Two Enumeration Problems about the Aztec Diamonds

by

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Abstract

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Abstract

An explicit enumeration of a class of figures called "Penta-Aztec Diamonds", which are related to the **Aztec Diamond** introduced by Elkies *et al* by a transformation, is given here. Also derived is a generating function that contains all the information about these tilings. Along the way, some previously known results about the tilings of the **Aztec Diamond** leading up to Stanley's multi-q-analog formula are rederived using the same general approach. Finally, the centrally symmetric tilings of **Aztec Diamonds** are also enumerated using the essentially same technique.

A Note of Appreciation

When I reflect upon my roller-coaster-like and stormy existence at MIT, the images of the many individuals who deserve "thank you" for their invaluable help, almost too many to list here, float in my memory: Professor Richard Stanley who, as a good advisor and a nice guy, ranks high up on the list; Professors Hung Cheng, a friend indeed in the many times of need, and Jim Propp who restored in me the confidence to survive; my schoolmates Clara Chan, Ellen Randall, Richard Ehrenborg, Wayne Goddard, Art Duval, and Leonard Schulman, for encouragement and assistance (the last three for proof-reading this document!); my neighbors and ex-neighbors S. Emilie Chang, Ibrahim El Sanhouri, Kristina Kreutzer, Nancy Lee, Erik Meyer, Debbie Swenson, and that inimitable scoundrel William Jockusch (friend, co-worker, one-time rival, bridge partner, roommate, and terrible cook) to be thanked and appreciated for their friendship and company; Professor Vernon and Dr. Beth Ingram, Housemasters of Ashdown House, remembered for their forever sympathetic ear and willing assistance to one of their many wards; Drs. Eric Chivian and Bethany Block of the Medical Department, instrumental in keeping me alive; my good friends from auld lang syne (Alice Liu, "Micro" Huang, Ru-Ning Hsieh, Jong Tu, Ping-Ying Chang, and Sian-ming Perng) who lent strength from afar; my bridging and wargaming cronies (Mark Ospeck, Jack Duranceau, George Yu, Dave Rho, Dwight Brown, Dan Shoham, Matt Tomlinson, Tom Courtney, and the rest) to whom I clung for sanity in the most trying of times; the many others who were responsible, directly or indirectly, wittingly or unconsciously, for my being and well-being; and last but far from least, my family: always supportive, always ready to listen, and with whom I was spending X'mas when inspiration hit.

Like the graduate student who wrote in his thesis, "The Lord be praised for his love and inspiration, the latter especially with respect to Lemma 2.1.1", let me give a special thanks to Professor Herbert Wilf and Doron Zeilberger for Equations 1.6, and 1.7.

Contents

List of Figures

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0

Chapter 0

Introduction

In this century combinatorists have studied at length the enumeration of certain tilings of figures[1] (or equivalently, matchings of graphs) and the transformation of other enumeration problems into tiling problems. (A classical example of such work is that of the problem of the tilings of the $2n \times 2n$ chessboard as detailed by Lovasz in his classic work [8].)

G. Kuperberg's recent study on the symmetry classes ([5]) used the the technique of the method of Permanents and Determinants on the problem of enumerating symmetry classes of plane partitions by treating them as figure-tiling or graph-matching problems. Here, these same techniques are applied to several enumeration problems centered around the geometrical figure introduced in Elkies et al [1] as the "Aztec Diamond", defined below:

Definition 1 *The* **Aztec Diamond** *f order n, which we will denote by $AD(n)$, is the geometrical figure in the Euclidean x-y plane obtained by taking all the lattice squares which fall completely inside the boundaries*

$$|x| + |y| \leq n + 1.$$

(See Figure 0-1)

In the introductory Chapter 1, we will introduce most of our pertinent definitions and concepts. Also included is a elegant derivation of Stanley's formula for the generating function, or the multi-q-analog for the tilings of the Aztec Diamond, via the "shuffle" manuever introduced

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[1]Defined abstractly as a union of closed dimension 2 subsets of the plane, each of which called a component, such that the intersection of any two component is of dimension at most one.
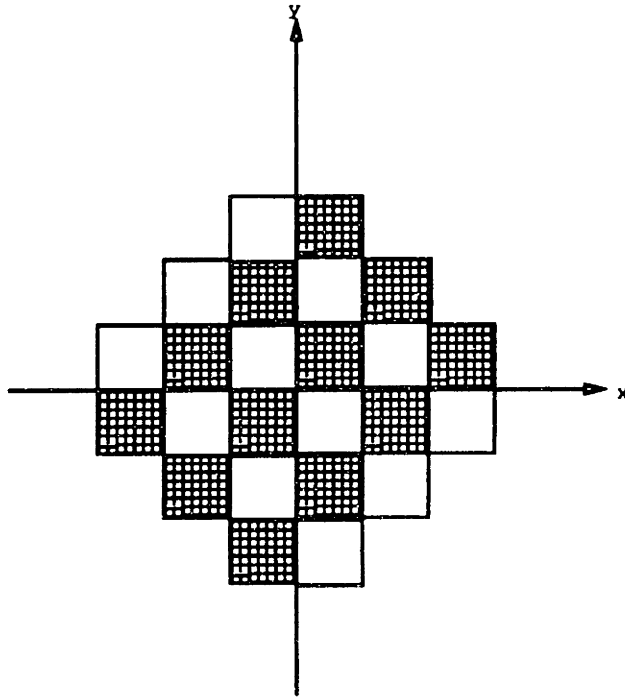
Figure 0-1: Order-3 Aztec Diamond

by J Propp in [1] to prove Theorem 2.5 which states that $ad(n)$, the number of dimer tilings of $AD(n)$, is given by

$$ad(n) = 2^{n(n+1)/2}.$$

In Chapter 2, we introduce our general approach and use it on the Aztec Diamonds, re-deriving Theorem 2.5 and culminating in Stanley's formula, Theorem 1.2, reproduced here:

$$ad(n; x, y, z, w) = \prod_{j \leq k} (x_j w_k + y_j z_k).$$

Here x, y, z, w are the weights of edges defined in Section 1.4. (cf. figures 1-14, 1-13)

In Chapter 3, we introduce the figure known as the Penta-Aztec Diamonds:

Definition 2 *The* **Penta Aztec Diamond** *of order* n, *denoted* $PD(n)$, *is a geometrical figure on*

2

the Euclidean plane defined as follows: take the collection of lattice squares $\bigcup_{j,k=1...n}[j-1,j] \times [k-1,k]$ (a gridded $n \times n$ square) and subdivide each square into four right triangles by cutting twice diagonally[2]. Furthermore, we attach to every other (quarter-sized) triagular tile on each edge (of the original square) its mirror image with respect to the boundary of the square, with the restriction that each of the corners must have both or neither of the possible add-on tiles. See Figure 0-2
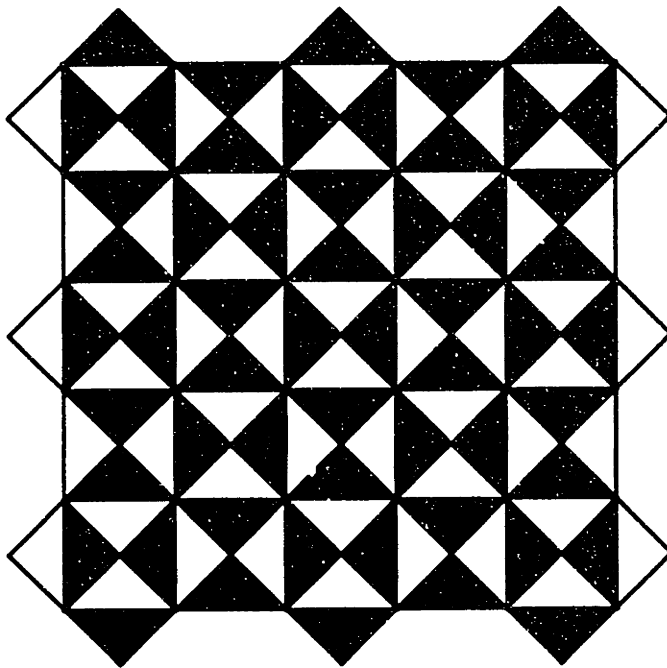
Figure 0-2: $PD(5)$

Note that it is immediately obvious that for odd-ordered **Penta Aztec Diamond**'s there are two inequivalent forms. For the convenience of the discussion to follow, we will only consider the version in which the left top corner always has the extra tiles attached.

With a trick we transform the problem of enumerating the tilings $PD(n)$ into the problem

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[2]*i.e.* draw all the lines defined by $x+y=j$ (where $j=1,2...2n-1$) and $x-y=k$ (where $-n < k < n$).

3

of enumerating the matchings of another graph related to the **Aztec Diamonds**. This is worked out by the same general approach used in Chapter 2. As a final result, we obtain the number of tilings of the figure $PD(n)$ (denoted $pd(n)$) in Theorem 3.1, reproduced here:

$$pd(2n) = 5^{n^2};$$ (0.1)

$$pd(4n+1) = 5^{2n(2n+1)};$$ (0.2)

$$pd(4n-1) = 2 \times 5^{2n(2n-1)}.$$ (0.3)

In the rest of Chapter 3, we obtain a multi-q-analog generating function of this result in Theorem 3.2.

In the last chapter, we will obtain $ad_2(n)$, the number of centrally symmetric tilings of $AD(n)$. The statement of the main result of the chapter is Theorem 4.1:

$$ad_2(2n) = 2^n ad_2(2n-1);$$ (0.4)

$$ad_2(4n-1) = 2^{2n^2-2n+1} \frac{H_4(4n+3)H_4(4n-1)(H(n)H(n-1))^2}{(H_2(2n-1)H_2(2n+1))^3};$$ (0.5)

$$ad_2(4n+1) = 2^{2n^2+1} \frac{(H_4(4n+3))^2(H(n))^4}{(H_2(2n+1))^6}.$$ (0.6)

Here $H_j(n)$, the step-factorial function, is defined as

$$H_j(n) \equiv \prod_{1 \leq k < n/j} (n-jk)!$$

and $H(n) \equiv H_1(n)$ (definition by Kuperberg in [5]).

It is certainly possible to try to enumerate the other symmetry classes of tilings using the Hafnian-Pfaffian Method, an extension of the Permanent-Determinant Method to non-bipartite graphs. However, since the answers do not seem to be integers of high factorizability, the result would have to be obtained in terms of some sum of closed-form functions, which has eluded the author so far. There are many related problems which merit further investigations and which hopefully will provide more material for future publication.

Chapter 1

Preliminaries

1.1 The Aztec Diamond

We start out with a few definitions.

Definition 3 *A* dimer tiling *of a planar figure,*[1] *is a partition of the set of components into pairs each of whose components are* in contact *in the sense of sharing an edge (dimension 1 boundary)*

Definition 4 *A* weighted *graph* G *is a triple* (V, E, W), *where* V *is a set called the* vertices; $E \in 2^{\left(\binom{V}{2}\right)}$, *a set of 2 element multisets of* V, *called* edges, *and we call an edge whose two vertices are identical a* loop; *and* $W : E \to R$ *is the* weight function, *where* R *is a commutative ring, normally the real or complex numbers .*

Definition 5 *A (perfect)* matching *M of a weighted graph* $G = (V, E, W)$ *that has no loops is a collection of edges* $M \subset E$ *such that all vertices in* V *are covered by exactly one edge in the collection. The* weight *associated with the matching* M, *or more generally any set of edges, is the product of the weight associated with every edge in the set, or*

$$w(M) = \prod_{e \in M} w(e)$$

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[1] Represented as a disjoint sum of components, as mentioned in the Introduction.
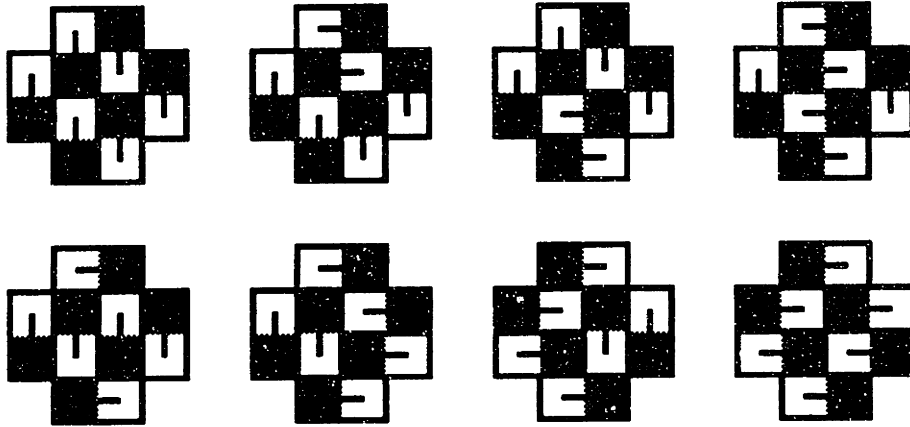
Figure 1-1: The 8 tilings of the order 2 Aztec Diamond

and the total weight of matchings of G is defined as $M(G) \equiv \sum_M w(M)$, where M ranges over all matching of G.

We will henceforth use the term the number of matchings[2] synonymously with the total weight of matchings.

Note that for a bipartite graph to have any matchings at all the number of "black" vertices must be the same as the number of "white" vertices. Therefore, when we talk about matchings of bipartite graphs it is implicitly assumed that they satisfy this condition.

Definition 6 The connectivity graph of a planar figure Γ, is the graph $G(\Gamma)$ whose vertex set is the set of components of the figure, such that an edge (of weight 1) exists between two vertices if and only if the two corresponding components are in contact, defined as intersecting in a set of dimension 1.

Definition 7 A domino tiling, or simply tiling, of a lattice region(a union of lattice squares) is a collection of dimer tiles or dominoes[3] whose mutual intersection is at most 1-dimensional

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[2]Even though this is a misnomer when the edges are not all weighted one.

[3]defined as two lattice squares in contact.
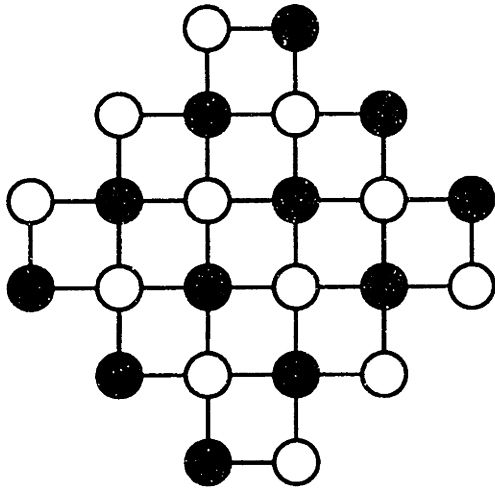
Figure 1-2: The connectivity graph of order 3 **Aztec Diamond**.

and union of the squares in each tile contains the whole region. When context permits, we will also identify the region by the tiling.

It is obvious from the above that there is a bijection between dimer tilings of a figure and matchings of its connectivity graph. In fact, we will in the future use the two interchangeably.

Elkies *et al* in [1] has obtained the number of dimer tilings for $AD(n)$. In fact, a generating function was obtained that provided rather more information. The proof, a simplified version of which we present below (it being short and elegant) uses the manuever known as the "shuffle".

To envision the shuffle, picture the plane as being colored in black and white in a checkerboard pattern, with the diagonal line of squares on the *NE or upper-right side* of the diamond being *black* as depicted above.

Definition 8 *The* shuffle *is the operation on a tiling that moves each tile one unit length in the direction parallel to the short sides of the tile such that the white square in the tile is on the right if the move is horizontal and left when the move is vertical, considering each tile as a unit. See Figure 1.2.*

moves of the shuffle.
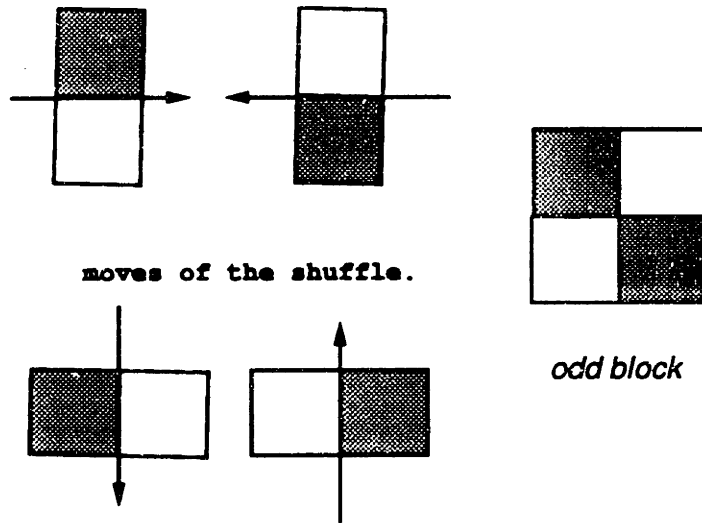
odd block

Figure 1-3: Shuffles and an Odd Block

Definition 9 *A lattice point is an* odd vertex *if it lies on the upper-left corner of a white square and an* even vertex *otherwise.*

Definition 10 *An* odd block *is a pair of tiles forming a 2 × 2 square that has white squares on the upper-right and lower-left. Equivalently, the region has an odd vertex at the center. Similarly define* even blocks.

It is easily seen that a pair of tiles forming an odd block will simply exchange places under the shuffle.

Definition 11 *An* odd-deficient tiling *of a (lattice) region is a collection of dimer tiles to which (a finite number of) pairs of dimer tiles that form odd blocks can be added so as to form a tiling of the region. An* even-deficient *tiling is defined similarly.*

8

Here, we observe that a region covered by an odd-deficient tiling can have only odd vertices on the corners of its boundary.

Lemma 1.1.1 *The process of shuffling is a involution between odd-deficient tilings on the whole checkerboarded plane.*

Proof: (From [1]) Let $\tilde{T}$ be a odd-deficient tiling on the plane, resulting from removing the odd blocks from T, a true tiling. Let T_1 be the collection of tiles obtained as the result of a shuffle on the tiles of $\tilde{T}$, to be identified as a odd-deficient tiling.

First we show that all the tiles in $\tilde{T}$ in fact stay disjoint after shuffling.

Proceeding by *reductio ad absurdum*, we consider, without loss of generality, a white square s that has two tiles covering it after the shuffle. (Figure 1.4)
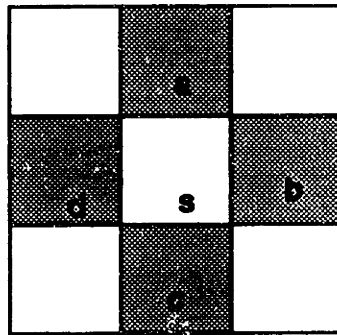
Figure 1-4:

Each of the possible tiles that cover s must also cover a black square adjacent to s so we will characterize those tiles by those four squares, as labeled above by a through d. Consider the six possible combination of tiles that cover the white square:

a, b Then the inverse images of the tiles overlap, contradiction.

c, d As above.

a, c The tiling T must have a tile that covers s, but that tile cannot cover b or d since they conflict with the original position of the two tiles that now covers s after the shuffle.

However, a tile covering a and s will form an odd block with the tile that shuffles to that same position, and equivalently for c and s. **contradiction**.

b, d As above.

a, d The geometry is different from above but the reasoning does not change.

b, c Same.

Now we proceed to prove that T_1 only has odd vertices on the corners. Suppose that v is a vertex on the corner of T_1. It is easily shown by the enumeration of cases that v must have an unequal number of black and white squares around it that are in T_1. Suppose that v is even; then a tile may cover, of the four squares around v, one black, one white, or one of each. In these three cases, the (inverse) image of the tile would cover one black square, one white square, or none at all around v. Therefore, v can only be a corner of T_1 if it is already a corner of $\tilde{T}$. But since $\tilde{T}$ is a odd deficient tiling, this is impossible. $\square$

With the help of the preceeding lemma, we can prove the following:

Theorem 1.1 *If we remove all odd blocks from a tiling of the* **Aztec** *Diamond of order* $n - 1$ *and shuffle, then we obtain a tiling of the* **Aztec** *Diamond of order* n *with holes in them that are "even" blocks.*

Proof: Using the same notation as above, we assume that $\tilde{T}$ is an odd-deficient tiling of the order $AD(n - 1)$. So T_1 lies completely inside $AD(n)$. Consider the complement of the order $AD(n - 1)$ tiled in such a way that there are no odd blocks and added to $\tilde{T}$ to obtain a odd-deficient tiling T^+ of the entire plane, the result of shuffling T^+ is another odd-deficient tiling of the plane. Some of the missing odd-blocks would lie in the semi-infinite strips of width 2 around $AD(n)$ and the rest falls completely inside $AD(n)$. Since none of these cross the boundary of adding those inside to T_1 gets us a complete tiling of $AD(n)$. $\square$

Corollary 1.1.1 *The number of tilings of $AD(n)$, hereafter also referred to as $ad(n)$, is $2^{n(n+1)/2}$*

Proof: Consider a tiling T of $AD(n - 1)$. Take out the odd blocks, storing their orientation somewhere. Now shuffle to get an odd-deficient tiling of $AD(n)$. Put the odd blocks taken out

before back in (by some set ordering, say from top to bottom, left to right). Now we have n even[4] blocks still to fill in, which we can orient in 2^n ways. It is easy to check that mapping is injective, so that is the ratio of the number of tilings. The rest follows by mathematical induction □

After we introduce a few other concepts we will prove a generating function relating to the Aztec Diamond tilings as well as some pertinent facts about how the Aztec Diamond relates to other combinatorial topics.

1.2   The Permanent-Determinant Method

The Permanent-Determinant Method is a method used to compute the number of matchings of bipartite graphs, used by G. Kuperberg in [5] to prove certain formulas for the number of plane partitions with various symmetries.

The simple gist of the method is as follows:

Definition 12 *For a (weighted) graph Z that is bipartite, we build a matrix for which the rows represent "black" vertices and the column represent "white" vertices. Each entry of the matrix is the weight of the edge connecting the black vertex of its row and the row vertex of its column. This we call the* bipartite connectivity matrix *of Z and denote by $B(Z)$.*

Notice that since the definition deliberately uses the two partitions of vertices as labels of rows and columns, the matrix so obtained is only unique up to the permutation of rows and columns, corresponding to changing the labeling of the vertices in the graph. However, the value of the permanent of the matrix is independent of the labeling used.

---

[4] Notice that by our coloring what's even for $AD(n)$ is odd for $AD(n \pm 1)$, and vice versa.

11

Example 1.2.1

$$B(AD(3)) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

if we number all vertices top-to-bottom then left-to-right.

It is easily seen that this matrix has a simple connection to the number of matchings of the graph, namely

$$M(Z) = \text{Per}(B(Z)),$$

where $\text{Per}(A)$ is the *permanent* of A, defined in the same way as the determinant except that there are no signatures on any term, *i.e.*

$$\text{Per}(A) = \sum_{\sigma \in S_m} \prod_{i=1}^{m} A_{i,\sigma(i)}$$

where m is the size of the matrix (half the number of vertices). However, this in itself is not of much use, since permanents are notoriously hard to calculate. However, there is a way to simplify the calculation, as follows:

Definition 13 *For any (weighted) planar bipartite graph T, an alternating modification $\vec{T}$ (not unique, but all denoted by the same symbol) is a weighted graph with the same vertex set and edge set of T and a modified weight-function $\vec{T}(e) \equiv w(e)T(e)$ such that if the edges $e_1, e_2, \ldots e_n$*

12

taken in order form a irreducible cycle ("face") we have that

$$w(e_1)w(e_3)\ldots w(e_{n-1}) = (-1)^{(n-2)/2}w(e_2)w(e_4)\ldots w(e_n)$$
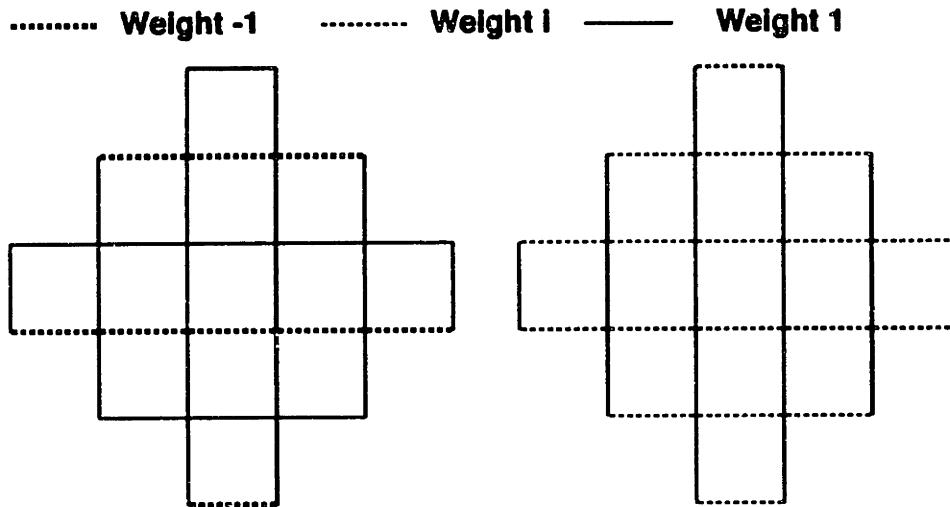
Figure 1-5: Two Alternating Modifications of the **Aztec Diamond** of order 3

Lemma 1.2.1 *If $\vec{T}$ is an alternating modification of T then the bipartite connectivity matrices of the two graphs satisfy*

$$\text{Per}(B(T)) = \pm \det(B(\vec{T}))$$

Proof: Take the two expressions term-by-term. Making an elementary move on a matching (see Fig. 1-6) [5] changes the signature of the permutation by $(-1)^{(l+1)}$ when the circuit is composed of $2l$ edges. So terms in the determinant of $B(\vec{T})$ that corresponds to matchings of

---

[5] If the edges $e_1, e_2, \ldots e_{2n}$ taken in order form a irreducible loop, then taking the edges $e_1, e_3, \ldots e_{2n-1}$ instead of $e_2, e_4, \ldots e_{2n}$ in a matching is called an elementary move or face rotation. It is shown for example in [8] that any two matchings of a graph can be modified into each other by a series of elementary moves.
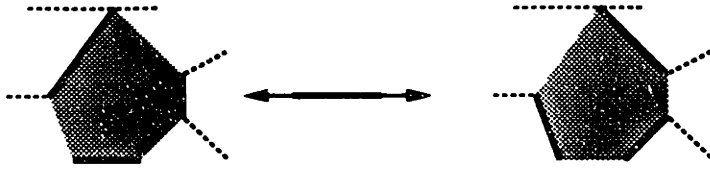
Figure 1-6: An elementary move

T all have the correct signature, up to a negative sign for every term. □

It can be shown (see [6]) that for all planar bipartite graphs an alternating modification can be found in a consistent manner. In the main text we will use this technique to solve several tiling/matching problems heretofore unresolved.

Here we will also mention the fact that this method can be generalized to one that deals with graphs that are not bipartite, called the "Hafnian-Pfaffian" transformation but since we will not deal with any such graph we will omit the discussions here and refer readers to [5] and [6].

1.3 The method of W-Z pairs for proving combinatorial identities

Oftentimes in solving combinatorial problems, it is far easier to prove a conjectured result than to envision one. This is the case here as well.

One method of proving conjectured equalities is Proof by WZ Rational Certificates. This is a general method of proving combinatorial identities of a certain form developed by Herbert Wilf and Doron Zeilberger and summed up in [2].

The basic idea is as follows: We have a conjectured identity

$$\sum_k F(n,k) \stackrel{?}{=} \text{constant, independent of } n > n_0,$$

where the summation over k is in theory over all integers[6] and n is also an argument with integer value. (We denote by Δ_n and Δ_k the forward differential operators in n and k, etc.)

If we can obtain a function $G(n,k)$ such that

$$\Delta_n F(n,k) = \Delta_k G(n,k) \tag{1.1}$$

$$\lim_{k \to \pm\infty} G(n,k) = 0 \tag{1.2}$$

then we see that

$$
\begin{aligned}
\Delta_n \sum_k F(n,k) &= \sum_k \Delta_n F(n,k) \\
&= \sum_k \Delta_k G(n,k) \\
&= G(n,+\infty) - G(n,-\infty) = 0.
\end{aligned}
$$

Hence the sum is in fact independent of n and we can say that the desired equality is in fact true[7].

Note that from the condition of the WZ pair, with the proviso

$$\lim_{n \to \infty} F(n,k) \equiv f_k \quad \text{exists and is finite;}$$

$$\lim_{L \nearrow \infty} \sum_n G(n,-L) = 0$$

we can deduce the relation

$$\sum_{n \geq 0} G(n,k) = \sum_{j < k} (f_j - F(0,j)),$$

But we will not utilize this part of the theorem here.

Usually, a combinatorial identity has some non-constant term as the right-hand side. Before the WZ method is used, the whole equation should be divided through by its RHS so that the prospective identity is a sum coming out constant. $F(n,k)$ is then usually set to be the summand of the left-hand-side.

---

[6]But often $F(n,k)$ is such that a finite number of (n,k) gives a nonzero $F(n,k)$.

[7]After checking that it is true for some initial value of n, which would be trivial in the case of a equality conjectured through empirical evidence.

In practice, we set

$$R(n,k) = \frac{G(n,k)}{F(n,k-1)}$$

and solve for R. With the help of a symbolic mathematics computer program such as Mathematica, R can usually be obtained within a reasonably short time by trial-and-error. R. W. Gosper has built an algorithm in MACSYMA which can be used to find (or prove the non-existence of) closed-form-sums, and there probably are programs in existence[8] which can be used to find these R's in an automated manner.

If F is *of closed form* ($F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n;k)$ being rational functions of n and k) then $R(n,k)$ will normally be a rational function of reasonble simplicity. This R is called a "certificate" of the equality and in essence it is all that is needed in a proof. Of course, we also need to check an initial condition, since this is really just an advanced form of mathematical induction, but that is relatively easy!

This is only the bare bones of the methods outlined by [2]. For a detailed description see [2] or [3].

With this powerful tool, we immediately can prove many equalities. For example:

Example 1.3.1

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$$

Proof:

$$R(n,k) = -\frac{3n - 2k + 3}{2(2n+1)}$$

□

This is somewhat trivial and easily provable by more straightfoward means, but there are less trivial ones:

---

[8]However, being less advanced, I carried out all the computations in this section by hand with some help from Mathematica.

Example 1.3.2 (Dixon)

$$\sum_k (-1)^k \binom{a+b}{a+k}\binom{c+a}{c+k}\binom{b+c}{b+k} = \frac{(a+b+c)!}{a!b!c!} \tag{1.3}$$

Proof: Here a, b, c is symmetric, and we will take only one of them, a say, as the independent variable n. As a matter of fact, we have

$$R(a,k) = \frac{(c+1-a)(b+1-a)}{2(a+k)(a+b+c+1)}$$

□

Example 1.3.3 (Vandermonde)

$$\sum_k \binom{a}{k}\binom{b}{k} = \binom{a+b}{a}. \tag{1.4}$$

Proof:

$$R(b,k) = -\frac{(a-k+1)}{a+b+1}.$$

Notice that a and b are quite symmetric. □

Example 1.3.4

$$-\sum_{j=-1}^{n} 4^{n-j}\binom{2n-j+1}{j+1}\frac{(2n-2j-1)!!(2j-1)!!}{(2n+1)!!}$$

$$= \frac{(4n+3)!!}{[(2n+1)!!]^2}, \tag{1.5}$$

$$-\sum_{j=-1}^{n} 4^{n-j}\binom{2n-j-1}{j-1}\frac{(2n-2j-1)!!(2j-1)!!}{(2n+1)!!}$$

$$= \frac{(4n-1)!!}{[(2n+1)!!]^2}. \tag{1.6}$$

Proof: It is easily shown *via* mathematical induction that the two are equivalent. More specifically, take Eqn. 1.5, substitute $n-1$ for n, and substract four times the resulting

equation from the original Equation 1.5 and we obtain Equation 1.6, which can then can be proven by the certificate

$$R(n,j) = \frac{4j^2 - 1}{(4n + 1)(4n + 3)}.$$

□

Example 1.3.5

$$\sum_{j=-1}^{n} \sum_{k} (-1)^{j+k} \binom{2n - j + 1}{j + 1}$$
$$\times \frac{(2n - 2j - 1)!!(2j - 1)!!}{(2n + 1)!!} \binom{2n + 1}{k} \binom{2j + 1}{2n - 2m - j - 2k}$$
$$= 0, \forall n \geq m. \tag{1.7}$$

Proof:

$$R(n,j) = \frac{2j(2j - 1)(2j + 1)}{(2n + 1)(2n - 2m - j - 2k + 2)}.$$

Notice that using $R(n,k)$ would not have worked.[9] □

This is just an outline of an extraordinarily powerful method. It has its limitations, but with some help from Mathematica or the like can be extremely useful.

1.4 Alternating Sign Matrices and a Generating Function of Aztec Diamond Tilings

Definition 14 *An ASM (Alternating Sign Matrix) of order n, (the set of which henceforth is denoted by $ASM(n)$), is a square matrix that has all entries being ± 1 or 0 with the property that all rows and columns add up to 1 and the $+$'s and $-$'s alternate in each column and row.*

Surprisingly, Alternating Sign Matrices, which have many interesting combinatorial properties, also have an interesting connection to Aztec Diamond tilings.

---

[9]Double sums in general must be treated case by case, and many sources such as [3] outlines ways to deal with huge combinatorial equalities.

The following result was proved in [1]

Lemma 1.4.1 *We can obtain a many-to-one mapping of $ASM(n)$ to the tilings of $AD(n)$ such that each ASM corresponds with 2^k tilings where k is the number of $+1$'s in the ASM. Alternately, we can say that there is a bijection from $ASM(n)$ to even-deficient tilings of $AD(n)$.*

Proof: Let's take any ASM (for demonstrative purposes, we will use the matrices

$$M_1 = \begin{pmatrix} & + & \\ + & - & + \\ & + & \end{pmatrix}, M_2 = \begin{pmatrix} & + & & \\ & & + & \\ + & & - & + \\ & & + & \end{pmatrix}, M_3 = \begin{pmatrix} & + & \\ & + & - & + \\ & & + & \\ + & & & \end{pmatrix}.$$
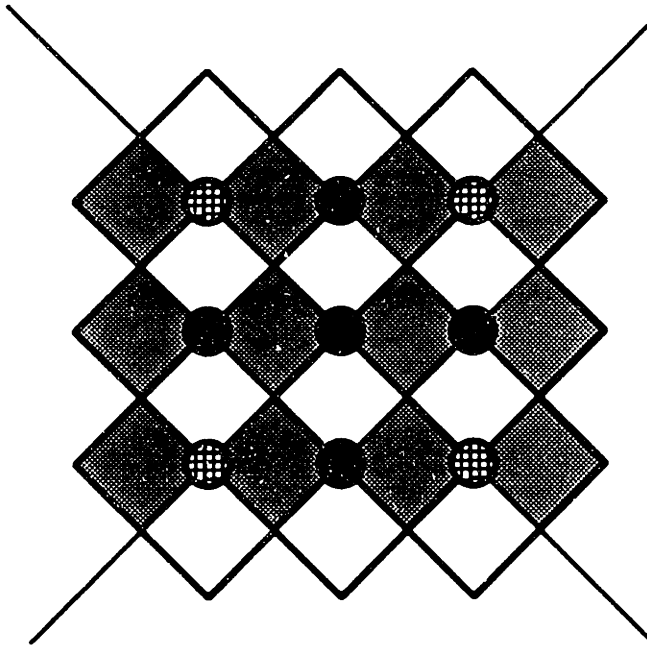
Figure 1-7: The ASM M_1 mapped to tilings of **Aztec Diamond's**.

See the figures 1-7 and 1-8. The way to envision the mapping is to picture the $n \times n$ square of the even vertices in the $AD(n)$, and envision these as the entries of the ASM thusly:

- If the even vertex is the center of an even block in the tiling, then the corresponding entry

19
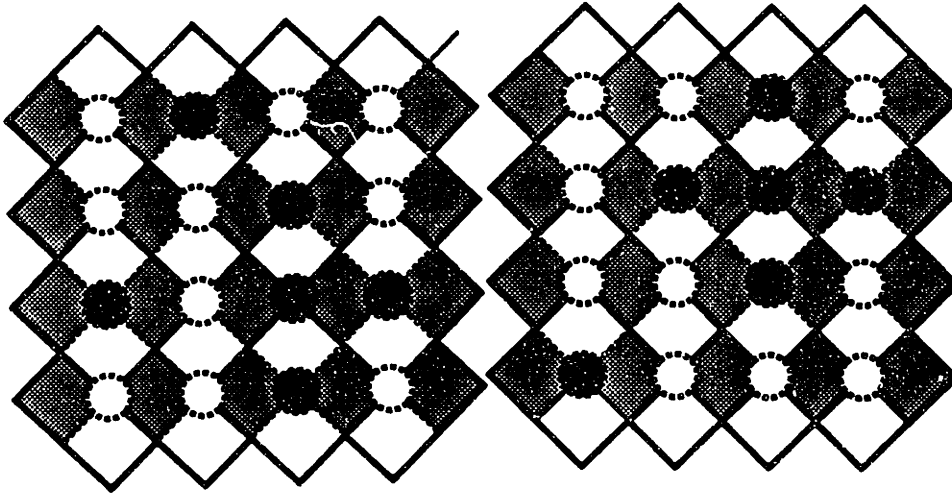
Figure 1-8: M_2, M_3 mapped to **Aztec Diamond's**.

is $+1$.

- If the even vertex is the intersection of *four* tiles, then the corresponding entry is -1.

- Otherwise, (i.e. the even vertex is the intersection of *three* tiles,) then the corresponding entry is 0.

To see that this is indeed a bijection, consider any given tiling of an **Aztec Diamond**. Then in order we have:

1. Between any even vertex with four intersecting tiles and the edge of the **Aztec Diamond** there must be a even block. Suppose A is an even vertex with four incident tiles as above; then two of them form a inward concave angle (the one which is the boundary of square U in the figure). Since the square U cannot be tiled with either of the squares S and T next to A, we can without loss of generality assume that U is tiled with V next to it. Then the other two tiles next to that vertex B must either be tiled together (case 2) or tiled separately (case 1). In the former case, we have a even block right there. In the latter, we have formed another angle that is inward concave at B, and recursively we have shown that there must be an end to the (case 1)'s we can encounter, so there must be a

20
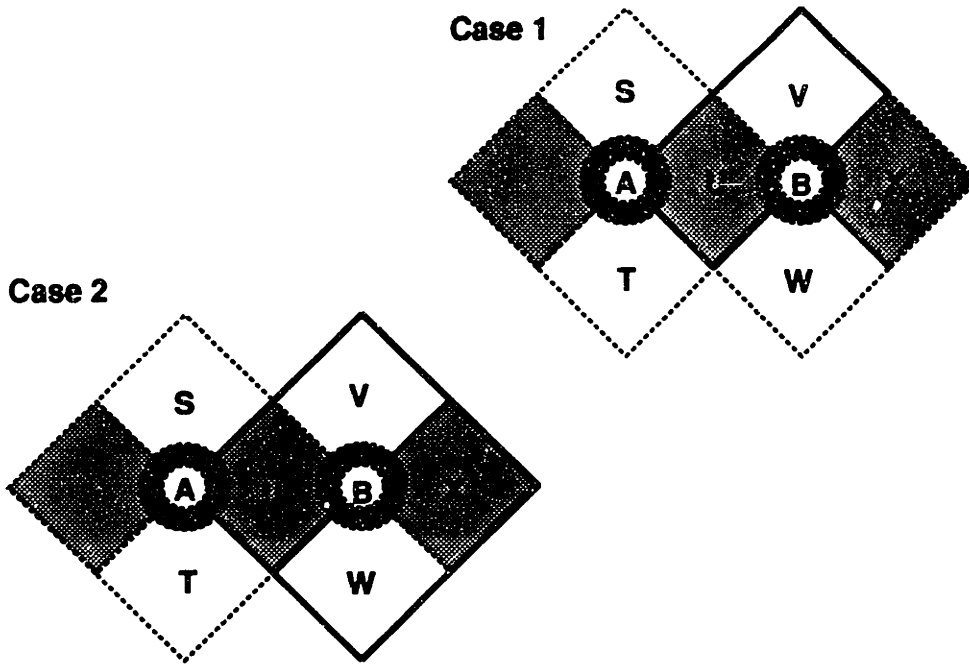
Figure 1-9:

even block somewhere along the way.

2. Between any two even blocks on the same line there must be a vertex with four incident tiles. We look at the next figure:
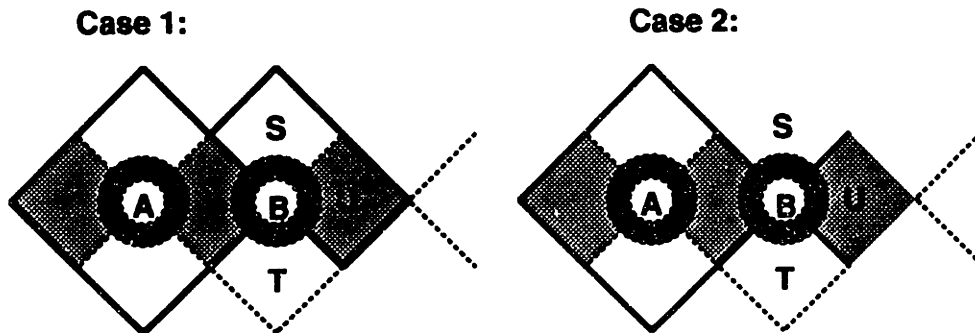
Figure 1-10:

Here, either we have that the next vertex B has four incident tiles (when square U is tiled with S or T (case 2) or we get an angle facing the same way (case 1). And recursively we can eventually find an even vertex with four incident tiles. As a result of the above, we

21

have proved that this mapping is well-defined.

3. Two distinct even-deficient tilings of the **Aztec Diamond** will map to two different ASM's: Following the above, for every vertex we draw boundaries of tiles as above. *I.e.* the incident edges at an even vertex will run NE, NW, and SW if the partial row-sum and partial column-sum of the ASM are both equal to 1, (see figure 1-11) etc. We have shown that the mapping is bijective.

□

[Entry in ASM of same order]   (Entry in ASM of one lower order)
(for the mapping of the lower-order ASM flip colors)

[+1]

(+1)   [-1]   (-1)

[0: row.sum=0, col.sum=1]   [0:row.sum=1, col.sum=1]
(0:row.sum=1, col.sum=0)   (0:row.sum=0, col.sum=0)

(0:row.sum=0, col.sum=1)   (0:row.sum=1, col.sum=1)
[0: row.sum=1, col.sum=0]   [0:row.sum=0, col.sum=0]
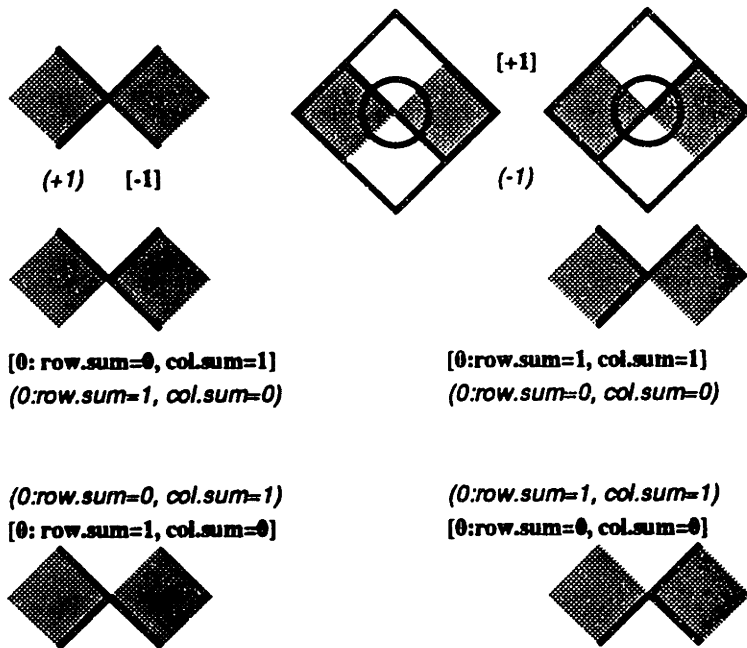
Figure 1-11: Entries in ASM mapping to even vertices of **Aztec Diamond** tiling.

In a completely identical fashion, we can show that the minus signs correspond to the odd blocks removed from the previous **Aztec Diamond** from which the **Aztec Diamond** under consideration can be shuffled; or, rather, we have the following.

Corollary 1.1.2 *There is a bijection from $ASM(n-1)$ to odd-deficient tilings of $AD(n)$ in a completely analogous manner to the above.*

22
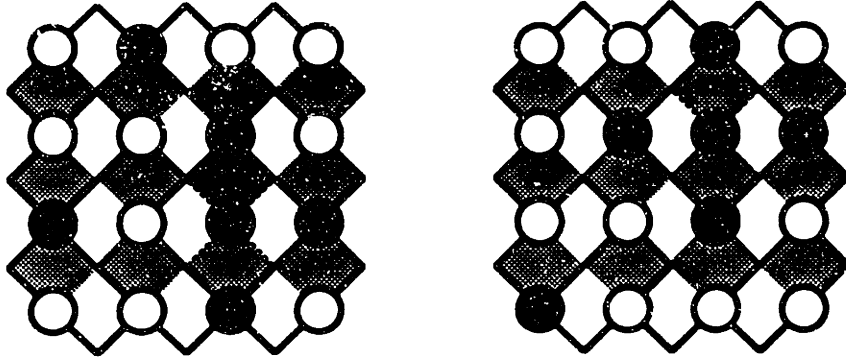
Figure 1-12: The ASM's M_2, M_3 mapped to tilings of $AD(3)$.

The immediate result of this is the following connection between two very fascinating combinatorial objects:

$$ad(n) = \sum_{\sigma \in ASM(n)} 2^{s(\sigma)} \tag{1.8}$$

where $s(\sigma)$ is the number of minus signs in σ. As a matter of fact, a direct extension of the above immediately gets us:

Theorem 1.2 (R. Stanley, 1989) *If we give weighting to the tiles of a tiling of $AD(n)$ in the following manner: those located as to the extreme upper-right are weighted x_1 (running NE) and y_1 (running NW) and the next row z_1 and w_1, and the next row x_2 and y_2 (see figure 1-13) etc.; then the total weight of the tilings is*

$$ad(n; x, y, z, w) = \prod_{j \leq k} (x_j w_k + y_j z_k)$$

Proof: Consider the total weight corresponding to an odd-deficient tiling of $AD(n)$. It will be of this form:

$$\prod(x, y, z, w) \prod O_j$$

Where $O_j \equiv (w_{j-1} x_j + z_{j-1} y_j)$ is the weight for the odd block removed in the shuffling. Now consider the effects of the shuffle on $AD(n)$, transforming t'.e result into an even-deficient tiling of $AD(n + 1)$; we get some factors corresponding to the even blocks, as well as the factors corresponding to shuffled tiles. Let's look at the effect of the shuffle on those tiles (see figure 1-

23
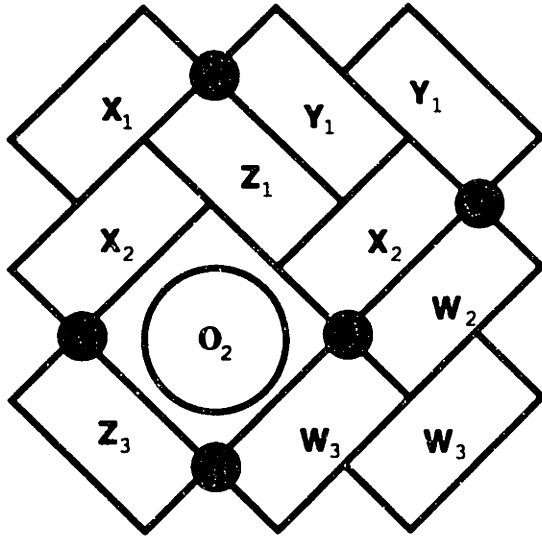
Figure 1-13: Odd-deficient tiling of $AD(3)$ and weights of tiles.

14:

$$x_j \quad \rightarrow \quad \bar{x}_j \tag{1.9}$$

$$y_j \quad \rightarrow \quad \bar{y}_j \tag{1.10}$$

$$z_j \quad \rightarrow \quad \bar{z}_{j+1} \tag{1.11}$$

$$w_j \quad \rightarrow \quad \bar{w}_{j+1} \tag{1.12}$$

Also for each even block we add in on row k, we get a factor of

$$\bar{E}_k \equiv \bar{x}_k \bar{w}_k + \bar{y}_k \bar{z}_k$$

Looking at the formula for the transforms above, and calculating the transformed(shuffled) image of O_j, we know that for each odd block removed we remove with it (in the corresponding term) a factor of

$$\bar{x}_k \bar{w}_k + \bar{y}_k \bar{z}_k = \bar{E}_k.$$

But from the properties of the ASM, we know that the number of pluses in each row is exactly one more plus than the number of minuses. This means that after we cancel the factors, we get

24
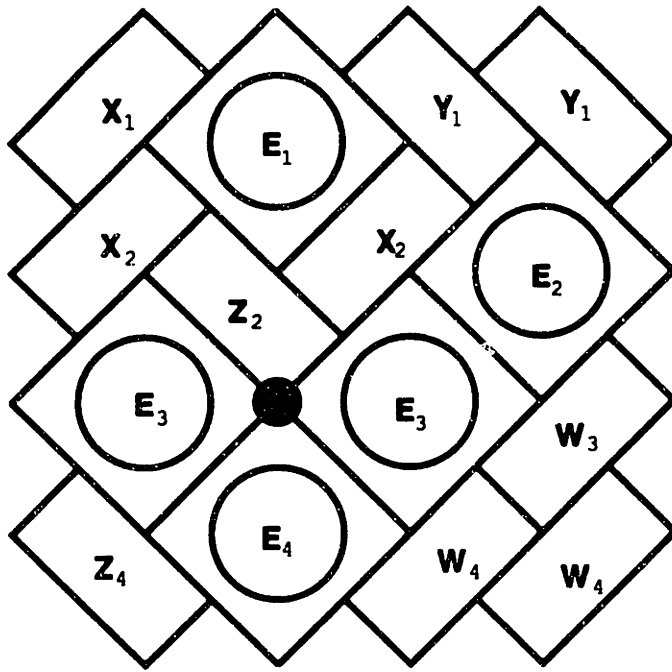
Figure 1-14: Even-deficient tiling of $AD(4)$, and weights of tiles.

that the result is a simple multiplication by $\prod_j \bar{E}_j$.

The rest is a simple induction. We transform the formula above for $ad(n; x, y, z, w)$ and multiply by $\prod_j \bar{E}_j$ to get exactly the same formula for $ad(n+1; x, y, z, w)$. $\square$

So, this theorem is proved in an elegant manner(this particular proof is discovered independently by several people, including the author and J Propp among others). Of course, the original proof was obtained by a different method (see [7]). In the next chapter, we will arrive at this generating function *via* a different method.

Chapter 2

The Aztec Diamond Tilings

2.1 The Aztec Diamond tilings, by the Propp-Kuperberg method

To obtain a solution to the problem of enumerating **Aztec Diamond** tilings via the Permanent-Determinant Method, it is necessary to obtain an alternating modification of the connectivity graph. We will now do so. Before we proceed, we explain the conventions and notations used in this chapter:

1. For clarity and convenience we have rotated the **Aztec Diamond** by forty-five degrees.

2. We have also identified the figure $AD(n)$ with its connectivity graph $G(AD(n))$ throughout where no confusion is likely .

3. We will refer to the bipartite connectivity matrix of the graph $\vec{G}(AD(n))$ as $B(AD(n))$ or simply $B(n)$, the number of tilings of $AD(n)$ as $ad(n)$, a function of n. [1]

4. We number *the row-vertices* as follows: start with number 1 at the top of the rightmost row, and circle around in a clockwise fashion. After vertex $2n$ in the lower-left corner has been reached, start again at the top right corner and repeat until all row-vertices has been numbered.

---

[1] We henceforth will identify with lowercase letters the "total number of tiling/matching" where the same upper-case letters would denote the graph or figure.

5. The column-vertices are numbered in a way symmetric to the above, starting at left-bottom and going counterclockwise. (See Figure 2-1.)
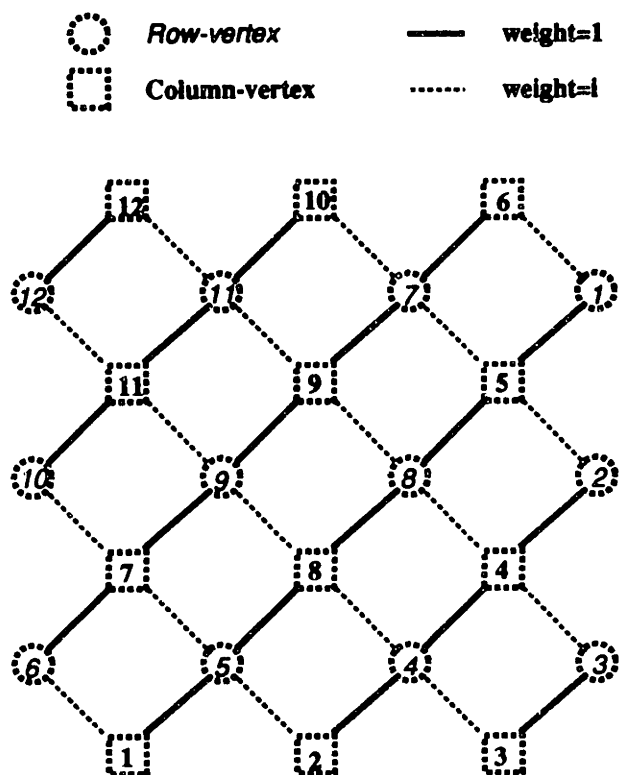
Figure 2-1: $\vec{G}(AD(3))$ and numbering of its vertices

Having done the above, now we need to give each edge a weight. For reasons of symmetry, we do so in this manner: all edges running northeast($\nearrow$) will be given weighting 1 and those running northwest($\nwarrow$) will be given weighting i.

This clearly forms an alternating modification of the original connectivity graph, since each square "face" has two edges that have weighting 1 and the other two i.

2.2 Solving the Enumeration problem of tiling $AD(n)$

Now we obtain the modified connectivity matrix $B(AD(n)) =$

$$
\begin{bmatrix}
 & & & & & & 1 & i & & & & \cdots \\
 & & & & & 1 & i & & & & & \\
 & & & & \cdot & \cdot & & & & & & \\
 & & & \cdot & \cdot & & & & & & & \\
 & & \cdot & \cdot & & & & & & & & \\
 & & 1\square & i & & & & & & & & \\
 & \cdot & i & 1 & & & & & & i & & \\
 & \cdot & \cdot & & & & & & \cdot & 1 & & \\
 & \cdot & \cdot & & & & & & & & & \\
 1 & \cdot & & & & & & i & \cdot & & \\
 1 & i & & & & & & i & 1 & & \\
 i & & & & & \star & 1 & & & & \\
 & & & i & 1 & \star & & & & & \\
 & & i & 1 & & & \star & & & & \\
 & \cdot & \cdot & & & & & \star & & & \\
 & \cdot & \cdot & & & & & & \star & & \\
 & i & 1 & & & & & & & \star & \\
 \vdots & & & & & & & & & & \ddots
\end{bmatrix}
$$

The entry above marked with $\square$ is (n, n); the one marked with a $\star$ $(2n, 2n)$; and the last displayed row/column the $(3n - 1)$th; in the lower right corner, the stars stand for entries that go into the $B(AD(n - 1))$.

Now, we attempt to reduce the determinant of the matrix to one that is related to the matrix of $AD(n - 1)$.

And so, we carry out the reduction in the follwing steps:

1. $(-i) \times$ (Row $2n - 1$) is added to (Row $2n$).

2. $(-i) \times$ (Column $2n - 1$) is added to (Column $2n$).

28

3. $(-1)\times$ (Row $2n - 2$) is added to (Row $2n$).

4. $(-1)\times$ (Column $2n - 2$) is added to (Column $2n$).

5. $(+i)\times$ (Row $2n - 3$) is added to (Row $2n$).

6. $(+i)\times$ (Column $2n - 3$) is added to (Column $2n$).

......

$(-i)^k\times$ (Row $2n - k$) is added to (Row $2n$).

$(-i)^k\times$ (Column $2n - k$) is added to (Column $2n$).

......

$2n - 1$. $(-i)^n\times$ (Row n) is added to (Row $2n$).

$2n$. $(-i)^n\times$ (Column n) is added to (Column $2n$).

After this we do all elementary eliminations. And this is the result:

$$\left[\begin{array}{c|c|cccccc} & 1 & & & & & & \\ & \cdot & & & & & & \\ & \cdot & & & & & & \\ 1 & & & & & & & \\ \hline & -2(-1)^n & 2 & -2i & \cdots & 2(-i)^{n-2} & 0 & \cdots \\ \hline & 2 & * & & & & & \\ & -2i & & * & & & & \\ & \vdots & & & * & & & \\ & 2(-i)^{n-2} & & & & * & & \\ & 0 & & & & & * & \\ & \vdots & & & & & & * \end{array}\right]$$

where the stars again denotes the entries in the order-$(n - 1)$ **Aztec Diamond**. That the entries in the above matrix is correct is easily verified.

The idea is that we find co-efficients a_j where j corresponds to the columns of $B(n-1)$ such that

$$\sum_{j=1}^{n(n-1)} a_j R_j = [2, -2i, -2, \ldots, 2(-i)^{n-2}, 0, \ldots]$$

(R_i being the row in $B(n-1)$ corresponding to the vertex numbered i).

And then we have (eliminating the residues in column $2n$ above)

$$\frac{ad(n)}{ad(n-1)} = \left| -2(-1)^n - (2a_1 - 2ia_2 - 2a_3 \ldots + 2(-i)^{n-2}a_{n-1}) \right|. \tag{2.1}$$
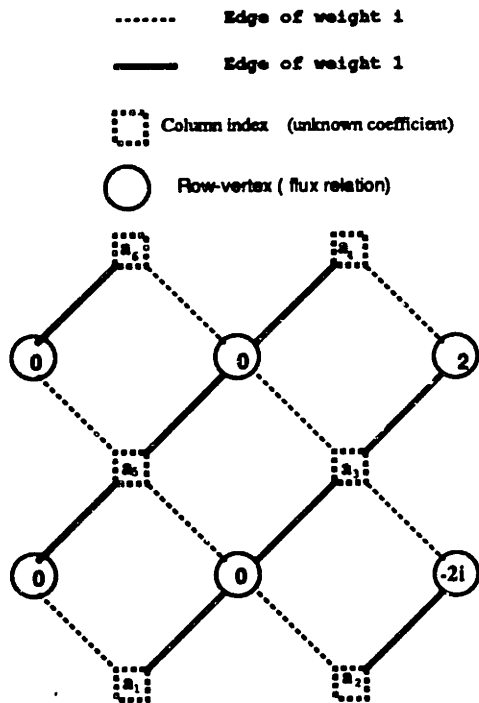
Figure 2-2: Original Propp-Kuperberg graph of $AD(3)$

To visualize the problem in another light, let's lay out $G(AD(n-1))$ with each coefficient a_i marked on the corresponding column-vertex and each number from the column $2n$ above entered on the corresponding row-vertex. (see Figure 2-2)

By the definition of the connectivity matrix, we can summarize the relation of the numbers in the graph as follows: the entry on each row-vertex is related to its neighbors by what we will call a "flux relation": For each edge going into that row-vertex we calculate the product of the weight of the edge edge (think of as transmission coefficient) and the entry on the column-vertex on the other end (think of as source) and sum over all the edges to obtained the entry on the row-vertex (total flux), i.e.

$$f = a_1 W_1 + a_2 W_2 + a_3 W_3 + a_4 W_4$$
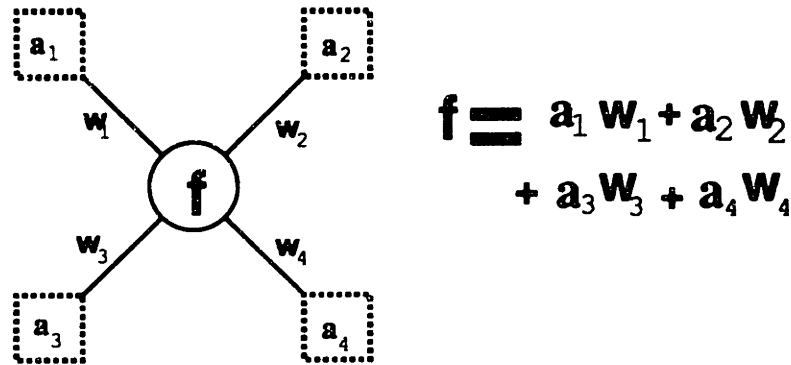
Figure 2-3: A flux relation in a P-K diagram

We can make another amendment to the above: imagine another row of squares being added to the right of $G(AD(n-1))$ and its edges be given orientations as before, we find that by putting in the numbers

$$-1, i, 1, \ldots - (-i)^{n-1}$$

on these vertices, we can eliminate all the non-zero row-vertices in the diagram.

(see Figure 2-4) The reason that this happens to work is the way the reduction functioned. As a matter of fact, this manuever can be carried through in all the problems dealt with in this thesis. This extension we will call the *augmented Propp-Kuperberg diagram* of $AD(n)$.

As a result, the whole problem is reduced to one in which it is easy to obtain a solution via generating functions.

To see that this is the case, consider two adjacent rows of the augmented P-K diagram, below: Here we will introduce a convention that will be used throughout the rest of the text. A row of numbers is regarded as equivalent to a polynomial function whose constant term is the first (leftmost) one in the row, the linear term the second, etc. An independent variable,

31
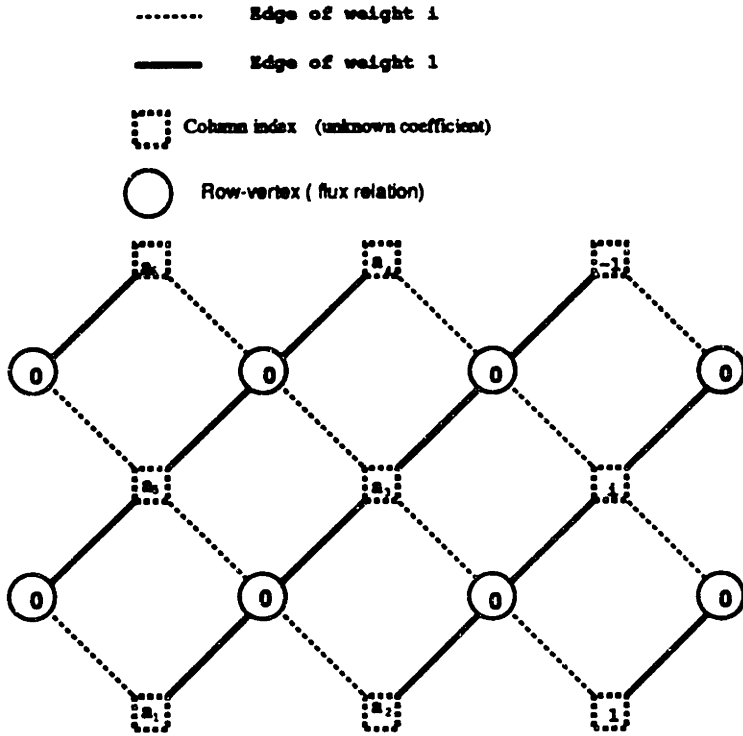
Figure 2-4: Augmented Propp-Kuperberg graph of $AD(3)$

usually t, will be specified most of the time. In particular, the top row of an (augmented) P-K diagram will be referred to as $g_n(t)$, where n is the order of the original figure.

In the diagram, we have

$$A_0 = -iB_0$$

$$iA_0 + A_1 = -(B_0 + iB_1)$$

$$iA_1 + A_2 = -(B_1 + iB_2)$$

$$\cdots \quad \cdots$$

$$iA_{n-2} + A_{n-1} = -(B_{n-2} + iB_{n-1})$$

$$iA_{n-1} = -B_{n-1}$$
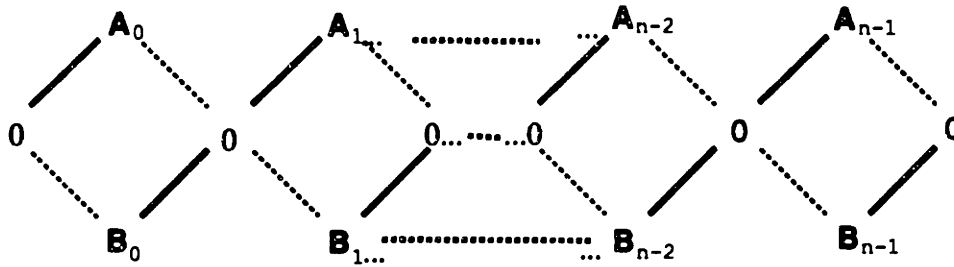
Figure 2-5: Two rows of augmented K-P diagram

Set

$$A(t) \equiv \sum_{j=0}^{n-1} A_j t^j;$$

$$B(t) \equiv \sum_{j=0}^{n-1} B_j t^j.$$

We have

$$A(t)(1 + it) = -B(t)(t + i),$$

or rather,

$$A(t) = iB(t)\frac{t+i}{t-i} \tag{2.2}$$

.

In this instance, if we call the bottom row of the (extended) P-K graph $f_n(t)$ then we have, recursively

$$g_n(t) = i^{n-1} f_n(t) \left(\frac{t+i}{t-i}\right)^{n-1}. \tag{2.3}$$

Since f and g are polynomials of degree $(n - 1)$, and we know that $-g_n$ is monic, we can deduce that

$$f_n(t) = -(-i)^{n-1}(t - i)^{n-1}. \tag{2.4}$$

33

But Equation 2.1 tells us that

$$\frac{AD(n)}{AD(n-1)} = \left| -2 \sum_{j=1}^{n-1} a_j (-i)^{j-1} - 2(-1)^{n-1} \right| = |-2f_n(-i)| = 2^n, \tag{2.5}$$

which means that

$$AD(n) = 2^{n(n+1)/2}.$$

This is what shuffling did for us in the introductory chapter. If this is all this technique can do then this is not worth much, but (in the next two chapters) we will demonstrate that this same technique is useful for other problems, while shuffling has limited application.

2.3 A multi-Q-analog of the same result

This multi-q-analog, in the form of a generating function involving four sets of independent variables, was first discovered by R. Stanley via the technique used in [7]. The proof by shuffling, given earlier and discovered by (at least) this author and J Propp independently, is relatively straightforward, but so is an extension of the last section.

Instead of having 1's as edge weights, Stanley's formula deals with an **Aztec Diamond** that has as weights on the jth row of diamonds the four variables x_j, y_j, z_j, w_j.

The graph of $AD(4)$, every q_i, p_i is alternately modified. We will use the same numbering of vertices that we used in the last section, so the problem is really not changed except for the changed edge-weights:
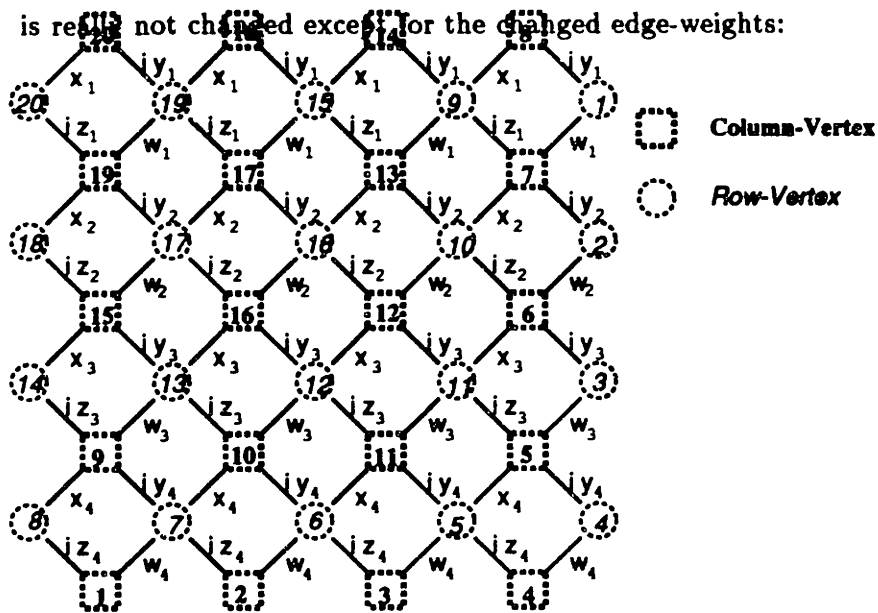
Figure 2-6: The graph of the Q-analog of $AD(4)$, modified

Here we write down the matrix $B(AD(n; x, y, z, w))$ for the (modified) graph:

$$
\begin{bmatrix}
 & & & & w_1 & iy_1 & & & & \cdots & \\
 & & & & w_2 & iy_2 & & & & & \\
 & & & \cdot & \cdot & & & & & & \\
 & & & \cdot & \cdot & & & & & & \\
 & & \cdot & \cdot & & & & & & & \\
 & w_n\square & iy_n & & & & & & & & \\
 & \cdot & iz_n & x_n & & & & & iy_n & & \\
 & \cdot & \cdot & & & & & & \cdot & x_n & \\
 & \cdot & \cdot & & & & & & & \cdot & \cdot \\
 w_n & \cdot & & & & & & iy_n & \cdot & & \\
 w_n & iz_n & & & & & & iy_n & x_n & & \\
 iz_n & & & & & \star & x_n & & & & \\
 & & & & iz_1 & x_1 & \ast & & & & \\
 & & & iz_2 & x_2 & & & \ast & & & \\
 & & \cdot & \cdot & & & & & \ast & & \\
 & & \cdot & \cdot & & & & & & \ast & \\
 & z_{n-1} & x_{n-1} & & & & & & & & \ast \\
 \vdots & & & & & & & & & & \ddots
\end{bmatrix}
$$

In the above matrix again the entry marked with $\square$ is (n, n), the one marked with a $\star$ $(2n, 2n)$, and the last displayed row/column the $(3n - 1)$th.

The reduction is similar to what we went through in the last section, so we will simply enumerate the steps:

1. $(-i\frac{y_1}{w_1})\times$ (Row $2n - 1$) added to (Row $2n$).

2. $(-i\frac{z_n}{w_n})\times$ (Column $2n - 1$) added to (Column $2n$).

3. $(-\frac{y_1 y_2}{w_1 w_2})\times$ (Row $2n - 2$) added to (Row $2n$).

4. $(-\left(\frac{z_n}{w_n}\right)^2 \times$ (Column $2n - 2$) added to (Column $2n$).

36

5. $(i\frac{y_1 y_2 y_3}{w_1 w_2 w_3}) \times$ (Row $2n - 3$) added to (Row $2n$).

6. $(i\left(\frac{z_n}{w_n}\right)^3) \times$ (Column $2n - 3$) added to (Column $2n$).

. . . . . .

$(-i)^k \frac{y_1 y_2 \cdots y_k}{w_1 w_2 \cdots w_k} \times$ (Row $2n - k$) added to (Row $2n$).

$\left(-i\frac{z_n}{w_n}\right)^k \times$ (Column $2n - k$) added to (Column $2n$).

. . . . . .

$2n - 1.$ $(-i)^n \frac{y_1 y_2 \cdots y_n}{w_1 w_2 \cdots w_n} \times$ (Row n) added to (Row $2n$).

$2n.$ $(-i)^n \left(\frac{z_n}{w_n}\right)^n \times$ (Column n) added to (Column $2n$).

At the end of which we obtain:

						C	A_1	A_2	$\cdots$	A_{n-1}	0	$\cdots$

Matrix with diagonal entries $w_1, w_2, \ldots, w_n, \ldots, w_n$ in upper-left block; bottom rows labeled C, B_1 (with $*$), B_2 (with $*$), $\vdots$ (with $*$), B_{n-1} (with $*$), 0 (with $*$), $\vdots$ (with $*$).

where we have

$$A_j = \left(x_n + \frac{y_n z_n}{w_n}\right)\left(-i\frac{z_n}{w_n}\right)^{j-1};$$

$$B_j = \left(x_j + \frac{y_j z_j}{w_j}\right)(-i)^{j-1}\prod_{l<j}\frac{y_l}{w_l};$$

$$C = -\left(x_n + \frac{y_n z_n}{w_n}\right)\left(-\frac{z_n}{w_n}\right)^{n-1}\left(\frac{y_1 y_2 \cdots y_{n-1}}{w_1 w_2 \cdots w_{n-1}}\right).$$

(2.6)

In the same manner as before, we try to obtain $a_1, a_2, \cdots$ such that

$$\sum_{j=1}^{n(n-1)} a_j R_j = [B_1, B_2, B_3 \ldots, B_{n-1}, 0, \ldots],$$

where R_i is the row in $B(n-1; x, y, z, w)$ corresponding to i.

We will obtain our answer in

$$\frac{ad(n; x, y, z, w)}{ad(n-1; x, y, z, w)} = w_1 w_2 \ldots w_{n-1} w_n^n \left| C - \sum_{j=1}^{n-1} A_j a_j \right|$$

(2.7)

We see that the Equations 2.6 means that

1. We can augment the P-K diagram as before with (-1) at the upper right corner (because of B_j's properties). And the rest of the added vertices have weights

$$v_1 = i\frac{y_1}{w_1}, v_2 = \frac{y_1 y_2}{w_1 w_2}, \ldots, v_{n-1} = -\prod_{j=1}^{n-1}\left(-i\frac{y_j}{w_j}\right).$$

2. If we identify the top and bottom rows of the (augmented) P-K diagram with $g_n(t)$ and $f_n(t)$ respectively, then we have

$$C - \sum_j A_j a_j = \left(x_n + \frac{y_n z_n}{w_n}\right) f_n\left(-i\frac{z_n}{w_n}\right)$$

(2.8)

So, we can take a look at adjacent rows of the P-K diagram:

38
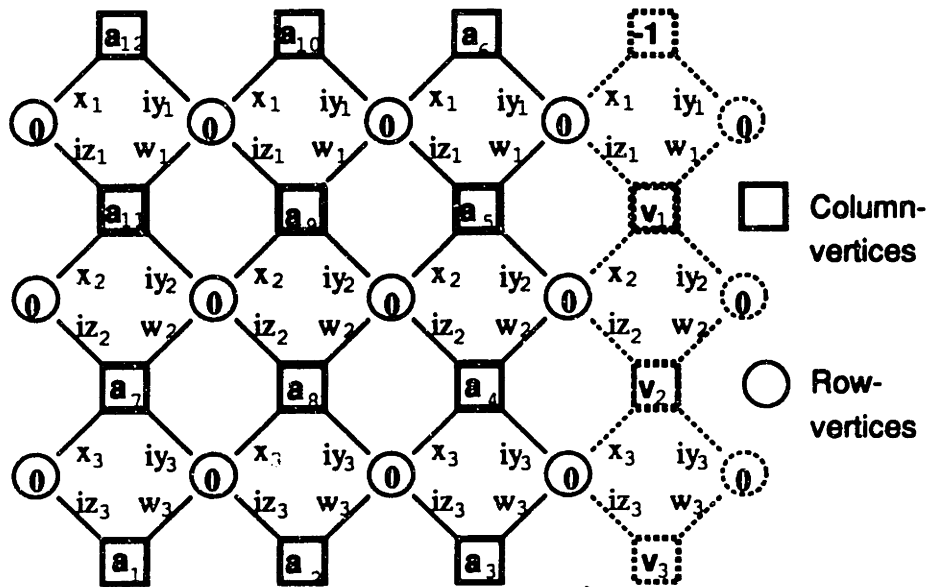
Figure 2-7:

Here, similar arguments to those in the last section the last section results in

$$A(t)(x_j + iy_j t) + B(t)(iz_j + w_j t) = 0.$$

In almost the same way, we get

$$f_n(t) = \prod_{j=1}^{n-1} \left(-\frac{x_j + iy_j t}{w_j} \right) \tag{2.9}$$

Now we get from Equation 2.8,

$$\frac{ad(n;)}{ad(n-1;)} = \left| w_1 w_2 \ldots w_{n-1} w_n^n \left(x_n + \frac{y_n z_n}{w_n} \right) f_n \left(-i\frac{z_n}{w_n} \right) \right|$$

$$= \prod_{j=1}^{n} (x_j w_n + y_j z_n). \tag{2.10}$$

39

Chapter 3

The Penta-Aztec-Diamond

3.1   The Description of the Penta-Aztec Diamond

The number of tilings of the **Aztec Diamond** of order n is a high power of 2. A inquisitive mind is motivated to investigate the possible existence of other sequences of graphs whose number of tilings relate to powers of other low primes. J Propp investigated the enumeration of dimer tilings of the figure obtained by cutting the square $[0, n] \times [0, n]$ by all the lines $x = k$, $y = k$, $x + y = k$, and $x - y = k$, [1] and found factors of high powers of 5. A slight modification of these figures yielded $PD(n)$ as introduced in Chapter 0.

Following the previous chapter, the function $pd(n)$ will be taken to mean the number of tilings of $PD(n)$.

Note that there are exactly 5 tilings of the **Penta Aztec Diamond** of order 2, which is a partial figure that will appear many times when calculating the number of tilings for higher ordered **Penta Aztec Diamonds** by hand and which appear to play the same role as the odd (even) block in the **Aztec Diamonds**. The prefix "penta" came from this fact. Although attempts at a purely combinatorial proof along the line of the "shuffle" proved fruitless, we can and will use a slight modification of the procedure in the last chapter to show:

---

[1] This "Aztec Square" is an analogy to the $2n \times 2n$ checkerboard tilings discussed in [8] and will be the object of future investigation.

41

Theorem 3.1

$$pd(2n) = 5^{n^2};$$

$$pd(4n+1) = 5^{2n(2n+1)};$$

$$pd(4n-1) = 2 \times 5^{2n(2n-1)}. \tag{3.1}$$

Note that for the "other" version of $PD(4n \pm 1)$ it can be shown via the identical chain of attack that the number of tilings is equal to twice the answer shown above, which is reasonable since each add-on actually cuts down the number of possible choices. Note the factor of 2 in
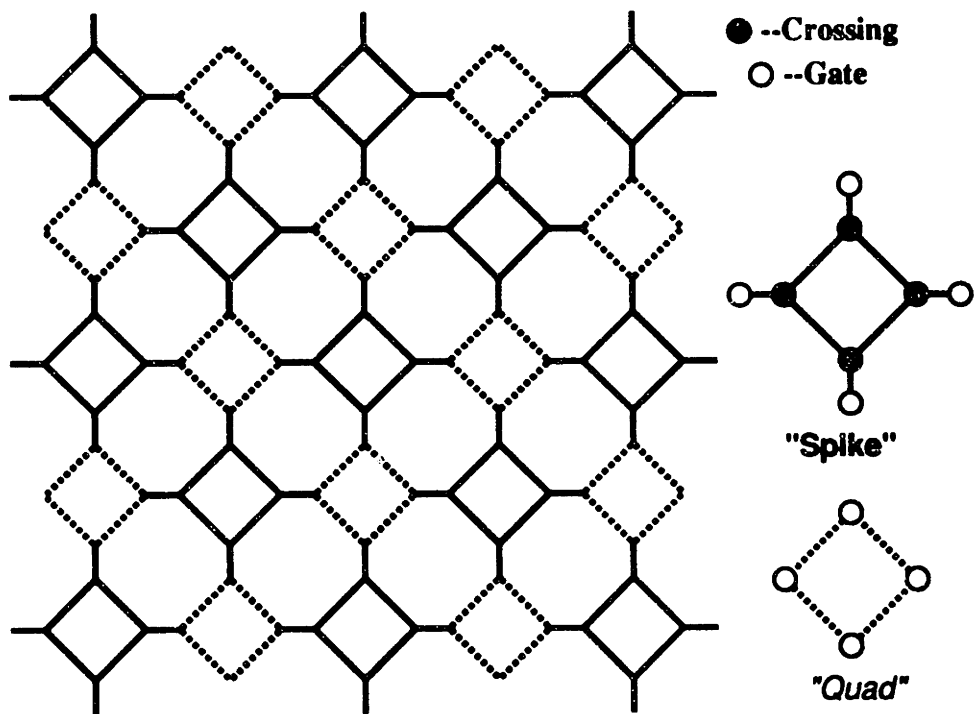
Figure 3-1: The connectivity graph $G(PD(5))$

the last formula above. The last two formulas above, enumerating matchings of $PD(4n \pm 1)$, figures with C_4 (rotation by 90 degrees) symmetries around the center (see Figure 3-1), is in accordance with the theorem of W. Jockusch which states that all planar graphs with a C_4

symmetry will have as the number of matchings either a square (when the number of vertices is divisible by 8) or twice a square.

3.2   Transformation of Cities

Here we borrow an idea from statistical mechanics and crystalline physics to transform the problem into a more familiar form. In that setting:

Definition 15 *A city C in a graph $G = (V, E, w)$ is a triplet (S, V_g, V_c) satisfying*

1. *S is a set of connected edges (called* streets*).*

2. *The disjoint union of V_g (the set of* gates*) and V_c (the set of* crossings*) is the set of vertices covered by the edges in S.*

3. *Any vertex $v_c \in V_c$ (called a* crossing*) satisfies*

$$v \in V_c \cup V_g, \forall \{v_c, v\} \in E.$$

We define an equivalence relation by treating two cities (S, V_g, V_c) and (S', V_g', V_c'), which may possibly be in two different graphs, as equivalent if there exist bijections $\phi_g : V_g \leftrightarrow V_g'$ and $\phi_c : V_c \leftrightarrow V_c'$ such that the bijection $\phi : V_g \cup V_c \leftrightarrow V_g' \cup V_c'$ they induce also induces a bijection $S \leftrightarrow S'$ that preserves the weight function in the original graphs.

Note that the terms "gate" is relative to the actual graph, and it is quite possible to have a gate that leads nowhere[2]. A city is not merely a subgraph (e.g. Fig. 3.2).

Definition 16 *A gate g in a city $C = (S, V_g, V_c)$ is "closed" in a matching m when the edge $\{g, v\} \in m$ covering g is in S and open if it is not (hence $v \notin V_g \cup V_c$). A quotient matching of the graph with respect to the city C is an imperfect matching[3] $m_q \subset E$ such that*

1. *$\forall v \in V - (V_g \cup V_c), \exists e \in m_q$ covering v;*

2. *$\forall v \in V_c, v$ is not covered by any edge in m_q;*

---

[2] By definition a crossing cannot be exposed to the outside.
[3] Defined as a collection of disjoint edges, as used by L. Lovasz in [8].
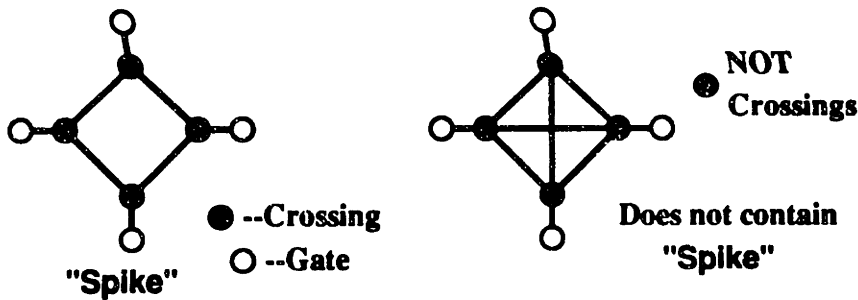
Figure 3-2: A graph that does *not* contain "Spike"

3. the gates of C may or may not be covered ("open") by m_q. A gate g that is covered by (an edge in) a quotient matching m_q is called open, and g is called closed if g is not open.

It is easy to see from the above that if a gate is closed in a quotient matching which can be extended to a (perfect) matching by adding a collection of edges E_c the gate will be closed in any such extension, and vice versa. We will define the (relative) weight , denoted $r(m_q)$ (or $r(m_q; G)$ if necessary) as

$$\sum_{E_c} \prod_{e \in E_c} w(e),$$

where E_c ranges over all possible extensions.

As a result of that, we can make the observation that

$$M(G) = \sum_{m_q} r(m_q) w(m_q) \tag{3.2}$$

where m_q ranges over all quotient matchings of G with respect to C.

In a graph G that contains a city C, we can perform the operation of substituting another city C' for C (provided that the two cities have the same number of gates) in an intuitive manner. It is easy to see there is an intuitively obvious bijection between the quotient matchings of the two graphs under consideration.

After all the above definitions, we finally get this lemma:

Lemma 3.2.1 *The number of matchings of any graph G that contains the "spiked" city S in Figure 3-1 is twice that of a graph $\tilde{G}$ that is the same except for the substitution for S with a simple quadrangular city ("1/2 quad") that has edge weights $\frac{1}{2}$.*

44

Proof: We take any quotient matching of G and consider the possible extensions to complete matchings of G and $\tilde{G}$ (See Figure 3.2). Obviously this only depends on which gates of the cities are open. We can see that according to the status of the gates there are these cases:

The extensions in "spike"   The extensions in 1/2 quad
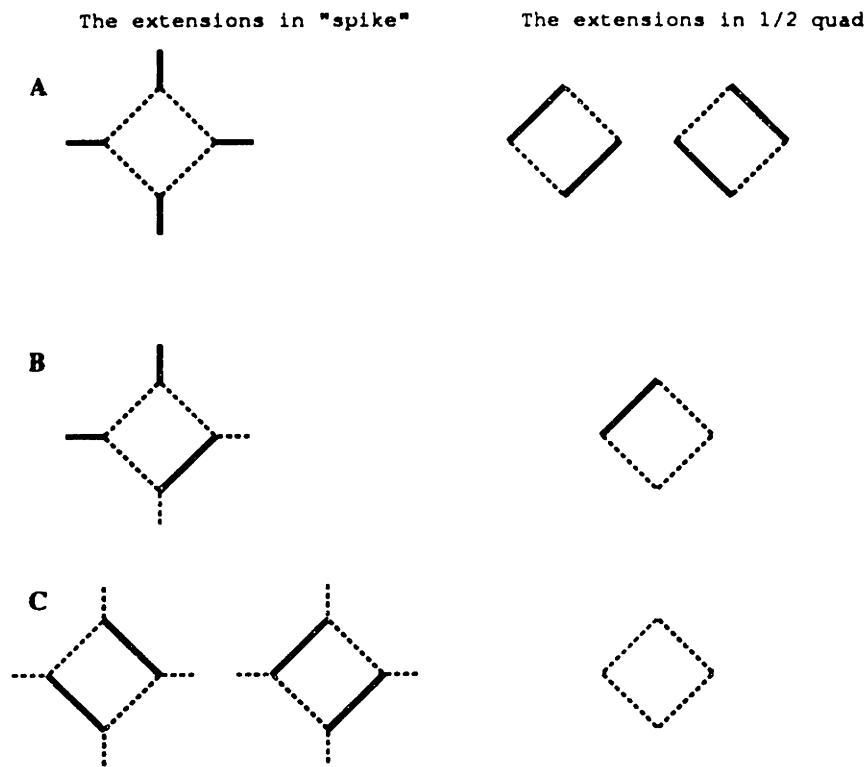
Figure 3-3: Correspondence between *cities*

1. In case A, all 4 gates are closed. There is only one extension of m_q, with weight 1, to G, while there are two possible extensions to $\tilde{G}$ each with weight $\frac{1}{4}$.

2. In case C, all gates are open. There is are two extensions to G each with relative weight 1 and only one extension to $\tilde{G}$ which also has relative weight 1.

3. In case B, there are two adjacent open gates and one possible extension each to G and $\tilde{G}$, but the extension to G has weight 1 while the extension to $\tilde{G}$ has weight $\frac{1}{2}$.

4. There no possible extensions if there is 1 or 3 open gates, or 2 open gates which are not adjacent.

45

So, since for all three pertinent cases $r(m_q; G) = r(m_q; \tilde{G})$, according to Equation 3.2, we have

$$M(G) = 2M(\tilde{G}).$$

☐

We will define $PAD(n)$, the *Polymorphed (penta-)Aztec Diamond*, to be the graph that is identical to $G(AD(n))$ except that every other square has edges weighted $\frac{1}{2}$ instead of 1 (see Fig. 3.2). Going along with previous notation, $pad(n)$ will be used to denote the number
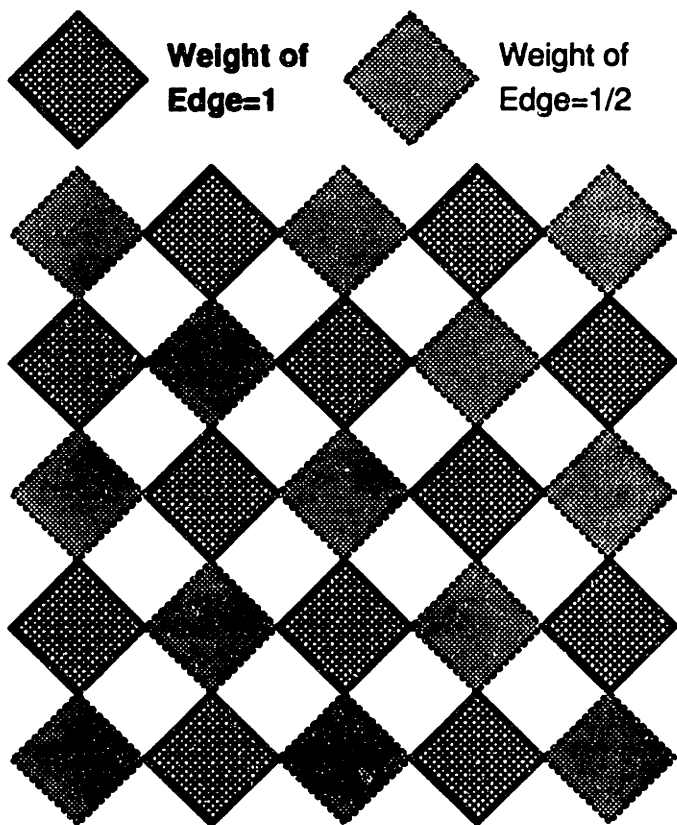
Figure 3-4: Transformed Penta Aztec Diamond of order 5.

of matchings of $PAD(n)$, and $\Phi(n)$ the connectivity matrix (of an alternating modification) $C(\overrightarrow{PAD}(n))$.

So, with the above, we have:

46

Lemma 3.2.2 *The tiling problem for* $PD(n)$ *can be converted as follows into the matching problem for for the polymorphed Aztec diamond:*

$$
\begin{aligned}
pd(2n) &= 2^{n(2n+1)}pad(2n) \\
pd(2n+1) &= 2^{2n(n+1)}pad(2n+1)
\end{aligned}
\tag{3.3}
$$

For the rest of this chapter we will devote our attention to the problem of enumerating matchings of $PAD(n)$.

3.3 Reducing $\Phi(n)$

We will adhere to our notation and numbering of vertices as in the last chapter. However, the matrix $\Phi(n)$ have different forms depending on the parity of the order, so we give both here:

$$\Phi(n) = \text{(see next page)}$$

$$
\begin{bmatrix}
& & & & & & & \tfrac{1}{2} & \tfrac{i}{2} & & & & \cdots \\
& & & & & & 1 & i & & & & & \\
& & & & & \cdot & \cdot & & & & & & \\
& & & & \cdot & \cdot & & & & & & & \\
& & & \tfrac{1}{2}\square & \tfrac{i}{2} & & & & & & & i & \\
& & \cdot & \tfrac{i}{2} & \tfrac{1}{2} & & & & & & \cdot & 1 & \\
& \cdot & \cdot & & & & & & & \cdot & \cdot & & \\
1 & \cdot & & & & & & & i & \cdot & & & \\
\tfrac{1}{2} & i & & & & & & & \tfrac{i}{2} & \tfrac{1}{2} & & & \\
\hline
\tfrac{i}{2} & & & & & & & \star & \tfrac{1}{2} & & & & \\
\hline
& & & & & \tfrac{i}{2} & \tfrac{1}{2} & \ast & & & & & \\
& & & & & i & 1 & & \ast & & & & \\
& & & & \cdot & \cdot & & & \ast & & & & \\
& & & \cdot & \cdot & & & & \ast & & & \\
& & i & 1 & & & & & \ast & & \\
\vdots & & & & & & & & & \ddots \\
\end{bmatrix}
$$

$n = 2m + 1 \text{ (odd } n),$

$$\begin{bmatrix}
 & & & & & & 1 & i & & & & \cdots \\
 & & & & & \tfrac{1}{2} & \tfrac{i}{2} & & & & & \\
 & & & & \cdot & \cdot & & & & & & \\
 & & & \cdot & \cdot & & & & & & & \\
 & & \cdot & \cdot & & & & & & & & \\
 & 1\square & i & & & & & & & & & \\
 \cdot & i & 1 & & & & & & i & & & \\
 \cdot & \cdot & & & & & & \cdot & 1 & & & \\
 \cdot & \cdot & & & & & & \cdot & \cdot & & & \\
 \tfrac{1}{2} & \cdot & & & & & \tfrac{i}{2} & \cdot & & & & \\
 1 & \tfrac{i}{2} & & & & i & \tfrac{1}{2} & & & & & \\
\hline
 i & & & & \star & 1 & & & & & & \\
\hline
 & & & i & 1 & \ast & & & & & & \\
 & \tfrac{i}{2} & \tfrac{1}{2} & & & \ast & & & & & & \\
 & \cdot & \cdot & & & & \ast & & & & & \\
 & \cdot & \cdot & & & & & \ast & & & & \\
 i & 1 & & & & & & & \ast & & & \\
 \vdots & & & & & & & & & \ddots & &
\end{bmatrix}$$

$n = 2m$ (even n).

Here as before entry (n, n) is marked with the $\square$; entry $(2n, 2n)$ is marked with the $\star$, and the last row (column) displayed is the $(3n - 1)$th.

Since it may not be clear to the reader what each entry is, we list the non-zero entries in the matrices:

$$\Phi(n)_{k,2n-k} = \begin{cases} 1 & n + k = \text{odd} \\ \tfrac{1}{2} & n + k = \text{even} \end{cases}, k = 1 \ldots 2n - 1;$$

$$\Phi(n)_{2n+1-k,k} = \begin{cases} i & n + k = \text{odd} \\ \tfrac{i}{2} & n + k = \text{even} \end{cases}, k = 1 \ldots n - 1$$

$$= \Phi(n)_{k,2n+1-k};$$

$$\Phi(n)_{n+1,n+1} \quad = \quad \begin{cases} \frac{1}{2} & n = \text{odd} \\ 1 & n = \text{even} \end{cases} \quad ;$$

$$\Phi(n)_{2n-k,2n+k} \quad = \quad \begin{cases} i & n+k = \text{odd} \\ \frac{i}{2} & n+k = \text{even} \end{cases} \quad ,k = 1\ldots n-1$$

$$= \quad \Phi(n)_{2n+k,2n-k};$$

$$\Phi(n)_{2n+k,2n+1-k} \quad = \quad \begin{cases} 1 & n+k = \text{odd} \\ \frac{1}{2} & n+k = \text{even} \end{cases} \quad ,k = 1,2,\ldots n-1$$

$$= \quad \Phi(n)_{2n+1-k,2n+k}.$$

Now we carry out the reduction of the matrix, in exactly the same manner as before:

1. $(-i)\times$ (Row $2n-1$) is added to (Row $2n$).

2. $(-i)\times$ (Column $2n-1$) is added to (Column $2n$).

3. $(-1)\times$ (Row $2n-2$) is added to (Row $2n$).

4. $(-1)\times$ (Column $2n-2$) is added to (Column $2n$).

5. $(+i)\times$ (Row $2n-3$) is added to (Row $2n$).

6. $(+i)\times$ (Column $2n-3$) is added to (Column $2n$).

. . . . . .

$(-i)^k\times$ (Row $2n-k$) is added to (Row $2n$).

$(-i)^k\times$ (Column $2n-k$) is added to (Column $2n$).

. . . . . .

$2n-1.$ $(-i)^n\times$ (Row n) is added to (Row $2n$).

$2n.$ $(-i)^n\times$ (Column n) is added to (Column $2n$).

After these operations, we obtain the reduced matrix:

$$
\left[
\begin{array}{c|c|ccccccc}
 & & & & & 1 & & & \\
 & & & & \tfrac{1}{2} & & & & \\
 & & & \cdot & & & & & \\
 & & \cdot & & & & & & \\
 & \tfrac{1}{2} & & & & & & & \\
1 & & & & & & & & \\
\hline
 & 1 & 1 & -2i & -1 & \cdots & 2(-1)^m i & 0 & \cdots \\
\hline
 & 1 & * & & & & & & \\
 & -2i & & * & & & & & \\
 & -1 & & & * & & & & \\
 & \vdots & & & & * & & & \\
 & 2(-1)^m i & & & & & * & & \\
 & 0 & & & & & & * & \\
 & \vdots & & & & & & & *
\end{array}
\right]
$$

n odd $(n = 2m + 1)$,

$$
\left[
\begin{array}{cc|cccccccc}
 & \frac{1}{2} & & & & & & \\
 1 & & & & & & & \\
 & \ddots & & & & & & \\
 1 & & & & & & & \\
 \frac{1}{2} & & & & & & & \\
\hline
 & -2 & 2 & -i & -2 & \cdots & 2(-1)^{m-1} & 0 & \cdots \\
\hline
 & 1 & * & & & & & \\
 & -2i & & * & & & & \\
 & -1 & & & * & & & \\
 & \vdots & & & & * & & \\
 & 2(-1)^{m-1} & & & & & * & \\
 & 0 & & & & & & * \\
 & \vdots & & & & & & & *
\end{array}
\right]
$$

n even ($n = 2m$).

3.4 The calculation of $pad(n)$

Now we will carry out largely the same manuevers as has been done before in the previous chapter, although they are in general more complicated than before.

First let's note that the $(2n)$th row, on which each number can be obtained from the one before by multiplication of $(-i)$ or $(-\frac{i}{2})$ (alternatingly), is consistent with an extension of the Propp-Kuperberg diagram such that one can eliminate ?ll the non-zero entries in the row-vertices. Given the way our elimination worked this is to be expected; sometimes, as in the next chapter, it is not possible to get a P-K diagram where there are no non-zero entries.

Our task is again to find numbers $a_1, a_2, \ldots$ such that we can cancel out the numbers in column $2n$ with multiples ᴏf the other columns. Once we found these coefficients, we may find the answer via

$$
\frac{pad(n)}{pad(n-1)} = \begin{cases} 2^{-n} |1 - (a_1 - 2ia_2 - a_3 + 2ia_4 + - \ldots)|, & \text{odd } n; \\ 2^{-(n-1)} |-2 - (2a_1 - ia_2 - 2a_3 + ia_4 + - \ldots)|, & \text{even } n. \end{cases} \tag{3.4}
$$

Let's look at two adjacent rows again in the extended P-K diagram of $PAD(n)$ (see figure 3.4).

Case (a) "half" square on the left
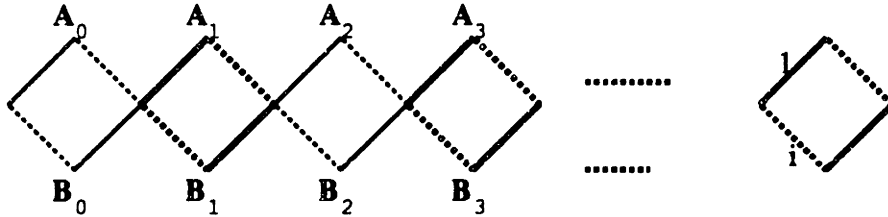
Case (b) "full" square on the left
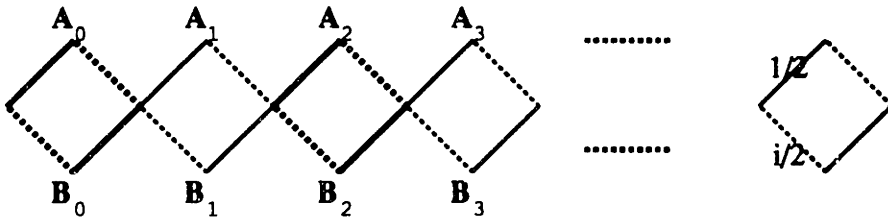
Figure 3-5: Rows of the augmented PK graph.

Let's consider case (a) as depicted above. This time, we will vary our approach a little bit. For any function (power series) $F(t)$ we define the *even part* F^+ and *odd part* F^- by

$$F^{\pm}(t) = (F(t) \pm F(-t))/2,$$

so if $F(t) = \sum_j F_j t^j$ then we have $F^+(t) = \sum_j F_{2j} t^{2j}$ and $F^-(t) = \sum_j F_{2j+1} t^{2j+1}$.

Defining $A(t)$ and $B(t)$ to be the polynomials associated with the numbers A_j and B_j in Fig. 3.4, we have

$$A^+(t)(\tfrac{1}{2} + \tfrac{i}{2}t) + A^-(t)(1 + it) + B^+(t)(\tfrac{i}{2} + \tfrac{1}{2}t) + B^-(t)(i + t) = 0$$

This we can separate into odd and even parts to obtain

$$\tfrac{1}{2}A^+ + itA^- + \tfrac{i}{2}B^+ + tB^- = 0$$

53

$$\tfrac{i}{2}tA^+ + A^- + \tfrac{1}{2}tB^+ + iB^- = 0 \tag{3.5}$$

We can write the above equations in matrix form as

$$\begin{pmatrix} A^+ \\ A^- \end{pmatrix} = \frac{-1}{1+t^2} \begin{pmatrix} i(1-t^2) & 4t \\ t & i(1-t^2) \end{pmatrix} \begin{pmatrix} B^+ \\ B^- \end{pmatrix}$$

$$\begin{pmatrix} B^+ \\ B^- \end{pmatrix} = \frac{-1}{1+t^2} \begin{pmatrix} -i(1-t^2) & 4t \\ t & -i(1-t^2) \end{pmatrix} \begin{pmatrix} A^+ \\ A^- \end{pmatrix} \tag{3.6}$$

Similarly, for case (b):

$$\begin{pmatrix} A^+ \\ A^- \end{pmatrix} = \frac{-1}{1+t^2} \begin{pmatrix} i(1-t^2) & t \\ 4t & i(1-t^2) \end{pmatrix} \begin{pmatrix} B^+ \\ B^- \end{pmatrix}$$

$$\begin{pmatrix} B^+ \\ B^- \end{pmatrix} = \frac{-1}{1+t^2} \begin{pmatrix} -i(1-t^2) & t \\ 4t & -i(1-t^2) \end{pmatrix} \begin{pmatrix} A^+ \\ A^- \end{pmatrix} \tag{3.7}$$

Now suppose that n is odd, that is, $n = 2m + 1$. We can identify the expression inside the bars in Equation 3.4 as $(-f^+(-i) - 2f^-(-i))$. The center row in the K-P diagram we can immediately guess to be (a constant multiple of) $(1 + t^2)^m$. This will insure that we can find a consistent set of numbers to fill in all the nodes.

Let us digress for a moment. In the extended P-K diagram here, we have n^2 unknowns and $(n + 1)(n - 1) = n^2 - 1$ relations, so all the numbers in the diagram are, determined up to a multiplicative constant. Since we know that the upper right corner is -1, we know that there can be only one set of numbers that satisfy all the relations, so once we find *one* compatible set of numbers it must be *the one*.

Returning to our calculation, we see that we need not calculate all the numbers in the P-K diagram immediately. Instead, we cancel out all the factors of $(1 + t^2)$ before substituting $(-i)$

54

for t. Using Equation 3.6 and Equation 3.7 we find that for the case $n = 4m + 1$ we have

$$\frac{pad(4m+1)}{pad(4m)} = \left| (-1,-2) \overbrace{\begin{pmatrix} 2i & i \\ 4i & 2i \end{pmatrix} \cdots \begin{pmatrix} 2i & i \\ 4i & 2i \end{pmatrix}}^{2m} \begin{pmatrix} 2i & 4i \\ i & 2i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} 2^{-(4m+1)} (-(-i)^{2m}) \right|$$

$$= 5^m (1,2) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix} 2^{-(4m+1)}$$

$$= 5^{2m} 2^{-(4m+1)}. \tag{3.8}$$

Similarly, for $n = 4m + 3$ we have

$$\frac{pad(4m+3)}{pad(4m+2)} = \left| (-1,-2) \overbrace{\begin{pmatrix} 2i & i \\ 4i & 2i \end{pmatrix} \cdots \begin{pmatrix} 2i & 4i \\ i & 2i \end{pmatrix}}^{2m+1} \begin{pmatrix} 2i & i \\ 4i & 2i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} 2^{-(4m+3)} (-(-i)^{2m+1}) \right|$$

$$= 5^m (1,2) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}^m \begin{pmatrix} 2 \\ 4 \end{pmatrix} 2^{-(4m+3)}$$

$$= 5^{2m+1} 2^{-(4m+3)}. \tag{3.9}$$

Now suppose n is even, and $n = 2m$. We can thus identify the expression inside the bars in Equation 3.4 we seek as $(-2f^+(-i) - f^-(-i))$. Here the problem is slightly more complicated, but we have the still relatively obvious first guess that the below-center row is $(1+t^2)^{m-1}(a+bt)$, for constants a, b.

So we have for $n = 4m + 2$

$$\begin{pmatrix} a(1+t^2)^{2m} \\ -(-i)^{2m+1} t(1+t^2)^{2m} \end{pmatrix} = \frac{-1}{1+t^2} \begin{pmatrix} -i(1-t^2) & 4t \\ t & -i(1-t^2) \end{pmatrix} \begin{pmatrix} \bar{a}(1+t^2)^{2m} \\ -(-i)^{2m} t(1+t^2)^{2m} \end{pmatrix}. \tag{3.10}$$

This equation is easily solved to obtain

$$a = 2i(-i)^{2m+1};$$

55

$$\bar{a} \;=\; 2i(-i)^m.$$

Now we make the substitution $t = -i$ after factoring out the factors of $(1 + t^2)$, and use Equations 3.6 and 3.7 to obtain

$$\frac{pad(4m+2)}{pad(4m+1)}$$

$$= \left| (-2,-1) \overbrace{\begin{pmatrix} 2i & 4i \\ i & 2i \end{pmatrix} \cdots \begin{pmatrix} 2i & 4i \\ i & 2i \end{pmatrix}}^{2m} \begin{pmatrix} 2i & i \\ 4i & 2i \end{pmatrix} \begin{pmatrix} 2i \\ i \end{pmatrix} 2^{-(4m+1)}(-(-i)^{2m+1}) \right|$$

$$= 5^m(2,1) \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}^m \begin{pmatrix} 2 \\ 1 \end{pmatrix} 2^{-(4m+1)}$$

$$= 5^{2m+1} 2^{-(4m+1)} \tag{3.11}$$

For the last case $n = 4m$, similar to Equation 3.10, we can get

$$\begin{pmatrix} a(1+t^2)^{2m-1} \\ -(-i)^{2m}t(1+t^2)^{2m-1} \end{pmatrix} = \frac{-1}{1+t^2} \begin{pmatrix} -i(1-t^2) & t \\ 4t & -i(1-t^2) \end{pmatrix} \begin{pmatrix} \bar{a}(1+t^2)^{2m-1} \\ -(-i)^{2m}t(1+t^2)^{2m} \end{pmatrix}.$$

This we can solve to get

$$\alpha = \tfrac{1}{2}(-i)^{2m-1}$$

Finally, similar to the case for $n = 4m + 2$ we get

$$\frac{pad(4m)}{pad(4m-1)}$$

$$= \left| (-2,-1) \overbrace{\begin{pmatrix} 2i & 4i \\ i & 2i \end{pmatrix} \cdots \begin{pmatrix} 2i & i \\ 4i & 2i \end{pmatrix}}^{2m-1} \begin{pmatrix} 2i & 4i \\ i & 2i \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} 2^{-(4m-1)}(-(-i)^{2m+1}) \right|$$

$$= 5^m(2,1) \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}^{m-1} \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} 2^{-(4m-1)}$$

$$= 5^{2m} 2^{-4m}. \tag{3.12}$$

Putting Equations 3.12, 3.8, 3.11, and 3.9 together, and using Lemma 3.2.2 we prove Theorem 3.2.2 by mathematical induction.

3.5 A q-analog of the Penta Aztec Diamond tilings

It is not unnatural to attempt a generalization of enumerations to generating functions. Here we construct a rather more general generating function which contains more of the information given about the matchings of $PAD(n)$.

We construct a new graph by modifying the weight of the edges in the connectivity graph $C(AD(n))$, which is a $n \times n$ square arrangement of lozenges, as follows:

- The four edges of the $(2k+1)$th lozenge in the jth row are weighted (clockwise from the upperleft) $x_j, y_j^{-1}, w_j, z_j^{-1}$

- The four edges of the $(2k)$th lozenge in the jth row are weighted (clockwise from the upperleft) $x_j^{-1}, y_j, w_j^{-1}, z_j$.

See the next figure for an example ($n = 4$).

As before, we will adhere to the same orientation and numbering of vertices. We obtain

$$\Phi(n; x, y, z, w) = \text{(see next page)}$$

Row-vertices ○   Column-vertices □
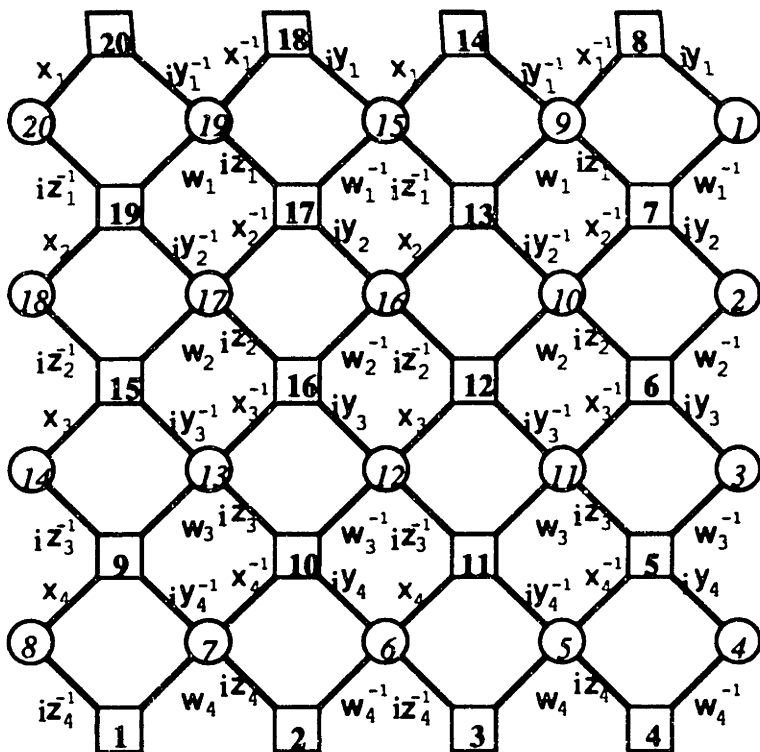
Figure 3-6: Order-4 Penta Aztec Diamond, multi-Q analog

$$
\begin{bmatrix}
 & & & & & w_1^{-1} & iy_1 & & & \cdots \\
 & & & & w_2^{-1} & iy_2 & & & & \\
 & & & & \ddots & & & & & \\
 & & & \ddots & & & & & & \\
 & & w_n^{-1}\square & iy_n & & & & & & \\
 & \cdot & iz_n & x_n^{-1} & & & & iy_n^{-1} & & \\
 & & & & & & & \ddots & x_n & \\
 & & & & & & & & \ddots & \\
 w_n^{-1} & \cdot & & & & iy_n & \cdot & & & \\
 w_n & iz_n & & & & iy_n^{-1} & x_n^{-1} & & & \\
\hline
 iz_n^{-1} & & & & \star & x_n & & & & \\
\hline
 & & iz_1 & x_1^{-1} & * & & & & & \\
 & & iz_2 & x_2^{-1} & & * & & & & \\
 & & & & & & * & & & \\
 & & & & 58 & & & * & & \\
 & & iz_{n-1} & x_{n-1}^{-1} & & & & * & & \\
 \vdots & & & & & & & & \ddots &
\end{bmatrix}
$$

Even:$n = 2m$;

$$
\left[
\begin{array}{ccccc|cc|ccccc}
 & & & & w_1 & iy_1^{-1} & & & & & & \cdots \\
 & & & w_2 & iy_2^{-1} & & & & & & & \\
 & & \cdot & \cdot & & & & & & & & \\
 & \cdot & & \cdot & & & & & & & & \\
 & \cdot & & \cdot & & & & & & & & \\
 & w_n\square & iy_n^{-1} & & & & & & & & & \\
 & \cdot & iz_n^{-1} & x_n & & & & & iy_n & & & \\
 & & \cdot & \cdot & & & & \cdot & x_n^{-1} & & & \\
 & & \cdot & \cdot & & & & & & \cdot & \cdot & \\
 w_n^{-1} & \cdot & & & & & iy_n & & \cdot & & & \\
 w_n & iz_n & & & & & iy_n^{-1} & x_n^{-1} & & & & \\
\hline
 iz_n^{-1} & & & & & \star & x_n & & & & & \\
\hline
 & & & & iz_1^{-1} & x_1 & \star & & & & & \\
 & & & iz_2^{-1} & x_2 & & & \star & & & & \\
 & & \cdot & \cdot & & & & & \star & & & \\
 & \cdot & & \cdot & & & & & & \star & & \\
 & iz_{n-1}^{-1} & x_{n-1} & & & & & & & & \star & \\
 \vdots & & & & & & & & & & & \ddots
\end{array}
\right]
$$

Odd:$n = 2m + 1$.

Again in the formulas above we mark the positions $\square = (n, n)$; $\star = (2n, 2n)$. The last row/column displayed is the $(3n - 1)$th.

We give the non-zero elements of $\Phi(n)$:

$$
\Phi(n)_{k, 2n-k} = \begin{cases} w_k & n = \text{odd} \\ w_k^{-1} & n = \text{even} \end{cases} , k = 1 \ldots n - 1
$$

$$
\Phi(n)_{2n-k, k} = \begin{cases} w_n & k = \text{odd} \\ w_n & k = \text{even} \end{cases} , k = 1 \ldots n
$$

59

$$\Phi(n)_{k,2n+1-k} = \begin{cases} iy_k^{-1} & n = \text{odd} \\ iy_k & n = \text{even} \end{cases} , k = 1 \ldots n - 1$$

$$\Phi(n)_{2n+1-k,k} = \begin{cases} iz_n^{-1} & k = \text{odd} \\ iz_n & k = \text{even} \end{cases} , k = 1 \ldots n - 1$$

$$\Phi(n)_{n+1,n+1} = \begin{cases} x_n & n = \text{odd} \\ x_n^{-1} & n = \text{even} \end{cases}$$

$$\Phi(n)_{2n-k,2n+k} = \begin{cases} iy_n^{-1} & k = \text{odd} \\ iy_n & k = \text{even} \end{cases} , k = 1 \ldots n - 1$$

$$\Phi(n)_{2n+k,2n-k} = \begin{cases} iz_k^{-1} & n = \text{odd} \\ iz_k & n = \text{even} \end{cases} , k = 1 \ldots n - 1$$

$$\Phi(n)_{2n+k,2n+1-k} = \begin{cases} x_k & n = \text{odd} \\ x_k^{-1} & n = \text{even} \end{cases} , k = 1 \ldots n - 1$$

$$\Phi(n)_{2n+1-k,2n+k} = \begin{cases} x_n & k = \text{odd} \\ x_n^{-1} & k = \text{even} \end{cases} , k = 1 \ldots n - 1$$

This is how we reduce the matrix when n is even:

1. $(-i)w_n^{-1}z_n^{-1} \times$ (Row $2n - 1$) is added to (Row $2n$).

2. $(-i)w_1 y_1 \times$ (Column $2n - 1$) is added to (Column $2n$).

3. $(-1) \times$ (Row $2n - 2$) is added to (Row $2n$).

4. $(-1)w_1 w_2 y_1 y_2 \times$ (Column $2n - 2$) is added to (Column $2n$).

5. $(+i)w_n^{-1}z_n^{-1} \times$ (Row $2n - 3$) is added to (Row $2n$).

6. $(+i)w_1 w_2 w_3 y_1 y_2 y_3 \times$ (Column $2n - 3$) is added to (Column $2n$).

$\ldots \ldots$

$$\left\{ (-i)^k \begin{array}{l} k = \text{odd}: w_n^{-1}z_n^{-1} \\ k = \text{even}(-1)^k \end{array} \right\} \times \text{(Row } 2n - k) \text{ is added to (Row } 2n).$$

$(-i)^k w_1 w_2 \ldots w_k y_1 y_2 \ldots y_k \times$ (Column $2n - k$) is added to (Column $2n$).

. . . . . .

$2n - 1.$ $\prod_{k=1}^{n}(-iw_k y_k) \times$ (Row n) is added to (Row $2n$).

$2n.$ $(-i)^n \times$ (Column n) is added to (Column $2n$).

This is how the reduction runs when n is odd:

1. $(-i)w_n z_n \times$ (Row $2n - 1$) is added to (Row $2n$).

2. $(-i)w_1^{-1} y_1^{-1} \times$ (Column $2n - 1$) is added to (Column $2n$).

3. $(-1) \times$ (Row $2n - 2$) is added to (Row $2n$).

4. $(-1)(w_1 w_2 y_1 y_2)^{-1} \times$ (Column $2n - 2$) is added to (Column $2n$).

5. $(+i)w_n z_n \times$ (Row $2n - 3$) is added to (Row $2n$).

6. $(+i)(w_1 w_2 w_3 y_1 y_2 y_3)^{-1} \times$ (Column $2n - 3$) is added to (Column $2n$).

. . . . . .

$$\left\{ \begin{array}{l} k = \text{odd:}(-i)^k w_n z_n \\ k = \text{even:}(-1)^k \end{array} \right\} \times \text{(Row } 2n - k) \text{ is added to (Row } 2n).$$

$(-i)^k (w_1 w_2 \ldots w_k y_1 y_2 \ldots y_k)^{-1} \times$ (Column $2n - k$) is added to (Column $2n$).

. . . . . .

$2n - 1.$ $\prod_{k=1}^{n}(-iw_k y_k)^{-1} \times$ (Row n) is added to (Row $2n$).

$2n.$ $(-i)^n w_n z_n \times$ (Column n) is added to (Column $2n$).

After all this row reduction, we obtain the result: if n is even,

$$
\left[
\begin{array}{ccccccc|c|cccccc}
 & & & & & w_1^{-1} & & & & & & & & \\
 & & & & w_2^{-1} & & & & & & & & & \\
 & & & \ddots & & & & & & & & & & \\
 & & w_n^{-1} & & & & & & & & & & & \\
 & w_n & & & & & & & & & & & & \\
 \ddots & & & & & & & & & & & & & \\
 w_n^{-1} & & & & & & & & & & & & & \\
 w_n & & & & & & & & & & & & & \\
\hline
 & & & & & & & c_n & a_1 & a_2 & a_3 & \cdots & a_{n-1} & 0 \;\cdots \\
\hline
 & & & & & & & b_1 & * & & & & & \\
 & & & & & & & b_2 & & * & & & & \\
 & & & & & & & b_3 & & & * & & & \\
 & & & & & & & \vdots & & & & * & & \\
 & & & & & & & b_{n-1} & & & & & * & \\
 & & & & & & & 0 & & & & & & * \\
 & & & & & & & \vdots & & & & & & \;* \\
\end{array}
\right] ;
$$

where

$$
a_j = (x_j^{-1} + y_j z_j w_j) \prod_{k=1}^{j-1} (-i w_k y_k),
$$

$$
b_{2j+1} = (-1)^j (x_n + y_n^{-1} z_n^{-1} w_n^{-1}),
$$

$$
b_{2j} = (-1)^j i (y_n + x_n^{-1} z_n^{-1} w_n^{-1}),
$$

$$
c = -(y_n + x_n^{-1} z_n^{-1} w_n^{-1}) \prod_{j=1}^{n-1} (w_j y_j);
$$

if n is odd,

$$
\left[
\begin{array}{ccccc|c|cccccc}
 & & & & w_1 & & & & & & & \\
 & & & w_2 & & & & & & & & \\
 & & \cdot & & & & & & & & & \\
 & \cdot & & & & & & & & & & \\
 & w_n & & & & & & & & & & \\
 w_n^{-1} & & & & & & & & & & & \\
 \cdot & & & & & & & & & & & \\
 & w_n^{-1} & & & & & & & & & & \\
 w_n & & & & & & & & & & & \\
\hline
 & & & & & \bar{c}_n & \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_{n-1} & 0 & \cdots \\
\hline
 & & & & & b_1 & * & & & & & & \\
 & & & & & b_2 & & * & & & & & \\
 & & & & & b_3 & & & * & & & & \\
 & & & & & \vdots & & & & * & & & \\
 & & & & & b_{n-1} & & & & & * & & \\
 & & & & & 0 & & & & & & * & \\
 & & & & & \vdots & & & & & & & *
\end{array}
\right] ;
$$

where,

$$
\bar{a}_j = (x_j + y_j^{-1} z_j^{-1} w_j^{-1}) \prod_{k=1}^{j-1} (-i w_k^{-1} y_k^{-1}),
$$

$$
b_j = \text{(as above)},
$$

$$
\bar{c} = (x_n + y_n^{-1} z_n^{-1} w_n^{-1}) \prod_{j=1}^{n-1} (w_j^{-1} y_j^{-1}).
$$

We will simply note that the numbers a_j make it possible for us to augment the resulting P-K diagram and hence enter analysis as before:

Here, we have, if we define $f_n(t)$ as before (cf. sections 3.4, 2.2 and 2.3),

$$\frac{pad(n;)}{pad(n-1;)} = \begin{cases} |w_1 w_2 \ldots w_n[(x_n + y_n^{-1}z_n^{-1}w_n^{-1})f_n^+(-i) + (x_n^{-1}z_n^{-1}w_n^{-1} + y_n)f_n^-(-i)]|, & n = \text{odd}; \\ |w_1^{-1}w_2^{-1} \ldots w_{n-1}^{-1}[(x_n + y_n^{-1}z_n^{-1}w_n^{-1})f_n^+(-i) + (x_n^{-1}z_n^{-1}w_n^{-1} + y_n)f_n^-(-i)]|, & n = \text{even}. \end{cases} \tag{3.13}$$
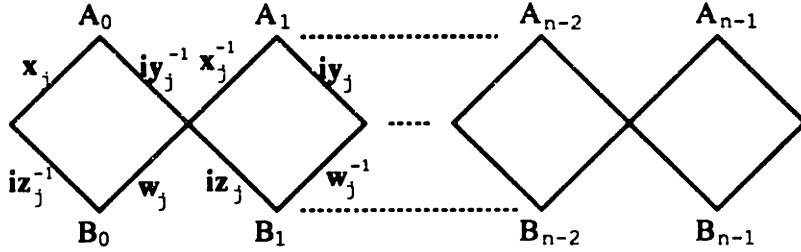

Figure 3-7: Rows j-1 and j of the KP diagram of $PAD(n)$

For the two adjacent rows depicted in the figure above, we have:

$$x_j A^+ + iy_j tA^- = -(iz_i^{-1}B^+ + w_j^{-1}B^-), \tag{3.14}$$

$$iy_j t^{-1} A^+ + x_j^{-1} A^- = -(w_j tB^+ + iz_j B^-); \tag{3.15}$$

or, written in matrix form:

$$\begin{pmatrix} B^+ \\ B^- \end{pmatrix} = \frac{-1}{1+t^2} \begin{pmatrix} -i(x_j z_j - y_j^{-1}w_j^{-1}t^2) & t(y_j z_j + x_j^{-1}w_j^{-1}) \\ t(x_j w_j + y_j^{-1}z_j^{-1}) & -i(x_j^{-1}z_j^{-1} - y_j w_j t^2) \end{pmatrix} \begin{pmatrix} A^+ \\ A^- \end{pmatrix}. \tag{3.16}$$

So, for the case of odd $n = 2m + 1$ we can guess the middle row to be

$$\left(-\prod_{j=1}^m (y_j^{-1}w_j^{-1}) \right) (-i(1+t^2))^m.$$

Hence, after cancelling factors of $(1 + t^2)$ and carrying out the substitution $t = -1$, (we

64

define $E_j \equiv 1 + x_j y_j z_j w_j$, $F_j \equiv x_{j+1} z_j + w_j y_{j+1}$):

$$\frac{PAD(2m+1;)}{PAD(2m;)} = w_1 w_2 \ldots w_{2m+1} z_{2m+1}^{-1} (y_{2m+1}^{-1} z_{2m+1}^{-1} w_{2m+1}^{-1}, x_{2m+1}^{-1} z_{2m+1}^{-1} w_{2m+1}^{-1})$$

$$\times \prod_{j=2m}^{(m+1)} \left[E_j \begin{pmatrix} y_j^{-1} w_j^{-1} & x_j^{-1} w_j^{-1} \\ y_j^{-1} z_j^{-1} & x_j^{-1} z_j^{-1} \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \prod_{j=1}^{m} (y_j w_j)^{-1}$$

If we take into account that

$$\begin{pmatrix} y_j^{-1} w_j^{-1} & x_j^{-1} w_j^{-1} \\ y_j^{-1} z_j^{-1} & x_j^{-1} z_j^{-1} \end{pmatrix} \begin{pmatrix} w_{j-1}^{-1} \\ z_{j-1}^{-1} \end{pmatrix} = (y_j^{-1} w_{j-1}^{-1} + x_j z_{j-1}^{-1}) \begin{pmatrix} w_j^{-1} \\ z_j^{-1} \end{pmatrix},$$

we can simplify the last expression to give

$$\frac{pad(2m+1;)}{pad(2m;)} = w_{m+1} \ldots w_{2m+1} z_{2m+1}^{-1} \left[\prod_{j=m+1}^{2m+1} E_j \prod_{j=m+1}^{2m} (w_j^{-1} y_{j+1}^{-1} + z_j^{-1} x_{j+1}^{-1}) \right] y_1^{-1} \ldots y_{m+1}^{-1}$$

$$= \prod_{j=m+1}^{2m+1} E_j \prod_{k=m+1}^{2m} F_k \prod_{k=m+2}^{2m+1} x_k^{-1} \prod_{j=1}^{2m+1} y_j^{-1} \prod_{k=m+1}^{2m+1} z_k^{-1}. \tag{3.17}$$

When the order is even ($n = 2m$) we can guess (as in the previous section) that the lower-middle-row of the K-P diagram is

$$A = \begin{pmatrix} A^+ \\ A^- \end{pmatrix} = \begin{pmatrix} -y_m z_m \left[\prod_{j=1}^{m-1}(iy_j z_j) \right] (1+t^2)^{m-1} \\ \left[\prod_{j=1}^{m}(iy_j w_j) \right] t(1+t^2)^{m-1} \end{pmatrix}.$$

After we finish all our operations we will get

$$\frac{pad(2m;)}{pad(2m-1;)} \tag{3.18}$$

$$= w_1^{-1} \ldots w_{2m-1}^{-1} E_{2m} z_{2m}^{-1} w_{2m}^{-1} (y_{2m}^{-1}, x_{2m}^{-1})$$

$$\left[\prod_{j=2m+1}^{(m+1)} E_j \begin{pmatrix} y_j^{-1} w_j^{-1} & x_j^{-1} w_j^{-1} \\ y_j^{-1} z_j^{-1} & x_j^{-1} z_j^{-1} \end{pmatrix} \right] \begin{pmatrix} z_m y_m \\ w_m y_m \end{pmatrix} \prod_{j=1}^{m-1} (y_j w_j)$$

$$= w_{m+1}^{-1} w_{m+2}^{-1} \ldots w_{2m}^{-1} \prod_{j=m+1}^{2m} E_j y \times y_1 \ldots y_m z_m z_{2m}^{-1} \prod_{j=m+1}^{2m} (y_j^{-1} w_{j-1}^{-1} + x_j^{-1} z_{j-1}^{-1})$$

65

$$= \prod_{j=m+1}^{2m} E_j \prod_{k=m}^{2m-1} F_k \prod_{j=m+1}^{2m} x_j^{-1} \prod_{k=m+1}^{2m} (y_k^{-1} y_{2m+1-k}) \prod_{j=m+1}^{2m} z_j^{-1} \prod_{k=m+1}^{2m} (w_j^{-1} w_{j-1}^{-1}).$$

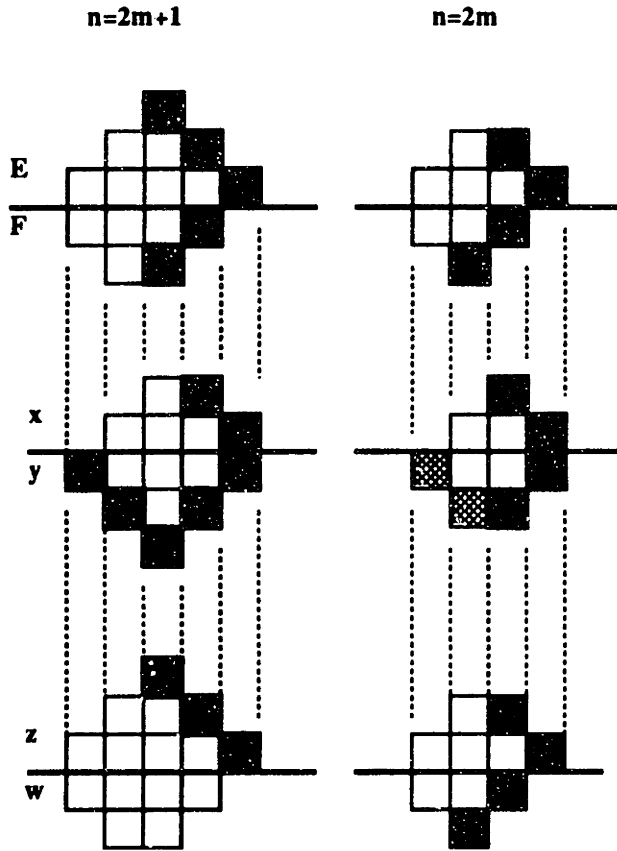Taking the last two together[4], we get

n=2m+1 **n=2m**

Figure 3-8: How the various terms change as order goes up

Theorem 3.2

$$PAD(2m; x, y, z, w) = \prod_{j=1}^{2m} \left(\frac{1 + x_j y_j z_j w_j}{z_j w_j} \right)^{\max(j, 2m+1-j)}$$
$$\prod_{k=1}^{2m-1} \left(\frac{x_{k+1} z_k + y_{k+1} w_k}{x_{k+1} y_{k+1}} \right)^{\max(k, 2m-k)} \tag{3.19}$$

---

[4]see next figure for a pictorial demonstration of how the exponents of various terms shift as we go up from order to order.

$$PAD(2m+1; x, y, z, w) = \prod_{j=1}^{2m+1} \left(\frac{1 + x_j y_j z_j w_j}{y_j z_j} \right)^{\max(j, 2m+2-j)}$$

$$\prod_{k=1}^{2m} \left(\frac{x_{k+1} z_k + y_{k+1} w_k}{x_{k+1} w_k} \right)^{\max(k, 2m+1-k)} \tag{3.20}$$

An interesting fact about these numbers: for the analogous Stanley's formula in the case of **Aztec Diamond** tilings, all $4n$ variables x_j, y_j, z_j, w_j are independent, and essentially there are $4n$ degrees of freedom in determining the generating function. In this formula, there are actually only $2n - 1$ independent variables corresponding to the factors between the big parentheses. There are constructions with more degrees of freedom but they are not included here because they have little symmetry to speak of.

(In case the reader have some trouble figuring out why these factors lead to factors of 5 in the original problem, it is necessary to point out that when we put the graph for the transformed version of the **Penta Aztec Diamond** into the form used in the generating function, some of the variables will be $\sqrt{2}$ and some $1/\sqrt{2}$. As a result of all this, all the E_j's will be 2, and all the F_k's $\frac{5}{2}$, hence the powers of 5.)

What this means is that there is less information available in a systemic manner from the matchings of this family of graphs. This is not at all surprising, since it is quite in line with the fact that this generating function come from a graph without the horizontal reflexive symmetry in the graph associated with Stanley's formula.

Thus concludes our little study of this family of tilings.

It is to be noted that attempted proofs by shuffling does not yield any useful results. Perhaps it is possible, but it has so far eluded the author.

Chapter 4

The Symmetric Tilings of the Aztec Diamond

4.1 Overview

From the problem of enumerating plane partitions, combinatorists have gone on to the problems of enumerations of partitions with certain symmetries. A systemic study of enumeration of the symmetry classes of plane partitions using the same methodology of this paper can be found in [5]. In an analogous manner, one immediate question to be asked after studying the tilings of **Aztec Diamonds** is the following: What about tilings with certain symmetries? There are five symmetry classes of **Aztec Diamond** tilings;

1. All tilings. $(ad(n))$

2. C_2: Centrally Symmetric tilings (the number of the tilings will be hereafter referred to as $ad_2(n)$).)

3. R_2: Reflectively Symmetric tilings (to one of the coordinate axes). [1]

4. K_4: Tilings invariant under reflections with respect to both coordinate axes.

5. C_4: Tilings invariant under rotation by 90 degrees.

---

[1] In crystography, R is used to denote reflective symmetries that change the parity of the coordinate system, as opposed to C_2 which is a rotation and preserves the outer product.

It happens that the last three cannot be represented as tilings of bipartite graphs, and hence cannot be treated using only methods discussed here. As a matter of fact, if these last three have express formulas, they will be of *non-closed form* construction. The second problem however has a closed form solution and is the main subject of this chapter.

Let me acknowledge here that much of this material is obtained in collaboration with William Jockusch.

The main result, is the following:

Theorem 4.1 (Conjectured by W. C. Jockusch, 1990) *The number of centrally symmetric tilings of the order n Aztec Diamond is given by:*

$$ad_2(2n) = 2^n ad_2(2n - 1) \tag{4.1}$$

$$ad_2(4n - 1) = 2^{2n^2 - 2n + 1} \frac{H_4(4n + 3)H_4(4n - 1)(H(n)H(n - 1))^2}{(H_2(2n - 1)H_2(2n + 1))^3} \tag{4.2}$$

$$ad_2(4n + 1) = 2^{2n^2 + 1} \frac{(H_4(4n + 3))^2(H(n))^4}{(H_2(2n + 1))^6} \tag{4.3}$$

Here $H_j(n)$, the step-factorial function, is defined as

$$H_j(n) \equiv \prod_{1 \leq k < n/j} (n - jk)!$$

and $H(n) \equiv H_1(n)$ (definition by G. Kuperberg in [5]).

Or, to phrase it a bit differently,

$$\frac{ad_2(4n + 3)}{ad_2(4n + 1)} = 2^{2n} \frac{(4n + 3)!(n!)^2}{((2n + 1)!)^3}$$

$$= 2^{2n-1} \frac{(4n + 3)!!}{((2n + 1)!!)^2}; \tag{4.4}$$

$$\frac{ad_2(4n + 1)}{ad_2(4n - 1)} = 2^{2n} \frac{(4n - 1)!((n - 1)!)^2}{((2n - 1)!)^3}$$

$$= 2^{2n-1} \frac{(4n - 1)!!}{((2n - 1)!!)^2}$$

$$= 4 \frac{ad_2(4n - 1)}{ad_2(4n - 3)} \tag{4.5}$$

We begin with some generalities. As before, we seek to obtain a recursive formula for $ad_2(n)$ by using the Permanent-Determinant method.

Consider the connectitivy graph $G(AD(n))$. Suppose we identify each vertex in the lower half with its image under $180°$ rotation in the top half. We rotate the figure by 45^0 so that the top of the resulting pyramid shape is on the upper-left corner, and number the vertices as follows: the row-vertices start from the extreme upper-right tip, and go counter-clockwise on the outside and repeat until all row-vertices are numbered. The column-vertices start from the lower-left and go clockwise in symmetry to the above.

The alternating modification is a bit more awkward. The approach used for $AD(n)$ is not quite usable in the sense that there will be a problem with the symmetry. Therefore, we adopt the following weights:

1. All edges running $\nearrow$ or $\diagup$ are given weight 1.

2. All edges running $\diagdown$ from row-vertex to column-vertex are given weight 1.

3. All edges running $\diagdown$ from row-vertex to column-vertex are given weight -1.

4. The two vertices in the center are connected by two edges of weight 1, with the other edges running outside alternating in sign.

(See figure 4.1.)

Before we carry out the calculations, please notice that the Equation 4.1 is easily obtained from a shuffling argument as set out in the preliminary chapter. However, further progress via an shuffling argument proved impossible despite many attempts, because of the "singularity" at the center of the **Aztec Diamond**. This also will be an important point later in the chapter. We will from now on use the notation $AD_2(n)$ to denote the weighted graph we just obtained, and $\Psi(n)$ its connectivity matrix.

Figure 4-1: The alternating modificated graph $AD_2(6)$

4.2 The preliminary calculations of reducing $\Psi(n)$

As a result of the numbering made in the last section, we find that $\Psi(n) =$

$$
\left(
\begin{array}{ccccc|c|c}
(-1)^{n+1} & & & & & 1 & \cdots \\
 & & & & 1 & 1 & \\
 & & & \cdot & \cdot & & \\
 & & & \cdot & \cdot & & \\
 & & 1 & 1 & & & \\
 & \cdot & -1 & 1\square & & & 1 \\
 & \cdot & \cdot & & & & \cdot \ 1 \\
 & \cdot & \cdot & & & & \cdot \cdot \\
 1 & -1 & & & & & 1 \ 1 \\
\hline
 1 & -1 & & & & (-1)^n{\star} & 1 \\
\hline
 & & & -1 & 1 & * \\
 & & -1 & 1 & & & * \\
 & & \cdot & \cdot & & & * \\
 & & \cdot & \cdot & & & * \\
 \vdots & & -1 & 1 & & & \ddots
\end{array}
\right)
$$

(In the matrix $\star$ is again position $(2n-1, 2n-1)$ and $\square$ is $(n+1, n+1)$.)

And the reduction process (very similar to the ones used before) proceeds thus:

1. Add $(-1)^n \times$ Row 1 to Row $2n-1$.

2. Add $(-1)^n \times$ Column 1 to Column $2n-1$.

3. Add Row $2n-2$ to Row $2n-1$.

4. Subtract Column $2n-2$ from Column $2n-1$.

5. Add Row $2n-3$ to Row $2n-1$

6. Add Column $2n-3$ to Column $2n-1$.

$\ldots$

Add Row $2n-k$ to Row $2n-1$.

Add Column $2n-k$ times $(-1)^{k+1}$ to Column $2n-1$.

$\ldots$

Add Row n to Row $2n-1$.

Add Column n times $(-1)^{n-1}$ to Column $2n-1$.

After the reduction we obtain the reduced matrix

$$
\left(
\begin{array}{c|c|c}
\begin{matrix} (-1)^{n+1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix} & & \\
\hline
& 4(-1)^n & \begin{matrix} 2 & 2 & \ldots & 2 & 0 & \ldots \end{matrix} \\
\hline
& \begin{matrix} 2 \\ -2 \\ \vdots \\ 2(-1)^{n-3} \\ 0 \\ \vdots \end{matrix} & \begin{matrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \\ & & & & & \ddots \end{matrix}
\end{array}
\right) .
$$

We can immediately make the following observation: to reduce this matrix, it is necessary, as before (cf. section 2.2), to fill in the P-K diagram. We give a few examples in figure 4-2. However, this diagram does not even come close to being easily worked on, so we are going to modify it as follows:

1. A row of 1's is extended on the left hand side.

2. The whole connectivity graph of $AD(n)$ is drawn and the numbers from the P-K diagram of $ad_2(n)$ entered on the appropriate vertices.

3. The number on each column-vertex in the P-K diagram is copied on its image under 180° rotation, differing by at most a sign.

4. In the extension, for even n, we make the top row anti-symmetric, the next row symmetric, etc; and vice versa, that is top row symmetric, next row anti-symmetric, etc, for odd n.

Here, we observe that the main reason that so much symmetry exists in the extended P-K diagram is that we could have re-written the relations in the (unextended) P-K diagram in such a fashion that the numbers shown will be written on the vertices in the upper half of the
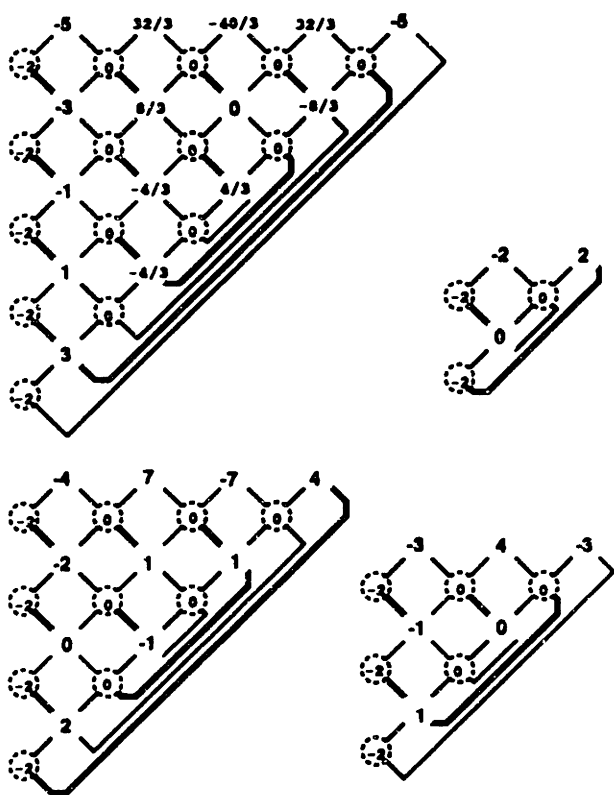
73

Figure 4-2: P-K diagrams for symmetric tilings of **Aztec Diamonds**, order 4 to 7

Aztec Diamond, instead of the upper-left half, thus obtaining an transposed version of the same diagram. (See figure 4-4 and 4.2 for an illustration of the process). It will then be clear then that reflecting the whole diagram with respect to the middle vertical line followed by a little shuffling of signs would get us the flux relations satisfied by (mostly[2]) the same numbers. The same goes for vertical symmetries. Thus we can show that not only that all rows are symmetric or anti-symmetric, we also have that all columns in the K-P diagram of an odd-ordered ad_2 will be symmetric or antisymmetric, and that all columns in the K-P diagram of an even-ordered ad_2 will either be symmetric or anti-symmetric except for the central element.

We can check that the diagram thus extended has zero flux on all row-vertices with very few exceptions. More specifically, the exceptions are all restricted to the right-hand half of one row(the middle row for odd orders and the lower-middle row for even orders), and then for the odd-ordered ones only half of it, namely the vertices that are even-numbered counted from the

---

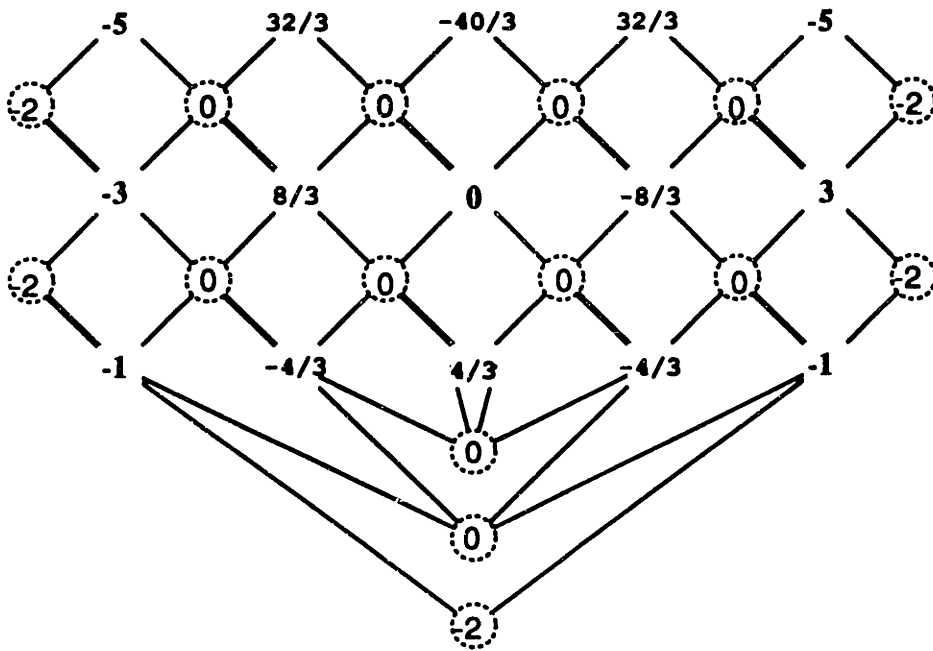[2]There is a sign change for some entries.

74

Figure 4-3: Transposed P-K graph for the symmetric tilings of **Aztec Diamond** order 7

left: See the figures 4-6 and 4-5.

It is easy to show that it is impossible to construct a diagram such that all flux relations are zero since that will over-determine the diagram.

Also, we can check that the number of degrees of freedom is correct: there are always $\binom{n}{2}$ independent entries in the augmented diagram and $\frac{(n+1)(n-2)}{2}$ relations and hence the diagram is determined up to a constant factor. In this calculations, of course the upperleft corner is always 1 since that's how we extended it. Again we will call the polynomial corresponding to the top row of the extended K-P diagram $g_n(t)$. And of course, we have:

$$\frac{ad_2(n)}{ad_2(n-2)} = |g_n(-1)|. \tag{4.6}$$

If $A(t)$ and $B(t)$ correspond to two adjacent rows, we have

$$\frac{A(t)}{B(t)} = \frac{1-t}{1+t},$$

unless the row of row-indices between A and B is the one where the exceptions to the flux
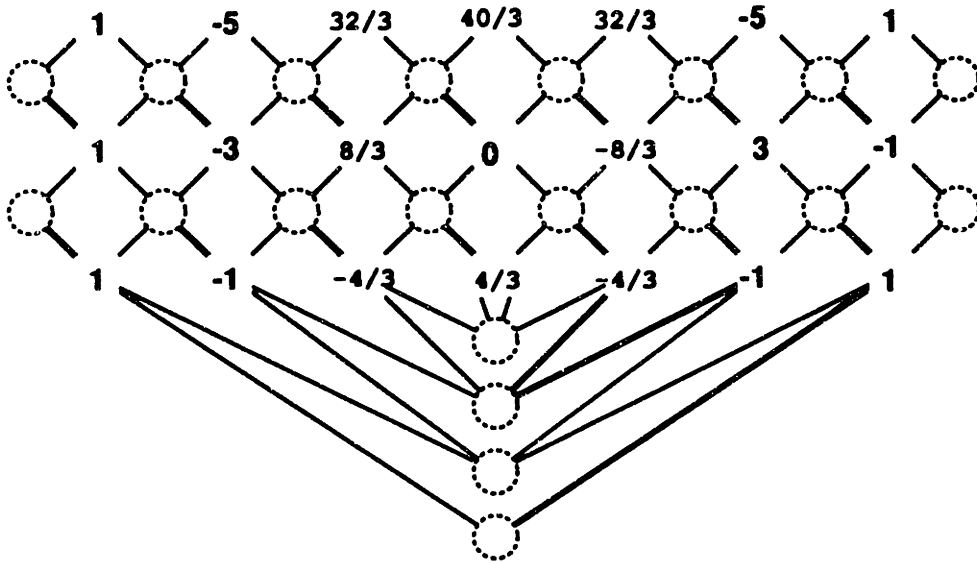
Figure 4-4: The last graph with the extended rows on either side

relations occur.

Lemma 4.2.1 *For $j = 0, 1, 2$ we have*

$$g_{4n+j}(t) = (1 - t)^j g_{4n-1}(t) \tag{4.7}$$

Proof: We can simply enumerate the cases and see that when $g_m(t)$ leads to a consistent PK diagram for $ad_2(m)$, $g_{m+1}(t) \equiv g_m(t)(1 - t)$ leads to a consistent diagram for $ad_2(m + 1)$ except when $m = 4k + 2$. When m is odd, it is easily seen as follows: we insert the row of numbers corresponding to $(1 - t) \times g_m(t)$ at the top and fill the rest of the upper half (down to and including the middle row) of the extended P-K diagram for $ad_2(m + 1)$ row by row, each time using the equation above to determine the next row.

Now, using the symmetry of the columns, we know that the bottom row of the original order m diagram corresponds to $g_m(-t)$. We fill in the bottom row with $g_m(-t)(1 + t)$ and
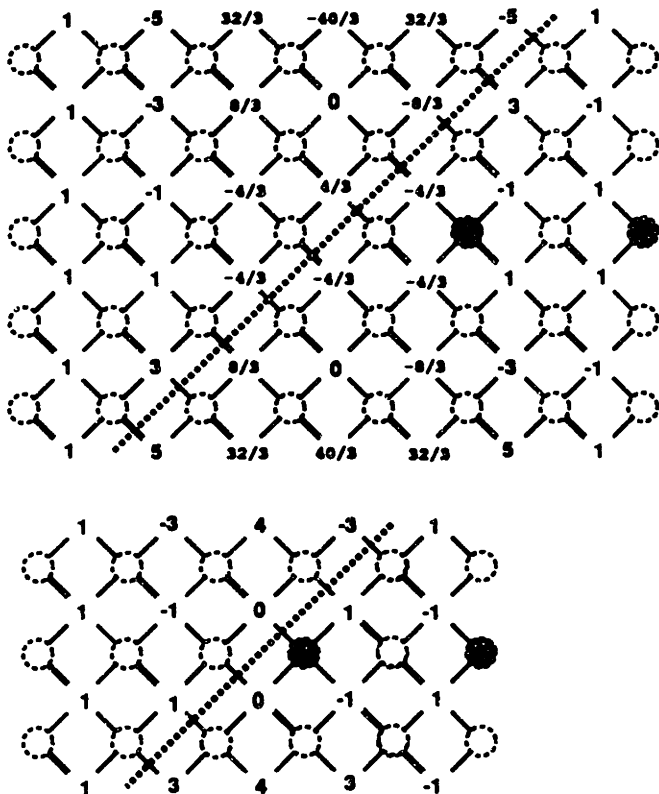
Figure 4-5: Augmented P-K diagrams for symmetric tilings of Aztec Diamonds, odd order

then proceed to fill in the rest of the rows upwards. Now all we have to do is check that the flux relations in the left-hand side between the middle row and the below-middle rows hold. Since these two rows correspond by construction to the two orignal center rows multiplied by $1 + t$, we can deduce that the flux relations in question (which reduce to the flux relations in the original graph) hold also.[3]

When $m = 4k$ the same argument will cover everything except the one flux relation that corresponds to the lower-central vertex in the original Aztec Diamond connectivity graph. This flux relation is dealt with as follows: the column containing the left-center column-veitex is anti-symmetric by our reasoning given above, so the number on this vertex is 0 and hence by symmetry the number on the other central column-vertex. This gives us one extra flux relation,

---

[3]Consider the original relations as corresponding to a statement that certain terms of a polynomial, including all the terms which are of degree no higher than half the degree of the polynomial, are zero. As a immediate consequence, all the terms of this polynomial times $(1 + t)$ which are of degree no higher than half the degree of the original polynomial vanishes. This correspond to the flux relations in the "problem row" we are investigating.
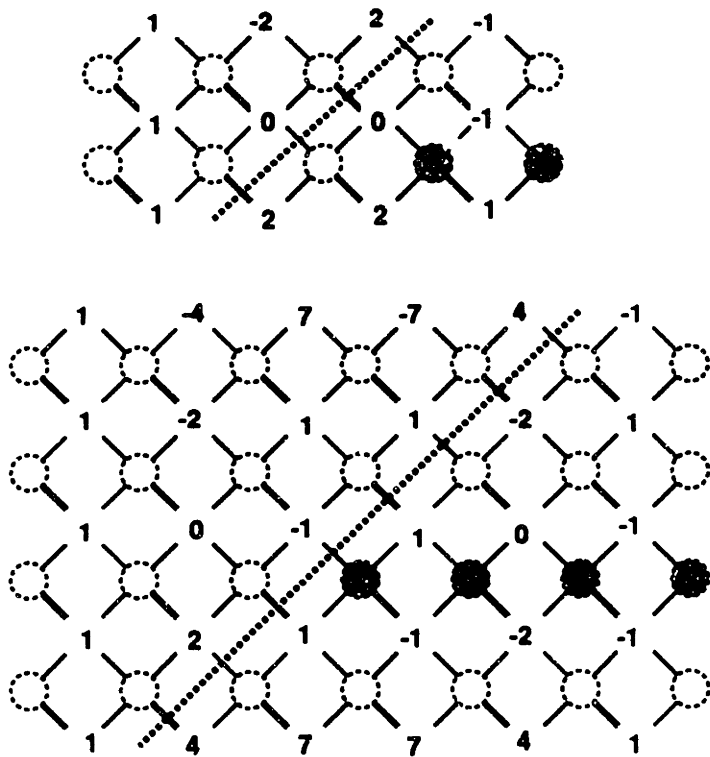
77

Figure 4-6: Augmented P-K diagrams for symmetric tilings of **Aztec** Diamonds, even order

and so the same reasoning will cover this case. □

As a result, our problem is to find $g_{4n-1}(t)$ or rather to find $g_{4n-1}(-1)$.

4.3 The Final Reduction

We define the above-center row of the P-K diagram for $ad_2(4n + 3)$ to correspond to the polynomial $f_n(t)$ of degree $4n + 2$, and also make another definition

$$h_k(t) \equiv \frac{g_{4k+1}(t)}{(1 - t)^{2k}}.$$

It is an immediate consequence from the symmetry of the diagram that both f_n and h_n are palindromic.

Similar to what we had before we have (cf. Equation 2.3):

$$\frac{g_{4n+3}(t)}{f_n(t)} = \left(\frac{1-t}{1+t}\right)^{2n}.$$

From the symmetry, we know that the below-center row corresponds to $f_n(-t)$.

It is seen that $f_n(t)$ satisfies the condition that

$$f_n(t)(1+t) - f_n(-t)(1-t)$$

(which is a odd polynomial by definition) has no terms of degree lower than $2n + 1$.

All these mean that we need to find a set of numbers $a_0 = 1, a_2, a_4 \ldots a_{2n}$ (which are the co-efficients of f_n) such that

$$(1+t)^{2n} | \sum_{j=0}^{n-1} a_{2j}(t^{2j} - t^{2j+1} - t^{4n-2j+1} + t^{4n-2j+2}) + a_{2n}(t^{2n} - t^{2n+1} + t^{2n+2}) \qquad (4.8)$$

This leads to a set of very symmetric equations that is elegant but not a particularly nice set of equations to solve at all!

Finally, it is seen that since f_n, and hence h_n is palindromic, it is advisable to look at one of the possible explicitly palindromic expansions of $h_n(t)$. More precisely, the expansion eventually considered is

$$h_n(t) = \sum_{j=-1}^{n} r_{n,j}(1-t)^{2n-2j} t^{1+j},$$

which would mean that

$$\frac{ad_2(4n+3)}{ad_2(4n+1)} = 2^{2n}(-1)^{n+1} \sum_{j=-1}^{n} r_{n,j}(-4)^{n-j}$$

And then we computed numerically that

$$r_{n,-1} = 1$$
$$r_{n,0} = -1$$
$$r_{n,1} = \frac{n}{2n+1}$$

$$r_{n,2} = \frac{n-1}{2n+1}$$

$$r_{n,n} = \frac{1}{2n+1} \qquad (4.9)$$

along with other special values of $r_{n,k}$.

After a lot of guesswork, it was guessed[4] that

$$r_{n,j} = (-1)^j \binom{2n-j+1}{j+1} \frac{(2j-1)!!(2n-2j-1)!!}{(2n+1)!!} \qquad (4.10)$$

Please note that under the usual convention that $(-1)!! = 1$ and $(-3)!! = -1$ the formula does agree with the equations 4.9.

We showed in the preliminary chapter using the WZ technique(see Section 3, Equation 1.7)

$$
\begin{aligned}
& [t^{2(n-m)+1}](f_n(t)(1+t)) \\
= \ & [t^{2(n-m)+1}] \sum_{j=-1}^{n} r_{n,j}(1-t)^{2n-2j}t^{1+j}(1+t)^{2n+1} \\
= \ & [t^{2(n-m)+1}] \sum_{j=-1} n \sum_{k} \binom{2n-j+1}{j+1} \frac{(2j-1)!!(2n-2j-1)!!}{(2n+1)!!}t^{1+j}(1-t^2)^{2n-2j}(1+t)^{2j+1} \\
= \ & \sum_{j=-1}^{n} \sum_{k} (-1)^{j+k} \binom{2n-j+1}{j+1} \frac{(2j-1)!!(2n-2j-1)!!}{(2n+1)!!} \binom{2n-2j}{k} \binom{2j+1}{2n-2m-2k-j} \\
= \ & 0
\end{aligned}
$$

Finally, it was found that

$$(-1)^{n-1} \sum_{j=-!}^{n} r_{n,j}(-4)^{n-j}$$

is equal to

$$\frac{(4n+3)!!}{((2n+1)!!)^2}$$

which was Equation 1.6, also proved in the same section.

Thus our theorem is established.

---

[4]Unfortunately, the obvious conjecture that leaps to one's mind was wrong! Normally, in these cases it is a lot harder to come up with an answer than to prove it. And so it proved here.

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