Bounds on Urysohn width

by

Alexey Balitskiy

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Author . . . .
Department of Mathematics
April 20, 2021

Certified by . . . .
Larry Guth
Claude Shannon Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Davesh Maulik
Chair, Department Committee on Graduate Theses

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Abstract

The Urysohn d-width of a metric space quantifies how closely it can be approximated by a d-dimensional simplicial complex. Namely, the d-width of a space is at most w if it admits a continuous map to a d-complex with all fibers of diameter at most w. This notion was introduced in the context of dimension theory, used in approximation theory, appeared in the work of Gromov on systolic geometry, and nowadays it is a metric invariant of independent interest. The main results of this thesis establish bounds on the width, relating local and global geometry of Riemannian manifolds in two contexts. One of them is bounding the global width of a manifold in terms of the width of its unit balls. The other one is waist-like inequalities, when a manifold is sliced into a family of (singular) surfaces, and the global width is related to the supremal width of the slices.

Thesis Supervisor: Larry Guth
Title: Claude Shannon Professor of Mathematics

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I am deeply indebted to my advisor Larry Guth for all his help and support. Besides him, I could mention dozens of people to whom I am grateful; I owe a lot to my friends, relatives, colleagues and music teachers. Nevertheless, I would like to keep this section short and meaningful, so let me just specifically mention Sasha Berdnikov, my roommate and coauthor. I truly admire his geometric intuition from which I benefited a lot. It is my pleasure and honor to work with him.

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Chapter 1

Introduction

Almost a century ago Pavel Samuilovich Urysohn, a Soviet mathematician of a dramatically short but prolific career, made several contributions that made him one of the founding fathers of dimension theory and general topology. This thesis is devoted to one particular notion, introduced by Urysohn in 1923 under the Russian name *poperechnik*. Now it is commonly called the *Urysohn width*.

Roughly speaking, the d-width measures how well a metric space can be approximated by a d-dimensional complex. A compact metric space is said to have *d-width* at most w, if there is a continuous map from the space to a d-dimensional simplicial complex with all fibers having diameter at most w. Informally, a space is "approximately d-dimensional" if its d-width is small, and it is "essentially $(d+1)$-dimensional" if its d-width is considerable.

The d-width of a $(d+1)$-dimensional space (named the *flattening coefficient* by Urysohn) is perhaps the most interesting metric invariant among the widths. It plays a role in systolic geometry, and overall it seems to be an upper bound for more or less any interesting invariant measuring the "size" of a space in length units.

On the other hand, the widths (the flattening coefficient as well as the others) often behave counterintuitively. As we will see,

- the product of two essentially 2-dimensional spaces can be approximately 3-dimensional;

- the unit balls in an essentially 2-dimensional surface can all be approximately 1-dimensional;
- there is an essentially 4-dimensional topological ball that can be sliced into a 1-dimensional family of approximately 2-dimensional fibers.

1.1 Outline

Chapter 2 collects the background on the notion of Urysohn width, including its history and significance in metric geometry. Several equivalent definitions are given and their properties are discussed.

Chapter 3 deals with the relation between local Urysohn width bounds in a Riemannian manifold and its global width. The results of this chapter constitute the preprint [7] joint with Aleksandr Berdnikov. We bound the 1-width of a Riemannian manifold in terms of its first homology and the supremal width of its unit balls. Answering a question of Larry Guth, we give examples of n-manifolds of considerable $(n-1)$-width in which all unit balls have arbitrarily small 1-width. In other words, the locality of topological dimension is not robust. We also give examples of topologically simple manifolds that are locally nearly low-dimensional.

Chapter 4 explains a simple proof of Karasev's topological centerpoint theorem (the topological counterpart of Rado's centerpoint theorem), based on a certain property of the Urysohn width, and using little to no topology.

Chapter 5 is dedicated to various questions of the following kind: for a continuous map $X \to Y$ from a compact metric space to a simplicial complex, can one guarantee the existence of a fiber large in the sense of Urysohn width? The results of this chapter, which form another joint preprint [8] with Aleksandr Berdnikov, include the following.

1. Any piecewise linear map $f : [0,1]^{m+2} \to Y^m$ from the unit euclidean $(m+2)$-cube to an m-polyhedron must have a fiber of 1-width at least $\frac{1}{2\beta m+m^2+m+1}$, where $\beta = \sup_{y\in Y} \operatorname{rk} H_1(f^{-1}(y))$ measures the topological complexity of the map.

2. There exists a piecewise smooth map $X^{3m+1} \to \mathbb{R}^m$, with X a Riemannian $(3m+1)$-manifold of large $3m$-width, and with all fibers being topological $(2m+1)$-balls of arbitrarily small $(m+1)$-width.

1.2 Notation

Symbol	Meaning
M^n	an n-dimensional Riemannian manifold (the indication of dimension is often omitted)
$U_r(S)$	the open r-neighborhood of S in a metric space
$\lesssim, \gtrsim$	inequalities that hold up to a multiplicative factor depending on dimension only
$A \sim B$	$A \lesssim B$ and $A \gtrsim B$

Table 1.1: Notation

Chapter 2

Urysohn width: history, context, properties

What we now call the *Urysohn width* was introduced under the Russian name *poperechnik* in early 1920s by Pavel Urysohn. Since then it appeared in the literature translated as "waist", "diameter" or "width", and we stick to the latter as the one most commonly used nowadays.

The d-width quantifies the failure of a space to be d-dimensional. The original definition of Urysohn was given in terms of closed coverings, and we give an overview of different equivalent ways of defining width in Section 2.3.

The goal of Section 2.1 is to give an overview of different notions of "waist"/"width" of a space. The earliest publications on the matter were written in 1920s and 1930s, and they are mostly in French and German. After that, much of the literature on waists was written in Russian, but even those sources sometimes define and attribute some of the waists incorrectly. Our hope is to clear out some of the confusion. It was Gromov who in 1980s brought the Urysohn width to the attention of the English-speaking mathematical community, and in a completely different context of metric geometry, which we review in Section 2.2. Nowadays it is considered as a metric invariant of independent interest.

Section 2.4 explains some simple properties of the width, but also contains a few examples that might be new; those examples show that the width does not respect taking products and covering maps.

Section 2.5 collects very few examples for which we know the width exactly.

2.1 Early notions of waist

I use the word "waist" as an umbrella term for multiple "size" invariants of spaces, quantifying their dimensional properties.

Pavel Urysohn proposed the historically first notion of waist around 1923, but it was only published posthumously by Pavel Alexandrov [3], based on Urysohn's records and their communication.

Definition 2.1.1. The *Urysohn d-width* (or *d-waist*, or *d-diameter*) of a compact metric space X is

$$\mathrm{UW}_d(X) = \inf_{\bigcup C_i = X} \sup_i \mathrm{diam}(C_i),$$

where the infimum is taken over all finite closed covers of S of multiplicity at most $d+1$. (A cover is said to have multiplicity, or ply, at most $d+1$ if no point of the space belongs to more than $d+1$ sets of the cover.)

In the publication [3] the new invariant does not carry any specific name, except for the codimension 1 case, when $d = \dim X - 1$.[1] This codimension 1 width is named *coefficient d'aplatissement* (the "flattening coefficient"). In the Russian translation [57] the name *poperechnik* is added.

Urysohn introduced the waist in the context of dimension theory. There are several ways to define the dimension of a topological space, introduced and studied by Lebesgue, Brouwer, Urysohn, and Menger. A theorem of Urysohn [56] says that in compact metric spaces the notions of inductive dimension (both small and large) and covering dimension coincide. A topological space is said to have *covering dimension*[2] at most d if any open cover admits a refining cover of multiplicity at most $d+1$. For compact spaces, one can use finite closed covers instead of open ones. The covering

[1] The dimension here and everywhere is the topological dimension, see more on that below.
[2] Also called the Lebesgue dimension or topological dimension.

dimension of a compact metric space coincides [56] with the *metric dimension*: it is at most d if for any $\varepsilon > 0$ there is a cover with multiplicity at most $d+1$ consisting of sets with diameter $< \varepsilon$. Informally, a d-dimensional space, when covered by "small" sets, must have a point covered by $d+1$ sets. Urysohn used this as a starting point to define the "deficit of the d-dimensionality" of a space as the infimal size of sets in the covering that has multiplicity $d+1$. Alexandrov notes that Urysohn's idea of considering finite closed covers was the predecessor for the Lusternik–Schnirelmann category.

Urysohn used the flattening coefficients in order to formulate the following result on the "continuity of dimension".

Theorem 2.1.2 (Urysohn [3])**.** *Let $F_1 \supset F_2 \supset \dots$ be a nested sequence of d-dimensional closed sets in a compact metric space. Then the intersection $\bigcap F_i$ has dimension d if and only if $\lim_{i\to\infty} \mathrm{UW}_{d-1}(F_i) > 0$.*

It is folklore that Urysohn's definition can be equivalently reformulated in terms of mappings (this is explained in Section 2.3). The equivalence is most likely due to Alexandrov, who actually invented the machinery for that proof, including the notion of the *nerve* of a cover.

Definition 2.1.3. The *Alexandrov d-cowidth* (or *d-codiameter*) of a compact metric space X is

$$\mathrm{AW}^d(X) = \inf_{p: X\to Z} \sup_{z\in Z} \mathrm{diam}(p^{-1}(z)),$$

where the infimum is taken over all continuous maps p from X to any metrizable topological space Z of dimension at most d.

Let me reiterate: $\mathrm{UW}_d(X) = \mathrm{AW}^d(X)$, and this is the most common definition of the Urysohn width in the recent literature. This definition is implicit in [4]; it was explicitly introduced in [55] as parallel (and dual, in a sense) to the Alexandrov width (see below), and the word *cowidth* is my translation of the Russian *kopoperechnik* appearing in later publications (e.g., [37], where it is translated as *codiameter*).

The following definition was given by Alexandrov [4] in the case $Y = \mathbb{R}^n$.

Definition 2.1.4. Let X be a compact metric space isometrically embedded in a metric space Y. The (relative) *Alexandrov d-width* (or *d-waist*, or *d-diameter*) of X in Y is

$$\mathrm{AW}_d(X,Y) = \inf_{p:X\to Z} \sup_{x\in X} \mathrm{dist}_Y(x, p(x)),$$

where the infimum is taken over all continuous maps p from X to any subspace $Z \subset Y$ of dimension at most d.

It is easy to see that $\mathrm{AW}^d(X) \le 2\,\mathrm{AW}_d(X,Y)$. Less obvious but true is the inequality $\mathrm{AW}_d(X,Y) \le \mathrm{UW}_d(X)$ provided that Y is a Banach space (so that the nerve of a covering of X can be geometrically realized in Y, possibly with self-intersections). Together with the equality $\mathrm{UW}_d(X) = \mathrm{AW}^d(X)$ these imply that the Alexandrov width (relative to a Banach space) and cowidth differ by a factor of at most 2.

Some authors [5, 31, 45] use the terms "Alexandrov diameter" and "Alexandrov width" for slightly different notions but I remark that Alexandrov did not introduce those.

Alexandrov's motivation was to introduce the analogues of Urysohn's concept building up on the alternative definitions of dimension based on homology with coefficients. Alexandrov used an early version of homology due to Vietoris, and we give here a counterpart with singular homology, in a general ambient space Y (Alexandrov used $Y = \mathbb{R}^n$).

Definition 2.1.5. Let X be a compact metric space isometrically embedded in a metric space Y. The *homological d-width* of X in Y is

$$\mathrm{HW}_d(X,Y) = \sup_{Z\in\mathcal{B}_d(X)} \inf\{r \ge 0 \mid Z \text{ bounds a } (d+1)\text{-chain of } Y \text{ in } U_r(\operatorname{supp} Z)\}.$$

Here $\mathcal{B}_d(X)$ is the space of d-boundaries (null-homologous singular d-cycles) in X; $\operatorname{supp} Z$ is the subset of X where the images of the singular simplices of Z land; $U_r(\cdot)$ is the r-neighborhood taken in Y.

We do not specify the homology coefficients here; one can consider any coefficients, and in fact, Alexandrov pays specific attention to the "mod 1" coefficients, $\mathbb{Q}/\mathbb{Z}$ or

$\mathbb{R}/\mathbb{Z}$. Alexandrov remarks that $\mathrm{HW}_d(X,Y) \le \mathrm{AW}_d(X,Y) \le \mathrm{UW}_d(X)$ in his case of interest ($Y = \mathbb{R}^n$), so I think it natural to consider this definition over ambient spaces Y in which this estimate holds; for example, over Banach spaces.[3] He also says that he is not aware of any inequalities of the form $\mathrm{UW}_d(X) \lesssim \mathrm{HW}_d(X,Y)$.[4] The ideas of this thesis can be used to build an example showing that no such inequality holds for $d = 1$, $Y = \mathbb{R}^4$, and $\mathbb{Z}$ coefficients; this example is a modification of the surface built in the proof of Theorem 2.4.4; it can be embedded in $\mathbb{R}^4$.

Another early notion of waist was introduced by Kolmogorov [38] in the context of approximation theory.[5] It can be regarded as a linear version of Alexandrov's width.

Definition 2.1.6. Let Y be a normed space with the unit ball B. The *Kolmogorov d-width* (or *d-waist*, or *d-diameter*) of a compact set $X \subset Y$ is

$$\mathrm{KW}_d(X,Y) = \inf\{r \mid X \subset rB + L, \text{for some affine } d\text{-subspace } L\}.$$

It took quite a few years after the work of Kolmogorov until the interest to waists has resumed. A plethora of other waists, which we do not consider here, were introduced in the '60s; Tikhomirov's systematic treatment [55] is a nice reference, but it only exists in Russian. Tikhomirov distinguishes between waists (arising in approximation problems) and cowaists (arising in optimal recovery), and establishes a correspondence between those. For instance, Definition 2.1.4 and Definition 2.1.3 give such a pair; this parallelism is perhaps one of the reasons why the term "Alexandrov width" is misused often.

The work of Tikhomirov [54] initiated the study of the dependence of relative waists, such as the Alexandrov width, on the ambient space Y. Expanding Y decreases the width, and one can ask whether there is a universal metric extension

[3] It is easy to show that $\inf_Y \mathrm{HW}_d(X,Y) = \mathrm{HW}_d(X, L^\infty(X))$, where $X \subset L^\infty(X)$ is the Kuratowski embedding (see below). Indeed, if a d-cycle of X bounds in its r-neighborhood inside Y, on can project the filling cycle via the 1-Lipschitz map $Y \overset{\text{Kurat.}}{\to} L^\infty(Y) \overset{\text{restr.}}{\to} L^\infty(X)$. This way one could define the *absolute homological d-width* as $\mathrm{HW}_d(X, L^\infty(X))$.

[4] In [21] Gromov asked this question for $d = 1$, and $X = Y$ being a path metric space; Aleksandr Berdnikov constructed counterexamples for several different homology coefficients (for example, with $\mathbb{Z}$ coefficients a certain connected sum of projective planes will do).

[5] It could be used, for instance, to quantify how well a class of functions can be approximated by trigonometric polynomials of bounded complexity.

minimizing the relative width. The following definition is due to Ismagilov [28], except that he worked with Banach isometric extensions, but that does not change the result.

Definition 2.1.7. Let X be a compact metric space. The *absolute Alexandrov d-width* of X is

$$\overline{\mathrm{AW}}_d(X) = \inf_Y \mathrm{AW}_d(X, Y),$$

where the infimum is taken over metric spaces Y admitting isometric embedding $X \hookrightarrow Y$.

Theorem 2.1.8. *The absolute Alexandrov width of X is attained for a specific metric extension $X \hookrightarrow \widetilde{X}$:*

$$\overline{\mathrm{AW}}_d(X) = \mathrm{AW}_d(X, \widetilde{X}).$$

Moreover, $\widetilde{X}$ can be taken a Banach space.

If X is a subset of a Banach space Y, one can take $\widetilde{X}$ to be the space of bounded functions on the unit ball of the dual space Y^* ([28, 55]). To apply this for an abstract metric space X, one can first use the Kuratowski isometric embedding [39]:

$$\begin{aligned} X &\to L^\infty(X), \\ x &\mapsto \mathrm{dist}_X(\cdot, x), \end{aligned}$$

where $L^\infty(X)$ denotes the space of bounded functions on X with the sup-norm. It turns out though that a simpler construction is possible: one can just take $\widetilde{X} = L^\infty(X)$ with the Kuratowski embedding $X \hookrightarrow \widetilde{X}$. The equality $\overline{\mathrm{AW}}_d(X) = \mathrm{AW}_d(X, \widetilde{X})$ is implicitly proven in [16] as well as the following earlier-known [55] equality, which can be taken as yet another definition of the Urysohn width. We give an argument proving both of these.

Theorem 2.1.9.

$$\overline{\mathrm{AW}}_d(X) = \frac{1}{2}\,\mathrm{AW}^d(X).$$

Proof of Theorem 2.1.8 and Theorem 2.1.9. It is clear that $\mathrm{AW}^d(X) \le 2\,\mathrm{AW}_d(X,Y)$ for any isometric extension Y; therefore, $\mathrm{AW}^d(X) \le 2\,\overline{\mathrm{AW}}_d(X)$. We will show that $2\,\mathrm{AW}_d(X, L^\infty(X)) \le \mathrm{AW}^d(X)$, and both results will follow. Suppose that $\mathrm{AW}^d(X) < w$, certified by a map $p : X \to Z$, $\dim Z \le d$, $\delta = \sup_z \operatorname{diam} p^{-1}(z) < w$.

To show that $\mathrm{AW}_d(X, L^\infty(X)) < w/2$ we need to come up with a map $X \to L^\infty(X)$ with at most d-dimensional image and $w/2$-close to the Kuratowski embedding. Here is a naïve attempt. To each $z \in Z$ assign the following function in $L^\infty(X)$:

$$f_z(\cdot) = \frac{\delta}{2} + \operatorname{dist}(\cdot, p^{-1}(z)).$$

The map $x \mapsto f_{p(x)}(\cdot)$ is $w/2$-close to the Kuratowski embedding, but the problem is that the assignment $z \mapsto f_z(\cdot)$ is discontinuous. We employ the standard "skeletal" argument to bypass this issue.

Cover Z with small open sets U_i with multiplicity $\le d+1$; "small" here means that $\sup_i \operatorname{diam} p^{-1}(U_i) < w$. Associated to this cover, we have a map $\varphi : X \to N$ to the nerve of the cover with the property that the preimage of any open star[6] S_v of vertex $v \in N$ has diameter $< w$. Under the Kuratowski embedding $X \to L^\infty(X)$, this preimage $\varphi^{-1}(S_v)$ gets sent to a set of diameter $< w$, and in $L^\infty(X)$ it is possible to cover it by a ball of radius $w/2$; let c_v be its center. Now we can define the piecewise linear map $c : N \to L^\infty(X)$ that sends v to c_v and extends linearly to N. The composition $X \xrightarrow{\varphi} N \xrightarrow{c} L^\infty(X)$ shows that $\mathrm{AW}_d(X, L^\infty(X)) < w/2$. □

2.2 Waists in metric geometry

As mentioned before, early notions of waists were introduced in the context of dimension theory and approximation theory. In the '80s, Gromov published extremely influential works [16, 19], in particular, introducing the Urysohn width to the context of metric geometry. The Urysohn d-width appears in [16, Appendix 1] under the name "d-diameter", and in [19] under the name "codimension d diameter". The Alexandrov

[6] The *open star* of a vertex of a simplicial complex is the union of the relative interiors of all faces containing the given vertex.

d-width appears in [16, Appendix 1] under the name "d-radius of an embedding", and the relation between the Urysohn width and the Alexandrov width of the Kuratowski embedding is explained there. Another proof of Theorem 2.1.9 can be found in [19].

Among other contributions, works [16, 19] establish relations between the width and other metric invariants, and pose several interesting questions about the behavior of the width.

Relation to the filling radius

The (absolute) *filling radius* of a closed Riemannian manifold M^n is the infimal $r \geq 0$ such that inside the r-neighborhood of the Kuratowski image of M in $L^\infty(M)$, the fundamental class $[M]$ bounds an $(n+1)$-chain.

Lemma 2.2.1 (Gromov [16])**.**

$$\mathrm{FillRad}(M^n) \leq \frac{1}{2}\,\mathrm{UW}_{n-1}(M^n).$$

For a short proof, see [21, 4.C].

Relation to the systole

The *systole* of a closed Riemannian manifold M^n with non-trivial fundamental group is the length of the shortest non-contractible loop. Gromov proved [16, Section 1.2] that the systole does not exceed six times the filling radius, provided that the manifold is *essential*, that is, the map $M \to K(\pi_1(M), 1)$ sends $[M]$ to a non-zero class in $H_n(K(\pi_1(M), 1))$. Together with Lemma 2.2.1 this bounds the systole in terms of the flattening coefficient. A more direct argument (with a better constant) can be found in [45]; its idea goes back at least to Švarc [53].

Lemma 2.2.2. *In an essential manifold,*

$$\mathrm{sys}(M^n) \leq 2\,\mathrm{UW}_{n-1}(M^n).$$

Relation to the volume

Larry Guth proved the following estimate conjectured by Gromov.

Theorem 2.2.3 (Guth [25]). *For a closed Riemannian manifold,*

$$\mathrm{UW}_{n-1}(M^n) \lesssim \mathrm{vol}(M^n)^{1/n}.$$

Together with Lemma 2.2.2 this gives another proof of Gromov's systolic inequality [16]. Nabutovsky gave a transparent proof [45] of this theorem with best known constants, building on the ideas from [47, 23].

For generalizations of Theorem 2.2.3 with weaker assumptions and relation to the Hausdorff content, see introduction to Chapter 3.

The following result on the product of widths was conjectured by Gromov [19] and proved by Perelman [48].

Theorem 2.2.4 (Perelman [48]). *For a closed Riemannian n-manifold with non-negative sectional curvature,*

$$\prod_{d=0}^{n-1} \mathrm{UW}_d(M^n) \sim \mathrm{vol}(M^n).$$

Relation to the curvature

Gromov conjectured [19] that a closed Riemannian manifold M^n whose scalar curvature is everywhere $\geq \sigma^2$ has codimension 2 width $\lesssim \frac{1}{\sigma}$. For $n = 2$, it was already understood by Bonnet. For $n = 3$, it is proven in [18] under additional homological assumptions.[7] A complete proof for $n = 3$ follows from the results of [13, 43]. The first of those papers uses the machinery of μ-bubbles (a generalization of minimal surfaces), and the second one uses (a version of) mean curvature flow. A simpler argument, which I learned from Yevgeny Liokumovich, could be given following the ideas from [21], and I sketch it below. For $n \geq 4$, the conjecture is open. It is not even known if the codimension 1 width could be bounded from above.

[7] Those assumptions are not mentioned in [18] but $H_1(M^3; \mathbb{Q}) = 0$ is enough; see also [16, Corollary E_2'] and [35] for different sufficient assumptions.

Theorem 2.2.5. *For a closed Riemannian manifold M^3 whose scalar curvature is ≥ 1,*

$$\mathrm{UW}_1(M^3) \lesssim 1.$$

Proof sketch. **Step 1** (cf. [21, Section 6]). It is possible to find finitely many non-intersecting stable minimal surfaces Σ_i such that the homology of the complement $H_1(M \setminus \bigcup \Sigma_i)$ is pure torsion. This is done inductively. After picking several surfaces, cut the manifold along them to get a manifold M' with boundary; if $H_2(M', \partial M') \neq 0$, find a stable minimal surface representing a non-zero class, and take it as the next Σ_i. Since M' has mean convex boundary, Σ_i does not intersect $\partial M'$. It also follows that the diameter of Σ_i is bounded (since the positive curvature of M' restricts to positive curvature of Σ_i). Eventually we arrive at the situation $H_2(M', \partial M') = 0$, and so $H_1(M')$ is pure torsion (by the Poincaré duality plus the universal coefficients theorem).

Step 2. Having M', a manifold with positive curvature and mean convex boundary with $H_1(M'; \mathbb{Q}) = 0$, one repeats the argument from [18, Corollary 10.11]: consider the distance levels from any point p and show that their connected components have controlled diameter, so that $\mathrm{UW}_1(M')$ is bounded. It might seem at this point that we are done, since all Σ_i have bounded diameters as well, but there is a caveat here. It might happen that boundary components of M' are connected to each other along the distance levels; then we would have to merge them all in the computation for $\mathrm{UW}_1(M)$; it can make it arbitrarily large. A possible solution here might be to "slow down" the distance function $d(\cdot) = \mathrm{dist}_{M'}(\cdot, p)$ near the boundary components. Blow up the metric in the small collar neighborhood of the boundary, and let $\widetilde{d}(\cdot)$ be the distance function from p in this new metric. It can be shown that any level set of $\widetilde{d}(\cdot)$ lies in a connected component of $d^{-1}([r-c, r])$, for some absolute constant c and some number r depending on the level set. Such components can be shown to have bounded diameter (cf. [43, Lemma 4.1]). □

Relation to the hypersphericity

It was observed by Gromov [19, Proposition F_1] that a closed Riemannian manifold M^n with $\mathrm{UW}_{n-1}(M) < 1$ does not admit a 1-Lipschitz map to the unit sphere S^n of non-zero degree. I refer to [22, Section 5]) for a detailed explanation of this result. One defines the *hypersphericity* [18, 17] of a closed Riemannian manifold M^n as the supremal radius $R \geq 0$ such that the manifold admits a 1-Lipschitz map of non-zero degree to the round n-sphere of radius R. Gromov observed that the hypersphericity is bounded from above by the codimension 1 width. The argument uses the trick that I call "fiber contraction". It will be used repeatedly in the sequel (Lemma 2.2.7, Theorem 3.0.2, etc.).

Paper [22] explains that for a Riemannian metric on the 2-sphere, the hypersphericity and the 1-width differ by a factor of no more than 7.

The following generalized definition of hypersphericity was suggested to me by Larry Guth.

Definition 2.2.6. The *k-th hypersphericity* of an n-dimensional manifold M, endowed with a metric, is

$$\mathrm{HS}_k(M) = \sup\{R \mid \exists \text{1-Lipschitz map } g : M \to \mathrm{Ell}(\underbrace{R, \ldots, R}_{k}, \underbrace{\varepsilon, \ldots, \varepsilon}_{n+1-k}) \text{ of non-zero degree, for some } \varepsilon > 0\}.$$

Here

$$\mathrm{Ell}(\lambda_0, \ldots, \lambda_n) = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} \left(\frac{x_i}{\lambda_i}\right)^2 = 1 \right\}$$

is the n-dimensional ellipsoid with the semi-axes λ_i with the intrinsic metric induced from $\mathbb{R}^{n+1}$.

In the extreme case $k = n$ the definition of $\mathrm{HS}_n(M^n)$ matches the conventional hypersphericity up to a constant factor, since the ellipsoid $\mathrm{Ell}(\underbrace{1, \ldots, 1}_{n}, \varepsilon)$ is bi-Lipschitz diffeomorphic to the unit n-sphere.

Lemma 2.2.7 (Folklore)**.**

$$\mathrm{HS}_k(M^n) \le 30\,\mathrm{UW}_{k-1}(M^n).$$

Proof. Let $g : M \to E = \mathrm{Ell}(\underbrace{R, \dots, R}_{k}, \underbrace{\varepsilon, \dots, \varepsilon}_{n+1-k})$ be a 1-Lipschitz map of non-zero degree, for some small $0 < \varepsilon < \frac{R}{100}$. We will prove that $R \le 30\,\mathrm{UW}_{k-1}(M^n)$. Assume the contrary, and let $p : M \to Y^{k-1}$ be a map asserting that $\mathrm{UW}_{k-1}(M^n) < R/30$. Every fiber $p^{-1}(y), y \in Y$, is of controlled diameter $< R/30$. Since g is 1-Lipschitz, the set $g(p^{-1}(y))$ has diameter $< R/30$. We will homotope g by deformation-retracting some of the sets $g(p^{-1}(y))$ onto certain $(n-k)$-dimensional sets. This will allow us to break its surjectivity (since $\dim Y + (n-k) < n$), contradicting the assumption on the degree of g.

Cover the ellipsoid E by open sets

$$U = \left\{ (x_0, \dots, x_n) \in E \subset \mathbb{R}^{n+1} \;\middle|\; \sqrt{x_0^2 + \ldots + x_{k-1}^2} < \frac{15R}{30} \right\}$$
$$V = \left\{ (x_0, \dots, x_n) \in E \subset \mathbb{R}^{n+1} \;\middle|\; \sqrt{x_0^2 + \ldots + x_{k-1}^2} > \frac{14R}{30} \right\}$$

and let functions $\varphi, \psi : E \to [0,1]$ form a subordinate partition of unity, $\varphi + \psi = 1$, $\operatorname{supp} \varphi \subset U$, $\operatorname{supp} \psi \subset V$.

Denote Y_U the subset of those $y \in Y$ such that $g(p^{-1}(y)) \cap U \ne \varnothing$. For each $y \in Y_U$ introduce an $(n-k)$-sphere $S_y \subset E$ of the form

$$S_y = \left\{ (x_0, \dots, x_n) \in E \subset \mathbb{R}^{n+1} \;\middle|\; x_0 = a_0, \dots, x_{k-1} = a_{k-1} \right\}$$

by choosing the numbers $a_0, \dots, a_{k-1}$ so that S_y and $g(p^{-1}(y))$ are close to one another (e.g., they intersect). Ideally, we would like $a(y) = (a_0, \dots, a_{k-1})$ to depend continuously on $y \in Y_U$, but a priori this is not true, so we apply a standard argument to construct $a : Y_U \to \left[-\frac{19R}{30}, \frac{19R}{30}\right]^k$ using a fine cell structure on Y. Without loss of generality we can assume that Y is a simplicial complex: if not, we replace Y by the nerve of the cover $\{p^{-1}(W_i)\}$, for a fine enough cover $\{W_i\}$ of Y. We can refine

Y further so that $p^{-1}(\sigma)$ has diameter $< R/30$ for every simplex $\sigma \subset Y$. Extend Y_U slightly so that it becomes a simplicial subcomplex of Y. For each vertex $v \in Y_U$, contained in Y_U, choose $a(y)$ arbitrarily (but so that $S_y \cap g(p^{-1}(y)) \neq \varnothing$). Extend a onto Y_U linearly. Now S_y is defined for all $y \in Y_U$. It is straightforward to check that the Hausdorff distance between S_y and $g(p^{-1}(y))$ is less than $8R/30$ for each $y \in Y_U$. Indeed, if y lies in a simplex $\sigma \subset Y_U$ with vertices $y_1, \dots, y_k$, then all the spheres $S_{y_1}, \dots, S_{y_k}$ intersect the set $g(p^{-1}(\sigma))$ of diameter $< R/30$. Then for the Hausdorff distance we have

$$\operatorname{dist}_H(S_y, g(p^{-1}(y))) \leq \underbrace{\operatorname{dist}_H(S_y, S_{y_1})}_{<2\|a(y)-a(y_1)\|_{\mathbb{R}^k}} + \underbrace{\operatorname{dist}_H(S_{y_1}, g(p^{-1}(y_1)))}_{<R/30} + \underbrace{\operatorname{dist}_H(g(p^{-1}(y_1)), g(p^{-1}(y)))}_{<R/30},$$

and

$$\|a(y) - a(y_1)\|_{\mathbb{R}^k} \leq \max_{2 \leq i \leq k} \|a(y_i) - a(y_1)\|_{\mathbb{R}^k} \leq \max_{2 \leq i \leq k} \operatorname{dist}_H(S_{y_i}, S_{y_1})$$
$$\leq \max_{2 \leq i \leq k} \left(\underbrace{\operatorname{dist}_H(S_{y_i}, g(p^{-1}(y_i)))}_{<R/30} + \underbrace{\operatorname{dist}_H(g(p^{-1}(y_i)), g(p^{-1}(y_1)))}_{<R/30} + \underbrace{\operatorname{dist}_H(g(p^{-1}(y_1)), S_{y_1})}_{<R/30} \right) < \frac{3R}{30}$$

It follows that the geodesic projection $\pi_y : g(p^{-1}(y)) \to S_y$ is well-defined for every $y \in Y_U$. Define $\Pi_y : g(p^{-1}(y)) \times [0,1] \to E$ as the linear homotopy connecting $\mathrm{id} : g(p^{-1}(y)) \to E$ and π_y, that is, $\Pi_y(z,t)$ is the point on the geodesic segment $[z, \pi_y(z)]$ dividing it in the ratio $t : (1-t)$.

Finally, we build the homotopy $H : M \times [0,1] \to E$ as

$$H(x,t) = \Pi_{p(x)}(g(x), \varphi(f(x))t).$$

It connects the map g of non-zero degree to the map $H(\cdot, 1) : M \to E$, which is not surjective, because the points $(x_0, \dots, x_n) \in E$ with $\max_{0 \leq i \leq k-1} |x_i| < R/30$ cannot be covered by a $(k-1)$-dimensional family of $(n-k)$-spheres. □

2.3 Equivalent definitions of the Urysohn width

In this section and the following one, X denotes a compact metric space. The diameter of a set is measured using the distance function in X: $\operatorname{diam} A = \sup_{a,a' \in A} \operatorname{dist}_X(a, a')$. For a map $p : S \to Z$, $S \subset X$, the quantity $\mathrm{W}(p) = \sup_{z \in Z} \operatorname{diam}(p^{-1}(z))$ will be called the *width* of p.

Definition 2.3.1. The *Urysohn d-width* of a closed subset S of a compact metric space X can be defined in either of the following ways.

$$\mathrm{UW}_d(S) = \inf_{\substack{\bigcup C_i = S \\ \text{mult.}\{C_i\} \le d+1}} \sup_i \operatorname{diam}(C_i), \tag{UC}$$

where the infimum is taken over all finite closed covers of S of multiplicity at most $d + 1$.

$$\mathrm{UW}_d(S) = \inf_{\substack{\bigcup U_i \supset S \\ \text{mult.}\{U_i\} \le d+1}} \sup_i \operatorname{diam}(U_i), \tag{UO}$$

where the infimum is taken over all open covers of S of multiplicity at most $d + 1$.

$$\mathrm{UW}_d(S) = \inf_{\bigcup_{i=0}^{d} W_i \supset S} \sup_i \sup_{\substack{U \text{ conn.} \\ \text{comp. of } W_i}} \operatorname{diam}(U), \tag{UO'}$$

where the infimum is taken over all open covers of S colored in at most $d + 1$ colors so that not two sets of the same color intersect.

$$\mathrm{UW}_d(S) = \inf_{\substack{p: S \to Z \\ Z \text{ simpl.} \\ \dim Z \le d}} \mathrm{W}(p), \tag{UM$^\mathrm{s}$}$$

where the infimum is taken over all continuous maps p from S to any simplicial complex Z of dimension at most d.

$$\mathrm{UW}_d(S) = \inf_{\substack{p: S \to Z \\ Z \text{ metriz.} \\ \dim Z \le d}} \mathrm{W}(p), \tag{UM$^\mathrm{m}$}$$

where the infimum is taken over all continuous maps p from S to any metrizable topological space Z of dimension at most d.

$$\mathrm{UW}_d(S) = \inf_{\substack{p:S\to Z\\ Z \text{ Hausd.}\\ \dim Z\le d}} \mathrm{W}(p), \tag{UM$^\mathrm{h}$}$$

where the infimum is taken over all continuous maps p from S to any Hausdorff topological space Z of (covering) dimension at most d.

$$\mathrm{UW}_d(S) = 2 \inf_{Y \overset{\text{iso}}{\supset} X} \inf_{\substack{p:S\to Z\\ Z\subset Y\\ \dim Z\le d}} \sup_{x\in S} \mathrm{dist}_Y(x, p(x)), \tag{UA}$$

where the infimum is taken over all isometric extensions Y of X and all continuous maps p from S to any subspace $Z \subset Y$ of dimension at most d.

$$\mathrm{UW}_d(S) = 2 \inf_{\substack{p:S\to Z\\ Z\subset L^\infty(X)\\ \dim Z\le d}} \sup_{x\in S} \mathrm{dist}_{L^\infty(X)}(k(x), p(x)), \tag{U$\overline{\mathrm{A}}$}$$

where the infimum is taken over all continuous maps p from S to any subspace $Z \subset L^\infty(X)$ of dimension at most d, and $k(x)$ denotes the Kuratowski image of x in $L^\infty(X)$.

Compare (UC) with Definition 2.1.1, (UM$^\mathrm{m}$) with Definition 2.1.3, (UA) with Definition 2.1.7, and (U$\overline{\mathrm{A}}$) with the conclusion of Theorem 2.1.9 and Theorem 2.1.8.

Proof of the equivalence of different definitions of the Urysohn width.
Denote by $w_\mathrm{c}, w_\mathrm{o}, w_{\mathrm{o}'}, w_\mathrm{s}, w_\mathrm{m}, w_\mathrm{h}, w_\mathrm{a}, w_{\overline{\mathrm{a}}}$ the width of a set $S \subset X$ measured as in (UC), (UO), (UO'), (UM$^\mathrm{s}$), (UM$^\mathrm{m}$), (UM$^\mathrm{h}$), (UA), (U$\overline{\mathrm{A}}$), respectively.

(UO $\le$ UC) Given a finite closed covering $S = \bigcup C_i$, we can use compactness to argue that

$$\delta_{i_0,\dots,i_{d+1}} = \min_{x\in X} \max_{0\le k\le d+1} \mathrm{dist}(x, C_{i_k})$$

is attained and positive. Take $\varepsilon > 0$ smaller then each $\delta_{i_0,\dots,i_{d+1}}$ over all collections of indices $i_0 < \ldots < i_{d+1}$, and consider the open covering $\{U_i\}$, where

$U_i = U_\varepsilon(C_i)$ is the ε-neighborhood of C_i. It has the same multiplicity as the covering $\{C_i\}$, and $\max\operatorname{diam} U_i \le \max\operatorname{diam} C_i + 2\varepsilon$. Taking $\varepsilon \to 0$, we get $w_\mathrm{o} \le \max\operatorname{diam} C_i$. Therefore, $w_\mathrm{o} \le w_\mathrm{c}$.

($\mathrm{UC} \le \mathrm{UM}^\mathrm{h}$) Suppose we are given a map $p : S \to Z^d$ to a Hausdorff space. Recall that the width of p is defined as $\mathrm{W}(p) = \sup_{z\in Z} \operatorname{diam}(p^{-1}(z))$. Fix a small number $\varepsilon > 0$. For each point $z \in p(S)$, one can select its open neighborhood V_z such that $\operatorname{diam}(p^{-1}(V_z)) < \operatorname{diam}(p^{-1}(z))+\varepsilon$; one can just pick $V_z = Z\setminus p(S\setminus U_{\varepsilon/2}(p^{-1}(z)))$. Indeed, $S \setminus U_{\varepsilon/2}(p^{-1}(z))$ is closed, hence compact (as S is compact); then $p(S \setminus U_{\varepsilon/2}(p^{-1}(z)))$ is compact, hence closed (as Z is Hausdorff). Therefore, V_z is an open neighborhood of z whose preimage is $U_{\varepsilon/2}(p^{-1}(z))$. By definition of dimension and compactness of $p(S)$, there is a finite open covering $\{V_i\}$ of $p(S)$, refining $\{V_z\}$, and with multiplicity at most $d + 1$. Then the open sets $U_i = p^{-1}(V_i)$ have diameter less than $\mathrm{W}(p) + \varepsilon$, and cover S with multiplicity at most $d + 1$. It follows from Lebesgue's number lemma that there is a closed covering $\{C_i\}$ of S with $C_i \subset U_i$. Repeating this with arbitrarily small ε, one gets $w_\mathrm{c} \le \mathrm{W}(p)$. Since this is true for all p, we conclude $w_\mathrm{c} \le w_\mathrm{h}$.

($\mathrm{UM}^\mathrm{h} \le \mathrm{UM}^\mathrm{m}$) $w_\mathrm{h} \le w_\mathrm{m}$ holds trivially.

($\mathrm{UM}^\mathrm{m} \le \mathrm{UA}$) $w_\mathrm{m} \le w_\mathrm{a}$ holds trivially.

($\mathrm{UA} \le \mathrm{U}\overline{\mathrm{A}}$) $w_\mathrm{a} \le w_{\overline{\mathrm{a}}}$ holds trivially.

($\mathrm{U}\overline{\mathrm{A}} \le \mathrm{UO}$) This was essentially shown in the proof of Theorem 2.1.9 and Theorem 2.1.8. Assume we are given an open cover $S \subset \bigcup U_i$ (which we can assume finite by compactness) with multiplicity at most $d + 1$ and with $\sup_i \operatorname{diam} U_i < w_\mathrm{o} + \varepsilon$, for an arbitrarily small $\varepsilon > 0$. Taking any partition of unity subordinate to this cover, one gets the map $\varphi : S \to N$ to the nerve. The vertices of N can be labeled so that vertex v_i corresponds to the set U_i. The preimage of the open star of $v_i \in N$ has diameter $< w_\mathrm{o} + \varepsilon$, since it is contained in U_i. Under the Kuratowski embedding $X \to L^\infty(X)$, the set U_i gets sent to a set of diameter $< w_\mathrm{o} + \varepsilon$, and in $L^\infty(X)$ it is possible to cover it by a ball of radius

$< \frac{1}{2}(w_{\mathrm{o}} + \varepsilon)$; let c_i be its center. Now one can define the piecewise linear map $c : N \to L^\infty(X)$ that sends v_i to c_i and extends linearly to N. The composition $S \xrightarrow{\varphi} N \xrightarrow{c} L^\infty(X)$ shows that $w_{\bar{\mathrm{a}}} \leq w_{\mathrm{o}} + \varepsilon$.

($\mathrm{UM}^{\mathrm{m}} \leq \mathrm{UM}^{\mathrm{s}}$) $w_{\mathrm{m}} \leq w_{\mathrm{s}}$ holds trivially.

($\mathrm{UM}^{\mathrm{s}} \leq \mathrm{UO}$) Given an open cover, one considers the mapping to the nerve. The preimage of every point is entirely contained in some U_i, hence $\mathrm{W}(\varphi) \leq \sup \operatorname{diam} U_i$. Therefore, $w_{\mathrm{s}} \leq w_{\mathrm{o}}$.

(UO ≤ UO') $w_{\mathrm{o}} \leq w_{\mathrm{o'}}$ holds trivially.

(UO' ≤ UO) This trick is folklore; the earliest reference I found is [46]. Suppose we are given an open cover of S of multiplicity at most d. First, consider the mapping to the nerve, at most d-dimensional complex. Second, take the barycentric subdivision of the nerve; its vertices are colored naturally with at most $d+1$ colors. Finally, for each vertex of the subdivision, take the preimage of its open star. This way we get a refinement of the original cover colored using at most $d+1$ colors. Therefore, $w_{\mathrm{o'}} \leq w_{\mathrm{o}}$.

□

Definition 2.3.1 was given for a compact space. We adopt the following convention: the width of a (not necessarily compact) space is defined in terms of open covers, (UO).

2.4 Properties of the width

Monotonicity

1. The width is monotone in dimension: $\mathrm{UW}_0(X) \geq \mathrm{UW}_1(X) \geq \mathrm{UW}_2(X) \geq \cdots$.

2. The width is monotone with respect to inclusion: if $S_1 \subset S_2$ then $\mathrm{UW}_d(S_1) \leq \mathrm{UW}_d(S_2)$ for all d. More generally, the width is monotone under distance-increasing maps: if $X \to Y$ does not decrease distances, then $\mathrm{UW}_d(X) \leq \mathrm{UW}_d(Y)$ for all d.

Non-triviality

The n-width of a closed Riemannian n-manifold is clearly zero. It is less obvious that the $(n-1)$-width of a closed Riemannian n-manifold is greater than zero, which is basically the corollary of Brouwer's invariance of dimension. Explicitly, this can be deduced from the Lebesgue covering theorem (discovered by Lebesgue [40] and first proven by Brouwer [12]), or from the Knaster–Kuratowski–Mazurkiewicz theorem [36]. Some tight bounds on the width can be deduced from those theorems, as it is explained in Section 2.5 below.

Continuity

Urysohn's argument [3] for the continuity of dimension, Theorem 2.1.2, basically implies the following statement.

Theorem 2.4.1. *Let $F_1 \supset F_2 \supset \dots$ be a nested sequence of closed sets in a compact metric space. Then*

$$\lim_{i\to\infty} \mathrm{UW}_d(F_i) = \mathrm{UW}_d\left(\bigcap F_i\right).$$

We also state another continuity statement that will be used in Chapter 5.

Lemma 2.4.2. *Let $f : X \to Y$ be a continuous map from a compact metric space X to a metrizable topological space Y. The function*

$$y \mapsto \mathrm{UW}_d(f^{-1}(y))$$

is upper semi-continuous for any d. Namely,

$$\mathrm{UW}_d(f^{-1}(y)) \geq \limsup_{y'\to y} \mathrm{UW}_d(f^{-1}(y')).$$

Proof. If a fiber $f^{-1}(y)$ is covered by open sets $U_i \subset X$, with diameters $< \mathrm{UW}_d(f^{-1}(y))+\varepsilon$ and multiplicity at most $d+1$, then these open sets in fact cover neighboring fibers $f^{-1}(y')$ as well (for $y' \in V$, where V is a sufficiently small neighborhood of y). $\square$

This statement remains true with Y being just Hausdorff instead of metrizable.

Behavior with respect to taking products

A well-known "downside" of the topological dimension is that the "sub-logarithmic" inequality

$$\dim X \times Y \leq \dim X + \dim Y$$

(proven by Urysohn for compact metric spaces; see [10] for its history) might be strict in some pathological examples. The first such example with $\dim X = \dim Y = 2$ and $\dim X \times Y = 3$ was given by Pontryagin [49]. A version of this example can be used in order to show the following.

Theorem 2.4.3. *For any $\varepsilon > 0$, there are two-dimensional compact metric spaces X and Y (in fact, finite simplicial complexes) whose 1-width is ≥ 1 while $\mathrm{UW}_3(X \times Y) < \varepsilon$.*

Proof sketch. Start from a large round 2-sphere, and cut many small holes everywhere. To obtain X, attach a small cross-cap (a Möbius band) to each hole. To obtain Y, attach a small "triple cross-cap" to each hole; that is, take a cylinder, one of whose ends is used to attach it to the sphere, while the other is glued to itself via the degree three self-map. Each of the two can be embedded in $\mathbb{R}^4$ roughly preserving the large-scale geometry of the 2-sphere. We endow X and Y with Riemannian metrics inherited from these embeddings.

To estimate $\mathrm{UW}_1(X)$ or $\mathrm{UW}_1(Y)$ from below, one can use the "fiber contraction" argument (which will be explained in Section 3.1), and the fact that X and Y admit "fundamental" classes in $H_2(\cdot; \mathbb{Z}/2)$ and $H_2(\cdot; \mathbb{Z}/3)$, respectively (relative to the circle "boundaries"). Informally, assuming that there is a map with small fibers from X (or Y) to a graph, one can continuously pinch each of the fibers to a point (depending on a fiber); this homotopes the identity map $X \to X$ (or $Y \to Y$) to a map with one-dimensional image, resulting in a contradiction at the level of H_2.

To show that $\mathrm{UW}_3(X \times Y) < \varepsilon$, one starts with a lemma, asserting that the product "double cross-cap $\times$ triple cross-cap" can be embedded in $\mathbb{R}^8$ with prescribed boundary, and avoiding any given 4-plane (it follows from [27, Satz IXa]). Now embed $X \times Y$ in $\mathbb{R}^8$, and using the lemma perturb the embedding in order to move away from the 4-planes of a fine cubical grid of $\mathbb{R}^8$. Once it is done, $X \times Y$ projects with small fibers to the 3-dimensional skeleton of the dual grid. □

Behavior with respect to covering maps

The methods of Chapter 3 can be used in order to show that the Urysohn width of a space and of its covering space might have arbitrarily large discrepancy.

Theorem 2.4.4. *1. For any $\varepsilon > 0$, there is a two-dimensional non-orientable closed Riemannian manifold M^2 whose 1-width is ≥ 1 while $\mathrm{UW}_1(\overline{M}) < \varepsilon$, where $\overline{M}$ is the orientation double cover of M.*

2. For any $\varepsilon > 0$, there is a four-dimensional non-orientable closed Riemannian manifold M^4 whose 2-width is ≥ 1 while $\mathrm{UW}_2(\widetilde{M}) < \varepsilon$, where $\widetilde{M}$ is the universal cover of M.

Construction sketch. I only sketch the construction for the first part. Take a fine cubical grid with the nodes $\frac{1}{N}\mathbb{Z}^3 \subset \mathbb{R}^3$, for some large $N \gg 1$; let Z be its 1-skeleton, and let Z' be the 1-skeleton of the dual grid. Define M_0 as the set of points in $\mathbb{R}^3$ equidistant from Z and Z'; it is a PL manifold. Take the quotient of M_0 with respect to the translations $v_1 = (10, 0, 0)$, $v_2 = (0, 10, 0)$, $v_3 = \left(\frac{1}{2N}, \frac{1}{2N}, 10 + \frac{1}{2N}\right)$. This way we get a closed manifold M with Riemannian metric inherited from the ambient torus; it is non-orientable because one of the three translations swaps Z and Z'. Using non-orientability, it can be shown that $\mathrm{UW}_1(M) > 1$ (this is again a corollary of the fiber contraction argument). But the double cover $\overline{M}$ can be viewed as M_0 modded out by the translations $v_1, v_2, 2v_3$. This manifold can be projected to Z or Z' with fibers of size $\sim \frac{1}{N}$. □

I am not aware of any inequalities of the form $\mathrm{UW}_d(\widetilde{M}) \lesssim \mathrm{UW}_d(M)$.

2.5 Exact value in some examples

Euclidean cube

The evident projection map $[0,1]^n \to [0,1]^{n-1}$ has width 1. The Lebesgue covering lemma can be used to give the reverse estimate of cube's flattening coefficient.

Lemma 2.5.1. *Every continuous map $f : [0,1]^n \to Z^d$ from the unit euclidean n-cube to a d-dimensional simplicial complex, $d < n$, has a fiber $f^{-1}(y)$ meeting some two opposite facets of the cube.*

Therefore, $\mathrm{UW}_{n-1}([0,1]^n) = 1$. Observing the trivial equality $\mathrm{UW}_0([0,1]^n) = \sqrt{n}$, it is natural to conjecture that $\mathrm{UW}_d([0,1]^n) = \sqrt{n-d}$, but I do not know how to prove this.

Regular simplex

Let $\triangle^n$ be the regular euclidean n-simplex with edge length 1. Let Z be the union of all straight line segments between the center and the $(n-2)$-skeleton. It is easy to see that there is a map $\triangle^n \to Z$ with the largest fiber having diameter $1/n$. The Knaster–Kuratowski–Mazurkiewicz lemma can be used to show that this is the smallest possible width of a map from $\triangle^n$ to a lower dimensional space.

Lemma 2.5.2. *Every continuous map $f : \triangle^n \to Z^d$ from the unit-edge euclidean n-simplex to a d-dimensional simplicial complex, $d < n$, has a fiber $f^{-1}(y)$ meeting all facets of the simplex.*

Any set meeting all facets must have diameter at least $1/n$ by Jung's theorem[8]. Therefore, $\mathrm{UW}_{n-1}(\triangle^n) = 1/n$. I do not know if the values $\mathrm{UW}_d(\triangle^n)$, $1 \le d \le n-2$, are known.

[8] Jung's theorem [30] asserts that a set of diameter D in $\mathbb{R}^n$ can be covered by a ball of radius $D\sqrt{\frac{n}{2n+2}}$.

Euclidean ball

Let B^n be the radius 1 euclidean n-ball. The exact value $\mathrm{UW}_{n-1}(B^n) = \sqrt{\frac{2n+2}{n}}$ is known [2, Remark 6.10] and can be computed as follows. The "optimal" map $B^n \to Z$ is similar to the one for simplex; one takes the regular simplex inscribed into B^n, and defines Z as the union of all radial segments that intersect its $(n-2)$-skeleton. A sharp lower bound on $\mathrm{UW}_{n-1}(B^n)$ follows from Jung's theorem together with the following waist-type estimate.

Theorem 2.5.3 (cf. Sitnikov[9] [52])**.** *Every continuous map $f : B^n \to Z^d$ from the unit euclidean n-ball to a d-dimensional simplicial complex, $d < n$, has a fiber $f^{-1}(y)$ that cannot be covered by a ball of radius < 1.*

The values $\mathrm{UW}_d(B^n)$ for $n/2 \le d \le n-2$ are not known. For $d < n/2$, $\mathrm{UW}_d(B^n) = 2$, as it follows from the computation of $\mathrm{UW}_d(\partial B^n)$ below.

Round sphere

Let S^n be the sphere of radius 1 in $\mathbb{R}^{n+1}$. For the computation of the width, it does not matter whether we consider S^n with the geodesic metric or the extrinsic metric (there is a monotone bijection between the distances in the two metrics).

Theorem 2.5.4 (Ščepin [58])**.**

$$\mathrm{UW}_d(S^n) = \operatorname{diam} S^n, \text{ for } d \le n/2;$$

$$\mathrm{UW}_d(S^n) < \operatorname{diam} S^n, \text{ for } d > n/2.$$

This result was rediscovered multiple times in different contexts [29, 51]. The argument in [58] contains a flaw[10] in the estimate for $\mathrm{UW}_2(S^4)$, so I would like to outline the idea of a different proof[11] that $\mathrm{UW}_d(S^n) = \operatorname{diam} S^n$ when $d \le n/2$. Given any map $S^n \to Z^d$, one can embed $Z^d \hookrightarrow M^{2d}$ to a non-compact manifold; M^{2d} can

[9] The closely related result of Sitnikov regards the Alexandrov width of the ball, and the argument can be adjusted in order to prove this theorem.

[10] The proof uses that $\pi_{2d}(\bigvee S^d)$ is finite [26], which is false for $d = 2$.

[11] I learned it from Roman Karasev.

be obtained as $\mathbb{R}^{2d}$ with a few handles attached [59, Lemma 7.1]. There are versions of the Borsuk–Ulam theorem [14, Section 33] implying that the map $S^n \to M^{2d}$, $2d \leq n$, must send a pair of antipodal points to the same image.

For $d > n/2$, I do not know much about $\mathrm{UW}_d(S^n)$ apart from the fact that they are less than $\operatorname{diam} S^n$. For $\mathrm{UW}_{n-1}(S^n)$, the strongest lower bound I am aware of is sketched in [2, Remark 6.9]; the method is the same as for the ball: applying appropriate counterparts of Jung's and Sitnikov's theorem to the half-sphere. A weaker bound follows from the results of [34]. The exact value of the flattening coefficient seems to be unknown.

Chapter 3

Local-to-global estimates of the Urysohn width

In the paper [25] Larry Guth proves that, on a closed Riemannian manifold, local volume estimates translate into global information about the Urysohn width. This resolved a conjecture of Gromov [16], and provided an alternative way to prove the celebrated systolic inequality of Gromov. Guth also conjectured a generalization of his theorem, dealing with the Hausdorff content on compact metric spaces in place of volume, and his conjecture was established by Liokumovich, Lishak, Nabutovsky, and Rotman [42]. Shortly after that, a simple and clever proof was given by Panos Papasoglu [47], and the method employed there gives the simplest and cleanest proof [45] of Gromov's systolic inequality, with the best dimensional constants known so far.

In the same paper [25], Guth gives an example of a metric on S^3 with locally small but globally large 2-width [25, Section 4]. Further, he asks if there is a setting in which local Urysohn width bounds translate into global ones.

Question 3.0.1 ([25, Question 5.3]). Suppose that M^n is a Riemannian manifold such that each unit ball $B \subset M$ has $\mathrm{UW}_q(B) < \varepsilon$. If ε is sufficiently small, does this inequality imply anything about $\mathrm{UW}_{q'}(M)$ for some $q' \geq q$?

We answer this question in the negative (see Theorem 3.0.3 below), and investigate how additional topological complexity assumptions affect the answer.

Our first result is an estimate of 1-width of a closed Riemannian manifold M, depending on its topological complexity as well as the supremal width of its unit balls.

Theorem 3.0.2. *Let M^n be a closed Riemannian manifold with the first $\mathbb{Z}/2$-Betti number $\beta = \operatorname{rk} H_1(M;\mathbb{Z}/2)$. If every unit ball has 1-width less than 1/15, then* $\mathrm{UW}_1(M) < \beta + 1$.

The dependence on β does not seem optimal. The best example we know has $\mathrm{UW}_1(M) \sim \beta^{1/n}$ (see Figure 3-1). This example is constructed in our second theorem, which resolves Guth's question in the negative.

Theorem 3.0.3. *For any $\varepsilon > 0$, there exists a Riemannian manifold M^n with all unit balls of 1-width less than ε, and such that* $\mathrm{UW}_{n-1}(M) \geq 1$.

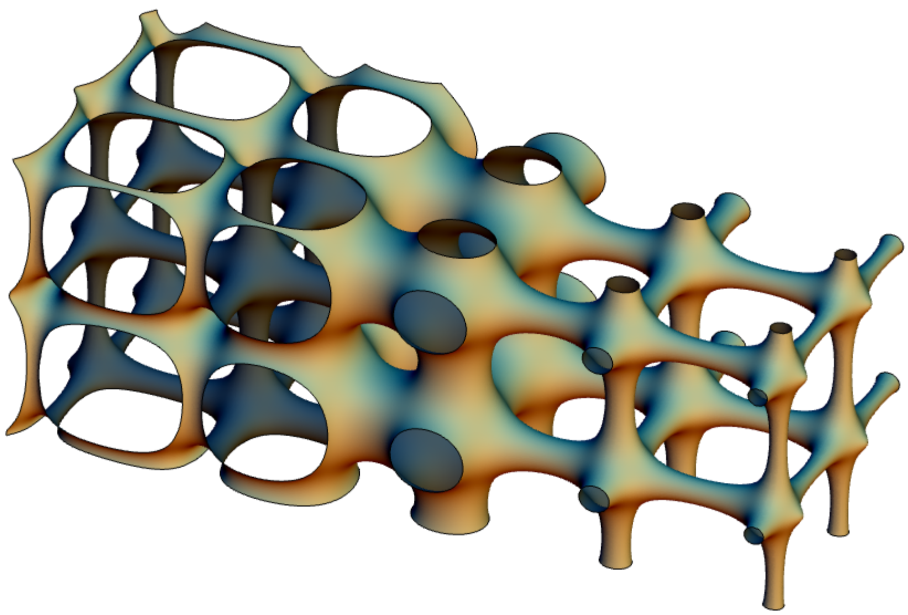


Figure 3-1: A piece of the surface from Theorem 3.0.3 for $n = 2$. The whole surface is made by replicating this piece periodically many times and closing up the ends. Roughly speaking, the left half of this surface has small Urysohn 1-width, as well as the right half, while the whole surface has large Urysohn 1-width

Note that the negative result with $q = 1$ and $q' = n - 1$ is the strongest possible over all choices of q, q'. Therefore, the answer to Question 3.0.1 is negative for all q, q'.

The example establishing this theorem has large Betti numbers. If one is looking for a topologically simple example, our third result gives it with M^n being a ball (but with a worse dimension in the local width bound).

Theorem 3.0.4. *For any $\varepsilon > 0$, there is a metric on the n-ball M^n (or n-sphere, or n-torus) such that its $(n-1)$-width is at least 1 but $\mathrm{UW}_{\lceil \log_2(n+1) \rceil}(B) < \varepsilon$ for every unit ball $B \subset M$.*

3.1 Bounding width from below

Before we get to the main results, let us discuss the main tools one can use to show a space has substantial Urysohn width.

The **first tool** is the Lebesgue covering lemma (discovered by Lebesgue [40] and first proved by Brouwer [12]), which can be used to show that the $(n-1)$-width of the unit euclidean n-cube equals 1.

Lemma 3.1.1. *Every continuous map $f : [0,1]^n \to Y^d$ from the unit n-cube to an d-dimensional simplicial complex, $d < n$, has a fiber $f^{-1}(y)$ meeting some two opposite facets of the cube.*

The **second tool** amounts to the "fiber contraction" argument, which goes back to Gromov [19, Proposition (F_1)]. A detailed exposition can be found in [22, Section 5]. We quote here a version of this argument due to Guth [22, Lemma 5.2].

Lemma 3.1.2. *Let W be a Riemannian manifold of convexity radius at least ρ; that is, any two points in a ball of radius $< \rho$ are connected by a unique minimal geodesic within this ball. Let $\pi : X \to Y$ be a map from a metric space X to a simplicial complex Y, such that all fibers of π have diameter less than ρ. Then any 1-Lipschitz map $f : X \to W$ is homotopic to a map factoring as $g \circ \pi$, for some $g : Y \to W$. Moreover, the homotopy moves each point of W by less than 2ρ.*

Corollary 3.1.3. *Let $f : X \to W$ be a 1-Lipschitz map from a metric space X to a Riemannian manifold W of convexity radius at least ρ. Suppose one of the following conditions holds for some n.*

1. *The induced map* $f_* : H_n(X) \to H_n(W)$ *is non-trivial.*

2. *For some closed subsets* $X_0 \subset X$, $W_0 \subset W$, f *sends* X_0 *to* W_0, *and the induced map* $f_* : H_n(X, X_0) \to H_n(W, U_{2\rho}(W_0))$ *is non-trivial. (Here* $U_{2\rho}(W_0)$ *is the neighborhood of* W_0 *of radius* 2ρ.*)*

Then $\mathrm{UW}_{n-1}(X) \geq \rho$.

3.2 Surface width estimates

Proof of Theorem 3.0.2. This proof follows closely the ideas from [22, Section 1], [16, Appendix 1, (E_1)-(E_1'')]. The main theorem in [22, Section 1] says, basically, that in the case $M \simeq S^2$, there is a universal way to measure the Urysohn 1-width: it is given by the map to the set of the connected components of distance spheres around any point. The largest diameter of such a component gives the value $\mathrm{UW}_1(S^2)$ within a factor of 7. We adapt this idea to higher dimensions, taking into account the topological complexity as well.

Pick any point $p \in M^n$. Consider the distance spheres $S_r(p)$. We show that $\mathrm{UW}_0(S_r(p)) < \beta + 1$ for each r. For $r < 1/2$ this is clear, so fix $r \geq 1/2$ and suppose that $\mathrm{UW}_0(S_r(p)) \geq \beta + 1$, so there are points x and y distance $\beta + 1$ apart in the same connected component of $S_r(p)$. Denote by γ a curve connecting x and y inside $S_r(p)$ (we can assume it exists by perturbing slightly the distance function $\mathrm{dist}(\cdot, p)$). Denote $x_0 = x$, $x_{\beta+1} = y$, and pick points $x_k \in \gamma$, $1 \leq k \leq \beta$, so that $\mathrm{dist}(x, x_k) = k$ (the distance is extrinsic, and not along γ). Notice that $\mathrm{dist}(x_i, x_j) \geq |i - j|$. Denote by g_k a minimal geodesic from p to x_k, for $0 \leq k \leq \beta + 1$. Denote by ℓ_k, $0 \leq k \leq \beta$, the loop formed by the curves g_k, g_{k+1} and the part of γ between x_k and x_{k+1}. The loops $\ell_0, \ldots, \ell_\beta$ cannot be independent in $H_1(M; \mathbb{Z}/2)$; hence, there exist indices $0 \leq i_1 < \ldots < i_r \leq \beta$ such that $[\ell_{i_1}] + \ldots + [\ell_{i_r}] = 0$ in $H_1(M; \mathbb{Z}/2)$.

The concatenation of $\ell_{i_1}, \ldots, \ell_{i_r}$ bounds a 2-chain D in M, which we also view as a closed subset of M. Assuming that 1 is a regular value of $\mathrm{dist}(\cdot, x_{i_1})$ on D (otherwise perturb this function slightly), one can view the intersection $D' = D \cap$

$B_1(x_{i_1})$ as a 2-chain as well. Now consider the map $f : M \to \mathbb{R}^2$ given by $f(\cdot) = (\mathrm{dist}(\cdot, p), \mathrm{dist}(\cdot, x_{i_1}))$ (see Figure 3-2). Note it is $\sqrt{2}$-Lipschitz. We will show that $f(D')$ covers a disk O of radius $\frac{\sqrt{2}-1}{2}$ in $\mathbb{R}^2$; formally speaking, the map $f : (D', \partial D') \to (\mathbb{R}^2, \mathbb{R}^2 \setminus \mathrm{int}\, O)$ is of degree 1 (mod 2). Then Corollary 3.1.3 can be applied to f (composed with a $1/\sqrt{2}$-homothety, to make it 1-Lipschitz), implying $\mathrm{UW}_1(D') \geq \frac{\sqrt{2}-1}{4\sqrt{2}}$. On the other hand, $\mathrm{UW}_1(D') \leq \mathrm{UW}_1(B_1(x_{i_1})) < 1/15$ since $D' \subset B_1(x_{i_1})$, which gives a contradiction.

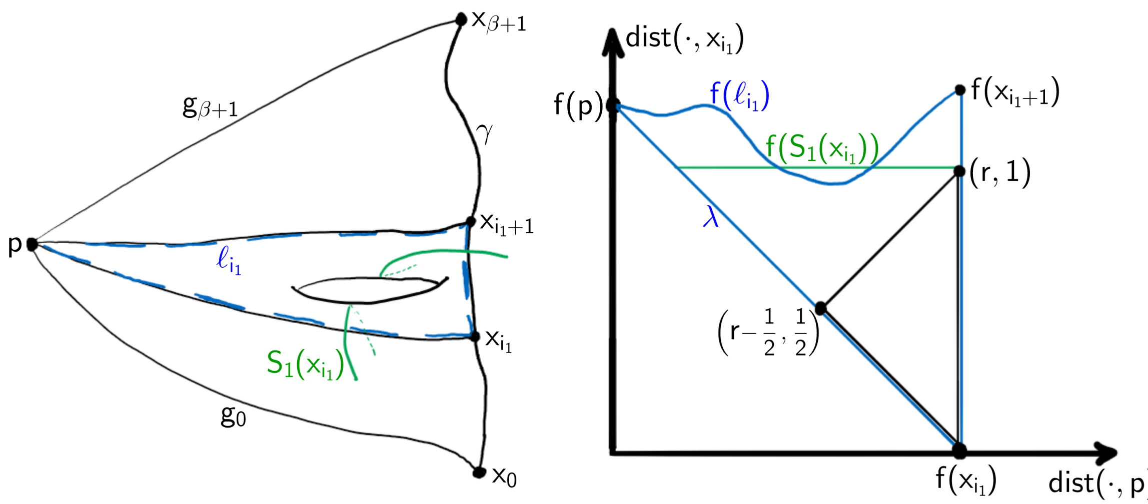


Figure 3-2: The map f covers a substantial triangular region

Observation 1. The image of f lies above the straight line λ connecting points $(r, 0)$ and $(0, r)$.

Observation 2. Consider the triangle Δ with the vertices $(r, 0)$, $(r, 1)$, and $(r - 1/2, 1/2)$, and observe that $f(\partial D') \cap \mathrm{int}\, \Delta = \varnothing$. Indeed, $\partial D' \subset \partial D \cup S_1(x_{i_1})$, so the image $f(\partial D')$ is contained in the union of the following curves:

- the line λ, where $f(g_{i_1})$ lies;
- the vertical straight line through $(r, 0)$, where $f(\gamma)$ lies;
- the horizontal line through $(r, 1)$, where $f(S_1(x_{i_1}))$ lies;
- the curves $f(g_k)$, $k > i_1$, each of which can be viewed as the graph of a 1-Lipschitz function of argument $\mathrm{dist}(\cdot, p)$; they all lie above the straight line connecting $(r, 1)$ and $(r - 1/2, 1/2)$.

Observation 3. Let $q = g_{i_1} \cap S_{1/2}(x_{i_1})$, and observe that the geodesic segment $[q, x_{i_1}] \subset g_{i_1}$ is present in the 1-chain $\partial D'$. The image $f([q, x_{i_1}])$ is the straight line segment between $(r, 0)$ and $(r - 1/2, 1/2)$ (traversed once). Other parts of $f(\partial D')$ are all contained in the union $f(\gamma) \cup f([p, q]) \cup f(S_1(x_{i_1})) \cup \bigcup_{k>i_1} f(g_k)$, avoiding this straight line segment. In view of the previous two observations, $f(\partial D')$ winds around Δ nontrivially. Therefore, the degree of $f : (D', \partial D') \to (\mathbb{R}^2, \mathbb{R}^2 \setminus \operatorname{int} \Delta)$ is $1 \pmod 2$.

Observation 4. The disk O inscribed in Δ is of radius $\frac{\sqrt{2}-1}{2}$.

This concludes the proof. □

Remark 3.2.1. Under the same assumptions (every unit ball in the surface M has 1-width less than $1/15$), one can show that the *homological systole* (the length of the shortest loop that is not null-homologous) is less than 2, regardless of genus. One way to show it is to adjust the proof of [22, Theorem 4.1].

3.3 Manifolds of small local but large global width

The constructions of this section are inspired by the mother of examples [19, Example H_1''].

3.3.1 Local join representation

A crucial ingredient for the constructions below is a decomposition of $\mathbb{R}^{2n-1}$ as the "local join" of several 1-dimensional complexes.

Lemma 3.3.1. *Fix $0 < \varepsilon < 1$. It is possible to triangulate $\mathbb{R}^{2n-1}$ by simplices with the following properties:*

1. *Each simplex is c_n-bi-Lipschitz to a regular simplex with edge length ε.*

2. *The vertices of triangulation can be colored by colors 1 through $2n$ so that each simplex receives all distinct colors.*

3. *The colored triangulation can be taken periodic with respect to n almost orthogonal translation vectors of length $\approx 10c_n$; hence the colored triangulation descends*

to the product $T^n \times \mathbb{R}^{n-1}$, where T^n is a flat torus with convexity radius at least 1.

Proof. In fact, one can take all simplices congruent to one another. For instance, one can take the (scaled) set of alcoves for the affine Weyl group $\widetilde{A}_n$ (see [50, Chapter 6]); this will give an example with a good value of c_n (perhaps, the best). If we are not chasing after good constants, much simpler constructions are possible. One way is to consider the cubical subdivision of $\mathbb{R}^{2n-1}$ with the set of vertices $\varepsilon\mathbb{Z}^{2n-1}$, split each ε-size cube into $(2n-1)!$ simplices, and then take the barycentric subdivision, which can be colored naturally. Either of these constructions can be made periodic easily. □

Definition 3.3.2. Let X^{2n-1} be $\mathbb{R}^{2n-1}$ or $T^n \times \mathbb{R}^{n-1}$. Triangulate it as in Lemma 3.3.1, and define Z_i, $1 \le i \le n$, to be the union of all edges of the triangulation between the vertices of colors $2i-1$ and $2i$. We say that X is the *ε-local join* of $Z_1, \ldots, Z_n$.

The motivation behind this definition is that every (top-dimensional) simplex σ of the triangulation can be written as the join $(\sigma \cap Z_1) * \ldots * (\sigma \cap Z_n)$; that is, any point $x \in \sigma$ can be written as

$$x = \sum_{i=1}^{n} t_i z_i, \quad \text{where } z_i \in \sigma \cap Z_i, \quad t_i \ge 0, \quad \sum_{i=1}^{n} t_i = 1.$$

The coefficients t_i are determined uniquely; if $t_i \ne 0$, the corresponding z_i is determined uniquely too. This defines a map $x \mapsto (t_1, \ldots, t_n)$ from σ to the standard $(n-1)$-dimensional simplex $\triangle^{n-1}$; for adjacent simplices of the triangulation, those maps agree on their intersection; hence, we have a well-defined map

$$\tau : X \to \triangle^{n-1},$$

which we call the *join map*. Note that $Z_i = \tau^{-1}(v_i)$, where $v_1, \ldots, v_n$ are the vertices of $\triangle^{n-1}$. For each vertex v_i, denote the opposite facet of $\triangle^{n-1}$ by $v_i^\vee$. For each complex Z_i, introduce its *dual* complex $Z_i^\vee = \tau^{-1}(v_i^\vee)$. In other words, $Z_i^\vee$ is the union of all $(2n-3)$-dimensional cell of our triangulation that do not intersect Z_i. There are natural retractions

$$\pi_i : X \setminus Z_i^\vee \to Z_i,$$

defined by sending $x = \sum_{i=1}^{n} t_i z_i \in \sigma$ to $z_i \in \sigma \cap Z_i$; they are well-defined since $t_i \neq 0$ whenever $x \notin Z_i^\vee$. Note that π_i moves each point by distance at most $\sup \operatorname{diam} \sigma \sim \varepsilon$.

3.3.2 Manifolds that are locally nearly one-dimensional

Proof of Theorem 3.0.3. Pick a torus T^n with convexity radius ≥ 1, as in Lemma 3.3.1. On scale ε, represent $X = T^n \times \mathbb{R}^{n-1}$ as the local join of one-dimensional complexes $Z_1, \ldots, Z_n$, as in Definition 3.3.2. The goal is to build a manifold $M^n \subset X$ so that on the large scale (~ 1) it resembles T^n homologically, but on the small ($\sim \varepsilon$) scale it will become porous in a way that makes its local 1-width small.

Recall the join map $\tau : X \to \triangle^{n-1}$ arising from the local join structure of X. In this proof, it will be convenient to think of the target simplex as a regular simplex of inradius 3, placed in $\mathbb{R}^{n-1}$ and centered at the origin. We make use of the join map $\tau : T^n \times \mathbb{R}^{n-1} \to \triangle^{n-1} \subset \mathbb{R}^{2n-1}$ to "perturb" the projection $p : T^n \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ onto the second factor:

$$\widetilde{p} := p - \tau/2.$$

The choice of the factor $1/2$ is not particularly important as long as it is less than 1. We only use that the p-term dominates the τ-term in the sense that $\widetilde{p}$ does not vanish outside of $T^n \times \operatorname{int} \triangle^{n-1}$.

Finally, define the "perturbation of $T^n = p^{-1}(0)$ by the $Z_i = \tau^{-1}(v_i)$":

$$M^n := \widetilde{p}^{-1}(0).$$

Note: as defined, M is a PL-manifold; but we can perturb τ slightly to make it smooth and to make 0 a regular value of $\widetilde{p}$; then M becomes a smooth manifold.

Observe that M is contained in $T^n \times \operatorname{int} \triangle^{n-1}$, so M is closed; it is also orientable by construction. See Figure 3-1 for an illustration of the case $n = 2$.

Now, within a unit ball $B_1(x) \subset M$, we want to find a projection on one of the Z_i with ε-small fibers. Recall the notation introduced after Definition 3.3.2: the dual complexes $Z_i^\vee = \tau^{-1}(v_i^\vee)$, and the retractions $\pi_i : X \setminus Z_i^\vee \to Z_i$. One of these retractions would do if we find i, depending on x, so that $B_1(x) \cap Z_i^\vee = \varnothing$. Pick i maximizing the distance between $\tau(x)$ and $v_i^\vee$ in $\triangle^{n-1}$; this distance is at least 3 by our choice of metric on $\triangle^{n-1}$. When we move x to $x' \in B_1(x)$, its p-projection changes by at most 1, whereas its τ-projection changes by at most 2 (since the value of $\widetilde{p}$ is fixed), so $\tau(x')$ never reaches $v_i^\vee$. We can now use the retraction $\pi_i : B_1(x) \to Z_i$, showing $\mathrm{UW}_1(B_1(x)) \lesssim \varepsilon$. (Notation $\lesssim$ means inequality that holds up to a factor depending on dimension only.)

To show that $\mathrm{UW}_{n-1}(M) \geq 1$, we use our second tool for estimating widths. Apply Corollary 3.1.3 to the 1-Lipschitz projection map $M \to T^n$ (the composition $M \hookrightarrow T^n \times \mathbb{R}^{n-1} \to T^n$), sending the fundamental class $[M] \in H_n(M)$ to $[T^n] \neq 0$. Indeed, (the Poincaré duals of) the classes of M and T^n are the same in $H^{n-1}(T^n \times \triangle^{n-1}, T^n \times \partial\triangle^{n-1})$ as zero level sets for homotopic mappings $\widetilde{p}$ and p, respectively; the homotopy $p - t\tau$, $t \in [0, 1/2]$, does not vanish on $T^n \times \partial\triangle^{n-1}$ since the p-term dominates the τ-term. □

Remark 3.3.3. In this argument, we used a torus as the "base space to be perturbed". In fact, this construction can be repeated for any reasonable base space, provided that it has sufficient convexity radius. One can easily adapt Lemma 3.3.1 for this case, and the rest of the proof goes unchanged. Morally, the outcome is that any manifold can be "homologically perturbed" to make its local (on the scale comparable with its convexity radius) 1-width arbitrarily small.

Remark 3.3.4. The parameters of the construction can be adjusted in order to get a manifold M^n of $\mathrm{UW}_{n-1}(M) \gtrsim \beta^{1/n}$, with all unit balls $B \subset M$ having $\mathrm{UW}_1(B) \lesssim 1$. Here β is the first Betti number of M. The adjustment is to start from a torus of convexity radius $\sim \beta^{1/n}$, and pick the triangulation scale $\varepsilon \sim 1$.

Question 3.3.5. Let M^2 be a closed Riemannian surface with the first $\mathbb{Z}/2$-Betti number β, and with every unit ball having 1-width less than ε, for some fixed small absolute constant ε. In the optimal bound $\mathrm{UW}_1(M) \lesssim f(\beta)$, what is the order of magnitude of the right hand side? It must be between $\beta^{1/2}$ (by Remark 3.3.4) and β (by Theorem 3.0.2).

3.3.3 Topologically simple n-manifolds that are locally nearly $\log n$-dimensional

The next result is an "amplification" of Guth's example [25, Section 4] of a 3-sphere with large 2-width but all unit balls UW_2-small. We start by taking a reasonable base space X^{2n-1} equipped with an ε-fine triangulation, as in Lemma 3.3.1. A particular choice of X is not really important; the only assumptions we need are its substantial codimension 1 width, and the existence of a colored triangulation (local join representation). For example, one can take X to be a unit cube in the euclidean space $\mathbb{R}^{2n-1}$; then $\mathrm{UW}_{2n-2}(X) \geq 1$ by the first tool. If one takes a unit euclidean ball, its codimension 1 width is known exactly (see [2, Remark 6.10]), but one can use the second tool to get a weaker bound $\geq 1/2$; then one can take X to be the ball of radius 2 in order to have $\mathrm{UW}_{2n-2}(X) \geq 1$. In both examples, a local join structure is given by Lemma 3.3.1. It is easy to modify the argument in order to take X a sphere, or a torus, etc.

The triangulation of X comes equipped with one-dimensional complexes $Z_1, \ldots, Z_n$, and the join map $\tau : X \to \triangle^{n-1}$ mapping Z_i to the i^{th} vertex of the simplex. Now we blow up the metric in X along the $\triangle^{n-1}$-direction and leave it unchanged along the fibers of τ. Formally speaking, endow $\triangle^{n-1}$ with an auxiliary metric making $\triangle^{n-1}$ a regular simplex with inradius 2 (this choice will be explained later), and add its pullback to the metric of X. The resulting metric on X is piecewise Riemannian, and after a slight smoothening, we get a Riemannian manifold X'.

Proposition 3.3.6. $\mathrm{UW}_{2n-2}(X') \geq 1$ *but* $\mathrm{UW}_n(B) \lesssim \varepsilon$ *for every unit ball* $B \subset X'$.

Proof. By assumption, $\mathrm{UW}_{2n-2}(X) \geq 1$. The metric on X' is even larger, so $\mathrm{UW}_{2n-2}(X') \geq$

1.

Now take a unit ball $B \subset X'$, and observe that $\mathrm{UW}_n(B) \leq \mathrm{UW}_n(\tau^{-1}(\tau(B)))$. By construction of the blown-up metric of X' (namely, by the choice of the auxiliary metric on $\triangle^{n-1}$), $\tau(B)$ misses at least one facet of $\triangle^{n-1}$, say, the i^{th} one. Then the retraction π_i (in the notation introduced after Definition 3.3.2) gives a map $\tau^{-1}(\tau(B)) \to Z_i$, whose fibers are small in the original metric of X. The map

$$B \to Z_i \times \triangle^{n-1}$$
$$x \mapsto (\pi_i(x), \tau(x))$$

gives a desired bound on $\mathrm{UW}_n(B)$. □

For $n = 2$ this construction recovers Guth's example. Let us rephrase this construction once again, since we are going to apply it inductively.

The blow-up construction. Start from a manifold X with metric g_X, and a piecewise smooth join map $\tau : X \to \triangle$ (obtained from a fine colored triangulation of X). Suppose $\triangle$ is equipped with an auxiliary metric $g_\triangle$, in which no unit ball $B^\triangle \subset \triangle$ meets all facets of $\triangle$. Consider the piecewise Riemannian metric $g_X + \tau^* g_\triangle$, and perturb it slightly to get a smooth metric g'_X. We say that g'_X is obtained from g_X by *blowing it up along the $\triangle$-direction, or across the join map*, via the auxiliary metric $g_\triangle$.

Observe that the distances in the blown-up metric g'_X do not decrease in comparison with the original metric g_X, hence the Urysohn width does not decrease either. By the same reason, if the space X itself was a simplex with the property "no unit ball meets all facets", the same holds true after the blow-up.

In the blow-up in Guth's example, the auxiliary metric on $\triangle^{n-1}$ was euclidean, making it a regular simplex of inradius 2. There is a more clever way to pick this auxiliary metric to get better estimates. Let $n = 2m$. We will start from a metric making $\triangle^{n-1}$ a regular simplex with inradius 2, and blow it up with the goal to have every unit ball $B^\triangle$ in $\triangle^{n-1}$ small in the sense of some width. Repeat the construction above: pick m skeleta, each of dimension 1, in a fine colored triangulation inside

$\triangle^{n-1}$, and blow up the metric of $\triangle^{n-1}$ across the join map in order to have maps $B^{\triangle} \to Y^m$ with small fibers, for every unit ball $B^{\triangle} \subset \triangle^{n-1}$ (this is the conclusion of Proposition 3.3.6). Now, using the modified metric on $\triangle^{n-1}$, we blow up the metric of X along the $\triangle^{n-1}$-direction. Call the resulting metric space X''.

Proposition 3.3.7. $\mathrm{UW}_{4m-2}(X'') \geq 1$ *but* $\mathrm{UW}_{m+1}(B) \lesssim \varepsilon$ *for every unit ball* $B \subset X''$.

Proof. $\mathrm{UW}_{m+1}(B) \leq \mathrm{UW}_{m+1}(\tau^{-1}(\tau(B)))$, where $\tau(B)$ lies in a unit ball $B^{\triangle} \subset \triangle^{n-1}$, missing, say, the i^{th} facet of $\triangle^{n-1}$. Then there is a map $\tau^{-1}(\tau(B)) \to Z_i \times Y^m$ with small fibers, defined as follows: a point x gets mapped to $(\pi_i(x), y) \in Z_i \times Y^m$, where y is the image of $\tau(x)$ under the map $B^{\triangle} \to Y^m$. $\square$

Iterating this procedure ℓ times, we arrive at the following conclusion.

Proposition 3.3.8. *For a unit euclidean cube X (or a regular simplex of inradius 2, or a ball, or a sphere, or a torus) of dimension $2^\ell k - 1$, there is a way to blow up the metric in order to get a space $X^{(\ell)}$ such that* $\mathrm{UW}_{2^\ell k-2}(X^{(\ell)}) \geq 1$ *but* $\mathrm{UW}_{k+\ell-1}(B) \lesssim \varepsilon$ *for every unit ball $B \subset X^{(\ell)}$.*

Proof. The original metric on X satisfies $\mathrm{UW}_{2^\ell k-2}(X) \geq 1$, and an ε-local join structure on X gives the join map $\tau : X \to \triangle^{2^{\ell-1}k-1}$. We blow up the metric of X across τ using a carefully chosen auxiliary metric on $\triangle^{2^{\ell-1}k-1}$. Inducting on ℓ, we may assume that there is a metric on $\triangle^{2^{\ell-1}k-1}$ such that no unit ball meets all its facets, and every unit ball is small in the sense of $\mathrm{UW}_{k+(\ell-1)-1}$. We use this metric to blow up the metric of X and this way get $X^{(\ell)}$. Arguing as in the proof of Proposition 3.3.7, one makes sure that every unit ball of $X^{(\ell)}$ is small in the sense of $\mathrm{UW}_{k+\ell-1}$. $\square$

Proof of Theorem 3.0.4. Let $\ell = \lceil \log_2(n+1) \rceil$, and apply Proposition 3.3.8 with $k = 1$ to get a ball X (or a sphere, or a torus) of dimension $2^\ell - 1$ with large global $(2^\ell - 2)$-width but small local ℓ-width. Now we can build M^n as a subspace of X. $\square$

Question 3.3.9. Let M^n be a Riemannian n-sphere, $n \geq 4$, with every unit ball having d-width less than ε_n, for some fixed small dimensional constant ε_n. What is the smallest $d = d(n)$ such that the assumption on local width would imply $\mathrm{UW}_{n-1}(M) \lesssim 1$? Theorems 3.0.2 and 3.0.4 imply $2 \leq d(n) < \log_2(n+1)$.

Chapter 4

Interlude: topological centerpoint

Throughout this chapter, the word *m-space* means "metrizable topological space of covering dimension at most m".

Theorem 4.0.1 (Karasev [31]). *Let $n = (m+1)r$. Every continuous map $f : \triangle^n \to Y^m$ from the n-simplex to an m-space has a fiber meeting all mr-faces of $\triangle^n$.*

This theorem subsumes the Knaster–Kuratowski–Mazurkiewicz theorem (for $r = 1$) and the Rado centerpoint theorem (for $Y^m = \mathbb{R}^m$ and affine f). The original proof [31] used the topological notion of $\mathbb{Z}/2$-index, and another gorgeous proof [32] used the symplectic moment map $\mathbb{C}P^n \to \triangle^n$ as well as cohomology structure of $\mathbb{C}P^n$. The goal of this chapter is to give a new proof using only Brouwer's invariance of dimension.

We remark that Theorem 4.0.1 holds with Y^m being just Hausdorff instead of metrizable. This was noticed in [15], and the proof below applies for this case as well.

4.1 Proof of Karasev's theorem via Urysohn width

The key fact used in the proof will be the following waist-type estimate.

Theorem 4.1.1 (Gromov [19, Corollary H'_1]). *Let $n = (m+1)(d+1)$. Every continuous map $f : X \to Y^m$ from a compact metric space to an m-space has a fiber $f^{-1}(y)$ of Urysohn d-width $\mathrm{UW}_d(f^{-1}(y)) \geq \mathrm{UW}_{n-1}(X)$.*

Proof. The assumptions on Y^m imply that $\mathrm{UW}_d(f^{-1}(y)) = \inf_{\text{open } V \ni y} \mathrm{UW}_d(f^{-1}(V))$.[1] Supposing the contrary to the statement of the theorem, and pulling back a fine open cover of Y, we obtain an open cover $\{U_i\}$ of X of multiplicity at most $m+1$, such that $\mathrm{UW}_d(U_i) < u := \mathrm{UW}_{n-1}(X)$ for all i. It follows from the definition of the d-width that every U_i admits an open cover $U_i = \bigcup_j U_{ij}$ of multiplicity at most $d+1$, with $\operatorname{diam} U_{ij} < u$. The cover $\{U_{ij}\}$ of X has multiplicity at most $(m+1)(d+1)$, and it can be assumed finite (by compactness), so we get $\mathrm{UW}_{n-1}(X) < u$, which is absurd. □

Let B^n be the unit ball in $\mathbb{R}^n$ with the euclidean metric. The only topological ingredient that we need is the fact that $\mathrm{UW}_{n-1}(B^n) > 0$. This follows easily from the Lebesgue covering theorem[2], as it was mentioned in Section 2.4. The exact value of $\mathrm{UW}_{n-1}(B^n)$ is known (see Section 2.5) but we will not need it here.

Lemma 4.1.2 (cf. Lemma 3.3.1)**.** *There is a triangulation T of $\mathbb{R}^n$ satisfying the following properties:*

- *T is preserved under the reflections in the hyperplanes spanned by the $(n-1)$-faces of T.*
- *The vertices of T split into $n+1$ orbits of the group generated by those reflections. In other words, if an n-simplex of T is being flipped (reflected in its facets) multiple times so that it ends up at the original place, then its orientation is unaltered.*

Proof. Such a triangulation comes from the type A root system and the corresponding affine Weyl group (see [50, Chapter 6] or [11, Chapter VI, §4]). □

Proof of Theorem 4.0.1. We regard the source space $\triangle^n$ as a simplex of the triangulation T from Lemma 4.1.2, and we denote G the corresponding reflection group. Suppose that for each fiber $f^{-1}(y)$, $y \in Y^m$, there is an mr-face $F_y \subset \triangle^n$ such that $y \notin f(F_y)$. Using the reflections in the facets of $\triangle^n$, we extend the map $f : \triangle^n \to Y^m$

[1] This is easy to check when Y^m is metrizable; but this is still true when Y^m is just Hausdorff, as explained in the next section.

[2] Or any other statement "equivalent" to Brouwer's fixed point theorem.

to get the map $\widetilde{f} : \mathbb{R}^n \to Y^m$; that is, $\widetilde{f}|_{\triangle_n} = f$ and $\widetilde{f}$ is G-invariant. The set $\widetilde{F}_y = G \cdot F_y = \bigcup_{g \in G} g \cdot F_y$ is a certain mr-subcomplex of T. Consider the "dual" subcomplex $\widetilde{F}_y^\vee = G \cdot F_y^\vee$, where $F_y^\vee$ is the $(r-1)$-face of $\triangle^n$ disjoint from F_y. Since $\widetilde{f}^{-1}(y)$ misses $\widetilde{F}_y$, there is a G-equivariant projection $p : \widetilde{f}^{-1}(y) \to G \cdot \widetilde{F}_y^\vee$, which moves every point by a distance at most $\operatorname{diam} \triangle^n$. On $\triangle^n$, it is defined as the composition $f^{-1}(y) \hookrightarrow \triangle^n \setminus F_y = (F_y * F_y^\vee) \setminus F_y \to F_y^\vee \hookrightarrow \widetilde{F}_y^\vee$, and then it is extended using the G-equivariance.

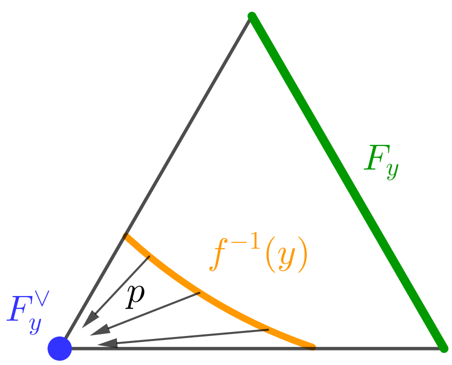


Figure 4-1: The projection map p inside $\triangle^n$

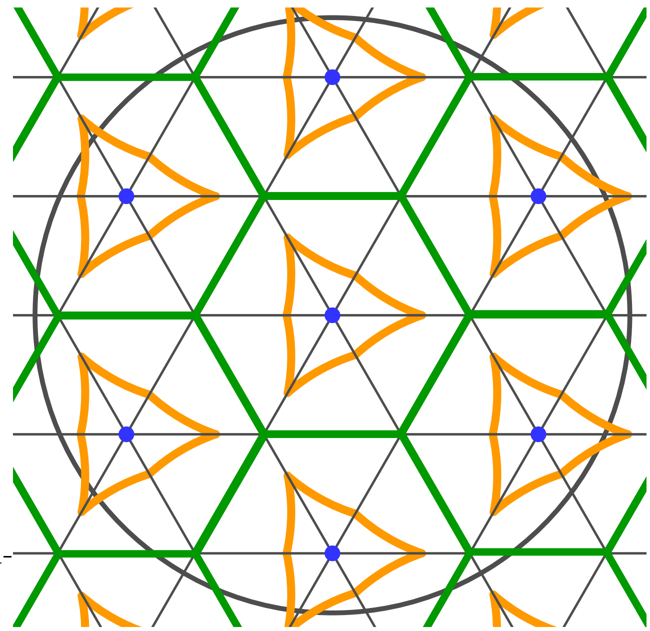


Figure 4-2: The "kaleidoscope" of reflections

Now pick a large number $R > \frac{2 \operatorname{diam} \triangle^n}{\mathrm{UW}_{n-1}(B^n)}$ and consider how $\widetilde{f}$ restricts on the ball $X = RB^n$ of radius R. The fiber of $\widetilde{f} : X \to Y^m$ over y admits a map of width $\leq 2 \operatorname{diam} \triangle^n$ to the $(r-1)$-dimensional space $\widetilde{F}_y^\vee$. But Theorem 4.1.1 outputs a fiber of the Urysohn width $\mathrm{UW}_{r-1}(\widetilde{f}^{-1}(y) \cap X) \geq \mathrm{UW}_{n-1}(X) = R \cdot \mathrm{UW}_{n-1}(B^n) > 2 \operatorname{diam} \triangle^n$, which yields a contradiction. $\square$

4.2 Extension for Hausdorff target spaces

Here we show that Theorem 4.1.1 holds true for any Hausdorff space Y. This gives another (different from the one in [15]) proof of Theorem 4.0.1 for the case of Hausdorff target space.

Recall that a topological space Y is said to satisfy the *Hausdorff separation axiom* if for any two points $y \neq y'$ in Y there are open neighborhoods $V \ni y, V' \ni y'$ such that $V \cap V' = \varnothing$.

Suppose we are given a map $f : X \to Y$ from a compact metric space to a Hausdorff space. In order to prove Theorem 4.1.1 in the Hausdorff setting, the only thing to be checked is $\mathrm{UW}_d(f^{-1}(y)) = \inf_{\text{open } V \ni y} \mathrm{UW}_d(f^{-1}(V))$, for any $y \in Y$. By the definition of width via covers we can assume that $f^{-1}(y)$ is covered with multiplicity at most $d+1$ by open sets U_i of diameter at most $\mathrm{UW}_d(f^{-1}(y))$. We need to show that in fact $\bigcup U_i$ covers a "fattened" fiber $f^{-1}(V)$, for some open neighborhood V of y. This is done as in the proof of ($\mathrm{UC} \leq \mathrm{UM}^{\mathrm{h}}$) in Section 2.3. In fact, $Y \setminus f(X \setminus f^{-1}(\bigcup U_i))$ is such a neighborhood.

Chapter 5

Waist measured via the Urysohn width

The question raised in this chapter is inspired by another famous Gromov's inequality, namely the waist of the sphere theorem [20]. It says that any generic smooth map $f : S^n \to \mathbb{R}^m$, $m < n$, has a fiber of $(n-m)$-volume at least the one of the $(n-m)$-dimensional "equatorial" subsphere. The target space can be replaced by any m-manifold [33], while it is not clear if one can replace it by an m-polyhedron Y^m. The only result in this direction we are aware of is [2, Theorem 7.3], saying that any generic smooth map $S^n \to Y^{n-1}$ has a fiber of length $\geq \pi$. A non-sharp version of the waist theorem, however, can be proved for any m-dimensional target space by induction using the Federer–Fleming isoperimetric estimate. This type of argument apparently goes back to Almgren, and it was used by Gromov in [16] (see the exposition in [24, Section 7], which applies to any target space, or in [1, Section 7]). A discrete version of this non-sharp estimate is proven in [44] along the same lines. For Riemannian metrics other than round, the case $n = 2$ is understood [41, 6], and the case $n = 3$ is investigated under additional curvature assumptions [43].

The Urysohn width itself is a waist-type invariant, in which the size of a fiber is measured via its diameter, instead of the volume. In this chapter, we investigate (non-sharp) waist theorems, where the size of a fiber is measured via the Urysohn width.

Prototype question. Fix integers n, m, d, such that $n > m + d$. Let $f : X^n \to Y^m$ be a continuous map from a compact Riemannian n-manifold to an m-dimensional simplicial complex. Let w be the supremal Urysohn d-width of fibers $f^{-1}(y)$, $y \in Y$, viewed as compact metric spaces with the extrinsic metric of X. Can one bound w from below in terms of the $(n-1)$-width of X? If not, can one bound w if the "topological complexity" of the fibers is restricted?

It is natural to expect that the answer should be affirmative in some sense when $n > m+d$ (if we hope that the corresponding property of the dimension [9] is robust). When $d = 1$, and the first Betty number of the fibers is bounded, this is indeed the case, as we will show in Section 5.2. However, in general this is far from true. In Section 5.3 it will be shown that even for $n = (m+1)(d-m) + 2m$ and topologically trivial fibers the answer is negative. In a sense, this shows the failure of the notion of the d-width to measure the "defect of d-dimensionality".

Let us describe the answers for the first four non-trivial cases of Prototype question. These four claims are the simplest special cases of the theorems explained in this chapter.

(A) *There is a map $f : [0,1]^3 \to [0,1]$ with all fibers having arbitrarily small 1-width.*

We describe this example ([19, Example H_1'']) briefly. Consider an ε-fine cubical grid in $\mathbb{R}^3$, and let Z_0 be its 1-skeleton. Let Z_1 be the 1-skeleton of the dual grid. Define f by setting $f(x) = \frac{\mathrm{dist}(x,Z_0)}{\mathrm{dist}(x,Z_0)+\mathrm{dist}(x,Z_1)}$. It can be checked that every fiber $\Sigma_y = f^{-1}(y)$, $y \in [0,1)$, retracts to Z_0 with every point moving by distance $\lesssim \varepsilon$; hence it has small 1-width. Similarly, the fibers over $y \in (0,1]$ are approximated by Z_1.

In Section 5.1, we explain how this example is generalized to higher dimensions, see Theorem 5.1.2. This might be known to experts, but we were not able to locate a reference.

(B) Notice that all regular fibers in the previous example have high genus. What happens if we bound their topological complexity?

Suppose that a piecewise linear map $f : [0,1]^3 \to [0,1]$ is such that all fibers $f^{-1}(y), y \in [0,1]$, are homeomorphic to $[0,1]^2$. Then there is a fiber $f^{-1}(y)$ of Urysohn 1-width at least $\frac{1}{3}$.

This is the baby case of one of our main results, Theorem 5.2.16. Here is the idea of the proof that will be developed in Section 5.2. Suppose that every fiber $X_y = f^{-1}(y)$ has width $\mathrm{UW}_d(X_y) < c$. So there are maps $X_y \to Z_y$ to graphs Z_y whose fibers are of diameter less than c. A naïve idea might be to assemble them together to get a map $[0,1]^3 \to \bigcup Z_y$. If there was a nice way to interpret $\bigcup Z_y$ as a two-dimensional space, then we would be done as long as $c < \mathrm{UW}_{n-1}(X)$. A careful argument might try to assemble the maps $X_y \to Z_y$ by induction on the skeletal structure of Y, subdivided finely. The newly built intermediate maps will have fibers with the size bounded in terms of c and the "topological complexity" of the X_y.

(C) The following is a special case of [19, Corollary H_1'], which we discuss in Section 5.1 (see Theorem 5.1.1).

Every continuous map $f : X^4 \to Y^1$ from a compact metric space to a graph has a fiber whose 1-width is at least the 3-width of X.

(D) Another major result of this paper is Theorem 5.3.1, a family of examples of maps with small and topologically trivial fibers; here is the simplest case.

There is a Riemannian metric on $[0,1]^4$ that has substantial 3-width but the fibers of the coordinate projection $f : [0,1]^4 \to [0,1]$ all have small 2-width.

We sketch roughly the idea of the construction. For each $y \in [0,1]$, the standard metric inside the fiber $f^{-1}(y) \simeq [0,1]^3$ is modified as follows. Consider the high-genus surface $\Sigma_y \subset f^{-1}(y)$, as in the example (A). In its small tubular neighborhood, blow up the metric in the normal direction; then, squeeze the metric everywhere outside the tubular neighborhood. The result can be mapped to the suspension of Z_0 or Z_1 with small fibers. However, the entire space $[0,1]^4$ can be shown to have substantial 3-width.

5.1 Waist of maps with arbitrary fibers

The first waist-type result for the Urysohn width follows directly from the definitions. It was observed by Gromov [19, Corollary H_1'], and we already saw its proof in the previous chapter (see Theorem 4.1.1).

Theorem 5.1.1. *Let X be a compact metric space, and let Y be a metrizable topological space of covering dimension m. Every continuous map $f : X \to Y$ has a fiber $f^{-1}(y)$ of d-width $\mathrm{UW}_d(f^{-1}(y)) \geq \mathrm{UW}_{n-1}(X)$, where $n = (m+1)(d+1)$.*

The relation between dimensions n, m, d in this theorem is optimal, as the following result (generalizing example (A) from the introduction) shows.

Theorem 5.1.2. *Let $n = (m+1)(d+1) - 1$, and let $\varepsilon > 0$ be any small number. There exists a continuous map $f : B^n \to \triangle^m$ from the unit euclidean n-ball to the m-simplex, whose fibers all have Urysohn d-width less than ε.*

The crucial tool used in the proof is the *local join representation* of $\mathbb{R}^n$, which will be also used in Section 5.3.

Lemma 5.1.3 (cf. Lemma 3.3.1)**.** *Fix $\varepsilon > 0$. There is a locally finite triangulation of $\mathbb{R}^n$ by simplices of diameter $< \varepsilon$, admitting a nice coloring: the vertices receive colors $0, 1, \ldots, n$ so that each simplex receives all distinct colors.*

Proof. In fact, there is such a triangulation with simplices congruent to one another, via the reflection in the facets. Such a triangulation can be obtained from the type A root system and the corresponding affine Coxeter hyperplane arrangement (see [50, Chapter 6]). (Of course, simpler constructions are also possible.) □

Definition 5.1.4 (cf. Definition 3.3.2)**.** Let $n = (m+1)(d+1) - 1$, and triangulate $\mathbb{R}^n$ by ε-small simplices, as in Lemma 5.1.3. Define Z_i, $0 \leq i \leq m$, to be the union of all simplices of the triangulation colored by colors $(d+1)i$ through $(d+1)i + d$. We say that $\mathbb{R}^n$ is the *ε-local join* of d-dimensional complexes $Z_0, \ldots, Z_m$.

The name is justified by the following observation: every (top-dimensional) simplex σ of the triangulation can be written as the join $(\sigma \cap Z_0) * \ldots * (\sigma \cap Z_m)$; that is, any point $x \in \sigma$ can be written as

$$x = \sum_{i=0}^{m} t_i z_i, \quad \text{where } z_i \in \sigma \cap Z_i, \quad t_i \geq 0, \quad \sum_{i=0}^{m} t_i = 1.$$

The coefficients t_i are determined uniquely, giving a well-defined *join map*

$$\tau : \mathbb{R}^n \to \triangle^m = \left\{ (t_0, \ldots, t_m) \;\middle|\; t_i \geq 0, \; \sum_{i=0}^{m} t_i = 1 \right\}.$$

Note that $Z_i = \tau^{-1}(v_i)$, where $v_0, \ldots, v_m$ are the vertices of $\triangle^m$. For each vertex v_i, denote the opposite facet of $\triangle^m$ by $v_i^\vee$. For each complex Z_i, its *dual* $(md + m - 1)$-dimensional complex is given by $Z_i^\vee = \tau^{-1}(v_i^\vee)$. There are natural retractions

$$\pi_i : \mathbb{R}^n \setminus Z_i^\vee \to Z_i,$$

defined by sending $x = \sum_{i=0}^{m} t_i z_i \in \sigma$ to $z_i \in \sigma \cap Z_i$; they are well-defined since $t_i \neq 0$ whenever $x \notin Z_i^\vee$. Note that π_i moves each point by distance $< \varepsilon$.

Proof of Theorem 5.1.2. Represent $\mathbb{R}^n$ as the $\varepsilon/2$-*local join* of d-dimensional complexes $Z_0, \ldots, Z_m$; let $\tau : \mathbb{R}^n \to \triangle^m$ be its join map. Take f to be the restriction of τ on the unit ball B^n. Let us check that the d-width of any fiber $F = f^{-1}(t_0, \ldots, t_m)$ is small. Fix any i for which $t_i \neq 0$. The (restricted) retraction map $\pi_i|_F : F \to Z_i$ has fibers of diameter $< \varepsilon$, so we are done. □

5.2 Waist of maps with fibers of bounded complexity

This section generalizes example (B) from the introduction. The main result, Theorem 5.2.16, which in particular implies the following waist inequality.

Any piecewise linear map $f : X^{m+2} \to Y^m$ from a metric $(m+2)$-polyhedron to an m-polyhedron must have a fiber of 1-width at least $\frac{\mathrm{UW}_{m+1}(X)}{2\beta m + m^2 + m + 1}$, where $\beta =$

$\sup_{y \in Y} \operatorname{rk} H_1(f^{-1}(y))$ *measures the topological complexity of the map.*

5.2.1 PL maps of polyhedra

We use the word *polyhedron* to refer to a topological space admitting a structure of a finite simplicial complex (together with rectilinear structure on each simplex), though we do not usually specify this structure. We say a continuous map $X \to Y$ of polyhedra is a *piecewise linear map*, or a *PL map*, if it is simplicial for some fine simplicial structures on X and Y.

A polyhedron with a metric space structure (giving the same topology) will be called a *metric polyhedron*. For example, it could be a polyhedron endowed with a smooth Riemannian metric on each maximal simplex, so that the metrics on adjacent simplices match in restriction to their common face.

For a map $f : X \to Y$, we sometimes denote the preimage $f^{-1}(A)$ of a subset $A \subset Y$ by X_A, if there is no confusion and f is understood from the context. If X and $A \subset Y$ are polyhedra, and f is a PL map, then X_A is naturally a polyhedron. If additionally X is metric, then X_A is metric as well (with the extrinsic metric).

Definition 5.2.1. We measure the *topological complexity* using the first Betty number. For a space X, we set $\operatorname{tc}(X) = \operatorname{rk} H_1(X;\mathbb{Z}) = \dim H_1(X;\mathbb{Q})$. For a map $f : X \to Y$, we set $\operatorname{tc}(f) = \sup_{y \in Y} \operatorname{tc}(X_y)$.

Remark 5.2.2. In fact, the estimates 5.2.11, 5.2.15, 5.2.16, 5.2.17 of this section hold in a stronger form, with $\operatorname{tc}(\cdot)$ replaced by a smaller quantity. Namely, we define $\operatorname{tc}'(X)$ as the largest number of linearly independent classes in $H^1(X;\mathbb{Q})$ with pairwise zero cup-products. Similarly, for a map $f : X \to Y$, we set $\operatorname{tc}'(f) = \sup_{y \in Y} \operatorname{tc}'(X_y)$. We formulate our results with $\operatorname{tc}(\cdot)$ for simplicity, but in the proofs we indicate the adjustments needed if we use $\operatorname{tc}'(\cdot)$.

Example 5.2.3. If X is a closed connected oriented surface of genus g, then $\operatorname{tc}(X) = 2g$ while $\operatorname{tc}'(X) = g$. If X is a connected oriented surface of genus g with $q > 0$ punctures, then $\operatorname{tc}(X) = \operatorname{tc}'(X) = 2g + q - 1$.

Lemma 5.2.4. *Every PL map $f : X \to Y$ of polyhedra satisfies the following regularity assumption. Fix a simplicial structure on Y for which f is simplicial. Fix a simplex $\triangle \subset Y$ (of any dimension), and let $\mathring{\triangle}$ be its relative interior. Then one can pick a PL map $\Psi_\triangle : \triangle \times \Sigma_\triangle \to X_\triangle$, for some polyhedron $\Sigma_\triangle$, such that*

- *$\Psi_\triangle$ is fibered over $\triangle$:*

$$\begin{array}{ccc} \triangle \times \Sigma_\triangle & \xrightarrow{\Psi_\triangle} & X_\triangle \\ & \searrow^{\text{projection}} & \downarrow f \\ & & \triangle \subset Y \end{array}$$

- *the restriction*

$$\Psi_\triangle|_{\mathring{\triangle} \times \Sigma_\triangle} : \mathring{\triangle} \times \Sigma_\triangle \to X_{\mathring{\triangle}}$$

  is a homeomorphism making f a fiber bundle over $\mathring{\triangle}$.

Proof. For $\Sigma_\triangle$, take the fiber over the center of $\triangle$, and the rest can be verified easily. $\square$

5.2.2 Connected maps

Definition 5.2.5. Let $f : X \to Y$ be a continuous map of topological spaces. It is called *connected* if the fibers $f^{-1}(z)$, $z \in Z$, are (nonempty and) path-connected. Every map f, connected or not, cannot be factored as

$$X \xrightarrow{\widetilde{f}} \widetilde{Y} \to Y,$$

with $\widetilde{f}$ connected, and with $\widetilde{Y}$ being the space of path-connected components of the fibers of f (topologized by the finest topology making $\widetilde{f}$ continuous). The map $\widetilde{f}$ is called the *associated connected map.*

If f is a PL map of polyhedra, then $\widetilde{f}$ is also PL, and $\widetilde{Y}$ is a polyhedron having the same dimension as $f(X)$.

Lemma 5.2.6. *Let $f : X \to Y$ be a connected PL map of polyhedra. If Y is connected then X is connected.*

Proof. Let $\gamma : [0,1] \to Y$ be a path in the base. Fix a simplicial structure of Y for which f is simplicial. Let us build a path $\widetilde{\gamma} : [0,1] \to X$ covering γ in the following weak sense: there is a monotone reparametrization map $r : [0,1] \to [0,1]$ such that $f(\widetilde{\gamma}(t)) = \gamma(r(t))$. First, split γ into arcs each of which belongs to a single cell of Y. Without loss of generality, there are finitely many of these arcs (this can be achieved by homotoping γ slightly, while fixing endpoints). For each such arc $[t', t''] \to Y$, one can lift γ by Lemma 5.2.4. If γ is lifted independently over $[t', t]$ and $[t, t'']$, the two lifted patches can be connected inside the fiber $f^{-1}(\gamma(t))$. This is how $\widetilde{\gamma}$ can be built. For the assertion of the lemma, having two points $x, x' \in X$, one can connect $f(x)$ to $f(x')$ in the base, and lift the path as above. The endpoints of the lifted path can be connected to x and x' in the corresponding fibers. This proves that X is connected. □

5.2.3 Foliations

Definition 5.2.7. Let Σ be a topological space. We use the word *foliation* to denote a continuous map $p : \Sigma \to Z$ to a graph (finite 1-dimensional simplicial complex), in the sense that Σ is foliated by the fibers $p^{-1}(z)$, $z \in Z$ (the *leaves*).

This is a non-standard use of the word "foliation". We could have used the word "slicing" as well in this context.

Definition 5.2.8. Let Σ be a polyhedron. We say a foliation $p : \Sigma \to Z$ is *simple* if it is a connected PL map.

For a foliation p of a compact metric space Σ, recall the notation $\mathrm{W}(p) = \sup_{z \in Z} \operatorname{diam} p^{-1}(z)$ for its width. The next lemma shows that, in a sense, any its foliation of width < 1 can be "simplified" while keeping its width < 1.

Lemma 5.2.9. *If a metric polyhedron Σ admits a foliation of width < 1, then it also admits a simple foliation width < 1.*

Proof. Let $p : \Sigma \to Z$ be a foliation of width < 1. Subdivide Z finely so that the preimage of the open star[1] S_v of every vertex $v \in Z$ has diameter < 1. Use the simplicial approximation theorem to approximate p by a simplicial (for some subdivision of Σ) map p' such that for each $x \in \Sigma$, $p'(x)$ belongs to the minimal closed cell of Z containing $p(x)$. It follows that each fiber of p' is contained in $p^{-1}(S_v)$ for some $v \in Z$, so p' has width < 1.

Next, replacing p' by the associated connected map $\widetilde{p}'$ (which is also PL), we arrive at the situation where the leaves $(\widetilde{p}')^{-1}(z)$ are (nonempty and) connected for all $z \in Z$, and have diameter < 1. □

5.2.4 Interpolation lemma

Definition 5.2.10. Let Σ be a topological space, and let $p_0 : \Sigma \to Z_0$, $p_1 : \Sigma \to Z_1$ be its foliations. An *interpolation* between these is a family of foliations $p_t : \Sigma \to Z_t$, $t \in [0, 1]$, continuous in the following sense.

- There are 2-dimensional polyhedron $Z_{[0,1]}$ together with a *parametrization* map $\pi : Z_{[0,1]} \to [0, 1]$, such that $\pi^{-1}(t) = Z_t \subset Z_{[0,1]}$.
- There is a continuous map $P : [0, 1] \times \Sigma \to Z_{[0,1]}$ fibered over $[0, 1]$, and giving p_t when restricted over $\{t\}$:

$$\begin{array}{ccc} [0,1]\times\Sigma & \xrightarrow{P} & Z_{[0,1]} \\ & \searrow^{\text{projection}} & \downarrow \pi \\ & & [0,1] \end{array} \qquad \begin{array}{ccc} \{t\}\times\Sigma & \xrightarrow{p_t} & Z_t \subset Z_{[0,1]} \\ & \searrow^{\text{projection}} & \downarrow \pi \\ & & \{t\} \end{array}$$

Lemma 5.2.11. *Let Σ be a metric polyhedron of topological complexity $\beta = \operatorname{tc}(\Sigma)$, and let $p_0 : \Sigma \to Z_0$, $p_1 : \Sigma \to Z_1$ be simple foliations. It is possible to interpolate between them through simple foliations of width at most $(\beta+2)\,\mathrm{W}(p_0)+(\beta+1)\,\mathrm{W}(p_1)$.*

We only outline the proof, since a more general statement will be proved in the next subsection. However, this outline illustrates the main method of this section.

[1] Recall that the *open star* of a vertex of a simplicial complex is the union of the relative interiors of all faces containing the given vertex. In a graph, the open star of a vertex is the vertex itself together with all incident open edges.

Lemma 5.2.12. *Given a (finite) connected graph Z (viewed as a topological space), there is a filtration by closed subspaces $Z^{(t)} \subset Z$, $t \in [0,1]$, such that*

- $Z^{(t)} = \alpha^{-1}([0,t])$*, for some continuous function* $\alpha : Z \to [1/2, 1]$*;*
- $Z^{(1/2)} = \alpha^{-1}(1/2)$ *consists of a single point;*
- *every preimage* $\alpha^{-1}(t)$*,* $t \in [1/2, 1]$*, consists of finitely many points (informally, this condition says that* $Z^{(t)}$ *depends continuously on* t*).*

One can also consider a satellite filtration by open subspaces $\mathring{Z}^{(t)} = \bigcup_{t' \in [0,t)} Z^{(t')} = \alpha^{-1}([0,t))$.

Proof. Such a filtration can be constructed using

$$\alpha(z) = \frac{\mathrm{dist}_Z(z_0, z)}{2 \sup\limits_{z' \in Z} \mathrm{dist}_Z(z_0, z')} + 1/2$$

for any fixed point $z_0 \in Z$ and any metrization of Z. □

Outline of the proof of Lemma 5.2.11. We can assume Σ connected (by dealing with each connected component separately).

The graph Z_1 is connected, since Σ is connected, and p_1 is simple (hence surjective). Filter Z_1 as in Lemma 5.2.12: $Z_1^{(0)} \subset \ldots \subset Z_1^{(t)} \subset \ldots \subset Z_1^{(1)}$, $t \in [0,1]$. We interpolate between p_0 and p_1 through foliations $p_t : \Sigma \to Z_t$, which can be roughly described as follows. To get a picture of p_t, first you draw the fibers of p_1 over $Z_1^{(t)}$. Then in the remaining room we draw the fibers of p_0 (their parts that fit). The resulting picture is interpreted as a foliation by connected leaves, and we call it p_t (see Figure 5-1).

Let us rigorously describe the space of leaves Z_t and the foliation map p_t.

- Define $Z_0^{(t)}$, $t \in [0,1]$, as the minimal closed subspace of Z_0 such that $p_0^{-1}(Z_0^{(t)}) \cup p_1^{-1}(\mathring{Z}_1^{(t)}) = \Sigma$; in other words,

$$Z_0^{(t)} = p_0\left(\Sigma \setminus p_1^{-1}(\mathring{Z}_1^{(t)})\right).$$

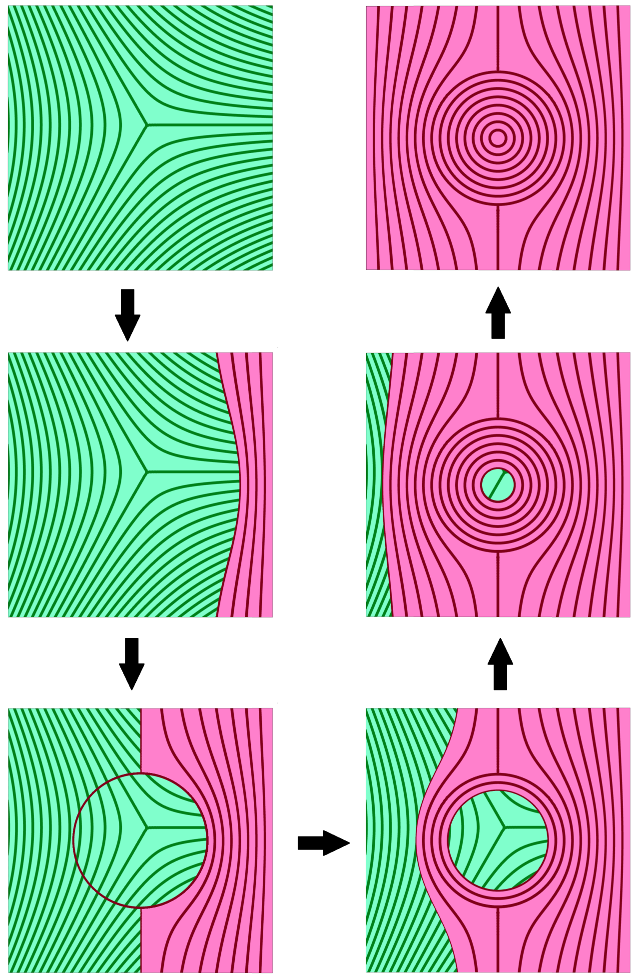


Figure 5-1: Interpolation between foliations. Each rectangle represents a foliation of Σ, given by a map to a graph. The foliations p_0 and p_1 are pictured in green and red, respectively

We write $\Sigma^{(t)} = \Sigma \setminus p_1^{-1}(\mathring{Z}_1^{(t)})$ for short.

- The map $p_0|_{\Sigma^{(t)}} : \Sigma^{(t)} \to Z_0^{(t)}$ might not have all fibers connected, so we factor it through its associated connected map:

$$\Sigma^{(t)} \overset{\widetilde{p}_0^{(t)}}{\to} \widetilde{Z}_0^{(t)} \to Z_0^{(t)}.$$

- The graph Z_t is defined as

$$\left(\widetilde{Z}_0^{(t)} \sqcup Z_1^{(t)}\right) / \overset{t}{\sim},$$

where $\overset{t}{\sim}$ is the following equivalence relation. Let us write $z \overset{t}{\approx} z'$ if $z \in \widetilde{Z}_0^{(t)}$, $z' \in Z_1^{(t)}$, and $(\widetilde{p}_0^{(t)})^{-1}(z)$ intersects $p_1^{-1}(z')$. Define $\overset{t}{\sim}$ to be the transitive closure of $\overset{t}{\approx}$. There are natural maps $\iota_0^{(t)} : \widetilde{Z}_0^{(t)} \to Z_t$ and $\iota_1^{(t)} : Z_1^{(t)} \to Z_t$.

- The map $p_t : \Sigma \to Z_t$ is defined as

$$p_t(x) = \begin{cases} \iota_1^{(t)}(p_1(x)), & \text{if } p_1(x) \in Z_1^{(t)} \\ \iota_0^{(t)}\left(\widetilde{p}_0^{(t)}(x)\right), & \text{otherwise.} \end{cases}$$

  Observe that for $t = 0, 1$ this agrees with the original foliations p_0 and p_1.

This describes the intermediate foliations p_t, but in order to describe the interpolation completely we also need to explain how the graphs Z_t assemble into a 2-complex $Z_{[0,1]}$, and how the maps p_t assemble into a continuous map $P : [0,1] \times \Sigma \to Z_{[0,1]}$. We do not give these details here, because a more general construction will be explained in the next subsection.

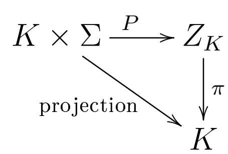
To finish the proof, we need to bound the size of the fibers of p_t. Why could it be possibly large? Because in the process of interpolating some vertices of the target graph merged under the $\overset{t}{\sim}$-identification, so multiple fibers of p_0 and p_1 might have been united. Consider a fiber of p_t. For this fiber, consider the longest chain of identifications

$$z_0 \overset{t}{\approx} z_1' \overset{t}{\approx} z_1 \overset{t}{\approx} z_2' \overset{t}{\approx} \dots$$

with $z_j \in \widetilde{Z}_0^{(t)}$, and with $z_j' \in Z_1^{(t)}$ all distinct. Suppose it has more than $1 + \operatorname{tc}(\Sigma)$ elements of $Z_1^{(t)}$. To every subchain $z_j' \overset{t}{\approx} z_j \overset{t}{\approx} z_{j+1}'$ assign an arc inside $(\widetilde{p}_0^{(t)})^{-1}(z_j)$ connecting some two points $x \in p_1^{-1}(z_j')$ and $y \in p_1^{-1}(z_{j+1}')$. This arc represents an element of relative homology $H_1(\Sigma, \Sigma_1)$, where we denoted $\Sigma_1 = p_1^{-1}(Z_1^{(t)})$. Recall that Σ_1 is connected by Lemma 5.2.6, so $\operatorname{rk} H_1(\Sigma, \Sigma_1) \leq \operatorname{tc}(\Sigma)$. There must be a relation between the classes of those arcs in $H_1(\Sigma, \Sigma_1)$. It follows that some z_j' repeats in the chain, which proves that such a chain has at most $1 + \operatorname{tc}(\Sigma)$ elements of $Z_1^{(t)}$, hence at most $2 + \operatorname{tc}(\Sigma)$ elements of $\widetilde{Z}_0^{(t)}$. We conclude that the diameter of a fiber of p_t is at most $(\beta + 2)\,\mathrm{W}(p_0) + (\beta + 1)\,\mathrm{W}(p_1)$. This finishes the proof if we measure the topological complexity with $\mathrm{tc}(\cdot)$. For the modified complexity $\mathrm{tc}'(\cdot)$, one can assign a class in $H^1(\Sigma)$ to each element z'_j (represented by the cochain that counts intersections with $p_1^{-1}(z'_j)$). One needs to verify that there is just one linear dependence between them (coming from the 0-cochain equal to the characteristic function of Σ_1), and that their products vanish; this will imply that some z'_j must repeat. □

5.2.5 Parametric interpolation lemma

Definition 5.2.13. Let Σ be a topological space, and let $\pi : Z_K \to K$ be a map of polyhedra such that every fiber is a (nonempty and) connected graph. A continuous map $P : K \times \Sigma \to Z_K$ is called a *parametric foliation over* K, or a *family of foliations parametrized by* K, if the composition $\pi \circ P : K \times \Sigma \to K$ is the projection onto the first factor:



We call Z_K the *space of leaves*, and π the *parametrization map*. For $s \in K$, the restriction $P|_{\{s\}\times\Sigma}$ can be viewed as a foliation $p_s : \Sigma \to \pi^{-1}(s)$, and we think of P as the family of foliations p_s parametrized by $s \in K$. We say that P is *simple* if it is PL and connected.

For a parametric foliation $P : K \times \Sigma \to Z_K$ of a metric space Σ, we keep using the notation $\mathrm{W}(P) = \sup\limits_{z \in Z_K} \operatorname{diam} P^{-1}(z)$ for the width.

Definition 5.2.14. Let Σ be a topological space. Let $P_0 : K \times \Sigma \to Z_K$ be a family of foliations, and let $p_1 : \Sigma \to Z_1$ be another foliation. An *interpolation* between them is a parametric foliation $P : CK \times \Sigma \to Z_{CK}$ over the cone $CK = ([0,1] \times K)/(\{1\} \times K)$, restricting to P_0 over the base $\{0\} \times K$ of CK, and to p_1 over the apex of CK.

We are in position to prove the principal lemma of this section.

Lemma 5.2.15 (Parametric interpolation)**.** *Let Σ be a metric polyhedron of topological complexity $\beta = \mathrm{tc}(\Sigma)$. Let $P_K : K \times \Sigma \to Z_K$ be a family of simple foliations over* *a d-dimensional complex K, and let $p_1 : \Sigma \to Z_1$ be a simple foliation. It is possible to interpolate between P_K and p_1 via a simple family $CK \times \Sigma \to Z_{CK}$ of width at most $(\beta + 2)\,\mathrm{W}(P_0) + (\beta + 1)\,\mathrm{W}(p_1)$.*

Proof. We can assume Σ connected (by dealing with each connected component separately).

The parametric foliation P_K splits into simple foliations $p_s : \Sigma \to Z_s$, where $Z_s = \pi^{-1}(s)$, $s \in K$, $\pi : Z_K \to K$ is the parametrization of the foliation base.

The proof idea is simple: for each $s \in K$, interpolate between p_s and p_1 as in Lemma 5.2.11, and make sure that the interpolation depends nicely on s, in order to assemble them altogether to a parametric interpolation. The details are pretty technical, and now we write them out.

The graph Z_1 is finite and connected, since Σ is compact and connected, and p_1 is simple (hence surjective). Filter Z_1 as in Lemma 5.2.12: $Z_1^{(0)} \subset \ldots \subset Z_1^{(t)} \subset \ldots \subset Z_1^{(1)}$, $t \in [0, 1]$. We interpolate between P_K and p_1 via a family $P : CK \times \Sigma \to Z_{CK}$ to be described. With a little abuse of notation, we use coordinates $(t, s) \in [0, 1] \times K$ on CK, with a convention that all points $(1, s)$ are identified with the apex of CK. The restriction $P|_{\{(t,s)\}\times\Sigma}$ is a foliation $p_{(t,s)} : \Sigma \to Z_{(t,s)}$, which can be pictured as follows. First, draw the fibers of p_1 over $Z_1^{(t)}$; then fill in the remaining room with the fibers of p_s (with their parts that fit). The resulting picture is interpreted as a foliation by connected leaves, and we call it $p_{(t,s)}$.

We now describe $P : CK \times \Sigma \to Z_{CK}$ formally.

- Define

$$
\begin{aligned}
P_0 : \quad [0, 1) \times K \times \Sigma &\to [0, 1) \times Z_K \\
(t, s, x) &\mapsto (t, p_s(x)) \\
P_1 : \quad CK \times \Sigma &\to CK \times Z_1 \\
(c, x) &\mapsto (c, p_1(x))
\end{aligned}
$$

- Define

$$\mathcal{Z}_1 = \bigcup_{(t,s)\in CK} Z_1^{(t)} \subset CK \times Z_1$$

where we think of $Z_1^{(t)}$ as sitting in $\{(t,s)\} \times Z_1$. The interior of $\mathcal{Z}_1$ is

$$\mathring{\mathcal{Z}}_1 = \bigcup_{(t,s)\in CK} \mathring{Z}_1^{(t)} \subset CK \times Z_1.$$

Define

$$\mathfrak{S} = ([0,1) \times K \times \Sigma) \setminus P_1^{-1}(\mathring{\mathcal{Z}}_1) \subset [0,1) \times K \times \Sigma$$

and

$$\mathcal{Z}_0 = P_0(\mathfrak{S}) \subset [0,1) \times Z_K.$$

- The map $P_0|_{\mathfrak{S}}$ might not be connected, so we factor it through its associated connected map:

$$\mathfrak{S} \xrightarrow{\widetilde{P}_0} \widetilde{\mathcal{Z}_0} \to \mathcal{Z}_0\,.$$

- The space of leaves is

$$Z_{CK} = \left(\widetilde{\mathcal{Z}_0} \sqcup \mathcal{Z}_1\right)/\!\sim,$$

where $\sim$ is the following equivalence relation. Let us write $z \approx z'$ if $z \in \widetilde{\mathcal{Z}_0}$, $z' \in \mathcal{Z}_1$, and $\widetilde{P}_0^{-1}(z)$ intersects $P_1^{-1}(z')$, as subsets of $CK \times \Sigma$. (Recall our convention for coordinates in a cone, in which $[0,1) \times K \subset CK$.) Define $\sim$ to be the transitive closure of $\approx$. There are natural maps $\iota_0 : \widetilde{\mathcal{Z}_0} \to Z_{CK}$ and $\iota_1 : \mathcal{Z}_1 \to Z_{CK}$.

- The parametric foliation P is defined as

$$\begin{aligned} P : CK \times \Sigma &\to Z_{CK} \\ \xi &\mapsto \begin{cases} \iota_1(P_1(\xi)), & \text{if } P_1(\xi) \in \mathcal{Z}_1 \\ \iota_0\left(\widetilde{P}_0(\xi)\right), & \text{otherwise.} \end{cases} \end{aligned}$$

It is easy to see that P indeed interpolates between P_K and p_1.

Clearly, P is connected. It is rather technical but straightforward to make sure that P is PL.

The analysis of the width was already done in Lemma 5.2.11. Any foliation from the family P interpolates between certain p_s, $s \in K$, and p_1, as in the construction of Lemma 5.2.11. Therefore, $\mathrm{W}(P) \leq (\beta+2)\,\mathrm{W}(P_0) + (\beta+1)\,\mathrm{W}(p_1)$. □

5.2.6 Waist of a PL map

Finally, we are ready to prove the main theorem of this section.

Theorem 5.2.16. *Let $f : X \to Y^m$ be a PL map from a metric polyhedron X to an m-dimensional polyhedron Y. Let $\beta = \mathrm{tc}(f)$ be its topological complexity, that is, $\beta = \sup_{y \in Y} \mathrm{tc}(f^{-1}(y))$. Then there is a fiber $X_y = f^{-1}(y)$ of Urysohn width*

$$\mathrm{UW}_1(X_y) \geq c(m, \beta)\,\mathrm{UW}_{m+1}(X),$$

for some positive constant c depending only on m and β.

Proof. Replacing f with its associated connected map, we can assume that f is connected. Even if f is not a fiber bundle, still locally this is almost the case by Lemma 5.2.4. For each simplex $\triangle \subset Y$ (of any dimension) in a fine triangulation of Y, the map f can be "almost" trivialized over $\triangle$ via a PL map

$$\Psi_\triangle : \triangle \times \Sigma_\triangle \to X_\triangle,$$

for some polyhedron $\Sigma_\triangle$; this map is a genuine trivialization over $\mathring{\triangle}$, the relative interior of $\triangle$. For each $y \in \mathring{\triangle}$, this map induces the distance function $d_y^\triangle$ on $\Sigma_\triangle$ defined as

$$d_y^\triangle(x, x') = \mathrm{dist}_X(\Psi_\triangle(y, x), \Psi_\triangle(y, x')).$$

Refining the triangulation of Y if needed, we can assume that all metrics $d_y^\triangle$ over $y \in \mathring{\triangle}$ are ε-close to one another in the following sense: the "layers" $\Psi_\triangle(\triangle \times \{x\})$ have diameter less than $\varepsilon/2$ for all $x \in \Sigma_\triangle$, hence for any $x, x' \in \Sigma_\triangle$ and any $y, y' \in \mathring{\triangle}$

we have $|d_y^{\triangle}(x,x') - d_{y'}^{\triangle}(x,x')| \leq \varepsilon$.

Suppose that $\mathrm{UW}_1(X_y) < w_0$, for all $y \in Y$, with $w_0 = c(m,\beta)\,\mathrm{UW}_{d+1}(X)$ to be specified later. We get a foliation of X_y of width less than w_0, which can be assumed simple by Lemma 5.2.9. The idea of the proof is to pick a dense discrete set of points in Y, and use those foliations to build a map $F : X \to Z^{m+1}$ of controlled width. This is done inductively on skeleta of Y.

At the zeroth step, for each vertex v of Y, pick a simple foliation $F_v : X_v \to Z_v$ of width less than w_0.

At the k^{th} step, $1 \leq k \leq m$, we assume that we already defined $F_{k-1} : X_{Y^{(k-1)}} \to Z_{Y^{(k-1)}}$, over the $(k-1)$-skeleton of Y, of width less than w_{k-1}, and we need to extend it over $Y^{(k)}$. Take a k-simplex $\triangle \subset Y$, and consider the corresponding "trivialization" $\Psi_\triangle : \triangle \times \Sigma_\triangle \to X_\triangle$. Pick a point y in the relative interior of $\triangle$, and a simple foliation p_y of $\Sigma_\triangle$ of $d_y^{\triangle}$-width $< w_0$. We would like to use Lemma 5.2.15 to build a parametric foliation $P_\triangle : \triangle \times \Sigma_\triangle \to Z_\triangle$ interpolating between p_y and the family of foliations

$$\partial\triangle \times \Sigma_\triangle \overset{\Psi_\triangle}{\to} X_{\partial\triangle} \overset{F_{k-1}}{\to} Z_{Y^{(k-1)}}$$

(here ∂ denotes the relative boundary). In order to apply that lemma, we need to fix a metric on $\Sigma_\triangle$, so we use $d_y^{\triangle}$ (recall that they are all ε-close). We get a map $P_\triangle : \triangle \times \Sigma_\triangle \to Z_\triangle$ of width less than $(\beta+2)w_{k-1} + (\beta+1)w_0$. The desired map $F_\triangle : X_\triangle \to Z_\triangle$ that we are looking for is already defined over $\partial\triangle$, so we specify it over $\mathring{\triangle}$:

$$X_{\mathring{\triangle}} \overset{\Psi_\triangle^{-1}}{\to} \mathring{\triangle} \times \Sigma \overset{P_\triangle}{\to} Z_\triangle.$$

The resulting map $F_\triangle$ is continuous. Repeating this over all k-simplices we get the map $F_k : X_{Y^{(k)}} \to Z_{Y^{(k)}}$ of width less than

$$w_k = (\beta+2)w_{k-1} + (\beta+1)w_0 + \varepsilon.$$

As $\varepsilon \to 0$, the solution of this recurrence tends to

$$w_k = (2(\beta+2)^k - 1)w_0.$$

Therefore, $\mathrm{UW}_{m+1}(X) \leq (2(\beta+2)^m - 1)c(m,\beta)\,\mathrm{UW}_{m+1}(X)$. Hence, for each $c < \frac{1}{2(\beta+2)^m-1}$, there is a fiber $X_{y(c)}$ of width at least $c\,\mathrm{UW}_{m+1}(X)$. Finally, send $c \to \frac{1}{2(\beta+2)^m-1}$, pick a limit point $\bar{y}$ of $\{y(c)\}$, and note that $\mathrm{UW}_1(X_{\bar{y}}) \geq \frac{\mathrm{UW}_{m+1}(X)}{2(\beta+2)^m-1}$ by upper semi-continuity of width (Lemma 2.4.2). □

This proof gives the value $c = \frac{1}{2(\beta+2)^m-1}$. A more careful analysis of the proof leads to a much better value, namely $c = \frac{1}{2\beta m+m^2+m+1}$, which we explain now.

The inductive interpolation step in the proof of Theorem 5.2.16 is done in a manner that allows us to split Y into m-simplices (basically according to the barycentric subdivision of the triangulation used in induction), so that over each simplex $\triangle$ we have the following picture. Over the vertices of $\triangle$, we have simple foliations $p_j : \Sigma_\triangle \to Z_j$, $j = 0, 1, \ldots, m$. Over a generic point of $\triangle$, we have a foliation $p : \Sigma_\triangle \to Z$ that looks as follows. First, draw the fibers of p_m over $Z_m^{(t_m)}$, a subgraph of Z_m (connected or empty). In the remaining room, draw (the parts of) the fibers of p_{m-1} over $Z_{m-1}^{(t_{m-1})}$, a subgraph of Z_{m-1}. Continue in the same fashion. At the last step, fill in the remaining room with (the parts of) the fibers of p_0. The touching fibers of different p_j get merged to a single fiber of p. What is the maximal length of a chain of merged fibers? We show that it can be bounded by $2\beta m + m^2 + m + 1$.

Lemma 5.2.17. *Let Σ be a metric polyhedron of topological complexity $\beta = \mathrm{tc}(\Sigma)$. Let $p_j : \Sigma \to Z_j$, $j = 0, 1, \ldots, m$, be simple foliations of width at most 1. Suppose a parametric foliation $P : \triangle \times \Sigma \to Z_\triangle$ over an m-simplex (restricting to p_j over the j^{th} vertex of $\triangle$) is obtained by applying Lemma 5.2.15 inductively; that is, one first interpolates between p_0 and p_1, then between the result and p_2, and so on. Then the width of P is at most $2\beta m + m^2 + m + 1$.*

Proof. As explained above, a generic foliation p in the family P is obtained by drawing fibers of p_j over $Z_j^{(t_j)}$, $j = 0, 1, \ldots, m$. We assume that every $Z_j^{(t_j)}$ is non-empty, otherwise the result follows by induction on m. Denote by Σ_j the closed subset of Σ covered (in the foliation p) by the fibers of $p_j, \dots, p_m$ (in particular, $\Sigma_0 = \Sigma$). Notice that for $1 \le j \le m$, Σ_j consists of at most $m - j + 1$ connected components, since each set $p_j^{-1}(Z_j^{(t_j)})$ is connected by Lemma 5.2.6. From the long exact sequence

$$\dots \to H_1(\Sigma) \to H_1(\Sigma, \Sigma_j) \to \tilde{H}_0(\Sigma_j) \to \dots$$

one finds that $\operatorname{rk} H_1(\Sigma, \Sigma_j) \le \operatorname{rk} H_1(\Sigma) + \operatorname{rk} \tilde{H}_0(\Sigma_j) \le \beta + m - j$.

We need to bound the number of fibers in a merged chain. Fix two points $x, y \in \Sigma$ in a single fiber $p^{-1}(z)$, and connect them by a path $\alpha : [0, 1] \to \Sigma$ inside this fiber. For each t, notice which of the regions $\Sigma_j \setminus \Sigma_{j+1}$ the point $\alpha(t)$ belongs to, and write down the corresponding index $J(t)$ (here Σ_{m+1} is assumed empty). We have a piecewise constant function $J : [0, 1] \to \{0, 1, \dots, m\}$. Denote the number of its discontinuities by D; without loss of generality, D is finite. Note that $\operatorname{dist}(x, y) \le D + 1$. We will transform α (while keeping it inside the same fiber of p, and fixing its endpoints x, y) to achieve $D \le (2\beta + m + 1)m$. Consider the following property, which α may or may not enjoy.

Desired property. For $1 \le j \le m$, we say that a path α is j-*nice* if the superlevel set $I^{\ge j} = \{t \in [0, 1] \mid J(t) \ge j\}$ consists of at most $\beta + m - j + 1$ components. We say that α is *nice* if it j-nice for all $1 \le j \le m$.

Suppose first α is not nice, and take the smallest index j such that α is not j-nice. Mark a point in each component of $I^{\ge j}$, so that we have marked points $t_1, \dots, t_k$, $k > \beta + m - j + 1$. Each arc $\alpha([t_i, t_{i+1}])$ represents an element of $H_1(\Sigma, \Sigma_j)$. Recall that $\operatorname{rk} H_1(\Sigma, \Sigma_j) \le \beta + m - j$. It follows that some two points $\alpha(t_i)$, $\alpha(t_{i'})$ can be connected inside $p^{-1}(z) \cap \Sigma_j$. Replace $\alpha([t_i, t_{i'}])$ with this new curve. We decreased the number of components of $I^{\ge j}$. Proceeding in the same fashion, we can make α j-nice. Repeating this procedure for larger j if needed, we make α nice.

Now that α is nice, we bound its number D of discontinuities. Clearly, D is bounded by the total number of the endpoints of all $I^{\geq j}$. Since α is nice,

$$D \leq \sum_{j=1}^{m} 2(\beta + m - j + 1) = (2\beta + m + 1)m.$$

□

We remark that the improved bound still does not seem sharp. In Gromov's example (example (A) of the introduction) the dependence on β is of order $\beta^{-1/3}$ while our bound only guarantees β^{-1}.

Remark 5.2.18. The proof of Theorem 5.2.16 together with the estimate of Lemma 5.2.17 hold with $\mathrm{tc}(\cdot)$ replaced by $\mathrm{tc}'(\cdot)$. Indeed, in the proof of Lemma 5.2.17, to each connected component of $p^{-1}(z) \cap \Sigma_j$ that α meets, one can assign a class in $H^1(\Sigma)$ in a way so that their products vanish, and their linear dependencies form at most $(m - j + 1)$-dimensional space (since they all arise from 0-cochains that are characteristic functions of the connected components of Σ_j).

5.3 Fibered manifolds with topologically trivial fibers of small width

The following result generalizes example (D) from the introduction.

Theorem 5.3.1. *For any non-negative integers m, k, and any $\varepsilon > 0$, there exists a map $X \to Y$ such that*

- *$X = F \times Y$, and the map is the trivial fiber bundle $F \times Y \to Y$;*
- *Y and F are closed topological balls of dimensions m and $mk + m + k$, respectively;*
- *X is endowed with a Riemannian metric with $\mathrm{UW}_{n-1}(X) \geq 1$, where $n = \dim X = mk + 2m + k$;*
- *for each $y \in Y$, the fiber $X_y \simeq F$ has $\mathrm{UW}_{k+m}(X_y) < \varepsilon$.*

Remark 5.3.2. Consider the trivial bundle $X' = F' \times Y' \to Y'$, where Y' is the euclidean m-ball of radius $\sim \varepsilon$, and F' is the euclidean $(mk+m+k)$-ball of radius $\sim \varepsilon$. The bundle X in the theorem will be constructed in a way so that near its boundary X will look exactly like X'. This allows to modify the construction to make X a closed manifold (e.g., a sphere or a torus), or to take the connected sum with other fibrations, etc.

Proof. To start with, take $Y = \mathbb{R}^m$, $F = \mathbb{R}^{mk+m+k}$, $X = F \times Y = \mathbb{R}^{mk+2m+k}$, and ignore for the moment that they are not closed balls. Let $p : X \to Y$ and $p_F : X \to F$ be the projection maps. We start from the euclidean metric on X, modify it, and then cut X to make it compact. Then the (restricted) map p will be the one we are looking for.

On the first factor $F = \mathbb{R}^{mk+m+k}$, consider the structure of the ε-local join of k-dimensional complexes $Z_0, \dots, Z_m$ in the sense of 5.1.4. The construction is based on the idea of blowing up the metric in between the Z_i. Let $\tau : F \to \triangle^m$ be the join map. We think of $\triangle^m$ as sitting in $\mathbb{R}^m$ with the center at the origin, scaled so that the inradius of $\triangle^m$ equals 3. Consider the "perturbation of the projection via the join map"

$$p^\tau : X \to Y, \quad p^\tau = p - \tau \circ p_F.$$

It will be useful to look at X in the coordinates $\Phi = (p_F, p^\tau)$. Namely, $\Phi : X \to X$ is the map given by $\Phi(x) = (p_F(x), p^\tau(x)) \in F \times Y = X$; its inverse is given by $x \mapsto (p_F(x), p(x) + \tau \circ p_F(x))$. It follows that the fibers of p^τ are PL homeomorphic to F.

Let $\phi_1 : [0, +\infty) \to \mathbb{R}$ be a monotone cut-off function that equals 1 on $[0,1]$ and 0 on $[1.1, \infty)$. Denote by $\phi_r^k : \mathbb{R}^k \to \mathbb{R}$ an r-sized bump function $\phi_r^k(x) := \phi_1(|x|/r)$; here $|\cdot|$ is the euclidean norm in $\mathbb{R}^k$. Let $g_X^{\text{euc}}, g_Y^{\text{euc}}$ be the standard metrics on the corresponding euclidean spaces, viewed as symmetric 2-forms. To define a new metric on X we take g_X^{euc}, blow it up transversely to $(p^\tau)^{-1}(x)$ for x close to the origin of

$\mathbb{R}^m$, and squeeze everywhere else. Formally,

$$g_X = \Phi^* g'_X, \text{ where } g'_X = \varepsilon g_X^{\text{euc}} + (1-\varepsilon)(\phi_2^m g_Y^{\text{euc}}) \times (\phi_2^{mk+m+k} g_F^{\text{euc}}).$$

In order for this to be well-defined, one might want to approximate Φ by a smooth map. From now on, we assume that X is endowed with metric g_X. To make X compact, one can replace it by its subset $B_3^{g_F^{\text{euc}}}(0) \times B_{3+m}^{g_Y^{\text{euc}}}(0)$. Radius $3+m$ here is chosen so that the 2.2-neighborhood of $\triangle^m$ is covered by $p(X)$. We write X' for the space $\Phi(X)$ with metric g'_X; clearly, X and X' are isometric.

Figure 5-2 depicts the case $m = 1, k = 0$: there, $X = \mathbb{R}^2$ is sliced by lines $p^{-1}(y)$ (bold black curves in the figure), each of which is the local join of a green point set Z_0 and a blue point set Z_1. On the left, the geometry of g_X is depicted by stretching X along the vertical direction, so that it corresponds to the value of p^τ. On the right, one sees X in the coordinates $\Phi = (p_f, p^\tau)$, with the pinching in the region where $|p^\tau(x)| > 2$.

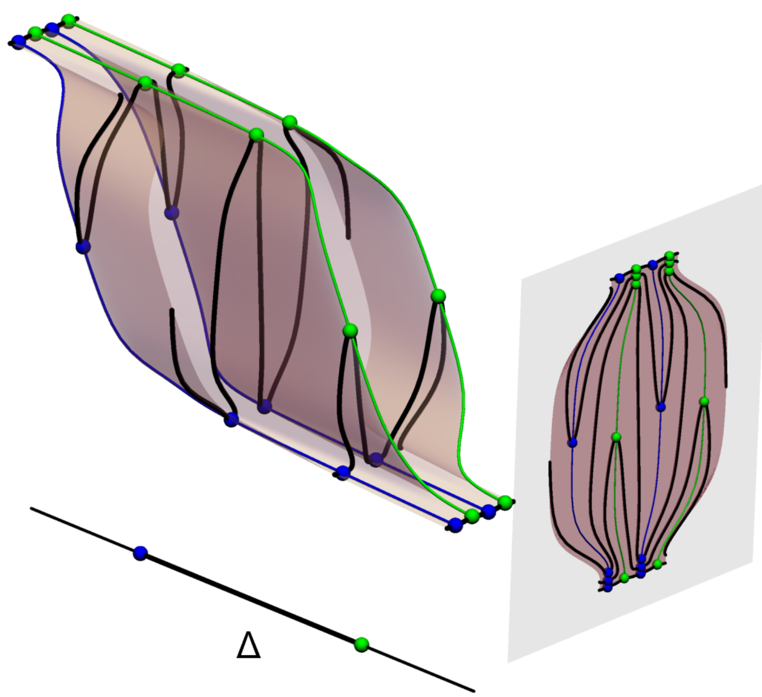


Figure 5-2: On the left: the map $p : X \to Y$, with X stretched vertically according to the values of p^τ. On the right: X viewed in the coordinates (p_f, p^τ)

Now let us verify the claimed properties of the metric g_X. To see that $\mathrm{UW}_{n-1}(X) \geq 1$, note that the unit ball $B_1^{g'_X}(0)$ is just the usual euclidean ball, and its width is > 1.

Finally, we show that the fibers of p have small width. Consider a fiber $X_y = p^{-1}(y)$, $y \in Y$, and the restriction of g_X on it. It equals $\varepsilon g_F^{\mathrm{euc}}$ plus a term supported in $\tau^{-1}(B_{2.2}^{g_Y^{\mathrm{euc}}}(y))$. The ball $B_{2.2}^{g_Y^{\mathrm{euc}}}(y)$ does not reach one of the faces $v_i^\vee$ of $\triangle^m$. We would like to use the retraction π_i (as in the discussion after Definition 5.1.4) to map $p^{-1}(y)$ to Z_i; this is not possible for the points in the dual complex $Z_i^\vee$, which is entirely contained in the squeezed zone, so we will not lose much if we just send it to a single point. Here is the map witnessing $\mathrm{UW}_{k+m}(X_y) \lesssim \varepsilon$:

$$X_y \simeq F \to (Z_i \times \triangle^m)/(Z_i \times v_i^\vee)$$
$$x \mapsto \begin{cases} (\pi_i(x), \tau(x)), & \text{if } x \notin Z_i^\vee \\ \star, & \text{otherwise.} \end{cases}$$

where $\star$ denotes the pinched copy of $Z_i \times v_i^\vee$ in the quotient. The fiber of this map over $\star$ is ε-small since the metric is squeezed around $Z_i^\vee$. Consider the fiber over any other point (z, t) of the quotient; since it is contained in $\tau^{-1}(t)$, its g_X-size does not exceed its g_F-size; since it is contained in $\pi_i^{-1}(z)$, its g_F-size is ε-small. □

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