag{5.8}$$

where $e_\pm(v)$ are the two edges attached to an unstable vertex v.

In the case that T is a simple k-tree, this can be identified with a stratum of the earlier compactification (5.5) of $\mathcal{F}$ in the case that the base parameter space $\mathcal{R}$ was a closed manifold. In the case that $\mathcal{T}$ is a tree of shape β, such a stratum corresponds to a nodal Riemann surface with two "stable" components homeomorphic to S_1^β and S_2^β, some number of unstable "neck" strip/cylinder components connecting them, and some number of strip/cylinder components on the ends. $\mathcal{F}_T(\{x_e\})$ is then the space of inhomogeneous pseudoholomorphic maps from this nodal surface to M, with asymptotics for each component determined by the decorations x_e.

Then, the compactified moduli space of maps from the degenerating family $\mathcal{S}$ with asymptotics $x_0, x_1, \ldots, x_k$ is

$$\overline{\mathcal{F}}(x_0; \ldots, x_k) = \bigsqcup_{\substack{T \in \mathcal{T}^B(w) \\ \text{decorations } \{x_e\}}} \mathcal{F}_T(\{x_e\}) \tag{5.9}$$

topologized by Gromov convergence. In the next chapter, we will prove that this is a smooth manifold with corners, then pay special attention to studying how the decompositions of (5.4) are compatible over the various boundary strata. The upshot is that $\overline{\mathcal{F}}$ has the structure of a $(\mathcal{R}, \{\mathcal{R}^\beta\})$-parameterized map of flow categories $\mathcal{M}_0 \longrightarrow (\mathcal{M}_1, \ldots, \mathcal{M}_k)$.

6. Smoothness of the moduli spaces

We will now explain how to equip the compactified moduli spaces of maps with the structure of a smooth manifold with corners. The main input is an estimate on the decay of the gluing maps, as explained by Fukaya-Oh-Ohto-Ono in [6], which allows us to show the smoothness of the coordinate change between two gluing charts for the same boundary stratum. To show the smoothness of the coordinate change between charts associated to different strata, we will reduce to the case of special gluing charts of geometric origin, coming from constraining our maps to have specified points pass through codimension one hypersurfaces.

For most of this section, we will focus on the moduli of maps from strips or cylinders satisfying Floer's equation. We will then end with an explanation of how to adapt the framework to the case of maps from surfaces with strip-like or cylindrical ends.

6.1. **Local hypersurfaces.** Suppose that (H_t, J_t) is a regular Floer datum, for either a pair of exact Lagrangians L_0, L_1, or for H_t-Hamiltonian Floer theory: our discussion will apply equally to moduli spaces of strips and cylinders. Throughout this section, fix x_-, x_+ to be elements of either $\mathcal{X}(L_0, L_1, H_t)$ (time 1 Hamiltonian chords) or $\mathcal{P}(H_t)$ (time 1 Hamiltonian orbits) in the open and closed case respectively. We will write Z to mean either $\mathbb{R} \times [0,1]$ or $\mathbb{R} \times S^1$ depending on context.

Suppose that $u \in W(x_-, x_+)$ is a solution to the (H_t, J_t)-Floer equation, of Maslov index $\mu(u) = d+1$, so that the component of $\mathcal{M}(x_-, x_+)$ containing u is d-dimensional. Choose $d+1$ distinct marked points $z_0, \ldots, z_d \in Z$ which are *regular* in the sense of [5]; write (s_i, t_i) for the coordinates of z_i. We will specify local coordinates on $W(x_-, x_+)$ and $\mathcal{M}(x_-, x_+)$ through a choice of local hypersurfaces through the marked points.

Definition 6.1. *A collection of codimension one open hypersurfaces* $\mathbf{H} = \{H_0, \ldots, H_d\} \subset M$ *is called a* complete system of transverse hypersurfaces *for* u *if*

(1) the evaluation map

$$\mathrm{ev}_{\mathbf{z}} = (\mathrm{ev}_{z_0}, \ldots, \mathrm{ev}_{z_d}) : W(x_-, x_+) \to M^{d+1}$$

is transverse to $H_0 \times \ldots H_d$ at u;

(2) For each $i = 0, \ldots, d$, $u^{-1}(H_i)$ and $\mathbb{R} \times \{t_i\} \subset Z$ intersect transversely at z_i, and this is the only intersection point of $\mathbb{R} \times \{t_i\}$ and $u^{-1}(\bar{H}_i)$ (where $\bar{H}_i$ is the closure of H_i).

In particular, an easy application of the implicit function theorem shows that there is an open neighbourhood $V_{\mathbf{H}} \subset W(x_-, x_+)$ of u, such that for each $v \in V$, there are unique points $z_i(v) = (s_i(v), t_i) \in \mathbb{R}$ with the same t-coordinate as z_i with $v(s_i(v), t_i) \in H_i$, and the resulting map

$$V \to \mathbb{R}^{d+1}, \qquad v \mapsto (s_0(v), \ldots, s_d(v))$$

is a diffeomorphism onto its image, yielding distinguished charts for $W(x_-, x_+)$ around u. Moreover, we can clearly arrange for $V_{\mathbf{H}}$ to be translation invariant; we obtain an open neighbourhood $U_{\mathbf{H}} \subset \mathcal{M}(x_-, x_+)$ such that

$$U_{\mathbf{H}} \to \mathbb{R}^d, \qquad v \mapsto (s_1(v) - s_0(v), s_2(v) - s_1(v), \ldots, s_d(v) - s_{d-1}(v)) \tag{6.1}$$

yields local charts on $U_{\mathbf{H}}$.

Note that the conditions of Definition 6.1 are open in u: we can choose the neighbourhood $U_{\mathbf{H}}$ small enough that $H_0, \ldots, H_d$ is also a complete system of transverse hypersurfaces for each $v \in U_{\mathbf{H}}$. Moreover, we can use this choice of hypersurfaces to give distinguished parametrizations for such v:

Definition 6.2. *We will say that* $\mathbf{H}$ *is* ordered *if*

$$s_0 < s_1 < \ldots < s_d;$$

in which case for $v \in \mathcal{M}(x_-, x_+)$ close to u, we say that $L_{\mathbf{H}}(v) = \frac{1}{2}(s_d(v) - s_0(v))$ is the length *of v with respect to* $\mathbf{H}$. *We will furthermore say that a parametrization of v is* $\mathbf{H}$-centered *if $s_0(v) = -L_{\mathbf{H}}(v)$ and $s_d(v) = L_{\mathbf{H}}(v)$.*

Note that when u is index one (so it is a discrete point in the moduli space), we only take one marked point $z_0(u)$, and the length $L_{\mathbf{H}}(u)$ is automatically zero.

When we perform gluing in the next section, we will put a metric on M. We will use a Z-dependent metric $\{g_z\}_{zinZ}$ (and in particular, not translationally invariant), in the following reasonably mild sense:

(i) in the case when Z is a strip, $g_z = g_0$ a constant metric along $\mathbb{R} \times \{0\}$ and likewise $g_z = g_1$ along $\mathbb{R} \times \{1\}$, so that L_0, L_1 are totally geodesic for g_0, g_1 respectively;

(ii) for sufficiently small $s \ll 0$ or sufficiently large $s \gg 0$, the metric is equal to some fixed s-independent metric g_t;

(iii) in some small neighbourhood around each marked point z_i, the metric is constant, so that each hypersurface H_i is totally geodesic.

This can be achieved by choosing g_{z_i} to be flat in a neighbourhood of $u(z_i)$, and then each H_i to be a flat disc.

Now, consider the Banach space $W^2_k(u^*TM)$ of class L^2_k vector fields ξ along u with $\xi(s,0) \in L_0, \xi(s,1) \in L_1$ for each $s \in \mathbb{R}$. There is then a codimension $d+1$ subspace

$$(6.2) \qquad W^2_{k,\mathbf{H}}(u^*TM) = \{\xi \in W^2_k(u^*TM) : \xi(z_i) \in TH_i \text{ for } i = 0, \ldots, n\} \subset W^2_k(u^*TM)$$

of vector fields tangent to each H_i at the marked points. The assumptions of Definition 6.1 then ensure that this space is in fact a complement for the kernel $\ker D_u$ for the linearization of the $\bar{\partial}$-operator

$$D_u : W^2_k(u^*TM) \to L^2_{k-1}(u^*TM).$$

In particular there is a unique bounded right inverse

$$(6.3) \qquad Q_u : L^2_{k-1}(u^*TM) \to W^2_k(u^*TM)$$

of D_u whose image is $W^2_{k,\mathbf{H}}(u^*TM)$.

6.2. **The pregluing map.** Suppose $\mathbf{u} = (u^1, \ldots, u^n) \in \mathcal{M}(x^0, x^1) \times \ldots \times \mathcal{M}(x^{n-1}, x^n)$ is a broken trajectory. Suppose, once at for all, that we have chosen a time-dependent metric g_t and totally geodesic and ordered complete systems of transverse hypersurfaces $\mathbf{H}^\ell = H^\ell_0, \ldots, H^\ell_{d_\ell}$ for each u^ℓ, where $\mu(u^\ell) = d_\ell + 1$ for each $\ell = 0, \ldots, n$, through regular marked points $z^\ell_0 = (s^\ell_0, t^\ell_0), \ldots, z^\ell_{d_\ell} = (s^\ell_{d_\ell}, t^\ell_{d_\ell})$. We also fix parametrizations which are $\mathbf{H}^\ell$-centered in the sense explained above.

Take the open neighbourhoods $U_{\mathbf{H}^\ell} \subset \mathcal{M}(x^{\ell-1}, x^\ell)$ as before, and we will fix the relatively compact open set

$$(6.4) \qquad U = U_{\mathbf{H}^1} \times \ldots \times U_{\mathbf{H}^n} \subset \mathcal{M}(x^0, x^1) \times \ldots \times \mathcal{M}(x^{n-1}, x^n).$$

The pregluing map will be a map

$$\text{pre}G : U \times [T_0, \infty)^{n-1} \to C^\infty(Z, M)$$

for some sufficiently large T_0 defined as follows. For each $\mathbf{u} \in U$ and a collection of *gluing lengths* $\mathbf{T} = (T_1, \ldots, T_{n-1}) \in [T_0, \infty)^{n-1}$, consider the strips

$$(6.5) \qquad Z_\ell(\mathbf{u}, \mathbf{T}) = [-T_{\ell-1} - L(u^\ell), L(u^\ell) + T_\ell] \times [0,1] \qquad \text{for } \ell = 1, \ldots, n$$

where by convention $T_0 = -\infty$ and $T_n = \infty$ (so that $Z_1 = (-\infty, L(u^1) + T_1] \times [0,1]$ and $Z_n = [-T_{n-1} - L(u^n), \infty))$, and the dependence of $L(u^\ell)$ on $\mathbf{H}^\ell$ is to be understood. We will also take the strips

$$W_\ell(T_\ell) = [-T_\ell, T_\ell] \times [0,1] \qquad \text{for } \ell = 1, \ldots, n-1.$$

Let us then consider infinite strip $Z = \mathbb{R} \times [0,1]$, decomposed as a union of the above strips glued along their "vertical" boundaries from left to right:

$$(6.6) \qquad Z = Z_1(\mathbf{u}, \mathbf{T}) \cup W_1(T_1) \cup Z_2(\mathbf{u}, \mathbf{T}) \cup W(T_2) \cup \ldots \cup W(T_{n-1}) \cup Z_n(\mathbf{u}, \mathbf{T}).$$

The pregluing pre$G(\mathbf{u},\mathbf{T})$ is then a strip $w : Z \to M$ constructed separately on each of the above regions of the strip. First, on the strips $Z_\ell(\mathbf{u},\mathbf{T})$, we have:

$$w|_{Z_\ell(\mathbf{u},\mathbf{T})} = u^\ell|_{[-T_{\ell-1}-L(u^\ell),L(u^\ell)+T_\ell]\times[0,1]} \tag{6.7}$$

On each $W(T_\ell)$, we glue $u^{\ell-1}$ and u^ℓ together using a cut-off function. Explicitly, for respectively $s \ll 0$ and $s \gg 0$ we can write

$$u^\ell(s,t) = \exp_{x^{\ell-1}}(\xi^\ell_-(s,t)), \qquad u^\ell(s,t) = \exp_{x^\ell}(\xi^\ell_+(s,t))$$

for $\xi^\ell_-(s,t) \in T_{x^{\ell-1}}M, \xi^\ell_+(s,t) \in T_{x^\ell}M$, and so as long as T_0 is chosen sufficiently large, we can define for $\ell = 1,\dots,n-1$

(6.8)

$$w|_{W_\ell(T_\ell}(s,t) = \exp_{x^\ell}\left(\beta_-(s)\xi^{\ell-1}_+(s+L(u^{\ell-1})+2T_\ell,t) + \beta_+(s)\xi^\ell_-(s-L(u^\ell)-2T_\ell,t)\right)$$

where $\beta_\pm : \mathbb{R} \to [0,1]$ are fixed smooth cut-off functions so that

$$\beta_-(s) = \begin{cases} 1 & \text{for } s \le -1; \\ 0 & \text{for } s \ge 0; \end{cases} \qquad \beta_+(s) = \begin{cases} 0 & \text{for } s \le 0; \\ 1 & \text{for } s \ge 1. \end{cases} \tag{6.9}$$

By construction, w then defines a smooth strip with boundaries on L_0, L_1 which decays to x^0, x^n at $-\infty, +\infty$ respectively, and w itself depends smoothly on $\mathbf{u},\mathbf{T}$.

Each $Z_\ell(\mathbf{u},\mathbf{T})$ carries marked points $z^\ell_i(u^\ell)$ for $i = 0,\dots,d_\ell$, and so Z inherits $\sum_\ell d_\ell + n$ marked points $z^\ell_i(\mathbf{u},\mathbf{T})$ for $\ell = 1,\dots,n$ and $i = 0,\dots,d_\ell$, and by construction we have $w(z^\ell_i(\mathbf{u},\mathbf{T})) \in H^\ell_i$.

Let us also record that $\bar\partial w$ is supported in the intervals $[-1,1]\times[0,1] \subset W_\ell(T_\ell)$ for $\ell = 1,\dots,n-1$; and by the exponential decay of each u^ℓ with all its derivatives, there is some $\delta > 0$ so that $||\bar\partial w|_{W_\ell(T_\ell)}||_{2,k-1}$ is bounded by a constant multiple of $e^{-\delta T_\ell}$; in particular for some constant C we have

$$||\bar\partial w||_{2,k-1} \le Ce^{-\delta \min(\mathbf{T})} \tag{6.10}$$

and thus we can consider w to be an approximate solution. We now go about the task of constructing an actual J_t-holomorphic strip near w.

Recall that the choice of metric g_t determines a linearization of the $\bar\partial$-operator

$$D_w : W^2_k(w^*TM) \to L^2_{k-1}(w^*TM)$$

along $w = w_{\mathbf{u},\mathbf{T}}$, which satisfies a well-known quadratic inequality: consider the non-linear mapping

$$\mathcal{F}_w : W^2_k(w^*TM) \to L^2_{k-1}(w^*TM), \qquad \xi \mapsto \left(\Pi^{\exp_w \xi}_w\right)^{-1}\bar\partial\exp_w(\xi) \tag{6.11}$$

where the exponential map and the parallel transport $\Pi^{\exp_w \xi}_w$ is to be taken with respect to the *time-dependent* metric g_t; we then have

Lemma 6.3. *Consider the error term in the linearization at* $w = w(\mathbf{u}, \mathbf{T})$

$$N_w(\xi) = \mathcal{F}_w(\xi) - \bar{\partial} w - D_w(\xi) \in L^2_{k-1}(w^*TM).$$

Then there is some constant $C > 0$*, independent of* $\mathbf{T}$*, such that whenever* $\xi_1, \xi_2 \in W^2_k(w^*TM)$ *each satisfy* $||\xi_i||_{2,k} \leq \frac{1}{C}$*, then*

$$||N_w(\xi_1) - N_w(\xi_2)||_{2,k-1} \leq C||\xi_1 - \xi_2||_{2,k}||\xi_1 + \xi_2||_{2,k}.$$

The key point of the construction is that as in (6.2), the choice of hypersurfaces singles out a complement for $\ker(D_w) \subset W^{2,k}(w^*TM)$. Indeed, consider the codimension $\sum_\ell d_\ell + n$ subspace

$$W^2_{k,\mathbf{H}}(w^*TM) = \{\xi \in W^2_k(w^*TM) : \xi(z^\ell_i(\mathbf{u}, \mathbf{T})) \in TH^i \text{ for each } \ell, i\}. \tag{6.12}$$

We claim that this subspace is a complement for $\ker(D_w)$:

Proposition 6.4. *For sufficiently large* $\mathbf{T}$*, the linearized operator* D_w *yields an isomorphism*

$$D_w|_{W^2_{k,\mathbf{H}}(w^*TM)} : W^2_{k,\mathbf{H}}(w^*TM) \xrightarrow{\sim} L^2_{k-1}(w^*TM)$$

and moreover the inverse $\left(D_w|_{W^2_{k,\mathbf{H}}(w^*TM)}\right)^{-1}$ *has operator norm uniformly bounded in* $\mathbf{u}, \mathbf{T}$*.*

Proof. We will follow the usual strategy: from the given right inverses with image (6.2)

$$Q_{u^\ell} : L^2_{k-1}((u^\ell)^*TM) \to W^2_{k,\mathbf{H}^\ell}((u^\ell)^*TM)$$

for the linearized operators along the component curves u^ℓ of $\mathbf{u}$, we will construct an approximate right inverse

$$Q_w : L^2_{k-1}(w^*TM) \to W^2_{k,\mathbf{H}}(w^*TM) \tag{6.13}$$

for D_w with image the subspace (6.12).

Choose smooth functions $\chi_+, \chi_- : \mathbb{R} \to [0, 1]$ so that

$$\chi_+(s) = \begin{cases} 0 & \text{for } s \leq -1; \\ 1 & \text{for } s \geq 1; \end{cases} \qquad \chi_-(s) = 1 - \chi_+(s). \tag{6.14}$$

We first define a "breaking map"

$$B : L^2_{k-1}(w^*TM) \to \bigoplus_{\ell=1}^{n} L^2_{k-1}((u^\ell)^*TM) \tag{6.15}$$

as follows. Given $\eta \in L^2_{k-1}(w^*TM)$, define $B_\ell(\eta) \in L^2_{k-1}((u^\ell)^*TM)$ piecewise on via the decomposition (6.6) of Z

$$B_\ell(\eta)(z) = \begin{cases} 0 & \text{for } z = (s,t) \in W_{\ell-1}(T_{\ell-1}) \text{ with } s \leq -1; \\ \Pi_w^{u^\ell} \chi_+(s)\eta & \text{for } z = (s,t) \in W_{\ell-1}(T_{\ell-1}) \text{ with } s \geq -1 \\ \eta & \text{for } z = (s,t) \in Z_\ell(\mathbf{u}, \mathbf{T}); \\ \Pi_w^{u^\ell} \chi_-(s)\eta & \text{for } z = (s,t) \in W_\ell(T_\ell) \text{ with } s \leq 1; \\ 0 & \text{for } z = (s,t) \in W_\ell(T_\ell) \text{ with } s \geq 1. \end{cases} \tag{6.16}$$

and $B_\ell(\eta) = 0$ elsewhere; it is thus supported in $[-1 - 2T_{\ell-1} - L(u^\ell), L(u^\ell) + 2T_\ell + 1]$, nothing that again these formulas also hold for $\ell = 1$ and $\ell = n$ by declaring $T_0 = T_n = \infty$.

We also define a "linear pregluing map"

$$\text{pre}dG : \bigoplus W^2_k((u^\ell)^*TM) \to W^2_k(w^*TM) \tag{6.17}$$

piecewise over Z using the decomposition (6.6) and the earlier cut-off functions β_-, β_+ as follows: given vector fields $\xi^\ell \in W^2_k((u^\ell)^*TM)$, we have

(6.18)

$$\text{pre}dG\left((\xi^\ell)_\ell\right)(z) = \begin{cases} \xi^\ell(z) & \text{for } z \in Z_\ell(\mathbf{u}, \mathbf{T}) \\ \beta_-(s)\Pi^w_{u^{\ell-1}}\xi^{\ell-1}(s + L(u^{\ell-1}) + 2T_\ell, t) & \text{for } z = (s,t) \in W_\ell(T_\ell) \\ \quad +\beta_+(s)\Pi^w_{u^\ell}\xi^\ell(s - L(u^\ell) - 2T_\ell, t) & \end{cases}$$

The approximate right inverse Q_w is then defined as the composition

$$Q_w = \text{pre}dG \circ (Q_{u^1} \oplus \ldots \oplus Q_{u^n}) \circ B : L^2_{k-1}(w^*TM) \to W^2_k(w^*TM) \tag{6.19}$$

By virtue of its construction, Q_w has image contained in $W^2_{k,\mathbf{H}}(w^*TM)$. Moreover, as in [4], it is uniformly bounded in $\mathbf{u}, \mathbf{T}$, and the operator norm of $D_w Q_w - \text{id}$ exponentially decays in $\min(\mathbf{T})$, and thus $\widetilde{Q}_w = Q_w(D_w Q_w)^{-1}$ is a true right inverse. □

Remark 6.5. *For proving the smoothness and exponential decay of the gluing map, it is considerably more useful to work directly with the approximate right inverse Q_w rather than $\widetilde{Q}_w$ in the Newton-Picard iteration.*

6.3. The gluing map.

From the implicit function theorem, we then immediately have:

Proposition 6.6. *For $\mathbf{T}$ sufficiently large, and some small $\varepsilon_0 > 0$ independent of $\mathbf{T}$, there exists a unique $\xi_{\mathbf{u},\mathbf{T}} \in B_{\varepsilon_0}(0) \cap \text{im}(Q_w) \subset W^{2,k}(w^*TM)$ so that*

$$\bar\partial \exp_w(\xi_{\mathbf{u},\mathbf{T}}) = 0$$

which moreover satisfies the estimate

$$||\xi_{\mathbf{u},\mathbf{T}}||_{2,k} \leq ||Q_w \bar\partial w||_{2,k}. \tag{6.20}$$

In light of (6.10), this bound can taken as exponential in $\min(\mathbf{T})$:

$$||\xi_{\mathbf{u},\mathbf{T}}||_{2,k} \leq Ce^{-\delta \min(\mathbf{T})}$$

for some constant $C > 0$. For sufficiently large $T_0 > 0$ we can then define the *gluing map*

$$G : U \times [T_0, \infty)^{n-1} \to \mathcal{M}(x^0, x^n), \qquad G(\mathbf{u}, \mathbf{T}) = \exp_w(\xi_{\mathbf{u},\mathbf{T}}). \tag{6.21}$$

The smoothness of the gluing map will be deduced as a corollary of the exponential decay estimates (6.23) for the Newton iteration used to construct $\xi_{\mathbf{u},\mathbf{T}}$

From the construction of the hypersurfaces H_i^ℓ, and in particular the geodesic assumption, the gluing map can be clearly characterized as follows: $G(\mathbf{u}, \mathbf{T})$ is the unique trajectory which admits a parametrized representative $v = \exp_w(\xi_{\mathbf{u},\mathbf{T}})$ for $\xi \in W^{2,k}(w^*TM)$ with $||\xi||_{2,k} < \varepsilon_0$, and so that $v(\mathbf{z}_i^\ell(\mathbf{u}, \mathbf{T})) \in H_i^\ell$.

This can be strengthened by the following observation.

Proposition 6.7. *For all sufficiently small $\varepsilon > 0$, there is an open neighbourhood $W \subset \mathcal{M}(x^0, x^n)$ of the broken trajectory $\mathbf{u}$ in the Gromov-Floer topology such that each element of W admits a parametrized representative v with:*

(1) $v = \exp_w(\xi_{\mathbf{u},\mathbf{T}})$ for some $||\xi||_{2,k} < \varepsilon$;
(2) there unique points $z_i^\ell(v) \in Z$, close to $z_i^\ell(\mathbf{u}, \mathbf{T})$, with the same t-coordinate t_i^ℓ, so that H_i^ℓ intersects $v(Z)$ transversely at $v(z_i^\ell(v))$.

Proof. It follows from Gromov compactness, combined with the exponential decay of v at the infinite ends, that for every $\eta > 0$, we can a neighbourhood W so that each element admits a parametrized representative v such that $v = \exp_w(\xi_{\mathbf{u},\mathbf{T}})$ for some $\mathbf{T}$, with $||\xi||_{C^1} < \eta$, from which the second property follows immediately. The first property follows from the exponential decay of long strips which are close to a critical point, in proved by Robbin-Salamon [9]. □

In particular, by choosing $T_0 > 0$ to be sufficiently large, we can find such open neighbourhoods W_1, W_2 with

$$W_1 \subset G(U \times [T_0, \infty)^{n-1}) \subset W_2.$$

This shows that G is surjective onto some open neighbourhood of $\mathbf{u}$. Moreover for $v \in W_2$, the s-coordinates of the marked points give rise to a well defined, translation-invariant smooth function

$$W_2 \to \mathbb{R}^{n-1}, \qquad v \mapsto (s_0^2(v) - s_{d_1}^1(v), s_0^3(v) - s_{d_2}^2(v), \dots, s_0^n(v) - s_{d_{n-1}}^{n-1}(v))$$

so that the composition

$$U \times [T_0, \infty)^{n-1} \xrightarrow{G} W_2 \to \mathbb{R}^{n-1}$$

is just the projection onto the $\mathbf{T}$ coordinates, which proves that G is a smooth embedding. To summarize:

Corollary 6.8. *For sufficiently large $T_0 > 0$, the gluing map $G : U \times [T_0, \infty)^{n-1} \to \mathcal{M}(x^0, x^n)$ is a smooth embedding onto some open neighbourhood W of the broken trajectory $\mathbf{u}$. Moreover, the gluing map is characterized by the following property: $G(\mathbf{u}, \mathbf{T})$ is the unique trajectory in W which admits a parametrization v so that*

$$v(\mathbf{z}_i^\ell(\mathbf{u}, \mathbf{T})) \in H_i^\ell$$

for each $\ell = 1, \ldots, n$ and $i = 0, \ldots, d_\ell$.

We now discuss the "associativity" of the constructed gluing maps. Let us first observe that the gluing estimates (6.20), (6.10) show that for each $\mathbf{u} \in U$ and $\mathbf{T} \in [T_0, \infty)^{n-1}$, the hyperplanes $\{H_i^\ell\}$ also satisfy the hypotheses of Definition 6.1 for the glued curve $G(\mathbf{u}, \mathbf{T})$, with marked points given by $z_i^\ell(\mathbf{u}, \mathbf{T})$.

In particular, choose some indices $0 = \ell_0 < \ell_1 < \ell_2 < \ldots < \ell_{r-1} < \ell_r = n$, and fix a value of $T_m \in [T_0, \infty)$ for each $m \notin \{\ell_1, \ldots, \ell_r\}$. For $\alpha = 1, \ldots, r$, consider the broken curves

$$\mathbf{u}^\alpha = (u^{\ell_{\alpha-1}+1}, \ldots, u^{\ell_\alpha}) \in \mathcal{M}(x^{\ell_{\alpha-1}}, \ldots, x^{\ell_\alpha})$$

and write $\mathbf{T}^\alpha = (T_{\ell_{\alpha-1}+1}, \ldots, T_{\ell_\alpha - 1})$.

Using the systems of transverse hypersurfaces $\mathbf{H}^\ell$ for $\ell = \ell_{\alpha-1}+1, \ldots, \ell_\alpha$, we can glue such broken curves using the gluing lengths $\mathbf{T}^\alpha$, to form a curve

$$G(\mathbf{u}^\alpha, \mathbf{T}^\alpha) \in \mathcal{M}(x^{\ell_{\alpha-1}}, x^{\ell_\alpha}).$$

Moreover, by the above observation, $\mathbf{H}^\alpha = \bigcup_{\ell=\ell_{\alpha-1}+1}^{\ell_\alpha} \{H_i^\ell\}$ is then itself a complete system of transverse hypersurfaces for this glued curve $G(\mathbf{u}^\alpha, \mathbf{T}^\alpha)$. In particular, we can then glue these curves together using gluing lengths $T_{\ell_1}, \ldots, T_{\ell_{r-1}}$. The following proposition states that gluing in this fashion is "associative":

Proposition 6.9. *For sufficiently large $T_{\ell_1}, \ldots, T_{\ell_{r-1}}$, the trajectory formed by this second gluing*

$$G\left((G(\mathbf{u}^\alpha, \mathbf{T}^\alpha))_{\alpha=1,\ldots,r}, (T_{\ell_1}, \ldots, T_{\ell_{r-1}})\right)$$

is equal to the original glued trajectory $G(\mathbf{u}, \mathbf{T})$.

This is an immediate consequence of Proposition 6.8, and the particular construction of the pregluing map we used: the marked points on each curve $G\left((G(\mathbf{u}^\alpha, \mathbf{T}^\alpha))_\alpha, (T_{\ell_\alpha})_\alpha\right)$ and $G(\mathbf{u}, \mathbf{T})$ coincide, and at these points both curves each go through the local hypersurfaces H_i^ℓ. The claimed equality then follows, since for sufficiently large $T_{\ell_1}, \ldots, T_{\ell_{r-1}}$, both these trajectories are in a neighbourhood $W \subset \mathcal{M}(x^0, x^n)$ of the broken curve $\mathbf{u}$ where Proposition 6.8 applies.

6.4. **Exponential decay of the gluing map.** Let us now discuss the derivatives of the gluing map. This has been discussed extensively by Fukaya-Oh-Ohto-Ono in [6]; accordingly while we shall outline the argument, an interested reader will find a much more complete account there. Our situation, on the other hand, is made significantly simpler for two reasons: for one, we are considering moduli spaces that are already regular, and so we do not have to build in obstruction bundles and Kuranishi maps into our estimates.

Perhaps more fundamentally, and with greater consequences for our results, we are only considering Lagrangians that intersect *transversely*, rather than the Morse-Bott-type situation studied by Fukaya-Oh-Ohto-Ono. Therefore, our Fredholm set-up *does not require Sobolev spaces with exponential weights*, and the estimates we obtain will be in the L^2_k-Sobolev norms, rather than a weighted $L^2_{k,\delta}$ norm.

Let us now state the main result. Observe that the preglued curves w, the vector fields $\xi_{\mathbf{u},\mathbf{T}}$, and the glued curves $G(\mathbf{u},\mathbf{T})$ are defined independently of the order of differentiability k used in the Sobolev estimates, and indeed we will need the following estimates at every order of differentiability $k > 2$. We will pick some S, and for each $\ell = 1,\ldots,n$ write $Z_\ell(S) = [-S,S]\times[0,1]$ considered as a substrip of $Z_\ell(\mathbf{u},\mathbf{T})$ for $\mathbf{T}$ with $\min(\mathbf{T}) > S$.

Proposition 6.10. *Fix the open neighbourhood U of* (6.4)*. Then for every order of differentiability $k > 2$ and $S > 0$, there are constants $T_0 > S$, C and $\delta > 0$ such that*

$$||(\nabla_{\mathbf{u}})^p \frac{\partial^{\mathbf{m}}}{\partial \mathbf{T}^{\mathbf{m}}}\left(\xi_{\mathbf{u},\mathbf{T}}|_{Z_\ell(S)}\right)||_{W^2_{k-p-|\mathbf{m}|}(Z_\ell(S))} \leq Ce^{-\delta\min(\mathbf{T})} \tag{6.22}$$

for all $\mathbf{u}\in U$ and $\mathbf{T}$ with $\min(\mathbf{T}) \geq T_0$, where $p \geq 0$ is an integer and $\mathbf{m} = (m_1,\ldots,m_{n-1})$ is a sequence of non-negative integers of sum $|\mathbf{m}| = m_1+\ldots+m_{n-1}$ so that $p+|\mathbf{m}| \leq k-1$.

Note that since $w|_{Z_\ell(S)} = u^\ell|_{[-S,S]\times[0,1]}$, the differentiation in the $\mathbf{T}$-direction takes place in a fixed Sobolev space $W^2_k([-S,S]\times[0,1],(u^\ell)^*TM)$. Also observe that when $p = 0$ and $\mathbf{m} = (0,\ldots,0)$, this is essentially the estimate obtained from (6.20) and (6.10). Like that bound, this estimate is proved by induction over the Newton-Picard iteration used to construct $\xi_{\mathbf{u},\mathbf{T}}$. Recalling the approximate right inverse Q_w of (6.19), $\xi_{\mathbf{u},\mathbf{T}}$ is constructed by setting

$$v_{(0)} = w_{\mathbf{u},\mathbf{T}}, \qquad \xi_{(0)} = 0 \in C^\infty(w^*TM)$$

and then iteratively defining $\xi_{(\alpha)}$ by

$$\xi_{(\alpha+1)} = \xi_{(\alpha)} + Q_w\left(\Pi^w_{v_{(\alpha)}}\right)\bar\partial v_{(\alpha)}, \qquad v_{(\alpha+1)} = \exp_w\left(\xi_{(\alpha+1)}\right).$$

Note that each $\xi_{(\alpha)}$ is a smooth vector field along w, well-defined independently of the Sobolev completion, by the construction of Q_w. It then suffices to show that for each k,S, there are constants $T_0 > S, C$ and δ so that

$$||(\nabla_{\mathbf{u}})^p \frac{\partial^{\mathbf{m}}}{\partial \mathbf{T}^{\mathbf{m}}}\left(\xi_{(\alpha)}|_{Z_\ell(S)}\right)||_{2,k-p-|\mathbf{m}|} \leq Ce^{-\delta\min(\mathbf{T})} \tag{6.23}$$

for each α. The result for the limit $\xi_{\mathbf{u},\mathbf{T}}$ then follows by the Arzela-Ascoli theorem. Indeed, a corollary of the above estimate is

Corollary 6.11. *The gluing map $G : U \times [T_0, \infty)^{n-1} \to \mathcal{M}(x^0, x^n)$ is smooth.*

The estimate (6.23) follows from induction over a slightly stronger bound, on larger regions of the strip than $Z_\ell(S)$. On each $Z_\ell(\mathbf{u}, \mathbf{T})$, the preglued strip is given by u^ℓ, and so we can think of $\xi_{\mathbf{u},\mathbf{T}}|_{Z_\ell(\mathbf{u},\mathbf{T})}$ as defining an element of $W^2_k(Z_\ell(\mathbf{u}, \mathbf{T}), (u^\ell)^*TM)$. Moreover, on each $W^\ell_{T_\ell}$, for sufficiently large T_ℓ the preglued curve w is very close to x^ℓ; and so for $\ell = 1, \dots, n-1$ we set

$$\zeta^\ell_{(\alpha)} = \exp^{-1}_{x^\ell}\left(v_{(\alpha)}|_{W_\ell(T_\ell)}\right) \in C^\infty(W_\ell(T_\ell), T_{x^\ell}M) \tag{6.24}$$

Let us also take for $\ell = 1, \dots, n$ the "error terms"

$$\eta^\ell_{(\alpha)} = B_\ell\left(\Pi^w_{v_{(\alpha)}} \bar\partial v_{(\alpha)}\right) \in C^\infty(Z, (u^\ell)^*TM) \tag{6.25}$$

where B_ℓ is as in (6.15).

Fix and order of differentiability $k > 2$, and take the Lipshitz constant $0 < \mu < 1$ for the contraction mapping $L^2_{k-1}(w^*TM) \to L^2_{k-1}(w^*TM)$ given by $\eta \mapsto \eta + \mathcal{F}_w Q_w \eta$, where $\mathcal{F}_w$ is as in (6.11). We claim that there are constants C_1, C_2, C_3 and $\delta > 0$ so that we have the inequalities for each ℓ and α: (compare (6.9), (6.10), (6.11) of [6]):

$$||(\nabla_{\mathbf{u}})^p \frac{\partial^{\mathbf{m}}}{\partial \mathbf{T}^{\mathbf{m}}}\left(\xi_{(\alpha)}|_{Z_\ell(\mathbf{u},\mathbf{T})}\right)||_{W^2_{k-p-|\mathbf{m}|}(Z_\ell(\mathbf{u},\mathbf{T}))} \le C_1(2-\mu^\alpha)e^{-\delta \min(\mathbf{T})} \tag{6.26}$$

$$||(\nabla_{\mathbf{u}})^p \frac{\partial^{\mathbf{m}}}{\partial \mathbf{T}^{\mathbf{m}}}\left(\zeta^\ell_{(\alpha)}|_{W^\ell(T_\ell)}\right)||_{W^2_{k-p-|\mathbf{m}|}(W_\ell(T_\ell))} \le C_2(2-\mu^\alpha)e^{-\delta \min(\mathbf{T})} \tag{6.27}$$

$$||(\nabla_{\mathbf{u}})^p \frac{\partial^{\mathbf{m}}}{\partial \mathbf{T}^{\mathbf{m}}}(\eta^\ell_{(\alpha)})||_{L^2_{k-1-p-|\mathbf{m}|}(Z)} \le C_3\mu^\alpha e^{-\delta \min(\mathbf{T})} \tag{6.28}$$

for all $\min(\mathbf{T}) > T_0$ sufficiently large, $\mathbf{u} \in U$ and $p, \mathbf{m}$ with $p + |\mathbf{m}| \le k-1$. Observe that whilst the dimensions of the strips $Z_\ell(\mathbf{u}, \mathbf{T})$ and $W_\ell(T_\ell)$ do depend on $\mathbf{u}, \mathbf{T}$, each of the vector fields $\xi_{(\alpha)}|_{Z_\ell(\mathbf{u},\mathbf{T})}$ and $\zeta^\ell_{(\alpha)}|_{W^\ell(T_\ell)}$ admit obvious extensions to a slightly wider strip; hence the derivatives with respect to $\mathbf{u}$ and $\mathbf{T}$ of these restrictions are still well defined.

For the base case $\alpha = 0$, the first two bounds , are automatic. The third bound follows since the error term $\eta = \bar\partial(w)$ is supported in the $n-1$ strips $[-1, 1] \times [0, 1] \subset W_\ell(T_\ell)$; on this region w is given by (6.8), and we can then appeal to the decay of $u^{\ell-1}(s, t) \to x^\ell$ as $s \to \infty$ and $u^\ell(s, t) \to x^\ell$ as $s \to -\infty$, which is exponential in all derivatives [9]. The "loss of differentiability" that appears in the above estimates arises since

$$\frac{\partial^m}{\partial T^m_\ell} u^{\ell-1}(s + L(u^{\ell-1}) + 2T_\ell, t) = 2^m \frac{\partial^m u^{\ell-1}}{\partial s^m}; \quad \frac{\partial^m}{\partial T^m_\ell} u^\ell(s - L(u^\ell) - 2T_\ell, t) = -2^m \frac{\partial^m u^\ell}{\partial s^m}$$

whilst

$$\nabla^p_{u^{\ell-1}} u^{\ell-1}(s+L(u^{\ell-1})+2T_\ell, t) = \nabla^p_{u^{\ell-1}} L(u^{\ell-1}) \frac{\partial^p u^{\ell-1}}{\partial s^p}; \quad \nabla^p_{u^\ell} u^\ell(s-L(u^\ell)-2T_\ell, t) = -\nabla^p_{u^\ell} L(u^\ell) \frac{\partial^p u^\ell}{\partial s^p}$$

and thus the $L^2_{k-1-p-|\mathbf{m}|}$ norm of

$$\nabla^p_{\mathbf{u}} \frac{\partial^{\mathbf{m}}}{\partial \mathbf{T}^{\mathbf{m}}}\left(\eta|_{W_\ell(T_\ell)}\right)$$

is controlled by the L^2_{k-1} norm of $\exp^{-1}_{x^\ell}(u^{\ell-1})$ and $\exp^{-1}_{x^\ell}(u^\ell)$.

The proof of the induction step for the estimates (6.4), (6.4) and (6.4) is lengthy and not particularly enlightening, and we refer the reader to [6] for a detailed account.

6.5. **Smoothness of the coordinate change.** We now address the question of placing the structure of a smooth manifold with corners on the compactified moduli space $\overline{\mathcal{M}}(x_-, x_+)$. This will involve a change of coordinates $T_\ell = \psi(r_\ell)$ for some diffeomorphism

$$\psi : (0, \varepsilon] \to [T_0, \infty)$$

so that $T_\ell \to \infty$ sufficiently fast as $r_\ell \to 0$. This choice of ψ is called *the gluing profile*,and the coordinates r_ℓ are referred to as *gluing parameters*, whilst the T_ℓ are *gluing lengths.* In this paper, we will always use the gluing profile

$$T_\ell = \psi(r_\ell) = \frac{1}{r_\ell}. \tag{6.29}$$

This will suffice for our purposes; the gluing profile $\psi(r) = e^{\frac{1}{r}}$ would work just as well. On the other hand, our results will *not* hold for $\psi(r) = -\log(r)$, the change of coordinates customarily used to "smooth the node" in algebraic Gromov-Witten theory.

Working again near a broken trajectory $(u^1, \ldots, u^n)$, we write

$$\mathbf{r} = (r_1, \ldots, r_{n-1}) \quad \text{and} \quad \mathbf{T} = (T_1, \ldots, T_{n-1}) = (\psi(r_1), \ldots, \psi(r_{n-1})).$$

Then given complete systems of transverse hypersurfaces $\mathbf{H}^\ell$ for each u^ℓ, we obtain open neighbourhoods $U_{\mathbf{H}^\ell} \subset \mathcal{M}(x^{\ell-1}, x^\ell)$ of each u^ℓ as before, and for $\varepsilon > 0$ sufficiently small, we can define

$$\mathcal{G}^n_{\mathbf{H}}(\mathbf{u}, \mathbf{r}) = G(\mathbf{u}, \mathbf{T}) \in \mathcal{M}(x^0, x^n) \quad \text{for } \mathbf{u} \in \Pi_\ell U_{\mathbf{H}^\ell}, \mathbf{r} \in (0, \varepsilon)^{n-1}.$$

We will say that $n-1$ is the *codimension* of $\mathcal{G}^n_{\mathbf{H}}$.

In view of Proposition 6.9, we then see that as long as $\varepsilon > 0$ is chosen small enough, this extends to a continuous map

$$\mathcal{G}^n_{\mathbf{H}} : \Pi_\ell U_{\mathbf{H}^\ell} \times [0, \varepsilon)^{n-1} \to \overline{\mathcal{M}}(x^0, x^n) \tag{6.30}$$

such that for each choice of indices $0 = \ell_0 < \ell_1 < \ldots < \ell_{m-1} < \ell_m = n$, writing $\mathbf{u}^\alpha = (u^{\ell_{\alpha-1}+1}, \ldots, u^{\ell_\alpha})$ and $\mathbf{r}^\alpha = (r_{\ell_{\alpha-1}+1}, \ldots, r_{\ell_\alpha - 1})$, whenever $r_{\ell_1} = \ldots = r_{\ell_m - 1} = 0$, we have

$$\mathcal{G}^n_{\mathbf{H}}(\mathbf{u}, \mathbf{r}) = \left(\mathcal{G}^{n_\alpha}_{\mathbf{H}^\alpha}(\mathbf{u}^\alpha, \mathbf{r}^\alpha)\right)_\alpha \in \Pi_\alpha \mathcal{M}(x^{\ell_{\alpha-1}}, x^{\ell_\alpha}) \subset \partial\overline{\mathcal{M}}(x^0, x^n)$$

where $n_\alpha = \ell_\alpha - \ell_{\alpha-1}$ and $\mathbf{H}^\alpha = \{H^\ell_i\}^{\ell_\alpha}_{\ell_{\alpha-1}+1}$ is the corresponding system of transverse hypersurfaces for $\mathbf{u}^\alpha$.

Before we state our main theorem, let us observe that our choice of gluing profile $\psi(r) = \frac{1}{r}$ ensures that we can rewrite the estimate of Proposition 6.10 as

$$||(\nabla_{\mathbf{u}})^p \frac{\partial^{\mathbf{m}}}{\partial \mathbf{r}^{\mathbf{m}}} \left(\xi_{\mathbf{u}, \mathbf{T}(\mathbf{r})}|_{Z_\ell(S)}\right) ||_{W^2_{k-p-|\mathbf{m}|}(Z_\ell(S))} \leq Ce^{-\delta_1 \frac{1}{\max(\mathbf{r})}} \tag{6.31}$$

for some constant C and some δ_1 smaller than the exponent δ of (6.31); this is precisely where the choice of a sufficiently fast growing gluing profile is important.

This estimate on its own is not enough to control the gluing map at the boundary strata: if $n \geq 3$, and we only send some sub-collection $r_{\ell_1}, \ldots, r_{\ell_{m-1}} \to 0$ and fix the $\mathbf{r}^\alpha$ as above, this estimate is somewhat useless. However, in view of "associativity of gluing" from Proposition 6.9, we can instead use the above estimates for gluing the fixed broken trajectory $\mathcal{G}^{n_\alpha}_{\mathbf{H}^\alpha}(\mathbf{u}^\alpha, \mathbf{r}^\alpha)$ using the gluing parameters $r_{\ell_1}, \ldots, r_{\ell_{m-1}}$.

Theorem 6.12. *The maps* $\mathcal{G}^n_{\mathbf{H}}$ *of* (6.30) *arising from choices of complete systems of transverse hypersurfaces* $\mathbf{H}$ *for each broken trajectory* $\mathbf{u}$ *from* x_- *to* x_+ *yield charts which endow* $\overline{\mathcal{M}}(x_-, x_+)$ *with the structure of a smooth manifold with corners.*

Proof. The restrictions of each $\mathcal{G}^n_{\mathbf{H}}$ to $\Pi_\ell U_{\mathbf{H}^\ell} \times (0, \varepsilon)^{n-1}$ are smooth by Corollary 6.11, so we must check the smoothness of the transition maps at the broken trajectories.

Given a complete system of transverse hypersurfaces $\mathbf{H}$ for a broken trajectory $\mathbf{u} \in \mathcal{M}(x_-, x^1) \times \ldots \times \mathcal{M}(x^{n-1}, x_+)$, and indices $0 < \ell_1 < \ldots < \ell_{m-1} < n$, writing $\mathbf{u}^\alpha = (u^{\ell_{\alpha-1}+1}, \ldots, u^{\ell_\alpha})$ and $\mathbf{r}^\alpha = (r_{\ell_{\alpha-1}+1}, \ldots, r_{\ell_\alpha - 1})$ as before, we see that the collection $\mathbf{H}^\alpha = \{H^\ell_i\}^{\ell_\alpha}_{\ell_{\alpha-1}+1}$ is also a complete system of transverse hypersurfaces for broken trajectory

$$\left(\mathcal{G}^{n_\alpha}_{\mathbf{H}_\alpha}(\mathbf{u}_\alpha, \mathbf{r}_\alpha)\right)_\alpha \in \mathcal{M}(x^{\ell_{\alpha-1}}, x^{\ell_\alpha}).$$

Proposition 6.9 then implies that near $\left(\mathcal{G}^{n_\alpha}_{\mathbf{H}_\alpha}(\mathbf{u}_\alpha, \mathbf{r}_\alpha)\right)_\alpha$, the transition functions between the two charts given by gluing all of $\mathbf{u}$ together

$$\Pi_\ell \mathcal{M}(x^{\ell-1}, x^\ell) \times [0, \varepsilon)^{n-1} \supset \Pi_\ell U_{\mathbf{H}^\ell} \times [0, \varepsilon)^{n-1} \to \overline{\mathcal{M}}(x_-, x_+)$$

and gluing first each $\mathbf{u}_\alpha$, and then each $\mathcal{G}^n_{\mathbf{H}_\alpha}(\mathbf{u}_\alpha, \mathbf{r}_\alpha)$

$$\Pi_\alpha \mathcal{M}(x^{\ell_{\alpha-1}}, x^{\ell_\alpha}) \times [0, \varepsilon)^{m-1} \supset \Pi_\alpha U_{\mathbf{H}^\alpha} \times [0, \varepsilon)^{m-1} \to \overline{\mathcal{M}}(x_-, x_+)$$

are smooth: indeed, the transition map is exactly

$$(\mathbf{u}, \mathbf{r}) \mapsto \left(\left(\mathcal{G}^{n_\alpha}_{\mathbf{H}^\alpha}(\mathbf{u}^\alpha, \mathbf{r}^\alpha)\right)_\alpha, (r_{\ell_1}, \ldots, r_{\ell_{m-1}})\right).$$

In other words, the transition functions between different codimension charts arising from the same hypersurfaces $\mathbf{H}$ are smooth.

We now much check the smoothness of transition functions between two charts arising from different choices of hypersurfaces $\mathbf{H}, \widetilde{\mathbf{H}}$. From what we have just shown, it suffices to check this smoothness between charts *of the same codimension* $n-1$. After appropriately shrinking the domains, this transition function is thus a map

$$\varphi : \Pi_\ell U_{\mathbf{H}^\ell} \times [0, \varepsilon_1)^{n-1} \to \Pi_\ell \widetilde{U}_{\widetilde{\mathbf{H}}^\ell} \times [0, \varepsilon_2)^{n-1}$$

so that $\mathcal{G}^n_{\mathbf{H}}(\mathbf{u}, \mathbf{r}) = \mathcal{G}^n_{\widetilde{\mathbf{H}}}(\varphi(\mathbf{u}, \mathbf{r}))$. By induction on the codimension, we can assume this is smooth over $\Pi_\ell U_{\mathbf{H}^\ell} \times \left([0, \varepsilon_1)^{n-1} \backslash \{(0, \ldots, 0)\}\right)$, so that it suffices to check smoothness at a broken trajectory $\mathbf{u}$ with n components.

We will deduce this smoothness from the characterization of the gluing maps given by Proposition 6.8. Write $z_i^\ell = (s_i^\ell, t_i^\ell)$ and $\widetilde{z}_i^\ell = (\widetilde{s}_i^\ell, \widetilde{t}_i^\ell)$ for the marked points induced by systems of hypersurfaces $\mathbf{H}, \widetilde{\mathbf{H}}$.

For $\mathbf{u}, \mathbf{r} \in U_{\mathbf{H}^\ell} \times [0, \varepsilon_1)^{n-1}$, take $Z_\ell(S) \subset Z_\ell(\mathbf{u}, \mathbf{T})$ in the decomposition (6.6) of the strip. There are then unique points $\widetilde{z}_i^\ell(\mathbf{u}, \mathbf{r}) = (\widetilde{s}_i^\ell(\mathbf{u}, \mathbf{r}), \widetilde{t}_i^\ell) \in Z_\ell(S)$ with the same t-coordinate as $\widetilde{z}_i^\ell$ so that

$$\mathcal{G}^n_{\mathbf{H}}(\mathbf{u}, \mathbf{r})|_{Z_\ell(S)}(\widetilde{z}_i^\ell) \in \widetilde{\mathbf{H}}_i^\ell$$

using the natural parametrization of $\mathcal{G}^n_{\mathbf{H}}(\mathbf{u}, \mathbf{r})|_{Z_\ell(S)}$. In particular, using the charts (6.1) on each $U_{\mathbf{H}^\ell}$, $\widetilde{U}_{\widetilde{\mathbf{H}}^\ell}$ given by the differences of successive s-coordinates of the marked points, the transition function φ is given by

$$\Pi_\ell U_{\mathbf{H}^\ell} \times [0, \varepsilon_1)^{n-1} \to \Pi_\ell \mathbb{R}^{d_\ell} \times [0, \varepsilon_2)^{n-1} \tag{6.32}$$

$$(\mathbf{u}, \mathbf{r}) \mapsto \left(\Pi_\ell (\widetilde{s}_i^\ell(\mathbf{u}, \mathbf{r}) - \widetilde{s}_{i-1}^\ell(\mathbf{u}, \mathbf{r}))_{i=1}^{d_\ell}, \widetilde{\mathbf{r}} \right) \tag{6.33}$$

where $\widetilde{r} = (\widetilde{r}_1, \ldots, \widetilde{r}_{n-1})$ is given in view of the construction of the gluing map and Proposition (6.8) by

$$\widetilde{r}_\ell = \psi^{-1} \left(\psi(r_\ell) + \frac{1}{4} (\widetilde{s}_0^{\ell+1}(\mathbf{u}, \mathbf{r}) - s_0^{\ell+1}(u^{\ell+1}) - \widetilde{s}_{d_\ell}^\ell(\mathbf{u}, \mathbf{r}) + s_{d_\ell}^\ell(u^\ell)) \right) \tag{6.34}$$

$$= \frac{r_\ell}{1 + \frac{r_\ell}{4} (\widetilde{s}_0^{\ell+1}(\mathbf{u}, \mathbf{r}) - s_0^{\ell+1}(u^{\ell+1}) - \widetilde{s}_{d_\ell}^\ell(\mathbf{u}, \mathbf{r}) + s_{d_\ell}^\ell(u^\ell))}. \tag{6.35}$$

It thus suffices to check that each $\widetilde{s}_i^\ell(\mathbf{u}, \mathbf{r})$ extends as a smooth function over $\mathbf{r} = (0, \ldots, 0)$; we claim that

$$\widetilde{s}_i^\ell(\mathbf{u}, \mathbf{r}) \to \widetilde{s}_i^\ell(u^\ell)$$

uniformly and exponentially with all derivatives as $\mathbf{r} \to 0$. However, this directly follows from the estimates (6.31) on the decay of the derivatives of the gluing map. Indeed, one can choose some smooth function $f : M \to \mathbb{R}$ whose zero set locally and transversely cuts out $\widetilde{H}_i^\ell$, and then take arbitrary derivatives of

$$f \left(\mathcal{G}^n_{\mathbf{H}}(\mathbf{u}, \mathbf{r})|_{Z_\ell(S)}(\widetilde{s}_i^\ell(\mathbf{u}, \mathbf{r}), \widetilde{t}_i^\ell) \right) = 0$$

in $\mathbf{u}$ and $\mathbf{r}$ to obtain the result. □

6.6. **Local coordinates for maps from surfaces.** We now turn to studying the moduli spaces $\mathcal{F}(\{x_\zeta\})$ maps from families of surfaces and their compactifications. As in our study of the moduli spaces of strips and cylinders, it is desirable to put local coordinates on the moduli space of geometric origin.

As earlier we assume that the parameter space of domain curves and perturbation data is either a closed manifold $\mathcal{R}$, or a manifold with boundary given by

$$\partial \mathcal{R} = \coprod_{\beta \in B} \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta.$$

The strata of the compactified moduli space are then index by weighted trees decorated by Floer data, which are either simple (have a single stable vertex) or β-shaped (have two stable vertices) for some $\beta \in B$. We will begin our study in the local neighbourhood of a point (p, u) in the top-dimensional (open, smooth) stratum of the moduli space, corresponding to the tree with a single vertex.

A potential source of coordinates comes from the forgetful map to the parameter space

$$U : \mathcal{F}(\{x_\zeta\}) \to \mathcal{R}; \tag{6.36}$$

however in general the derivative of this map is neither injective nor surjective

$$T_{(p,u)}\mathcal{F}(\{x_\zeta\}) = \ker(D^{\text{ext}}_{\mathcal{S}_p,u}) \to T_p\mathcal{R}. \tag{6.37}$$

Suppose for the moment that $p \in \mathcal{R}$ is in the collar neighbourhood of one of the boundary components, so it lies in the image of a gluing map

$$G^\beta : \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta \times (L, \infty] \longrightarrow \mathcal{R}.$$

The surface $\mathcal{S}_p$ can then be decomposed into a "thin" part, consisting of the image of all the ends, and either a strip $[-L_e, L_e] \times [0, 1]$ or a cylinder $[-L_e, L_e] \times S^1$ which is added during the gluing process (which we call the "neck"); and a "thick" part consisting of everything else.

Definition 6.13. *We will say that the neck is* transverse *for* (p, u) *if the composite of* (6.37) *with the inverse of the gluing map and the projection*

$$T_{(p,u)}\mathcal{F}(\{x_\zeta\}) \to T_p\mathcal{R} \xrightarrow{(dG^\beta)^{-1}} T\mathcal{R}_1^\beta \oplus T\mathcal{R}_2^\beta \oplus \mathbb{R} \to \mathbb{R} \tag{6.38}$$

is surjective. We will typically abuse notation slightly and label the neck by $\beta \in B$, and think of the indexing set B for the boundary components of $\partial\mathcal{R}$ as the set of necks.

A priori there is no reason for the neck β to be transverse for (p, u) even if p is in the corresponding collar neighbourhood; we will however later see that the gluing construction for maps is a natural source of them.

We now give two slightly different sources of natural coordinates on $\mathcal{F}$. Suppose that $\dim T_{(p,u)}\mathcal{F} = d$. First, in the direct generalization of the situation for strips and cylinders, choose d distinct marked points $\mathbf{z} = \{z_1, \ldots, z_d\} \subset \mathcal{S}_p$. Let us also choose a collection $\mathbf{H} = \{H_1, \ldots, H_d\}$ of small real codimension one hypersurfaces, with $u(z_i) \in H_i$.

Definition 6.14. *A collection of hypersurfaces* $\mathbf{H} = \{H_1, \ldots, H_d\}$ *is a* complete system of transverse hypersurfaces *for* (p, u) *through the marked points* $\mathbf{z} = (z_1, \ldots, z_d)$ *if:*

1. *each z_i lies on one of the ends of $\mathcal{S}_p$, and does not lie on the boundary;*
2. *in the coordinates (s, t) around z_i supplied by the strip-like or cylindrical end, we have $\partial_s u(z_i) \neq 0$, with $u^{-1}(H_i)$ transverse to the vector $\partial/\partial s$ at z_i;*
3. *the product of the evaluation at all the z_i and forgetful maps*

   $$ev_{\mathbf{z}} \times U : \mathcal{F}(\{x_\zeta\}) \to M^d$$

   is transverse to $H_1 \times \ldots \times H_d$ at (p, u).

Fix small enough discs around each $z_i \in \mathcal{S}_p$. Given a complete system of transverse hypersurfaces $\mathbf{H}$, we obtain a small neighbourhood $U_{\mathbf{H}} \subset \mathcal{F}(\{x_\zeta\})$ around (p,u), so that for each $v \in U_{\mathbf{H}}$, there are unique points $z_i(v) = (s_i(v), t_i)$ in these small discs around z_i, with the same t-coordinate at z_i, so that $v(z_i(v)) \in H_i$. In particular, by possibly shrinking $U_{\mathbf{H}}$, the map which records each $s_i(v)$

$$U_{\mathbf{H}} \to \mathbb{R}^d, \qquad v \mapsto (s_1(v), \dots, s_d(v)) \tag{6.39}$$

is a diffeomorphism onto its image, yielding distinguished charts for $\mathcal{F}(\{x_\zeta\})$ around (p,u). For each (p', u') in this local chart, the construction furnishes a collection of marked points $\mathbf{z}'$ close to the original $\mathbf{z}$, and after identifying the necks of $\mathcal{S}_{p'}$ with those of $\mathcal{S}_p$ we see that $\mathbf{H}$ is also a transverse system of necks and hypersurfaces for (p', u').

The second source of natural coordinates around (p,u) come from a combination of the neck lengths and a complete system of transverse hypersurfaces. Suppose that r is in the image of a collar neighbourhood of $\mathcal{R}^\beta \subset \mathcal{R}$, and choose additionally $d-1$ distinct marked points $\mathbf{z} = \{z_1, \dots, z_{d-1}\} \subset \mathcal{S}_p$, and a collection $\mathbf{H} = \{H_1, \dots, H_d\}$ of small real codimension one hypersurfaces, with $u(z_i) \in H_i$.

Definition 6.15. *The neck β and the hypersurfaces $\mathbf{H}$ are said to be a* complete system of necks and hypersurfaces *for (p,u) through the marked points $\mathbf{z}$ if:*

(1) each z_i is in the thin part of $\mathcal{S}_p$ (i.e. the ends of the neck), and does not lie on the boundary;
(2) in the coordinates (s,t) around z_i supplied by the gluing process, we have $\partial_s u(z_i) \neq 0$, with $u^{-1}(H_i)$ transverse to the vector $\partial/\partial s$ at z_i;
(3) the product of the evaluation at all the z_i and forgetful maps

$$ev_{\mathbf{z}} \times U : \mathcal{F}(\{x_\zeta\}) \to M^{d-1} \times \mathcal{R}$$

is transverse to $H_1 \times \ldots \times H_{d-k} \times G^\beta((L_0, \infty))$ at (p,u), where by $G^\beta((L_0,\infty))$ we mean the 1-dimensional submanifold of $\mathcal{R}$ through r obtained by fixing all the inputs of the gluing map G^β except the gluing length.

As before, with small enough discs around each z_i, we have a small neighbourhood $U_{\beta,\mathbf{H}} \subset \mathcal{F}(\{x_\zeta\})$ around (p,u), so that for each $v \in U_{\beta,\mathbf{H}}$, there are unique points $z_i(v) = (s_i(v), t_i)$ in these small discs around z_i, with the same t-coordinate at z_i, so that $v(z_i(v)) \in H_i$. In particular, by possibly shrinking $U_{\beta,\mathbf{H}}$, the map which records each $s_i(v)$ as well as the neck length L

$$U_{\beta,\mathbf{H}} \to \mathbb{R}^{d-1} \times \mathbb{R}, \qquad v \mapsto (s_1(v), \dots, s_{d-1}(v), L \tag{6.40}$$

is a diffeomorphism onto its image. For each (p', u') in this local chart, the construction furnishes a collection of marked points $\mathbf{z}'$ close to the original $\mathbf{z}$, as well as a complete system of transverse neck and hypersurfaces through this point.

As before in the case of gluing together strips and cylinders, we will also want to pick metrics on M. We will choose a $\mathcal{S}$-dependent metric $(g_{p,z})_{p \in \mathcal{R}, z \in \mathcal{S}_p}$ such that:

(i) it is equal to a fixed metric g_C for each component of the boundary $C \subset \partial\mathcal{S}_p$ which makes the corresponding Lagrangian L_C totally geodesic;
(ii) sufficiently far along each strip-like or tubular end (parametrized by (s,t)), it is equal to some fixed s-independent metric g_t;
(iii) in a small neighbourhood around each marked point z_i, the metric g_z is constant, so that the corresponding hypersurface H_i *totally geodesic.*

We will also need to choose a metric on $\mathcal{R}$, which we assume over the collar neighbourhoods of the boundary $\mathcal{R}_1^\beta \times \mathcal{R}_2^\beta \times (0,\infty]$ is a product metric; in particular the subspace obtained by holding the gluing length constant and varying $\mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$ is totally geodesic.

Finally, before we move on to discussing gluing itself, consider the Banach space $W_k^2(u^*TM)$ of class L_k^2 vector fields ξ along $\mathcal{S}_p$ with the appropriate Lagrangian boundary conditions. For some collection of marked points and hypersurfaces, there is a codimension $|\mathbf{z}| = |\mathbf{H}|$ subspace

$$W_{k,\mathbf{H}}^2(u^*TM) = \{\xi \in W_k^2(u^*TM) : \xi(z_i) \in TH_i \text{ for each } i\} \subset W_k^2(u^*TM) \tag{6.41}$$

of vector fields tangent to each H_i at the marked points. In the event that (p,u) admits a complete system of transverse hypersurfaces, as in the earlier case of strips and cylinders the direct sum

$$T_p\mathcal{R} \oplus W_{k,\mathbf{H}}^2(u^*TM) \subset T_p\mathcal{R} \oplus W_k^2(u^*TM)$$

is a complement of the kernel $\ker D_{\mathcal{S}_p,u}^{\text{ext}}$ of the extended linearized operator

$$D_{\mathcal{S}_p,u}^{\text{ext}} : T_p\mathcal{R} \oplus W_k^2(u^*TM) \to L_{k-1}^2(u^*TM).$$

Similarly, in the case whether there is a complete system of necks and transverse hypersurfaces, there is a codimension 1 subspace

$$T_{p,\beta} = T_p(\mathcal{R}^\beta \times \{L\}) \subset T_p\mathcal{R}$$

and the direct sum

$$T_{p,\beta} \oplus W_{k,\mathbf{H}}^2(u^*TM) \subset T_p\mathcal{R} \oplus W_k^2(u^*TM)$$

is a complement for the kernel $\ker D_{\mathcal{S}_p,u}^{\text{ext}}$ of the extended linearized operator. In either case, there is a unique bounded right inverse

$$Q_{p,u} : L_{k-1}^2(u^*TM) \to W_k^2(u^*TM) \tag{6.42}$$

of $D_{\mathcal{S}_p,u}^{\text{ext}}$ whose image is respectively $T_p\mathcal{R} \oplus W_{k,\mathbf{H}}^2(u^*TM)$ in the case with no transverse neck required, and $T_{p,\beta} \oplus W_{k,\mathbf{H}}^2(u^*TM)$ in the event of a transverse neck and hypersurface system.

6.7. **Gluing maps from surfaces.** We now take up the task of describing collar neighbourhoods of the boundary strata of the compactified moduli spaces of maps $\overline{\mathcal{F}}(\{x_\zeta\})$ from families of surfaces. To this end, let us fix a weighted tree $T \in \mathcal{T}^B(w)$ decorated by Floer generators $\{x_e\}$, and consider the corresponding boundary stratum

$$\mathcal{F}_T(\{x_e\}).$$

We will deal separately with the two cases where T is a simple tree (just one stable vertex), and where T is β-shaped (two stable vertices, connected by a "neck"). Let us first discuss the case where T is simple.

We work in the neighbourhood of a point $(p, \mathbf{u}) \in \mathcal{F}_T(\{x_e\})$: this is given by $p \in \mathcal{R}$, a map $(u : \mathcal{S}_p \to M) \in \mathcal{F}(\{x_e\})$, as well as a Floer trajectory for each unstable vertex $v \in V(T)$

$$(u^v : Z^v \to M) \in \mathcal{M}(x_{e^-}, x_{e^+})$$

where Z^v is understood to be the infinite strip or cylinder as appropriate.

Let us write d_v to be the dimension of the appropriate moduli space at the vertex v (stable or unstable). Fix, for each v unstable, a system of transverse hypersurfaces $\mathbf{H}^v$ through marked points $\mathbf{z}^v$ on the appropriate domain strip/cylinder, with families of metrics making them totally geodesic. At the stable vertex, choose either a system of transverse hypersurfaces $\mathbf{H}^v$ through d_v marked points $\mathbf{z}^v$ on the ends of the surface, or, supposing that p is in a collar neighbourhood of a boundary component $\mathcal{R}^\beta$ of $\mathcal{R}$, choose just $d_v - 1$ hypersurfaces $\mathbf{H}^v$ through marked points $\mathbf{z}^v$ in the "thin" part of the surface, such that $(\beta, \mathbf{H}^v)$ is a complete system of necks and hypersurfaces for (p, u^v).

We then allow $(p, \mathbf{u})$ to vary within the relatively compact open set

$$U = \prod_{v \text{ stable}} U_{\mathbf{H}^v} \times \prod_{v \text{ unstable}} U_{\mathbf{H}^v} \subset \mathcal{F}_T(\{x_e\})$$

in the case when no neck was used, or in the case when a system of transverse neck and hypersurfaces was chosen we take

$$U = \prod_{v \text{ stable}} U_{\beta, \mathbf{H}^v} \times \prod_{v \text{ unstable}} U_{\mathbf{H}^v} \subset \mathcal{F}_T(\{x_e\})$$

Writing $E = E(T)$ for the set of interior edges of the tree T, consider a collection of gluing lengths

$$\mathbf{T} = (T_e)_{e \in E} \in [T_0, \infty)^E$$

where we very much apologize for using T to denote both the tree and the gluing lengths, but we hope the difference will be clear in context.

Then, for $(p, \mathbf{u}, \mathbf{T}) \in U \times [T_0, \infty)^E$, we can construct the *preglued surface*

$$\text{pre}S(p, \mathbf{u}, \mathbf{T})$$

similarly to before: we cut off each end of each component surface at the s-coordinate of the furthest out marked point (in the event that, for the stable component, there is no marked point on a particular end, just cut off the entire end), and gluing back in strips/cylinders of length $4T_e$ for each edge $e \in E(T)$.

In this case, the preglued surface is canonically biholomorphic to $\mathcal{S}_p$ (since all we did was chop off and glue back on some strips/cylinders to the ends). It also automatically carries a collection of marked points $\mathbf{z}(p, \mathbf{u}, \mathbf{T})$. It also carries a distinguished collection of hypersurfaces $\mathbf{H}$ through these marked points, in the event that (p, u) had a complete system of necks and hypersurfaces, the preglued surface also has a distinguished "neck" region, and it always has a domain-dependent metric on M for which the hypersurfaces are totally geodesic around the marked points. By using a familiar procedure with cut-off functions, we also have a preglued map

$$\text{pre}G(\mathbf{p}, \mathbf{u}, \mathbf{T}) : \text{pre}S(\mathbf{p}, \mathbf{u}, \mathbf{T}) \to M$$

from which we can use Picard iteration to construct an actual solution to the equation.

Similarly, consider the case where T is a β-shaped tree. In this case, we have two stable vertices, v_1^β and $v_2\beta$. We work in a neighbourhood of a point $(p, \mathbf{u}) = (p_1, p_2, \mathbf{u}) \in \mathcal{F}_T(\{x_e\})$: this is given by two points $p_1 \in \mathcal{R}_1^\beta$ and $p_2 \in \mathcal{R}_2^\beta$, maps $u : (\mathcal{S}_i^\beta)_{p_i} \to M$ for $i = 1, 2$, as well as a Floer trajectory u^v for every unstable vertex $v \in V(T)$.

Choose, for each vertex $v \in V(T)$ (stable or unstable) a system of transverse hypersurfaces $\mathbf{H}^v$ through marked points $\mathbf{z}^v$, and allow $(p, \mathbf{u})$ to vary in some open subset where these hypersurfaces remain transverse. Given a collection $\mathbf{T} = (T_e)_{e \in E(T)} \in [T_0, \infty)^{E(T)}$ of gluing lengths, one can again construct the *preglued surface*

$$\text{pre}S(p, \mathbf{u}, \mathbf{T})$$

by cutting off each end at the s-coordinate of the furthest out marked point and gluing back in strips/cylinders of length $4T_e$ for each edge $e \in E(T)$. The resulting preglued surface inherits not only a system of hypersurfaces and marked points, but also carries a distinguished neck from the gluing process.

For ease of reading, let us in both the cases when T is simple and β-shaped write

$$S = \text{pre}S(p, \mathbf{u}, \mathbf{T}), \qquad w = \text{pre}G(p, \mathbf{u}, \mathbf{T}) : S \to M$$

to refer to the preglued surface and preglued map. Let us write $q \in \mathcal{R}$ for point in $\mathcal{R}$ representing S; the case that T is simple this is automatically $q = p$, but it may be different in the case that T is β-shaped (since the gluing parameters change the neck length).

We now turn again to Newton-Picard iteration to produce an actual solution close to the preglued map. We will not spell out the details, instead we will sketch the argument and highlight the minor differences to the earlier argument for gluing strips and cylinders.

We first note that, using the perturbation data (K, J) on S, so that Y is the Hamiltonian-vector-field-valued one-form associated to K, there is an a priori bound, exponential in $\min(\mathbf{T})$, on the L^2_{k-1} norm of

$$(Dw - Y)^{0,1} \in L^2_{k-1}(S, \Omega^{0,1}_S \otimes_{\mathbb{C}} w^*TM)$$

in view of the consistency assumptions on the perturbation data (this is slightly different from the earlier setting, where we could also conclude that $\bar{\partial} w$ was compactly supported in the gluing regions).

Second, we choose a local trivialization of $\mathcal{R}$ around S which makes the ends constant. Our given S-dependent metric on M, together with a metric on $\mathcal{R}$, determines a linearization of the map $w \mapsto (Dw - Y)^{0,1}$, an "extended linearized operator"

$$D_w^{\text{ext}} : T_q\mathcal{R} \oplus W_k^2(S, w^*TM) \to L_{k-1}^2(S, \Omega_S^{0,1} \otimes_{\mathbb{C}} w^*TM)$$

which satisfies a quadratic inequality analogous to that in Lemma 6.3.

Third, the choices of transverse necks and hyperplanes determine a specified finite codimension subspace of $T_q\mathcal{R} \oplus W_k^2(S, w^*TM)$, which is either of the form

$$\begin{cases} T_q\mathcal{R} \oplus W_{k,\mathbf{H}}^2(u^*TM) & \text{if } (q,w) \text{ has no distinguished neck;} \\ T_{q,\beta} \oplus W_{k,\mathbf{H}}^2(u^*TM) & \text{if } (q,w) \text{ has a distinguished neck } \beta. \end{cases}$$

In either case, this subspace is in fact a complement for $\ker(D_w^{\text{ext}})$, and the unique right inverse Q_w for D_w^{ext} with this as its image has operator norm uniformly bounded in $p, \mathbf{u}, \mathbf{T}$. This is again proved by constructing an approximate right inverse by gluing the relevant operators Q_u of (6.3) and Q_{p,u_v} of (6.42) for each unstable and stable vertex of T respectively.

Consequently for $\mathbf{T}$ sufficiently large, within some ball of size independent of $\mathbf{T}$, there is a unique pair

$$\begin{cases} \tau \in T_q\mathcal{R}, \xi \in W_{k,\mathbf{H}}^2(S, w^*TM) & \text{if no distinguished neck;} \\ \tau \in T_{q,\beta}\mathcal{R}, \xi \in W_{k,\mathbf{H}}^2(S, w^*TM) & \text{if there is a distinguished neck} \end{cases}$$

so that the map

$$\exp_w(\xi) : \mathcal{S}_{\exp_q(\tau)} \longrightarrow M$$

solves the equation $(Du - Y)^{0,1} = 0$.

In particular, for sufficiently large T_0, we can define the *gluing map*

$$G : U \times [T_0, \infty)^E \to \mathcal{F}(\{x_\zeta\}) \tag{6.43}$$

which sends $p, \mathbf{u}, \mathbf{T}$ to $(\exp_q(\tau), \exp_w(\xi))$ as above.

We then obtain a straight-forward generalization of Corollary 6.8 to this setting:

Proposition 6.16. *For sufficiently large T_0, the gluing map G is a smooth embedding onto some open neighbourhood W of $U \subset \partial\overline{\mathcal{F}}(\{x_\zeta\})$ in the Gromov-Floer topology. Moreover, the gluing map can be characterized by the following property: $G(p, \mathbf{u}, \mathbf{T})$ is the unique $(q, v) \in W$ for which:*

1. *if there is a distinguished neck, its length in $\mathcal{S}_q$ is equal to the length of the corresponding neck of* $\mathrm{pre}G(\mathbf{p}, \mathbf{u}, \mathbf{T})$*;*
2. *the image $u(z_i)$ of each marked point $z_i \in \mathbf{z}(\mathbf{p}, \mathbf{u}, \mathbf{T})$ lies in the corresponding hypersurface $H_i \in \mathbf{H}$.*

In particular, this implies an "associativity of gluing" property by the same argument as Proposition 6.9. We now give our main result for these moduli spaces of maps from surfaces, which is a straightforward generalization of the case of strips or cylinders. Choose a gluing profile

$$\psi : (0, \varepsilon] \to [T_0, \infty)$$

as we did before in (6.29), and set gluing parameters $\mathbf{r} = \psi^{-1}(\mathbf{T})$. Then, given either a complete system of transverse hypersurfaces $\mathbf{H}$ or a complete system of a distinguished neck and transverse hypersurfaces $\beta, \mathbf{H}$ for some small open set $U \subset \mathcal{F}_T(\{x_\zeta\})$, we set

$$\mathcal{G}(p, \mathbf{u}, \mathbf{r}) = G(p, \mathbf{u}, \mathbf{T})$$

which provided $\varepsilon > 0$ is chosen small enough extends to a continuous map

$$\mathcal{G} : U \times [0, \varepsilon)^E \to \mathcal{F}(\{x_\zeta\}) \to \overline{\mathcal{F}}(\{x_\zeta\}). \tag{6.44}$$

Theorem 6.17. *Considering now all the boundary strata indexed by extended trees, the maps $\mathcal{G}$ arising from choices of complete transverse systems hypersurfaces, possibly also with a distinguished neck, yield charts which endow $\overline{\mathcal{F}}(\{x_\zeta\})$ with the structure of a smooth manifold with corners.*

As before, there are two main steps of the proof. The first is to compare two gluing charts corresponding to different but nested strata $\mathcal{M}_{T_1} \subset \mathcal{M}_{T_2}$ which arise from the same selection of transverse necks and hypersurfaces; here the smoothness of the coordinate change is guaranteed by the geometric characterization of gluing in Proposition 6.16.

It then remains to compare two gluing charts $\mathcal{G}, \tilde{\mathcal{G}}$ on the same stratum $\mathcal{F}_T$, arising from different choices of necks and hypersurfaces. To do this, compare the transition map with respect to the two sets of charts (**??**) on $\mathcal{F}_T$ arising from the two choices of necks and surfaces. In other words, compare the relative positions of the marked points and neck lengths on *the same surface* which is the domain of $\mathcal{G}(p, \mathbf{u}, \mathbf{r}) = \tilde{\mathcal{G}}(\tilde{p}, \tilde{\mathbf{u}}, \tilde{\mathbf{r}})$ as $\mathbf{r} \to 0$. For the neck lengths, this comparison is tautologous since this data is intrinsic to $\mathcal{F}$.

For the marked points, consider a possible small extension of a body region $\mathcal{S}_p^{body}$ of one of the components of $\mathbf{u}$; this can also be considered a subset of $\mathcal{G}(p, \mathbf{u}, \mathbf{r})$ and thus of $\tilde{\mathcal{G}}(\tilde{p}, \tilde{\mathbf{u}}, \tilde{\mathbf{r}})$ as well; any one of the marked points $\tilde{\mathbf{z}}(\tilde{p}, \tilde{u}, \tilde{r})$ lying in this region then defines a point $\tilde{z}(\mathbf{u}, \mathbf{r})$ of $\mathcal{S}_u^{bulk}$. It then suffices to show that $\tilde{z}(\mathbf{u}, \mathbf{r}) \to \tilde{z}$ uniformly and exponentially with all derivatives as $\mathbf{r} \to 0$, where $\tilde{z}$ is corresponding the original marked point used to construct the chart $\tilde{\mathcal{G}}$. The crucial input which then concludes the proof is an estimate on the exponential decay of the derivatives of the gluing map in the bulk region, as in Proposition 6.10; we will leave the details to the interested reader.

7. The tangent bundles of the moduli spaces

Having shown that our compactified moduli spaces are smooth manifolds with corners, we now study their tangent bundles, and how these decompose at the boundary. Let us outline our concerns in a toy example: consider a codimension one stratum

$$\mathcal{M}_1 \times \mathcal{M}_2 = \mathcal{M}(x_-, \tilde{x}) \times \mathcal{M}(\tilde{x}, x_+) \subset \partial\overline{\mathcal{M}}(x_-, x_+) = \partial\mathcal{M}$$

of a moduli space of Floer strips or cylinders. From the manifold with boundary structure, we obtain a decomposition

$$T\mathcal{M}_1 \oplus T\mathcal{M}_2 \oplus \mathbb{R} \cong T\mathcal{M}|_{\partial\mathcal{M}} \tag{7.1}$$

after a choice of normal vector field. Each of the bundles $T\mathcal{M}_1, T\mathcal{M}_2, T\mathcal{M}$ are in turn computed by the kernels of the appropriate linearized operators:

$$T_{u^1}\mathcal{M}_1 \oplus \mathbb{R} \cong \ker(D_{u^1}), \qquad T_{u^2}\mathcal{M}_2 \oplus \mathbb{R} \cong \ker(D_{u^2}), \qquad T_u\mathcal{M} \oplus \mathbb{R} \cong \ker(D_u) \tag{7.2}$$

where the $\mathbb{R}$ factors arise from the translational degree of freedom (we suppress the choice of slice for the $\mathbb{R}$-action). Finally, if $u = G(u^1, u^2, T) = u^1 \#_T u^2$ is a gluing of u^1, u^2 with a sufficiently large gluing length T, the gluing property of the Fredholm index yields an isomorphism

$$\ker(D_{u^1 \#_T u^2}) \cong \ker(D_{u^1}) \oplus \ker(D_{u^2}). \tag{7.3}$$

The central problem we resolve in this section is the compatibility of these three isomorphisms (7.1), (7.2), (7.3).

The first point of order concerns the particular fashion in which we equipped our moduli spaces with smooth structures— we chose a gluing profile ψ to reparametrize our moduli spaces, and so we must ensure that in the appropriate sense the isomorphism (7.1) is unaffected by this procedure. We then turn to more generally addressing the problem for moduli spaces of strips and cylinders.

We will then turn to the moduli spaces $\overline{\mathcal{F}}$ of surfaces, where there is an additional challenge: recall that the tangent spaces are computed as

$$T_{(p,u)}\mathcal{F} = \ker(D^{\mathrm{ext}}_{\mathcal{S}_p,u})$$

where $D^{\mathrm{ext}}_{\mathcal{S}_p,u}$ is the *extended* linearized operator, taking into account variations of the conformal structure within the family $\mathcal{R}$. However, the analogue of the index gluing property (7.3) holds for the *pointwise* linearized operator $D_{\mathcal{S}_p,u}$ for which the Riemann surface $\mathcal{S}_p$ is taken to be fixed. In particular, we must then mediate the decomposition of the tangent bundle (7.1) with an identification

$$\ker(D^{\mathrm{ext}}_{\mathcal{S}_p,u}) \cong T_p\mathcal{R} \oplus \mathrm{ind}(D_{\mathcal{S}_p,u}) \tag{7.4}$$

of virtual vector bundles, where $\mathrm{ind}(D_{\mathcal{S}_p,u})$ is the index bundle of the pointwise linearized operator.

7.1. **The smooth and topological tangent bundles.** Our first concern is to control the behaviour of the tangent bundles $T\overline{\mathcal{M}}$ under the reparametrization procedure used to produce smooth structures in the previous section.

Suppose that we have a broken family of surfaces indexed by an weighted tree T decorated by Floer data; we can include in this analysis the case when T is just a string of unstable vertices so as to treat gluing for strips/cylinders as well as more general surfaces uniformly. Write E for the set of interior edges of T. Recall that we write $\mathcal{M}_T$ for the moduli space of maps from these broken surfaces; we will simply write $\mathcal{M}$ for the unbroken moduli space which we are gluing on to, $\overline{\mathcal{M}}$ for its compactification, and $\partial\mathcal{M}$ for the union of all its boundary strata.

Recall that in the previous section, we defined gluing maps

$$G : U \times (T_0, \infty]^E \to \overline{\mathcal{M}}$$

for small enough open sets $U \subset \mathcal{M}_T$ and very large T_0. These charts are smooth diffeomorphisms onto their image over $U \times (T_0, \infty)^E$. Suppose we then have two such charts, G_1 and G_2, defined using different choices of necks and hypersurfaces; after shrinking we assume they have the same domain $U \times (T_0, \infty]^E$. Consider the transition functions as all the gluing lengths $\mathbf{T} \in (T_0, \infty]^E$ are sent to infinity: the exponential decay of the gluing map implies that this transition function converges exponentially fast to

$$(\mathbf{u}, \mathbf{T}) \mapsto (\mathbf{u}, \mathbf{T} + \mathbf{f}(\mathbf{u})) \tag{7.5}$$

for some function $\mathbf{f} : U \to \mathbb{R}^E$ (we are suppressing the $\mathbf{p}$ coordinates from the notation, which parametrize the domain curves). In particular, under this change of coordinates, the transition map for the tangent space $T\mathcal{M}$ continuously extends over $U \times \big((T_0, \infty)^E \cup \{\infty\}^E\big)$, and over $U \times \{\infty\}^E$ is given by

$$\begin{pmatrix} \mathrm{id} & 0 \\ D\mathbf{f} & \mathrm{id} \end{pmatrix} : T\mathcal{M}_T \oplus \mathbb{R}^E \to T\mathcal{M}_T \oplus \mathbb{R}^E. \tag{7.6}$$

This defines canonical extensions of $T\mathcal{M}$, as a topological vector bundle, from the open stratum $\mathcal{M}$ to $\mathcal{M} \cup \mathcal{M}_T$ for each T. The same argument shows that whenever $\mathcal{M}_{T_2} \subset \overline{\mathcal{M}}_{T_1}$ is a nested pair of strata, there is a canonical extension of the tangent bundle $T\mathcal{M}_{T_1}$ from $\mathcal{M}_{T_1}$ to $\mathcal{M}_{T_1} \cup \mathcal{M}_{T_2}$. The associativity of the gluing maps associated to the same system of necks and hypersurfaces then allows us to fit all these extensions together, as follows.

Proposition 7.1. *Over each closed stratum $\overline{\mathcal{M}}_T$, we have a topological vector bundle called the* topological tangent bundle $T^{top}\overline{\mathcal{M}}_T$ *of the stratum. These have the properties:*

(1) On the interior $\mathcal{M}_T \subset \overline{\mathcal{M}}_T$ of each stratum, there is a canonical identification $T^{top}\overline{\mathcal{M}}_T|_{\mathcal{M}_T} \cong T\mathcal{M}_T$ with the smooth tangent bundle of $\mathcal{M}_T$.

(2) Under the product decomposition $\overline{\mathcal{M}}_T = \prod_{v \in V(T)} \overline{\mathcal{M}}_v$ (where $\overline{\mathcal{M}}_v$ is the appropriate compactified space of maps from the surface corresponding to a vertex v, with the appropriate Floer generators as asymptotics), we have

$$T^{top}\overline{\mathcal{M}}_T = \bigoplus_{v \in V(T)} T^{top}\overline{\mathcal{M}}_v.$$

(3) Whenever $\overline{\mathcal{M}}_{T_2} \subset \overline{\mathcal{M}}_{T_1}$ are a nested pair of strata, so that T_1 (with interior edges E_1) is obtained by contracting T_2 (with interior edges E_2) at the edges $E_2 \backslash E_1$, there is a canonical short exact sequence of vector bundles

$$0 \to \mathbb{R}^{E_2 \backslash E_1} \to T^{top}\overline{\mathcal{M}}_{T_1}|_{\overline{\mathcal{M}}_{T_2}} \to T^{top}\overline{\mathcal{M}}_{T_2} \to 0 \tag{7.7}$$

(4) *these short exact sequences are compatible under passing to deeper strata, in the following sense. Suppose that $\overline{\mathcal{M}}_{T_3} \subset \overline{\mathcal{M}}_{T_2} \subset \overline{\mathcal{M}}_{T_1}$ is a nested sequence of strata, so that T_1, T_2, T_3 are related by contraction, with edges E_1, E_2, E_3, then we have a commutative diagram of vector bundles* after restricting to $\mathcal{M}_{T_3}$*:*

$$\begin{array}{ccccc} & & & & \mathbb{R}^{E_3\setminus E_2} \\ & & & & \downarrow \\ \mathbb{R}^{E_2\setminus E_1} & \longrightarrow & T^{top}\overline{\mathcal{M}}_{T_1} & \longrightarrow & T^{top}\overline{\mathcal{M}}_{T_2} \\ \downarrow & & \| & & \downarrow \\ \mathbb{R}^{E_3\setminus E_1} & \longrightarrow & T^{top}\overline{\mathcal{M}}_{T_1} & \longrightarrow & T^{top}\overline{\mathcal{M}}_{T_3} \\ \downarrow & & & & \\ \mathbb{R}^{E_3\setminus E_2} & & & & \end{array} \tag{7.8}$$

where all the rows and columns are exact; the two rows and the right column are the canonical exact sequences of (7.7)*, and the left column is the standard presentation* $\mathbb{R}^{E_3\setminus E_1} \cong \mathbb{R}^{E_2\setminus E_1} \oplus \mathbb{R}^{E_3\setminus E_2}$.

The short exact sequences come immediately from the lower triangular form of the transition matrices in (7.6); their compatibility can be checked on any system of gluing maps arising from the same necks and hypersurfaces, and is a direct consequence of the associativity of gluing.

Let us now compare to the usual tangent bundle of the *smooth manifold with corners* $\overline{\mathcal{M}}$, which we denote as $T^{sm}\overline{\mathcal{M}}$. We consider again for some small open set $U \subset \mathcal{M}_T$ a pair of gluing charts

$$\mathcal{G}_1, \mathcal{G}_2 : U \times [0,\varepsilon)^E \to \overline{\mathcal{M}}$$

arising from different systems of necks and hypersurfaces. Again, we see that as the gluing parameters $\mathbf{r} \to 0$, the transition function converges to

$$(\mathbf{u}, \mathbf{r}) \mapsto \big(\mathbf{u}, \psi^{-1}(\psi(\mathbf{r}) + \mathbf{f}(\mathbf{u}))\big)$$

for the same function $\mathbf{f} : U \to \mathbb{R}^E$ as before. The difference now is that due to the gluing profile ψ, the transition map for the tangent space $T\mathcal{M}$ extends smoothly over $U \times [0,\varepsilon)^E$, and over $U \times \{0\}^E$ is given by the matrix

$$\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} : T\mathcal{M}_T \oplus \mathbb{R}^E \to T\mathcal{M}_T \oplus \mathbb{R}^E. \tag{7.9}$$

In particular, the smooth tangent bundle has a canonical, coordinate independent decomposition over each boundary stratum

$$T^{sm}\mathcal{M}|_{\overline{\mathcal{M}}_T} \cong T^{sm}\overline{\mathcal{M}}_T \oplus \mathbb{R}^E. \tag{7.10}$$

In other words: the parameter rescaling procedure used to produce the smooth structure picks out distinguished normal directions around each boundary stratum. The decompositions (7.10) are moreover compatible with passing to deeper strata, as can be checked by using a family of gluing charts from the same necks and hypersurfaces.

Proposition 7.2. *There exist continuous isomorphisms between the smooth and topological tangent bundles*

$$\phi_T : T^{sm}\overline{\mathcal{M}}_T \xrightarrow{\cong} T^{top}\overline{\mathcal{M}}_T$$

over each stratum, compatible with the relevant structures in the following sense:

1. *away from some small neighbourhood of the boundary $\partial\mathcal{M}_T$, the ϕ_T is given by the identifications $T^{sm}\overline{\mathcal{M}}_T = \mathcal{M}_T \cong T^{top}\overline{\mathcal{M}}_T$;*
2. *under the decomposition $\overline{\mathcal{M}}_T = \prod_{v\in V(T)} \overline{\mathcal{M}}_v$ and the resulting direct sum decomposition of $T^{sm}\overline{\mathcal{M}}_T$ and $T^{top}\overline{\mathcal{M}}_T$, the isomorphism ϕ_T is the direct sum of the isomorphisms ϕ_v for each vertex $v \in V(T)$;*
3. *if $\overline{\mathcal{M}}_{T_2} \subset \overline{\mathcal{M}}_{T_1}$ are a nested pair of strata, so that T_1 (with interior edges E_1) is obtained by contracting T_2 (with interior edges E_2) at the edges $E_2\backslash E_1$, then we have a commutative diagram*

$$\begin{array}{ccccc} \mathbb{R}^{E_2\backslash E_1} & \longrightarrow & T^{sm}\overline{\mathcal{M}}_{T_1}|_{\overline{\mathcal{M}}_{T_2}} & \longrightarrow & T^{sm}\overline{\mathcal{M}}_{T_2} \\ \| & & \downarrow{\scriptstyle \phi_{T_1}} & & \downarrow{\scriptstyle \phi_{T_2}} \\ \mathbb{R}^{E_2\backslash E_1} & \longrightarrow & T^{top}\overline{\mathcal{M}}_{T_1}|_{\overline{\mathcal{M}}_{T_2}} & \longrightarrow & T^{top}\overline{\mathcal{M}}_{T_2} \end{array} \tag{7.11}$$

   where the top row is the short exact sequence coming from the decomposition (7.10)*, and the bottom row is the short exact sequence* (7.7)*.*

The proof goes by constructing such a system of isomorphisms by induction over the dimension of the strata. In view of conditions (1) and (2), we are reduced to the case of defining $\phi : T^{sm}\overline{\mathcal{M}} \xrightarrow{\cong} T^{top}\overline{\mathcal{M}}$ over the top dimensional stratum. The key point here is that having already chosen ϕ_T over some boundary stratum $\overline{\mathcal{M}}_T$, at each point $\mathbf{u}$ on this boundary stratum there is a contractible space of possible choices of $\phi|_{\mathbf{u}}$ satisfying condition (3). This thus defines ϕ in a neighbourhood of $\partial\mathcal{M}$. Moreover, it is easy to see that over the image of a gluing chart $\mathcal{G} : U \subset (0,1)^E \to \mathcal{M}$, the natural identification $T^{sm}\overline{\mathcal{M}} = T\mathcal{M} \cong T^{top}\overline{\mathcal{M}}$ almost satisfies (7.11), except that instead of the identity map $\mathrm{id} : \mathbb{R}^{E_2\backslash E_1} \to \mathbb{R}^{E_2\backslash E_1}$, we have multiplication by a constant scalar (in fact by $-1/\psi'(r)$). However, the space of ϕ satisfying the analogous condition to (7.11) where the left-most vertical arrow is instead a positive real scalar is still contractible: we can thus deform this isomorphism so that it agrees with our already constructed map at the boundary.

Having proven that the structures of $T^{sm}\overline{\mathcal{M}}$ and $T^{top}\overline{\mathcal{M}}$ are essentially equivalent (and by the above proof, equivalent up to a contractible space of choices), we will cease to make any distinction between them, and drop the overline from the notation, and write $T\mathcal{M}$ for either one.

7.2. **The kernel of the linearized operator on strips and cylinders.** For this section, we consider a compactified moduli space $\overline{\mathcal{M}}(x_-, x_+)$ of Floer strips or cylinders with limits at Floer generators x_-, x_+. However to keep notation consistent between this and the sequel, we will still index the boundary strata $\mathcal{M}_T$ by weighted chains with Floer data, with one vertex for each component of the broken surface (of course, all these vertices are considered unstable). The open stratum $\mathcal{M}$ corresponds to the chain with just one vertex.

At each $\mathbf{u} = (u^v)_{v \in V(T)}$, we have a vector space

$$\ker(D_{\mathbf{u}}) := \bigoplus_{v \in V(T)} \ker(D_{u^v}) \tag{7.12}$$

where $D_{u^v} : W^2_k((u^v)^*TM) \to L^2_{k-1}((u^v)^*TM)$ is the associated linearized operator. By regularity, over each stratum $\mathcal{M}_T$, these fit into a vector bundle $\ker(D_{\mathbf{u}})$: this is precisely the *index bundle* of the corresponding family of operators. Our first task is to fit these together into a vector bundle over the whole compactification $\overline{\mathcal{M}}$.

First, consider a system of transverse hypersurfaces $\mathbf{H}$ through some marked points $\mathbf{z}(u)$ for some u in a small neighbourhood $U \subset \mathcal{M}$. Then take the projections

$$\pi_i : TH_i \to N_i = N_{H_i \subset M}$$

to the normal bundles of H_i; by evaluating a vector field $\xi \in \ker(D_u)$ at the marked points z_i and then projecting to N_i, we obtain an isomorphism

$$(\pi_i \circ \mathrm{ev}_{z_i})_i : \ker(D_u) \quad \xrightarrow{\sim} \quad \bigoplus_i N_i|_{u(z_i)} \tag{7.13}$$

of vector bundles over U. In particular, if we choose trivializations of each normal bundle $N_i \cong H_i \times \mathbb{R}$, we obtain a local trivialization of $\ker(D)$ over U.

Now, suppose that $U \subset \mathcal{M}_T$, and we use this system of transverse hypersurfaces $\mathbf{H}$ to produce a gluing map $G_T : U \times (0, \infty)^E \to \mathcal{M}$. Writing $d_v + 1 = \dim \ker(D_{u^v})$ and $d + 1 = \sum_v (d_v + 1)$, the above construction, applied to the broken curves $\mathbf{u} \in U$, as well as the glued curves $G_T(\mathbf{u}, \mathbf{T})$, yields:

(i) a trivialization over U of $\ker(D_{\mathbf{u}})$:

$$\ker(D_{\mathbf{u}}) = \bigoplus_v \ker(D_{u^v}) \cong \left(\bigoplus_v \mathbb{R}^{d_v+1} \right) \times U = \mathbb{R}^{d+1} \times U;$$

(ii) a trivialization over $U \times (0, \infty)^E$ of $G_T^* \ker(D)$.

Now, suppose we have two different systems of transverse hypersurfaces $\mathbf{H}_1, \mathbf{H}_2$ over $U \subset \mathcal{M}_T$; we can shrink U to assume it is of the form $U = \prod_{v \in V(T)} U^v$. Fix trivializations of the normal bundles of these hypersurfaces in M, and thus two trivializations of each vector bundle $\ker(D_{u^v})$. We thus obtain, for each vertex v of T, a family of transition matrices

$$\chi^v_{12} : \mathbb{R}^{d_v+1} \times U^v \to \mathbb{R}^{d_v+1} \times U^v$$

and thus a transition map $\chi_{12}^T : \mathbb{R}^{d+1} \times U \to \mathbb{R}^{d+1} \times U$ relating the two trivializations of $\ker(D_{\mathbf{u}})$ over U.

Let $G_1, G_2 : U \times (0,\infty)^E \to \mathcal{M}$ be the two associated gluing maps. Comparing the trivializations of $G_1^* \ker(D)$ and $G_2^* \ker(D)$, we then have a family of transition matrices χ_{12}

(7.14)

$$\begin{array}{ccccccc} G_1^* \ker(D) & \xrightarrow{\cong} & \mathbb{R}^{d+1} \times U \times (0,\infty)^E & \xrightarrow{\chi_{12}} & \mathbb{R}^{d+1} \times U \times (0,\infty)^E & \xleftarrow[\cong]{} & G_2^* \ker(D) \\ & & \downarrow & & \downarrow & & \\ & & U \times (0,\infty)^E & \xrightarrow{\phi_{12}} & U \times (0,\infty)^E & & \end{array}$$

covering the transition map ϕ_{12} between the charts G_1, G_2. Now, send $\min(\mathbf{T}) \to \infty$; the transition map ϕ_{12} extends continuously

$$U \times \left((0,\infty)^E \cup \{\infty\}^E\right) \to U \times \left((0,\infty)^E \cup \{\infty\}^E\right)$$

and is the identity on $U \times \{\infty\}^E$. Furthermore, the exponential decay of the gluing maps of Proposition 6.10 implies that χ_{12} also extends continuously to $U \times \left((0,\infty)^E \cup \{\infty\}^E\right)$, and over $U \times \{\infty\}^E$ is exactly equal to χ_{12}^T.

We conclude that the vector bundle $\ker(D)$ over $\mathcal{M}$ continuously extends over $\mathcal{M} \cup \mathcal{M}_T$, with a natural identification over $\mathcal{M}_T$ with $\ker(D_{\mathbf{u}})$. We can do this over every stratum simultaneously, and again appeal to the associativity of gluing to show that these extensions are compatible over inclusions of strata $\mathcal{M}_{T_2} \subset \overline{\mathcal{M}_{T_1}}$. To summarize:

Proposition 7.3. *Over each closed stratum $\overline{\mathcal{M}}_T$, there is a (topological) vector bundle $\ker(D_T)$, so that:*

1. *over the interior $\mathcal{M}_T$ of the stratum, there is an identification $\ker(D_T)|_{\mathcal{M}_T} = \ker(D_{\mathbf{u}})$ the vector bundle given by the kernels of the linearized operators;*
2. *under $\overline{\mathcal{M}}_T = \prod_{v \in V(T)} \overline{\mathcal{M}}_v$, we have identifications of $\ker(D_T)$ with $\bigoplus_v \ker(D_v)$, compatible with the identifications of (1), where $\ker(D_v)$ is corresponding vector bundle over $\overline{\mathcal{M}}_v$;*
3. *whenever $\overline{\mathcal{M}}_{T_2} \subset \overline{\mathcal{M}}_{T_1}$ is an inclusion of strata, we have an isomorphism $\ker(D_{T_1})|_{\overline{\mathcal{M}}_{T_2}} \cong \ker(D_{T_2})$; these isomorphisms are coherent under passing between deeper strata.*

We will usually just notate this vector bundle over the entire compactified moduli space $\overline{\mathcal{M}}$ as $\ker(D)$, and call it the *index bundle*.

We are now tasked with relating the index bundle and the tangent bundle. The basic relation is that over the top-dimensional stratum $\mathcal{M}$, there is a canonical short exact sequence

$$0 \to \mathbb{R} \to \ker(D) \to T\mathcal{M} \to 0 \tag{7.15}$$

where the trivial $\mathbb{R}$ factor of $\ker(D)$ is spanned by $\tau = \partial_s u$, the derivative of the translation action of $\mathbb{R}$ on the space $W(x_-, x_+)$ of parametrized solutions.

We must now understand how these exact sequences fit together along the compactification $\overline{\mathcal{M}}$; we will produce our results in the category of continuous vector bundles, using the topological tangent bundle described earlier as our model of $T\mathcal{M}$.

Consider a stratum $\mathcal{M}_T$; write V for the vertices and E for the interior edges of T, so that $|V| = |E| + 1$. Over $\mathcal{M}_T$, there is a canonical short exact sequence

$$0 \to \mathbb{R}^V \to \ker(D_T) \to T\mathcal{M}_T \to 0$$

obtained as the direct sum of each sequence (7.15) over each vertex. Now, consider a gluing map $G_T : U \times (0, \infty)^E \to \mathcal{M}$ associated to a transverse system of hypersurfaces $\mathbf{H}$, and consider $\tau = \partial_s u$ as a section of $G_T^* \ker(D)$ over $U \times (0, \infty)^E$. The exponential decay of the gluing map directly implies that this section continuously extends over $U \times \{\infty\}^E$, and on this fibre is given by $(\partial_s u^v)_{v \in V} \in \oplus_v \ker(D_{u^v}) = \ker(D_T)$. In particular, we can immediately verify in these local coordinates:

Proposition 7.4. *There is a commutative diagram of vector bundles over $\mathcal{M}_T$*

$$\begin{array}{ccccc} & & & & \mathbb{R}^E \\ & & & & \downarrow \\ \mathbb{R} & \longrightarrow & \ker(D)|_{\mathcal{M}_T} & \longrightarrow & T\mathcal{M}|_{\mathcal{M}_T} \\ \downarrow & & \downarrow \cong & & \downarrow \\ \mathbb{R}^V & \longrightarrow & \ker(D_T) & \longrightarrow & T\mathcal{M}_T \end{array} \tag{7.16}$$

where each row is the exact sequence of (7.15)*; the right column is the exact sequence* (7.7)*; the middle vertical arrow is the isomorphism of Proposition 7.3, and the left vertical arrow is the diagonal map $s \mapsto (s, \dots, s)$.*

In particular, there is a canonical identification of $\mathbb{R}^E$ with $\mathbb{R}^V/\mathbb{R}$; it is not hard to see that this is given by the map $\mathbb{R}^V \to \mathbb{R}^E$ which sends a tuple $(s_v)_{v \in V}$ to $(s_{v_+(e)} - s_{v_-(e)})_{e \in E}$, where $v_-(e), v_+(e)$ are the incoming and outgoing vertices of an edge e respectively.

Likewise, if $\overline{\mathcal{M}}_{T_2} \subset \overline{\mathcal{M}}_{T_1}$ is an inclusion of strata, we have a similar commutative diagram of vector bundles over $\mathcal{M}_{T_2}$

$$\begin{array}{ccccc} & & & & \mathbb{R}^{E_2 \setminus E_1} \\ & & & & \downarrow \\ \mathbb{R}^{V_1} & \longrightarrow & \ker(D_{T_1})|_{\mathcal{M}_{T_2}} & \longrightarrow & T\mathcal{M}_{T_1}|_{\mathcal{M}_{T_2}} \\ \downarrow & & \downarrow \cong & & \downarrow \\ \mathbb{R}^{V_2} & \longrightarrow & \ker(D_{T_2}) & \longrightarrow & T\mathcal{M}_{T_2} \end{array} \tag{7.17}$$

where the map $\mathbb{R}^{V_1} \to \mathbb{R}^{V_2}$ sends $(s_v)_{v \in V_1}$ to $(s'_w)_{w \in V_2}$ where s'_w is exactly s_v for the unique vertex of T_1 which w is associated to under the contraction of T_2 to T_1.

This is the precise sense, for moduli spaces of strips and cylinders, in which the isomorphisms of (7.1), (7.2), (7.3) are compatible.

7.3. **The kernel of the extended linearized operator.** We now wish to produce a similar result for the compactified moduli spaces $\overline{\mathcal{F}}$ of maps from families of surfaces. Let us fix a family of open-closed pointed surfaces $\mathcal{S} \to \mathcal{R}$, where either $\mathcal{R}$ is a closed manifold, or $\mathcal{R}$ has boundary components $\mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$ for $\beta \in B$ and the family degenerates to nodal surfaces along this boundary. Suppose that this family of surfaces carries Lagrangian labels, compatible Floer data, and perturbation data. Let us also fix Floer generators $\{x_\zeta\}$; then the moduli space of maps from these surfaces

$$\mathcal{F}(\{x_\zeta\})$$

admits a compactification with strata $\mathcal{F}_T$ indexed by weighted trees, either simple or β-shaped, decorated with Floer generators. Each stratum in turn decomposes as a product $\mathcal{F}_T = \prod_v \mathcal{M}_v$ over moduli spaces of maps from the individual component domain curves, indexed by the vertices of v. Write $V^{st}(T)$ and $V^{un}(T)$ for the stable and unstable vertices of T respectively. At $\mathbf{u} \in \mathcal{F}_T$, for $v \in V^{st}(T)$ recall we have an identification of the moduli space $\mathcal{F}_v$ of such u^v with the kernel of the extended linearized operator of (**??**):

$$T\mathcal{F}_v = \ker(D_{u^v}^{\text{ext}});$$

likewise at $v \in V^{un}$ an unstable vertex we have the exact sequence of (7.15) relating $T\mathcal{M}_v$ and the kernel of the usual linearized operator $\ker(D_{u^v})$. We will write

$$\ker(D_T^{\text{ext}})|_{\mathbf{u}} = \bigoplus_{v \in V^{st}(T)} \ker(D_{u^v}^{\text{ext}}) \quad \oplus \bigoplus_{v \in V^{un}(T)} \ker(D_{u^v})$$

for the resulting vector bundle formed over $\mathcal{F}_T$; we then have a canonical exact sequence of vector bundles over $\mathcal{M}_T$

$$0 \to \mathbb{R}^{V^{un}} \to \ker(D_T^{\text{ext}}) \to T\mathcal{F}_T \to 0. \tag{7.18}$$

We first describe how the vector bundles $\ker(D_T^{\text{ext}})$ fit together under the inclusion of strata; since we can decompose each $\mathcal{F}_T = \prod_v \mathcal{M}_v$, it suffices to describe how the vector bundle $\ker(D^{\text{ext}})$ on the open stratum $\mathcal{F}$ relates to $\ker(D_T^{\text{ext}})$ on $\mathcal{F}_T \subset \overline{\mathcal{F}}$.

To do this, we will again use systems of transverse hypersurfaces and possibly a neck as well. First, consider such a system $\mathbf{H}$ or $(\beta, \mathbf{H})$ for some marked points $\mathbf{z}(u)$ for some $u = (p, u)$ in a small neighbourhood $U \subset \mathcal{M}$. Recall that an element of $\ker(D_u^{\text{ext}})$ is a pair (ρ, ξ) where $\rho \in T_p\mathcal{R}$ and a vector field $\xi \in C^\infty(\mathcal{S}_p, u^*TM)$. Again, take the projections $\pi_i : TH_i \to N_i = N_{H_i \subset M}$; evaluating the vector field $\xi \in \ker(D^{\text{ext}})$ at $u(z_i)$ and projecting along π_i we obtain an element of N_i. Similarly, in the case where there is a distinguished neck, p lies in a collar neighbourhood $\mathcal{R}_1^\beta \times \mathcal{R}_2^\beta \times (L_0, \infty] \subset \mathcal{R}$, so we can project ρ to the $\mathbb{R}$ factor. By assumption, we then have an isomorphism over vector spaces

$$\ker(D_u^{\text{ext}}) \cong \begin{cases} \bigoplus N_i|_{u(z_i)} & \text{in the case with no distinguished neck;} \\ \mathbb{R} \oplus \bigoplus N_i|_{u(z_i)} & \text{in the presence of a distinguished neck.} \end{cases}$$

In particular, a choice of trivializations of each normal bundle $N_i \cong H_i \times \mathbb{R}$ leads to a trivialization of $\ker(D^{\text{ext}})$ over U.

Now suppose that $U \subset \mathcal{F}_T$ is small enough, and that for each vertex v of the weighted tree T we have chosen a system of hypersurfaces $\mathbf{H}^v$ and possibly also a neck. This then produces a gluing map $G_T : U \times (0,\infty)^E \to \mathcal{M}$ where E is the set of interior edges of T, so that on the image of the gluing map, we have a transverse system formed by the hypersurfaces $\mathbf{H} = \bigcup_v \mathbf{H}^v$, any distinguished necks of the component curves, as well as in the case that T is β-shaped the new neck formed by the gluing process. We will write E^{st} for the set of these new necks, since they correspond to the edges of the "underlying stable tree" of T obtained by deleting all the unstable vertices. Of course, we must E^{st} is either empty or a singleton, depending on whether T is a simple tree or β-shaped for some β.

In particular applying the above both to the broken curves $\mathbf{u} \in U$ and to the glued curves $G_T(\mathbf{u}, \mathbf{T})$ we have:

(i) a trivialization over U of $\ker(D_T^{\text{ext}})$;
(ii) a trivialization over $U \times (0,\infty)^E$ of $G_T^* \ker(D^{\text{ext}})$.

Suppose again we have two different systems of transverse necks and hypersurfaces over $U \subset \mathcal{F}_T$ as above, and assume U is of the form $U = \prod_v U^v$. Over each $v \in V(T)$ we have a family of transition matrices

$$\xi_{12}^v : \mathbb{R}^{\dim \ker(D_v^{\text{ext}})} \times U^v \to \mathbb{R}^{\dim \ker(D_v^{\text{ext}})} \times U^v$$

and thus a transition map χ_{12}^T between the two trivializations of $\ker(D_T^{\text{ext}})$ over U.

Again, let $G_1, G_2 : U \times (0,\infty)^E \to \mathcal{M}$ be the associated gluing maps; we then have a family of transition matrices

$$\chi_{12} : \mathbb{R}^{\dim \ker(D^{\text{ext}})} \times U \times (0,\infty)^E \to \mathbb{R}^{\dim \ker(D^{\text{ext}})} \times U \times (0,\infty)^E$$

relating the two trivializations of $\ker(D^{\text{ext}})$ over $\mathcal{M}$, covering the transition map ϕ_{12} between the charts G_1, G_2. As $\min(\mathbf{T}) \to \infty$, the transition map ϕ_{12} extends continuously to the identity $U \times \{\infty\}^E \to U \times \{\infty\}^E$. Let us look more carefully at χ_{12}, as $\min \mathbf{T} \to \infty$. The respective trivializations of $\ker(D^{\text{ext}})$ induced by the system of transverse hypersurfaces each take the form

$$\ker(D^{\text{ext}}) = \mathbb{R}^{E^{st}} \oplus \left(\bigoplus_{\substack{v \in V^{st} \\ \text{with a neck}}} \mathbb{R} \oplus \bigoplus_{v \in V(T)} \mathbb{R}^{\mathbf{H}^v} \right).$$

A familiar argument using the exponential decay of the gluing maps shows that as $\min(\mathbf{T}) \to \infty$, the transition functions converge to a matrix

$$\begin{pmatrix} \text{id} & 0 \\ Df & \chi_{12}^T \end{pmatrix} \tag{7.19}$$

where $D\mathbf{f}$ is the derivative of a smooth function $f : U \to \mathbb{R}^{E^{st}}$, and χ_{12}^T is exactly the transition map between the two associative trivializations of $\ker(D_T^{\text{ext}})$ over U.

In summary, we obtain a canonical continuous extension of $\ker(D^{\text{ext}})$ to $\mathcal{F} \cup \mathcal{F}_T$, so that over $\mathcal{F}_T$ we have a canonical short exact sequence of vector bundles

$$0 \to \mathbb{R}^{E^{st}} \to \ker(D^{\text{ext}})|_{\mathcal{M}_T} \to \ker(D_T^{\text{ext}}) \to 0 \tag{7.20}$$

where the term $\mathbb{R}^{E^{st}}$ is always $\mathbb{R}$ or 0, depending on whether T has two or one stable vertices. Once again we can do this over every stratum simultaneously, using the associativity of gluing to ensure that these extensions are compatible with each-other. This shows:

Proposition 7.5. *Over each closed stratum $\overline{\mathcal{F}}_T$, there is a topological vector bundle $\ker(D_T^{\text{ext}})$, so that:*

(1) over the interior $\mathcal{F}_T$ of each stratum, there is an identification with $\ker(D_{\mathbf{u}}^{\text{ext}})$ the vector bundle given by the kernels of the extended linearized operators;
(2) under $\overline{\mathcal{F}}_T = \prod_v \overline{\mathcal{M}}_v$, we have identifications of $\ker(D_T^{\text{ext}})$ with $\bigoplus_v \ker(D_v^{\text{ext}})$, compatible with the identifications of (1), where $\ker(D_v^{\text{ext}})$ is the corresponding vector bundle over $\overline{\mathcal{M}_v}$ (in particular, $\ker(D_v^{\text{ext}}) = \ker(D_v)$ for v unstable);
(3) whenever $\overline{\mathcal{F}}_{T_2} \subset \overline{\mathcal{F}}_{T_1}$ is an inclusion of strata, we have a short exact sequence of vector bundles over $\overline{\mathcal{F}}_{T_2}$

$$0 \to \mathbb{R}^{E_2^{st} \setminus E_1^{st}} \to \ker(D_{T_1}^{\text{ext}})|_{\overline{\mathcal{F}}_{T_2}} \to \ker(D_{T_2}^{\text{ext}}) \to 0; \tag{7.21}$$

These exact sequences are compatible, in the sense that whenever $\overline{\mathcal{F}}_{T_3} \subset \overline{\mathcal{F}}_{T_2} \subset \overline{\mathcal{F}}_{T_1}$, we have a commutative diagram of vector bundles after restricting to $\overline{\mathcal{F}}_{T_3}$:

$$\begin{array}{ccccc}
 & & & & \mathbb{R}^{E_3^{st}\setminus E_2^{st}} \\
 & & & & \downarrow \\
\mathbb{R}^{E_2^{st}\setminus E_1^{st}} & \longrightarrow & \ker(D_{T_1}^{\text{ext}}) & \longrightarrow & \ker(D_{T_2}^{\text{ext}}) \\
\downarrow & & \| & & \downarrow \\
\mathbb{R}^{E_3^{st}\setminus E_1^{st}} & \longrightarrow & \ker(D_{T_1}^{\text{ext}}) & \longrightarrow & \ker(D_{T_3}^{\text{ext}}) \\
\downarrow & & & & \\
\mathbb{R}^{E_3^{st}\setminus E_2^{st}} & & & &
\end{array} \tag{7.22}$$

where the two rows and the right column are the exact sequences of (7.21), and the left column is the standard presentation $\mathbb{R}^{E_3^{st}\setminus E_1^{st}} \cong \mathbb{R}^{E_2^{st}\setminus E_1^{st}} \oplus \mathbb{R}^{E_3^{st}\setminus E_2^{st}}$.

Equipped with an understanding of how the various vector bundles $\ker(D_T^{\text{ext}})$ on each stratum $\mathcal{F}_T$ fit together, we now connect this with our understanding of the tangent bundles $T\mathcal{F}_T$ (again, we will use the topological tangent bundle defined earlier as our model).

Let us, yet again, consider a gluing map $G : U \times (0, \infty)^E \to \mathcal{F}$ defined by a system of transverse necks and hypersurfaces for some $U \subset \mathcal{F}_T$. We will seek to fill in, using this chart, the dashed arrows in the following diagram:

$$
\begin{array}{ccccc}
 & & & & \mathbb{R}^{V^{un}} \\
 & & & & \downarrow \\
\mathbb{R}^{E^{st}} & \longrightarrow & \ker(D^{\text{ext}})|_{\mathcal{M}_T} & \longrightarrow & \ker(D_T^{\text{ext}}) \\
\downarrow & & \downarrow \cong & & \downarrow \\
\mathbb{R}^{E} & \longrightarrow & T\mathcal{F}|_{\mathcal{F}_T} & \longrightarrow & T\mathcal{F}_T
\end{array}
\tag{7.23}
$$

where the top row is the exact sequence of (7.21), the bottom row is the exact sequence of (7.7), and the right column is the exact sequence of (7.18). As a sanity check, it is not hard to see that $|E| = |E^{st}| + |V^{un}|$.

By inspection of the construction of $\ker(D^{\text{ext}})$ and $T\mathcal{F}$ over the boundary strata, under the gluing map G associated to the system of necks and hypersurfaces, the standard isomorphism $\ker(D^{\text{ext}}) \cong T\mathcal{F}$ over the open stratum $\mathcal{F} \cup \mathcal{F}_T$ extends to $\mathcal{F}_T$. We claim furthermore that under this map, the image of the sub-bundle $\mathbb{R}^{E^{st}}$ lies in the sub-bundle $\mathbb{R}^E$. To see this, write $(p, u) = G(\mathbf{u}, \mathbf{T})$, choose some edge $e \in E^{st}$, and consider the element $(\rho_e, \xi_e) \in \ker(D_u^{\text{ext}})$ where $\rho_e \in T_p\mathcal{R}$ is the vector representing the neck length corresponding to e, and ξ_e is tangent to each hypersurface H_i.

Consider the image of (ρ_e, ξ_e) under

$$\ker(D_u^{\text{ext}}) \to T_{(p,u)}\mathcal{F} \xrightarrow{dG^{-1}} T_{\mathbf{u}}\mathcal{F}_T \oplus \mathbb{R}^E.$$

Each ξ_e is tangent everywhere to H_i, and under ρ_e projected to the $\mathbb{R}$-subspace of $T_p\mathcal{R}$ corresponding to any neck length other than that corresponding to e is zero; when projected to this factor, it is 1. From our description of the gluing map and local coordinates on $T_{\mathbf{u}}\mathcal{M}_T$, we see that the further projection to $T_{\mathbf{u}}\mathcal{M}_T$ is zero.

In fact, we can give a more precise description of the image of (ρ_e, ξ_e) in the $\mathbb{R}^E$ factor as follows. If we write this image as $(\lambda_\eta)_{\eta \in E}$, then $\lambda_\eta = 0$ unless the underlying edge of the stable tree T^{st} which η collapses to is e, in which case we write $[\eta] = e$. Moreover, ρ_e is the tangent vector to $\mathcal{R}$ representing a unit change in neck length along e. Since ξ_e is everywhere tangent to H_i, in the direction of (ρ_e, ξ_e) the individual bulk regions of the domain do not change in length; the only change in length comes from changes in the gluing lengths. From this we see that in fact

$$\sum_{[\eta]=e} \lambda_\eta = 1.$$

In particular, the choice of gluing map defines over $U \times (0, \infty)^E$ a map of vector bundles

$$\mathbb{R}^{E^{st}} \times U \times (0, \infty)^E \to \mathbb{R}^E \times U \times (0, \infty)^E$$

which, on each fibre, sends each basis vector w_e corresponding to $e \in E^{st}$ to some $(\lambda^e_\eta)_{\eta \in E}$ so that

$$\lambda^e_\eta = 0 \quad \text{if } [\eta] \neq e, \qquad \sum_{[\eta]=e} \lambda^e_\eta = 1. \tag{7.24}$$

The key takeaway is that although these parameters (λ^e_η) are not invariant under a change of gluing map, fibrewise the space of such linear maps $\mathbb{R}^{E^{st}} \to \mathbb{R}^E$ is *contractible*; given such a map, so is the space of possible fillings for the middle dashed arrow in (7.23).

In particular, we can choose an isomorphism

$$\chi : \ker(D^{\text{ext}}) \xrightarrow{\cong} T\mathcal{F}$$

over all of $\mathcal{F} \cup \mathcal{F}_T$, which away from some small neighbourhood of the boundary $\mathcal{F}_T$ is just exactly the standard isomorphism, which furnishes the middle dashed map of a commutative diagram of the form (7.23). Then, working inductively over the strata, we have the following.

Proposition 7.6. *For each closed stratum $\overline{\mathcal{F}}_T \subset \overline{\mathcal{F}}$, there is a short exact sequence of vector bundles over $\overline{\mathcal{F}}_T$*

$$0 \to \mathbb{R}^{V^{un}(T)} \to \ker(D^{\text{ext}}_T) \xrightarrow{\chi_T} T\mathcal{F}_T \to 0 \tag{7.25}$$

satisfying the following:

(1) the first map $\mathbb{R}^{V^{un}} \to \ker(D^{\text{ext}}_T)$ is equal to that in the short exact sequence (7.18)*, and away from some small neighbourhood of the boundary, so is the second map χ_T;*

(2) under the decompositions $\overline{\mathcal{F}}_T = \prod_v \mathcal{M}_v$, $\ker(D^{\text{ext}}_T)$ and $T\mathcal{F}_T$, χ_T is the direct sum of the maps χ_v for each vertex v (and in particular for an unstable vertex v, χ_v is always given by the canonical projection map $\ker(D) \to T\mathcal{M}$ of (7.15)*);*

(3) if $\overline{\mathcal{F}}_{T_2} \subset \overline{\mathcal{F}}_{T_1}$ is an inclusion of strata, then we have a commutative diagram over vector bundles over $\overline{\mathcal{F}}_{T_2}$:

$$\begin{array}{ccccc} & & \mathbb{R}^{V_1^{un}} & \longrightarrow & \mathbb{R}^{V_2^{un}} \\ & & \downarrow & & \downarrow \\ \mathbb{R}^{E_2^{st} \setminus E_1^{st}} & \longrightarrow & \ker(D^{\text{ext}}_{T_1}) & \longrightarrow & \ker(D^{\text{ext}}_{T_2}) \\ \downarrow & & \downarrow{\scriptstyle \chi_{T_1}} & & \downarrow{\scriptstyle \chi_{T_2}} \\ \mathbb{R}^{E_2 \setminus E_1} & \longrightarrow & T\mathcal{F}_{T_1} & \longrightarrow & T\mathcal{F}_{T_2} \end{array} \tag{7.26}$$

where the maps are given by:

(i) the middle and right vertical rows are the short exact sequences of (7.25) *for T_1, T_2;*

(ii) the middle horizontal row is the exact sequence of (7.21)*;*

(iii) the bottom horizontal row is the exact sequence of (7.7)*;*

(iv) the top horizontal map sends a basis vector w_{v_1} corresponding to $v_1 \in V_1^{un}$ to the sum $\sum_{[v_2]=v_1} w_{v_2}$ over the basis vectors of $\mathbb{R}^{V_2^{un}}$ corresponding to vertices of T_2 which are contracted to v_1;

(v) the left vertical row is some linear map satisfying the condition of (7.24).

Moreover, the space of choices of such systems of maps $\{\chi_T\}$ is contractible.

8. Stable framings

In this section, we explain a setting in which the flow categories governing (exact) Hamiltonian and Lagrangian Floer theory admit structures analogous to *stable framings* in the sense of Definition 2.4; more precisely we will construct *spectral systems* in the sense of Definition 2.6 on them. For the entirety of this section, fix an exact symplectic manifold M of dimension $2n$. First, let us recall some terminology from the introduction.

Definition 8.1. *A* background framing *on M is a stable trivialization of TM as a symplectic vector bundle:*

$$\Phi_M : TM \oplus \mathbb{R}^{2N-2n} \cong \mathbb{R}^{2N} \tag{8.1}$$

for some $k \geq n$, where $\mathbb{R}^{2N}$ is the trivial symplectic vector bundle over M with fibre the standard $2k$-dimensional symplectic vector space.

Now, suppose that L is an exact Lagrangian submanifold of M. In the presence of background framing Φ_M on M, image $\Phi_M(TL \oplus \mathbb{R}^{N-n})$ is a Lagrangian sub-bundle of $L \times \mathbb{R}^{2N}$; in particular there is a *Gauss map* to the Lagrangian Grassmannian

$$\Gamma_L : L \longrightarrow U(N)/O(N). \tag{8.2}$$

Definition 8.2. *A* framed brane structure with respect to Φ_M *on L is a null-homotopy of the* stable Gauss map*, which is the composite*

$$\Gamma_L^\infty : L \longrightarrow U(k)/O(k) \longrightarrow U/O. \tag{8.3}$$

Equivalently, it is a Lagrangian sub-bundle Γ_L^ℓ of the trivial vector bundle $\mathbb{R}^{2N+2\ell}$ over $L \times [0,1]$ for some $\ell \geq 0$, which over $L \times \{0\}$ is exactly $\Phi_M(TL \oplus \mathbb{R}^N) \oplus \mathbb{R}^\ell$, and over $L \times \{1\}$ is the standard sub-bundle $\mathbb{R}^{N+\ell}$.

We will usually assume that N is chosen sufficiently large that the set of homotopy classes of null-homotopies of the stable Gauss map Γ_L^∞ is the same as those of the original Gauss map Γ_L. Indeed, $N \geq n+2$ suffices for this.

8.1. Spaces of abstract caps.

Let us first sketch the construction in the Hamiltonian Floer theory case. Consider $\mathcal{M}$, the flow category associated to the Floer theory of a time-dependent Hamiltonian $\{H_t\}$ on a Liouville domain with a choice of time-dependent almost complex structure $\{J_t\}$. The generators $x \in \mathrm{Ob}(\mathcal{M})$ are time 1 periodic orbits of the Hamiltonian flow. It will often be convenient to assume that for each $t \in [0,1]$ the almost complex structure J_t is constant in a Darboux neighbourhood of $x(t)$.

At each Floer generator x, and after fixing a connection on TM, there is an invertible operator from linearizing the Floer equation at the constant solution $u(s,t) = x(t)$:

$$D_x : W^2_1(\mathbb{R} \times S^1, x^*TM) \longrightarrow L^2(\mathbb{R} \times S^1, \Omega^{0,1} \otimes_J x^*TM) \tag{8.4}$$

$$D_x(\xi) = \bar{\partial}_{J_t}\xi + Y_t(\xi) \tag{8.5}$$

for a translation invariant $Y_t \in \Omega^{0,1}_{\mathbb{R}\times S^1} \otimes_{\mathbb{C}} \mathrm{End}(x^*TM)$.

The morphisms in the flow category $\mathcal{M}(x_0, x_1)$ are the moduli spaces of cylinders $u : \mathbb{R} \times S^1 \to M$ satisfying Floer's equation with x_0, x_1 as asymptotics. With the additional choice of a connection, the vector bundle $\mathbb{R} \oplus T\mathcal{M}$ is isomorphic to kernel of a linearized operator

$$D_u : W^2_1(\mathbb{R} \times S^1, u^*TM) \longrightarrow L^2(\mathbb{R} \times S^1, \Omega^{0,1} \otimes_J u^*TM) \tag{8.6}$$

$$D_u(\xi) = \bar{\nabla}\xi + Y(\xi) \tag{8.7}$$

for some $Y \in \Omega^{0,1}_{\mathbb{R}\times S^1} \otimes_J \mathrm{End}(u^*TM)$. This section Y moreover exponentially converges to a translation invariant form Y_t of (8.5) on either end $s \to \pm\infty$.

Now, assume that M comes equipped with a background framing Φ_M. We can then stabilize the operator D_u, replacing it with an operator

$$D_u : W^2_1(\mathbb{R} \times S^1, \mathbb{R}^{2N}) \longrightarrow L^2(\mathbb{R} \times S^1, \mathbb{R}^{2N}) \tag{8.8}$$

without changing either its kernel or cokernel (explicitly, we take the direct sum of the original D_u with the standard $\bar{\partial}$-operator for $L^{k,2}$-sections of $\mathbb{R}^{2N}$). In particular, we can absorb the dependence of the operator on a choice of connection into the endomorphism valued 1-form Y, so that the operator D_u takes the form

$$D_u\xi = \bar{\partial}_J\xi + Y(\xi).$$

Now, consider the Riemann surface

$$S_+ = \mathbb{C}P^1 \backslash \{0\} = \mathbb{C}^* \cup \{\infty\} \tag{8.9}$$

together with an "asymptotic marker" at the removed point $0 \in \mathbb{C}P^1$, that is a distinguished choice of real tangent ray $\mathbb{R}_{\geq 0} \subset \mathbb{C} \cong T_0\mathbb{C}P^1$.

We then let $\mathrm{Ends}(S_+)$ be the space of tubular ends for S_+ compatible with the asymptotic marker. Explicitly, these are maps

$$\varepsilon : (-\infty, 0] \times S^1 \hookrightarrow S_+ \subset \mathbb{C}P^1$$

which are biholomorphisms onto their image such that $\lim_{s\to\infty}(\varepsilon(s,t)) = 0$ and $\varepsilon((-\infty, 0] \times \{1\}))$ is tangent to the distinguished ray $\mathbb{R}_{\geq 0} \subset T_0\mathbb{C}P^1$. $\mathrm{Ends}(S_+)$ is topologized as a subspace of smooth maps, and moreover, $\mathrm{Ends}(S_+)$ is contractible.

Take one of the Floer generators $x \in \mathrm{Ob}(\mathcal{M}))$; this comes with a family of 1-periodic time-dependent complex structures J_t on $\mathbb{R}^{2N}$, a translation invariant $Y_t \in \Omega^{0,1}_{\mathbb{R}\times S^1} \otimes_{\mathbb{C}} \mathrm{End}(\mathbb{R}^{2N})$.

Definition 8.3. *The space of* abstract Floer tube-caps $\mathcal{U}(x)$ *at the Floer generator* $x \in \mathrm{Ob}(\mathcal{M})$ *is defined to be the set of tuples*

$$\mathcal{U}(x) = \{(\varepsilon, L, J, Y, g)\} \tag{8.10}$$

where

(i) $\varepsilon \in \mathrm{Ends}(S_+)$ *is a tubular end for* S_+ *compatible with the marker;*
(ii) $L > 0$ *is a real number;*
(iii) J *is an almost complex structure on* $S_+ \times \mathbb{R}^{2N}$*, which equals* J_t *over* $\varepsilon((-\infty, -L) \times S^1)$*;*
(iv) $Y \in \Omega^{0,1}_{S_+} \otimes_{\mathbb{C}} \mathrm{End}(\mathbb{R}^{2N})$*, which equals* Y_t *over* $\varepsilon((-\infty, -L) \times S^1)$*.*
(v) g *is a metric on* S_+*, which agrees with the standard metric on* $\mathbb{R} \times S^1$ *on the tubular end.*

For $v \in \mathcal{U}(x)$, we then have an "abstract linear Floer operator"

$$D_v : W^2_1(S_+, \mathbb{R}^{2N}) \longrightarrow L^2(S_+, \Omega^{0,1}_{S_+} \otimes_J \mathbb{R}^{2N}) \tag{8.11}$$

specified by this data. The key point is that the space of tube-caps $\mathcal{U}(x)$ is *contractible*, and thus the "virtual index bundle"

$$V(x) = \mathrm{ind}(D_v)$$

over $\mathcal{U}(x)$ yields a "well-defined" virtual vector space associated to each Floer generator $x \in \mathrm{Ob}(\mathcal{M})$. Moreover, by gluing the linearized operators over the moduli spaces $\mathcal{M}(x_0, x_1)$ to such abstract linear Floer operators, we will define maps

$$G : \mathcal{M}(x_0, x_1) \times \mathcal{U}(x_1) \longrightarrow \mathcal{U}(x_0) \tag{8.12}$$

and exhibit "isomorphisms" of virtual vector bundles

$$\mathrm{ind}(D_u) \oplus \mathrm{ind}(D_v) \cong \mathrm{ind}(D_{G(u,v)}) \qquad \text{for } (u, v) \in \mathcal{M}(x_0, x_1) \times \mathcal{U}(x_1) \tag{8.13}$$

from a gluing construction. The maps of (8.12) should fit into commutative diagrams

$$\begin{array}{ccc} \mathcal{M}(x_0, x_1) \times \mathcal{M}(x_1, x_2) \times \mathcal{U}(x_2) & \longrightarrow & \mathcal{M}(x_0, x_1) \times \mathcal{U}(x_1) \\ \downarrow & & \downarrow \\ \mathcal{M}(x_0, x_2) \times \mathcal{U}(x_2) & \longrightarrow & \mathcal{U}(x_0) \end{array} \tag{8.14}$$

where the left vertical arrow is the gluing map taking two Floer trajectories in $\mathcal{M}(x_0, x_1)$, $\mathcal{M}(x_1, x_2)$ to the corresponding broken trajectory in $\mathcal{M}(x_0, x_2)$; and the isomorphisms of (8.13) should be likewise associative.

In this setting of Liouville manifolds with background framings, this construction then endows the flow categories of Hamiltonian Floer theory with stable framings in the sense of Definition 2.4.

An analogous construction holds in Lagrangian Floer theory. Again fix a framed brane structure $\Phi_M : TM \oplus \mathbb{R}^{2N-2n} \cong \mathbb{R}^{2N}$ on a Liouville manifold M. Suppose L_0, L_1 are exact

Lagrangians (either compact, or conical and separated at infinity), together with framed brane structures: let us realize these by Lagrangian sub-bundles Γ_{L_i} of $L_i \times [0,1] \times \mathbb{R}^{2N}$ for $i = 0, 1$, which are the tangent sub-bundle $\Phi_M(TL_i) \oplus \mathbb{R}^{N-n}$ over $L_i \times \{0\}$ and the standard sub-bundle $\mathbb{R}^N$ over $L_i \times \{1\}$.

Now choose regular Floer data (H_t, J_t) for L_0, L_1, yielding a flow category $\mathcal{M}$; choose as well a (possibly time dependent) symplectic connection. Recall that at a generator $x \in \mathcal{M}$, which is a time 1 Hamiltonian chord from L_0 to L_1, we have an invertible operator

$$D_x : W^2_1(\mathbb{R} \times [0,1], (x^*TM, L_0|_{x(0)}, L_1|_{x_1})) \longrightarrow L^2(\mathbb{R} \times [0,1], \Omega^{0,1} \otimes_J x^*TM) \tag{8.15}$$

$$D_x(\xi) = \bar{\partial}_{J_t}\xi + Y_t(\xi) \tag{8.16}$$

where $W^2_1(\mathbb{R}\times[0,1], (x^*TM, TL_0|_{x(0)}, TL_1|_{x_1}))$ is the space of sections of x^*TM on $\mathbb{R}\times[0,1]$ which map $\mathbb{R} \times \{i\}$ into $TL_i|_{x(i)}$ for $i = 0, 1$. In the presence of background framing, we can take the direct sum of this operator with a standard invertible $\bar{\partial}$-operator on $\mathbb{R}^{2N-2n}$, and replace it by an operator

$$D_x : W^2_1(\mathbb{R} \times [0,1], (\mathbb{R}^{2N}, TL_0|_{x_0} \oplus \mathbb{R}^{N-n}, TL_1|_{x_1} \oplus \mathbb{R}^{N-n})) \longrightarrow L^2(\mathbb{R} \times [0,1], \Omega^{0,1} \otimes_J \mathbb{R}^{2N}).$$

Consider the Riemann surface with boundary $D_+ = D\backslash\{-1\}$ given by removing -1 from the unit disc; let us write $\mathrm{Ends}(D_+)$ for the (contractible) space of strip-like ends

$$\varepsilon : (-\infty, 0] \times [0,1] \longrightarrow D_+$$

around the boundary puncture.

Definition 8.4. *The space of* abstract Floer strip-caps $\mathcal{U}(x)$ *is the set of tuples*

$$\mathcal{U}(x) = \{(\varepsilon, L, J, Y, g, \Lambda, \Gamma)\}$$

where:

- *(i)* $\varepsilon \in \mathrm{Ends}(D_+)$ *is a strip-like end for* D_+*;*
- *(ii)* $L > 0$ *is a real number;*
- *(iii)* J *is an almost complex structure and symplectic connection on the trivial bundle* $\mathbb{R}^{2N}$ *over* D_+*, which over* $\varepsilon((-\infty, -L) \times [0,1])$ *agrees with* J_t*;*
- *(iv)* $Y \in \Omega^{0,1}_{D_+} \otimes_J \mathrm{End}(\mathbb{R}^{2N})$ *is such that over* $\varepsilon((-\infty, -L) \times [0,1])$ *it agrees with the translation invariant* Y_t *of* (8.16)*;*
- *(v)* g *is a metric on* D_+*, which agrees with the standard metric on* $\mathbb{R} \times [0,1]$ *on the strip-like end;*
- *(vi)* $\Lambda \to \partial D_+$ *is a Lagrangian sub-bundle of* $\partial D_+ \times \mathbb{R}^{2N}$*, which over* $\varepsilon((-\infty, -L) \times \{i\})$ *is the constant sub-bundle* $TL_i|_{x(i)} \oplus \mathbb{R}^{N-n}$ *for* $i = 0, 1$*;*
- *(vii)* $\Gamma \to \partial D_+ \times [0,1]$ *is a Lagrangian sub-bundle of the trivial bundle* $\partial D_+ \times [0,1] \times \mathbb{R}^{2N}$*, such that* $\Gamma|_{\partial D_+ \times \{0\}} = \Lambda$ *and* $\Gamma|_{\partial D_+ \times \{1\}}$ *is the standard sub-bundle* $\mathbb{R}^N$*, which moreover satisfies the condition that over* $\varepsilon((-\infty, -L) \times \{i\}) \times [0,1]$*,* Γ *is the sub-bundle* Γ_{L_i} *specified by the framed brane structures on* L_i *for* $i = 0, 1$*.*

For such a tuple $v \in \mathcal{U}(x)$, we then have an "abstract linear Floer operator"

$$D_v : W^2_1(D_+, \mathbb{R}^{2N}, \Lambda) \longrightarrow L^2(D_+, \Omega^{0,1}_{D_+} \otimes_J \mathbb{R}^{2N}) \tag{8.17}$$

specified by this data, where $W^2_1(D_+, \mathbb{R}^{2N}, \Lambda)$ is the space of class W^2_1 maps $D_+ \to \mathbb{R}^{2N}$ which send ∂D_+ to the Lagrangian sub-bundle Λ.

The key point again is that the space of strip-caps $\mathcal{U}(x)$ is contractible; in particular we can speak of a "well-defined" virtual vector space $V(x) = \mathrm{ind}(D_v)$ associated to the Floer generator x. These spaces come with gluing maps

$$G : \mathcal{M}(x_0, x_1) \times \mathcal{U}(x_1) \longrightarrow \mathcal{U}(x_0)$$

which are covered by identifications of the index bundles

$$\mathrm{ind}(D_u) \oplus \mathrm{ind}(D_v) \cong \mathrm{ind}(D_{G(u,v)}) \qquad \text{for } (u, v) \in \mathcal{M}(x_0, x_1) \times \mathcal{U}(x_1).$$

Let us in earnest construct these gluing maps and the above gluing isomorphisms of index bundles.

8.2. **Gluing Floer tubes/strips to abstract tube/strip-caps.** We first explain how to construct gluing maps $\mathcal{M}(x_0, x_1) \times \mathcal{U}(x_1) \to \mathcal{U}(x_0)$ which make the diagram (8.14) commute.

To begin with, we will not strictly speaking define these gluings on the whole of $\mathcal{M}(x_0, x_1)$. Instead, we will choose closed codimension zero sub-manifolds-with-corners

$$\mathring{\mathcal{M}}(x_0, x_1) \subset \mathcal{M}(x_0, x_1)$$

together with retractions $r : \mathcal{M}(x_0, x_1) \to \mathring{\mathcal{M}}(x_0, x_1)$ in the following fashion. For each sequence of Floer generators $x_0 > x_1 > \ldots > x_\ell$, we pick a smooth collar neighbourhood of the corresponding boundary stratum of $\mathcal{M}(x_0, x_\ell)$:

$$C(x_0, \ldots, x_\ell) : \mathcal{M}(x_0, x_1) \times \ldots \times \mathcal{M}(x_{\ell-1}, x_\ell) \times (T_0, \infty]^{\ell-1} \longrightarrow \mathcal{M}(x_0, x_\ell)$$

for some $T_0 > 0$; where the smooth structure on $(T_0, \infty]$ is the one determined by our choice of gluing profile

$$\psi : [0, \varepsilon) \to (T_0, \infty].$$

We say a system of collar neighbourhoods is *coherent* if for each $0 \le i < j \le \ell$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(x_0, x_1) \times \ldots (\mathcal{M}(x_i, x_{i+1}) \times \ldots \times \mathcal{M}(x_{j-1}, x_j)) \ldots \times \mathcal{M}(x_{\ell-1}, x_\ell) \times (T_0, \infty]^{\ell-1} & & \\ \downarrow & \searrow & \\ \mathcal{M}(x_0, x_1) \times \ldots \mathcal{M}(x_i, x_j) \ldots \times \mathcal{M}(x_{\ell-1}, x_\ell) \times (T_0, \infty]^{\ell-j+i} & \longrightarrow & \mathcal{M}(x_0, x_\ell). \end{array}$$

We also wish to pick a system of tube- or strip-like ends; for this purpose let us write $Z = \mathbb{R} \times S^1$ or $\mathbb{R} \times [0, 1]$ depending on whether we are considering the Hamiltonian or Lagrangian case, $Z_- = \mathbb{R}_{\le 0} \times S^1$ or $\mathbb{R}_{\le 0} \times [0, 1]$ and $Z_+ = \mathbb{R}_{\ge 0} \times S^1$ or $\mathbb{R}_{\ge 0} \times [0, 1]$, and $Z_{[a,b]} = [a, b] \times S^1$ or $[a, b] \times [0, 1]$.

Definition 8.5. *A* coherent system of collar neighbourhoods and tube-/strip-like ends *is a choice of coherent collar neighbourhoods $C(x_0, \ldots, x_\ell)$ for the whole flow category $\mathcal{M}$, and for each pair of Floer generators $x_- > x_+$ and $u \in \mathrm{Int}\mathcal{M}(x_-, x_+)$, a smoothly varying choice of positive real number $L(u)$ and a representative map u in the $\mathbb{R}$-translation class, such that:*

(i) if we decompose Z as

$$Z = Z_{(-\infty,-L]} \cup Z_{[-L,L]} \cup Z_{[L,\infty)}$$

then the image of $u(Z_{(-\infty,-L]})$ is sufficiently close to the Floer generator x_-, and the image of $u(Z_{[L,\infty)})$ is sufficiently close to x_+. We think of $Z_{(-\infty,-L]}$ and $Z_{[L,\infty)}$ as a choice of tube-/strip-like ends for u;

(ii) if $u = C(u_1, \ldots, u_\ell, T_1, \ldots, T_{\ell-1})$, then

(a) $L(u) = L(u_1) + 2T_1 + L(u_2) + \ldots + L(u_{\ell-1}) + 2T_{\ell-1} + L(u_\ell)$;

(b) for each $i = 1, \ldots, \ell - 1$, writing

$$\begin{aligned} L_i^- &= L(u_1) + 2T_1 + \ldots + L(u_i) = L(C(u_1, \ldots, u_i, T_1, \ldots, T_{i-1})) \\ L_i^+ &= L(u_{i+1}) + 2T_{i+1} + \ldots + L(u_\ell) = L(C(u_{i+1}, \ldots, u_\ell, T_{i+1}, \ldots, T_{\ell-1})), \end{aligned}$$

then we have, after suitable translations so that the intervals align, as $T_i \to \infty$ we have uniform convergence

$$u|_{Z_{[-L(u),-L(u)+2L_i^-]}} \longrightarrow C(u_1, \ldots, u_i, T_1, \ldots, T_{i-1})|_{Z_{[-L_i^-,L_i^-]}}$$

as well as

$$u|_{Z_{[L(u)-2L_i^+,L(u)]}} \longrightarrow C(u_{i+1}, \ldots, u_\ell, T_{i+1}, \ldots, T_{\ell-1})|_{Z_{[-L_i^+,L_i^+]}}$$

and moreover this convergence is exponentially fast in T_i.

In other words, this is a choice of tube-/strip- like ends on the domain Z for each $u \in \mathcal{M}(x_-, x_+)$, so that the domain of $C(u_1, \ldots, u_\ell, T_1, \ldots, T_{\ell-1})$ is built by applying the gluing process to this choice of tube-/strip-like ends for $u_1, \ldots, u_\ell$ with gluing lengths given by $T_1, \ldots, T_{\ell-1}$.

We then define, for some $T \gg 0$, $\mathring{\mathcal{M}}(x_-, x_+) \subset \mathcal{M}(x_-, x_+)$ to be the complements of the $(T, \infty]$ collar neighbourhoods, explicitly

(8.18)
$$\mathring{\mathcal{M}}(x_-, x_+) = \mathcal{M}(x_-, x_+) - \bigcup_{\substack{x_0 > x_1 \ldots > x_\ell \\ x_0 = x_-, x_\ell = x_+}} C\left(\mathcal{M}(x_0, x_1) \times \ldots \times \mathcal{M}(x_{\ell-1}, x_\ell) \times (T, \infty]^{\ell-1}\right).$$

By construction, this is itself a manifold with corners, with boundary faces of the form

$$C(\mathring{\mathcal{M}}(x_0, x_1) \times \mathring{\mathcal{M}}(x_1, x_2) \times \{T\}).$$

Moreover, there are natural continuous retractions $r : \mathcal{M}(x_0, x_1) \longrightarrow \mathring{\mathcal{M}}(x_0, x_1)$ which are compatible with gluing in that

$$\begin{array}{ccccc}
\mathcal{M}(x_0,x_1) \times \mathcal{M}(x_1,x_2) & \longrightarrow & & \longrightarrow & \mathcal{M}(x_0,x_2) \\
\downarrow{\scriptstyle r\times r} & & & & \downarrow{\scriptstyle r} \\
\mathring{\mathcal{M}}(x_0,x_1) \times \mathring{\mathcal{M}}(x_1,x_2) & = & C(\mathring{\mathcal{M}}(x_0,x_1) \times \mathring{\mathcal{M}}(x_1,x_2) \times \{T\}) & \hookrightarrow & \mathring{\mathcal{M}}(x_0,x_2)
\end{array}$$

commutes whenever $x_0 > x_1 > x_2$.

We can now define maps

$$G : \mathring{\mathcal{M}}(x_-, x_+) \times \mathcal{U}(x_+) \longrightarrow \mathcal{U}(x_-)$$

so that the diagram (8.14) commutes with $\mathring{\mathcal{M}}$ in place of $\mathcal{M}$.

The basic construction. For $u \in \mathcal{M}(x_-, x_+)$ and $v \in \mathcal{U}(x_+)$, to specify $G(u, v)$ we must furnish: a choice of end ε on S_+, a real number L, an almost complex structure, and an endomorphism-valued (0,1)-form $Y \in \Omega^{0,1}_{S_+} \otimes_J \mathrm{End}(\mathbb{R}^{2N})$, and a metric.

Let us first explain the choice of end. Recall that on the domain of u, we have tube-/strip-like ends $\varepsilon_\pm : Z_\pm \to Z$, and we have also have an end $\varepsilon : Z_- \to S_+$ (or in the Lagrangian case, D_+). Consider the Riemann surface $S(u, v)$ obtained by gluing

$$Z \backslash \varepsilon_+(Z_{(2T,\infty)}) \quad \text{to} \quad S_+ \backslash \varepsilon(Z_{(-\infty,-2L)}).$$

Examining $S(u, v)$ "left to right", it has a negative tube-/strip-like end $\varepsilon(u, v) : Z_- \to S$ inherited from ε_-, a "bulk region" biholomorphic to $Z_{[-L(u),L(u)]}$, a "neck" of length $2T + 2L$, and then a final region biholomorphic to $S_+ \backslash \varepsilon(Z_-)$. Choose a biholomorphism

$$f(u, v) : S(u, v) \longrightarrow S_+$$

which preserves the asymptotic marker; there is a contractible space of such choices. We then take the corresponding end $f(u, v) \circ \varepsilon(u, v)$ on S_+. The real number $L > 0$ which we take as our new "cut-off" point will just be the fixed choice $L = T$. The metric is then simply the metric on the glued surface, pushed forward to S_+ by the choice of biholomorphism.

Now, for the choice of almost complex structure J and 1-form Y. Recall we assumed that J_t is constant in a small neighbourhood of $x(t)$ for each Floer generator x. In particular, on the two ends on the domains of u, v identified by our gluing process the two choices of almost complex structures agree, and so yield an almost complex structure on the glued surface. In order to produce a glued 1-form, take the 1-form Y_u on Z; this converges exponentially fast to translation-invariant forms $Y_{x_\pm}$ at either end. So, take a 1-form $\tilde{Y}_u$ such that

$$\tilde{Y}_u = \begin{cases} Y_{x_-} & \text{on } Z_{(-\infty,-T-1)}; \\ Y_u & \text{on } Z_{(-T+1-L(u),L(u)+T-1)}; \\ Y_{x_+} & \text{on } Z_{(L(u)+T+1,\infty)} \end{cases}$$

and is given by a suitable interpolation of $Y_{x_\pm}$ and Y_u on the remaining regions. This $\tilde{Y}_u$ then glues to the 1-form on S_+ specified by v, thus all together defining an element $G(u,v) \in \mathcal{U}(x_-)$. This hence defines a map

$$\mathring{\mathcal{M}}(x_-, x_+) \times \mathcal{U}(x_+) \longrightarrow \mathcal{U}(x_-). \tag{8.19}$$

However, these maps do not satisfy the desired commutative diagrams (8.14). We rectify this problem by slightly modifying the maps, working inductively over the strata:

In the case that $\mathring{\mathcal{M}}(x_-, x_+)$ is without boundary, meaning that there does not exist a Floer generator x_1 with $x_- > x_1 > x_+$, we leave the map (8.19) unchanged.

The case that $\mathring{\mathcal{M}}(x_-, x_+)$ has boundary. By induction, we have already constructed a map on the boundary

$$\partial\mathring{\mathcal{M}}(x_-, x_+) \times \mathcal{U}(x_+) \longrightarrow \mathcal{U}(x_-). \tag{8.20}$$

This map does not exactly agree with the map (8.19) from the basic construction above, however:

(i) with appropriate choices of biholomorphisms from the glued surface to the standard S_+, the two maps (8.19) and (8.20) do yield the same tube-/strip-like end for S_+, by the assumptions of Definition 8.5.
(ii) the almost complex structure and 1-forms J, Y specified by (8.19) and (8.20) are extremely close; their difference in the appropriate norms is bounded by $Ce^{-\delta T}$ for some constants C, δ.

In particular, we can choose a linear interpolation between the almost complex structures and 1-forms arising from (8.19) and (8.20), and use this to define an extension of (8.20) over collar neighbourhoods of the form

$$\mathring{\mathcal{M}}(x_0, x_1) \times \ldots \times \mathring{\mathcal{M}}(x_{\ell-1}, x_\ell) \times [T-1, T]^{\ell-1} \times \mathcal{U}(x_+) \longrightarrow \mathcal{U}(x_-).$$

This map can then be glued to the basic construction of (8.19), to produce a map

$$\mathring{\mathcal{M}}(x_-, x_+) \times \mathcal{U}(x_+) \longrightarrow \mathcal{U}(x_-) \tag{8.21}$$

satisfying the commutative diagrams (8.14) on the boundary faces indexed by chains $x_- > x_1 > x_+$.

8.3. **Gluing the index bundles over the truncated moduli spaces.**

Our next task is to produce isomorphisms

$$\mathrm{ind}(D_u) \oplus \mathrm{ind}(D_v) \cong \mathrm{ind}(D_{G(u,v)})$$

of (virtual) vector bundles lying over the gluing maps $G : \mathring{\mathcal{M}}(x_-, x_+) \times \mathcal{U}(x_+) \longrightarrow \mathcal{U}(x_-)$ we have just constructed. Crucially, these isomorphisms will be required to moreover be *coherent*, in that the isomorphisms over commutative diagram of spaces (8.14) must commute.

Recall that in Section **??**, we showed that the index bundles $\ker(D)$ over the un-compactified moduli spaces extended continuously over the compactification by broken trajectories, where the bundle decomposes as a direct sum of the index bundles of the components, and that moreover this can be directly compared the decomposition of the tangent bundle $T\mathcal{M}$ over the boundary. This is, however, not immediately applicable to the gluing maps (8.19) of abstract caps which we have just constructed, since this uses a *finite length* gluing process.

To this end, it will be helpful for us to construct decompositions of the index bundle $\ker(D)$ over the boundaries of the *truncated* moduli spaces $\mathring{\mathcal{M}}(x_-, x_+)$. Explicitly, suppose we have $x_0 > x_1 > \ldots > x_\ell$ and $u_1, \ldots, u_\ell \in \mathring{\mathcal{M}}(x_0, x_1) \times \ldots \times \mathring{\mathcal{M}}(x_{\ell-1}, x_\ell)$. These in turn define an element

$$u = C(u_1, \ldots, u_\ell, \{T\}^{\ell-1}) \in \mathring{\mathcal{M}}(x_0, x_\ell);$$

we will usually just suppress the gluing length T from the notation and write $u = C(u_1, \ldots, u_\ell)$. Our immediate aim is to produce an isomorphism

$$\phi : \ker(D_{u_1}) \oplus \ldots \oplus \ker(D_{u_\ell}) \longrightarrow \ker(D_u)$$

which is moreover coherent under taking further gluings.

We will construct ϕ by taking a "pre-gluing" map using cut-off functions, and then using an L^2_1-orthogonal projection onto $\ker(D_u)$. Let us first fix cut-off functions: for each $x_- > x_+$ and $u \in \mathring{\mathcal{M}}(x_-, x_+)$, take the bump function χ_u on its domain Z that is 1 on $Z_{(-L(u)-T, L(u)+T)}$, and zero on $Z_{(-\infty, -L(u)-T-1)}$ and $Z_{(L(u)+T+1, \infty)}$. Now, consider the linear map

$$\text{pre}\phi : \ker(D_{u_1}) \oplus \ldots \oplus \ker(D_{u_\ell}) \longrightarrow L^2_1(Z, \mathbb{R}^{2N})$$

defined as follows: for $\text{pre}\phi(\xi_1, \ldots, \xi_\ell)$ is the section of $\mathbb{R}^{2N}$ obtained by gluing the cut-off sections $\chi_{u_1}\xi_1, \ldots, \chi_{u_\ell}\xi_\ell$ after they have been suitably translated to the regions of Z specified by its gluing decomposition into ℓ "bulks" of length $2L(u_i)$ and $\ell - 1$ "necks" of length $4T$.

Lemma 8.6. *If the original cut-off $T > 0$ for $\mathring{\mathcal{M}}$ is chosen sufficiently large, then the map* $\text{pre}\phi$ *is close to an isometry, and the composite of* $\text{pre}\phi$ *with the orthogonal projection* $\Pi_{\ker(D_u)}$ *onto* $\ker(D_u)$ *is close in operator norm to* $\text{pre}\phi$*: there is $\varepsilon_1 \ll 1$ such that*

$$(1 - \varepsilon_1)(||\xi_1||^2 + \ldots + ||\xi_\ell||^2) \leq ||\text{pre}\phi(\xi_1, \ldots, \xi_\ell)||^2 \leq (1 + \varepsilon_1)(||\xi_1||^2 + \ldots + ||\xi_\ell||^2)$$

and $\varepsilon_2 \ll 1$ such that

$$||\text{pre}\phi(\xi_1, \ldots, \xi_\ell) - \Pi_{\ker(D_u)}\text{pre}\phi(\xi_1, \ldots, \xi_\ell)||^2 \leq \varepsilon_2(||\xi_1||^2 + \ldots + ||\xi_\ell||^2).$$

The first property is immediate from the exponential decay of solutions ξ_i to $D_{u_i}\xi_i = 0$ along the ends. The second property comes from an a priori estimate on the operator norm of a right inverse to D_u, independent of the gluing length T.

In particular, the projection $\Pi_{\ker(D_u)}\text{pre}\phi$ defines an isomorphism of vector bundles

$$\tilde{\phi} : \ker(D_{u_1}) \oplus \ldots \oplus \ker(D_{u_\ell}) \longrightarrow \ker(D_u).$$

Now, observe that the space of linear maps of vector bundles $\psi : \oplus_i \ker(D_{u_i}) \to \ker(D_u)$ which satisfy the operator norm bound

$$||\psi - \mathrm{pre}\phi|| \leq \sqrt{\varepsilon_2} \tag{8.22}$$

is convex and thus contractible; each of its elements is automatically an isomorphism of vector bundles.

Now observe that if $1 \leq i < j \leq \ell$, then writing $u_{ij} = C(u_i, \ldots, u_j)$, for $(\xi_1, \ldots, \xi_\ell)$ we can also take the composite element of $\ker(D_u)$

$$\Pi_{\ker(D_u)}\mathrm{pre}\phi\left(\xi_1, \ldots, \xi_{i-1}, \Pi_{\ker(D_{u_{ij}})}\mathrm{pre}\phi(\xi_i, \ldots, \xi_j), \xi_{j+1}, \ldots, \xi_\ell\right).$$

It is not difficult to check that the composite map this defines also satisfies the operator norm bound (8.22), and so there is an essentially unique homotopy (through isomorphisms of vector bundles) between this composite and $\Pi_{\ker(D_u)}\mathrm{pre}\phi$.

In particular, we can then inductively choose, for each $x_0 > x_1 > \ldots > x_\ell$ and $(u_1, \ldots, u_\ell) \in \Pi_i \mathring{\mathcal{M}}(x_{i-1}, x_i)$, continuously varying isomorphisms

$$\phi : \ker(D_{u_1}) \oplus \ldots \oplus \ker(D_{u_\ell}) \longrightarrow \ker(D_{C(u_1,\ldots,u_\ell)})$$

satisfying $||\phi - \mathrm{pre}\phi|| \leq \sqrt{\varepsilon_2}$, and so that for each $1 \leq i < j \leq \ell$ the triangles

$$\begin{array}{ccc}
\ker(D_{u_1}) \oplus \ldots \underbrace{\ker(D_{u_i}) \ldots \ker(D_{u_j})} \ldots \oplus \ker(D_{u_\ell}) & & \\
\downarrow & \searrow & \\
\ker(D_{u_1}) \oplus \ldots \ker(D_{C(u_i,\ldots,u_j)}) \ldots \oplus \ker(D_{u_\ell}) & \longrightarrow & \ker(D_{C(u_1,\ldots,u_\ell)})
\end{array}$$

commute; this is precisely the desired coherence condition. Moreover, if one desired, one could fix ϕ so that away from some collar neighbourhood of the boundary of $\Pi_i \mathring{\mathcal{M}}(x_{i-1}, x_i)$ it coincides with $\Pi_{\ker(D_u)}\mathrm{pre}\phi$.

8.4. **Gluing the index bundles for abstract caps.** In order to use a similar technique to glue the index bundles over moduli spaces of Floer trajectories to the those from abstract Floer caps, let us first get a more concrete handle on the virtual index bundle $\mathrm{ind}(D) \to \mathcal{U}(x)$ for $x \in \mathrm{Ob}(\mathcal{M})$. To simplify this discussion, let us work in just the Hamiltonian Floer theory case (so we are considering the space of abstract tube-caps).

For each $q \geq 0$, consider the space of tuples

$$\mathcal{U}_q(x) = \{(\varepsilon, L, J, Y, g, f)\} \tag{8.23}$$

where

(i) $v = (\varepsilon, L, J, Y, g)$ is an element of $\mathcal{U}(x)$.

(ii) $f : \mathbb{R}^q \longrightarrow C^\infty(S_+, \Omega^{0,1}_{S_+} \otimes_J \mathbb{R}^{2N})$ is a linear "obstruction" map, such that for each $a \in \mathbb{R}^{2N}$, $f(a)$ is identically zero over the end $\varepsilon(Z_-) \subset S_+$, and so that the combined map

$$D_v + f : L^2_1(S_+, \mathbb{R}^{2N}) \oplus \mathbb{R}^q \longrightarrow L^2(S_+, \Omega^{0,1}_{S_+} \otimes_J \mathbb{R}^{2N}) \tag{8.24}$$

is surjective.

(iii) L is large enough so that the following holds. Write χ_L for the bump function on S_+ that is 1 on $S_+ \backslash \varepsilon(Z_{(-\infty,-L)})$ and 0 on $\varepsilon(Z_{(-\infty,-L-1]})$. Then, there exists a right inverse Q of $D_v + f$ so that for each $(\xi, a) \in \ker(D_v + f)$, we have

$$||Q(D_v \chi_L \xi + f(a))|| \leq \sqrt{\varepsilon_2}||\xi||. \tag{8.25}$$

The final condition may seem at first sight mystifying, but it is extremely convenient for our technique of constructing "gluings" of index bundles by first constructing a "pregluing" using cut-off functions, and then taking an orthogonal projection. Observe that for fixed $(\varepsilon, J, Y, g, f)$, a large enough L always exists: for each $(\xi, a) \in \ker(D_v + f)$, the 1-form $D_v \chi_L \xi + f(a)$ is supported in $\varepsilon(Z_{[-L-1,-L]})$; by choosing L large enough and appealing to the exponential decay of solutions ξ to the translation-invariant equation $\bar{\partial}\xi + Y_x(\xi) = 0$ we obtain the desired estimate.

Over $\mathcal{U}_q(x)$, the index bundle $\mathrm{ind}(D_v)$ is then realized as a formal difference of a vector bundle of locally constant rank and a trivial vector bundle:

$$\mathrm{ind}(D_v) = \ker(D_v + f) - \mathbb{R}^q.$$

There are clear "stabilization" embeddings $\mathcal{U}_q(x) \hookrightarrow \mathcal{U}_{q+1}(x)$ which send (ε, J, Y, f) to the tuple with the same $v = (\varepsilon, J, Y)$ and obstruction map given by

$$\Sigma f : \mathbb{R}^{q+1} \xrightarrow{project} \mathbb{R}^q \xrightarrow{f} C^\infty(S_+, \Omega^{0,1}_{S_+} \otimes_J \mathbb{R}^{2N})$$

and it is moreover clear that $\ker(D_v + \Sigma f) = \ker(D_v + f) \oplus \mathbb{R}$. There spaces also come equipped with obvious forgetful maps

$$\mathcal{U}_q(x) \longrightarrow \mathcal{U}(x)$$

which are compatible with the stabilizations.

The resulting map

$$\bigcup_{q \geq 0} \mathcal{U}_q(x) \longrightarrow \mathcal{U}(x) \tag{8.26}$$

is then a weak homotopy equivalence: this comes down to the fact that for a fixed integer m, the space of surjective maps $\mathbb{R}^q \longrightarrow \mathbb{R}^m$ is $(q-m-1)$-connected. We consequently use (8.26) to replace $\mathcal{U}(x)$ by the directed system $\{\mathcal{U}_q(x)\}$.

Let us also observe that the maps (8.19) admit natural lifts to $\mathcal{U}_q(x)$:

$$G : \mathring{\mathcal{M}}(x_-, x_+) \times \mathcal{U}_q(x_+) \longrightarrow \mathcal{U}_q(x_-) \tag{8.27}$$

defined in the following fashion: for $u \in \mathring{\mathcal{M}}(x_-, x_+)$ and $(v, f) \in \mathcal{U}_q(x_+)$, on the Riemann surface obtained by gluing Z and S_+ along their respective ends, extend $f : \mathbb{R}^q \to C^\infty(S_+, \Omega^{0,1} \otimes \mathbb{R}^{2N})$ by zero over the whole glued surface. Of course, in order for this extension $\tilde{f}$ by zero to define a bona fide element of $\mathcal{U}_q(x_-)$, we need to know that the operator $D_{G(u,v)} + \tilde{f}$ is surjective. However this follows by a standard gluing argument: choose right inverses for the operators D_u and $D_v + f$, and use cut-off functions to paste them together into an approximate right inverse for $D_{G(u,v)} + \tilde{f}$.

We now define gluings for the index bundles. Explicitly, for $x_0 > \ldots > x_\ell$ and $(u_1, \ldots, u_\ell) \in \mathring{\mathcal{M}}(x_0, x_1) \times \ldots \times \mathring{\mathcal{M}}(x_{\ell-1}, x_\ell)$, and $v \in \mathcal{U}_q(x_\ell)$, we seek to define an isomorphism

$$\phi : \ker(D_{u_1}) \oplus \ldots \oplus \ker(D_{u_\ell}) \oplus \ker(D_v + f) \longrightarrow \ker(D_{G(u_1,\ldots,u_\ell,v)} + \tilde{f}).$$

These isomorphisms should also satisfy two coherence conditions: for $1 < i \leq \ell$ the triangle

$$\begin{array}{ccc} \ker(D_{u_1}) \oplus \ldots \underbrace{\ker(D_{u_i}) \ldots \ker(D_{u_\ell}) \oplus \ker(D_v + f)} & & \\ \downarrow & \searrow & \\ \ker(D_{u_1}) \oplus \ldots \ker(D_{u_{i-1}}) \oplus \ker(D_{G(u_i,\ldots,u_\ell,v)} + \tilde{f}) & \longrightarrow & \ker(D_{G(u_1,\ldots,u_\ell,v)} + \tilde{f}) \end{array} \tag{8.28}$$

commutes; and for $1 \leq i < j \leq \ell$ the triangle

$$\begin{array}{ccc} \ker(D_{u_1}) \oplus \ldots \underbrace{\ker(D_{u_i}) \ldots \ker(D_{u_j})} \ldots \oplus \ker(D_v + f) & & \\ \downarrow & \searrow & \\ \ker(D_{u_1}) \oplus \ldots \ker(D_{C(u_i,\ldots,u_j)}) \ldots \oplus \ker(D_v + f) & \longrightarrow & \ker(D_{G(u_1,\ldots,u_\ell)}) \end{array} \tag{8.29}$$

commutes, where the vertical arrow is the earlier constructed gluing isomorphism for the index bundles over boundary strata of $\mathring{\mathcal{M}}$.

Write $w = G(u_1, \ldots, u_\ell, v)$. To construct ϕ, once again consider a map

$$\text{pre}\phi : \ker(D_{u_1}) \oplus \ldots \oplus \ker(D_{u_\ell}) \oplus \ker(D_v + f) \longrightarrow L^2_1((S_+, g_w), \mathbb{R}^{2N}) \oplus \mathbb{R}^q$$

where the target L^2_1 space is defined using the glued metric g_w on S_+, which takes a tuple

$$(\xi_1, \ldots, \xi_\ell, (\xi_v, a)) \in \ker(D_{u_1}) \oplus \ldots \oplus \ker(D_{u_\ell}) \oplus \ker(D_v + f)$$

to the element

$$(\chi_{u_1}\xi_1 + \ldots + \chi_{u_\ell}\xi_\ell + \chi_{L+T}\xi_v, a) \in L^2_1((S_+, g_w), \mathbb{R}^{2N}) \oplus \mathbb{R}^q$$

where $\chi_{u_1}\xi_1 + \ldots + \chi_{u_\ell}\xi_\ell + \chi_{L+T}\xi_v$ is the section of $\mathbb{R}^{2N}$ obtained by cutting off each $\xi_1, \ldots, \xi_\ell, \xi_v$ using the bump functions $\chi_{u_1}, \ldots, \chi_{u_\ell}$ and χ_{L+T}, and gluing them together on the glued surface (S_+, g_w). We then have the analogue of Lemma (8.6):

Lemma 8.7. *For $T > 0$ sufficiently large, the map* preϕ *is close an isometry, in that there is $\delta_1 \ll 1$ such that*

$$(1 - \delta_1)||(\xi_1, \ldots, \xi_\ell, \xi_v, a)||^2 \leq ||\text{pre}\phi(\xi_1, \ldots, \xi_\ell, \xi_v, a)||^2 \leq (1 + \delta_1)||(\xi_1, \ldots, \xi_\ell, \xi_v, a)||^2.$$

Moreover, the composite of $\mathrm{pre}\phi$ *with the orthogonal projection* $\Pi_{\ker(D_v+f)}$ *is close in operator norm to* $\mathrm{pre}\phi$*, in that there is* $\delta_2 \ll 1$ *such that*

$$||\mathrm{pre}\phi(\xi_1, \ldots, \xi_\ell, \xi_v, a) - \Pi_{\ker(D_v+f)}\mathrm{pre}\phi((\xi_1, \ldots, \xi_\ell, \xi_v, a))||^2 \leq \delta_2||(\xi_1, \ldots, \xi_\ell, \xi_v, a)||^2.$$

Again, this lemma follows from exponential decay estimates for the solutions ξ_i along the strip-like ends (where they solve translation-invariant equations), together with the a priori estimate (8.25) on the right inverse applied to cut-off solutions.

Once again, the space of all linear maps of vector spaces

$$\psi : \ker(D_{u_1}) \oplus \ldots \oplus \ker(D_{u_\ell}) \oplus \ker(D_v + f) \longrightarrow \ker(D_w + \tilde{f})$$

which satisfy the operator norm bound $||\psi - \mathrm{pre}\phi|| \leq \sqrt{\delta_2}$ is contractible; any such ψ is automatically an isomorphism. We can then, as in the construction of gluing maps for the index bundles over $\mathring{\mathcal{M}}$, inductively construct continuously varying isomorphisms

$$\phi : \ker(D_{u_1}) \oplus \ldots \oplus \ker(D_{u_\ell}) \oplus \ker(D_v + f) \longrightarrow \ker(D_{G(u_1,\ldots,u_\ell,v)} + \tilde{f})$$

satisfying this bound, so that the diagrams (8.28) and (8.29) commute.

This discussion completely carries through to the Lagrangian Floer theory case as well. There, the points of $\mathcal{U}_q(x)$ must be additionally defined to carry Lagrangian boundary conditions, as well as homotopies of these Lagrangian boundary conditions to the standard $\mathbb{R}^N \subset \mathbb{R}^{2N}$, but otherwise the discussion is the same.

The upshot of this is:

Proposition 8.8. *In the presence of a background framing (and in the Lagrangian case, framed brane structures), the flow categories associated to regular Floer data for Hamiltonian and Lagrangian Floer theory admit spectral systems in the sense of Definition 2.6. This is the structure of a spectrally enriched functor*

$$\mathfrak{o} : \mathcal{M}^{-I} \longrightarrow \mathrm{Sp}.$$

Moreover, these spectral systems have the property that for each Floer generator, $\mathfrak{o}(x)$ *is weak equivalent to a shift of the sphere spectrum* $\mathbb{S}^{\mu(x)}$*.*

Indeed, these are defined as the direct limit of Thom spectra

$$\mathfrak{o}(x) = \varinjlim \mathcal{U}_k(x)^{\mathrm{ind}(D)}. \tag{8.30}$$

The structure maps $\mathfrak{o}(x_0) \wedge \mathcal{M}^{-I}(x_0, x_1) \longrightarrow \mathfrak{o}(x_1)$ are the associated maps on Thom spectra associated to the gluing maps on the spaces of caps (8.12) and the index bundles over them (8.13).

8.5. **Framings for moduli spaces of maps from surfaces.** Having now endowed the flow categories associated to exact Hamiltonian and Lagrangian Floer theory with stable framings, we now explain how these results generalize to give framings over maps between such flow categories determined by moduli spaces of maps from genus zero punctured Riemann surfaces.

Consider a family $\mathcal{S} \to \mathcal{R}$ of open-closed pointed Riemann surfaces, with incoming points labelled $\Sigma^+ \cong \{1, 2, \ldots, k\}$ and a single outgoing point labelled $\Sigma^- \cong \{0\}$, equipped with Lagrangian labels, regular Floer data, and regular perturbation data. We can also allow this family to degenerate, in which case we take $\mathcal{R}$ to have boundary $\sqcup_\beta \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$, and we require the Lagrangian labels, Floer data and perturbation data to be consistent. We will, at this point, impose a restriction on the families $\mathcal{S}$ which has hitherto not been required:

- the family of surfaces $\mathcal{S} \to \mathcal{R}$ consists of *genus zero* punctured Riemann surfaces with boundary.

Write $\mathcal{M}_0, \ldots, \mathcal{M}_k$ for the flow categories associated to the regular Floer data at the ends of $\mathcal{S}$. The moduli spaces $\mathcal{F}(x_0; x_1, \ldots, x_k)$ of maps from $\mathcal{S}$ to M with these asymptotics and boundaries on the Lagrangian labels is then a $\mathcal{R}$-map of flow categories $\mathcal{M}_0 \longrightarrow (\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k)$.

Recall that for $(r, u) \in \mathcal{F}(x_0; x_1, \ldots, x_k)$, there are two associated Fredholm operators: on the one hand we have the pointwise linearized operator

$$D_{\mathcal{S},u} : W_1^2(\mathcal{S}_r, u^*TM) \longrightarrow L^2(\mathcal{S}_r, \Omega^{0,1} \otimes_J u^*TM)$$

as well as the extended linearized operator

$$D^{\text{ext}}_{\mathcal{S},u} : T_r\mathcal{R} \oplus W_1^2(\mathcal{S}_r, u^*TM) \longrightarrow L^2(\mathcal{S}_r, \Omega^{0,1} \otimes_J u^*TM).$$

The regularity condition ensures that the extended linearized operator is surjective, which is not true for the pointwise linearized operator. The virtual index bundle fo the pointwise linearized operator can be computed as the formal difference

$$\text{ind}(D_{\mathcal{S},u}) = \ker(D^{\text{ext}}_{\mathcal{S},u}) - T\mathcal{R};$$

to make this more concrete, choose a vector bundle E over $\mathcal{R}$ together with an isomorphism

$$E \oplus T\mathcal{R} \cong \mathbb{R}^m \tag{8.31}$$

for some m. In the case that $\mathcal{R}$ has boundary $\sqcup_\beta \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$, we fix such an E_1^β and E_2^β with

$$E_1^\beta \oplus T\mathcal{R}_1^\beta \cong \mathbb{R}^{m_1}, \qquad E_2^\beta \oplus T\mathcal{R}_2^\beta \cong \mathbb{R}^{m_2}$$

such that $m = m_1 + m_2$, together with an isomorphism

$$E_1^\beta \oplus E_2^\beta \cong E|_{\mathcal{R}^\beta} \oplus \mathbb{R}$$

so that the square

$$\begin{array}{ccc} E_1^\beta \oplus E_2^\beta \oplus T\mathcal{R}_1^\beta \oplus T\mathcal{R}_2^\beta & \longrightarrow & \mathbb{R}^{m_1} \oplus \mathbb{R}^{m_2} \\ \downarrow & & \downarrow \\ E|_{\mathcal{R}^\beta} \oplus \mathbb{R} \oplus T\mathcal{R}_1^\beta \oplus T\mathcal{R}_2^\beta \longrightarrow E|_{\mathcal{R}^\beta} \oplus T\mathcal{R}|_{\mathcal{R}^\beta} & \longrightarrow & \mathbb{R}^{m_1+m_2} \end{array}$$

commutes. In particular we then model the index bundle of the pointwise operator with the vector bundle

$$I(r,u) = E_r \oplus \ker(D^{\text{ext}}_{S,u}). \tag{8.32}$$

Observe that in fact $I(r,u)$ is the kernel of a surjective Fredholm operator

$$\mathbb{R}^m \oplus W^2_1(\mathcal{S}_r, u^*TM) \cong E_r \oplus T_r\mathcal{R} \oplus W^2_1(\mathcal{S}_r, u^*TM) \longrightarrow L^2(\mathcal{S}_r, \Omega^{0,1} \otimes_J u^*TM)$$

which on the $W^2_1(\mathcal{S}_r, u^*TM)$ factor is exactly the pointwise linearized Floer operator $D_{\mathcal{S},u}$.

In particular, consider for $q_1, q_2, \ldots, q_k$ the spaces of abstract caps $\mathcal{U}_{q_1}(x_1), \ldots, \mathcal{U}_{q_k}(x_k)$ as defined in the previous section. There are then gluing maps

$$G : \mathcal{F}(x_0; \ldots, x_k) \times \mathcal{U}_{q_1}(x_1) \times \ldots \times \mathcal{U}_{q_k}(x_k) \longrightarrow \mathcal{U}_{m+\sum q_i}(x_0) \tag{8.33}$$

well defined up to a contractible space of choices through exactly the same procedure as before. Moreover, over this gluing map there is likewise an isomorphism of the index vector bundles

$$\phi : I(r,u) \oplus \ker(D_{v_1} + f_1) \oplus \ldots \oplus \ker(D_{v_k} + f_k) \longrightarrow \ker(D_{G(u,v_1,\ldots,v_k)} + \tilde{f}) \tag{8.34}$$

These isomorphisms satisfy similar coherence conditions as in the case of gluing abstract caps onto flow categories of a Floer datum as explained earlier, as well as an additional coherence diagram arising from the decomposition of a boundary component $\mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$ of $\mathcal{R}$ and corresponding degeneration of the Riemann surfaces: the square

$$\begin{array}{ccc} \mathcal{F}_1^\beta(x_0; \ldots x_\beta \ldots) \times \mathcal{F}_2^\beta(x_\beta; \ldots) \times \prod \mathcal{U}_{q_i}(x_i) & \longrightarrow & \mathcal{F}(x_0; x_1, \ldots, x_k) \times \prod \mathcal{U}_{q_i}(x_i) \\ \downarrow & & \downarrow \\ \mathcal{F}_1^\beta(x_0; \ldots x_\beta \ldots) \times \prod \mathcal{U}_{q_i}(x_i) & \longrightarrow & \mathcal{U}_{m+\sum q_i}(x_0) \end{array}$$

commutes, where the left vertical map is the composition (8.33) for $\mathcal{F}_2^\beta$. Moreover, the isomorphisms ϕ of index bundles also satisfy the analogous coherence conditions to (8.28) and (8.29).

In particular, by taking the Thom spectra associated to (8.33) and (8.34) we have:

Proposition 8.9. *Given flow categories $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_k$ from regular Hamiltonian or Lagrangian Floer data, and a family of open-closed pointed Riemann surfaces $\mathcal{S} \to \mathcal{R}$ with consistent labels and perturbation data, where $\mathcal{R}$ is either a closed manifold or has a boundary $\sqcup_{\beta \in B} \mathcal{R}_1^\beta \times \mathcal{R}_2^\beta$ over which $\mathcal{S}$ degenerates, the spectral systems $\mathfrak{o}_i$ for each $\mathcal{M}_i$ of Proposition 8.8 admit an extension over the corresponding $\mathcal{R}$-map $\mathcal{F} : \mathcal{M}_0 \to (\mathcal{M}_1, \ldots, \mathcal{M}_k)$.*

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