Construction of Deligne categories through ultrafilters and its applications

by

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 2021

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Abstract

The present thesis is concerned with the study of Deligne categories and their application to various representation-theoretic problems. The lens that is used to view Deligne categories in this study is the one of ultrafilters and ultraproducts. As will be shown in our work, this approach turns out to be a very powerful one. Especially if one wants to solve such representation-theoretic problems as presented by P.Etingof in his papers on "Representation theory in complex rank" ([13, 14]).

The results are presented in two parts. In the first one (Chapters 2 and 3) an introduction to the theory of ultrafilters is given, and the construction of the Deligne categories through ultrafilters is presented. This also allows us to understand how one can make sense of Deligne categories as a limit in rank and characteristic.

The later part of the text describes two applications of this construction to actual representation-theoretic problems. In Chapter 4 the full classification of simple commutative, associative and Lie algebras in $\mathrm{Rep}(S_\nu)$ for $\nu \notin \mathbb{Z}_{\geq 0}$ is stated and proven. The second application, the construction of deformed double current algebras as a space of endomorphisms of a certain ind-object of $\mathrm{Rep}(S_\nu)$, is contained in Chapter 5. There it is also proven that this construction agrees with Guay's deformed double current algebra of type A if the rank $r \geq 4$ (Guay's algebra is presently only defined for such rank), and the presentation by generators and relations for the case of $r = 1$ is given.

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Acknowledgments

The person who provided the most essential help towards the completion of this Thesis is, naturally, my advisor, Pavel Etingof. I would like to express my deep graditude to him for providing me with the problems to work on and guiding me through all steps of the research process, starting from grasping the problem and ending with the editing and the publication of my papers. And I would also like to thank him for his kindness and care in all other matters that are not directly connected with research.

I would also like to thank all my co-authors, both professors and students, with whom I've got to work with during my time at MIT: Merrick Cai, Nate Harman, Lev Kruglyak, Andrei Mandelshtam, Eric Rains, David Vogan and Christopher Xu.

I am also grateful to my fellow graduate students: Aleksandr Petrov, Vasily Krylov and Aleksandra Utiralova for both mathematical and non-mathematical discussions we've got to share with each other, and all the staff of the mathematical department of MIT for all their work that made my life here much easier.

Finally, I would like to thank my parents and especially my mother for first introducing me to mathematics in my childhood and urging me on further on my mathematical journey throughout my adolescence.

Contents

Chapter 1

Introduction

Deligne categories, the main object of the studies contained here, were first introduced by P. Deligne and J. Milne in [8], and then studied in more detail by P. Deligne in his seminal work *La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel* ([7]). In this paper he showed how one can extend the usual categories of representation of symmetric group $\mathbf{Rep}(S_n)$ to the non-integer values of n. Thus, he was able to construct the category $\mathrm{Rep}(S_\nu)$ that will be studied in the present thesis. Moreover, he also provided an outline of the construction of other interpolation categories: $\mathrm{Rep}(GL_\nu), \mathrm{Rep}(SO_\nu)$ and $\mathrm{Rep}(Sp_\nu)$.

Since then the study of these categories was taken on by many mathematicians. A lot of important results on $\mathrm{Rep}(S_\nu)$ were presented in papers [4, 5] by J. Comes and V. Ostrik; the notion of interpolation category was also generalized in many ways through the work of F.Knop in [37, 38] and M. Mori in [42]. However the idea to try and transfer various representation-theoretic structures from finite rank to Deligne categories and study them in that setting, which forms the core of the approach to Deligne categories taken in the present thesis, was first fully expressed in the papers on *Representation theory in complex rank* by P. Etingof ([13, 14]). Since then many of his students worked on the problems originating in this milieu, including, but not limited to: I. Entova-Azenbud ([10, 11, 12]), N. Harman ([30, 31]), C. Ryba ([46, 47]), A. Utiralova ([54, 55]) and the author of the present thesis ([18, 31, 33, 34]).

Mathematicians have also applied different tools to study of Deligne categories. Initially (for examples in the papers by J. Comes and V. Ostrik) one worked with the original construction of $\mathrm{Rep}(S_\nu)$ as a Karoubian envelope of the additive envelope of a certain combinatorial skeletal category. However the solution of representation-theoretical problems in complex rank called for another way of working with Deligne categories. An approach was needed that would allow one to view Deligne categories more directly as the categories that interpolate usual finite rank categories $\mathbf{Rep}(S_n)$. An outline of one such approach that looks at Deligne categories as categories parametrized by the $\mathbb{A}^1$ scheme can be found in [41, 49]. Another approach that views Deligne categories as limits in rank and characteristic of the finite rank categories by using ultrafilters was first outlined by P. Deligne in his letter to V.Ostrik and later discussed in more detail by N. Harman in [30] and his graduate Thesis.

It is this last approach, which the author of the present thesis found most useful in his research, and which will be discussed and applied in what follows. What the reader will find below is the compilation of the three papers written by the author (together with co-authors) during his graduate studies, namely [31, 18] and [34]. All of these papers use the construction of $\mathrm{Rep}(S_\nu)$ through as an ultraproduct in order to solve other representation-theoretic problems. Another paper, [33], which applies the same ideas to the case of the Deligne category $\mathrm{Rep}(GL_\nu)$ and the Yangians in complex rank, is not included here for the sake of space.

In Part I of the thesis, *Construction of Deligne categories through ultrafilters*, the foundation for everything else discussed later is given. In Chapter 2 the notation used through the rest of the text is introduced, the original construction of the Deligne category $\text{Rep}(S_\nu)$ is given, and, finally, an overview of the theory of ultrafilters and ultraproducts together with many examples of their application in algebraic setting is presented. In Chapter 3 the construction of $\text{Rep}(S_\nu)$ as an ultraproduct is explained, together with some other related constructions of the same kind.

After that Part II, *Applications*, leads the reader, as the name suggests, to look on a few applications of the tools presented in Part I to some concrete representation-theoretic problems. The first of them, namely, to classify all possible simple commutative, associative and Lie algebras in $\text{Rep}(S_\nu)$ is discussed in Chapter 4. This can be seen as continuation of the work started by L. Sciarappa in [49], there he classified all simple commutative algebras in $\text{Rep}(S_\nu)$ for transcendental ν[1]. This problem is successfully solved by us in the case of associative and Lie algebras (see Theorems 4.2.1.2 and 4.3.2.5) and the conjecture is stated for the case of Lie superalgebras (see Conjecture 4.4.1.7).

The next application, presented in Chapter 5, is concerned with construction of deformed double current algebras (DDCAs) through Deligne categories. These algebras were studied at length by N. Guay with co-authors (see [27, 24, 25, 26, 28]) and other mathematicians (see for example [53]). These papers gave various presentations of the DDCA associated with $\mathfrak{gl}_r$ for $r \geq 4$. Through Deligne category we succeed, by using the notion of the extended Cherednik algebra recently introduced in [19], in constructing these algebras for any integer rank (i.e. value of r)[2]. In later

[1] N. Harman later outlined in [30], how one can extend Sciarappa's arguments to algebraic ν using the language of ultrafilters

[2] However it should be noted that it is not the only advantage of our approach to DDCA. Another consists in the fact, that by constructing DDCA as a spherical subalgebra of the extended Cherednik algebra in complex rank, we automatically obtain a large family of representations of DDCA. Indeed, any representation M of the extended Cherednik algebra in complex rank gives us a structure of a representation of the DDCA on the space of homomorphisms $\text{Hom}_{\text{Rep}(S_\nu)}(\mathbb{C}, M)$.

sections we prove that for $r \geq 4$ our construction gives the same algebra as the one studied by Guay. In the last sections of this Chapter we also give a presentation of DDCA in rank 1 by generators and relations by showing that this algebra is given by the most general flat filtered deformation of $U(\mathfrak{po})$, a universal enveloping algebra of the Poisson algebra of polynomials on the symplectic plane. Then, at last, we extend these considerations to the case of other, more general DDCA in rank one.

Part I

Construction of Deligne categories through ultrafilters

Chapter 2

Background and definitions

In this chapter we will provide an overview of the two main conceptual building blocks of the present thesis: Deligne Category $\mathrm{Rep}(S_\nu)$ and ultraproducts.

2.1 General notation and definitions

First, however, let us introduce a number of various definitions and pieces of notation that might not be entirely standard.

In what follows we will use a lot of different categories of representations. We will always denote the usual (“finite rank") categories of representations using the boldface font, and use the regular font for the interpolation categories (e.g. $\mathrm{Rep}(S_\nu)$).

For example we will use the following notation for the categories of representations of symmetric groups. For convenience set $\mathbb{F}_0 = \mathbb{Q}$.

Definition 2.1.0.1. By $\mathbf{Rep}(S_n; \Bbbk)$ denote the category of (possibly infinite dimensional) representations of the symmetric group S_n over $\Bbbk$. By $\mathbf{Rep}^f(S_n; \Bbbk)$ denote the full subcategory of finite dimensional representations. Also for $p \geq 0$ set

$\mathbf{Rep}_p(S_n) := \mathbf{Rep}(S_n; \overline{\mathbb{F}}_p)$ and $\mathbf{Rep}^f_p(S_n) := \mathbf{Rep}^f(S_n; \overline{\mathbb{F}}_p)$.

Note that for $p > n$ the latter category is semi-simple and the irreducible objects are the same as in characteristic 0. More precisely, irreducible representations over $\overline{\mathbb{F}}_p$ can be obtained as a reduction modulo p of irreducible $\overline{\mathbb{Q}}$-representations which sit inside irreducible $\mathbb{C}$-representations as a $\overline{\mathbb{Q}}$-lattice of the full rank. Below we will mostly work in the positive characteristic with $p > n$.

Definition 2.1.0.2. By $\mathcal{A}_n$ denote the subgroup of S_n consisting of even permutations.

Now we would like to introduce some notation concerned with Young diagrams.

Definition 2.1.0.3. For a Young diagram λ, by $l(\lambda)$ denote the number of rows of the diagram (the length), by $|\lambda|$ the number of boxes (the weight) and by $\mathrm{ct}(\lambda)$ the content of λ, i.e., $\mathrm{ct}(\lambda) = \sum_{(i,j)\in\lambda}(j-i)$, where (i,j) denotes the box of λ in row i and column j.

Now we can fix notation for irreducible representations of S_n.

Definition 2.1.0.4. For $p = 0$ or $p > n$ and a Young diagram λ such that $|\lambda| = n$ denote by $X_p(\lambda)$ the unique simple object of $\mathbf{Rep}_p(S_n)$ corresponding to λ.

For $n > 0$ and $p \geq 0$ denote by $\mathfrak{h}_n^p \in \mathbf{Rep}_p(S_n)$, or shortly by $\mathfrak{h}_n$ (if there is no ambiguity about the characteristic) the standard permutation representation of S_n.

Also, by *sgn* denote the 1-dimensional sign representation of S_n.

There is an important central element in $\Bbbk[S_n]$:

Definition 2.1.0.5. Denote the central element $\sum_{1\leq i<j\leq n} s_{ij} \in \Bbbk[S_n]$ by Ω_n.

Remark 2.1.0.6. Note that Ω_n acts on $X_p(\lambda)$ by $\mathrm{ct}(\lambda)$.

As another piece of notation, below we will frequently use the following operation on Young diagrams:

Definition 2.1.0.7. For a Young diagram λ and an integer $n \geq \lambda_1 + |\lambda|$ denote by $\lambda|_n$ the Young diagram $(n - |\lambda|, \lambda_1, \ldots, \lambda_{l(\lambda)})$, where λ_i is the length of the i-th row of λ.

In what follows we will often use the language of tensor categories. Here's what we mean by a tensor category (see Definition 4.1.1 in [16]):

Definition 2.1.0.8. A tensor category $\mathcal{C}$ is a $\Bbbk$-linear locally finite abelian rigid symmetric monoidal category, such that $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \simeq \Bbbk$.

We will also fix a notation for the symmetric structure:

Definition 2.1.0.9. For a couple of object X, Y of a tensor category $\mathcal{C}$, we will denote by $\sigma_{X,Y}$ the map from $X \otimes Y$ to $Y \otimes X$, given by the symmetric structure, i.e., the map permuting X and Y. When $Y = X$ we will also denote this map by σ_X. Oftentimes, when the object we are referring to is obvious from the context, we will denote it simply by σ.

We will also use the notion of the ind-completion of a category. For a general category ind-objects are given by diagrams in the category, with morphisms being morphisms between diagrams. However, in the case of a semisimple category there is a slightly more direct description.

Definition 2.1.0.10. For a semisimple category $\mathcal{C}$ with the set of simple objects $\{V_\alpha\}$ for $\alpha \in A$ the category[1] $\mathrm{IND}(\mathcal{C})$ is the category $\mathcal{D}$ with objects $\bigoplus_{\alpha \in A} M_\alpha \otimes V_\alpha$,

[1]We use all uppercase letters to denote IND, so as not to confuse it with the induction functors.

where M_α are (possibly infinite dimensional) vector spaces. The morphism spaces are given by:

$$\mathrm{Hom}_{\mathcal{D}}(\bigoplus_{\alpha\in A} M_\alpha \otimes V_\alpha, \bigoplus_{\beta\in A} N_\beta \otimes V_\beta) = \prod_{\alpha\in A} \mathrm{Hom}_{\mathrm{Vect}}(M_\alpha, N_\alpha).$$

Thus, in this case, we can think of ind-objects as infinite direct sums of objects of $\mathcal{C}$.

Next we would like to explain a way to define an ind-object of $\mathcal{C}$.

Construction 2.1.0.11. Suppose $0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_i \subset \ldots$ is a nested sequence of objects of $\mathcal{C}$. Then their formal colimit, which we denote by X, is an object of $\mathrm{IND}(\mathcal{C})$. We can write it down explicitly in terms of Definition 2.1.0.10.

Indeed, suppose we have $X_i = \bigoplus_{\alpha\in A} M_{i,\alpha} \otimes V_\alpha$. Then it follows that:

$$\bigcup_{i\in\mathbb{N}} X_i = X = \bigoplus_{\alpha\in A} \left(\bigcup_{i\in\mathbb{N}} M_{i,\alpha}\right) \otimes V_\alpha,$$

where $\bigcup_{i\in\mathbb{N}} X_i = \varinjlim X_i$ stands for the colimit along the diagram consisting of points numbered by $\mathbb{N}$ and arrows from i to $i+1$ for all i.

Remark 2.1.0.12. Suppose that X and Y are two objects constructed via Construction 2.1.0.11. Then:

$$\mathrm{Hom}_{\mathrm{IND}(\mathcal{C})}(X, Y) = \varprojlim_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

In case when X is actually an object of $\mathcal{C}$, this simplifies to:

$$\mathrm{Hom}_{\mathrm{IND}(\mathcal{C})}(X, Y) = \bigcup_{j\in\mathbb{N}} \mathrm{Hom}_{\mathcal{C}}(X, Y_j).$$

In other words, X is a compact object of $\mathrm{IND}(\mathcal{C})$.

Example 2.1.0.13. We have $\mathbf{Rep}_p(S_n) = \mathrm{IND}(\mathbf{Rep}_p^f(S_n))$. Indeed, this holds for the representation category of any finite dimensional algebra.

We will also use a notion of a bifiltered algebra below.

Definition 2.1.0.14. A bifiltered vector space V is a vector space together with a collection of subspaces $F^{i,j}V$ for $i, j \in \mathbb{Z}_{\geq 0}$ such that $F^{i,j}V \subset F^{i+1,j}V$ and $F^{i,j}V \subset F^{i,j+1}V$, and there exists a basis of V such that the intersection of this basis with $F^{i,j}V$ gives a basis of $F^{i,j}V$ (i.e. the filtrations $F^{i,\cdot}V$ and $F^{\cdot,j}V$ are compatible).

A bifiltered algebra A is an algebra which is bifiltered as a vector space such that $F^{i,j}A \cdot F^{i',j'}A \subset F^{i+i',j+j'}A$.

This structure also induces a few standard filtrations:

Remark 2.1.0.15. Notice that bifiltered structure on A induces two filtrations on A via restriction. The first one is given by $F_h^i A = F^{i,\bullet}$ and we will call it the horizontal filtration of A. The second one is given by $F_v^i A = F^{\bullet,i}A$ and we will call it the vertical filtration of A.

There is another filtration on A that we will call the total filtration on A. It is given by $F_t^l A = \bigcup_{i+j=l} F^{i,j}A$.

Also it's easy to see that to specify a bifiltration it is enough to specify the horizontal and vertical degree of each generator of A.

2.1.1 Wreath products $S_n \ltimes \Gamma^n$

Later on (to deal with DDCA of rank 1 with non-trivial Γ) we will need to use a certain interpolation of categories of representations of wreath products. Below we will state basic facts about representations of wreath products in finite rank.

Definition 2.1.1.1. For a finite group Γ, consider the action of S_n on Γ^n by permutations. The semidirect product $S_n \ltimes \Gamma^n$ is called the wreath product.

Remark 2.1.1.2. Outside of the present section we will be interested only in $\Gamma \subset \mathrm{SL}(2, \Bbbk)$. However the results stated in the present section hold for any Γ.

We have the following classification of irreducible representations of $S_n \ltimes \Gamma^n$.

Proposition 2.1.1.3. *Suppose $\Bbbk$ is an algebraically closed field of characteristic $char(\Bbbk) = p > n, |\Gamma|$ or $p = 0$. Suppose A is the set of indices which goes over all of the irreducible representations of Γ over $\Bbbk$, i.e., $\{W_\alpha\}_{\alpha \in A}$ is the set of irreducible representations of Γ. Then the set of all irreducible representations of $S_n \ltimes \Gamma^n$ over $\Bbbk$ is in 1-1 correspondence with functions:*

$$\lambda : A \to \text{Partitions},$$

such that $\sum_{\alpha \in A} |\lambda(\alpha)| = n$. The representation corresponding to fixed λ is given by:

$$X_p(\lambda) = \mathrm{Ind}_{(\prod_{\alpha\in A} S_{\lambda(\alpha)}) \ltimes \Gamma^n}^{S_n \ltimes \Gamma^n} (\bigotimes_{\alpha \in A} X_p(\lambda(\alpha)) \otimes W_\alpha^{\otimes |\lambda(\alpha)|}).$$

We will use the notations for the representation categories similar to the case of the symmetric group:

Definition 2.1.1.4. By $\mathbf{Rep}(S_n \ltimes \Gamma^n; \Bbbk)$ denote the category of representations of the wreath product $S_n \ltimes \Gamma^n$ over $\Bbbk$. By $\mathbf{Rep}^f(S_n \ltimes \Gamma^n; \Bbbk)$ denote the full subcategory of finite dimensional representations.

Also for $p \geq 0$ set

$$\mathbf{Rep}_p(S_n \ltimes \Gamma^n) := \mathbf{Rep}(S_n \ltimes \Gamma^n; \overline{\mathbb{F}}_p),\ \mathbf{Rep}_p^f(S_n \ltimes \Gamma^n) := \mathbf{Rep}^f(S_n \ltimes \Gamma^n; \overline{\mathbb{F}}_p).$$

2.2 Deligne category $\mathrm{Rep}(S_\nu)$

In this section we will give a standard definition and discuss a few important properties of the main category used in the present thesis – the Deligne category $\mathrm{Rep}(S_\nu)$. The well known construction that we will present here is due to Deligne [7]. For more on this topic see [4, 6, 5, 13, 14]. In this section we assume that $\Bbbk$ has characteristic 0.

We will start by introducing the system of vector spaces which is going to play a role of the homomorphism spaces in the corresponding skeletal category. Although these spaces are best understood using diagrams, we will omit this for the sake of space. We advise anyone seeing Deligne categories for the first time to see [4] for a much clearer diagrammatic construction of $\mathrm{Rep}(S_\nu)$.

Definition 2.2.0.1. Denote by $\Bbbk P_{n,m}$ a vector space over a field $\Bbbk$ with the basis given by all possible partitions of an $n+m$-element set. Diagrammatically an element of the basis is represented by two rows of $\bullet$'s, the first of length n and the second of length m, where all $\bullet$'s belonging to the same part of the partition are connected by edges. So, in other words, it is a graph on $n+m$ vertices, the set of connected components of which corresponds to a partition of $n+m$ (The graphs with the same set of connected components represent the same basis element).

Define a map $\phi_\nu^{n,m,k} : \Bbbk P_{m,k} \times \Bbbk P_{n,m} \to \Bbbk P_{n,k}$ for $\nu \in \Bbbk$ as follows. Consider two basis elements $\lambda \in \Bbbk P_{n,m}$ and $\mu \in \Bbbk P_{m,k}$. Take a vertical concatenation of the graphical representations of the corresponding partitions (the last one on top) and identify the rows of length m. After this we are left with a partition of three rows of $\bullet$'s of length n, m and k. Now let's denote by $l(\mu, \lambda)$ the number of connected components consisting purely of $\bullet$'s lying in the second row. Also regard a partition of rows n, k consisting of the same connected components as the partition of rows n, m, k but with elements of the second row deleted, and denote it by $\mu \cdot \lambda$. Then $\phi_\nu^{n,m,k}(\mu, \lambda) = \nu^{l(\mu,\lambda)} \mu \cdot \lambda$.

Define $\Bbbk P_n(\nu)$ to be $\Bbbk P_{n,n}$ with a structure of an algebra given by the map $\phi_\nu^{n,n,n}$. This algebra is called the *partition algebra* and it was introduced by Purdon in [45].

Remark 2.2.0.2. The spaces $\Bbbk P_{n,m}$ can be seen as limits of the homomorphism spaces $\mathrm{Hom}_{S_N}(\mathfrak{h}_N^{\otimes n}, \mathfrak{h}_N^{\otimes m})$, where $\mathfrak{h}_N$ is the permutation representation of S_N.

Using this we can define a preliminary skeletal[2] category $\mathrm{Rep}^0(S_\nu; \Bbbk)$:

Definition 2.2.0.3. For $\nu \in \Bbbk$ we denote by $\mathrm{Rep}^0(S_\nu; \Bbbk)$ a skeletal rigid symmetric monoidal $\Bbbk$-linear category with objects given by elements of $\mathbb{Z}_{\geq 0}$, which can be graphically represented by rows of $\bullet$'s, and denoted by $[n]$.

The set of morphisms $\mathrm{Hom}_{\mathrm{Rep}^0(S_\nu;\Bbbk)}([n], [m])$ is equal to $\Bbbk P_{n,m}$ and the composition maps are given by $\phi_\nu^{n,m,k}$.

Tensor product on objects is defined by the horizontal concatenation of rows and on morphisms by the horizontal concatenation of diagrams. All objects $[n]$ are self-dual.

Using this we can define the Deligne category $\mathrm{Rep}(S_\nu; \Bbbk)$ itself:

Definition 2.2.0.4. For $\nu \in \Bbbk$, the Deligne category $\mathrm{Rep}(S_\nu; \Bbbk)$ is the Karoubian envelope of the additive envelope of $\mathrm{Rep}^0(S_\nu; \Bbbk)$.

[2]Here "skeletal" means that all isomorphism classes of objects consist of exactly one object

This means that we add all possible direct sums and direct summands into our category.

Below we will list a few pieces of notation and results concerning Deligne categories. They are well known and can be found for example in [4, 13].

Definition 2.2.0.5. The object [1] is called the permutation representation and is denoted by $\mathfrak{h}$. The object [0] is called the trivial representation and is denoted by $\Bbbk$ (by a slight abuse of notation).

The important properties of $\mathrm{Rep}(S_\nu;\Bbbk)$ are listed below:

Proposition 2.2.0.6. ***a)*** *For $\nu \notin \mathbb{Z}_{\geq 0}$ $\mathrm{Rep}(S_\nu;\Bbbk)$ is a semisimple tensor category.*
b) *For $\nu \notin \mathbb{Z}_{\geq 0}$ simple objects of $\mathrm{Rep}(S_\nu;\Bbbk)$ are in 1-1 correspondence with Young diagrams of arbitrary size. They are denoted by $\mathcal{X}(\lambda)$. Moreover $\mathcal{X}(\lambda)$ is a direct summand in $[|\lambda|]$.*
c) *The categorical dimension of $\mathfrak{h}$ is ν and of $\Bbbk$ is 1.*
d) *All $\mathcal{X}(\lambda)$ are self-dual.*

The Deligne category enjoys a certain universal property:

Proposition 2.2.0.7. *(8.3 in [7]) For any $\Bbbk$-linear Karoubian symmetric monoidal category $\mathcal{T}$, the category of $\Bbbk$-linear symmetric monoidal functors from $\mathrm{Rep}(S_\nu;\Bbbk)$ to $\mathcal{T}$ is equivalent to the category $\mathcal{T}_\nu^f$ of commutative Frobenius algebras in $\mathcal{T}$ of dimension ν. The equivalence sends a functor F to the object $F(\mathfrak{h})$.*

The important consequence of this result is that for every commutative Frobenius algebra A in a Karoubian symmetric category $\mathcal{T}$ of dimension ν, we have a symmetric monoidal functor from $\mathrm{Rep}(S_\nu;\Bbbk)$ to $\mathcal{T}$ which sends $\mathfrak{h}$ to A.

Remark 2.2.0.8. Here by a commutative Frobenius algebra in $\mathcal{T}$ we mean an object A with the following structure. It is an associative commutative algebra with the corresponding algebraic structure given by $\mu_A, 1_A$, and if we define a map:

$$\mathrm{Tr} : A \xrightarrow{1\otimes \mathrm{coev}_A} A \otimes A \otimes A^* \xrightarrow{\mu_A \otimes 1} A \otimes A^* \xrightarrow{\mathrm{ev}_A} \mathbb{1},$$

then the pairing $A \otimes A \xrightarrow{\mu_A} A \xrightarrow{\mathrm{Tr}} \mathbb{1}$ is required to be non-degenerate, i.e., it corresponds to an isomorphism between A and A^* under the identification of the homomorphism space $\mathrm{Hom}_{\mathcal{T}}(A \otimes A, \mathbb{1})$ with $\mathrm{Hom}_{\mathcal{T}}(A, A^*)$.

In the rest of the thesis we will use Deligne categories over the following fields:

Definition 2.2.0.9. For $\nu \in \mathbb{C}$ set $\mathrm{Rep}(S_\nu) := \mathrm{Rep}(S_\nu; \mathbb{C})$. For $\nu \in \overline{\mathbb{C}(\nu)}$ set $\mathrm{Rep}^{\mathrm{ext}}(S_\nu) := \mathrm{Rep}(S_\nu; \overline{\mathbb{C}(\nu)})$.

Remark 2.2.0.10. Note that although $\mathbb{C}$ and $\overline{\mathbb{C}(\nu)}$ are isomorphic as fields, such isomorphism is not canonical. Thus it will be convenient to distinguish them in the following discussions.

2.3 Ultrafilters and ultraproducts

In this section we will introduce the reader to the elements of the theory of ultrafilters and ultraproducts relevant for our presentation. We will start with the definitions of both of these concepts, provide a number of examples and then move on to define a notion of a restricted ultraproduct.

2.3.1 Basic definitions

We will begin by defining what an ultrafilter actually is:

Definition 2.3.1.1. An ultrafilter $\mathcal{F}$ on a set X is a subset of 2^X satisfying the following properties:

- $X \in \mathcal{F}$;
- If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$;
- If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
- For any $A \subset X$ either A or $X \backslash A$ belongs to A, but not both.

For any X, there is an obvious family of examples of ultrafilters. Indeed, taking $\mathcal{F}_x = \{A \in 2^X | x \in A\}$ for any $x \in X$ gives us an ultrafilter. Such ultrafilters are called principal. Using Zorn's lemma one can show that non-principal ultrafilters $\mathcal{F}$ exist iff the cardinality of X is infinite. However the proof is non-constructive.

From now on we will only work with non-principal ultrafilters on $X = \mathbb{N}$.

Definition 2.3.1.2. For the rest of the thesis we will denote by $\mathcal{F}$ a fixed non-principal ultrafilter on $\mathbb{N}$.

Note that it doesn't matter which non-principal ultrafilter to take, and all our results do not depend on this choice. Also note that all cofinite sets belong to $\mathcal{F}$. Indeed, if some cofinite set wouldn't belong to $\mathcal{F}$, it would follow that a finite set belongs to $\mathcal{F}$. But from this one can conclude that $\mathcal{F}$ is a principal ultrafilter for one of the elements of this set.

Throughout our presentation we will use the following shorthand phrase.

Definition 2.3.1.3. By the statement "A holds for almost all n", where A is a logical statement depending on n, we will mean that A is true for some subset of natural numbers U, such that $U \in \mathcal{F}$.

The following is an important lemma describing what happens with the conjuction and disjunction of statements which "hold for almost all n".

Lemma 2.3.1.4. *1) If for two logical statements A and B we know that A holds for almost all n and B holds for almost all n, then $A \wedge B$ holds for almost all n.*

2) If for a finite number of logical statements A_i, for $i \in I$, we know that $\bigvee_{i \in I} A_i$ holds for almost all n, then there is $j \in I$ such that A_j holds for almost all n.

Proof. 1) Indeed, we know that there is a set $U_A \in \mathcal{F}$ such that A holds for all $n \in U_A$, and the corresponding set for B. Now by definition of the ultrafilter $U_A \cap U_B \in \mathcal{F}$, and $A \wedge B$ holds for all $n \in U_A \cap U_B$.

2) Suppose that none of the statements A_i hold for almost all n. This means that the sets on which A_i hold do not belong to $\mathcal{F}$. Thus by definition of the ultrafilter, the sets $V_i = \{n \in \mathbb{N} | \ A_i \text{ does not hold}\}$ are in $\mathcal{F}$. Thus $V = \bigcap_{i \in I} V_i \in \mathcal{F}$. But for any $n \in V$ we know that all of the statements A_i do not hold. Hence for any $n \in V$ we know that $\bigvee_{i \in I} A_i$ does not hold. But the set $U = \{n \in N | \bigvee_{i \in I} A_i\}$ belongs to $\mathcal{F}$ by assumption. So we have V and $\mathbb{N} \backslash V$ belonging to $\mathcal{F}$. A contradiction. □

We will use these elementary observations quite frequently, sometimes without even mentioning it.

Now, define the notion of an ultraproduct.

Definition 2.3.1.5. Suppose we have a sequence of sets E_n labeled by natural numbers. Consider the set $\prod'_{\mathcal{F}} E_n$ consisting of the sequences $\{e_n\}_{n \in A}$ for a set $A \in \mathcal{F}$ and $e_n \in E_n$. i.e., $\prod'_{\mathcal{F}} E_n$ consists of sequences of elements of E_n which are defined for almost all n. Then $\prod_{\mathcal{F}} E_n$ is the quotient of $\prod'_{\mathcal{F}} E_n$ by the following relation: $\{e_n\}_{n \in A} \sim \{e'_n\}_{n \in A'}$ iff $e_n = e'_n$ for almost all n (i.e., on $B \subset A' \cap A$, such that $B \in \mathcal{F}$). The set $\prod_{\mathcal{F}} E_n$ is called the ultraproduct of the sequence $\{E_n\}_{n \in \mathbb{N}}$.

Remark 2.3.1.6. Thus in a nutshell the ultraproduct consists of “germs" of sequences of elements which are defined for almost all n. Because of this in what follows we will use “sequence” to mean “sequence defined for almost all n”.

Remark 2.3.1.7. Note that for any finite set C, the ultraproduct of its copies $\prod_{\mathcal{F}} C_i$ with $C_i = C$ is equal to C. Indeed, for any sequence $\{c_n\}_{n\in A}$, for some $A \in \mathcal{F}$, we can define $U_d = \{n \in A | d = c_n\}$ for any $d \in C$. Then we have $\bigcup_{d\in C} U_d = A$, thus one of the U_d's must belong to $\mathcal{F}$. So it follows that $\{c_n\}_{n\in A} \sim \{d\}_{n\in A}$ for this particular d.

Oftentimes we use the following notation:

Definition 2.3.1.8. For a sequence $\{E_n\}_{n\in\mathbb{N}}$, denote an element $\{e_n\}_{n\in\mathbb{N}} \in \prod_{\mathcal{F}} E_n$ by $\prod_{\mathcal{F}} e_n$.

This construction is interesting for us, because it, in a certain sense, preserves a lot of algebraic structures. We will explore this dimension of ultraproducts below.

Example 2.3.1.9. First, note that the ultraproduct inherits any operation or any relation which is defined on a sequence of sets E_n for almost all n. For example, suppose we are given a sequence of k-ary operations $\circ_n$ defined for almost all n. Let $E := \prod_{\mathcal{F}} E_n$ and consider the k-ary operation $\circ : E \times E \times \cdots \times E \to E$ defined as

$$\circ(e^1, e^2, \ldots, e^k) = \circ(\prod_{\mathcal{F}} e_n^1, \ldots, \prod_{\mathcal{F}} e_n^k) = \prod_{\mathcal{F}} \circ_n (e_n^1, \ldots, e_n^k).$$

Note that this is the same as taking $\circ = \prod_{\mathcal{F}} \circ_n \in \prod_{\mathcal{F}} \mathrm{Hom}_{\mathrm{Sets}}(E^{\times k}, E)$, so we can call $\circ$ an ultraproduct of $\circ_n$. Now if we have any sequence of relations r_n given for almost all n, they can be written as a sequence of k-ary maps with Boolean values. And one can define r to be a relation on E in a similar way

$$r(e^1, e^2, \ldots, e^k) = r(\prod_{\mathcal{F}} e_n^1, \ldots, \prod_{\mathcal{F}} e_n^k) = \prod_{\mathcal{F}} r_n(e_n^1, \ldots, e_n^k) \in \prod_{\mathcal{F}} \mathbf{2} = \mathbf{2}.$$ [3]

[3] Here $\mathbf{2}$ stands for the Boolean set $\{0, 1\}$.

For the same reason we can call the relation r the ultraproduct of the relations r_n. Note that this means that if the relation r_n was true for almost all n (i.e., $\mathrm{Im}(r_n) = \{1\}$ for almost all n), it follows that r is also true.

One can easily check for oneself that the above examples (2.3.1.9) can be extended to any collections of sequences of sets, maps between them and relations between maps. That means that if we have a collection of sequences of sets with a certain algebraic structure defined by maps between them, we can form the ultraproducts of these sets and these maps. Moreover if the sequences of maps satisfy a certain collection of relations, the ultraproduct will satisfy them too.

These observations may be formulated in the following way:

Theorem 2.3.1.10. ***Łoś's theorem*** *(Theorem 2.3.2 in [48])*

Suppose we have a collection of sequences of sets $E_i^{(k)}$ for $k = 1, \dots, m$, a collection of sequences of elements $f_i^{(r)}$ for $r = 1, \dots, l$, and a formula of a first order language $\phi(x_1, \dots, x_l, Y_1, \dots, Y_m)$ depending on some parameters x_i and sets Y_j. Denote by $E^{(k)} = \prod_{\mathcal{F}} E_n^{(k)}$ and $f^{(r)} = \prod_{\mathcal{F}} f_n^{(r)}$. Then $\phi(f_n^{(1)}, \dots, f_n^{(l)}, E_n^{(1)}, \dots, E_n^{(m)})$ is true for almost all n iff $\phi(f^{(1)}, \dots, f^{(l)}, E^{(1)}, \dots E^{(m)})$ is true.

This theorem might seem very abstract in the form it is given above, but one can grasp it quite easily by working through a few simple examples of its application. This is exactly what we are going to do in the next section. Many of these examples will also be used in the rest of the thesis.

2.3.2 Examples of ultraproducts

Example 2.3.2.1. If E_n is a sequence of monoids/groups/rings/fields then $\prod_{\mathcal{F}} E_n$ with operations given by taking the ultraproduct of the operations as elements of the corresponding sets of set-theoretical maps gives us correspondingly a structure of a monoid/group/ring/field by Łoś's theorem.

Example 2.3.2.2. If V_i are finite dimensional vector spaces over a field $\Bbbk$, then $\prod_{\mathcal{F}} V_n$ is a vector space over $\prod_{\mathcal{F}}\Bbbk$, which is not necessarily finite dimensional, since the property of being finite dimensional cannot be written in a first-order language. But if the dimensions of V_n are bounded, then they are the same for almost all n and hence V has the same dimension (for example, because the ultraproduct of bases is a basis).

Example 2.3.2.3. Take the ultraproduct of a countably infinite number of copies of $\overline{\mathbb{Q}}$. By Łoś's theorem $\prod_{\mathcal{F}} \overline{\mathbb{Q}}$ is a field, which is algebraically closed. It has characteristic zero since $\forall k \in \mathbb{Z}$ such that $k \neq 0$ it follows that $k = \prod_{\mathcal{F}} k \neq 0$. Also it is easy to see that its cardinality is continuum. Hence by Steinitz's theorem[4] $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \simeq \mathbb{C}$. However, there is no canonical isomorphism. We will now use this fact to show that we can in fact choose such an isomorphism so that $\prod_{\mathcal{F}} n$ maps into any transcendental element of $\mathbb{C}$.

Indeed, consider the ultraproduct of integers $\prod_{\mathcal{F}} n$. Via the isomorphism constructed in the previous paragraph this is an element of $\mathbb{C}$. Notice that this element cannot satisfy any nontrivial polynomial equation over $\mathbb{Q}$ (indeed, the corresponding polynomial must have infinitely many roots), hence $\prod_{\mathcal{F}} n$ is a transcendental element of $\mathbb{C}$. By an automorphism of $\mathbb{C}$ we can send this element into any transcendental element of $\mathbb{C}$.

Thus we conclude that for any transcendental element $\nu \in \mathbb{C}$ there is an isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \simeq \mathbb{C}$, such that $\prod_{\mathcal{F}} n = \nu$.

Also notice that by Steinitz's theorem it follows that $\overline{\mathbb{C}(x)} \simeq \mathbb{C}$, since they

[4] This theorem tells us that two uncountable algebraically closed fields are isomorphic iff their characteristic and cardinality are the same. It is proven in [51].

have the same cardinality. Thus we can also conclude that there is an isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \simeq \overline{\mathbb{C}(x)}$ such that $\prod_{\mathcal{F}} n = x$.

Example 2.3.2.4. Take the ultraproduct of $\overline{\mathbb{F}}_{p_n}$ for some sequence of distinct prime numbers p_n. As before, by Łoś's theorem $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n}$ is a field, which is algebraically closed. Also it has cardinality continuum. Now for any natural number k, we have $k = \prod_{\mathcal{F}} k \neq 0$, since it is equal to zero for at most a finite number of n. Hence $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} \simeq \mathbb{C}$ by Steinitz's theorem, again not in a canonical way. Below we will show that we can pick a particular isomorphism that satisfies a certain important property.

Suppose we are given an algebraic number $\nu \in \mathbb{C}$. Let us show that there exists a sequence of integers ν_n and prime numbers p_n such that $\nu_n < p_n$ and $\prod_{\mathcal{F}} \nu_n = \nu$ inside $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} \simeq \mathbb{C}$. This important fact will be used extensively, when we will apply ultraproducts to Deligne categories.

Let $q(x) \in \mathbb{Z}[x]$ be the minimal polynomial for ν. We would like to find an infinite number of pairs ν_n, p_n such that $q(\nu_n) = 0 \mod p_n$. Let us show that there is an infinite number of primes dividing the collection of numbers $q(l)$ for $l \in \mathbb{N}$, from this it would follow that there is an infinite number of pairs since only a finite number of primes divide each $q(l)$. Suppose it is not so, and there are only k such primes. Fix C such that we have $q(l) < C \cdot l^{\deg(q)}$ for all positive integer values of l. Denote by Q the number of integers of the form $q(l)$ for $l \in \mathbb{Z}_{\geq 0}$ such that $q(l) < L$. By the above inequality (that is $q(l) < C \cdot l^{\deg(q)}$) Q is at least $\frac{1}{C} \cdot L^{\frac{1}{\deg(q)}}$. On the other hand the number P of numbers less than L divisible only by k fixed primes is less or equal to $\log_2(L)^k$, since each prime number is at least 2. Hence for big enough L we have $P < Q$, which contradicts the hypothesis[5].

[5]This proof is also written by Nate Harman in the proof of Prop. 2.2 in [30].

Hence we can take a sequence of distinct primes p_n and a sequence of integers ν_n tending to infinity such that $q(\nu_n) = 0$ in $\mathbb{F}_{p_n}$ and $\nu_n < p_n$. It follows that $\prod_{\mathcal{F}} \nu_n$ in $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n}$ is a root of $q(x)$. Hence by an automorphism of $\mathbb{C}$ we can send $\prod_{\mathcal{F}} \nu_n$ into ν.

Example 2.3.2.5. Suppose $\mathcal{C}_n$ is a sequence of (locally small) categories. We can define the ultraproduct category $\widehat{\mathcal{C}} = \prod_{\mathcal{F}} \mathcal{C}_n$ as the category whose objects are sequences of objects in $\mathcal{C}_n$. For clarity we will denote the ultraproduct of objects by[6] $\prod_{\mathcal{F}}^{C}$. The morphisms in $\widehat{\mathcal{C}}$ are given by

$$\mathrm{Hom}_{\widehat{\mathcal{C}}}(\prod_{\mathcal{F}}^{C} X_n, \prod_{\mathcal{F}}^{C} Y_n) = \prod_{\mathcal{F}} \mathrm{Hom}_{\mathcal{C}_n}(X_n, Y_n),$$

and the composition maps are given by the ultraproducts of the composition maps, i.e., $(\prod_{\mathcal{F}} f_n) \circ (\prod_{\mathcal{F}} g_n) = \prod_{\mathcal{F}} (f_n \circ g_n)$. By Łoś's theorem this data satisfies the axioms of a category. If the categories $\mathcal{C}_n$ have some "algebraic" structures, for example the structures of an abelian or monoidal category, then $\widehat{\mathcal{C}}$ also has these structures[7].

Usually $\widehat{\mathcal{C}}$ is too big and it is interesting to consider a certain full subcategory $\mathcal{C}$ in there, for example by only considering the ultraproducts of sequences of objects of $\mathcal{C}_i$ bounded in some sense. This will be discussed in more detail in the next subsection.

Remark 2.3.2.6. Note that taking the ultraproduct of a sequence of algebraic objects as such is different from considering their ultraproduct as a sequence of objects in certain categories.

For example, consider a sequence of countably-dimensional vector spaces V_n over $\Bbbk$. By Łoś's theorem $\prod_{\mathcal{F}} V_n$ is a vector space (although its dimension is more than countable). However, we can also regard V_n as objects of the categories $\mathcal{C}_n = \mathrm{Vect}_{\Bbbk}$

[6] The superscript C stands for "category".

[7] But the finite-length property, for example, does not survive, as it cannot be formulated as a first-order logical statement.

and construct $\prod_{\mathcal{F}}^{C} V_n \in \prod_{\mathcal{F}} \mathrm{Vect}_{\Bbbk}$. The category $\prod_{\mathcal{F}} \mathrm{Vect}_{\Bbbk}$ is not equivalent to the category of vector spaces (for example, it is rigid and can have objects of non-integer dimension), so $\prod_{\mathcal{F}}^{C} V_n$ is not a vector space in any sense.

Also frequently it is possible to think about an ultraproduct as a certain kind of a limit as $n \mapsto \infty$, where n becomes a "free" parameter. This way of thinking is actually partially applicable to the setting of Deligne categories that we will discuss in the next Chapter. In the next example we will show how this is possible in the case of the ultraproduct of finite-dimensional algebras.

Example 2.3.2.7. Consider a sequence of finite dimensional algebras A_n over $\overline{\mathbb{Q}}$ with a sequence of fixed vector space isomorphisms $A_n \simeq V$. Equivalently, this means that we have a sequence of binary operations $\mu_n : V \otimes V \to V$ which satisfy all the axioms of an algebra. Suppose in some basis (and hence in any basis) the matrices of μ_n have entries which depend polynomially on n.

Consider $A = \prod_{\mathcal{F}} A_n$. By Example 2.3.2.3 this is an algebra over $\overline{\mathbb{C}(x)}$. Since A_i are finite dimensional and all isomorphic to V via a fixed isomorphism, we can also conclude that the binary operation on A, which we denote by μ, is given by $\prod_{\mathcal{F}} \mu_n$. Since μ_n depended polynomially on n and $x = \prod_{\mathcal{F}} n$, it follows that μ is given by the same formulas as the sequence μ_n with n substituted by x. In other words, if $c_{\alpha,\beta}^{\gamma}(n)$ are the structure constants of μ_n in a certain basis then $c_{\alpha,\beta}^{\gamma}(x)$ are the structure constants of μ. I.e., n becomes a formal parameter in A.

2.3.3 Restricted ultraproducts

When one works with a sequence of objects which are in some sense infinite dimensional, it's sometimes useful to consider a subobject in the ultraproduct consisting of the sequences of elements which are in a some way bounded. This can be called a *restricted ultraproduct.* We have already mentioned this in the case of categories in Example 2.3.2.5. For example, the Deligne category $\mathrm{Rep}(S_\nu)$ itself will be constructed as a full subcategory in a certain ultraproduct category.

In this section we will outline the definitions of the restricted ultraproduct which makes sense in the case of filtered or graded vector spaces and categories.

Definition 2.3.3.1. For a sequence of vector spaces E_n with an increasing filtration $F^0E_n \subset F^1E_n \subset \cdots \subset F^kE_n \subset \ldots$, define the restricted ultraproduct $\prod_{\mathcal{F}}^r E_n$ to be equal to $\bigcup_{k=0}^{\infty} \prod_{\mathcal{F}} F^kE_n \subset \prod_{\mathcal{F}} E_n$.

Definition 2.3.3.2. For a sequence of vector spaces E_n with a grading $E_n = \bigoplus_{k=0}^{\infty} \mathrm{gr}^k E_n$, define the restricted ultraproduct $\prod_{\mathcal{F}}^r E_n$ to be equal to $\bigoplus_{k=0}^{\infty} \prod_{\mathcal{F}} \mathrm{gr}^k E_n \subset \prod_{\mathcal{F}} E_n$. Note that by taking $F^kE_n = \bigoplus_{i=0}^{k} \mathrm{gr}^i E_n$, this construction matches the construction of Definition 2.3.3.1.

We will use this notion in the case when the dimensions of the space F^kE_n are finite and stabilize as $n \to \infty$ for fixed k. Let us give a few examples.

Example 2.3.3.3. Consider a countable-dimensional vector space V over $\Bbbk$. Consider a sequence of copies of V, i.e., $V_n = V$. Also consider an increasing filtration F^jV by finite dimensional subspaces and the same filtration on all V_n. We can calculate the restricted ultraproduct of this sequence:

$$\prod_{\mathcal{F}}^r V_n = \bigcup_{k=0}^{\infty} \prod_{\mathcal{F}} F^kV_n = \bigcup_{k=0}^{\infty} F^kV = V.$$

Whereas the usual ultraproduct $\prod_{\mathcal{F}} V_n$ is more than countable-dimensional.

Now let us apply the notion of restricted ultraproduct to the extension of Example 2.3.2.7. This further shows how one can think about an ultraproduct as a certain "limit" $n \to \infty$.

Example 2.3.3.4. Consider A_n, a sequence of (possibly infinite dimensional) algebras over $\overline{\mathbb{Q}}$ with an increasing filtration by finite dimensional subspaces, such that for every $k \in \mathbb{N}$ there is N_k such that for $n > N_k$ all $F^k A_n$ are isomorphic as vector spaces to a fixed vector space $F^k A_\infty$ via fixed isomorphisms. I.e., every filtered component stabilizes after a certain point.

This means that we have a collection of sequences of coherent multiplication maps $\mu_n^{k,l} : F^k A_\infty \times F^l A_\infty \to F^{k+l} A_\infty$ defined for almost all n. Let's also suppose that this sequence depends polynomially on n.

Consider $A = \prod_{\mathcal{F}}^r A_n$. Note that as a vector space the restricted ultraproduct equals to:

$$\prod_{\mathcal{F}}^r A_n = \bigcup_{k=0}^{\infty} \prod_{\mathcal{F}} F^k A_n = \bigcup_{k=0}^{\infty} F^k A_\infty,$$

since $F^k A_n = F^k A_\infty$ for almost all n.

Now as in Example 2.3.2.7 the ultraproducts $\mu^{k,l} = \prod_{\mathcal{F}} \mu_n^{k,l}$ define a coherent collection of multiplication maps, the union of which defines a map $\mu : A \times A \to A$. The structure constants of this multiplication can also be obtained by taking the structure constants of A_n and plugging in x instead of n.

Note that the same construction works if the structure constants depend on n as rational functions.

This example shows better why it makes sense to think about the ultraproduct as a limit.

We also would like to introduce a related construction, which we will also call a restricted ultraproduct. This will take place in the setting of the ultraproducts of categories. Suppose $\{\mathcal{D}_i\}$ is a sequence of artinian abelian categories and $\mathcal{D} = \prod_{\mathcal{F}} \mathcal{D}_i$ is their ultraproduct (an abelian category which is, in general, not artinian). Suppose $\mathcal{C}$ is a full artinian subcategory of $\mathcal{D}$. Using Construction 2.1.0.11 we can obtain ind-objects of $\mathcal{C}$ in the following way.

Construction 2.3.3.5. Suppose we have a sequence of ind-objects $X_n \in \mathrm{IND}(\mathcal{D}_n)$ such that each X_n is equipped with a filtration by objects of $\mathcal{D}_n$. I.e., we have $F^0X_n \subset F^1X_n \subset \dots \subset F^iX_n \subset \dots$, where all $F^iX_n \in \mathcal{D}_n$ and $X_n = \bigcup_{i\in\mathbb{N}} F^iX_n$. Also suppose that for each $i \geq 0$, $\prod_{\mathcal{F}}^C F^iX_n \in \mathcal{C}$. Denote $\prod_{\mathcal{F}}^C F^iX_n$ by F^iX_∞. It is clear that we have injections $F^iX_\infty \hookrightarrow F^{i+1}X_\infty$.

It follows that the sequence F^iX_∞ defines an object $X_\infty \in \mathrm{IND}(\mathcal{C})$ as:

$$X_\infty = \bigcup_{i\in\mathbb{N}} F^iX_\infty = \bigcup_{i\in\mathbb{N}} \prod\nolimits_{\mathcal{F}}^C F^iX_n.$$

We will use a special notation for this construction:

Definition 2.3.3.6. In the setting of Construction 2.3.3.5, call X_∞ the restricted ultraproduct of X_n with respect to the fixed filtration. We will write

$$X_\infty = \prod\nolimits_{\mathcal{F}}^{C,r} X_n.$$

Remark 2.3.3.7. Let $\widetilde{F}^\bullet$ be another filtration on the sequence $\{X_n\}$ such that $\prod_{\mathcal{F}}^C F^iX_n \in \mathcal{C}$, and let $\widetilde{X}_\infty$ be the corresponding restricted ultraproduct. Let us say that $F, \widetilde{F}$ are equivalent if for any i there exist $r(i), s(i)$ such that $F^iX_n \subset \widetilde{F}^{r(i)}X_n$ and $\widetilde{F}^iX_n \subset F^{s(i)}X_n$ for almost all n. If $F, \widetilde{F}$ are equivalent, then we have maps $F^iX_\infty \to \widetilde{F}^{r(i)}X_\infty$ and $\widetilde{F}^iX_n \to F^{s(i)}X_\infty$, which give rise to maps $X_\infty \to \widetilde{X}_\infty$ and $\widetilde{X}_\infty \to X_\infty$ which are clearly inverse to each other; thus X_∞ and $\widetilde{X}_\infty$ are naturally isomorphic. This shows that X_∞ depends only on the equivalence class of the filtration F.

However, not all filtrations are equivalent. E.g., if $X_n = \Bbbk^n$, F^iX_n is spanned by the first $i+1$ standard basis vectors for $i \leq n-1$, $g_n \in GL(n, \Bbbk)$ and $\widetilde{F} = g_n(F)$ on

X_n then in general $F, \widetilde{F}$ are not equivalent. Thus, without specifying a filtration (at least up to equivalence), we cannot define the restricted ultraproduct of X_n.

Remark 2.3.3.8. Note that we can easily define the restricted ultraproduct of a series of bifiltered algebras A_n, with finite-dimensional filtration components as $\bigcup_{i,j\geq 0} \prod_{\mathcal{F}} F^{i,j} A_n$. Note that the result is the same as the restricted product taken with respect to the total filtration of A_n.

The same goes for the sequence of bifiltered ind-objects of artinian categories similarly to Construction 2.3.3.5.

Thus below we will use these two operations interchangeably.

Chapter 3

Construction of Deligne categories through ultrafilters

In this chapter we will show how we can use the notion of ultraproduct introduced in the previous chapter to construct the Deligne category $\mathrm{Rep}(S_\nu)$ as a "limit" of finite rank categories $\mathbf{Rep}(S_n)$. We will also show how this can be further used to study Ind-objects of $\mathrm{Rep}(S_\nu)$ and the ν-tensor powers of unital vector spaces in $\mathrm{Rep}(S_\nu)$. In the end we will also outline the construction of the Deligne category of wreath products $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$. The constructions of this chapter will serve as the foundation for all the applications we will discuss in the later chapters.

3.1 Deligne category $\mathrm{Rep}(S_\nu)$ as an ultraproduct

We will begin by showing how one can construct the category $\mathrm{Rep}(S_\nu)$ itself using ultraproducts, and then discuss some important consequences of this construction: construction of simple objects, restriction and induction functors etc. through ultraproducts. The main ideas of this approach were contained in [7],[30][1]. However we spell it out in more detail here.

The main idea of this approach is to construct the category $\mathrm{Rep}(S_\nu)$ for non-integer ν as a full subcategory in the ultraproduct category following Example 2.3.2.5 with $\mathcal{C}_n$ being representation categories of the symmetric group in the finite rank. We have the following result (See the introduction of [7] or Theorem 1.1 in [30]):

Theorem 3.1.0.1. ***a)*** *Suppose $\nu \in \mathbb{C}$ is transcendental. Consider $\widehat{\mathcal{C}} = \prod_{\mathcal{F}} \boldsymbol{Rep}_0^f(S_n)$. Set $\mathfrak{h}_\nu := \prod_{\mathcal{F}}^C \mathfrak{h}_n$. Fix an isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \simeq \mathbb{C}$ such that $\prod_{\mathcal{F}} i = \nu$. Then the full subcategory of the $\prod_{\mathcal{F}} \overline{\mathbb{Q}}$-linear category $\widehat{\mathcal{C}}$ generated by $\mathfrak{h}_\nu$ under taking tensor products, direct sums and direct summands is equivalent to the $\mathbb{C}$–linear category $Rep(S_\nu)$, in a way consistent with the fixed isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \simeq \mathbb{C}$.*

b) *Suppose $\nu \in \mathbb{C}$ is algebraic but not a nonnegative integer. Fix a sequence of distinct primes p_n, a sequence of integers ν_n, and an isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} \simeq \mathbb{C}$ such that $\prod_{\mathcal{F}} \nu_n = \nu$. Set $\widehat{\mathcal{C}} := \prod_{\mathcal{F}} \boldsymbol{Rep}_{p_n}^f(S_{\nu_n})$. Set $\mathfrak{h}_\nu := \prod_{\mathcal{F}}^C \mathfrak{h}_{p_n}^{\nu_n}$. Then the full subcategory of the $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n}$-linear category $\widehat{\mathcal{C}}$ generated by $\mathfrak{h}_\nu$ under taking tensor products, direct sums and direct summands is equivalent to the $\mathbb{C}$-linear category $Rep(S_\nu)$, in a way consistent with the fixed isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} \simeq \mathbb{C}$.*

Proof. **a)** The required isomorphism of fields exists by Example 2.3.2.3. So we have a Karoubian symmetric monoidal category $\widehat{\mathcal{C}}$ linear over $\mathbb{C}$, with an object $\prod_{\mathcal{F}}^C \mathfrak{h}_n$ of dimension ν. Since every $\mathfrak{h}_n$ is a commutative Frobenius algebra, it follows by Łoś's theorem that $\mathfrak{h}_\nu$ is also a commutative Frobenius algebra. Hence by Proposition 2.2.0.7 we obtain a symmetric monoidal functor $F : \mathrm{Rep}(S_\nu) \to \widehat{\mathcal{C}}$ which takes $\mathfrak{h}$ to $\mathfrak{h}_\nu$. Since $\mathrm{Rep}(S_\nu)$ is generated by $\mathfrak{h}$ under taking tensor products, direct sums and direct summands, it follows that the image of $\mathrm{Rep}(S_\nu)$ under F is the full subcategory $\mathcal{C}$ in

[1] For the similar discussion about $\mathrm{Rep}(GL_\nu)$ see [7], [30], [33].

$\widehat{\mathcal{C}}$ generated by $\mathfrak{h}_\nu$ under taking tensor products, direct sums and direct summands. So we know that $F : \mathrm{Rep}(S_\nu) \to \mathcal{C}$ is essentially surjective. Now it is enough to prove that it is fully faithful.

Note that it is enough to prove that

$$\prod_{\mathcal{F}} \mathrm{Hom}_{S_n}(\mathfrak{h}_n^{\otimes r}, \mathfrak{h}_n^{\otimes s}) = \mathrm{Hom}_{\mathrm{Rep}(S_\nu)}([r],[s]),$$

and that the composition maps are the same. Indeed, if this is true, both categories can be obtained as the Karoubian envelopes of the additive envelopes of the categories consisting of all $[r]$ or $\mathfrak{h}_\nu^{\otimes r}$ respectively.

But this follows from Theorem 2.6 in [4]. Indeed, there it is stated that there is an isomorphism between $\overline{\mathbb{Q}}P_{r,s}$ and $\mathrm{Hom}_{S_n}(\mathfrak{h}_n^{\otimes r}, \mathfrak{h}_n^{\otimes s})$ for $n > r + s$. So for almost all n we have $\mathrm{Hom}_{S_n}(\mathfrak{h}_n^{\otimes r}, \mathfrak{h}_n^{\otimes s}) = \overline{\mathbb{Q}}P_{n,m}$. Also Proposition 2.8 in the same article states that under this isomorphism the composition rule on $\mathrm{Hom}_{S_n}(\mathfrak{h}_n^{\otimes r}, \mathfrak{h}_n^{\otimes s})$ transforms into the composition rule on $\overline{\mathbb{Q}}P_{r,s}$ in the definition of $\mathrm{Rep}^0(S_\nu)$. So it follows that, indeed, $\prod_{\mathcal{F}} \mathrm{Hom}_{S_n}(\mathfrak{h}_n^{\otimes r}, \mathfrak{h}_n^{\otimes s}) = \mathrm{Hom}_{\mathrm{Rep}(S_\nu)}([r],[s])$, and the composition rule is the same.

b) Again the required isomorphism exists by Example 2.3.2.4. The rest of the proof is the same since the representation theory of S_n is the same in zero characteristic and in characteristic $p > n$, and $p_n > \nu_n$ for almost all n. □

Remark 3.1.0.2. Note that for the purposes of this theorem we could also have used the categories $\mathbf{Rep}_{p_n}(S_{\nu_n})$.

We can also formulate a similar result for $\mathrm{Rep}^{\mathrm{ext}}(S_\nu)$:

Corollary 3.1.0.3. *Fix an isomorphism* $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \simeq \overline{\mathbb{C}(\nu)}$ *such that* $\prod_{\mathcal{F}} n = \nu$. *Set* $\widehat{\mathcal{C}} = \prod_{\mathcal{F}} \boldsymbol{Rep}_0^f(S_n)$. *Set* $\mathfrak{h}_\nu = \prod_{\mathcal{F}}^C \mathfrak{h}_n$. *Then the full subcategory of the* $\prod_{\mathcal{F}} \overline{\mathbb{Q}}$*-linear* *category $\widehat{\mathcal{C}}$ generated by $\mathfrak{h}_\nu$ under taking tensor products, direct sums and direct summands is equivalent to the $\overline{\mathbb{C}(\nu)}$-linear category Rep$(S_\nu)$, in a way consistent with the fixed isomorphism $\prod_\mathcal{F} \overline{\mathbb{Q}} \simeq \overline{\mathbb{C}(\nu)}$.*

Proof. This follows from the above Theorem and the fact that $\mathbb{C} \simeq \overline{\mathbb{C}(\nu)}$ (see Example 2.3.2.3). $\square$

Remark 3.1.0.4. As mentioned in the beginning of Section 2.1, to treat the algebraic and transcendental cases simultaneously, it's useful to agree on the convention that by $\overline{\mathbb{F}}_0$ we will mean $\overline{\mathbb{Q}}$, and so the case $\nu_n = n$, $p_n = 0$ in the setting of part (b) of the Theorem 3.1.0.1 gives us transcendental ν. Also below we will always assume that the sequences p_n and ν_n are the sequences from Theorem 3.1.0.1 or Corollary 3.1.0.3 corresponding to the given ν. Finally, we will work only with $\nu \in \mathbb{C} \backslash \mathbb{Z}_{\geq 0}$.

Remark 3.1.0.5. Note that for any finite group G we can characterize $\text{Rep}(S_\nu) \boxtimes \mathbf{Rep}(G;\mathbb{C})^2$ as a full subcategory in $\prod_\mathcal{F} \mathbf{Rep}_{p_n}(S_{\nu_n}) \boxtimes \mathbf{Rep}_{p_n}(G)$, which consists of sequences of objects $\prod_\mathcal{F} V_n = \prod_\mathcal{F} \sum_k U_{n,k} \otimes Y_k$, where Y_k runs over all the irreducible objects of G (with $p_n > |G|$), such that each $\prod_\mathcal{F} U_{n,k}$ is an object of $\text{Rep}(S_\nu)$. Indeed on the level of abelian categories it follows from the fact that $\mathbf{Rep}_{p_n}(G)$ splits into a finite sum of categories of vector spaces, and it's easy to check that the tensor structure agrees (it's the same for almost all n).

Now we would like to explain why this construction of the Deligne categories is quite useful. To begin with, we would like to construct the simple objects $\mathcal{X}(\lambda)$ as ultraproducts. This is easy to do, using the notation from Definition 2.1.0.7:

Proposition 3.1.0.6. *The irreducible object $\mathcal{X}(\lambda)$ of* $\text{Rep}(S_\nu)$ *can be obtained as an ultraproduct of irreducible objects of* $\mathbf{Rep}^f_{p_n}(S_{\nu_n})$ *as* $\mathcal{X}(\lambda) = \prod^C_\mathcal{F} X_{\nu_n}(\lambda|_{\nu_n})$.

[2] Here, we use $\boxtimes$ to denote a Deligne tensor product of locally finite abelian categories, for the definition see 1.11 in [16].

Proof. From Section 3.3 of [4] we know that the algebras $\Bbbk P_r(\nu)$ for $\nu \neq 0, 1, \ldots, 2r$ have the same set of idempotents obtained by specialization from idempotents of $\Bbbk(x)P_r(x)$. Now by construction all simple objects of $\mathrm{Rep}(S_\nu)$ are given by the primitive idempotents of $\mathrm{End}_{\mathrm{Rep}^0(S_\nu;\Bbbk)}([r]) = \Bbbk P_r(\nu)$. And by Theorem 3.1.0.1, $\Bbbk P_r(\nu) \simeq \prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} P_r(\nu_n)$ in such a way that basis elements are ultraproducts of basis elements. Thus it follows that idempotents in $\Bbbk P_r(\nu)$ are given by the ultraproducts of the same idempotents for almost all n. And so the claim follows. □

Note that Theorem 3.1.0.1 does not specify which sequences of finite rank representations actually give us an object of the Deligne category. Proposition 3.1.0.6 allows us to reformulate the definition of $\mathrm{Rep}(S_\nu)$ as an ultraproduct so that this becomes apparent.

Proposition 3.1.0.7. *In the notation of Theorem 3.1.0.1 the category* $\mathrm{Rep}(S_\nu)$ *can be described as the full subcategory of* $\widehat{\mathcal{C}} = \mathbf{Rep}^f_{p_n}(S_{\nu_n})$ *consisting of sequences of objects* $Y_n = \bigoplus_{\alpha \in A_n} X_{p_n}(\lambda_{n,\alpha})$ *for some indexing sets* A_n *and Young diagrams* $\lambda_{n,\alpha}$ *such that both the sequence of* $|A_n|$ *and the sequence of* $\max_{\alpha \in A_n}(|\lambda_{n,\alpha}| - (\lambda_{n,\alpha})_1)$*, where* $(\lambda_{n,\alpha})_1$ *is the length of the first row, are bounded for almost all* n.

Proof. We know that $\mathrm{Rep}(S_\nu)$ is a full subcategory of $\widehat{\mathcal{C}}$ so we just need to match the objects.

On the one hand, suppose $Y \in \mathrm{Rep}(S_\nu)$. We know that for some set of Young diagrams μ_α with $\alpha \in A$, a finite indexing set, we have $Y = \bigoplus_{\alpha \in A} \mathcal{X}(\mu_\alpha)$, so from Proposition 3.1.0.6 it follows that $Y = \prod^C_{\mathcal{F}} \bigoplus_{\alpha \in A} X_{p_n}(\mu_\alpha|_{\nu_n})$. Thus we have a required sequence with $A_n = A$ and $\lambda_{\alpha,n} = \mu_\alpha|_{\nu_n}$. The sequence $|A_n| = A$ is constant, hence so is the sequence $\max_{\alpha \in A_n}(|\lambda_{n,\alpha}| - (\lambda_{n,\alpha})_1) = \max_{\alpha \in A}(|\mu_\alpha|)$.

On the other hand, suppose we have a sequence described in the statement of the Theorem. Since we know that $|A_n|$ is bounded for almost all n, there is a finite number of options for the cardinality of $|A_n|$ for almost all n, thus from part 2 of Lemma 2.3.1.4 it follows that for almost all n the cardinality is the same. Fix A to be a set of this cardinality. So, for almost all n we have $Y_n = \bigoplus_{\alpha \in A} X_{p_n}(\lambda_{n,\alpha})$. Suppose $\max_{\alpha \in A_n}(|\lambda_{n,\alpha}| - (\lambda_{n,\alpha})_1)$ is bounded by L. Now each $\lambda_{n,\alpha}$ is a Young diagram of weight ν_n with at most L boxes in the rows above the first one. I.e., for n big enough (namely, $\nu_n > 2L$), it follows that each $\lambda_{n,\alpha} = \mu_{n,\alpha}|_{\nu_n}$ where $\mu_{n,\alpha}$ is a Young diagram of weight at most L. So for almost all n each Y_n is uniquely determined by a collection of $|A|$ Young diagrams of weight at most L. Notice that there is only a finite number of such collections. So by the same Lemma it follows that for almost all n the collection is the same. Denote it by $\{\mu_\alpha\}_{\alpha \in A}$. Hence, for almost all n up to a permutation we have $Y_n = \bigoplus_{\alpha \in A} X_{p_n}(\mu_\alpha|_{\nu_n})$. Hence we have $\prod_{\mathcal{F}}^C Y_n = \bigoplus_{\alpha \in A} \mathcal{X}(\mu_\alpha)$ which is indeed an object of $\mathrm{Rep}(S_\nu)$. □

So, as promised in Example 2.3.2.5, $\mathrm{Rep}(S_\nu)$ can indeed be described as given by ultraproducts bounded in a certain sense.

We will also need to explain how to interpolate the central element $\Omega_n \in \Bbbk[S_n]$ to $\mathrm{Rep}(S_\nu)$. Recall that we can consider the central elements of $\Bbbk[S_{\nu_n}]$ as endomorphisms of the identity functor of $\mathbf{Rep}_{p_n}(S_{\nu_n})$.

Definition 3.1.0.8. Denote by Ω the endomorphism of the identity functor of $\mathrm{Rep}(S_\nu)$ given by the restriction of the endomorphism $\prod_{\mathcal{F}} \Omega_{\nu_n}$.

One can easily calculate the action of Ω on simple objects.

Proposition 3.1.0.9. *[13] The action of Ω on an object $\mathcal{X}(\lambda)$ is given by:*

$$\Omega|_{\mathcal{X}(\lambda)} = \left(ct(\lambda) - |\lambda| + \frac{(\nu - |\lambda|)(\nu - |\lambda| - 1)}{2} \right) 1_{\mathcal{X}(\lambda)}.$$

Proof. Since $\mathcal{X}(\lambda) = \prod_{\mathcal{F}}^{C} X_{p_n}(\lambda|_{\nu_n})$, one needs to calculate $\prod_{\mathcal{F}} \mathrm{ct}(\lambda|_{\nu_n})$. It's easy to see that each box of λ contributes an extra -1 to the content of $\lambda|_{\nu_n}$, also $\nu_n - |\lambda|$ new boxes in the first row contribute $0 + 1 + \cdots + (\nu_n - |\lambda| - 1)$ to the content of $\lambda|_{\nu_n}$, thus we have:

$$\prod\nolimits_{\mathcal{F}} \mathrm{ct}(\lambda|_{\nu_n}) = \prod\nolimits_{\mathcal{F}} \left(\mathrm{ct}(\lambda) - |\lambda| + \frac{(\nu_n - |\lambda|)(\nu_n - |\lambda| - 1)}{2} \right) =$$

$$= \mathrm{ct}(\lambda) - |\lambda| + \frac{(\nu - |\lambda|)(\nu - |\lambda| - 1)}{2},$$

which is exactly the value in the statement of the proposition. □

Remark 3.1.0.10. Note that all of the results of this Section work mutatis mutandis for $\mathrm{Rep}^{\mathrm{ext}}(S_\nu)$ (see Definition 2.2.0.9).

We also need to describe the generalizations of the induction and the restriction functors. First let's define the latter using the universal property of $\mathrm{Rep}(S_\nu)$.

Definition 3.1.0.11. Consider the category $\mathrm{Rep}(S_{\nu-k}) \boxtimes \mathbf{Rep}(S_k; \mathbb{C})$ for an integer k, and in it the object $\mathcal{X} \otimes \mathbb{C} \oplus \mathbb{C} \otimes X_k$. This object is a commutative Frobenius algebra, and has dimension ν, so by the universal property we have a functor $\mathrm{Rep}(S_\nu) \to \mathrm{Rep}(S_{\nu-k}) \boxtimes \mathbf{Rep}(S_k; \mathbb{C})$. This functor is called the restriction functor and is denoted by $\mathrm{Res}^{S_\nu}_{S_{\nu-k} \times S_k}$.

Now we want to describe it in terms of ultraproducts.

Proposition 3.1.0.12. *For all $\nu \notin \mathbb{Z}_{\geq 0}$, the functor $Res^{S_\nu}_{S_{\nu-k} \times S_k}$ is equal to $\prod_{\mathcal{F}} Res^{S_{\nu_n}}_{S_{\nu_n - k} \times S_k}$, where the latter functors are the regular restriction functors for the finite groups.*

Proof. Recall that the Littlewood-Richardson coefficient $c^\lambda_{\xi,\mu}$ equals to the number of Littlewood-Richardson tables of the skew shape λ/ξ and of weight μ. We will use the fact that we have:

$$\mathrm{Res}^{S_{|\lambda|}}_{S_{|\lambda|-k}\times S_k}(X(\lambda)) = \bigoplus_{|\xi|=|\lambda|-k,\ |\mu|=k} c^{\lambda}_{\xi,\mu} X(\xi)\otimes X(\mu)\ .$$

A priori the above ultraproduct functor is a not a functor between $\mathrm{Rep}(S_\nu)$ and $\mathrm{Rep}(S_{\nu-k})\boxtimes \mathbf{Rep}(S_k;\mathbb{C})$, but between the bigger categories of unrestricted ultraproducts. Indeed this functor acts bewteen:

$$\prod_{\mathcal{F}} \mathrm{Res}^{S_{\nu_n}}_{S_{\nu_n-k}\times S_k} : \prod_{\mathcal{F}} \mathbf{Rep}_{p_n}(S_{\nu_n}) \to \prod_{\mathcal{F}} \left(\mathbf{Rep}_{p_n}(S_{\nu_n-k})\boxtimes \mathbf{Rep}_{p_n}(S_k)\right)\ .$$

Now by Remark 3 after the Theorem 1.4.1 we know how to characterize the category $\mathrm{Rep}(S_{\nu-k})\boxtimes \mathbf{Rep}(S_k;\mathbb{C})$ inside the image. So we need to check that if we restrict it to $\mathrm{Rep}(S_\nu)$, we will indeed get objects of $\mathrm{Rep}(S_{\nu-k})\boxtimes \mathbf{Rep}(S_k;\mathbb{C})$.

So consider $\mathcal{X}(\lambda) = \prod_{\mathcal{F}} X(\lambda|_n)$. Now

$$\left(\prod_{\mathcal{F}} \mathrm{Res}^{S_{\nu_n}}_{S_{\nu_n-k}\times S_k}\right)(\mathcal{X}(\lambda)) = \prod_{\mathcal{F}} \mathrm{Res}^{S_{\nu_n}}_{S_{\nu_n-k}\times S_k}(X(\lambda|_n)) =$$

$$= \prod_{\mathcal{F}} \bigoplus_{|\mu|=k, |\xi|=t_n-k} c^{\lambda|_n}_{\xi,\mu} X(\xi)\otimes X(\mu)\ ,$$

where the c's are the Richardson-Littlewood coefficients.

So if ν_n is sufficiently big, the gap between the first and the second rows of $\lambda|_{\nu_n}$ is bigger than k. For such ν_n the skew shapes μ/ξ for admissible ξ are all disconnected – there is a part above the first row and the part in the first row. Note that we also can put any sequence of numbers in the part lying in the first row. Hence if we denote by $M(\mu)$ the set of weights μ' (not necessarily partitions) such that $0 \le \mu'_i \le \mu_i$ it follows that the previous expression equals

$$\prod_{\mathcal{F}} \bigoplus_{|\mu|=k, \mu' \in M(\mu), \xi} c^{\lambda}_{\xi,\mu'} X(\xi|_{\nu_n}) \otimes X(\mu) = \bigoplus_{|\mu|=k, \mu' \in M(\mu), \xi} c^{\lambda}_{\xi,\mu'} \mathcal{X}(\xi) \otimes X(\mu) \ ,$$

where $c^{\lambda}_{\xi,\mu'}$ is the number of skew-shapes λ/ξ of weight μ'. Indeed in this formula we just first summed over the possible choices of fixing the length and the content of the first row (by fixing μ') and then the rest. So the image of $\mathcal{X}(\lambda)$ under the ultraproduct functor indeed lies in $\mathrm{Rep}(S_{\nu-k}) \boxtimes \mathbf{Rep}(S_k;\mathbb{C})$.

Now since $\prod_{\mathcal{F}} \mathrm{Res}^{S_{\nu_n}}_{S_{\nu_n-k}\times S_k}(\mathcal{X}) = \mathcal{X}\otimes\mathbb{C}\oplus\mathbb{C}\otimes X_k$, it follows that the ultraproduct functor sends $\mathcal{X}$ to $\mathcal{X}\otimes\mathbb{C}\oplus\mathbb{C}\otimes X_k$. So by universality we conclude that the functors are the same. □

Corollary/Definition 3.1.0.13. *There is a functor from Rep*$(S_{\nu-k}) \boxtimes \boldsymbol{Rep}(S_k;\mathbb{C})$ *to Rep*(S_ν)*, which is biadjoint to* $Res^{S_\nu}_{S_{\nu-k}\times S_k}$*, and which is denoted by* $Ind^{S_\nu}_{S_{\nu-k}\times S_k}$*. This functor equals to* $\prod_{\mathcal{F}} Ind^{S_{\nu_n}}_{S_{\nu_n-k}\times S_k}$.

Proof. It can be proven in the same way as above that the ultraproduct of the induction functors defines a functor into the Deligne category.

After we know this, by Łoś's theorem it follows that this functor is biadjoint to the restriction functor, since it is true in the finite rank. □

Remark 3.1.0.14. Note that this allows us to define the restriction and the induction functors for any subgroup of S_k in the following way:

$$\mathrm{Ind}^{S_\nu}_{S_{\nu-k}\times G} = \mathrm{Ind}^{S_\nu}_{S_{\nu-k}\times S_k} \circ \mathrm{Ind}^{S_{\nu-k}\times S_k}_{S_{\nu-k}\times G} \ ,$$

where the later functor is defined to be $\mathrm{Ind}^{S_{\nu-k}\times S_k}_{S_{\nu-k}\times G} = \left(\mathrm{Id} \boxtimes Ind^{S_k}_G\right)$. The same thing also holds for restrictions.

Now we would like to give the reader a general idea of how this can be used to transfer constructions and facts from representation theory in finite rank to the context of Deligne categories.

Suppose we have a representation-theoretic structure $\mathcal{Y}_n$ in each $\mathbf{Rep}_{p_n}(S_{\nu_n})$ which can be constructed uniformly in an element-free way for every n. Then we can try define the same structure $\mathcal{Y}$ in $\mathrm{Rep}(S_\nu)$ using the analogs of the same objects and maps, we would only need to prove that these objects and maps do indeed lie in $\mathrm{Rep}(S_\nu)$. Since the definitions are the same, it would follow that $\mathcal{Y} = \prod_{\mathcal{F}} \mathcal{Y}_n$. Now one can try to transfer the properties of $\mathcal{Y}_n$ to $\mathcal{Y}$. For some it can be as easy as a direct application of Łoś's theorem . Others require quite a bit of technical work before it becomes possible can do that.

Oftentimes the structure $\mathcal{Y}$ might include some ind-objects of $\mathrm{Rep}(S_\nu)$, for example if the structures $\mathcal{Y}_n$ were of infinite length as S_{ν_n}-modules. In the present thesis this will happen, for example, when we will try to define the rational Cherednik algebra and its various generalization in $\mathrm{Rep}(S_\nu)$. Thus we need to understand how deal with ind-objects in the ultraproduct setting. This and some other related constructions will be discussed in the next section.

3.2 Related constructions

In this section we will discuss several further constructions related to the Deligne category $\mathrm{Rep}(S_\nu)$ and how these constructions can be thought of in terms of ultraproducts.

3.2.1 Ind-objects of $\mathbf{Rep}(S_\nu)$ as restricted ultraproducts

First we are going to explain how ind-objects of $\mathrm{Rep}(S_\nu)$ can be obtained as restricted ultraproducts, thus extending Theorem 3.1.0.1 in a certain way.

To do that, we will use the result of Construction 2.1.0.11.

Proposition 3.2.1.1. *Suppose we have a sequence of representations $M_n \in \mathbf{Rep}_{p_n}(S_{\nu_n})$, with fixed filtration by subrepresentations of finite length. i.e., we have $F^i M_n \in \mathbf{Rep}^f_{p_n}(S_{\nu_n})$ such that $\bigcup_{i\in\mathbb{N}} F^i M_n = M_n$. Also suppose that $\prod^C_{\mathcal{F}} F^i M_n \in \mathrm{Rep}(S_\nu)$. Then it follows that $M = \prod^{C,r}_{\mathcal{F}} M_n = \bigcup_{i\in\mathbb{N}} \prod^C_{\mathcal{F}} F^i M_n$ is an object of* $\mathrm{IND}(\mathrm{Rep}(S_\nu))$.[3]

Proof. This follows from Construction 2.3.3.5. □

Remark 3.2.1.2. Note that, using Remark 2.1.0.12, we can conclude that if $M \in \mathrm{IND}(\mathrm{Rep}(S_\nu))$ has finite length, then for any $N \in \mathrm{IND}(\mathrm{Rep}(S_\nu))$ constructed via Proposition 3.2.1.1, we have:

$$\mathrm{Hom}_{\mathrm{IND}(\mathrm{Rep}(S_\nu))}(M, N) = \bigcup_{j\in\mathbb{N}} \mathrm{Hom}_{\mathrm{Rep}(S_\nu)}(M, F^j N) =$$

$$= \bigcup_{j\in\mathbb{N}} \prod\nolimits_{\mathcal{F}} \mathrm{Hom}_{\mathbf{Rep}_{p_n}(S_{\nu_n})}(M_n, F^j N_n) = \prod\nolimits^r_{\mathcal{F}} \mathrm{Hom}_{\mathbf{Rep}_{p_n}(S_{\nu_n})}(M_n, N_n),$$

with the filtration arising from the filtration on N.

[3]One can also define, through a more involved construction, the category $\mathrm{IND}(\mathrm{Rep}(S_\nu))$ as a subcategory of $\prod_{\mathcal{F}} \mathbf{Rep}_{p_n}(S_{\nu_n})$. Note that this subcategory will not be full. In this way one would also be able to consider $\prod^C_{\mathcal{F}} \bigcup_{i\in\mathbb{N}} F^i M_n$, i.e., take the ultraproduct directly. It can be shown that this would define the same object M.

3.2.2 Unital vector spaces and complex tensor powers

In this section we will discuss the construction of the complex tensor powers of the unital vector space in the Deligne category $\mathrm{Rep}(S_\nu)$ and a related construction of the symmetric tensor power of a unital algebra.

Below we will use the notion of a unital vector space. For details see [13].

Definition 3.2.2.1. A unital vector space V is a vector space together with a unit, i.e., a distinguished non-zero vector denoted by $1 \in V$.

In [13] it is shown that given a finite dimensional unital vector space V, one can functorially define an ind-object $V^{\otimes \nu} \in \mathrm{Rep}(S_\nu)$. The idea behind this is that, although there is no way to algebraically define what x^ν is, there is, on the other hand, a way to define what $(1+x)^\nu$ is. Namely, $(1+x)^\nu := \sum_{m \geq 0} \binom{\nu}{m} x^m$.

We can also construct this object via an ultraproduct. Anyone not familiar with [13] might regard this as definition for the purposes of the present thesis.

Proposition 3.2.2.2. *For a finite dimensional unital vector space V, the ind-object $V^{\otimes \nu}$ is given by:*

$$V^{\otimes \nu} = \prod\nolimits_{\mathcal{F}}^{C,r} V^{\otimes \nu_n}.$$

Proof. Using the notation of [13], we have:

$$V^{\otimes \nu_n} = \bigoplus_{\lambda} S^{\lambda|_{\nu_n}} V \otimes X(\lambda|_{\nu_n}),$$

where $S^{\lambda|_{\nu_n}}$ are the corresponding Schur functors. Thus we can define a filtration on each $V^{\otimes \nu_n}$ as

$$F^i V^{\otimes \nu_n} = \bigoplus_{|\lambda| \leq i} S^{\lambda|_{\nu_n}} V \otimes X(\lambda|_{\nu_n}).$$

Thus, taking the restricted ultraproduct with respect to this filtration, we obtain

$$\prod_{\mathcal{F}}^{C,r} V^{\otimes \nu_n} = \bigcup_i \bigoplus_{|\lambda| \le i} \left(\prod_{\mathcal{F}} S^{\lambda|_{\nu_n}} V \right) \otimes \mathcal{X}(\lambda) = \bigoplus_{\lambda} S^{\lambda,\infty} V \otimes \mathcal{X}(\lambda) = V^{\otimes \nu},$$

as needed.

Note that we could have also used the filtration on $V^{\otimes \nu_n}$ induced by the filtration on V given by $F^0 V = \Bbbk \cdot 1$ and $F^1 V = V$. I.e. the filtration, there the i-th term is spanned by all tensor monomials with no more than i elements in the product that are not equal to 1. Indeed this filtration is a sub-filtration of the filtration used above in the proof.

□

Symmetric powers of a unital algebra

In this section we will discuss a related construction in the case of the unital algebra. Here we will be concerned not with the tensor, but with symmetric powers of the unital vector space. Since the space of invariants of $V^{\otimes \nu}$ is an actual vector space, these objects will be usual vector spaces and not the objects of the Deligne category.

We will discuss the following class of algebras:

Definition 3.2.2.3. Consider A – a unital algebra. We will consider this algebra as a unital vector space with a unit given by the unit of the algebra. We call A a filtered unital algebra if there is an ascending $\mathbb{Z}_{\geq 0}$-filtration by finite-dimensional subspaces such that $\Bbbk \cdot 1 \subset F^0 A$. We will also suppose that such an algebra has a fixed vector space decomposition $A = \Bbbk \cdot 1 \oplus A'$.

To make things clearer we will start with considering everything for transcendental ν. I.e. we have $\Bbbk = \overline{\mathbb{Q}}$.

We would like to consider symmetric powers of a filtered unital algebra. I.e. we want to study the structure of $S^n(A)$. First of all note that this algebra admits a bifiltration.

Definition 3.2.2.4. For a filtered unital algebra A, introduce a standard bifiltration of the algebra $S^n(A)$ in the following way. Consider $S^n(A)$ as $(A^{\otimes n})^{S_n}$. Introduce a bifiltration on $A^{\otimes n}$ via the following formulas for horizontal and vertical degrees:

$$\deg_h(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = |\{i | a_i \notin \Bbbk \cdot 1\}| \ ,$$

$$\deg_v(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \sum_i \deg(a_i) \ .$$

It is easy to see that this bifiltration restricts on the space of invariants of S_n.

Now we can prove the following Proposition.

Proposition 3.2.2.5. *The associated graded algebra of the symmetric power $S^n(A)$ with respect to the horizontal filtration $\mathrm{gr}_h(S^n(A))$ is isomorphic to $\bigoplus_{i=0}^n S^i(A')$ as a vector space.*

Proof. Taking the associated graded with respect to the horizontal filtration allows us to use the standard splitting $A = \Bbbk \cdot 1 \oplus A'$. This allows us to view $A^{\otimes n}$ as $(\Bbbk \cdot 1 \oplus A')^{\otimes n}$. I.e. we have a decomposition of $\mathrm{gr}_h(S^n(A))$ into a direct sum $\mathrm{gr}_h(S^n(A)) = \bigoplus_{i=0}^n \mathrm{gr}_h(S^k(A))_i$, where $\mathrm{gr}_h(S^n(A))_i$ consists of symmetric tensors, the tensor monomials of which have exactly i components in A' and the rest $n-i$ components are scalars 1. I.e. we have:

$$\mathrm{gr}_h(S^n(A))_i = [\bigoplus_{\sigma \in Sh(i,n-i)} C(\sigma(1)) \otimes C(\sigma(2)) \otimes \cdots \otimes C(\sigma(n))]^{S_n} \ ,$$

where $C(1) = \cdots = C(i) = A'$, $C(i+1) = \cdots = C(n) = \Bbbk \cdot 1$ and $Sh(i, n-i)$ is the group of shuffles of i and $n-i$. Hence:

$$\mathrm{gr}_h(S^n(A))_i \simeq [A'^{\otimes i}]_i^S \otimes (\Bbbk \cdot 1)^{\otimes(n-i)} \simeq S^i(A') \ ,$$

under the symmetrizing isomorphism. Hence we conclude that:

$$\mathrm{gr}_h(S^n(A)) \simeq \bigoplus_{i=0}^{n} S^i(A') \ .$$

Notice that the horizontal grading on the l.h.s. translates exactly into the grading by the degree of symmetric power on the r.h.s and the vertical filtration on l.h.s. translates into the filtration by the sum of degrees with respect to A of the elements of the term in the symmetric product. □

Now we would like to consider an ultraproduct of such algebras:

Definition 3.2.2.6. For a filtered unital algebra A over $\Bbbk = \overline{\mathbb{Q}}$, define $S^\nu(A)$ to be equal to an algebra $\prod_{\mathcal{F}}^r S^n(A)$ over $\mathbb{C}$, where the restricted ultraproduct is taken with respect to the total filtration of the bifiltered algebras.

Obviously this algebra inherits a bifiltration from $S^n(A)$. Thus we can consider $\mathrm{gr}_h(S^\nu(A))$. We can understand this algebra with the help of the following Proposition.

Proposition 3.2.2.7. *We have a bifiltered vector space isomorphism between*

$$\mathrm{gr}_h(S^\nu(A)) \simeq S^\bullet(A') \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \ .$$

Proof. Indeed we have:

$$\mathrm{gr}_h(S^\nu(A)) = \prod\nolimits_{\mathcal{F}}^{r} \mathrm{gr}_h(S^n(A)) \simeq \bigoplus_{i=1}^{\infty} \prod\nolimits_{\mathcal{F}}^{r} \mathrm{gr}_h(S^n(A))_i \ ,$$

where the last restricted ultraproduct is taken with the respect to the filtration on $\mathrm{gr}_h(S^n(A))_i$ induced by the vertical filtration.

Now for each $n > i$, $\mathrm{gr}_h(S^n(A))_i$ has a filtered isomorphism with the same vector space $S^i(A')$. Hence $\prod_{\mathcal{F}}^{r} \mathrm{gr}_h(S^n(A))_i = S^i(A') \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. Thus we conclude:

$$\mathrm{gr}_h(S^\nu(A)) \simeq S^\bullet(A') \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \ .$$

□

To characterize this algebra more precisely we need to construct a certain map from A to each $S^n(A)$.

Proposition 3.2.2.8. *There is a map of Lie algebras $\delta_n : A \to S^n(A)$ (where the structure of the Lie algebra on both sides is given by the commutator) that sends $a \in A$ to $\sum_{i=1}^n a_i$, where $a_i = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1$, where a is on the i-th place. This gives rise to an algebra map $\delta : A \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to S^\nu(A)$ that sends $1_A \mapsto \nu \cdot 1_{\mathcal{A}}$.*

Proof. Indeed δ_n is a well-defined map and it's a standard fact that it indeed gives us a map of Lie algebras. This map also respects the bifiltration if we consider the horizontal filtration of A to be given by $F_h^0 A = \Bbbk \cdot 1$ and $F_h^1 A = A$ and use the usual filtration on A as the vertical one. Hence, taking an ultraproduct $\prod_{\mathcal{F}} \delta_n$ we obtain a well-defined map $\delta : A \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to S^\nu(A)$.

Now notice that under this map δ_n we have $\delta_n(1) = n \cdot 1 \otimes 1 \otimes \cdots \otimes 1$. Now, the element $1 \otimes \cdots \otimes 1 \in S^n(A)$ is the unity of this algebra. Thus $\prod_{\mathcal{F}} 1 \otimes \cdots \otimes 1$ is the unity of $S^\nu(A)$. So, we conclude that:

$$\delta(1_A) = \prod\nolimits_{\mathcal{F}} \delta_n(1_A) = \left(\prod\nolimits_{\mathcal{F}} n\right) \cdot \prod\nolimits_{\mathcal{F}} 1 \otimes \cdots \otimes 1 = \nu \cdot 1_{\mathcal{A}} \ .$$

□

This map allows us to define a map from $U(A)$:

Definition 3.2.2.9. Denote by Δ_n a map from the universal enveloping algebra $U(A)$ to $S^n(A)$ arising from the map δ_n.

Now note that there is a bifiltration on $U(A)$ which comes from the bifiltration on $T^\bullet(A)$ arising from the bifiltration on A and given by the same formulas as in Definition 3.2.2.4. With this filtration each Δ_n is a bifiltered morphism. This allows us to take their ultraproduct:

Lemma 3.2.2.10. *The ultraproduct $\Delta = \prod_{\mathcal{F}} \Delta_n$ is a well defined bifiltered morphism from $U(A) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ to $S^\nu(A)$.*

Now we would like to prove that Δ is a surjective map.

Lemma 3.2.2.11. *The map Δ is surjective.*

Proof. It's enough to prove that all Δ_n are surjective and so it is enough to prove that $S^n(A)$ is generated by the image of δ_n. We will do so by induction on the degree of the horizontal filtration.

Now $F^{1,\bullet}S^n(A)$ is precisely the image of δ_n so the base of induction is clear.

Suppose that $F^{i-1,\bullet}S^n(A)$ is generated by the image of δ_n. Suppose $f \in F^{i,\bullet}S^n(A)$. Now using the isomorphism of $\mathrm{gr}_h(S^n(A))$ with $\sum_{j=0}^n S^j(A')$, we may assume that $f = \widetilde{f} + g$, where $\widetilde{f} = \sum \widetilde{f}_l$ and each $\widetilde{f}_l = a_1^{(l)} \otimes a_2^{(l)} \otimes \cdots \otimes a_n^{(l)} \otimes 1 \otimes \cdots \otimes 1 + \text{shuffles}$, where each $a_i^{(l)} \in A'$ and $g \in F^{i-1,\bullet}S^n(A)$.

But now

$$h_l = \delta(a_1^{(l)})\delta(a_2^{(l)})\dots\delta(a_n^{(l)}) =$$

$$= a_1^{(l)}\otimes a_2^{(l)}\otimes\cdots\otimes a_n^{(l)}\otimes 1\otimes\cdots\otimes 1 + \text{shuffles} + \text{lower order terms in horizontal filtration}\ .$$

Hence $f - \sum h_l \in F^{i-1,\bullet}S^k(A)$ and we are done. □

Now since we know that $\Delta(1_A) = \nu \cdot 1_{\mathcal{A}}$ it follows that $1_A - \nu \in \ker(\Delta)$.

Proposition 3.2.2.12. *The map* $\widetilde{\Delta} : U(A)/(1_A - \nu) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to S^\nu(A)$ *is a filtered algebra isomorphism.*

Proof. We already know that this map is surjective. Now this map induces a graded map of the associated graded algebras with respect to the horizontal filtration. We know that $\mathrm{gr}_h(S^\nu(A)) = S^\bullet(A')$. Now $\mathrm{gr}_h(U(A)/(1_a - \nu))$ is isomorphic to $\mathrm{gr}(U(A')) \simeq S^\bullet(A')$. Hence, since the map is surjective, it also has to be injective. Hence $\widetilde{\Delta}$ is an isomorphism. □

Remark 3.2.2.13. The same construction can be repeated in the case of algebraic ν. In order to do so we should consider a lattice filtered unital algebra $A_{\mathbb{Z}}$ defined over $\mathbb{Z}$ and the sequence of algebras $A_n = A_{\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_{p_n}$.

In this case as we know $\nu_n < p_n$, all of the constructions which use the isomorphisms related to symmetric invariants work in the same way and we can still define the ν-symmetric power as $S^\nu(A) = \prod_{\mathcal{F}}^r S^{\nu_n}(A_n)$. Everything else can be repeated and we obtain a similar isomorphism $\widetilde{\Delta} : U(A_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C})/(1_A - \nu) \to S^\nu(A)$.

3.2.3 The category $\mathbf{Rep}(S_\nu \ltimes \Gamma^\nu)$

In this section we will explain how the category of representations of the wreath product in complex rank can be constructed.

There are several ways to approach this problem. One construction was developed by Knop in [38]. Another approach can be found in [42]. However, in the present thesis we will use a different approach, outlined in [13], that uses the notion of a unital vector space outlined in the previous section. For brevity we will only address the case of transcendental ν in this section, although with slight modifications the results can be extended to the algebraic case as well.

Consider a finite subgroup $\Gamma \subset \mathrm{SL}(2, \overline{\mathbb{Q}})$. Proposition 3.2.2.2 allows us to define the following algebra:

Definition 3.2.3.1. An ind-object $\mathbb{C}[\Gamma]^{\otimes \nu}$ is constructed via Proposition 3.2.2.2 starting with $\overline{\mathbb{Q}}[\Gamma]$ as a unital vector space. It has the structure of the algebra given by the ultraproduct of the algebra structures on $\overline{\mathbb{Q}}[\Gamma]^{\otimes n}$.

Using this, one can define the category $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$ in the following way:

Definition 3.2.3.2. The category $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$ is the category of $\mathbb{C}[\Gamma]^{\otimes \nu}$-modules in $\mathrm{Rep}(S_\nu)$. I.e., its objects are objects of $\mathrm{Rep}(S_\nu)$) with the structure of a $\mathbb{C}[\Gamma]^{\otimes \nu}$-module, and its morphisms are morphisms in $\mathrm{Rep}(S_\nu)$ which commute with the module structure.

It can be shown that $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$ is equivalent to the wreath product category defined by Knop.

We can construct some of the objects of $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$ as ultraproducts.

Proposition 3.2.3.3. *Consider a sequence of modules $M_n \in \mathbf{Rep}_0(S_n \ltimes \Gamma^n)$ whose ultraproduct as S_n-modules is a well-defined object of Rep(S_ν). Then, this ultraproduct also lies in Rep$(S_\nu \ltimes \Gamma^\nu)$.*

Proof. Denote $M = \prod_{\mathcal{F}}^C M_n$. Indeed since M_n has a structure of a $\overline{\mathbb{Q}}[\Gamma]^{\otimes n}$-module in

$\mathbf{Rep}_0(S_n)$, it follows that M has a structure of $\prod_{\mathcal{F}}^{C}\overline{\mathbb{Q}}[\Gamma]^{\otimes n} = \mathbb{C}[\Gamma]^{\otimes \nu}$-module. Hence it is an object of $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$. $\square$

In this way we can interpolate irreducible objects of $\mathbf{Rep}_0(S_n \ltimes \Gamma^n)$.

Definition 3.2.3.4. In the notation of Proposition 2.1.1.3, consider λ to be any function:

$$\lambda : A \to \{\text{Partitions}\}.$$

Denote by $\mathcal{X}(\lambda)$ the object of $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$ defined as:

$$\mathcal{X}(\lambda) = \prod_{\mathcal{F}}^{C} X(\lambda_n),$$

where $\lambda_n(\text{triv}) = \lambda(\text{triv})|_n$ and $\lambda_n(\alpha) = \lambda(\alpha)$ for all other irreducibles α of Γ.

It follows that $\mathcal{X}(\lambda)$ is irreducible.

Remark 3.2.3.5. We leave out the proof of the fact that these ultraproducts indeed define an object of $\mathrm{Rep}(S_\nu)$. This can be done using the results of [38], but this is not required for the results we are going to establish in the present thesis.

Part II

Applications

Chapter 4

Classification of simple algebras in $\mathrm{Rep}(S_\nu)$

This chapter will concern the first application of the construction of $\mathrm{Rep}(S_\nu)$ through ultraproducts. We will show how one can use this construction in order to classify all simple commutative, associative and Lie algebras in $\mathrm{Rep}(S_\nu)$.

We will begin by providing a number of technical lemmas concerned with representation theory of symmetric group of big enough rank. Then we will use it to classify simple associative (and commutative as a special case) algebras in $\mathrm{Rep}(S_\nu)$. This will also allow us to state an interesting result concerning the existence of symmetric monoidal functors between Deligne categories. Then in the last sections we will classify simple Lie algebras in $\mathrm{Rep}(S_\nu)$ and provide a conjecture regarding the classification of Lie super-algebras in the same category.

4.1 Technical results on representations of S_N.

In this section we prove a few rather technical lemmas which we will use extensively in our proofs of the classification. The reader can skip this section at first, and then go back to it when the need arises.

4.1.1 Facts about small-index subgroups in S_N.

In this subsection we will prove that under some restrictions on the index of a subgroup of S_N it is conjugate to either $\mathcal{A}_n \times H$ or $S_n \times H$, where H is a subgroup of S_{N-n}.

So suppose $N > 10$, r an integer less then $N/2$, and G a subgroup of S_N of index less than $\binom{N}{r}$. First following Theorem 5.2 in [9] we have the following proposition:

Proposition 4.1.1.1. *Under the above assumptions, up to a conjugation, G contains the group $\mathcal{A}_{N-j}$ with $j < r$, where $\mathcal{A}_{N-j}$ is the group of even permutations of the first $N-j$ elements.*

Now we have the second result:

Proposition 4.1.1.2. *Suppose G is a subgroup of S_N which contains $\mathcal{A}_{N-j}$ and $N > 2j+7$, then G is conjugate to either $S_{N-j'} \times H$ or $\mathcal{A}_{N-j'} \times H$ for some $H \subset S_{j'}$ and $j' \leq j$.*

Proof. Let's consider the standard action of S_N on N elements. Consider the orbit of the first element under the action of G. By assumption it contains the first $N-j$ elements. Up to taking a group conjugate to G (we conjugate by an element fixing the first $N-j$ elements) we may assume that the orbit of the first element under G is equal to the first $N-j'$ elements for $j' \leq j$. We want to prove that G contains

$\mathcal{A}_{N-j'}$. To do this, it is enough to prove that any 3-cycle consisting of the first $N-j'$ elements belongs to G.

Let's denote by B the set of the first $N-j$ elements and by C the set of the $j-j'$ elements directly after B. So we need to consider 3-cycles of four types.

The first case is a 3-cycle consisting solely of elements of B. It is trivial by assumption.

The second case is a 3-cycle permuting elements x, y, z such that $x, y \in B$ and $z \in C$. Also let's denote the first element by 1. Since z belongs to the orbit of 1 under G, it follows that $\exists g \in G$ such that $g(z) = 1$. Now since $|B| = N - j$ is bigger than j by at least 6, it follows by the pigeonhole principle that there exist two elements $a, b \in B$ such that $g(a), g(b) \in B$ and all $a, b, g(a), g(b), x, y$ are distinct. Now consider a double transposition τ which interchanges $a \leftrightarrow x$ and $b \leftrightarrow y$, it belongs to $\mathcal{A}_{N-j}$ and hence to G. Now consider a 3-cycle π permuting $g(a), g(b)$ and 1. It also belongs to $\mathcal{A}_{N-j}$ and hence G. Now $\tau g^{-1} \pi g \tau \in G$ is a 3-cycle permuting x, y, z. Indeed if $c \in N$ is not equal to a, b, x, y, z then both π and τ act trivially, hence c maps to c under the above map. The elements a and m first map to x, y accordingly, then under g they map to something on which π acts trivially, so they are mapped back and then back to a and b. Now (x, y, z) first map to (a, b, z) then to $(g(a), g(b), 1)$, then to $(g(b), 1, g(a))$, then to (b, z, a) and to (y, z, x).

The third case is $x \in B$ and $y, z \in C$. Again suppose $g \in G$ maps z to 1. Again by the pigeonhole principle there are $a, b, c \in B$ such that all three $g(a), g(b), g(c) \in B$ and $a, b, c, g(a), g(b), g(c), x$ are distinct. By τ denote the double transposition interchanging $a \leftrightarrow x$ and $b \leftrightarrow c$, as before $\tau \in G$ by the assumptions. Now using two previous cases it follows that a 3-cycle π which permutes $g(a), 1, g(z)$ belongs to G. Hence by the same logic as above $\tau g^{-1} \pi g \tau \in G$ is the required 3-cycle.

The final case is $x, y, z \in C$. As before, fix $g \in G$ mapping z to 1. By the above cases there is a 3-cycle $\pi \in G$ permuting $g(y), g(z), 1$. Then $g^{-1}\pi g$ is the required cycle.

Hence $\mathcal{A}_{N-j'} \subset G'$, where G' is a group conjugate to G. Since the orbit of 1 consists of the first $N - j'$ elements, it follows that $G' \subset S_{N-j'} \times S_{j'}$. By the above discussion we are limited to the two cases: $G' = \mathcal{A}_{N-j'} \times H$ or $G' = S_{N-j'} \times H$, where $H \subset S_{j'}$. □

Now we are ready to state the main theorem of this section:

Theorem 4.1.1.3. *Suppose $G \subset S_N$ has index less than $\binom{N}{r}$ for $N > 2r + 8$. Then G is conjugate either to $S_{N-j} \times H$ or $\mathcal{A}_{N-j} \times H$ for some $H \subset S_j$ and $j \leq r$.*

Proof. Using Proposition 4.1.1.1 we conclude that the conjugate group G' contains $\mathcal{A}_{N-j'}$ for $j' < r$. Now using the Proposition 4.1.1.2 we conclude that the conjugate group G'' is equal to either $S_{N-j} \times H$ or $\mathcal{A}_{N-j} \times H$ for some $H \subset S_j$ and $j \leq j' \leq r$, since $N > 2r + 7 \geq 2j' + 7$. □

4.1.2 Lemmas on ultraproducts of representations of S_{ν_n}.

Now we will prove some other results concerned with sequences of representations of S_{ν_n} that give us an element of $\mathrm{Rep}(S_\nu)$.

Lemma 4.1.2.1. *Suppose V is an object of $Rep(S_\nu)$ such that $V = \prod_{\mathcal{F}} V_n$ and $V_n = Ind_{G_n}^{S_{\nu_n}}(W_n)$ for some subgroup $G_n \subset S_{\nu_n}$. Then it follows that $G_n = S_{\nu_n - j} \times H$ for some $j \in \mathbb{Z}_{>0}$ and $H \subset S_j$, for almost all n. Also $W = \prod_{\mathcal{F}} W_n$ is an object of $Rep(S_{\nu-j}) \boxtimes \boldsymbol{Rep}(H; \mathbb{C})$, hence $V = Ind_{S_{\nu-j} \times H}^{S_\nu}(W)$.*

Proof. Suppose V is equal to the sum of $l(V)$ simple objects of $\mathrm{Rep}(S_\nu)$ such that each one is a subobject of $[m]$ with $m \leq m(V)$. Then for almost all n we have V_n being equal to the sum of $l(V)$ irreducible representations included in V^m for $m \leq m(V)$. Hence for almost all n we have $\dim V_n \leq l(V) \cdot (\nu_n)^{m(V)}$. But since $V_n = \mathrm{Ind}_{G_n}^{S_{\nu_n}}(W_n)$ for $G_n \subset S_{\nu_n}$, we know that $\dim V_n = \dim W_n \cdot |S_{\nu_n}|/|G_n| \geq |S_{\nu_n}|/|G_n|$. Hence we obtain the following inequality:

$$l(V) \cdot (\nu_n)^{m(V)} \geq |S_{\nu_n}|/|G_n| \ .$$

So we have a subgroup $G_n \subset S_{\nu_n}$ with the index bounded by $l(V) \cdot \nu_n^{m(V)}$. Since $\binom{\nu_n}{m(V)+1}$ is a polynomial of degree $m(V)+1$ with the highest term being equal to $\frac{\nu_n^{m(V)+1}}{(m(V)+1)!}$ it follows that all but finite number of ν_n we have $\binom{\nu_n}{m(V)+1} \geq l(V) \cdot (\nu_n)^{m(V)}$. Hence for almost all n, G_n satisfies the condition of Theorem 4.1.1.3 with $N = \nu_n$ and $r = m(V)+1$. Thus for almost all n we have, after a conjugation, $G_n' = S_{\nu_n - j_n} \times H_n$ or $G_n' = \mathcal{A}_{\nu_n - j_n} \times H_n$ for $j \leq m(V)+1$ and $H_n \subset S_{j_n}$.

For conjugate subgroups G and G' the objects $\mathrm{Ind}_G^{S_N}(U)$ and $\mathrm{Ind}_{G'}^{S_N}(U)$ are isomorphic for the correct choice of the action of G and G' on U. Hence we may suppose that $V_n = \mathrm{Ind}_{G_n'}^{S_{\nu_n}}(W_n)$. But there is a finite number of subgroups H in S_j for $j \leq m(V)+1$, hence there is a finite number of ways to choose G_n' for every n. Thus (note that here we Lemma 2.3.1.4) for almost all n we have the same $j_n = j$ and $H_n = H \subset S_j$, and $G_n = S_{\nu_n - j} \times H$ or $G_n = \mathcal{A}_{\nu_n - j} \times H$ for almost all n. First we need to rule out the possibility $G_n = \mathcal{A}_{\nu_n - j} \times H$.

So suppose $G_n = \mathcal{A}_{\nu_n - j} \times H$. Then

$$V_n = \mathrm{Ind}_{G_n}^{S_{\nu_n}}(W_n) = \mathrm{Ind}_{S_{\nu_n - j} \times S_j}^{S_{\nu_n}}(U_n) \ ,$$

where U_n is an $S_{\nu_n - j} \times S_j$-module with an action of $S_{\nu_n - j}$ induced from the action of $\mathcal{A}_{\nu_n - j}$. But any such representation is equivalent to itself tensored with the sign representation, hence if a partition λ appears in the decomposition, so does its conjugate λ^*.

However for any partition we have that $l(\lambda) \times l(\lambda^*) \geq |\lambda|$, so in particular $\max\{l(\lambda), l(\lambda^*)\} \geq \sqrt{|\lambda|}$. Therefore any representation induced from $\mathcal{A}_{\nu_n - j}$ to $S_{\nu_n - j}$ contains an irreducible component corresponding to a partition of length at least $\sqrt{\nu_n - j}$. So

$$U_n = \sum_i X(\lambda_i) \otimes X(\mu_i) \ ,$$

where $|\lambda_i| = \nu_n - j$, $|\mu_i| = j$ and at least one of λ_i is of length at least $\sqrt{\nu_n - j}$.

So we have:

$$V_n = \bigoplus_{i,\zeta} X(\zeta)^{\oplus c^{\zeta}_{\lambda_i,\mu_i}}, \tag{4.1}$$

where ζ are partitions of ν_n and c's are the Littlewood-Richardson coefficients.

Suppose λ_j is of length at least $\sqrt{\nu_n - j}$. Then there is ζ such that $c^{\zeta}_{\lambda_j,\mu_j} \neq 0$ and hence ζ contains λ_i and thus $l(\zeta) \geq l(\lambda_j) \geq \sqrt{\nu_n - j}$. So the lengths of Young diagrams appearing in V_n are unbounded. But this contradicts V being an object of the Deligne category, because all the simple objects appearing in V lie in $\mathcal{X}^{\otimes m}$ for some bounded m, and hence the length of the Young diagrams appearing in V_n should be bounded for almost all n. Hence $G_n = S_{\nu_n - j} \times H$.

So $V_n = \mathrm{Ind}^{S_{\nu_n}}_{S_{\nu_n - j} \times H}(W_n)$. The last thing to check is that $\prod_{\mathcal{F}} W_n$ is an object of the Deligne category. By the Remark 3.1.0.5, writing $W_n = \oplus W_n^k \otimes U_k$, where U_k is all possible irreducible representations of H, it's enough to check that each sequence W_n^k gives an object of $\mathrm{Rep}(S_\nu)$. As we know from our previous discussions this is true iff the number of irreducible representations in the sequence W_n^k for each k is bounded and the number of boxes in the corresponding Young diagrams in all the rows except the first one is bounded too. But note that when we induce, each representation in W_n^k gives us at least one irreducible representation in the resulting object, so if the number of irreducible representation is unbounded here it is also unbounded in V_n. Also if the number of boxes in all the rows except the first one is unbounded, then it follows that the number of boxes in the irreducible components of V_n is also unbounded. Indeed by Littlewood-Richardson rule we only add boxes to diagrams when applying induction. Hence for V to lie in $\mathrm{Rep}(S_\nu)$, W also should lie in $\mathrm{Rep}(S_{\nu-j}) \boxtimes \mathbf{Rep}(H;\mathbb{C})$. So we are done and $V = \mathrm{Ind}_{S_{\nu-j}\times H}^{S_\nu}(W)$. $\square$

The next lemma concerns the projective representations of S_n. Denote by $\widehat{S}_n$ the double cover of S_n. We may regard projective representations of S_n as a linear representations of $\widehat{S}_n$. We will need the following result ([36]):

Theorem 4.1.2.2. *Suppose $n > 12$ and $p \neq 2$, then any irreducible projective representation of S_n, which is faithful as a $\widehat{S}_n$-representation has dimension at least:*

$$\min\left(2^{\lfloor\frac{n-1-\kappa_n}{2}\rfloor}, 2^{\lfloor\frac{n-2-\kappa_{n-1}}{2}\rfloor}(n-2-\kappa_n-2\kappa_{n-1})\right) ,$$

where κ_n is 1 if $p|n$ and 0 otherwise.

Under the same assumptions, any irreducible projective representation of $\mathcal{A}_n$, which is faithful as a $\widehat{\mathcal{A}}_n$-representation has dimension at least:

$$\min\left(2^{\lfloor\frac{n-2-\kappa_n}{2}\rfloor}, 2^{\lfloor\frac{n-3-\kappa_{n-1}}{2}\rfloor}(n-2-\kappa_n-2\kappa_{n-1})\right) ,$$

where κ_n is the same.

Since the only nontrivial normal subgroups of $\widehat{S}_n$ are $\widehat{\mathcal{A}}_n$ and the central subgroup, it follows that any non-linear representation of S_n satisfies the condition of the above theorem. Now we can apply this to obtain the following lemma:

Lemma 4.1.2.3. *Suppose W_n is a sequence of projective representations of S_{ν_n}(or $\mathcal{A}_{\nu_n}$) for some unbounded sequence ν_n, such that* $\dim W_n \leq M\nu_n^L$. *Then almost all W_n are actually linear representations of S_{ν_n} (or $\mathcal{A}_{\nu_n}$).*

Proof. Suppose the action of S_{ν_n} on W_n is non-linear for almost all n. Then by the above theorem it follows that for almost all n we have $\dim W_n \geq 2^{\lfloor \frac{\nu_n - 4}{2} \rfloor}$ and hence $\dim W_n \geq 2^{\nu_n - 5}$. So we get that for almost all n $M\nu_n^l \geq 2^{\nu_n-5}$, which is a contradiction since this inequality holds only for a finite number of n.

The same proof with $\dim W_n \geq 2^{\nu_n - 6}$ instead of $\dim W_n \geq 2^{\nu_n-5}$ holds for $\mathcal{A}_{\nu_n}$. □

To prove the last lemma we will need to use another result, namely Lemma 2.8 from [13]:

Lemma 4.1.2.4. *For each $C > 0$ and $k \in \mathbb{Z}_+$ there exists $N(C,k) \in \mathbb{Z}_+$ such that for each $m > N(C,k)$, if $X(\mu)$ is an irreducible representation of S_m which has dimension* $\dim X(\mu) \leq Cm^k$, *then either the first row or the first column of μ has length $\geq m - k$.*

Now to state our lemma we will need the following definitions:

Definition 4.1.2.5. a) For an object W of a symmetric tensor category define $\mathfrak{gl}(W)$ to be the object $W \otimes W^*$. It has a structure of an associative algebra given by $1 \otimes ev \otimes 1 : \mathfrak{gl}(W) \otimes \mathfrak{gl}(W) \to \mathfrak{gl}(W)$, and thus has a structure of a Lie algebra given by a commutator. **b)** For an object W of a symmetric tensor category define $\mathfrak{sl}(W)$ to be the Lie algebra given by the kernel of the map $ev : \mathfrak{gl}(W) \to \mathbf{1}$. In case of the category $Vect$ this algebra is simple iff the map $\mathbf{1} \to \mathbf{1}$ given by the composition of the evaluation and the coevaluation maps for W is not zero.
c) For an object W of a symmetric tensor category such that the above map $\mathbf{1} \to \mathbf{1}$

is zero, define $\mathfrak{psl}(W)$ to be the cokernel of the map $coev : \mathbf{1} \to \mathfrak{sl}(W)$. In case of the category $Vect$ this algebra is simple.

d) For an object W of a symmetric tensor category equipped with a (skew-)symmetric non-degenerate bilinear form (an isomorphism $\psi : W \to W^*$), define $\mathfrak{so}(W)(\mathfrak{sp}(W))$ to be the Lie subalgebra in $\mathfrak{gl}(W)$ given by the kernel of $\sigma \circ \psi \otimes \psi^{-1} + Id$. In the case of the category $Vect$ this algebra is simple.

Lemma 4.1.2.6. *Suppose V is an object of $Rep(S_\nu)$ given by the ultraproduct of $V_n \in \boldsymbol{Rep}_{p_n}(S_{\nu_n})$, almost all of which are isomorphic to $End(W_n)$ (or $\mathfrak{sl}(W_n)$, $\mathfrak{psl}(W_n)$, $\mathfrak{so}(W_n)$, $\mathfrak{sp}(W_n)$ if these objects are defined), for some $W_n \in \boldsymbol{Rep}(S_{\nu_n})$. Then there exist $W_n' \in \boldsymbol{Rep}(S_{\nu_n})$ such that $End(W_n) \simeq End(W_n')$(or the corresponding Lie algebras are defined and isomorphic) and $W = \prod_{\mathcal{F}} W_n'$ is an object of $Rep(S_\nu)$. Hence $V = W \otimes W^*,\ \mathfrak{sl}(W),\ \mathfrak{psl}(W),\ \mathfrak{so}(W), \mathfrak{sp}(W)$.*

Proof. Suppose $V = \sum_{i=1}^{M} \mathcal{X}(\lambda_i)$, where all $\mathcal{X}(\lambda_i)$ lie in $[k]$ for $k \leq L$. Then for almost all n ($n > 3$)

$$\dim W_n \leq \dim V_n \leq M(\nu_n)^L,$$

so by Lemma 4.1.2.4 there is a number $N(M, L)$ such that for any $n > N(M, L)$ any irreducible representation $X(\mu)$ appearing in W_n is such that either the first row or the first column of μ has length $\nu_n - L$.

Hence for almost all n we have $W_n = \oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)}$, with each $\mu_j^{(n)}$ having the first row of length at least $\nu_n - L$, and $\mathcal{E}_j^{(n)}$ being either a one-dimensional trivial or a sign representation. Let's denote by W_n' a representation equal to $W_n' = \oplus X(\mu_j^{(n)})$.

We need to check several cases.

- $V_n = \mathrm{End}(W_n)$

Since the number of summands in $V_n = W_n \otimes W_n$ is bounded and is bigger or equal than the number of summands in W_n it follows that the number of summands in the latter representation is bounded. Hence, since there is a finite number of ways to put L boxes into the rows of a Young diagram, it follows that for almost all n we have $W_n = \oplus X(\mu_j|_n) \otimes \mathcal{E}_j^{(n)}$, for some partitions μ_j of weight at most L. But now since there are also a finite number of ways to assign to each j either a trivial or a sign representation it follows that for almost all n the same assigment is used and so we can write $W_n = \oplus X(\mu_j|_n) \otimes \mathcal{E}_j$. Now, if one of $\mathcal{E}_j$ is the sign representation and another $\mathcal{E}_i$ is the trivial representation then the ultraproduct of $X(\mu_j|_n) \otimes X(\mu_i|_n) \otimes \mathcal{E}_j \otimes \mathcal{E}_i = (\oplus X(\eta_j|_n)) \otimes sgn$ is not an object of $\mathrm{Rep}(S_\nu)$ since the number of rows is unbounded. But this contradicts V being the object of $\mathrm{Rep}(S_\nu)$, hence $\mathcal{E}_j$ are all trivial or all sign representations. In the latter case taking W'_n instead of W_n does not change $W_n \otimes W_n = W'_n \otimes W'_n$ (since $W'_n \otimes W'_n = W_n \otimes W_n \otimes sgn^{\otimes 2}$). But hence $W = \prod_{\mathcal{F}} W'_n = \oplus \mathcal{X}(\mu_j)$ is well defined.

- $\mathfrak{sl}$ or $\mathfrak{psl}$

In the first case we subtract one trivial representation of S_{ν_n} and in the second case two trivial representations, so it follows that the number of summands in W_n is still bounded. Using the same reasoning as above it also follows that $\mathcal{E}_j$ are all trivial or are all sign representations since subtracting trivial representations from $W_n \otimes W_n$ cannot delete a representation with a big number of rows. So it follows that we again can take W'_n and get the same $\mathrm{End}(W_n) = \mathrm{End}(W'_n)$ and hence the same Lie algebra.

- $\mathfrak{so}$

Let's write W_n as $W_n = \oplus X(\lambda_j^{(n)}) \otimes U_j$. There all $\lambda_j^{(n)}$ are different for different j and U_j are trivial (but not necessarily one-dimensional) representations of S_{ν_n}. Now, an invariant symmetric bilinear form on W_n is given by an isomorphism $\phi : W_n \to W_n^*$. Since all irreducible representations of S_n are real and self-dual, by Schur's lemma it follows that an isomorphism $\phi : W_n \to W_n^*$ decomposes to the sum of isomorphisms $\phi_j : X(\lambda_j^{(n)}) \otimes U_j \to X^*(\lambda_j^{(n)}) \otimes U_j^*$ given by the tensor product of the isomorphism

$X(\lambda_j^{(n)}) \to X^*(\lambda_j^{(n)})$ and an isomorphism $\psi_j : U_j \to U_j^*$. So our symmetric invariant bilinear form is the sum of products of the symmetric invariant forms on $X(\lambda_j^{(n)})$ and invariant forms on U_j. Thus the forms on U_j also should be symmetric. But then up to change of basis we can assume that the invariant bilinear form pairs $\oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)}$ to itself in the decomposition $W_n = \oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)}$. Hence $\mathfrak{so}(W_n)$ contains copies of all tensor products of $\oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)} \otimes \oplus X(\mu_i^{(n)}) \otimes \mathcal{E}_i^{(n)}$ for $i \neq j$. This means first that the number of summands in W_n is bounded. And that the previous argument can be again used to prove that all $\mathcal{E}_j$ are either trivial or sign. So we can again take W_n' instead of W_n to obtain the same Lie algebra.

- $\mathfrak{sp}$

Here the discussion in the previous paragraph can be repeated, but the invariant bilinear form on U_j should be skew-symmetric. Hence all U_j are even-dimensional. Again up to the change of basis in U_j, we can write down W_n as $\oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)} \otimes \overline{\mathbb{F}}_{p_n}^2$ where invariant form is given by sum of products of invariant forms on $\oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)}$ with standard skew-symmetric form on $\overline{\mathbb{F}}_{p_n}^2$. Thus again it follows that $\mathfrak{sp}(W_n)$ contains copies of $\oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)} \otimes \oplus X(\mu_i^{(n)}) \otimes \mathcal{E}_i^{(n)}$ for $i \neq j$. So all the previous arguments can be repeated. □

4.2 Classification of simple associative algebras in $\mathrm{Rep}(S_\nu)$ and functors between Deligne categories.

Now we have everything we need to classify simple associative algebras in $\mathrm{Rep}(S_\nu)$. This will in fact lead us to also prove a result concerning existence of symmetric monoidal functor between $\mathrm{Rep}(S_\nu)$ and $\mathrm{Rep}(S_\nu')$ for $\nu, \nu' \in \mathbb{C}$.

4.2.1 Classification.

To classify simple associative algebras in $\mathrm{Rep}(S_\nu)$ let us first remind ourselves of the classification of such algebras in $\mathbf{Rep}_p(S_N)$. There is the following way of constructing such algebras. Fix $G \subset S_N$ and a simple associative algebra $\mathrm{Mat}_m(\overline{\mathbb{F}}_p)$ with an action of G. From this information we can construct the algebra $\mathrm{Fun}_G(S_N, \mathrm{Mat}_m(\overline{\mathbb{F}}_p))$ in the category $\mathbf{Rep}_p(S_N)$, which is equal to $\mathrm{Ind}_G^{S_N}(\mathrm{Mat}_m(\overline{\mathbb{F}}_p))$ as a representation. We have the following theorem (see for example [15], where it is formulated for any group):

Theorem 4.2.1.1. *Fix an algebraically closed field* $\Bbbk$. *Any simple associative algebra in* $Rep(S_N, \Bbbk)$ *is isomorphic to* $Fun_G(S_N, Mat_m(\Bbbk))$ *and all such algebras are simple. Moreover* G *is defined up to conjugation in* S_N *and the action of* G *on* $Mat_m(\Bbbk)$ *up to conjugation in* $Aut(Mat_m(\Bbbk))$.

Now by Łoś's theorem simple associative algebras in $\mathrm{Rep}(S_\nu)$ are given by ultraproducts of simple associative algebras in $\mathbf{Rep}_{p_n}(S_{\nu_n}) = \mathcal{C}_n$ such that their ultraproduct as objects of $\mathcal{C}_n$ lies in $\mathrm{Rep}(S_\nu)$.

So suppose $A \in \mathrm{Rep}(S_\nu)$ is a simple associative algebra in $\mathrm{Rep}(S_\nu)$, which is equal to the ultraproduct of $\mathrm{Ind}_{G_n}^{S_{\nu_n}}(B_n)$, where B_n are matrix algebras. Then we can apply Lemma 4.1.2.1 to conclude that for almost all n we have $G_n = S_{\nu_n - j} \times H$, $B = \prod_{\mathcal{F}} B_n$ is an object of $\mathrm{Rep}(S_{\nu-j}) \boxtimes \mathbf{Rep}(H; \mathbb{C})$ and $A = \mathrm{Ind}_{S_{\nu-j}\times H}^{S_\nu}(B)$.

For the next step, we need to understand which sequences of B_n are admissible and wat can we obtain as the result of taking their ultraproduct. We know that $B_n = \mathrm{Mat}_{m_n}(\overline{\mathbb{F}}_{p_n})$ with a structure of a representation of $S_{\nu_n - j} \times H$. Let's slightly change the notation and denote $B_n = \mathrm{End}(V_n)$, where V_n are some finite-dimensional spaces over $\overline{\mathbb{F}}_{p_n}$. Since $S_{\nu_n - j} \times H$ acts by algebra automorphisms on B_n, we have a homomorphism $S_{\nu_n - j} \times H \to \mathrm{Aut}(B_n) = PGL(V_n)$. So we have a structure of a projective representation of $S_{\nu_n - j} \times H$ on V_n. But note that $\dim V_n \le \dim B_n$ which is bounded by some $M(\nu_n - j)^L$ since $B = \prod_{\mathcal{F}} B_n$ is an object of the Deligne category. So Lemma 4.1.2.3 can be applied, and hence we conclude that the structure of the representation of $S_{\nu_n - j}$ is linear and not projective.

Thus each V_n is a representation of $S_{\nu_n - j}$ together with a projective action of H. Now we want to prove that $\prod_{\mathcal{F}} V_n$ as a representation of $S_{\nu_n - j}$ is a well-defined object of $\mathrm{Rep}(S_{\nu - j})$. But this follows from Lemma 4.1.2.6. So indeed we have $V = \prod_{\mathcal{F}} V_n$ an object of $\mathrm{Rep}(S_{\nu - j})$.

Now we are ready to state the classification theorem:

Theorem 4.2.1.2. *Suppose A is a simple associative algebra in Rep(S_ν), then it is isomorphic to $Ind_{S_{\nu-j} \times H}^{S_\nu}(B)$, where $j \in \mathbb{Z}_+$, $H \subset S_j$ and B equals to $V \otimes V^*$, where V is an object of Rep($S_{\nu-j}$), together with an action of H on $V \otimes V^*$ by algebra automorphisms. Any algebra obtained in this way is a simple associative algebra in Rep(S_ν).*

Moreover, H is defined uniquely up to conjugation in S_j, and the structure of a H-representation on $V \otimes V^*$ is defined uniquely up to conjugation inside the automorphism space $Aut_{Ass-alg}(V \otimes V^*)$.

Proof. From the above discussion it follows that $A = Ind_{S_{\nu-j} \times H}^{S_\nu}(B)$, for $B = \prod_{\mathcal{F}} B_n$ and that $B_n = V_n \otimes V_n^*$ for V_n with a projective action of H and linear action of $S_{\nu_n - j}$ which commute with each other. Moreover $\prod_{\mathcal{F}} V_n = V$ is a well-defined object of $\mathrm{Rep}(S_{\nu - j})$.

Suppose $V = \bigoplus \mathcal{X}(\lambda_j) \otimes \mathbb{C}^{k_j}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$, then a projective action of H on each V_n is given by the map $\rho_n : H \to \left(\bigoplus GL(\mathbb{C}^{k_j})\right)/(\{c \cdot Id\})$. But there is a finite number of such maps up to conjugacy, hence for almost all n they are the same and we get the projective action of H on V itself. Which is the same as the action of H on $V \otimes V^*$ by algebra automorphisms.

Also any such algebra is simple by Łoś's theorem .

Now we only need to check the uniqueness statement. Suppose we have two algebras $\mathrm{Ind}^{S_\nu}_{S_{\nu-j}\times H}(V\otimes V^*)$ and $\mathrm{Ind}^{S_\nu}_{S_{\nu-j'}\times H'}(W\otimes W^*)$. These algebras are isomorphic iff almost all algebras in the corresponding ultraproducts are isomorphic. But by Theorem 4.2.1.1 it follows that this is only possible iff $S_{\nu_n-j} \times H$ and $S_{\nu_n-j'} \times H'$ are conjugate in S_{ν_n} for almost all n, $\mathrm{End}(V_n) = \mathrm{End}(W_n)$ and the actions of $S_{\nu_n-j} \times H$ and $S_{\nu_n-j'} \times H'$ are conjugate in $\mathrm{Aut}(\mathrm{End}(V_n))$. So it follows that $j = j'$ (for $\nu_n > 2\max(j, j')$) for almost all n and hence H is conjugate to H' inside S_j (since the conjugation should leave S_{ν_n-j} invariant). Also it follows that V_n and W_n must have the same dimension, and since the action of S_{ν_n} on them is the same up to conjugation, we can assume that $V_n = W_n$ and they lead to the same object of $\mathrm{Rep}(S_\nu)$. Hence the last requirement is that the actions of H and H' on $\mathrm{End}(V_n)$ are conjugate. Hence by Łoś's theorem the statement of our Theorem follows.

□

Remark 4.2.1.3. This gives us a classification of simple commutative algebras in $\mathrm{Rep}(S_\nu)$ given in [49], [30] as a special case, where we restrict ourselves to B being 1-dimensional.

4.2.2 Symmetric monoidal functors between Deligne categories

In this section we will show how our result about the classification of commutative algebras can help us classify symmetric tensor functors $\mathrm{Rep}(S_\nu) \to \mathrm{Rep}(S_{\nu'})$ and also their generalization $\mathrm{Rep}(S_\nu) \to \mathrm{Rep}(S_{\nu'_1}) \boxtimes \cdots \boxtimes \mathrm{Rep}(S_{\nu'_k})$.

We will start with functors $\mathrm{Rep}(S_\nu) \to \mathrm{Rep}(S_{\nu'})$, for $\nu, \nu' \notin \mathbb{Z}_{\geq 0}$. From Proposition 2.2.0.7 we know that all such functors are classified by commutative Frobenius algebras of dimension ν in $\mathrm{Rep}(S_{\nu'})$. We will start with the following lemma.

Lemma 4.2.2.1. *Any commutative Frobenius algebra $A \in Rep(S_\nu)$ is isomorphic to the direct sum of simple commutative algebras.*

Proof. Using previous notation, we know from Łoś's theorem that A corresponds to a sequence of objects $A_n \in \mathcal{C}_n$, with almost all of them being commutative Frobenius algebras. Now note that such an algebra cannot have a non-trivial radical. Indeed since $\sqrt{0}$ lies in every maximal ideal, it also lies in the kernel of Tr. But then for any element $N \in \sqrt{0}$, we have $\mathrm{Tr}(a \cdot N) = 0$, hence the form is degenerate, which is a contradiction. So almost all A_n are semisimple as commutative algebras in the category of vector spaces. Also all of them are finite-dimensional, since they are objects of $\mathcal{C}_i$.

Thus it follows that the action of S_{ν_n} on such an A_n arises from the action of S_{ν_n} on $\mathrm{mspec}(A_n)$. And now if the action of S_{ν_n} on $\mathrm{mspec}(A_n)$ has l_n orbits with stabilizers $H_1, \ldots, H_{l_n}$, it follows that $A_n = \bigoplus \mathrm{Fun}_{H_j}(S_{\nu_n}, \overline{\mathbb{F}_{p_n}})$.

So almost all A_n are semisimple commutative algebras as objects of $\mathcal{C}_i$, so by Łoś's theorem the same holds for A. □

We also need another lemma.

Lemma 4.2.2.2. *Any semisimple commutative algebra $A \in Rep(S_\nu)$ is a Frobenius algebra.*

Proof. We know that $A = \bigoplus_{i=1}^{N} A_i$, where A_i are simple algebras. Note that Tr_A is equal to $(\mathrm{Tr}_{A_1}, \ldots, \mathrm{Tr}_{A_N})$, so if we prove that each A_i is a Frobenius algebra it will follow that A is too.

So consider $A = \mathrm{Ind}_{S_{\nu-j} \times H}^{S_\nu}(\mathbf{1})$. We want to prove that $\mathrm{Tr} \circ \mu$ is a non-degenerate form on A. To do that it is enough to prove that for n such that $\nu_n > j$ the corresponding form on $A_n = \mathrm{Ind}_{S_{\nu_n - j} \times H}^{S_{\nu_n}}(\mathbf{1}) = \mathrm{Fun}_{S_{\nu_n-j} \times H}(S_{\nu_n}, \mathbf{1})$ is non-degenerate.

Consider functions h_i which are zero everywhere except one conjugacy class of $S_{\nu_n - j} \times H$, where they are equal to 1. Such functions give us a basis of A_n. Now obviously $h_i h_j = \delta_{ij} h_i$. Hence $\mathrm{Tr}(h_i) = 1$, and $\mathrm{Tr}(h_i h_j) = \delta_{ij}$. So the form is indeed non-degenerate, and we are done. □

From these two lemmas and Proposition 2.2.0.7 it follows:

Proposition 4.2.2.3. *All $\mathbb{C}$-linear symmetric tensor functors between the Deligne categories Rep$(S_\nu) \to$ Rep$(S_{\nu'})$ for $\nu, \nu' \notin \mathbb{Z}_{\geq 0}$ are in 1-1 correspondence with semisimple commutative algebras in Rep$(S_{\nu'})$ of dimension ν.*

Proof. From Proposition 2.2.0.7 we know that such functors are in 1-1 correspondence with commutative Frobenius algebras of dimension t, but from Lemmas 4.2.2.1 and 4.2.2.1 we know that any commutative Frobenius algebra is a semisimple commutative algebra and vice versa. □

Now since the dimension of $\mathrm{Ind}_{S_{\nu_n-k} \times H}^{S_{\nu_n}}(\mathbb{C})$ is $\binom{\nu_n}{k}\frac{k!}{|H|}$, it follows that dimensions of simple commutative algebras in $\mathrm{Rep}(S_\nu)$ are multiples of $\binom{\nu}{k}$ for some integer k, and thus all possible dimensions of commutative Frobenius algebras are positive integer linear combinations of $\binom{\nu}{k}$. Let us define a corresponding algebraic structure:

Definition 4.2.2.4. For $\nu \in \mathbb{C}$ denote by $R_+(\nu)$ a set of all non-negative integer linear combinations of binomial coefficients in ν.

With this definition we can formulate the following Corollary:

Corollary 4.2.2.5. *Symmetric monoidal functors between the Deligne categories Rep$(S_\nu') \to$ Rep(S_ν) with $\nu, \nu' \notin \mathbb{Z}_{\geq 0}$ exist iff $\nu' \in R_+(\nu)$.*

Remark 4.2.2.6. One can obtain a similar description for $\mathbb{C}$-linear symmetric tensor functors $\mathrm{Rep}(S_\nu) \to \mathrm{Rep}(S_{\nu'_1}) \boxtimes \cdots \boxtimes \mathrm{Rep}(S_{\nu'_k})$. Such functors are in 1-1 correspondence with finite sums of external tensor products of simple commutative algebras in $\mathrm{Rep}(S_{\nu'_i})$. And such a functor exists iff ν is the positive integer linear combination of products of binomial coefficients in $\nu'_1, \ldots, \nu'_k$. I.e. if $\nu \in R_+(\nu'_1) \cdot \cdots \cdot R_+(\nu'_k)$

It turns out that for algebraic ν it is possible to describe $R_+(\nu)$ with a good degree of exactness, which makes it easier to apply Corollary 4.2.2.5 in concrete examples. This was done in [35] by Andrei Mandelshtam and the author of the present thesis. The result is as follows:

Theorem 4.2.2.7. *For an algebraic number $\nu \in \mathbb{C}\backslash\mathbb{Z}_{\geq 0}$ and each prime integer p define a set $\mathcal{P}_p$ that consists of prime ideals $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{Q}(\nu)}$ over (p) such that the ramification index of $\mathfrak{p}$ is 1, $|\mathcal{O}_{\mathbb{Q}(\nu)}/\mathfrak{p}| = p$, and $v_{\mathfrak{p}}(\nu) \geq 0$. Then $R_+(\nu)$ is a subring of $\mathcal{O}_{\mathbb{Q}(\nu)}$ consisting of $x \in \mathbb{Q}(\nu)$ such that $v_{\mathfrak{p}}(x) \geq 0$ for all $\mathfrak{p} \in \mathcal{P}_p$ for all primes p.*

For $\nu \in \mathbb{Z}_{\geq 0}$, we have $R_+(\nu) = \mathbb{Z}_{\geq 0}$.

4.3 Classification of simple Lie algebras in $\mathrm{Rep}(S_\nu)$

In this section we will provide the classification of simple Lie algebras in $\mathrm{Rep}(S_\nu)$. The steps of the proof are similar to that for the case of associative algebras. However we need a bit more preliminary lemmas in order to understand what kind of simple algebras in the finite rank can appear in the ultraproduct sequence.

First let us state the classification theorem for Lie algebras in characteristic p. See chapter 4 of [52].

Theorem 4.3.0.1. *Suppose $\mathfrak{g}$ is a simple finite-dimensional Lie algebra over an algebraically closed field of characteristic $p > 5$. Then it is either of classical or*

Cartan type.

Now we need to explain what classical and Cartan type means. First, classical type Lie algebras can be obtained in the following way. Take any Dynkin diagram C and define a Lie algebra $\mathfrak{g}_C$ as a vector space spanned by Chevalley basis corresponding to C with the ordinary Chevalley relations taken modulo p. It turns out that this algebra is simple for any C except A_{kp-1} for a positive integer k. In this case we also need to take quotient by 1-dimensional center of $\mathfrak{sl}_{kp}$ spanned by scalar matrices and we get a simple algebra $\mathfrak{psl}_{kp}$.

Algebras of Cartan type form four series of simple Lie algebras, namely $W(m, \underline{n})$, $S(m, \underline{n})$, $H(m, \underline{n})$ and $K(m, \underline{n})$, where $m \in \mathbb{Z}_{>0}$ and $\underline{n} \in \mathbb{Z}_{>0}^m$ (in the last case m is odd, in the second to last case it is even). We will discuss some of their properties in the next subsection.

The result analogous to Theorem 4.2.1.1 also holds in the case of Lie algebras, we only need to exchange word “associative" to “Lie" in the statement of the Theorem ([15]).

Theorem 4.3.0.2. *Fix an algebraicallly closed field $\mathbb{k}$. Any simple Lie algebra in $Rep(S_N, \mathbb{k})$ is isomorphic to $Fun_G(S_N, \mathfrak{h})$, for a Lie algebra $\mathfrak{h}$ simple in the category of vector spaces. All such algebras are simple. Moreover G is defined up to conjugation in S_N and the action of G on $\mathfrak{h}$ up to conjugation in $Aut(Mat_m(\mathbb{k}))$.*

We can now state the following Proposition:

Proposition 4.3.0.3. *Any simple Lie algebra $\mathfrak{g} \in Rep(S_\nu)$ is equal to $Ind_{S_{\nu-j} \times H_j}^{S_\nu} \mathfrak{h}$, for some $j \in \mathbb{Z}_{>0}$ and $H \subset S_j$, where $\mathfrak{h}$ is a simple Lie algebra given by the ultraproduct of simple Lie algebras $\mathfrak{h}_n \in \mathcal{C}_n$ which remain simple under the forgetful functor $Res : \mathcal{C}_n \to Vec$.*

Proof. Since we know that $\mathfrak{g} = \prod_{\mathcal{F}} \mathfrak{g}_n$ and $\mathfrak{g}_n = \mathrm{Ind}_{G_n}^{S_{\nu_n}}(\mathfrak{h}_n)$, where $\mathfrak{h}_n$ is a simple Lie algebra as an object of the category of vector spaces, the result follows from Lemma 4.1.2.1 and Łoś's theorem . □

4.3.1 Ultraproducts of Lie algebras of Cartan type

Now, first, we would like to rule out the possibility of almost all $\mathfrak{h}_n$ being of Cartan type.

To do this let us first explain what $W(m, \underline{n})$ actually is (see chapter 4.2 in [52] for details). First, we need to define $\mathcal{O}(m)$ and $\mathcal{O}(m, \underline{n})$.

Definition 4.3.1.1. By $\mathcal{O}(m)$ denote a commutative algebra over $\overline{\mathbb{F}}_q$ with a basis $x_1^{(a_1)} \dots x_m^{(a_m)}$, for $a_i \in \mathbb{Z}_{\geq 0}$ with multiplication defined by:

$$x_1^{(a_1)} \dots x_m^{(a_m)} \cdot x_1^{(b_1)} \dots x_m^{(b_m)} = \binom{a_1 + b_1}{a_1} \dots \binom{a_m + b_m}{a_m} x_1^{(a_1+b_1)} \dots x_m^{(a_m+b_m)} .$$

By $\mathcal{O}(m, \underline{n})$ denote a subalgebra of $\mathcal{O}(m)$ spanned by $x_1^{(a_1)} \dots x_m^{(a_m)}$ with $0 \leq a_i < p^{n_i}$.

Using this, the Witt algebra $W(m, \underline{n})$ can be obtained in the following way.

Definition 4.3.1.2. By $W(m, \underline{n})$ denote a simple Lie algebra given as follows:

$$W(m, \underline{n}) = \sum_{i=1}^{m} \mathcal{O}(m, \underline{n}) \partial_i .$$

All other simple algebras $S(m, \underline{n})$, $H(m, \underline{n})$ and $K(m, \underline{n})$ can be realized as subalgebras in $W(m, \underline{n})$. We will need an important proposition about the automorphism groups of such algebras, see chapter 7.3 of [52].

Proposition 4.3.1.3. *There is an isomorphism $\phi : Aut_{\mathcal{C}}(\mathcal{O}(m,\underline{n})) \to Aut(W(m,\underline{n}))$ from a certain subgroup $Aut_{\mathcal{C}}(\mathcal{O}(m,\underline{n})) \subset Aut(\mathcal{O}(m,\underline{n}))$, to the group of automorphisms of Witt algebra given by:*

$$\sigma \to (D \mapsto \sigma \circ D \circ \sigma^{-1}) ,$$

where D is an arbitary element of $W(m,\underline{n})$. Moreover it restricts to give an isomorphism between $S(m,\underline{n})$, $H(m,\underline{n})$ and $K(m,\underline{n})$ and certain subgroups of $Aut_{\mathcal{C}}(\mathcal{O}(m,\underline{n}))$.

So from this proposition it follows what we need to understand the structure of the group $\mathrm{Aut}_{\mathcal{C}}(\mathcal{O}(m,\underline{n}))$. This is done in [57]. See Corollary 1 and Theorem 2.

Proposition 4.3.1.4. *Take any isomorphism $\sigma \in Aut_{\mathcal{C}}(\mathcal{O}(m,\underline{n}))$, denote by y_i the images of $x_i = x_i^{(1)}$ under this morphism. By $\bar{y}_i$ denote the linear part of y_i. It follows that the map $x_i \mapsto \bar{y}_i$ defines an element of $GL(m,\overline{\mathbb{F}}_p)$. Also there is an exact sequence:*

$$0 \to \mathcal{B} \to Aut_{\mathcal{C}}(\mathcal{O}(m,\underline{n})) \to GL(m,\overline{\mathbb{F}}_p) ,$$

where $\mathcal{B}$ is solvable and the last morphism is as described above.

Since the automorphism groups of all Cartan type Lie algebras are subgroups in $\mathrm{Aut}_{\mathcal{C}}(\mathcal{O}(m,\underline{n}))$[1] it follows that such an exact sequence holds for any $\mathrm{Aut}(X(m,\underline{n}))$ with $X = W, S, H, K$. Namely we have:

$$0 \to \mathcal{B}_X \to \mathrm{Aut}(X(m,\underline{n})) \to GL(m,\overline{\mathbb{F}}_p) , \tag{4.2}$$

for different $\mathcal{B}_X$.

[1] Theorem 7.3.2 [52].

Also we will need to know the dimension formulas for Cartan type Lie algebras, they are summarized in the following proposition (see [52] section 4.2).

Proposition 4.3.1.5. *The dimension of Cartan type algebras are given by the formulas*

$$\dim(W(m,\underline{n})) = mp^{\sum n_i} \ ,$$

$$\dim(S(m,\underline{n})) = (m-1)(p^{\sum n_i}-1) \ , \dim(H(m,\underline{n})) = p^{\sum n_i}-2 \ ,$$

$$\textit{and } \dim(K(m,\underline{n})) = p^{\sum n_i} \textit{ or } p^{\sum n_i}-1 \textit{ depending on } m \textit{ mod } p \ .$$

Now we have everything we need to move on. So let us prove the following proposition.

Proposition 4.3.1.6. *In Proposition 4.3.0.3 almost all $\mathfrak{h}_n$ are of classical type.*

Proof. Suppose that almost all $\mathfrak{h}_n$ in Proposition 4.3.0.3 are of Cartan type. Let's denote $\mathfrak{h}_n = X_n(m_n, \underline{N_n})$ ($X_n = W, S, H, K$), then we have a homomorphism $S_{\nu_n - j} \to \mathrm{Aut}(\mathfrak{h}_n)$. Hence, because of (2), we have $S_{\nu_n-j} \to GL(m, \overline{\mathbb{F}}_{p_n})$. There are two possibilities here. Either for almost all n this homomorphism is trivial or not.

First suppose it is trivial for almost all n. Then for almost all n, $S_{\nu_n-j} \to \mathcal{B}_n$, but since the latter group is solvable, it follows that for almost all n the kernel of this morphism contains $\mathcal{A}_n$. But then $\mathfrak{h}_n$ contains only one-dimensional representations of S_{ν_n-j}. But since the dimension of $\mathfrak{h}_n$ is bigger than $p_n - 3$ it follows that the length of $\mathfrak{h}_n$ as a representation of S_{ν_n-j} is unbounded, hence its ultraproduct does not define an object of the Deligne category.

The only other option is that this morphism is non-trivial for almost all n. Note that this morphism cannot have $\mathcal{A}_n$ as its kernel or the previous argument can be repeated. Hence, since the lowest dimension of a S_{ν_n-j}-representation which is not trivial or sign is $\nu_n - j - 1$, it follows that $m_n \geq \nu_n - j - 1$, and thus the dimension of $\mathfrak{h}_n$ is at least $p_n^{\nu_n-j-1} - 3$. So it grows exponentially. But the dimension of any sequence of representations defining an element of the Deligne category grows polynomially. So again ultraproduct of $\mathfrak{h}_n$ does not belong to $\mathrm{Rep}(S_{\nu-j})$.

Thus the result follows. □

4.3.2 Ultraproducts of classical Lie algebras

Now we can move on with our classification assuming that for almost all n, the algebras $\mathfrak{h}_n$ are of classical type. Here we again have two possibilities. Either the size of the Dynkin diagram coresponding to $\mathfrak{h}_n$ is bounded for almost all n or it is not. Let's start with the first case.

Proposition 4.3.2.1. *If the size of the Dynkin diagram corresponding to $\mathfrak{h}_n$ is bounded, then $\mathfrak{h}$ has trivial action of $S_{\nu-j}$.*

Proof. Since there is a finite number of Dynkin diagrams of bounded size, it follows that for almost all n the corresponding Dynkin diagram is the same, so as a Lie algebra in the category of vector spaces, $\mathfrak{h}_n$ is of the same type (again see Lemma 2.3.1.4). But then its automorphism group is a subgroup in some $GL_N(\overline{\mathbb{F}}_{p_n})$. So since the lowest dimension of an irreducible S_{ν_n-j}-representation which is not trivial or sign is $\nu_n - j - 1$ it follows that almost all $\mathfrak{h}_n$ are sums of one-dimensional representations of S_{ν_n-j}. But it cannot contain the sign representations for almost all n, or the ultraproduct wouldn't lie in $\mathrm{Rep}(S_{\nu-j})$. Hence for almost all n, $\mathfrak{h}_n$ is a trivial representation of S_{ν_n-j}. Thus the corresponding ultra-product is the same classical Lie algebra corresponding to the Dynkin diagram with a trivial action of S_ν, i.e. equal to the sum of the copies of the unit object in $\mathrm{Rep}(S_{\nu-j})$. □

In the second case the size of the Dynkin diagram is unbounded. But the number of infinite series of Dynkin diagrams is finite, so we may assume that for almost all n the type of the Dynkin diagram is the same, and it is either A, B, C or D. To proceed further we need to know something about the automorphism groups of these algebras. This information can be found in [50], it is summarized in the following Proposition.

Proposition 4.3.2.2. *The group of automorphisms of the Lie algebra of type A_{n-1} (both in the case $p|n$ and $p \nmid n$) is the semi-direct product of $PSL(n)$ by $\mathbb{Z}/2\mathbb{Z}$, where the second group acts by $X \mapsto -X^t$. We will denote the generator of this group by τ.*

The group of automorphisms of the Lie algebra of type B_n is $PSp(n)$.

The group of automorphisms of the Lie algebra of type C_n is $PSO(2n+1)$.

The group of automorphisms of the Lie algebra of type D_n is $PO(2n)$, for $n > 4$.

We have an additional complication in the A_{n-1} case, so let us deal with this case first.

Proposition 4.3.2.3. *If $\mathfrak{h}_n$ is a simple Lie algebra of type A for almost all n, then, for almost all n, $S_{\nu_n - j}$ maps into the subgroup $PSL(n)$ of automorphisms of $\mathfrak{h}_n$.*

Proof. Suppose that the map $S_{\nu_n-j} \to \mathbb{Z}/2\mathbb{Z}$, which is obtained as a composition of maps $S_{\nu_n-j} \to \mathrm{Aut}(\mathfrak{h}_n)$ and $\mathrm{Aut}(\mathfrak{h}_n) \to \mathbb{Z}/2\mathbb{Z}$, is non-trivial for almost all n. Note that the map $\mathcal{A}_{\nu_n-j} \to \mathbb{Z}/2\mathbb{Z}$ obtained by restriction of this map is trivial, so $\mathcal{A}_{\nu_n-j}$ maps into $PSL(N_n)$. Suppose this map is non-trivial. By Lemma 4.1.2.3 it follows that this map gives us a linear representation of $\mathcal{A}_{\nu_n-j}$ on $V = \overline{\mathbb{F}_{p_n}}^{N_n}$. By choosing a bilinear $\mathcal{A}_{\nu_n-j}$-invariant form on V, we can suppose that $V^* \simeq V$ as a representation of $\mathcal{A}_{\nu_n-j}$. But then since τ acts on an element of $V \otimes V^*$ as $\tau(v \otimes w) = -w \otimes v$, it follows that τ commutes with the action of $\mathcal{A}_{\nu_n-j}$. So the action of τ on the isomorphic image of $\mathcal{A}_{\nu_n-j}$ is actually trivial, and hence the image of S_{ν_n-j} is actually a direct product of $\mathcal{A}_{\nu_n-j}$ and $\mathbb{Z}/2\mathbb{Z}$, which is absurd. So $\mathcal{A}_{\nu_n-j}$ is in the kernel of the map $S_{\nu_n-j} \to \mathrm{Aut}(\mathfrak{h}_n)$. So for almost all n, $\mathfrak{h}_n$ decomposes as the sum of one-dimensional representations of S_{ν_n-j}, but their ultraproduct lies in the Deligne category, so almost all of them can not contain any sign representations. Hence the action of S_{ν_n-j} on $\mathfrak{h}_n$ is actually trivial. So we get a contradiction. □

From this Proposition it follows that in each case S_{ν_n-j} maps into a projective group of the corresponding group of linear transformations of a vector space. But from Lemma 4.1.2.3 it follows that in each case we have an honest map from S_{ν_n-j} to the corresponding group of linear transformations, i.e. a representation of S_{ν_n-j} on the corresponding vector space.

We have the following Proposition:

Proposition 4.3.2.4. *If the size of the Dynkin diagram corresponding to $\mathfrak{h}_n$ is unbounded, then for almost all n, $\mathfrak{h}_n = \mathfrak{x}(V_n)$ for the same $\mathfrak{x}$ ($\mathfrak{x} = \mathfrak{sl}, \mathfrak{psl}, \mathfrak{sp}, \mathfrak{so}$). Also there exist V_n' such that $\mathfrak{x}(V_n) = \mathfrak{x}(V_n')$ and such that $V = \prod_{\mathcal{F}} V_n'$ is an object of Rep($S_{\nu-j}$), hence $\mathfrak{h} = \mathfrak{x}(V)$.*

Proof. We have established the first part of the proposition. Also from the discussion above it follows that we have an action of S_{ν_n-j} on V_n, which leaves the corresponding bilinear form invariant. Now using Lemma 4.1.2.6 we conclude that such V_n' indeed exist. □

Now we can formulate the following classification theorem.

Theorem 4.3.2.5. *Every simple Lie algebra in the category Rep(S_ν) is isomorphic to the one given by $Ind_{S_{\nu-j}\times H}^{S_\nu}(\mathfrak{h})$, where $j > 0$ is an integer, H a subgroup of S_j*

which acts on a Lie algebra $\mathfrak{h}$ in Rep$(S_{\nu-j})$ by Lie algebra automorphisms, and $\mathfrak{h}$ is of one of the following kinds:

· An exceptional Lie algebra which is equal to the sum of copies of the unit object of Rep$(S_{\nu-j})$.

· $\mathfrak{sl}(V)$ for any V of non-zero dimension, or $\mathfrak{psl}(V)$ for any V of dimension zero.

· $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$ for any V with a (skew)-symmetric non-degenerate bilinear form.

Finally, such a simple Lie algebra is determined uniquely by the above data up to conjugation of H inside S_j and conjugation of the action of H inside of Aut$(\mathfrak{h})$.

Proof. We have already checked that $\mathrm{Ind}_{S_{\nu-j}\times H}^{S_\nu}(\mathfrak{h})$ defined using the data described above gives us a simple Lie algebra, since the resulting algebra $\mathfrak{g}$ is an ultraproduct of the Lie algebras which are simple due to Theorem 4.3.0.2.

Now from Propositions 4.3.0.3, 4.3.1.6, 4.3.2.1 and 4.3.2.4 we conclude that any simple Lie algebra can be obtained in this way. Indeed from these propositions we know that such $\mathfrak{h}$ exists and is either a trivial representation of $S_{\nu-j}$ or it is given by $\mathfrak{x}(V)$. Now note that if $\mathfrak{h} = \mathfrak{sl}(V)$, the dimension of almost all V_n are not divisible by p_n, or the algebra $\mathfrak{sl}(V_n)$ would not be simple for almost all n, hence the dimension of V, which can be obtained through the isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} = \mathbb{C}$ is non-zero. In the case of $\mathfrak{psl}(V)$ on the other hand it is divisible by p_n for almost all n and hence the dimension of V is zero. In the case $\mathfrak{x} = \mathfrak{so}$, $\mathfrak{sp}$ the S_{ν_n-j}-module V_n has an S_{ν_n-j}-invariant (skew)-symmetric bilinear form for almost all n, hence it gives us a (skew)-symmetric bilinear form on V in Rep(S_ν). So indeed every simple Lie algebra can be obtained in the specified way.

The proof of the uniqueness is the same as in Theorem 4.2.1.2.

□

4.4 A conjecture concerning the classification of simple Lie superalgebras in $\mathbf{Rep}(S_\nu)$

In this section we will state a conjectural extension of the above classifications to the setting of Lie superalgebras, and outline a possible approach to generalize their proofs. The textbook reference about the theory of Lie superalgebras is [43], it contains the classification of simple Lie superalgebras over $\mathbb{C}$ and their construction (Chapters 1-2 and 4). See the original paper of Kac [32] for the classification.[2] How these results generalize to the modular case with big enough p can be found in [2] Section 2.3, [39] and [40] Section 10.

4.4.1 Lie superalgebras in tensor categories and their simplicity in $Vect$

First let us describe the classification results regarding simple Lie superalgebras in the category of vector spaces.

Below we assume that for every Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the $\mathfrak{g}_1$ component is non-zero. If $\mathfrak{g}_1$ is zero, then $\mathfrak{g}$ is a regular Lie algebra and the result of the previous section applies.

First we will need some definitions.

Definition 4.4.1.1. Fix V, W to be non-zero objects of a symmetric rigid tensor category.

a) Define the Lie superalgebra $\mathfrak{gl}(V|W)$ to be the object $(V \oplus W) \otimes (V^* \oplus W^*)$ with

[2]The classification problem of finite dimensional simple Lie superalgebras in zero and positive characteristic has a long history, which we in no way attempt to review; thus many important references are not given here.

the $\mathbb{Z}/2\mathbb{Z}$-grading given by

$$\mathfrak{gl}(V|W)_0 = V \otimes V^* \oplus W \otimes W^* \text{ and } \mathfrak{gl}(V|W)_1 = V \otimes W^* \oplus W \otimes V^* \ .$$

The superbracket $[\ ,\ ]_{i,j} : \mathfrak{gl}(V|W)_i \otimes \mathfrak{gl}(V|W)_j \to \mathfrak{gl}(V|W)_{i+j}$ is given by the map

$$[\ ,\ ]_{i,j} = \mu - (-1)^{ij} \mu \circ \sigma \ ,$$

where μ is the associative algebra multiplication and $\sigma = \sigma_{\mathfrak{gl}(V|W)}$ (see Def. 2.1.0.9).
b) Define the Lie superalgebra $\mathfrak{sl}(V|W)$ to be the subalgebra in $\mathfrak{gl}(V|W)$ given by the kernel of the map

$$str : \mathfrak{gl}(V|W) \to \mathfrak{gl}(V|W)_0 \xrightarrow{(ev_V\ ,\ -ev_W)} \mathbf{1} \ .$$

c) Consider the map

$$l : \mathbf{1} \xrightarrow{coev_V \oplus coev_W} \mathfrak{gl}(V|W)_0 \ .$$

The image of this map lies in $\mathfrak{sl}(V|W)$ iff $\dim V = \dim W$. In this case define the Lie superalgebra $\mathfrak{psl}(V|W)$ to be the cokernel of $l : \mathbf{1} \to \mathfrak{sl}(V|W)$.
d) Fix a bilinear form on $V \oplus V^*$ specified by the identity map

$$\psi : V \oplus V^* \to (V \oplus V^*)^* = V^* \oplus V \ .$$

Define the Lie superalgebra $\mathfrak{p}(V)$ to be the subalgebra in $\mathfrak{sl}(V|V^*)$ preserving this form, i.e. the kernel of the map

$$\mathfrak{gl}(V|V^*) \xrightarrow{1+\sigma\circ(\psi\otimes\psi^{-1})} \mathfrak{gl}(V|V^*) \ .$$

e) Consider a morphism $c : V \oplus V \to V \oplus V$, given by the matrix

$$\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} .$$

This morphism can also be considered as an element (i.e. a map $\mathbf{1} \to \mathfrak{gl}(V|V)$) of $\mathfrak{gl}(V|V)$ using evaluation and coevaluation maps. The Lie superalgebra $\hat{\mathfrak{q}}(V)$ is defined as the centralizer of this element, i.e. as the kernel of the map

$$\mathfrak{gl}(V|V) \to \mathfrak{gl}(V|V) \otimes \mathbf{1} \xrightarrow{Id \otimes c} \mathfrak{gl}(V|V) \otimes \mathfrak{gl}(V|V) \xrightarrow{[\ ,\ ]} \mathfrak{gl}(V|V) .$$

Next we define the Lie superalgebra $\tilde{\mathfrak{q}}(V)$ as the kernel of the map $\hat{\mathfrak{q}}(V) \to \mathbf{1}$ given by the restriction of the map $\mathfrak{gl}(V|V)_1 \xrightarrow{(ev_V\ ,\ ev_V)} \mathbf{1}$. Then there is a non-zero map $l : \mathbf{1} \xrightarrow{coev \oplus coev} \tilde{\mathfrak{q}}(V)_0$. The cokernel of this map it the Lie superalgebra $\mathfrak{q}(V)$.

f) Suppose there is a symmetric non-degenerate bilinear form on V and a skew-symmetric non-degenerate bilinear form on W. Denote the corresponding maps $\psi_V : V \to V^*$ and $\psi_W : W \to W^*$. Then $\mathfrak{osp}(V|W)$ is the following subalgebra in $\mathfrak{gl}(V|W)$. We define $\mathfrak{osp}(V|W)_0$ to be equal to the kernel of the map

$$\mathfrak{gl}(V|W)_0 \xrightarrow{(Id + \sigma \circ (\psi_V \otimes \psi_V^{-1})) \oplus (Id + \sigma \circ (\psi_W \otimes \psi_W^{-1}))} \mathfrak{gl}(V|W)_0 ,$$

and $\mathfrak{osp}(V|W)_1$ to be equal to the kernel of the map

$$\mathfrak{gl}(V|W)_1 = V \otimes W^* \oplus W \otimes V^* \xrightarrow{(\sigma \circ (\psi_V \otimes 1)\ ,\ 1 \otimes \psi_W)} W^* \otimes V^* .$$

Remark 4.4.1.2. These definitions mimic the standard definitions of the above Lie superalgebras in an element-free fashion. It is straightforward to check that this definition agrees with the usual definitions for the category of supervector spaces, and that the superbracket descends onto the various kernels and cokernels used in the definition.

Now we want to know when exactly these superalgebras are simple for the categories $Vect_{\Bbbk_0}$ and $Vect_{\Bbbk_p}$. This is explained by the various classification results. Here $\Bbbk_0$ stands for an algebraically closed field of characteristic 0 and $\Bbbk_p$ stands for an algebraically closed field of characteristic p.

Proposition 4.4.1.3. *(Theorem 1.3.1 in [43]) Suppose V, W are non-zero objects of $Vect_{\Bbbk_0}$.*

a) *The Lie superalgebra $\mathfrak{sl}(V|W)$ is simple in $Vect_{\Bbbk_0}$ iff $\dim(V) \neq \dim(W)$.*

b) *In $Vect_{\Bbbk_0}$ if $\dim(V) = \dim(W) > 1$ the Lie superalgebra $\mathfrak{psl}(V|W)$ is defined and is simple.*

c) *The Lie superalgebra $\mathfrak{osp}(V|W)$ is simple in $Vect_{\Bbbk_0}$.*

d) *The Lie superalgebra $\mathfrak{q}(V)$ is simple in $Vect_{\Bbbk_0}$ iff $\dim(V) \geq 2$.*

e) *The Lie superalgebra $\mathfrak{p}(V)$ is simple in $Vect_{\Bbbk_0}$ iff $\dim(V) \geq 2$.*

Proposition 4.4.1.4. *(Section 10 in [40], Section 6 in [3], Section 4.1 in [1]) Suppose V, W are non-zero objects of $Vect_{\Bbbk_p}$.*

a) *The Lie superalgebra $\mathfrak{sl}(V|W)$ is simple in $Vect_{\Bbbk_p}$ iff $\dim(V) \neq \dim(W)$ mod p (See Section 10 in [40]).*

b) *In $Vect_{\Bbbk_p}$ if $\dim(V) = \dim(W)$ mod p and $\dim(V), \dim(W) > 1$, the Lie superalgebra $\mathfrak{psl}(V|W)$ is defined and is simple (See Section 10 in [40]).*

c) *The Lie superalgebra $\mathfrak{osp}(V|W)$ is simple in $Vect_{\Bbbk_p}$. (See Section 10 in [40]).*

d) *The Lie superalgebra $\mathfrak{q}(V)$ is simple in $Vect_{\Bbbk_p}$ iff $\dim(V) \geq 2$.*

e) *The Lie superalgebra $\mathfrak{p}(V)$ is simple in $Vect_{\Bbbk_p}$ iff $\dim(V) \geq 2$.*

We have some results about the classification of all such superalgebras.

Theorem 4.4.1.5. *(Theorem 1.3.1 in [43], Section 4.2 in [32]) Let $\mathfrak{g}$ be a finite dimensional simple Lie superalgebra over $\Bbbk_0$. Then it is either given by one of the examples of Proposition 5.1.2 or by one of the exceptional Lie superalgebras $\mathfrak{d}(2,1;\alpha)$ for $\alpha \in \Bbbk_0$, $\alpha \neq 0,1$, $\mathfrak{f}(4)$ or $\mathfrak{g}(3)$ or by one of the Cartan type superalgebras $W(n)$, $S(n)$, $\tilde{S}(n)$ and $H(n)$.*

Remark 4.4.1.6. The Lie superalgebras $\mathfrak{d}(2,1;\alpha)$ form a one-parametric series of superalgebras of the same dimension.

Conjecture 4.4.1.7. *(Conjecture 1 in [39]) Let $\mathfrak{g}$ be a finite dimensional simple Lie superalgebra over $\Bbbk_p$ with $p \geq 7$. Then it is either given by one of the examples of Proposition 4.4.1.4 or by one of the exceptional Lie superalgebras or by a certain algebra of Cartan type.*

4.4.2 Lie superalgebras in Rep(S_ν)

Now we can explain how these results can lead to a classification of simple Lies superalgebras in Deligne category Rep(S_ν).

Using Definition 4.4.1.1 we can construct the Lie superalgebras in the category Rep(S_ν) as follows. Fix an integer $j > 0$, a subgroup $H \subset S_j$ and a Lie superalgebra $\mathfrak{h}$ in Rep($S_{\nu-j}$) of one of the following kinds:

· An exceptional or a Cartan type Lie superalgebra which as an object of Rep($S_{\nu-j}$) is equal to the sum of the copies of the unit object.

· $\mathfrak{sl}(V|W)$ for V, W such that $\dim(V) \neq \dim(W)$ and $V, W \neq 0$.

· $\mathfrak{psl}(V|W)$ for V, W such that $\dim(V) = \dim(W)$ and both objects are not trivial or isomorphic to $\mathbf{1}$.

· $\mathfrak{osp}(V|W)$ for any pair of non-zero objects V, W with a non-degenerate billinear form, which is symmetric and skew-symmetric respectively.

· $\mathfrak{q}(V)$ for V not 0 or $\mathbf{1}$.

· $\mathfrak{p}(V)$ for V not 0 or $\mathbf{1}$.

Also fix an action of H on $\mathfrak{h}$ by Lie superalgebra automorphisms. Then we can denote $\mathfrak{g}$ in $\mathrm{Rep}(S_\nu)$ as $\mathrm{Ind}_{S_{\nu-j}\times H}^{S_\nu}(\mathfrak{h})$. This is obviously a Lie superagbera since it can be presented as an ultraproduct of Lie superalgebras. The fact that this is simple follows from the statement simmilar to Theorem 4.3.0.2 for the case of Lie superalgebras, which also follows from the general statement for operads given in [15].

Now using this we can state a conjecture similar to Theorem 4.3.2.5.

Conjecture 4.4.2.1. *Any simple Lie superalgebra in $Rep(S_\nu)$ is isomorphic to the one obtained as $Ind_{S_{\nu-j}\times H}^{S_\nu}(\mathfrak{h})$ from the data $j, H, \mathfrak{h}$ as described above.*

Such a simple Lie superalgebra is determined uniquely by the above data up to conjugation of H inside S_j and conjugation of the action of H inside of $Aut(\mathfrak{h})$.

Sketch of Proof. Below is a rough sketch of the proof for transcendental ν. In the case of algebraic ν the proof might be similar but relies on Conjecture 4.4.1.7.

The steps of the proof are the same as in Theorem 4.3.2.5.

First using the analogue of Proposition 4.3.0.3 it is easy to see that, indeed, we have $\mathfrak{g} = \mathrm{Ind}_{S_{\nu-j}\times H}^{S_\nu}(\mathfrak{h})$ for some simple Lie superalgebra $\mathfrak{h}$ which is given by the ultraproduct of simple Lie superalgebras $\mathfrak{h}_n$ which remain simple when considered as an object of $Vect$.

The next step is a little bit more vague, since we need to rule out the possibility of almost all $\mathfrak{h}_n$ being of Cartan type with a non-trivial action of S_{ν_n-j}(analogue of Proposition 4.3.1.6). This requires some work in the case of algebraic t, but is straightforward for transcendental ν. Indeed in this case all $\mathfrak{h}_n$ are Lie superalgebras over a field of characteristic 0 and the dimension of Cartan type superalgebras grows exponentially with n (hence their rank is bounded, hence almost all actions of $S_{\nu_n - j}$ are trivial).

Next as an analogue of Proposition 4.3.2.1 it is easy to show that if the dimensions of $\mathfrak{h}_n$ are bounded, then the action of $S_{\nu_n - j}$ will be trivial for almost all n.

Another step is to show that $S_{\nu_n - j} \times H$ acts by inner automorphisms of $\mathfrak{h}_n$ for almost all n. This is done in a way similar to Proposition 4.3.2.3 since the groups of outer automorphisms of these superalgebras are some small groups.

The last step is the analogue of Proposition 4.3.2.4 which is based on the Lemma 4.1.2.6. To prove this one needs to extend Lemma 4.1.2.6 to cover some more examples relevant for the superalgebras case, which can be done in a similar way.

After all this is done the claim will follow in the same way as in Theorem 4.3.2.5.

□

Chapter 5

Construction of Deformed Double Current Algebras in Rep(S_ν)

In this chapter we will present the construction of the deformed double current algebras (DDCA) of type A as an algebra of endomorphisms of a certain ind-object of Rep(S_ν). We will also show that this algebra of endomorphisms can be obtained as an ultraproduct of spherical subalgebras of an extended Cherednik algebra. This will allow us to prove that starting with rank 4 our DDCA is isomorphic to the one constructed by Guay in [25].

In the last section of this chapter we will study the case of rank 1 in more detail. Namely we will give the presentation of this DDCA by generators and relations. We will conclude by showing how these results can be extended to other DDCAs of rank 1, especially the one of type B.

5.1 Symplectic reflection algebras and extended Cherednik algebras

Our construction of DDCA will make use of the notions of extended Cherednik algebras (type A, higher rank DDCA) and symplectic reflection algebras (other types, rank 1 DDCAs). For this reason we will provide the definitions and basic properties of this algebras in the present section. However, since both classes of algebras are the generalization of the rational Cherednik algebra of type A (which we will just call Cherednik algebra in the present thesis), we will start with recalling its definition and properties.

5.1.1 The Cherednik algebra

Below we give a definition of the rational Cherednik algebras of type A. For the definition and theory of general rational Cherednik algebras, see [20].

Definition 5.1.1.1. The rational Cherednik algebra of type A and rank n over a field $\Bbbk$, denoted by $H_{t,k}(n, \Bbbk) = H_{t,k}(n)$, where $t, k \in \Bbbk$, is defined as follows. Consider the standard representation of S_n acting by permutations on $\mathfrak{h} = \Bbbk^n$ with the basis given by $y_i \in \mathfrak{h}$, and the dual basis $x_i \in \mathfrak{h}^*$. Then $H_{t,k}(n)$ is the quotient of $\Bbbk[S_n] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the following relations:

$$[x_i, x_j] = 0, \ [y_i, y_j] = 0, \ [y_i, x_j] = \delta_{ij}(t - k \sum_{m \neq i} s_{im}) + (1 - \delta_{ij}) k s_{ij},$$

where s_{ij} denotes the transposition of i and j.

In other words, this is the rational Cherednik algebra corresponding to the root system A_{n-1}.

This algebra has a filtration determined by $\deg(x_i) = \deg(y_i) = 1$ and $\deg(g) = 0$ for any group element g. The associated graded algebra is:

$$\mathrm{gr}(H_{t,k}(n)) = \Bbbk[S_n] \ltimes S(\mathfrak{h} \oplus \mathfrak{h}^*).$$

This follows from the fact that the analog of the PBW theorem holds for this algebra:

Proposition 5.1.1.2. *The natural map $H_{0,0}(n) \to \mathrm{gr}(H_{t,k}(n))$ is a vector space isomorphism.*

Another important object is the spherical subalgebra of the rational Cherednik algebra.

Definition 5.1.1.3. If $\mathrm{char}(\Bbbk) = p > n$ or $p = 0$, denote by $B_{t,k}(n)$ the subalgebra $\mathbf{e}H_{t,k}(n)\mathbf{e}$ of $H_{t,k}(n)$, where $\mathbf{e} \in \Bbbk[S_n]$ is the averaging idempotemt.

Note that:

$$\mathrm{gr}(B_{t,k}(n)) = S(\mathfrak{h} \oplus \mathfrak{h}^*)^{S_n} = \Bbbk[x_1, \dots, x_n, y_1, \dots, y_n]^{S_n}.$$

Remark 5.1.1.4. One can construct the spherical subalgebra in another way. Indeed, regard $\Bbbk$ as the trivial representation of S_n and apply to it the induction functor $\mathrm{Ind}_{S_n}^{H_{t,k}(n)}(\Bbbk)$. It's easy to see that this representation is in fact $H_{t,k}(n)\mathbf{e}$. Now the spherical subalgebra is given as follows:

$$B_{t,k}(n) = \mathbf{e}H_{t,k}(n)\mathbf{e} = \mathrm{Hom}_{S_n}(\Bbbk, H_{t,k}(n)\mathbf{e}) = \mathrm{End}_{H_{t,k}(n)}(\mathrm{Ind}_{S_n}^{H_{t,k}(n)}(\Bbbk)).$$

Now we can introduce the corresponding categories of representations.

Definition 5.1.1.5. By $\mathbf{Rep}(H_{t,k}(n);\Bbbk)$ denote the category of (possibly infinite dimensional) representations of the rational Cherednik algebra $H_{t,k}(n) = H_{t,k}(n,\Bbbk)$. Also set $\mathbf{Rep}_p(H_{t,k}(n)) = \mathbf{Rep}(H_{t,k}(n), \overline{\mathbb{F}}_p)$.

5.1.2 Symplectic reflection algebras

Now let us define the notion of symplectic reflection algebras that in a certain way generalizes that of the Cherednik algebra given above. We are going to use it to construct DDCA with non-trivial Γ in rank 1. Below we will give some basic definitions, needed for our purposes. For more on this topic see [17].

The symplectic reflection algebra is defined as follows:

Definition 5.1.2.1. Fix a finite subgroup $\Gamma \subset \mathrm{SL}(2;\Bbbk)$. Fix numbers $t, k \in \Bbbk$. Fix numbers $c_C \in \Bbbk$ for every conjugacy class $C \subset \Gamma$; we will denote the collection of these numbers by c. For every conjugacy class C, set $T_C := \frac{1}{2}\mathrm{Tr}|_{\Bbbk^2}\gamma$, where $\gamma \in C$ is an element of the conjugacy class and we take the trace over the tautological representation. Consider $V = (\Bbbk^2)^n$, the tautological representation of the wreath product $S_n \ltimes \Gamma^n$. Note that this space has a natural symplectic structure, which we will denote by ω. Let Σ stand for the set of elements of $S_n \ltimes \Gamma^n$ conjugate to a transposition. For a conjugacy class $C \subset \Gamma$, let Σ_C be the set of all elements conjugate to $(1, 1, \ldots, 1, \gamma)$ for $\gamma \in C$.

The symplectic reflection algebra $H_{t,k,c}(n,\Gamma)$ is the quotient of $\Bbbk[S_n \ltimes \Gamma^n] \ltimes T(V)$ by the relations:

$$[y,x] = t\omega(y,x) - k\sum_{s\in\Sigma}\omega(y,(1-s)x)s - \sum_C \frac{c_C}{1-T_C}\sum_{s\in\Sigma_C}\omega((1-s)y,(1-s)x)s,\ x,y \in V.$$

We can also define the spherical subalgebra of this algebra:

Definition 5.1.2.2. The spherical subalgebra of the symplectic reflection algebra $H_{t,k,c}(n,\Gamma)$ is denoted by $B_{t,k,c}(n,\Gamma)$ and is given by:

$$B_{t,k,c}(n,\Gamma) = \mathbf{e}H_{t,k,c}(n,\Gamma)\mathbf{e},$$

where $\mathbf{e}$ is the symmetrizer for $S_n \ltimes \Gamma^n$.

Remark 5.1.2.3. As before we have:

$$B_{t,k,c}(n,\Gamma) = \mathrm{Hom}_{S_n \ltimes \Gamma^n}(\Bbbk, \mathrm{Ind}_{S_n \ltimes \Gamma^n}^{H_{t,k,c}(n,\Gamma)}(\Bbbk)).$$

We will use the same notation for the categories of representations:

Definition 5.1.2.4. By $\mathbf{Rep}(H_{t,k,c}(\Gamma,n);\Bbbk)$ denote the category of representations of the symplectic reflection algebra $H_{t,k,c}(\Gamma,n)$ over $\Bbbk$. Also for $p \geq 0$ denote

$$\mathbf{Rep}_p(H_{t,k,c}(\Gamma,n)) = \mathbf{Rep}(H_{t,k,c}(\Gamma,n);\overline{\mathbb{F}}_p) .$$

Remark 5.1.2.5. Notice that when $\Gamma = 1$ we get back the case of rational Cherednik algebra of type A, i.e., $H_{t,k,\emptyset}(n,1) = H_{t,k}(n)$. Also, in the case $\Gamma = \mathbb{Z}/2\mathbb{Z}$ we get the rational Cherednik algebra of type B.

5.1.3 Extended Cherednik algebra

In this section we will introduce the notion of the extended Cherednik algebra that in yet another way generalizes the notion of the Cherednik algebra and will discuss its basic properties. This algebra is going to be used by us to construct DDCA of type A in higher rank. This algebra was introduced in [19], see this paper for more information regarding it. Everywhere we suppose that $\mathrm{char}(\Bbbk) > n$.

Definition 5.1.3.1 (Definition 2.4 in [19])**.** For $t, k \in \Bbbk$ and $n, r \in \mathbb{Z}_{>0}$ define the extended Cherednik algebra $H_{t,k}(n, r)$ to be a quotient of the semi-direct product:

$$\Bbbk[S_n] \ltimes [\Bbbk\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle \otimes (\mathrm{End}(\Bbbk^r))^{\otimes n}] \ ,$$

where S_n acts by permuting x_i, y_i and the copies of $\mathrm{End}(\Bbbk^r)$. The quotient is taken by the ideal generated by the following relations:

$$[x_i, x_j] = 0 \ , \ [y_i, y_j] = 0 \ ,$$
$$[y_i, x_j] = \delta_{ij}(t - k \sum_{m \neq i} s_{im}\sigma_{im}) + (1 - \delta_{ij})k s_{ij}\sigma_{ij} \ ,$$

where s_{ij} are the transpositions from S_n viewed as elements of $\Bbbk[S_n]$ and σ_{ij} is the following element of $\mathrm{End}(\Bbbk^r)^{\otimes n}$:

$$\sigma_{ij} = \sum_{\alpha,\beta} (E_{\alpha\beta})_i (E_{\beta\alpha})_j \ .$$

Here by $(g)_i$ for $g \in \mathrm{End}(\Bbbk^r)$ we denote an element of $\mathrm{End}(\Bbbk^r)^{\otimes n}$ which is equal to $1 \otimes \cdots \otimes g \otimes \cdots \otimes 1$ with g on the i-th place[1]. Notice that σ_{ij} as an operator acting on $(\Bbbk^r)^{\otimes n}$ is exactly the operator which transposes the i-th and j-th spaces.

Remark 5.1.3.2. Obviously for $r = 1$ the algebra $H_{t,k}(n, 1)$ is just the usual rational Cherednik algebra of type A_{n-1}. I.e $H_{t,k}(n, 1) = H_{t,k}(n)$.

Now there is also an analogue of the polynomial representation for $H_{t,k}(n, r)$.

Proposition 5.1.3.3 (Proposition 2.7 in [19])**.** *Consider the vector space given by* $V(n, r) = \Bbbk[x_1, \ldots, x_n] \otimes (\Bbbk^r)^{\otimes n}$. *It has a natural action of* $H_{t,k}(n, r)$ *given by the*

[1]We use the extra brackets around g here, since in what follows we will consider the cases where the elements of $\mathrm{End}(\Bbbk^r)$ we are going to use are equal to elementary matrices.

following formulas:

$$x_i \mapsto x_i \cdot \otimes 1 \ , \ s_{ij} \mapsto s^x_{ij} \otimes \sigma_{ij} \ , (g)_i \mapsto 1 \otimes (g)_i$$
$$y_i \mapsto \partial_i \otimes 1 - k \sum_{j \neq i} \frac{s^x_{ij} \otimes 1}{x_i - x_j} \ ,$$

where s^x_{ij} is the transposition acting on $\Bbbk[x_1, \dots, x_n]$.

Proof. Notice that $s^x_{ij} = s_{ij}\sigma_{ij}$ in this representation. Using this it is easy to see that all the operators satisfy the required relations. □

Corollary 5.1.3.4 (Proposition 2.8 in [19])**.** *The algebra $H_{t,k}(n, r)$ enjoys the PBW-property in the sense that the multiplication map*

$$\Bbbk[x_1, \dots, x_n] \otimes [\Bbbk[S_n] \otimes (End(\Bbbk^r))^{\otimes n}] \otimes \Bbbk[y_1, \dots, y_n] \to H_{t,k}(n, r)$$

is an isomorphism. Moreover the multiplication maps for any other ordering of tensor multiples are also isomorphisms.

Proof. This follows from the fact that the polynomial representation introduced above is faithful and the image of $H_{t,k}(n, r)$ in $\mathrm{End}(V(n, r))$ is the subalgebra of $\Bbbk[S_n] \ltimes (\mathcal{D}^{reg}(\mathbb{A}^n) \otimes \mathrm{End}(\Bbbk^r)^{\otimes n})$, where $\mathcal{D}^{reg}(\mathbb{A}^n)$ is the algebra of differential operators on the regular locus of $\mathbb{A}^n$. □

We can also define the spherical subalgebra of $H_{t,k}(n, r)$.

Definition 5.1.3.5. For $t, k \in \Bbbk$ and $n, r \in \mathbb{Z}_{>0}$ define the spherical subalgebra of the extended Cherednik algebra to be $B_{t,k}(n, r) = \mathbf{e}H_{t,c}(n, r)\mathbf{e}$, where $\mathbf{e}$ is a symmetrizing element $\mathbf{e} = \frac{1}{n!}\sum_{s\in S_n} s$.

There is a natural vector space bifiltration on $H_{t,k}(n, r)$.

Definition 5.1.3.6. Assign to an element $\prod_i x_i^{n_i} s \bigotimes_i (g_i)_i \prod_i y_i^{m_i} \in H_{t,k}(n,r)$ the following bidegree. Denote by $H = |\{i \in \{1,\ldots,n\} \mid n_i = 0,\ g_i \notin \Bbbk \cdot \mathrm{Id}_{\Bbbk^r},\ m_i = 0\}|$, and by $V = \sum_i (n_i + m_i)$. Then $\deg(\prod_i x_i^{n_i} s \bigotimes_i (g_i)_i \prod_i y_i^{m_i}) = (n - H, V)$. Define the bifiltration on $H_{t,k}(n,r)$ using this formula.

I.e. the horizontal degree tells us how many indices actually appear in the monomial, and the vertical degree is the total polynomial degree of the monomial. Note that this is not an algebra bifiltration. The same vector space bifiltration restricts to the spherical subalgebra.

However note that the associated graded of $H_{t,k}(n,r)$ with respect to the vertical filtration is simply $\mathrm{gr}_v(H_{t,k}(n,r)) \simeq \Bbbk[S_n] \ltimes (\Bbbk[x_1,\ldots,x_n,y_1,\ldots,y_n] \otimes \mathrm{End}(\Bbbk^r)^{\otimes n})$. Now the vector space bifiltration of $H_{t,k}(n,r)$ restricts to $\mathrm{gr}_v(H_{t,k}(n,r))$ and makes it a bifiltered algebra.

Moreover, the associated graded of the spherical subalgebra $B_{t,k}(n,r)$ is given by a similar formula $\mathrm{gr}_v(B_{t,k}(n,r)) \simeq (\Bbbk[x_1,\ldots,x_n,y_1,\ldots,y_n] \otimes (\mathrm{End}(\Bbbk^r))^{\otimes n})^{S_n}$. And again this associated graded is a bifiltered algebra.

But now this is simply $\mathrm{gr}_v(B_{t,k}(n,r)) \simeq S^n(\mathrm{End}(\Bbbk^r)[x,y])$. And the bifiltration on $\mathrm{gr}_v(B_{t,k}(n,r))$ coincides exactly with the standard bifiltration of $S^n(\mathrm{End}(\Bbbk^r)[x,y])$ arising from the fact that $\mathrm{End}(\Bbbk^r)[x,y]$ is a filtered unital algebra with the filtration given by the total degree of the polynomial. I.e. we are now in the setting of Section 3.2.2 with $A = \mathrm{End}(\Bbbk^r)[x,y]$. We will use the results of that Section below to construct a generating set of $B_{t,k}(n,r)$ and then later, when we take the ultraproduct of $B_{t,k}(n,r)$ to obtain the DDCA.

Remark 5.1.3.7. As the final remark of this section note that we can also construct the spherical subalgebra using the induction functor in the following way:

$$B_{t,k}(n,r) = \mathrm{Hom}_{S_n}(\Bbbk, H_{t,k}(n,r)\mathbf{e}) =$$

$$= \mathrm{End}_{H_{t,k}(n,r)}(H_{t,k}(n,r)\mathbf{e}) = \mathrm{End}_{H_{t,k}(n,r)}(\mathrm{Ind}_{S_k}^{H_{t,k}(n,r)}(\Bbbk)) \ .$$

Generating set of $B_{t,k}(n,r)$

Here we would like to present a way to construct a generating set for $B_{t,k}(n,r)$. It will later turn out that this generating set can be used to construct a basis of the corresponding DDCA.

Pick a basis in $\mathrm{End}(\Bbbk^r)$, which contains $\mathrm{Id}_{\Bbbk^r}$ as an element. Let us denote this basis by $\alpha_i \in \mathrm{End}(\Bbbk^r)$ with i going from 1 to r^2 and $\alpha_1 = \mathrm{Id}_{\Bbbk^r}$. Now we can define the following elements in $B_{t,k}(n,r)$.

Definition 5.1.3.8. Define the elements $T_{r,q,n}(g)$, for $g \in \mathrm{End}(\Bbbk^r)$ by the following formula (here $L = r + q$):

$$\sum_{r,q\geq 0, r+q=L} T_{r,q,n}(g)\frac{u^r}{r!}\frac{v^q}{q!} = \sum_{i=1}^{n}(g)_i\frac{(ux_i + vy_i)^L}{L!}\mathbf{e},$$

where u, v are formal variables.

They are defined for $\mathrm{char}(\Bbbk) > r + q$ or zero characteristic.

More explicitly, $T_{r,q,n}(g)$ is proportional to the sum of all shuffles of r copies of x_i and q copies of y_i multiplied by $(g)_i$ and summed over all i from 1 to n.

Note that the highest term of $T_{r,q,n}$ with respect to the vertical filtration is:

$$T_{r,q,n}(g) = \sum_i (g)_i x_i^r y_i^q + \text{lower order terms} \ .$$

Suppose that $\mathrm{char}(\Bbbk) = 0$. Now note that we have established that we have an isomorphism $\mathrm{gr}_v(B_{t,k}(n,r)) \simeq S^n(\mathrm{End}(\Bbbk^r)[x,y])$. From Proposition 3.2.2.8 we know that there is a map $\delta_n : \mathrm{End}(\Bbbk^r)[x,y] \to S^n(\mathrm{End}(\Bbbk^r)[x,y])$. Note that under this map $\delta_n(g \cdot x^r y^q) = \sum_i (g)_i x_i^r y_i^q$. I.e. the images of $T_{r,q,n}(g)$ in the associated graded span exactly the image of the map δ_n. More precisely it is enough to consider all $T_{r,q,n}(\alpha_l)$ for $r, q \in \mathbb{Z}_{\geq 0}$ and $\alpha_l \in \{1, \dots, r^2\}$ to span this image, since $\alpha_l \cdot x^r y^q$ for all such r, q, l are the basis of $\mathrm{End}(\Bbbk^r)[x,y]$.

Now we can define something like the shuffled products of the above elements.

Definition 5.1.3.9. Denote by $\mathbf{m}$ the collection of integers $m_{r,q,l}$, for $r, q \in \mathbb{Z}_{\geq 0}$ and $l \in \{1, \dots, r^2\}$. Denote $|\mathbf{m}| = \sum_{r,q,l} m_{r,q,l}$ and $w(\mathbf{m}) = \sum_{r,q,l}(r+q)m_{r,q,l}$. We define $T_n(\mathbf{m})$ with $M = |\mathbf{m}|$, by the following formula:

$$\sum_{\mathbf{m}, |\mathbf{m}|=M} T_n(\mathbf{m}) \prod_{r,q,l} \frac{z_{r,q,l}^{m_{r,q,l}}}{m_{r,q,l}!} = \frac{(\sum_{r,q,l} z_{r,q,l} T_{r,q,n}(\alpha_l))^M}{M!} ,$$

here $z_{r,q,l}$ are formal variables.

Note that these elements are defined for $w(\mathbf{m}) < \mathrm{char}(\Bbbk)$ or zero characteristic.

Note that with respect to the total filtration

$$T_n(\mathbf{m}) = \prod_{r,q,l} T_{r,q,n}(\alpha_l))^{m_{r,q,l}} + \text{lower order terms} .$$

Suppose we are in the case $\mathrm{char}(\Bbbk) = 0$.

Since $T_{r,q,n}(\alpha_l)$ span the image of δ_n in the associated graded algebra with respect to the vertical filtration, it follows that $T_n(\mathbf{m})$ span everything which is generated by δ_n inside $\mathrm{gr}_v(B_{t,k}(n,r))$. If we use the notation of Definition 3.2.2.9, we can state this by saying that the images of $T_n(\mathbf{m})$ in the associated graded span the image of Δ_n.

But from Proposition 3.2.2.11 we know that this image covers the whole algebra. So the following Proposition follows.

Proposition 5.1.3.10. *Suppose* $char(\Bbbk) = 0$*. The elements* $T_n(\mathbf{m})$ *for all choices of* $\mathbf{m}$ *form a generating set of* $B_{t,k}(n, r)$*.*

It's a bit trickier to show that these elements give us a generating set in positive characteristic for big enough $\mathrm{char}(\Bbbk)$. Luckily, we will not need this fact to prove that these elements give us a basis of the corresponding DDCA. However, the proof of this fact for $r = 1$ can be found in Section 4.1.2 of [18].

5.2 Extended Cherednik algebras and symplectic reflection algebras in complex rank

In this section we will explain how one can extend the notion of the algebras introduced in the last section from the finite rank to the Deligne category $\mathrm{Rep}(S_\nu)$. We will also show how these constructions are related to each other through ultraproducts.

5.2.1 Extended Cherednik algebras in complex rank

In this section we will explain how to work with the extended Cherednik algebras in the complex rank. First we will define a category of representations of $H_{t,k}(\nu, r)$.

In order to do this we need to explain a few things about the central elements in $\mathrm{Rep}(S_\nu)$. This will build on the discussion around Definition 2.1.0.5.

Construction 5.2.1.1. Let us define the action of the central element Ω on objects of $\mathrm{Rep}(S_\nu)$. Consider $E_2 \subset \Bbbk[S_\nu]$ as defined in [13]. This is the interpolation of the subspaces spanned by transpositions in the group algebra. Then we have a map $\Delta_{E_2} : E_2 \to E_2 \otimes E_2$ that interpolates the usual coproduct map $\Delta(s_{ij}) = s_{ij} \otimes s_{ij}$. Also we have a map $\omega : \Bbbk \to E_2$ interpolating the central element inclusion map $1 \mapsto \sum s_{ij} = \Omega$. We also automatically have an action map $a_{E_2} : E_2 \otimes V \to V$ for any object $V \in \mathrm{Rep}(S_\nu)$. Thus we get the alternative way to define the map $\Omega : V \to V$ given by the identity functor endomorphism. More precisely, this map is given by $a_{E_2} \circ (\omega \otimes 1)$.

For our purposes we need to slightly upgrade this central element.

Construction 5.2.1.2. Note that there is a map $i_{E_2} : E_2 \to \mathfrak{h} \otimes \mathfrak{h}$, which interpolates the map $s_{ij} \mapsto \frac{x_i \otimes x_j + x_j \otimes x_i}{2}$. Also consider a map $coev_{\mathrm{End}(\Bbbk^r)} : \Bbbk \to \mathrm{End}(\Bbbk^r) \otimes \mathrm{End}(\Bbbk^r)$ (i.e. we have $1 \mapsto \sum_{i,j} E_{ij} \otimes E_{ji}$). Now we can construct $\omega_{\mathrm{End}(\Bbbk^r)}$ as follows:

$$\omega_{\mathrm{End}(\Bbbk^r)} = (1 \otimes \sigma_{\mathrm{End}(\Bbbk^r),\mathfrak{h}} \otimes 1) \circ (1 \otimes i_{E_2} \otimes 1) \circ (1 \otimes \Delta_{E_2}) \circ (1 \otimes \omega) \circ tw_{\mathrm{End}(\Bbbk^r)} \ ,$$

which takes $\Bbbk \to \mathrm{End}(\Bbbk^r) \otimes \mathfrak{h} \otimes \mathrm{End}(\Bbbk^r) \otimes \mathfrak{h} \otimes E_2$.

Now suppose V is an object of $\mathrm{Rep}(S_\nu)$, with a fixed map $\alpha : \mathrm{End}(\Bbbk^r) \otimes \mathfrak{h} \otimes V \to V$. Then we can define $\Omega_{\mathrm{End}(\Bbbk^r)} : V \to V$ as

$$\Omega_{\mathrm{End}(\Bbbk^r)} = \alpha \circ (1 \otimes \alpha) \circ (1 \otimes a_{E_2}) \circ (\omega_{\mathrm{End}(\Bbbk^r)} \otimes 1) \ .$$

So we have another "central element" for special objects of $\mathrm{Rep}(S_\nu)$.

Definition 5.2.1.3. The category $\mathrm{Rep}(H_{t,k}(\nu, r))$ is defined as follows. The objects are given by triples (M, x, y, α), where M is an ind-object of $\mathrm{Rep}(S_\nu)$, x is a map $x : \mathfrak{h}^* \otimes M \to M$, y a map $y : \mathfrak{h} \otimes M \to M$ and α is a yet another map $\alpha : (\mathrm{End}(\Bbbk^r) \otimes \mathfrak{h}) \otimes M \to M$, all of which are morphisms in $\mathrm{IND}(\mathrm{Rep}(S_\nu))$. They are required to satisfy the following conditions:

$$x \circ (1 \otimes x) - x \circ (1 \otimes x) \circ (\sigma \otimes 1) = 0,$$

as a map from $\mathfrak{h}^* \otimes \mathfrak{h}^* \otimes M$ to M;

$$y \circ (1 \otimes y) - y \circ (1 \otimes y) \circ (\sigma \otimes 1) = 0,$$

as a map from $\mathfrak{h} \otimes \mathfrak{h} \otimes M$ to M;

$$\alpha \circ (1 \otimes \alpha) - \alpha \circ (1 \otimes \alpha) \circ (\sigma_{\mathrm{End}(\Bbbk^r)\otimes\mathfrak{h}} \otimes 1) =$$
$$= \alpha \circ (\mu_{\mathrm{End}(\Bbbk^r)} \otimes 1 - [\mu_{\mathrm{End}(\Bbbk^r)} \otimes 1] \circ [\sigma_{\mathrm{End}(\Bbbk^r)} \otimes 1]) \circ (1 \otimes \pi_{\mathrm{diag}} \otimes 1) \circ (1 \otimes \sigma_{\mathfrak{h},\mathrm{End}(\Bbbk^r)} \otimes 1),$$

as a map from $\mathrm{End}(\Bbbk^r) \otimes \mathfrak{h} \otimes \mathrm{End}(\Bbbk^r) \otimes \mathfrak{h} \otimes M \to M$, where $\mu_{\mathrm{End}(\Bbbk^r)}$ is multiplication in $\mathrm{End}(\Bbbk^r)$ and $\pi_{\mathrm{diag}} : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ is the interpolation of the projection $x_i \otimes x_j \mapsto \delta_{ij} x_i$;

$$\alpha \circ (\iota_{\mathrm{End}(\Bbbk^r)} \otimes 1) - 1 \otimes \mathrm{Tr}_{\mathfrak{h}} \otimes 1 = 0,$$

as a map $\Bbbk \otimes \mathfrak{h} \otimes M \to M$, where $\iota_{\mathrm{End}(\Bbbk^r)}$ is the unit map of $\mathrm{End}(\Bbbk^r)$ and $\mathrm{Tr}_{\mathfrak{h}} : \mathfrak{h} \to \Bbbk$ is the trace, the interpolation of the map $x_i \mapsto 1$;

$$y \circ (1 \otimes x) - x \circ (1 \otimes y) \circ (\sigma \otimes 1) = t \cdot \mathrm{ev}_{\mathfrak{h}} \otimes 1 - k \cdot (\mathrm{ev}_{\mathfrak{h}} \otimes 1) \circ (\Omega^3_{\mathrm{End}(\Bbbk^r)} - \Omega^{1,3}_{\mathrm{End}(\Bbbk^r)}),$$

as a map $\mathfrak{h} \otimes \mathfrak{h}^* \otimes M$ to M, where $\Omega_{\mathrm{End}(\Bbbk^r)}$ is a central element from Construction 5.2.1.2, and indices indicate the spaces on which $\Omega_{\mathrm{End}(\Bbbk^r)}$ acts in the tensor product $\mathfrak{h} \otimes \mathfrak{h}^* \otimes M$.

The morphisms of $\mathrm{Rep}(H_{t,k}(\nu))$ are the morphisms of $\mathrm{IND}(\mathrm{Rep}(S_\nu))$ which commute with the action-maps x,y and α.

Remark 5.2.1.4. Some comments are in order to explain why this is indeed the correct generalization of Definition 5.1.3.1. To see that one needs to understand that Definition 5.2.1.3 above, if used in the finite rank, gives us the usual category of representations of the extended Cherednik algebra $H_{t,k}(n,r)$. Indeed, note that since M is already an object of the category of representations of symmetric group, we do not need to define its action. Now maps x and y determines the action of elements x_i and y_i. The map α determines the action of elements $(g)_i$. The first two formulas tell us that x_i commute with each other and so also y_i. The third formula gives us the commutation relation between $(g)_i$ and $(h)_j$ (i.e. $[(g)_i, (h)_j] = \delta_{ij}([g,h])_i$). The fourth tells us that all $(1)_i$ act trivially. And, finally, the fifth formula, if expanded, gives us the correct commutation relation between x_i and y_j.

Now we would like to show how we can construct some of the objects of the category $\mathrm{Rep}(H_{t,k}(\nu,r))$ as ultraproducts.

Remark 5.2.1.5. Below we will denote by t_n, k_n the elements of $\overline{\mathbb{F}}_{p_n}$ such that $\prod_{\mathcal{F}} t_n = t$ and $\prod_{\mathcal{F}} k_n = k$ under the fixed isomorphism of $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} \simeq \mathbb{C}$. We will use a similar notation for all other parameters of algebras used in the thesis.

Lemma 5.2.1.6. *Suppose M_n is a sequence of objects of $\mathbf{Rep}_{p_n}(H_{t_n,k_n}(\nu_n,r))$ such that their (restricted) ultraproduct as objects of $\mathbf{Rep}_{p_n}(S_{\nu_n})$ lies in $\mathrm{IND}(\mathrm{Rep}(S_\nu))$. Suppose x_n, y_n and α_n are the maps which define the action of generators of the corresponding Cherednik algebra on M_n. Then $(\prod_{\mathcal{F}}^{C,r} M_n, \prod_{\mathcal{F}} x_n, \prod_{\mathcal{F}} y_n, \prod_{\mathcal{F}} \alpha_n)$ defines an object of $\mathrm{Rep}(H_{t,k}(\nu,r))$.*

Proof. It's easy to see that the data $(\prod_{\mathcal{F}}^{C,r} M_n, \prod_{\mathcal{F}} x_n, \prod_{\mathcal{F}} y_n, \prod_{\mathcal{F}} \alpha_n)$ is well defined. Since x_n, y_n and α_n satisfy the same conditions in finite rank and complex rank it follows that by Łoś's theorem this is indeed an object of $\mathrm{Rep}(H_{t,k}(\nu,r))$. $\square$

Now we would like to construct an interpolation of the functors $\mathrm{Ind}_{S_{\nu_n}}^{H_{t_n,k_n}(\nu_n,r)}$. It is possible to construct the full functor as an ultraproduct, but this functor would a priori have $\prod_{\mathcal{F}} \mathbf{Rep}_{p_n}(H_{t_n,k_n}(\nu_n,r))$ as its target category, so we would need to explain why the functor really gives us objects of $\mathrm{Rep}(H_{t,k}(\nu,r))$. Instead we will construct this functor explicitly, which will also show that it agrees with the ultraproduct functor when applied to objects of $\mathrm{Rep}(S_\nu)$.

The idea is, following the PBW theorem, to think about "$H_{t,k}(\nu,r)$" as "the direct sum $\bigoplus_{i,j\geq 0} S^i(\mathfrak{h}^*)\otimes S^j(\mathfrak{h})\otimes(\mathrm{End}(\Bbbk^r))^{\otimes\nu}\otimes\mathbb{C}[S_\nu]$" and take the tensor product with $V\in\mathrm{Rep}(S_\nu)$ "over $\mathbb{C}[S_\nu]$".

Before the actual construction we need to note several things.

Construction 5.2.1.7. Denote $A=\mathrm{End}(\Bbbk^r)$. First, since A is a unital algebra with the standard filtration $F^0A=\Bbbk\cdot 1$ and $F^1A=A$, we have an induced filtration on $A^{\otimes\nu}$. Note that $A^{\otimes\nu}$ as an algebra is generated by its first filtration component $F^1A^{\otimes\nu}$. This component itself is actually a subobject of $\mathfrak{h}\otimes A$, more precisely to obtain it we need to throw out a subobject $\mathcal{X}((1))\otimes F^0A$ from $\mathfrak{h}\otimes A$ (note $\mathfrak{h}=\mathcal{X}((1))\oplus\Bbbk$). It follows that there are maps $i_{l,A}: F^lA^{\otimes\nu}\to(\mathfrak{h}\otimes A)^{\otimes l}$ and $\pi_{l,A}:(\mathfrak{h}\otimes A)^{\otimes l}\to F^lA^{\otimes\nu}$. Let us denote the multiplication map

$$\pi_{l+1,A}\circ 1\otimes i_{l,A}:(\mathfrak{h}\otimes A)\otimes F^lA^{\otimes\nu}\to F^{l+1}A^{\otimes\nu}$$

by $\mu_{l,A}$.

Also note that $S^{i+1}(\mathfrak{h})$ is isomorphic to a direct summand of $\mathfrak{h}\otimes S^i(\mathfrak{h})$, let's denote the corresponding inclusion and projection as $\iota_{i+1,y}$ and $\pi_{i+1,y}$ respectively. The same is true for $\mathfrak{h}^*$, the corresponding morphisms are $\iota_{i+1,x}$ and $\pi_{i+1,x}$.

With this we can proceed to construct the induction functor.

Construction 5.2.1.8. For an object $V \in \mathrm{Rep}(S_\nu)$, consider an ind-object given by $I_V = \oplus_{i,j\geq 0} I_{i,j}$, where $I_{i,j} = S^i(\mathfrak{h}^*) \otimes S^j(\mathfrak{h}) \otimes A^{\otimes\nu} \otimes V$, and maps $x_V : \mathfrak{h}^* \otimes I_V \to I_V$, $y_V : \mathfrak{h} \otimes I_V \to I_V$ and $\alpha_V : (\mathfrak{h} \otimes A) \otimes I_V \to I_V$, which are defined as follows.

First let us define $\alpha_V|_{I_{i,j}} : (\mathfrak{h} \otimes A) \otimes I_{i,j} \to I_{i,j}$. We will do so by considering the action of this map on each filtration component $F^l I_{i,j} = S^i(\mathfrak{h}^*) \otimes S^j(\mathfrak{h}) \otimes F^l A^{\otimes\nu} \otimes V$. Now we can define the action of $\alpha_V|_{F^l I_{i,j}} : (\mathfrak{h} \otimes A) \otimes F^l I_{i,j} \to F^{l+1} I_{i,j}$ to be equal to

$$\alpha_V|_{F^l I_{i,j}} = (1 \otimes \mu_{l,A} \otimes 1) \circ (\sigma_{\mathfrak{h}\otimes A, S^i(\mathfrak{h}^*)\otimes S^j(\mathfrak{h})} \otimes 1) \ .$$

Now define $x_V|_{I_{i,j}} : \mathfrak{h}^* \otimes I_{i,j} \to I_{i+1,j}$ to be equal to $\pi_{i+1,x} \otimes 1$ for all i, j. Also define $y_V|_{I_{0,j}} : \mathfrak{h} \otimes I_{0,j} \to I_{0,j+1}$ as $\pi_{j+1,y} \otimes 1$. And lastly we also define the map $y_V|_{I_{i,j}} : \mathfrak{h} \otimes I_{i,j} \to I_{i,j+1} \oplus I_{i-1,j}$ by induction in i as:

$$\Big[(x \otimes 1) \circ (1 \otimes y \otimes 1) \circ (\sigma_{\mathfrak{h},\mathfrak{h}^*} \otimes 1) + t \cdot \mathrm{ev}_{\mathfrak{h}} \otimes 1 - k \cdot (\mathrm{ev}_{\mathfrak{h}} \otimes 1) \circ (\Omega_A^{I_{i-1,j}} - \Omega_A^{\mathfrak{h}, I_{i-1,j}})\Big] \circ (1 \otimes \iota_{i,x} \otimes 1).$$

Now we would like to show that this defines an object of $\mathrm{Rep}(H_{t,k}(\nu, r))$. Indeed:

Lemma 5.2.1.9. *In the notations of Construction 5.2.1.8, the tuple $(I_V, x_V, y_V, \alpha_V)$ defines an object of Rep$(H_{t,k}(\nu, r))$.*

Proof. Indeed, the first two formulas of Definition 5.2.1.3 are satisfied by the properties of symmetric powers, and we defined the action of y_V by induction in such a way that the last equation is also satisfied. The equations for α_V are satisfied in a straightforward way.

Another way to see that is to note that in the finite rank case this construction amounts to $H_{t_n,k_n}(\nu_n) \otimes_{S_{\nu_n}} V_n$, and so by Łoś's theorem , we do get a correct structure of an "$H_{t,k}(\nu)$-module". □

Now we need to construct the action of the induction functor on morphisms.

Construction 5.2.1.10. In the notation of Construction 5.2.1.8, given a morphism $\phi : V \to U$, define a morphism $I_\phi : I_V \to I_U$ in the following way:

$$(I_\phi)|_{S^i(\mathfrak{h}^*)\otimes S^j(\mathfrak{h})\otimes A^{\otimes\nu}\otimes V} := 1 \otimes \phi \ .$$

Lemma 5.2.1.11. *In the notation of Constructions 5.2.1.8 and 5.2.1.10, I_ϕ is a morphism in* $\mathrm{Rep}(H_{t,k}(\nu,r))$.

Proof. This is easy to see both straight from the definition, or by the ultraproduct argument, since in finite rank this defines an actual $H_{t_n,k_n}(\nu_n,r)$-module morphism.

□

Now we can define the actual functor:

Definition 5.2.1.12. Define a functor $\mathrm{Ind}_{S_\nu}^{H_{t,k}(\nu,r)} : \mathrm{Rep}(S_\nu) \to \mathrm{Rep}(H_{t,k}(\nu,r))$ in the following way. On objects it takes V to the triple $(I_V, x_V, y_V, \alpha_V)$ from Construction 5.2.1.8. And on morphisms it takes $\phi : V \to U$ to I_ϕ from Construction 5.2.1.10. This is a well defined functor by Lemmas 5.2.1.9 and 5.2.1.11.

The next Corollary follows by construction and the above lemmas:

Corollary 5.2.1.13. *For any object $V \in \mathrm{Rep}(S_\nu)$ such that $V = \prod_{\mathcal{F}} V_n$ we have:*

$$\mathrm{Ind}_{S_\nu}^{H_{t,k}(\nu,r)} V = \prod\nolimits_{\mathcal{F}}^{C,r} \mathrm{Ind}_{S_{\nu_n}}^{H_{t_n,k_n}(\nu_n,r)} V_n,$$

where the filtration on $\mathrm{Ind}_{S_{\nu_n}}^{H_{t_n,k_n}(\nu_n,r)} V_n$ is obtained from the vector space bifiltration of $H_{t_n,k_n}(\nu_n,r)$ (which can be seen to be S_{ν_n}-invariant).

Note that the elements of the above definition and constructions in the case of $r = 1$, i.e. in the case of the regular Cherednik algebra, can be found in the paper [10] by Inna Entova-Aizenbud. A very similar exposition for rank 1 can also be found in [18], a later paper by Pavel Etingof, Eric Rains and the author of the present thesis.

5.2.2 Symplectic reflection algebras in complex rank

In this section we will briefly restate some of the results of the previous section in the context of symplectic reflection algebras. As in Section 3.2.3, we will work for transcendental ν for simplicity. Also as in that section, we fix a finite group $\Gamma \subset \mathrm{SL}(2, \overline{\mathbb{Q}})$.

Below we will define the category $\mathrm{Rep}(H_{t,k,c}(\nu, \Gamma))$ following the lines of Definition 5.2.1.3. To do this, we need to find the analog of V in Definition 5.1.2.1.

Proposition 5.2.2.1. *The ultraproduct $\prod_{\mathcal{F}}^{C} (\overline{\mathbb{Q}}^2)^n$ as $S_n \ltimes \Gamma^n$-modules defines an object of* $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$*.*

Proof. Indeed as S_n-modules, each $(\overline{\mathbb{Q}}^2)^n = \mathfrak{h}_n \oplus \mathfrak{h}_n$, hence their ultraproduct is given by $\mathfrak{h}^{\oplus 2}$ as an object of $\mathrm{Rep}(S_\nu)$. Thus by Proposition 3.2.3.3 it follows that it is also an object of $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$. The symplectic pairing is given by the ultraproduct of symplectic pairings. $\square$

We will denote this object by V and call the fundamental representation of "$S_\nu \ltimes \Gamma^\nu$". Also V carries a natural symplectic pairing ω. Now we are ready to define the category itself.

Definition 5.2.2.2. Consider t, k, c_C, T_C as in Definition 5.1.2.1 with $\Bbbk = \mathbb{C}$. Let $\nu \in \mathbb{C}$ be a transcendental number. The objects of the category $\mathrm{Rep}(H_{t,k,c}(\nu, \Gamma))$ are given by pairs (M,y), where M is an object of $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$ and y is a map:

$$y : V \otimes M \to M,$$

such that the following holds:

$$y\circ(1\otimes y)\circ((1-\sigma)\otimes 1) = (\omega\otimes 1)\circ\left(t - k(\Omega^3 - \Omega^{1,3}) - \sum_C \frac{c_C}{1-T_C}(\Omega^3_C - \Omega^{13}_C - \Omega^{23}_C + \Omega^{123}_C)\right),$$

as a map from $V \otimes V \otimes M$ to M, where Ω is an endomorphism from Definition 3.1.0.8 and Ω_C is the endomorphism obtained in a similar way as the ultraproduct of endomorphisms of the identity functor arising from the sum of elements of the group belonging to the conjugacy class C.

The morphisms are given by morphisms in $\mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$ which commute with y.

In a fashion similar to the discussion after Definition 5.2.1.3 one can see that this definition is the same as in finite rank, written in an element free way. Thus for the same reasons one obtains the following statement, which generalizes Proposition 3.2.3.3 and Lemma 5.2.1.6.

Proposition 5.2.2.3. *Suppose M_n are $H_{t_n,k_n,c_n}(n,\Gamma)$-modules whose ultraproduct $\prod_{\mathcal{F}}^{C,r} M_n$ is a well defined object of* $\mathrm{IND}(\mathrm{Rep}(S_\nu))$. *Suppose y_n denotes the corresponding map $y_n : (\overline{\mathbb{Q}}^2)^n \otimes M_n \to M_n$. Then $(\prod_{\mathcal{F}}^{C,r} M_n, \prod_{\mathcal{F}} y_n)$ is an object of* $\mathrm{Rep}(H_{t,k,c}(\nu,\Gamma))$.

Also repeating the steps of Section 5.2.1 we can construct the induction functor. Since the construction is almost literally the same, we just state the result.

Proposition 5.2.2.4. *There is a functor*

$$\mathrm{Ind}_{S_\nu\ltimes\Gamma^n}^{H_{t,k,c}(\nu,\Gamma)} : \mathrm{Rep}(S_\nu \ltimes \Gamma^\nu) \to \mathrm{Rep}(H_{t,k,c}(\nu,\Gamma)) \ ,$$

such that, if $M \in \mathrm{Rep}(S_\nu \ltimes \Gamma^\nu)$ *is an object given by ultraproduct of* $S_n \ltimes \Gamma^n$*-modules, i.e.,* $M = \prod_{\mathcal{F}}^C M_n$*, then:*

$$\mathrm{Ind}_{S_\nu \ltimes \Gamma^n}^{H_{t,k,c}(\nu,\Gamma)}(M) = \prod\nolimits_{\mathcal{F}}^{C,r} \mathrm{Ind}_{S_n \ltimes \Gamma^n}^{H_{t_n,k_n,c_n}(n,\Gamma)}(M_n).$$

5.3 DDCA in Deligne Categories

In this section we will define and study the Deformed Double Current Algebra of rank r and type A as an algebra of endomorphisms of an ind-object of $\mathrm{Rep}(S_\nu)$. We will also provided a basis of this algebra. In the last part of this section we will say a few extra words about the case of rank 1 that we will treat in more detail later on.

5.3.1 Definition

First let us define the Deformed Double Current algebra of rank r. We will construct it as an algebra of endomorphisms in $\mathrm{Rep}(H_{t,k}(\nu, r))$.

Definition 5.3.1.1. For $r \in \mathbb{Z}_{>0}$, $\nu \in \mathbb{C}\backslash\mathbb{Z}_{\geq 0}$ and $t, k \in \mathbb{C}$, define the DDCA of rank r, denoted $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$, as:

$$\widetilde{\mathcal{D}}_{t,k,\nu}(r) = \mathrm{End}_{\mathrm{Rep}(H_{t,k}(\nu,r))}(\mathrm{Ind}_{S_\nu}^{H_{t,k}(\nu,r)}(\mathbb{C}))\ .$$

This is obviously an interpolation of Remark 5.1.3.7. This can be made precise using Corollary 5.2.1.13:

Proposition 5.3.1.2. *The algebra* $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$ *is equal to a restricted ultraproduct of* $B_{t_n,k_n}(\nu_n, r)$ *with respect to the total filtration.*

Proof. Indeed since

$$\widetilde{\mathcal{D}}_{t,k,\nu}(r) = \mathrm{End}_{\mathrm{Rep}(H_{t,k}(\nu,r))}(\mathrm{Ind}_{S_\nu}^{H_{t,k}(\nu,r)}(\mathbb{C})) = \mathrm{Hom}_{\mathrm{Rep}(S_\nu)}(\mathbb{C}, \mathrm{Ind}_{S_\nu}^{H_{t,k}(\nu,r)}(\mathbb{C}))$$

by Corollary 5.2.1.13, it follows that:

$$\widetilde{\mathcal{D}}_{t,k,\nu}(r) = \prod_{\mathcal{F}}^{r} \mathrm{Hom}_{\mathbf{Rep}_{p_n}}(\overline{\mathbb{F}}_{p_n}, \mathrm{Ind}_{S_{\nu_n}}^{H_{t_n,k_n}(\nu_n,r)}(\overline{\mathbb{F}}_{p_n})) = \prod_{\mathcal{F}}^{r} B_{t_n,k_n}(\nu_n, r)\ .$$

□

Remark 5.3.1.3. Note that $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$ has a vector space bifiltration which it inherits through the ultraproduct construction from the similar filtrations on $B_{t_n,k_n}(\nu_n, r)$.

Remark 5.3.1.4. Also note that all of the above can be repeated *verbatim* for the case of $\mathrm{Rep}^{\mathrm{ext}}(S_\nu)$ from Definition 2.2.0.9. In this case we obtain the algebra $\widetilde{\mathcal{D}}^{\mathrm{ext}}_{t,k,\nu}(r)$ over $\overline{\mathbb{C}(\nu)}$.

5.3.2 Basis of DDCA

Now we would also like to generalize the elements from the Section 5.1.3 to the DDCA. And in this way construct a basis of this algebra.

Construction 5.3.2.1. Consider elements $T_{\nu_n}(\mathbf{m})$ of $B_{t_n,k_n}(\nu_n, r)$ as maps belonging to $\mathrm{Hom}_{\mathbf{Rep}_{p_n}(S_{\nu_n})}(\overline{\mathbb{F}}_{p_n}, H_{t_n,k_n}(\nu_n, r)\mathbf{e})$. Since these elements are defined for big enough characteristic, they are defined for almost all n. And since their degree as maps is bounded, it follows that the ultraproduct $T(\mathbf{m}) = \prod_{\mathcal{F}} T_{\nu_n}(\mathbf{m})$ is a well-defined element of $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$. And the same hold for $\widetilde{\mathcal{D}}^{\mathrm{ext}}_{t,k,\nu}(r)$.

Now let us consider the associated graded of $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$ with respect to the vertical filtration.

Proposition 5.3.2.2. *The associated graded algebra* $\mathrm{gr}_v(\widetilde{\mathcal{D}}_{t,k,\nu}(r))$ *is isomorphic to* $S^\nu(End(\Bbbk^r)[x,y])$ *as a bifiltered algebra.*

Proof. We have $\mathrm{gr}_v(\widetilde{\mathcal{D}}_{t,k,\nu}(r)) = \prod_{\mathcal{F}}^r \mathrm{gr}_v(B_{t_n,k_n}(\nu_n, r))$. But since we know what the associated graded of the spherical subalgebra is, it follows that:

$$\mathrm{gr}_v(\widetilde{\mathcal{D}}_{t,k,\nu}(r)) = \prod_{\mathcal{F}}^r S^{\nu_n}(\mathrm{End}(\overline{\mathbb{F}}_{p_n}^r)[x,y]) = S^\nu(\mathrm{End}(\Bbbk^r)[x,y]) \ .$$

□

By Remark 3.2.2.13 it follows that there is an isomorphism:

$$\widetilde{\Delta} : U(\mathrm{End}(\Bbbk^r)[x,y])/(1_{\mathrm{End}(\Bbbk^r)} - \nu) \simeq \mathrm{gr}_v(\widetilde{\mathcal{D}}_{t,k,\nu}(r)) \ .$$

From this we can derive the following statement about the basis of $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$.

Proposition 5.3.2.3. *The set* $\{T(\mathbf{m})\}$ *for all* $\mathbf{m}$ *such that for all* $(r,q,l) \neq (0,0,1)$ *we have* $m_{r,q,l} \in \mathbb{Z}_{\geq 0}$ *and* $m_{0,0,1} = 0$*, forms a basis of* $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$ *(and of* $\widetilde{\mathcal{D}}^{\mathrm{ext}}_{t,k,\nu}(r)$*).*

Proof. Indeed, since $\widetilde{\Delta}$ is an isomorphism, it follows that the images of the basis of $U(\mathrm{End}(\Bbbk^r)[x,y])/(1_{\mathrm{End}(\Bbbk^r)} - \nu)$ form a basis of the DDCA.

Now let us pass to the associated graded with respect to the horizontal filtration. We know that $\mathrm{gr}_h(U(\mathrm{End}(\Bbbk^r)[x,y])/(1_{\mathrm{End}(\Bbbk^r)} - \nu)) = S^\bullet(\mathrm{End}(\Bbbk^r)[x,y]/\Bbbk \cdot 1_{\mathrm{End}(\Bbbk^r)})$. Hence the basis of this vector space is given by $\prod_{r,q,l}(\alpha_l x^r y^q)^{m_{r,q,l}}$ for all $\mathbf{m}$ specified in the statement of the problem. But now under $\mathrm{gr}_h(\widetilde{\Delta})$ these elements map exactly into the images of $T(\mathbf{m})$ in the associated graded $\mathrm{gr}_h(\mathrm{gr}_v(\widetilde{\mathcal{D}}_{t,k,\nu}(r)))$.

Hence $T(\mathbf{m})$ form a basis of $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$. □

5.3.3 DDCA extended by a central element

In the previous section we have seen that $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$ has a certain basis which arises from the fact that this DDCA is a deformation of $U(\mathrm{End}(\Bbbk^r)[x,y])/(1_{\mathrm{End}(\Bbbk^r)} - \nu)$. Here we would like to extend this construction to the case of $U(\mathrm{End}(\Bbbk^r)[x,y])$. We can do this if we turn ν into a central element instead of a scalar.

In order to do this let us start with $\widetilde{\mathcal{D}}^{\mathrm{ext}}_{t,k,\nu}(r)$ which is an algebra over $\overline{\mathbb{C}(\nu)}$. If we can find a certain $\mathbb{C}[\nu]$-lattice in $\widetilde{\mathcal{D}}^{\mathrm{ext}}_{t,k,\nu}(r)$ which is closed under multiplication, this would allow us to consider this lattice as an algebra over $\mathbb{C}$, making ν a new central element. It turns out that such a lattice can be found, and it is actually given precisely by the $T(\mathbf{m})$-basis of $\widetilde{\mathcal{D}}^{\mathrm{ext}}_{t,k,\nu}(r)$. So in the following Proposition we will show that the structure constants of $\widetilde{\mathcal{D}}^{\mathrm{ext}}_{t,k,\nu}(r)$ in this basis are polynomial in ν.

Proposition 5.3.3.1. *The product of $T(\mathbf{m}_1)$ and $T(\mathbf{m}_2)$ in $\widetilde{\mathcal{D}}^{\mathrm{ext}}_{t,k,\nu}(r)$ can be expressed as a linear combination of $T(\mathbf{m})$ with coefficients in $\mathbb{C}[\nu]$ for $\mathbf{m}$ such that $m_{0,0,1} = 0$.*[2]

Proof. Since $T(\mathbf{m}_i) = \prod_{\mathcal{F}} T_{\nu_n}(\mathbf{m}_i)$ we can instead prove that for n big enough $T_n(\mathbf{m}_1) \cdot T_n(\mathbf{m_2})$ can be expressed as a linear combination of $T_n(\mathbf{m}')$ with coefficients which depend polynomially on n. Recall that $T_n(\mathbf{m})$ are the elements of $B_{t,k}(\nu, r)$.

In order to do so we will first introduce a notion of an admissible sum:

Definition 5.3.3.2. For a collection of functions $a : [\lambda] \to \{x, y\}$, $u : [\lambda] \to [k]$ and $\gamma : [k] \to [r^2]$, construct an element:

$$\sum_{i_1,\dots,i_k=1}^{n} (\alpha_{\gamma(1)})_{i_1} (\alpha_{\gamma(2)})_{i_2} \dots (\alpha_{\gamma(k)})_{i_k} a(1)_{i_{u(1)}} a(2)_{i_{u(2)}} \dots a(l)_{i_{u(l)}} \mathbf{e} \ .$$

We will call all such elements admissible sums. Call $|k|$ the width and $|\lambda|$ the weight of the admissible sum.

[2]The proof of this fact in the case of $r = 1$ is due to T. Schedler.

Note that the product of admissible sums is an admissible sum. Indeed if we have two admissible sums with the data (a_1, u_1, γ_1) and (a_2, u_2, γ_2), their product is an admissible sum with the function a given by concatenation of a_1 and a_2, i.e. $a : [\lambda_1 + \lambda_2] \to \{x, y\}$ such that $a(i) = a_1(i)$ for $i \leq \lambda_1$ and $a(i) = a_2(i - \lambda_1)$ for $i > \lambda_1$; with the function u given by concatenation of u_1 and u_2 in the sense that $u : [\lambda_1 + \lambda_2] \to [k_1 + k_2]$ maps $i \leq \lambda_1$ to $u(i) = u_1(i)$ and maps $i > \lambda_1$ to $u(i) = u_2(i - \lambda_1) + k_1$; with the function γ given by concatenation of γ_1 and γ_2, i.e. $\gamma : [k_1 + k_2] \to [r^2]$, i.e. $\gamma(i) = \gamma_1(i)$ for $i \leq k_1$ and $\gamma(i) = \gamma_2(i - k_1)$ for $i > k_1$. This follows easily from the fact that $(g)_i$ commutes with both x_j and y_j. Note that we see that the weights and widths of the admissible sums add up when we take their product.

Now also note that we have:

$$T_{r,q,n}(\alpha_l) = \frac{(r)!(q)!}{(r+q)!} \sum_{\substack{a:[r+q]\to\{x,y\}, \\ |a^{-1}(x)|=r}} \sum_{i=1}^{n} (\alpha_l)_i a(1)_i a(2)_i \dots a(r+q)_i \mathbf{e} \ .$$

I.e., we see that $T_{r,q,n}(\alpha_l)$ is equal to the linear combination of admissible sums with width 1 and weight $r + q$ with n-independent coefficients.

Since $T_n(\mathbf{m})$ is the linear combination of the products of $T_{r,q,n}(\alpha_l)$ with n-independent coefficients it follows that $T_n(\mathbf{m})$ itself is a linear combination of admissible sums with n-independent coefficients. Hence $T_n(\mathbf{m}_1)T_n(\mathbf{m}_2)$ is also such a linear combination.

Now if we prove that any admissible sum can written down as a linear combination of $T_n(\mathbf{m})$ with coefficients depending polynomially on n for $\mathbf{m}$ such that $m_{0,0,1} = 0$, we would prove our Proposition.

Let us prove this result by inducting on the sum of the weight and the width of the admissible sum.

As the base of our induction suppose we have an admissible sum of weight 0 and width 0. Then the sum is just 1, so we are done, since $T(\mathbf{m})$ with $m_{r,q,l} = 0$ for all r, q, l is equal to 1.

Now for the induction step suppose we have proven our hypothesis for all admissible sums with the sum of weight and width less than N.

Suppose we have an admissible sum S of weight λ and width k given by functions a, u, γ, such that $\lambda + k = N$. First suppose that $\mathrm{Im}(u)$ does not cover the set $\gamma^{-1}(1)$. It follows that there is $j \in [k]$ such that i_j does not appear as a subscript of x or y and only appears as a subscript of $(\alpha_{\gamma(j)})_{i_j} = (1)_{i_j}$, but $(1)_{i_j} = 1$, so we can take this sum, gaining a multiple of n and reducing our problem to the admissible sum with smaller width, for which the problem is already solved. Hence in this case we are done.[3]

So we can suppose that there are no $j \in [k]$ such that $\gamma(j) = 1$ and $j \notin \Im(u)$. Now let us define $\mathbf{m}$ in the following way. Set

$$m_{r,q,l} = \left|\{j \in [k] |\ \gamma(j) = l,\ R_j = r,\ Q_j = q\}\right| ,$$

where

$$R_j = \left|\{i \in [\lambda] |\ u(i) = j,\ a(i) = x\}\right| \text{ and } Q_j = \left|\{i \in [\lambda] |\ u(i) = j,\ a(i) = y\}\right| .$$

Notice that we have $m_{0,0,1} = 0$ by our requirement.

Now note that $T_n(\mathbf{m})$ is proportional with an n-independent coefficient to the linear combination of admissible sums which differ from S only by the permutation of $[\lambda]$ and $[k]$. If we prove that when we permute elements in the admissible sum the

[3]Note that this is precisely where the polynomial dependence on n comes from.

only extra terms we get are admissible sums with smaller sum of width and weight with n-independent coefficients, we are done. Indeed, then it would follow that for some n-independent constant $S - T_n(\mathbf{m})$ is the linear combination of admissible sums with the sum of weight and width $< N$ for which the hypothesis is known.

So let us prove this assertion. Since $(g)_i$ commutes with both x_j and y_j, x_i commute among themselves and y_i commute too, we need to consider three cases: 1)what happens when we commute $(\alpha_l)_{i_j}$ and $(\alpha_{l'})_{i_{j'}}$ in the sum; 2)what happens when we commute x_{i_j} and $y_{i_{j'}}$ in the sum; 3) what happens when we commute x_{i_j} and y_{i_j} in the sum.

In the first case we use the fact that $[(g)_{i_j}, (h)_{i_{j'}}] = \delta_{i_j, i_{j'}}([g,h])_{i_j}$. So it follows that the extra term in the sum we get is as follows:

$$\sum_{\ldots, i_j, \ldots, i_{j'}, \cdots = 1}^{n} \ldots \delta_{i_j, i_{j'}}([\alpha_l, \alpha_{l'}])_{i_j} \cdots = \sum_{\ldots, i_j, \ldots, \cap i_{j'}, \cdots = 1}^{n} \ldots ([\alpha_l, \alpha_{l'}])_{i_j} \ldots \ .$$

So in this case, since $[\alpha_l, \alpha_{l'}]$ can be written as a linear combination of α_i with n-independent coefficients, it follows that we get admissible sums with smaller width, as required.

In the second case we know that $[x_i, y_j] = \delta_{ij}(t - k\sum_{m \neq i} s_{im}\sigma_{im} - k) + (ks_{ij}\sigma_{ij})$, when we insert this into our sum somewhere, first of all the weight drops by two. Then in the first term, which is proportional to $\delta_{i_j, i_{j'}}(t - k\sum_{m \neq i_j} s_{i_j,m}\sigma_{i_j,m} - k)$, we delete the sum over $i_{j'}$ (this forces us to take the product of two α_l in the $\mathrm{End}(\Bbbk^r)$ part of the admissible sum after some commutation, but this by the above remarks doesn't cause a problem). Then we also are required to commute all S_n elements to the right to be absorbed into $\mathbf{e}$, which only changes the function u in the admissible sum, and to move all σ's to the left, where by acting they permute $(\alpha_l)_i$, changing the function γ. The second term is proportional to $(ks_{i_j, i_{j'}}\sigma_{i_j, i_{j'}})$. And here again we just commute S_n elements to the right and σ's to the left.

Now the final case is when we commute x_{i_j} with y_{i_j}. Since it holds that $[x_i, y_i] = t - k\sum_{m\neq i} s_{im}\sigma_{im}$, we again see that the weight drops by 2 and all of the preceding remarks apply to make all extra terms into the linear combinations of admissible sums with lower width plus weight with n-independent coefficients.

Thus we have proven the induction step and the proposition follows. □

From this proposition it follows that the $\mathbb{C}[\nu]$-lattice given by the direct sum $\bigoplus_{\mathbf{m}, m_{0,0,1}\neq 0} \mathbb{C}[\nu]\cdot T(\mu)$ forms a subalgebra in $\widetilde{\mathcal{D}}^{\text{ext}}_{t,k,\nu}(r)$. So we can define:

Definition 5.3.3.3. Define the DDC algebra $\mathcal{D}_{t,k}(r)$ over $\mathbb{C}$ to be equal to the $\mathbb{C}[\nu]$-lattice $\bigoplus_{\mathbf{m}, m_{0,0,1}\neq 0} \mathbb{C}[\nu]\cdot T(\mu) \subset \widetilde{\mathcal{D}}^{\text{ext}}_{t,k,\nu}(r)$.

Now in this algebra ν becomes a central element which we will call K. Note that before we had $\prod_{\mathcal{F}} T_{0,0,\nu_n}(1) = \prod_{\mathcal{F}} \nu_n = \nu$. Now in this algebra it becomes K – an independent element, so it makes sense to also denote $T_{0,0}(1) = K \in \mathcal{D}_{t,k}(r)$.

We can also see that this extends the isomorphism

$$\widetilde{\Delta} : U(\mathrm{End}(\Bbbk^r)[x,y])/(1_{\mathrm{End}(\Bbbk^r)} - \nu) \simeq \mathrm{gr}_v(\widetilde{\mathcal{D}}_{t,k,\nu}(r))$$

to the isomorphism:

$$\widetilde{\Delta} : U(\mathrm{End}(\Bbbk^r)[x,y]) \simeq \mathrm{gr}_v(\mathcal{D}_{t,k}(r)) \ ,$$

which fully explains the name "deformed double current algebra".

Thus we can conclude that:

Corollary 5.3.3.4. *The set $\{T(\mathbf{m})\}$ for all $\mathbf{m}$ forms a basis of $\mathcal{D}_{t,k}(r)$.*

Notice that we also have the following important Corollary which connects the DDC algebra $\mathcal{D}_{t,k}(r)$ with $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$.

Corollary 5.3.3.5. *The DDC algebra $\widetilde{\mathcal{D}}_{t,k,\nu}(r)$ is isomorphic to $\mathcal{D}_{t,k}(r)/(K-\nu)$.*

5.3.4 DDCA of rank 1

In this section we will say a few words about how the above results translate to the case $r=1$. This will be helpful for the discussions in Section 5.5 and 5.6.

First let us introduce a simplified notation for these DDCAs.

Definition 5.3.4.1. For $t,k,\nu \in \mathbb{C}$ denote $\widetilde{\mathcal{D}}_{t,k,\nu} = \widetilde{\mathcal{D}}_{t,k,\nu}(1)$, $\widetilde{\mathcal{D}}^{\mathrm{ext}}_{t,k,\nu} = \widetilde{\mathcal{D}}^{\mathrm{ext}}_{t,k,\nu}(1)$ and $\mathcal{D}_{t,k} = \mathcal{D}_{t,k}(1)$.

We will denote the elements $T(\mathbf{m})$ constructed for $r=1$ in the same way, since it will be obvious to which algebra these elements belong from the context. However note that since $r=1$ the only value l (in $m_{r,q,l}$) can take is 1.

Similarly if we consider the corresponding Cherednik algebra $H_{t,k}(n)$, we don't have to speak about bifiltration anymore, since the horizontal filtration turns out to be trivial. Hence, instead of using the bifiltration from Definition 5.1.3.6 we can define:

Definition 5.3.4.2. Define the filtration on $H_{t,k}(n)$ by assigning the following weight to the generators $\deg(x_i) = \deg(y_i) = 1$ and $\deg(s_i) = 0$.

Remark 5.3.4.3. This filtration restricts to the filtration of $H_{t,k}(n)\mathbf{e}$ by S_n-modules. Under taking the ultraproducts it corresponds to the filtration of $H_{t,k}(\nu)\mathbf{e}$ by objects of $\mathrm{Rep}(S_\nu)$ given by:

$$F^m H_{t,k}(\nu)\mathbf{e} = \sum_{l=0}^{m}\sum_{i=0}^{l} S^i(\mathfrak{h}^*)\otimes S^{l-i}(\mathfrak{h})\otimes \mathbb{C}\ ,$$

in the language of Construction 5.2.1.8 for $r = 1$ (hence $A = \mathbb{C}$).

Now Corollary 5.3.3.4 can also be strengthened a bit in this case. We have:

Proposition 5.3.4.4. *For $L \leq n$ and char$(\Bbbk) = 0$ or large compared to n, the vector space $F^L B_{t,k}(n)/F^{L-1}B_{t,k}(n)$ has a basis $\{T_n(\mathbf{m})|\ w(\mathbf{m}) = L\}$.*

Proof. This in fact follows directly from theory of invariants, since this quotient space is isomorphic to $\Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n]_L^{S_n}$, and $T_n(\mathbf{m})$, as elements of the quotient, transform into the products of symmetric polynomials $P_{r,q,n} = \sum_{i=1}^n x_i^r y_i^q$. See [18] for details. □

5.4 Isomorphism with Guay's construction.

The DDC algebras were constructed by Guay and co-authors first for type A in [25] and then for any simple Lie algebra in [28]. In this section we will explain how our algebra is connected with the one constructed by Guay. Note that in this section we always have $r \geq 4$, since Guay's DDC algebras are not defined for smaller rank (the paper by Guay and Yang on the construction of DDCA for $r = 2$ and 3 is however in preparation).

5.4.1 Guay's DDCA of type A.

First, let us recall one of the main definitions of Guay's DDCA.

Definition 5.4.1.1. The Guay's DDC algebra $\mathbb{D}_{\lambda,\beta}(r)$ for $\lambda, \beta \in \mathbb{C}$ is an algebra generated by elements $z, K(z), Q(z), P(z)$, where $z \in \mathfrak{sl}_r$, which satisfy the following relations. The subalgebra generated by z and $K(z)$ is isomorphic to $U(\mathfrak{sl}_r[u])$, i.e. there is a map $U(\mathfrak{sl}_r[u]) \to \mathbb{D}_{\lambda,\beta}(r)$. Similarly the subalgebra generated by z and

$Q(z)$ is isomorphic to $U(\mathfrak{sl}_r[v])$. Also, $P(z)$ is linear in z and $[y, P(z)] = P([y, z])$. And if we consider $1 \leq a, b, c, d \leq r$ such that $(a, b) \neq (d, c)$ and $a \neq b, c \neq d$ we have:

$$[K(E_{ab}), Q(E_{cd})] = P([E_{ab}, E_{cd}]) + (\beta - \tfrac{\lambda}{2})(\delta_{bc}E_{ad} + \delta_{ad}E_{bc}) + \frac{\lambda}{4}(\delta_{ad} + \delta_{cb})S(E_{ab}, E_{cd}) +$$

$$+ \frac{\lambda}{4} \sum_{1 \leq i \neq j < n} S([E_{ab}, E_{ij}], [E_{ji}, E_{cd}]) \ ,$$

where $S(z, y) = zy + yz$.

Remark 5.4.1.2. Note that if $[E_{ab}, E_{cd}] = 0$ (i.e. $b \neq c$ and $a \neq d$) the last relation simplifies to:

$$[K(E_{ab}), Q(E_{cd})] = -\lambda E_{ad}E_{cb} \ ,$$

since only the last term for $(i, j) = (b, d)$ or $(i, j) = (c, a)$ survives.

5.4.2 Construction of the homomorphism $\mathbb{D}_{\lambda,\beta}(r) \to \mathcal{D}_{t,k}(r)$

In this section we will construct a map from Guay's DDCA to our DDCA, i.e. we will construct the elements in $\mathcal{D}_{t,k}(r)$ that satisfy the relations of Definition 5.4.1.1.

First we need to establish a convenient way to perform calculations in $\mathcal{D}_{t,k}(r)$. Since this algebra is defined as an ultraproduct of a family of other algebras, we will use that for our calculations.

Definition 5.4.2.1. Suppose $X \in \widetilde{\mathcal{D}}_{t,k,\nu}(r)$ is an element of DDCA. We have $X = \prod_{\mathcal{F}} X_n$, where $X_n \in B_{t_n,k_n}(\nu_n, r)$. We will denote this correspondence by $X \sim X_n$.

A similar correspondence exists for $\mathcal{D}_{t,k}(r)$. The only difference is that here $\nu_n \sim K$ instead of $\nu_n \sim \nu$.

Note also that all the elements of $B_{t_n,k_n}(\nu_n, r)$ actually have a multiple of $\mathbf{e}$ on the right. We will omit this for brevity.

Now we can construct a map between the DDC algebras.

Proposition 5.4.2.2. *There is a map* $\psi : \mathbb{D}_{k,-\frac{t}{2}-\frac{k(r-2)}{4}}(r) \to \mathcal{D}_{t,k}(r)$ *given by:*

$$\psi(z) = T_{0,0}(z) \ , \ \psi(K(z)) = T_{1,0}(z) \ , \ \psi(Q(z)) = T_{0,1}(z) \ , \ \psi(P(z)) = T_{1,1}(z) \ ,$$

where $z \in \mathfrak{sl}_r$.

Proof. The above expressions can be rewritten as

$$\psi(z) \sim \sum_{i=1}^{\nu_n}(z)_i \ , \ \psi(K(z)) \sim \sum_{i=1}^{\nu_n}(z)_i \cdot x_i \ , \ \psi(Q(z)) \sim \sum_{i=1}^{\nu_n}(z)_i \cdot y_i \ \text{ and}$$

$$\psi(P(z)) \sim \sum_{i=1}^{\nu_n}(z)_i \cdot \frac{x_i y_i + y_i x_i}{2} \ .$$

We only need to check that the images of $z, K(z), Q(z)$ and $P(z)$ satisfy the required relations.

We will start with relations between z and $K(z)$. Note that their image in $\mathcal{D}_{t,k}(r)$ is contained in the ultraproduct of subalgebras of $B_{t_n,k_n}(\nu_n, r)$ generated by x_i and g. Since x_i commute with each other, these subalgebras are equal to

$$(\mathrm{End}(\Bbbk^r)^{\otimes \nu_n} \otimes \Bbbk[x_1, \ldots, x_{\nu_n}])^{S_{\nu_n}} = ((\mathrm{End}(\Bbbk^r)[x])^{\otimes \nu_n})^{S_{\nu_n}} = S^{\nu_n}(\mathrm{End}(\Bbbk^r)[x]) \ .$$

But by Proposition 3.2.2.12 we know that

$$\prod_{\mathcal{F}} S^{\nu_n}(\mathrm{End}(\Bbbk^r)[x]) \simeq U(\mathrm{End}(\Bbbk^r)[x])/(1_{\mathrm{End}(\Bbbk^r)[x]} - \nu) \ .$$

And the construction of this isomorphism also shows that under it $\psi(z) \mapsto z$ and $\psi(K(z)) \mapsto z \cdot x$. Also note that:

$$[\psi(K(z_1)), \psi(K(z_2))] \sim \sum_{i,j} [(z_1)_i \cdot x_i, (z_2)_j \cdot x_j] \mapsto \sum_i ([z_1, z_2])_i (x_i)^2 \sim [z_1, z_2] \cdot x^2$$

under the above isomorphism. Thus these elements generate

$$U(\mathfrak{sl}_r[x]) \subset U(\mathrm{End}(\mathbb{k}^r)[x])/(1_{\mathrm{End}(\mathbb{k}^r)[x]} - \nu).$$

The same holds for z and $Q(z)$.

Now, since $\psi(P(z)) \sim \sum_{i=1}^{\nu_n} (z)_i \cdot \frac{x_i y_i + y_i x_i}{2}$ it follows that it is linear in z and

$$[\psi(y), \psi(P(z))] = \psi(P[y, z]) \ ,$$

since $[(y)_j, (z)_i] = \delta_{ij}([y, z])_i$.

We need to check the last relation of Definition 5.4.1.1.

So let us first calculate $[\psi(K(E_{ab})), \psi(Q(E_{cd}))]$. We have:

$$[\psi(K(E_{ab})), \psi(Q(E_{cd}))] \sim$$

$$\sim [\sum_{i=1}^{\nu_n} (E_{ab})_i \cdot x_i, \sum_{j=1}^{\nu_n} (E_{cd})_j \cdot y_j] = \sum_{i,j=1}^{\nu_n} [(E_{ab})_i, (E_{cd})_j] \cdot x_i y_j + \sum_{i,j=1}^{\nu_n} (E_{cd})_j (E_{ab})_i \cdot [x_i, y_j] =$$

$$= \sum_{i=1}^{\nu_n} ([E_{ab}, E_{cd}])_i \cdot \left(\frac{x_i y_i + y_i x_i + [x_i, y_i]}{2} \right) +$$

$$+ \sum_{i,j=1}^{\nu_n} \left(\frac{(E_{cd})_j (E_{ab})_i + (E_{ab})_i (E_{cd})_j - [(E_{ab})_i, (E_{cd})_j]}{2} \right) \cdot [x_i, y_j] =$$

$$= \psi(P([E_{ab}, E_{cd}])) + \sum_{i,j=1}^{\nu_n} \frac{(E_{cd})_j (E_{ab})_i + (E_{ab})_i (E_{cd})_j}{2} \cdot [x_i, y_j] \ .$$

Now we need to work with the last term. We will expand it using the commutator relation in the extended Cherednik algebra and we note that elements of S_{ν_n} disappear into the assumed $\mathbf{e}$ term in the formula:

$$\sum_{i,j=1}^{\nu_n} \frac{(E_{cd})_j (E_{ab})_i + (E_{ab})_i (E_{cd})_j}{2} \cdot [x_i, y_j] =$$

$$= -\sum_{i \neq j} \frac{(E_{cd})_j (E_{ab})_i - (E_{ab})_i (E_{cd})_j}{2} k_n \sigma_{ij} + \sum_i \frac{(E_{cd})_i (E_{ab})_i + (E_{ab})_i (E_{cd})_i}{2} (t_n - k_n \sum_{m \neq i} \sigma_{im}) \ .$$

Notice that $(E_{\alpha\beta})_i (E_{\gamma\delta})_j \sigma_{ij} = (E_{\alpha\delta})_i (E_{\gamma\beta})_j$, so the above expression becomes:

$$-k_n \sum_{i \neq j} (E_{ad})_i (E_{cb})_j - \frac{t_n}{2} \sum_i (\delta_{ad} (E_{cb})_i + \delta_{bc} (E_{ad})_i) + \frac{k_n}{2} \sum_{i \neq m} (\delta_{ad} (E_{cb})_i \sigma_{im} + \delta_{bc} (E_{ad})_i \sigma_{im}) \ .$$

Now we need to calculate what $\sum_{i \neq m} (E_{\alpha\beta})_i \sigma_{im}$ is equal to. We have:

$$\sum_{i \neq m} (E_{\alpha\beta})_i \sigma_{im} = \sum_{i \neq m} \sum_{\gamma,\delta} (E_{\alpha\beta}) i (E_{\gamma\delta})_i (E_{\delta\gamma})_m = \sum_{i \neq m} \sum_{\delta} (E_{\alpha\delta})_i (E_{\delta\beta})_m \ .$$

So the answer is:

$$[\psi(K(E_{ab})), \psi(Q(E_{cd}))] \sim$$

$$\sim \psi(P([E_{ab}, E_{cd}])) - k_n \sum_{i,j=1, i \neq j}^{\nu_n} (E_{ad})_i (E_{cb})_j - \frac{t_n}{2} \sum_i^{\nu_n} (\delta_{ad} (E_{cb})_i + \delta_{bc} (E_{ad})_i) -$$

$$+ \frac{k_n}{2} \sum_{m,i=1, m \neq i}^{\nu_n} \sum_{e=1}^{r} (\delta_{ad} (E_{ce})_i (E_{eb})_m + \delta_{bc} (E_{ae})_i (E_{ed})_m) \ .$$

Now we need to calculate the image of the r.h.s. of the same relation. The first term is clear. The second term contains elements like $\psi(E_{\alpha\beta}) \sim \sum_i (E_{\alpha\beta})_i$. The third term is more complex. We have:

$$\psi(S(E_{ab},E_{cd}))\sim\sum_{i,j}((E_{ab})_i(E_{cd})_j+(E_{cd})_j(E_{ab})_i)=$$

$$=2\sum_{i\neq j}(E_{ab})_i(E_{cd})_j+\sum_i(\delta_{bc}(E_{ad})_i+\delta_{ad}(E_{cb})_i)\ .$$

Now we want to transform the last term:

$$\sum_{\alpha\neq\beta}S([E_{ab},E_{\alpha\beta}],[E_{\beta\alpha},E_{cd}])\ .$$

Before we calculate its image we can rewrite it as follows:

$$-2S(E_{ad},E_{cb})+\delta_{ad}\sum_{\alpha\neq a}S(E_{\alpha b},E_{c\alpha})+\delta_{bc}\sum_{\alpha\neq b}S(E_{a\alpha},E_{\alpha d})\ .$$

Now, since $c\neq d$ and $a\neq b$, it follows that:

$$\psi(S(E_{ad},E_{cb}))\sim 2\sum_{i\neq j}(E_{ad})_i(E_{cb})_j\ .$$

And since $\delta_{ad}\delta_{bc}=0$ in our situation, it follows:

$$\psi(\delta_{ad}\sum_{\alpha\neq a}S(E_{\alpha b},E_{c\alpha}))\sim 2\delta_{ad}\sum_{i\neq j}\sum_{\alpha\neq a}(E_{\alpha b})_i(E_{c\alpha})_j+(r-1)\delta_{ad}\sum_i(E_{cb})_i\ ,$$

and similarly:

$$\psi(\delta_{bc}\sum_{\alpha\neq b}S(E_{a\alpha},E_{\alpha d}))\sim 2\delta_{bc}\sum_{i\neq j}\sum_{\alpha\neq b}(E_{a\alpha})_i(E_{\alpha d})_j+(r-1)\delta_{bc}\sum_i(E_{ad})_i\ .$$

Now we can assemble all the formulas to obtain that the r.h.s. of the relation equals to:

$$\psi(P([E_{ab},E_{cd}]))+$$

$$+\left[\beta-\frac{\lambda}{2}\right]\left(\delta_{ad}\sum_i(E_{ad})_i+\delta_{ad}\sum_i(E_{cb})_i\right)+\frac{\lambda}{2}(\delta_{ad}+\delta_{bc})\sum_{i\neq j}(E_{ab})_i(E_{cd})_j+$$

$$+\frac{\lambda}{4}\left(\sum_i\delta_{bc}(E_{ad})_i+\sum_i\delta_{ad}(E_{cb})_i\right)-\lambda\sum_{i\neq j}(E_{ad})_i(E_{cb})_j+$$

$$+\frac{\lambda}{2}\left(\delta_{ad}\sum_{i\neq j}\sum_{\alpha\neq a}(E_{\alpha b})_i(E_{c\alpha})_j+\delta_{bc}\sum_{i\neq j}\sum_{\alpha\neq b}(E_{a\alpha})_i(E_{\alpha d})_j\right)+$$

$$+\frac{\lambda(r-1)}{4}\left(\delta_{ad}\sum_i(E_{cb})_i+\delta_{bc}\sum_i(E_{ad})_i\right)=$$

$$=\psi(P([E_{ab},E_{cd}]))+\left[\beta-\frac{\lambda}{2}+\frac{\lambda}{4}+\frac{\lambda(r-1)}{4}\right]\left(\delta_{ad}\sum_i(E_{ad})_i+\delta_{ad}\sum_i(E_{cb})_i\right)-$$

$$-\lambda\sum_{i\neq j}(E_{ad})_i(E_{cb})_j+\frac{\lambda}{2}\left(\delta_{ad}\sum_{i\neq j}\sum_{\alpha}(E_{\alpha b})_i(E_{c\alpha})_j+\delta_{bc}\sum_{i\neq j}\sum_{\alpha}(E_{a\alpha})_i(E_{\alpha d})_j\right).$$

We can see that these two formulas are the same if $\lambda=k$ and $\beta=-\frac{t}{2}-\frac{k}{4}(r-2)$. □

From now on fix $\lambda=k$ and $\beta=-\frac{t}{2}-\frac{k}{4}(r-2)$.

5.4.3 Surjectivity of ψ

We have successfully constructed a morphism between Guay's DDCA and our DDCA. Now we would like to prove that this is in fact an isomorphism. In this section we would like to start by proving that ψ is a surjective map.

Proposition 5.4.3.1. *For $t+rk\neq 0$, the map ψ defined in Proposition 5.4.2.2 is surjective.*

Proof. Since $T(\mathbf{m})$ form a basis of $\mathcal{D}_{t,k}(r)$ and they themselves are given by the linear combinations of the products of $T_{r,q}(z)$ for all $z \in \mathrm{End}(\Bbbk^r)$ it follows that it is enough to prove that $T_{r,q}(z)$ lie in the image of ψ. More precisely to prove that ψ surjects onto $F_v^N \mathcal{D}_{t,k}(r)$ it is enough to prove that all $T_{r,q}(z)$ for $r+q \leq N$ are in the image of ψ.

We would like to prove the last statement by inducting on N. But our induction will be slightly more involved than one could hope for.

Nevertheless we would like to start with proving the base case. Namely that all $T_{0,0}(z)$ are in the image of ψ. Indeed we know that for all $z \in \mathfrak{sl}_r$ $\psi(z) = T_{0,0}(z)$, so we only need to show that $K = T_{0,0}(\mathrm{Id})$ is in the image. Denote $H = E_{11} - E_{22}$ and consider $[\psi(K(H)), \psi(Q(H))]$:

$$[\psi(K(H)), \psi(Q(H))] \sim \sum_{i,j} (H)_i (H)_j [x_i, y_j] \ .$$

Now we will calculate this modulo the image of ψ (we will denote this by $\underset{\psi}{\sim}$). So after we apply the same operations to the last term as in Proposition 5.4.2.2 and then note that $H^2 = E_{11} + E_{22}$, we have:

$$[\psi(K(H)), \psi(Q(H))] \sim -k_n \sum_{i \neq j} (H)_i (H)_j \sigma_{ij} - t_n \sum_i (E_{11}+E_{22})_i + k_n \sum_{i \neq j} (E_{11}+E_{22})_i \sigma_{ij} \ .$$

This time we will calculate the last terms by inserting the identity $1 = (E_{11} + \dots E_{rr})_k$. We get:

$$\sum_{i \neq j} (E_{11} + E_{22})_i \sigma_{ij} =$$

$$= \sum_{i \neq j} (E_{11})_i (E_{11})_j + \sum_{i \neq j} (E_{22})_i (E_{22})_j + \sum_{\alpha \neq 1} \sum_{i \neq j} (E_{1\alpha})_i (E_{\alpha 1})_j + \sum_{\alpha \neq 2} \sum_{i \neq j} (E_{2\alpha})_i (E_{\alpha 2})_j \ .$$

Putting this into original formula we get:

$$[\psi(K(H)),\psi(Q(H))]=$$

$$=-k_n\sum_{i\neq j}((E_{11})_i(E_{11})_j+(E_{22})_i(E_{22})_j-(E_{12})_i(E_{21})_j-(E_{21})_i(E_{12})_j)-t_n\sum_i(E_{11}+E_{22})_i-$$

$$+k_n\sum_{i\neq j}((E_{11})_i(E_{11})_j+(E_{22})_i(E_{22})_j)+k_n\sum_{\alpha\neq 1}\sum_{i\neq j}(E_{1\alpha})_i(E_{\alpha 1})_j+k_n\sum_{\alpha\neq 2}\sum_{i\neq j}(E_{2\alpha})_i(E_{\alpha 2})_j\ .$$

Now notice that $\psi(z_1)\psi(z_2)=\sum_{i\neq j}(z_1)_i(z_2)_j+\sum_i(z_1\cdot z_2)_i$, so it follows that:

$$[\psi(K(H)),\psi(Q(H))]\underset{\psi}{\sim}$$

$$\underset{\psi}{\sim}-(t_n+rk_n)\sum_i(E_{11}+E_{22})_i=-(t_n+rk_n)\sum_i(2+z)_i\underset{\psi}{\sim}-2(t+rk)K\ ,$$

for some $z\in\mathfrak{sl}_r$. Hence we know that K is in the image of ψ.

Now we will prove the surjectivity in general by induction. For each m we will be proving that all $T_{r,q}(z)$ with $r+q\leq m+1$ and $z\in\mathfrak{sl}_r$ are in the image of ψ and that all $T_{r,q}(1)$ with $r+q\leq m$ are in the image of ψ (so that ψ surjects onto $F_v^m\mathcal{D}_{t,k}(r)$). From this statement it will follow that ψ is surjective.

Now the base for $m=0$ holds since we have just proved that $T_{0,0}(1)=K$ is in the image of ψ and also we know that $T_{0,1}(z)$ and $T_{1,0}(z)$ are in the image.

So we are ready to prove the induction step. Suppose the statement holds for m and we want to prove it for $m+1$. We need to prove that all $T_{r,q}(z)$ with $z\in\mathfrak{sl}_r$ and $r+q=m+2$ lie in the image of ψ and also that all $T_{r,q}(1)$ for $r+q=m+1$ lie there.

We will start with the first statement. Note that we already know that $T_{m+2,0}(z)$ and $T_{0,m+2}(z)$ are in the image, since $\psi(z)$ and $\psi(K(z))$ generate $U(\mathfrak{sl}_r[x])\subset\mathcal{D}_{t,k}(r)$

and the analogous statement holds for $\psi(z)$ and $\psi(Q(z))$. It is enough to prove that, for example $T_{m+2-k,k}(E_{13})$ is in the image for each k from 1 to $m+1$, since then by taking commutators with $T_{0,0}(z)$ we can obtain any other $T_{m+2-k,k}(z')$. Let's calculate the commutator of $T_{m+2-k,k-1}(E_{12})$ and $T_{0,1}(H)$, both of which are in the image. To do that, we will denote by $f_{r,q}(i)$ the polynomial in x_i and y_i which appears in $T_{r,q}(E_{12}) \sim \sum_i (E_{12})_i f_{r,q}(i)$. We have:

$$[T_{m+2-k,k-1}(E_{12}), T_{0,1}(H)] \sim$$

$$\sim -2\sum_i (E_{12})_i \frac{f_{m+2-k,k-1}(i)y_i + y_i f_{m+2-k,k-1}(i)}{2} + \sum_{i,j} \frac{(E_{12})_i (H)_j + (H)_j (E_{12})_i}{2} [f_{m+2-k,k-1}(i), y_j] \, .$$

Now since we know that ψ surjects onto $F_v^m \mathcal{D}_{t,k}(r)$ we would like to calculate the above commutator modulo degree m. The last term is zero modulo degree m since it contains at least one commutator of x and y which decreases the degree by 2. Now also modulo degree m the monomials in the first term commute. So, we have:

$$[T_{m+2-k,k-1}(E_{12}), T_{0,1}(E_{23})] \underset{\psi}{\sim} -2\sum_i (E_{12})_i x_i^{m+2-k} y_i^k \underset{\psi}{\sim} -2T_{m+2-k,k}(E_{13}) \, .$$

Now we only need to prove that $T_{r,q}(1)$ for $r+q=m+1$ are in the image. To do that let us calculate the commutator of $T_{r,q+1}(H)$ and $T_{1,0}(H)$. We have:

$$[T_{1,0}(H), T_{r,q+1}(H)] \underset{\psi}{\sim} \sum_{i,j} (H)_i (H)_j [x_i, f_{r,q+1}(j)] \, .$$

We need to calculate this term modulo degree m. Hence we can commute the terms in $f_{r,q+1}(j)$ under the commutator. I.e. we have:

$$[T_{1,0}(H), T_{r,q+1}(H)] \underset{\psi}{\sim} \sum_{i,j} (H)_i (H)_j [x_i, x_j^r y_j^{q+1}] \underset{\psi}{\sim} \sum_{i,j} (H)_i (H)_j x_j^r [x_i, y_j^{q+1}] \underset{\psi}{\sim}$$

$$\underset{\psi}{\sim} \sum_{i,j}\sum_{l=0}^{q}(H)_i(H)_j x_j^r y_j^l [x_i, y_j] y_j^{q-l} \underset{\psi}{\sim}$$

$$\underset{\psi}{\sim} -k_n \sum_{i\neq j}\sum_{l=0}^{q}(H)_i(H)_j \sigma_{ij} x_j^r y_j^l y_i^{q-l} - t_n(q+1)\sum_i (H^2)_i x_i^r y_i^q + k_n \sum_{i\neq j}\sum_{l=0}^{q}(H^2)_j \sigma_{ij} x_j^r y_j^l y_i^{q-l} \underset{\psi}{\sim}$$

$$\underset{\psi}{\sim} -k_n \sum_{l=0}^{q}\sum_{i\neq j}\left[(E_{11})_i(E_{11})_j + (E_{22})_i(E_{22})_j - (E_{12})_j(E_{21})_i - (E_{21})_j(E_{12})_i\right] x_j^r y_j^l y_i^{q-l} -$$

$$-t_n(q+1)\sum_i (H^2)_i x_i^r y_i^q +$$

$$+k_n \sum_{l=0}^{q}\sum_{i\neq j}\left((E_{11})_i(E_{11})_j + (E_{22})_i(E_{22})_j + \sum_{\alpha\neq 1}(E_{1\alpha})_j(E_{\alpha 1})_i + \sum_{\alpha\neq 2}(E_{2\alpha})_j(E_{\alpha 2})_i\right) x_j^r y_j^l y_i^{q-l} \underset{\psi}{\sim}$$

$$\underset{\psi}{\sim} -t_n(q+1)\sum_i (H^2)_i x_i^r y_i^q +$$

$$+k_n \sum_{l=0}^{q}\sum_{i\neq j}\left((E_{12})_j(E_{21})_i + (E_{21})_j(E_{12})_j + \sum_{\alpha\neq 1}(E_{1\alpha})_j(E_{\alpha 1})_i + \sum_{\alpha\neq 2}(E_{2\alpha})_j(E_{\alpha 2})_i\right) x_j^r y_j^l y_i^{q-l} .$$

Now note the following formula:

$$\sum_{i\neq j}(z_1)_j(z_1)_i x_j^{r_1} y_j^{q_1} x_i^{r_2} y_i^{q_2} \sim$$

$$\sim T_{r_1,q_1}(z_1)T_{r_2,q_2}(z_2) - \sum_i (z_1 \cdot z_2) x_i^{r_1+r_2} y_i^{q_1+q_2} \text{ modulo } F_v^{r_1+q_1+r_2+q_2-1}\mathcal{D}_{t,k}(r) .$$

In our case $r_1+q_1+r_2+q_2 = r+q = m+1$. Since we know that $F_v^m\mathcal{D}_{t,k}(r) \subset \mathrm{Im}(\psi)$, it follows that we can use this formula. Also notice that since everywhere there we can use the above formula $z_1, z_2 \in \mathfrak{sl}_r$ and $r_i+q_i < m+1$, it follows that $T_{r_i,q_i}(z_i) \in \mathrm{Im}(\psi)$. Thus it follows that:

$$[T_{1,0}(H), T_{r,q+1}(H)] \underset{\psi}{\sim}$$

$$\underset{\psi}{\sim} -t_n(q+1)\sum_i (H^2)_i x_i^r y_i^q - k_n(q+1)\sum_i \left[(E_{11})_i + (E_{22})_i + (r-1)(E_{11})_i + (r-1)(E_{22})_i\right] x_i^r y_i^q \underset{\psi}{\sim} .$$

$$\underset{\psi}{\sim} -(q+1)(t_n+rk_n)\sum_i (E_{11}+E_{22})_i x_i^r y_i^q \underset{\psi}{\sim} -2(q+1)(t_n+rk_n)\sum_i x_i^r y_i^q \underset{\psi}{\sim} -2(q+1)(t_n+rk_n)T_{r,q}(1) .$$

So we have proven the inductive step and hence it follows that ψ is surjective. □

5.4.4 Injectivity of ψ

In this subsection we are going to show that, if $t + rk \neq 0$, ψ is injective and, hence, it is an isomorphism. In order to do that we will show that $\mathbb{D}_{\lambda,\beta}(r)$ has a faithful representation $\mathbb{D}_{\lambda,\beta}(r) \to \mathrm{End}(M)$, such that M is also a $\mathcal{D}_{t,k}(r)$-module and the action map for $\mathbb{D}_{\lambda,\beta}(r)$ factors through ψ.

Here, we will extensively use the results of [25]. First of all we need to define an alternative presentation of Guay's DDCA – $D_{\lambda,\beta}(r)$, which will be isomorphic to $\mathbb{D}_{\lambda,\beta}(r)$. This presentation is quite involved and its exact form isn't important for us, so we will state an abbreviated version of it.

Definition 5.4.4.1 (Definition 8.1 in [25])**.** The algebra $D_{\lambda,\beta}(r)$ is generated by elements $X_{i,0}^{\pm}, X_{i,1}^{\pm}, H_{i,0}, H_{i,1}$ for $i \in \{1, \ldots, r-1\}$ and $X_{0,0}^{+}, X_{0,1}^{+,\pm}$, which satisfy a number of relations.

Also there are two specific elements in this algebra, denoted by $\omega_0^{+,\pm}$ (see Section 9 of [25]).

Another result which is important to us is the explicit structure of the isomorphism between $D_{\lambda,\beta}(r)$ and $\mathbb{D}_{\lambda,\beta}(r)$.

Theorem 5.4.4.2 (Theorem 15.1 in [25])**.** *Define a map* $\zeta : D_{\lambda,\beta}(r) \to \mathbb{D}_{\lambda,\beta}(r)$ *to be* *equal to:*

$$\zeta(X^{\pm}_{i,0}) = E^{\pm}_i,\ \zeta(H_{i,0}) = H_i,\ \zeta(X^{\pm}_{i,1}) = Q(E^{\pm}_i),\ \zeta(H_{i,1}) = Q(H_i),$$

$$\zeta(X^{+}_{0,0}) = K(E_{-\theta}),\ X^{+,\pm}_{0,1} = P(E_{-\theta}) - \lambda\omega^{+,\pm}_0\ ,$$

where $E^+_i = E_{i,i+1},\ E^-_i = E_{i+1,i},\ E_\theta = E_{1,r},\ E_{-\theta} = E_{r,1}$ *and* $H_i = E_{i,i} - E_{i+1,i+1}$. *This map is an isomorphism.*

Another set of results that Guay proved in [25] are concerned with constructing a family of $D_{\lambda,\beta}(r)$-modules.

Proposition 5.4.4.3 (Section 9 of [25])**.** *For any* $l \in \mathbb{Z}_{\geq 0}$ *the vector space* $\mathbf{V}_l = H_{-t,-k}(l,1) \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^r)^{\otimes l}$ *has a structure of* $D_{\lambda,\beta}(r)$*-module given by the following formulas. For* $m \in \mathbf{v} \in H_{-t,-k}(l,1) \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^r)^l$ *we have*

$$X^{\pm}_{i,r}(m\otimes\mathbf{v}) = \sum_{j=1}^{l} m y^r_j \otimes (E^{\pm}_i)_j\mathbf{v},\ H_{i,r}(m\otimes\mathbf{v}) = \sum_{j=1}^{l} m y^r_j \otimes (H_i)_j\mathbf{v},$$

$$X^{+}_{0,0}(m\otimes\mathbf{v}) = \sum_{j=1}^{l} m x_j \otimes (E_{-\theta})_j\mathbf{v},$$

$$X^{+,\pm}_{0,1}(m\otimes\mathbf{v}) = \sum_{j=1}^{l} m\frac{x_j y_j + y_j x_j}{2} \otimes (E_{-\theta}) f_j\mathbf{v} - \lambda\omega^{+,\pm}_0(m\otimes\mathbf{v})\ .$$

Remark 5.4.4.4. Note that in our case the parameters of the Cherednik algebra has to be $-t, -k$ as opposed to Guay's t, k. This discrepancy arises from us using a different sign in one of the commutators which define the Cherednik algebra, and also because of the different signs in the formulas which connect t, k and λ, β in our expositions. We will also see that these signs arise naturally because in the definition above we are using a right action on the Cherednik algebra side of the tensor product.

Guay also proved a PBW property for his DDCA in [25]. As a by-product of his proof he arrived at the following result.

Proposition 5.4.4.5. *For $\beta \neq \frac{r\lambda}{4} + \frac{\lambda}{2}$ (equivalently $t + rk \neq 0$) and for any element $x \in D_{\lambda,\beta}(r)$, there exists $l \in \mathbb{Z}_{>0}$ such that the map $\rho_l : D_{\lambda,\beta}(r) \to End(\mathbf{V}_l)$ specified above sends x to a non-zero operator, i.e. $\rho_l(x) \neq 0$.*

In other words it follows that $\bigoplus_{l>0} \mathbf{V}_l$ gives us a faithful representation of $D_{\lambda,\beta}(r)$.

Now to prove that ψ is injective we will construct a $\mathcal{D}_{t,k}(r)$-module structure on $\mathbf{V}_l$. In order to do that we first want to show that for any l there is a surjective map from $\mathcal{D}_{t,k}(r)$ to $B_{t,k}(l,r)$.

Proposition 5.4.4.6. *There is a surjective map $\pi_l : \mathcal{D}_{t,k}(r) \to B_{t,k}(l,r)$ that sends*

$$T(\mathbf{m}) \mapsto T_l(\mathbf{m})$$

including $K \mapsto l$.

Proof. Since $T(\mathbf{m})$ form a basis, these formulas define a vector space map from $\mathcal{D}_{t,k}(r)$ to $B_{t,k}(l,r)$. Now from Section 5.3.3 we know that a product of $T_l(\mathbf{m}_1)$ and $T_l(\mathbf{m}_2)$ is a linear combination of $T_l(\mathbf{m})$ with coefficients being polynomial in l. And the same statement holds for $T(\mathbf{m})$ but we need to substitute K for l in these polynomials. Hence this map is a map of algebras. It is surjective since $T_l(\mathbf{m})$ form a generating set of $B_{t,k}(l,r)$.

□

Now let us first construct a representation of $H_{t,k}(l,r)$ on $H_{-t,-k}(l,1) \otimes (\mathbb{C}^r)^l$. To do this we use the same ideas as in Proposition 5.1.3.3.

Proposition 5.4.4.7. *For any $l \in \mathbb{Z}_{>0}$, there is a structure of representation of $H_{t,k}(l,r)$ on $H_{-t,-k}(l,1) \otimes (\mathbb{C}^r)^l$ given by:*

$$x_i(m \otimes \mathbf{v}) = mx_i \otimes \mathbf{v},\ y_i(m \otimes \mathbf{v}) = my_i \otimes \mathbf{v},\ (g)_i(m \otimes \mathbf{v}) = m \otimes (g)_i\mathbf{v},$$

$$s_{ij}(m \otimes v) = ms_{ij} \otimes \sigma_{ij}\mathbf{v}\ .$$

Proof. We just need to check that these formulas define a representation. This is easy to do. Indeed for example:

$$[y_i, x_i](m \otimes \mathbf{v}) = m[x_i, y_i] \otimes \mathbf{v} = m(t - k\sum_{j\neq i} s_{ij}) \otimes \mathbf{v} =$$

$$= t \cdot m \otimes \mathbf{v} - k\sum_{j\neq i} ms_{ij} \otimes \sigma_{ij}^2\mathbf{v} = (t - k\sum_{j\neq i} s_{ij}\sigma_{ij})(m \otimes \mathbf{v})\ .$$

There we can see that the opposite signs for t and k come from the use of the right action. The other commutators can be checked in the similar fashion. □

Note that we can derive the following Corollary from this result:

Corollary 5.4.4.8. *For any $l \in \mathbb{Z}_{>0}$, there is a structure of a representation of $B_{t,k}(l,r)$ on $\mathbf{V}_l = H_{-t,-k}(l,1) \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^r)^l$ obtained by restriction of the representation of $H_{t,k}(l,r)$ on $H_{-t,-k}(l,1) \otimes (\mathbb{C}^r)^l$.*

We will denote the corresponding map by $\tau_l : B_{t,k}(l,r) \to \mathrm{End}(\mathbf{V}_l)$.

Proof. Indeed this follows from the fact that $B_{t,k}(l,r) = \mathbf{e}H_{t,k}(l,r)\mathbf{e}$ and the fact that the action of $\mathbb{C}[S_l] \subset H_{t,k}(l,r)$ on $H_{-t,-k}(l,1) \otimes (\mathbb{C}^r)^l$ is right on $H_{-t,-k}(l,1)$ and left on $(\mathbb{C}^r)^l$. Hence the averaging operator $\mathbf{e}$ ensures that we stay within $\mathbf{V}_l$. □

It follows that we have the following diagram:

$$\begin{array}{ccccc}
\mathrm{End}(\mathbf{V}_l) & & \xleftarrow{\quad\tau_l\quad} & & B_{t,k}(l,r) \\
\rho_l \big\uparrow & & & & \big\uparrow \pi_l \\
D_{\lambda,\beta}(r) & \xrightarrow{\zeta} & \mathbb{D}_{\lambda,\beta}(r) & \xrightarrow{\psi} & \mathcal{D}_{t,k}(r)
\end{array}$$

We want to show that this diagram is commutative:

Proposition 5.4.4.9. *For any $l \in \mathbb{Z}_{>0}$ it holds that $\rho_l = \tau_l \circ \pi_l \circ \psi \circ \zeta$.*

Proof. It is enough to check this identity on the generators of $D_{\lambda,\beta}(r)$. This is easy to do. We have

$$(\pi_l \circ \psi \circ \zeta)(X^{\pm}_{0,r}) = (\pi_l \circ \psi)(E^{\pm}_i) = \pi_l(T_{0,0}(E^{pm}_i)) = T_{0,0,l}(E^{pm}_i),$$

and hence:

$$(\tau_l \circ \pi_l \circ \psi \circ \zeta)(X^{\pm}_{0,r})(m \otimes \mathbf{v}) = \sum_j m \otimes (E^{\pm}_i)_j \mathbf{v} = \rho_l(X^{\pm}_{0,r})(m \otimes \mathbf{v}) \ .$$

And the same holds for $H_{i,0}$.

Now $(\pi_l \circ \psi \circ \zeta)(X^{\pm}_{1,r}) = T_{0,1,l}(E^{\pm}_i)$, hence

$$(\tau_l \circ \pi_l \circ \psi \circ \zeta)(X^{\pm}_{1,r})(m \otimes \mathbf{v}) = \sum_j m y_j \otimes (E^{\pm}_i)_j \mathbf{v} = \rho_l(X^{\pm}_{1,r})(m \otimes \mathbf{v}) \ .$$

And again the same holds for $H_{i,1}$.

For $X^{+}_{0,0}$ we have $(\pi_l \circ \psi \circ \zeta)(X^{+}_{0,0}) = T_{1,0,l}(E_{-\theta})$ and so:

$$(\tau_l \circ \pi_l \circ \psi \circ \zeta)(X^{+}_{0,0})(m \otimes \mathbf{v}) = \sum_j m x_j \otimes (E_{-\theta})_j \mathbf{v} = \rho_l(X^{+}_{0,0})(m \otimes \mathbf{v}) \ .$$

Lastly

$$(\pi_l \circ \psi \circ \zeta)(X_{0,1}^{+,\pm}) = T_{1,1,l}(E_{-\theta}) - \lambda(\pi_l \circ \psi \circ \zeta)\omega_0^{+,\pm} \ .$$

Now since $\omega_0^{+,\pm}$ lies in the subspace generated by $X_{i,0}^{\pm}$ and $H_{i,0}$ it follows that

$$(\tau_l \circ \pi_l \circ \psi \circ \zeta)(\omega_0^{+,\pm}) = \rho_l(\omega_0^{+,\pm})$$

holds as proved by the previous formulas. Hence we have:

$$(\tau_l \circ \pi_l \circ \psi \circ \zeta)(X_{0,1}^{+,\pm})(m \otimes \mathbf{v}) =$$

$$= \sum_j m \frac{x_j y_j + y_j x_j}{2} \otimes (E_{-\theta})_j \mathbf{v} - \lambda \omega_0^{+,\pm}(m \otimes \mathbf{v}) = \rho_l(X_{0,1}^{+,\pm})(m \otimes \mathbf{v}) \ .$$

And so the result follows. □

And so we can formulate the result which we wanted to prove in this section.

Theorem 5.4.4.10. *For $t + kr \neq 0$, the map $\psi : \mathbb{D}_{\lambda,\beta}(r) \to \mathcal{D}_{t,k}(r)$ constructed in Proposition 5.4.2.2 is an isomorphism.*

Proof. We know surjectivity from Proposition 5.4.3.1. Now take any non-zero element $x \in \mathbb{D}_{\lambda,\beta}(r)$. Since ζ is an isomorphism there is $y \in D_{\lambda,\beta}(r)$ such that $\zeta(y) = x$. Now by Proposition 5.4.4.5 there exists l such that $\rho_l(y) \neq 0$. Hence by Proposition 5.4.4.9 it follows that $(\tau_l \circ \pi_l \circ \psi \circ \zeta)(y) = (\tau_l \circ \pi_l)(\psi(x)) \neq 0$. Hence $\psi(x) \neq 0$. Thus ψ is injective and so it is an isomorphism. □

5.5 DDCA of rank 1 type A

In this section we will study the DDC algebra $\mathcal{D}_{1,k}$ of rank 1 and type A in more detail. Namely we will present a presentation of this algebra by generators and relations.

This will be done by studying the flat filtered deformation of the universal enveloping algebra of a Lie algebra of polynomials on symplectic plane with a Poisson bracket. Hence we will begin this section by defining and exlporing that algebra.

5.5.1 The Lie algebra $\mathfrak{po}$

Let us start by giving a definition of $\mathfrak{po}$, the Lie algebra of polynomials on symplectic plane:

Definition 5.5.1.1. By $\mathfrak{po}$ denote the Lie algebra over $\Bbbk$ which, as a vector space, is given by $\Bbbk[p, q]$ and the structure of Lie algebra of which is determined by the following bracket:

$$[q^k p^l, q^m p^n] = (lm - nk)q^{k+m-1}p^{l+n-1}.$$

We will denote the element $1 \in \Bbbk[p, q]$ by K.

In other words, this Lie algebra is given by the standard Poisson bracket on $\Bbbk[p, q]$ determined by $\{p, q\} = 1$.

This algebra admits the following grading:

Definition 5.5.1.2. Endow the Lie algebra $\mathfrak{po}$ with a grading given by the formula $\deg(q^k p^l) = k + l - 2$. In this grading the bracket has degree 0.

Note that $(-\frac{q^2}{2}, pq, \frac{p^2}{2})$ constitutes an $\mathfrak{sl}_2$-triple. Hence we conclude that $\mathfrak{po}_0 \simeq \mathfrak{sl}_2$. This endows $\mathfrak{po}$ with a structure of an $\mathfrak{sl}_2$-module. It is easy to see that $\mathfrak{po}_i$ is isomorphic to the simple highest weight module V_{i+2} of highest weight $i + 2$.

Definition 5.5.1.3. Denote by $\mathfrak{n}$ the Lie subalgebra of $\mathfrak{po}$ given by $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{po}_i$.

As an $\mathfrak{sl}_2$-module we have:

$$\mathfrak{n} = V_3 \oplus V_4 \oplus V_5 \oplus \dots.$$

5.5.2 A presentation of $\mathfrak{po}$ by generators and relations.

In order to find presentation of $\mathcal{D}_{1,k}$, we first need to find that of $\mathfrak{po}$ and $U(\mathfrak{po})$. Now, in order to find this presentation of $\mathfrak{po}$ by generators and relations, it is enough to find the corresponding presentation of $\mathfrak{n}$. The rest will follow easily. This was done in [56] using a computer calculation of the cohomology spaces of $\mathfrak{n}$ to obtain a minimal set of generators and relations. We will reproduce this result below. We will also present a direct proof of this result in Appendix A.

First, it's easy to find the generators:

Definition 5.5.2.1. The Lie algbera $\mathfrak{n}$ is generated by $\mathfrak{n}_1$.

Proof. Indeed, this easily follows by induction from the formulas $p^k q^l = [\frac{p^{k+1}q^{l-2}}{k+1}, \frac{q^3}{3}]$ for $l \geq 2$, $p^k q = [\frac{p^k}{k}, \frac{pq^2}{2}]$ and $p^k = [\frac{p^{k-1}}{k-1}, p^2 q]$. □

So it follows that the algebra $\mathfrak{n}$ is a quotient of the free Lie algebra $L(\mathfrak{n}_1)$, where $\mathfrak{n}_1 \simeq V_3$. The Lie algebra $L(\mathfrak{n}_1)$ has a grading determined by $\deg(\mathfrak{n}_1) = 1$.

To describe the relations in a language of $\mathfrak{sl}_2$-modules we will first have to introduce a few definitions.

Definition 5.5.2.2. Fix an isomorpism of $\mathfrak{n}_1$ with V_3 with the highest weight vector specified as $c_1 = \frac{q^3}{6}$.

Consider $\Lambda^2\mathfrak{n}_1 = L(\mathfrak{n}_1)_2$. As $\mathfrak{sl}_2$-modules we have $\Lambda^2\mathfrak{n}_1 \simeq V_4 \oplus V_0$. Denote the submodule of $\Lambda^2\mathfrak{n}_1$ isomorophic to V_0 by ϕ_1 and the submodule isomorphic to V_4 by ϕ_2. Fix an isomorphism of ϕ_1 with V_0 with the highest weight vector specified as

$c_1 \wedge c_4 - c_2 \wedge c_3$, where $c_i = f^{i-1} c_1$. Fix an isomorphism of ϕ_2 with V_4 with the highest weight vector specified as $d_1 = c_2 \wedge c_1$.

Consider $\phi_2 \otimes \mathfrak{n}_1 \subset L(\mathfrak{n}_1)_3$. We have $\phi_2 \otimes \mathfrak{n}_1 \simeq V_7 \oplus V_5 \oplus V_3 \oplus V_1$. Denote the submodule isomorphic to V_1 by ψ_1, the submodule isomorphic to V_3 by ψ_2, the submodule isomorphic to V_5 by ψ_3 and submodule isomorphic to V_7 by ψ_4. Also fix an isomorphism of ψ_1 with V_1 with the highest weight vector specified as $-4d_1 \otimes c_4 + 3d_2 \otimes c_3 - 2d_3 \otimes c_2 + d_4 \otimes c_1$, where $d_i = f^{i-1} d_1$.

Consider $\wedge^2 \phi_2 \subset L(\mathfrak{n}_1)_4$. We have $\wedge^2 \phi_2 = V_6 \oplus V_2$. Denote the submodule isomorphic to V_2 by χ_1. Fix an isomorphism of χ_1 with V_2 with the highest weight vector specified as $3d_3 \wedge d_2 - 2d_4 \wedge d_1$.

We have the following proposition.

Proposition 5.5.2.3. *The Lie algebra $\mathfrak{n}$ is isomorphic to the quotient of the free Lie algebra $L(\mathfrak{n}_1)$ by the ideal generated by the $\mathfrak{sl}_2$-modules ϕ_1, ψ_4, ψ_1 and χ_1. This is a minimal set of relations.*

Proof. As stated in the beginning of this section, one can find a proof of this result by a computer computation in [56]. See Appendix B for a more direct proof. □

Now we can move to the description of the whole algebra. First let us introduce the notation for the remaining part of $\mathfrak{po}$:

Definition 5.5.2.4. Denote by $\mathfrak{b}$ the Lie subalgebra of $\mathfrak{po}$ given by $\mathfrak{po}_{-2} \oplus \mathfrak{po}_{-1} \oplus \mathfrak{po}_0$. We have $\mathfrak{po} = \mathfrak{b} \oplus \mathfrak{n}$.

We will also need a little more notation:

Definition 5.5.2.5. Fix an isomorphism of $\mathfrak{b}_0$ with $\mathfrak{sl}_2$ given by $e \mapsto b_1 = -\frac{q^2}{2}$ and $f \mapsto b_3 = \frac{p^2}{2}$. Fix an isomorphism of $\mathfrak{b}_{-1}$ with V_1 with the highest weight vector specified as $a_1 = q$. Fix an isomorphism of $\mathfrak{b}_{-2}$ with V_0 with the highest weight vector specified as K.

Consider the free Lie algebra $L(\mathfrak{b} \oplus \mathfrak{n}_1)$. Consider $\Lambda^2\mathfrak{b}_{-1} \subset L(\mathfrak{b} \oplus \mathfrak{n}_1)_2$, we have $\Lambda^2\mathfrak{b}_{-1} \simeq V_0$. Fix an isomorphism of $\Lambda^2\mathfrak{b}_{-1}$ with V_0 with the highest weight vector specified as $a_1 \wedge a_2$.

Consider $\mathfrak{n}_1 \otimes \mathfrak{b}_{-1} \subset L(\mathfrak{b} \oplus \mathfrak{n}_1)_2$. We have $\mathfrak{n}_1 \otimes \mathfrak{po}_{-1} \simeq V_4 \oplus V_2$. Denote the submodule isomorphic to V_2 by α_1 and the submodule isomorphic to V_4 by α_2. Fix an isomorphism of α_1 with V_2 with the highest weight vector specified as $c_2 \otimes a_1 - 2c_1 \otimes a_2$.

Proposition 5.5.2.6. *The Lie algebra $\mathfrak{po}$ is generated by $\mathfrak{b} \oplus \mathfrak{n}_1$ with the following set of relations:*

$$\mathfrak{b}_{-2} \simeq V_0 \textit{ is central}, \ \mathfrak{b}_0 \simeq \mathfrak{sl}_2, \ \mathfrak{b}_{-1} \simeq V_1 \textit{ as an } \mathfrak{sl}_2\textit{-module}, \ \Lambda^2\mathfrak{b}_{-1} = \mathfrak{b}_{-2},$$
$$\mathfrak{n}_1 \simeq V_3 \textit{ as an } \mathfrak{sl}_2\textit{-module}, \ \alpha_2 = 0, \ \alpha_1 = \mathfrak{b}_0,$$
$$\phi_1 = 0, \ \psi_4 = 0, \ \psi_1 = 0, \ \chi_1 = 0,$$

where we use the isomorphisms from Definition 5.5.2.2 and Definition 5.5.2.5. And by $\lambda X \simeq \mu Y$ for two $\mathfrak{sl}_2$-submodules of $L(\mathfrak{b} \oplus \mathfrak{n}_1)$ with two fixed isomorphisms with V_j and two numbers λ, μ we mean that we take the quotient by the image of the map

$$V_j \xrightarrow{(\lambda, -\mu)} V_j \oplus V_j \simeq X \oplus Y \subset L(\mathfrak{b} \oplus \mathfrak{n}_1).$$

Proof. This easily follows from Proposition 5.5.2.3. Indeed, the first line of relations ensures that the subalgebra generated by $\mathfrak{b}$ is indeed $\mathfrak{b}$, the third line ensures that the subalgebra generated by $\mathfrak{n}_1$ is isomorphic to $\mathfrak{n}$. The second line fixes the adjoint action of $\mathfrak{b}$ on $\mathfrak{n}_1$ making sure that nothing more is generated. □

One can also give a more explicit presentation, without using the language of $\mathfrak{sl}_2$-modules.

Proposition 5.5.2.7. *The Lie algebra $\mathfrak{po}$ is generated by elements K of degree -2, $q = a_1$ and $p = a_2$ of degree -1, $e := b_1 = -\frac{q^2}{2}$ and $f := b_3 = \frac{p^2}{2}$ of degree 0, and $r := c_1 = \frac{q^3}{6}$ of degree 1, with defining relations:*

$$\begin{gathered}
[K, X] = 0 \text{ for any } X,\ [p, q] = K,\ [f, q] = p,\ [p, f] = 0,\ [e, p] = q, \\
[[f, e], f] = 2f, \\
[r, p] = e,\ [e, r] = 0,\ \mathrm{ad}_f^4(r) = 0,\ [e, [f, r]] = 3r, \\
[r, \mathrm{ad}_f^3(r)] - [\mathrm{ad}_f(r), \mathrm{ad}_f^2(r)] = 0, \qquad (5.1) \\
\mathrm{ad}_r^3(f) = 0, \\
4[\mathrm{ad}_f^3(r), \mathrm{ad}_r^2(f)] - 3[\mathrm{ad}_f^2(r), \mathrm{ad}_f \mathrm{ad}_r^2(f)] + 2[\mathrm{ad}_f(r), \mathrm{ad}_f^2 \mathrm{ad}_r^2(f)] - [r, \mathrm{ad}_f^3 \mathrm{ad}_r^2(f)] = 0, \\
3[\mathrm{ad}_f^2 \mathrm{ad}_r^2(f), \mathrm{ad}_f \mathrm{ad}_r^2(f)] - 2[\mathrm{ad}_f^3 \mathrm{ad}_r^2(f), \mathrm{ad}_r^2(f)] = 0.
\end{gathered}$$

Proof. In order to get this presentation from the one given in Proposition 5.5.2.6, to start with, we need to throw out some of the generators. Indeed, in the formulation we threw out the generator corresponding to h in the $\mathfrak{sl}_2$-triple of $\mathfrak{b}_0$ and we have only taken one generator from the whole of $\mathfrak{n}_1$ – the highest-weight vector r. This is obviously enough, since we can generate the whole of $\mathfrak{sl}_2$ using e and f, and then generate the rest of $\mathfrak{n}_1$ by the action of $\mathfrak{b}_0$ on r.

Now, it's easy to see that the first line of the relations in Proposition 5.5.2.6 transforms into the first two lines of relations (5.5.2.7) and the second line of the relations in Proposition 5.5.2.6 transforms into the third line of the relations (5.5.2.7). We only need to keep the highest-weight vectors of the third line of the relations in Proposition 5.5.2.6, since the rest of the relations can be generated by the action of

$\mathfrak{b}_0$. These four highest-weight vectors are given in the last lines of relations (5.5.2.7) in the same order as the corresponding $\mathfrak{sl}_2$-modules in Proposition 5.5.2.6.

For the details of these calculations see Appendix A. □

Remark 5.5.2.8. Using this we can also write down a presentation of $\mathfrak{po}$ with just three generators. Indeed, the Lie algebra $\mathfrak{po}$ is generated by elements p, f and r of degrees $-1, 0, 1$ respectively, with defining relations:

$$[\mathrm{ad}_p^3(r), X] = 0 \text{ for any } X,$$

$$[[[p,r],f],p] = p, \; [p,f] = 0, \; \text{(degree -1)}$$

$$[[[p,r],f],f] = 2f, \; \text{(degree 0)}$$

$$\mathrm{ad}_r^2(p) = 0, \; \mathrm{ad}_f^4(r) = 0, \; [[[p,r],f],r] = 3r, \; \text{(degree 1)}$$

$$[r, \mathrm{ad}_f^3(r)] - [\mathrm{ad}_f(r), \mathrm{ad}_f^2(r)] = 0, \; \text{(degree 2)}$$

$$\mathrm{ad}_r^3(f) = 0, \; \text{(degree 3)}$$

$$4[\mathrm{ad}_f^3(r), \mathrm{ad}_r^2(f)] - 3[\mathrm{ad}_f^2(r), \mathrm{ad}_f\mathrm{ad}_r^2(f)] + 2[\mathrm{ad}_f(r), \mathrm{ad}_f^2\mathrm{ad}_r^2(f)] - [r, \mathrm{ad}_f^3\mathrm{ad}_r^2(f)] = 0,$$

$$3[\mathrm{ad}_f^2\mathrm{ad}_r^2(f), \mathrm{ad}_f\mathrm{ad}_r^2(f)] - 2[\mathrm{ad}_f^3\mathrm{ad}_r^2(f), \mathrm{ad}_r^2(f)] = 0 \text{ (degree 4)}.$$

5.5.3 Flat filtered deformations of $U(\mathfrak{po})$

In the beginning of Section 5.5.1 we've mentioned that $\mathcal{D}_{1,k}$ is going to be isomorphic to a flat filtered deformation of $U(\mathfrak{po})$. For this reason in this section we will formulate a result on flat filtered deformations of $U(\mathfrak{po})$ obtained via computer calculations and then present a known flat filtered deformation of $U(\mathfrak{po})$.

Using computer calculation one can arrive at the following proposition about the deformations of $U(\mathfrak{po})$. Again, before we can formulate the relations in terms of $\mathfrak{sl}_2$-modules we need to introduce some notations:

Definition 5.5.3.1. Consider a free associative algebra $T(\mathfrak{b} \oplus \mathfrak{n}_1)$. Denote the subspace $S^2\mathfrak{b}_{-1} \subset T(\mathfrak{b} \oplus \mathfrak{n}_1)_2$ isomorphic to V_2 as $\mathfrak{sl}_2$-module by β_1. Fix an isomorphism of $S^2\mathfrak{b}_{-1}$ with V_2 with the highest weight vector specified by a_1^2.

Also for any $\mathfrak{sl}_2$-submodule $\gamma \subset T(\mathfrak{b} \oplus \mathfrak{n}_1)$, denote by $K^i\gamma$ the submodule $\gamma \otimes \mathfrak{b}_{-2}^{\otimes i}$. If γ had a fixed isomorphism with V_j with the highest weight vector specified by v_γ, fix an isomorphism of $\gamma \otimes \mathfrak{b}_{-2}^{\otimes i}$ with V_j with the highest weight vector specified by $v_\gamma \otimes K^{\otimes i}$.

We are ready to state the main result of the section.

Proposition 5.5.3.2. *Suppose U is a flat filtered deformation of $U(\mathfrak{po})$ as an associative algebra (up to an automorphism), such that $U(\mathfrak{b})$ is still a subalgebra of U, and the action of $U(\mathfrak{b})$ on $\mathfrak{b} \oplus \mathfrak{n}_1$ is not deformed. Then U is isomorphic to A_{s_1,s_2} defined below for some values of s_1 and s_2. The algebra A_{s_1,s_2} is generated by $\mathfrak{b} \oplus \mathfrak{n}_1$ with the set of relations given by the first two lines of Proposition 5.5.2.6 and the following relations, which substitute the last line in Proposition 5.5.2.6:*

$$\phi_1 = -\frac{s_1 K}{2},\ \psi_4 = 0,\ \psi_1 \simeq 15 s_1 \mathfrak{b}_{-1},\ \chi_1 \simeq 3((30 s_1 + 14 s_2 K)\mathfrak{b}_0 + 7 s_2 \beta_1),$$

where $s_1, s_2 \in \mathbb{C}[K]$, "$\simeq$" means the same thing as in Proposition 5.5.2.6, and all the submodules of $L(\mathfrak{b} \oplus \mathfrak{n}_{-1})$ are interpreted as submodules of $T(\mathfrak{b} \oplus \mathfrak{n}_{-1})$ via the map $L(\mathfrak{b} \oplus \mathfrak{n}_{-1}) \to T(\mathfrak{b} \oplus \mathfrak{n}_{-1})$ which sends the elements of the free Lie algebra into the corresponding commutators in the free associative algebra.

Proof. Note that our requirement on the type of deformation effectively means that we consider such deformations of relations in Proposition 5.5.2.6 which change only the last four relations, augmenting them by some lower order terms.

The computer calculation that classified all possible deformations of this type was preformed by Eric Rains. An outline of this calculation can be found in Proposition 4.2.13 of [18]. □

Remark 5.5.3.3. Note that we can specialize the central element K to a number, which will give a 3-parameter flat family of algebras $A_{s_1,s_2,K}$, with $s_1, s_2, K \in \mathbb{C}$. These parameters have degrees $4, 6, -2$, respectively; alternatively, we may view this deformation as one with four deformation parameters $s_1, s_2, s_1' = s_1 K, s_2' = s_2 K$ of degrees $4, 6, 2, 4$, respectively, which are constrained by the relation $s_1 s_2' = s_2 s_1'$; i.e., deformations are parametrized by a quadratic cone in $\mathbb{C}^4$. Also, we see that up to rescaling there are only two essential parameters, $s_1^* = s_1 K^2$ and $s_2^* = s_2 K^3$.

As before, this presentation can be formulated more explicitly as follows:

Proposition 5.5.3.4. *The algebra A_{s_1,s_2} is generated by the same generators as $\mathfrak{po}$ and the same set of relations as in Proposition 5.5.2.7, with the last four relations deformed as follows:*

$$\begin{gathered}
[r, \mathrm{ad}_f^3(r)] - [\mathrm{ad}_f(r), \mathrm{ad}_f^2(r)] = -\frac{s_1 K}{2}, \\
\mathrm{ad}_r^3(f) = 0, \qquad (5.2) \\
4[\mathrm{ad}_f^3(r), \mathrm{ad}_r^2(f)] - 3[\mathrm{ad}_f^2(r), \mathrm{ad}_f \mathrm{ad}_r^2(f)] + 2[\mathrm{ad}_f(r), \mathrm{ad}_f^2 \mathrm{ad}_r^2(f)] - [r, \mathrm{ad}_f^3 \mathrm{ad}_r^2(f)] = 15 s_1 q, \\
3[\mathrm{ad}_f^2 \mathrm{ad}_r^2(f), \mathrm{ad}_f \mathrm{ad}_r^2(f)] - 2[\mathrm{ad}_f^3 \mathrm{ad}_r^2(f), \mathrm{ad}_r^2(f)] = 3((30 s_1 + 14 s_2 K)e + 7 s_2 q^2),
\end{gathered}$$

where $s_1, s_2 \in \mathbb{C}[K]$.

Proof. This is easy to see following the proof of Proposition 5.5.2.7. □

Remark 5.5.3.5. We can also rewrite the above relations (Proposition 5.5.3.4) using the set of generators of Remark 5.5.2.8. Indeed, the algebra A_{s_1,s_2} is generated by the same set of generators as $\mathfrak{po}$ in Remark 5.5.2.8 (i.e., p, f, r) and the same set of relations as in Remark 5.5.2.8, with the last four (degrees $2, 3, 4$) deformed as follows:

$$[r, \mathrm{ad}_f^3(r)] - [\mathrm{ad}_f(r), \mathrm{ad}_f^2(r)] = -\frac{s_1 K}{2}, \tag{5.3}$$

$$\mathrm{ad}_r^3(f) = 0,$$

$$4[\mathrm{ad}_f^3(r), \mathrm{ad}_r^2(f)] - 3[\mathrm{ad}_f^2(r), \mathrm{ad}_f\mathrm{ad}_r^2(f)] + 2[\mathrm{ad}_f(r), \mathrm{ad}_f^2\mathrm{ad}_r^2(f)] - [r, \mathrm{ad}_f^3\mathrm{ad}_r^2(f)] = 15s_1\mathrm{ad}_p^2(r),$$

$$3[\mathrm{ad}_f^2\mathrm{ad}_r^2(f), \mathrm{ad}_f\mathrm{ad}_r^2(f)] - 2[\mathrm{ad}_f^3\mathrm{ad}_r^2(f), \mathrm{ad}_r^2(f)] = 3(7s_2\mathrm{ad}_p^2(r)^2 - (30s_1 + 14s_2K)\mathrm{ad}_p(r)),$$

where $K = \mathrm{ad}_p^3(r)$ and $s_1, s_2 \in \mathbb{C}[K]$.

Below we will show that the universal enveloping algebra of the Lie algebra $\mathbb{C}[x, \partial]$ gives us an example of such a deformation. This result is well-known, see [21].

Definition 5.5.3.6. Denote by $\mathbb{C}[x, \partial]$ the Lie algebra of polynomial differential operators, with a Lie bracket given by the commutator.

Consider a grading on $\mathbb{C}[x, \partial]$ given by $\deg(x^k\partial^l) = k + l - 2$. We have a decomposition $\mathbb{C}[x, \partial] = \bigoplus_{i=-2} \mathbb{C}[x, \partial]_i$. It's easy to see that with this grading the Lie bracket decreases filtration degree at least by 2 and preserves degree modulo 2:

$$[,] : \mathbb{C}[x, \partial]_i \otimes \mathbb{C}[x, \partial]_j \to \mathbb{C}[x, \partial]_{i+j} \oplus \mathbb{C}[x, \partial]_{i+j-2} \oplus \ldots .$$

Indeed, when we compute the commutator we use the identity $[\partial, x] = 1$ at least once, and each time it decreases the grading by 2.

Lemma 5.5.3.7. *The associated graded Lie algebra of $\mathbb{C}[x, \partial]$ is isomorphic to $\mathfrak{po}$.*

Proof. Writing down the commutator of basis elements, we have:

$$[x^k\partial^l, x^m\partial^n] = (lm - nk)x^{k+m-1}\partial^{l+n-1} + \ldots .$$

So by taking the associated graded of $\mathbb{C}[x,\partial]$ and denoting the image of x by q and the image of ∂ by p, we end up with $\mathfrak{po}$. □

And we have the following corollary:

Corollary 5.5.3.8. *$\mathbb{C}[x,\partial]$ is a non-trivial flat filtered deformation of $\mathfrak{po}$ as a Lie algebra.*

Proof. The flatness follows from Lemma 5.5.3.7 and the fact that the graded dimensions of the two Lie algebras are the same.

The fact that this deformation is non-trivial (which is not hard to check directly) is known as the van Hove-Groenewold's theorem in quantum mechanics, which says that classical infinitesimal symmetries deform nontrivially under quantization. See Theorem 13.13 in [29]. □

Now from Proposition 5.5.3.2 it follows that $U(\mathbb{C}[x,\partial])$ must be isomorphic to A_{s_1,s_2} for some choice of s_1 and s_2. Let us now compute these parameters.

Proposition 5.5.3.9. *The algebra $U(\mathbb{C}[x,\partial])$ is isomorphic to $A_{1,0}$.*

Proof. From Proposition 5.5.3.2 we know that $U(\mathbb{C}[x,\partial]) \simeq A_{s_1,s_2}$. Since this deformation actually comes from the Lie algebra deformation, we can conclude that s_2 must be equal to zero. Now we can consider the Lie algebra $\mathfrak{a}_{s_1}$ given by the generators and relations of Proposition 5.5.3.4 with $s_2 = 0$. So we know that $\mathbb{C}[x,\partial] \simeq \mathfrak{a}_{s_1}$. Let's denote this isomorphism by $\varepsilon : \mathfrak{a}_{s_1} \to \mathbb{C}[x,\partial]$. Since ε is determined up to a constant, we can set the image of K under ε to be $\varepsilon(K) = 1$. Now since $\mathfrak{a}_{s_1}$ is a deformation of $\mathrm{gr}(\mathbb{C}[x,\partial])$, we know that $\varepsilon(q) = x + \dots$, $\varepsilon(p) = \partial + \dots$, $\varepsilon(e) = -\frac{x^2}{2} + \dots$, $\varepsilon(f) = \frac{\partial^2}{2} + \dots$ and $\varepsilon(r) = \frac{x^3}{6} + \dots$, where "$\dots$" stand for the lower order terms. Also note that since the commutator is deformed in degrees starting with -2, it follows that the lower order terms also can appear only starting with degrees -2. Hence $\varepsilon(q) = x$ and $\varepsilon(p) = \partial$. Suppose $\varepsilon(e) = -\frac{x^2}{2} + c_1$ and $\varepsilon(f) = \frac{\partial^2}{2} + c_2$, it follows that $[\varepsilon(e), \varepsilon(f)] = x\partial + \frac{1}{2}$. Now by calculating $[[\varepsilon(e), \varepsilon(f)], \varepsilon(e)] = [x\partial, -\frac{x^2}{2}] = -x^2$, we conclude that c_1 must be equal to 0. The same holds true for c_2. Now suppose $\varepsilon(r) = \frac{x^3}{6} + d_1 x + d_2 \partial$. Now $[\varepsilon(q), \varepsilon(r)] = -d_2$, hence $d_2 = 0$. And $[\varepsilon(p), \varepsilon(r)] = \frac{x^2}{2} + d_1$, hence $d_1 = 0$. So we know the images of the commutators. Now it's enough to calculate one of the relations.

We compute $\mathrm{ad}_f(r) = [\frac{\partial^2}{2}, \frac{x^3}{6}] = \frac{x^2\partial + x}{2}$, $\mathrm{ad}_f^2(r) = [\frac{\partial^2}{2}, \frac{x^2\partial + x}{2}] = x\partial^2 + \partial$ and $\mathrm{ad}_f^3(r) = [\frac{\partial^2}{2}, x\partial^2] = \partial^3$. So it follows that:

$$[\frac{x^3}{6}, \partial^3] - [\frac{x^2\partial + x}{2}, x\partial^2 + \partial] = -\frac{3}{2}x^2\partial^2 - 3x\partial - 1 + \frac{3}{2}x^2\partial^2 + 3x\partial + \frac{1}{2} = -\frac{1}{2}.$$

Thus we conclude that $s_1 = 1$. □

We also have a corollary:

Corollary 5.5.3.10. *The deformation $A_{1,0}$ is flat.*

Remark 5.5.3.11. Of course we could have proved that $\mathbb{C}[x, \partial]$ is isomorphic to $\mathfrak{a}_1$ without using computer computation and Proposition 5.5.3.2. Indeed, one just needs to check that 1, x, ∂, $-\frac{x^2}{2}$, $\frac{\partial^2}{2}$ and $\frac{x^3}{6}$ satisfy the required relations, which is easy to do.

5.5.4 The deformed double current algebra of type A as a flat filtered deformation of $U(\mathfrak{po})$

Now, in this section we would like to show that the generic choice of parameters s_1 and s_2 for the deformation of $U(\mathfrak{po})$ gives us the DDC algebra $\mathcal{D}_{1,k}$.

Below we will need to compute things in $\mathcal{D}_{1,k}$. To do so we will make use of the same method as in Definition 5.4.2.1. I.e. we will use $\sim$ to denote the transition from an element of $\mathcal{D}_{1,k}$ to a corresponding sequence of elements of $B_{1,k}(n)$.

With this tool we are ready to continue:

Proposition 5.5.4.1. *The algebra $\mathcal{D}_{1,k}$ is a flat filtered deformation of $U(\mathfrak{po})$.*

Proof. Indeed, we know that the basis in this algebra is given by $T(\mathbf{m})K^i$. Also recall the natural filtration we considered in the previous section (so that the element $T(\mathbf{m})K^i$ belongs to $(\mathcal{D}_{1,k})_{w(\mathbf{m})}$). Since by Proposition 5.3.4.4 we know that $\mathrm{gr}B_{1,k_n}(\nu_n) = \overline{\mathbb{Q}}[P_{r,q,\nu_n}]_{r,q\geq 0, 0<r+q}$ in sufficiently low degrees (where P_{r,q,ν_n} are the symmetric polynomials mentioned in the proof of that Proposition), where the associated graded is taken with respect to the filtration from Definition 5.3.4.2, it follows that

$$\mathrm{gr}\mathcal{D}_{1,k} = (\prod\nolimits_{\mathcal{F}} \overline{\mathbb{Q}}[P_{r,q,\nu_n}]_{r,q\geq 0, 0<r+q\leq \nu_n})|_{\nu=K} = \mathbb{C}[P_{r,q}]_{r,q\geq 0},$$

where $P_{r,q} = \mathrm{gr}^{r+q}(T_{r,q})$ and $P_{0,0} = \mathrm{gr}^0(K)$.

Now the bracket $[,]$ acts as follows:

$$[,] : (\mathcal{D}_{1,k})_n \otimes (\mathcal{D}_{1,k})_m \to (\mathcal{D}_{1,k})_{m+n-2} \oplus (\mathcal{D}_{1,k})_{m+n-4} \oplus \ldots,$$

where we consider the grading of the algebra as a vector space. Indeed, this follows from the fact that $[T(\mathbf{m}), T(\mathbf{n})] \sim [T_{\nu_n}(\mathbf{m}), T_{\nu_n}(\mathbf{n})]$, and to calculate the latter expression we need to use the commutator $[x_i, y_j]$ at least once, which, each time we use it, lowers the degree by 2. We would like to calculate the leading term of the commutator. To calculate $\mathrm{gr}^{w(\mathbf{m})+w(\mathbf{n})-2}([T(\mathbf{m}), T(\mathbf{n})])$ it is enough to compute it via $\sim$, commuting freely elements within $T_{\nu_n}(\mathbf{m})$ and leaving only the highest term in the commutator of $[x_i, y_j] = \delta_{ij} + \dots$. So:

$$\mathrm{gr}^{w(\mathbf{m})+w(\mathbf{n})-2}([T(\mathbf{m}), T(\mathbf{n})]) \sim \left[\prod_{r,q\geq 0, r+q>0} P_{r,q,\nu_n}^{m_{r,q}}, \prod_{r,q\geq 0, r+q>0} P_{r,q,\nu_n}^{n_{r,q}}\right] =$$

$$= \prod_{r,q\geq 0, r+q>0} P_{r,q,\nu_n}^{m_{r,q}+n_{r,q}} \sum_{r_1,r_2,q_1,q_2} \frac{m_{r_1,q_1} n_{r_2,q_2}}{P_{r_1,q_1,\nu_n} P_{r_2,q_2,\nu_n}} [P_{r_1,q_1,\nu_n}, P_{r_2,q_2,\nu_n}].$$

But now:

$$[P_{r_1,q_1,\nu_n}, P_{r_2,q_2,\nu_n}] = \sum_{i,j=1}^{\nu_n} [x_i^{r_1} y_i^{q_1}, x_j^{r_2} y_j^{q_2}] =$$

$$= (q_1 r_2 - q_2 r_1) P_{r_1+r_2-1, q_1+q_2-1, \nu_n},$$

where we use $P_{0,0,\nu_n}$ to denote ν_n.

These formulas show us that $\mathrm{gr}\mathcal{D}_{1,k}$ is isomorphic to a deformation of $U(\mathfrak{po})$ after identification of $T_{i,j}$ with $p^i q^j$. So it follows that $\mathcal{D}_{1,k}$ is a deformation of $U(\mathfrak{po})$. Moreover it is a flat filtered deformation, by virtue of the fact that $T(\mathbf{m})K^i$ constitute a basis of $\mathcal{D}_{1,k}$. $\square$

Since we know all possible flat filtered deformations of $U(\mathfrak{po})$, it follows that $\mathcal{D}_{1,k}$ is isomorphic to A_{s_1,s_2} for some choice of constants. We would also like to calculate the exact correspondence.

Proposition 5.5.4.2. *The DDC-algebra $\mathcal{D}_{1,k}$ is isomorphic to A_{s_1,s_2} with*

$$s_1 = 1 + k(k+1)(1-K) \text{ and } s_2 = k(k+1).$$

Proof. We know that $\mathcal{D}_{1,k} \simeq A_{s_1,s_2}$ for some $s_1, s_2 \in \mathbb{C}[K]$. Denote this isomorphism by $\beta : A_{s_1,s_2} \to \mathcal{D}_{1,k}$. It is enough to calculate s_1, s_2 via evaluating one of the commutators. We will largely follow the steps of the proof of Proposition 5.5.3.9.

First, we will define another shorthand notation for our calculations:

Definition 5.5.4.3. Recall the notation of Definition 5.4.2.1. Consider $X \in \mathcal{D}_{1,k}$ which is represented by a sequence $X \sim X_n$. Also suppose that $X = \beta(Y)$ for an element $Y \in A_{s_1,s_2}$. Consider the faithful polynomial representation:

$$\pi_n : B_{1,k_n}(\nu_n) \to \overline{\mathbb{Q}}(x_1, \dots, x_{\nu_n})[\partial_1, \dots, \partial_{\nu_n}]^{S_{\nu_n}}.$$

Denote $Y \sim \pi_n(X_n)$.

I.e. with the help of this notation we will be able to preform calculation in the algebras of polynomial operators.

Note that the highest orders of generators are as follows. We have[4] $\mathrm{gr}^0(K) = K$, $\mathrm{gr}^1(T_{1,0}) = q$, $\mathrm{gr}^1(T_{0,1}) = p$, $\mathrm{gr}^2(T_{2,0}) = q^2$, $\mathrm{gr}^2(T_{1,1}) = pq$, $\mathrm{gr}^2(T_{0,2}) = p^2$ and $\mathrm{gr}^3(T_{3,0}) = q^3$. Thus it follows that

$$\beta(K) = K + \dots\ ,\ \beta(q) = T_{1,0} + \dots\ ,\ \beta(p) = T_{0,1} + \dots\ ,\ \beta(e) = -\frac{T_{2,0}}{2} + \dots\ ,$$

$$\beta(f) = \frac{T_{0,2}}{2} + \dots \text{ and } \beta(r) = \frac{T_{3,0}}{6} + \dots\ ,$$

where "$\dots$" stand for lower order terms. Note that since commutator has additional terms only 2 degrees lower, it follows that additional terms in β also can only be $2k$ degrees lower for a positive integer k.

Thus it follows that there are no additional terms in the action of β on K, q and p. Suppose $\beta(e) = -\frac{T_{2,0}}{2} + \gamma_1$ and $\beta(f) = \frac{T_{0,2}}{2} + \gamma_2$ for some $\gamma_i \in \mathbb{C}[K]$. First let's

[4] Here we slightly abuse the notation and denoting by p, q both the elements q, p in $\mathfrak{po}$ and the generators of A_{s_1,s_2}. Since these elements lie in different spaces this shouldn't cause any confusion.

calculate $[\beta(e),\beta(f)]$:

$$[e,f]\sim_\beta \frac{1}{4}[\sum_i D_i^2,\sum_j x_j^2]=\frac{1}{4}[\sum_i \partial_i^2-c(c+1)\sum_{i\neq k}\frac{1}{(x_i-x_k)^2},\sum_j x_j^2]=$$

$$=\frac{1}{4}\sum_{i,j}[\partial_i^2,x_j^2]=\frac{1}{4}\sum_i(4x_i\partial_i+2)\sim_\beta \beta^{-1}(T_{1,1}),$$

so we conclude that $[\beta(e),\beta(f)]=T_{1,1}$. Now we want $[[\beta(e),\beta(f)],\beta(e)]=-\frac{1}{2}[T_{1,1},T_{2,0}]$ to be equal to $2\beta(e)$. We calculate:

$$[[e,f],e]\sim_\beta \frac{1}{2}[\sum_i x_i\partial_i,\sum_j x_j^2]\sim_\beta \beta^{-1}\left(\frac{1}{2}T_{2,0}\right).$$

Thus we conclude $\gamma_1=0$. A similar calculation results in $\gamma_2=0$.

Now we can write $\beta(r)=\frac{T_{3,0}}{6}+\delta_1T_{1,0}+\delta_2T_{0,1}$ for $\delta_i\in\mathbb{C}[K]$ (we only need to add elements of the lower degrees which have the same parity). Let's calculate $[\beta(r),\beta(q)]$ and $[\beta(r),\beta(p)]$. To do this, we need to calculate $[T_{3,0},T_{1,0}]$ and $[T_{0,3},T_{0,1}]$. The first one is obviously zero. So we have $[\beta(r),\beta(a_1)]=\delta_2[T_{0,1},T_{1,0}]=\delta_2K$. But this commutator should be zero. Hence $\delta_2=0$. Now for the other one:

$$[T_{0,1,n},T_{3,0,n}]=\sum_{i,j}[\partial_i,x_j^3]=3T_{2,0,n},$$

Thus $[\beta(r),\beta(p)]=[\frac{T_{3,0}}{6}+\delta_1T_{1,0},T_{0,1}]=\frac{T_{2,0}}{2}-\delta_1K=-\beta(e)-\delta_1K$. Hence $\delta_1=0$. Thus we have successfully calculated the images of all the generators.

Now we need to calculate the image of $3[\mathrm{ad}_f^2\mathrm{ad}_r^2(f),\mathrm{ad}_f\mathrm{ad}_r^2(f)]-2[\mathrm{ad}_f^3\mathrm{ad}_r^2(f),\mathrm{ad}_r^2(f)]$.

Indeed, this is the only relation where both s_1 and s_2 are present. We calculate:

$$\mathrm{ad}_r(f) = -[f,r] \sim_\beta -\frac{1}{12}[\sum_i \partial_i^2, \sum_j x_j^3] = -\frac{1}{2}\sum_i (x_i^2\partial + x_i),$$

and

$$\mathrm{ad}_r^2(f) \sim_\beta \frac{1}{12}[\sum_i x_i^2\partial_i, \sum_j x_j^3] = \frac{1}{4}\sum_i x_i^4.$$

Similarly we can compute the results of the action of powers of ad_f. The differential operator part is quite straightforward, but we will write down the part depending on c in more detail. Denoting $X = \mathrm{ad}_r^2(f)$ and $\kappa = k(k+1)$, we have:

$$\mathrm{ad}_f(X) \sim_\beta \frac{1}{8}[\sum_i \partial_i^2, \sum_j x_j^4] = \sum_i \left(x_i^3\partial_i + \frac{3}{2}x_i^2 \right),$$

$$\mathrm{ad}_f^2(X) \sim_\beta \frac{1}{4}[\sum_i \partial_i^2, \sum_j (2x_j^3\partial_j + 3x_j^2)] - \frac{\kappa}{2}[\sum_{i\neq j} \frac{1}{(x_i-x_j)^2}, \sum_m x_m^3\partial_m] =$$

$$= \sum_i \left(3x_i^2\partial_i^2 + 6x_i\partial_i + \frac{3}{2} \right) - \frac{\kappa}{2}\left(\sum_{i\neq j} \frac{2x_i^3 - 2x_j^3}{(x_i-x_j)^3} \right).$$

Now, transforming the last sum, we have:

$$\sum_{i\neq j} \frac{2x_i^3 - 2x_j^3}{(x_i-x_j)^3} = \sum_{i\neq j} \frac{2x_ix_j - x_i^2 - x_j^2 + 3x_i^2 + 3x_j^2}{(x_i-x_j)^2} = 3\sum_{i\neq j} \frac{x_i^2 + x_j^2}{(x_i-x_j)^2} - n(n-1).$$

So in total we have:

$$\mathrm{ad}_f^2(X) \sim_\beta \sum_i (3x_i^2\partial_i^2 + 6x_i\partial_i) - \frac{3}{2}\kappa \sum_{i\neq j} \frac{x_i^2 + x_j^2}{(x_i-x_j)^2} + \frac{3}{2}n + \frac{\kappa n(n-1)}{2}.$$

The next one is

$$\mathrm{ad}_f^3(X) \sim_\beta \frac{1}{2}[\sum_i \partial_i^2, \sum_j (3x_j^2\partial_j^2 + 6x_j\partial_j)] - \frac{3}{4}\kappa[\sum_m \partial_m^2, \sum_{i\neq j} \frac{x_i^2 + x_j^2}{(x_i - x_j)^2}] -$$

$$-\frac{\kappa}{2}[\sum_{i\neq j} \frac{1}{(x_i - x_j)^2}, \sum_m (3x_m^2\partial_m^2 + 6x_m\partial_m)].$$

The second commutator in this formula amounts to:

$$\sum_{k,i\neq j} [\partial_k^2, \frac{x_i^2 + x_j^2}{(x_i - x_j)^2}] =$$

$$= \sum_{i\neq j} \left(2\frac{2x_i\partial_i + 2x_j\partial_j}{(x_i - x_j)^2} + 2\frac{(x_i^2 + x_j^2)(-2\partial_i + 2\partial_j)}{(x_i - x_j)^3} + \frac{2+2}{(x_i - x_j)^2} + \right.$$

$$\left. +2\frac{-4x_i + 4x_j}{(x_i - x_j)^3} + \frac{(x_i^2 + x_j^2)(6+6)}{(x_i - x_j)^4} \right) =$$

$$= 4\sum_{i\neq j} \left(\frac{(x_i + x_j)(x_i\partial_j - x_j\partial_i)}{(x_i - x_j)^3} - \frac{1}{(x_i - x_j)^2} + 3\frac{(x_i^2 + x_j^2)}{(x_i - x_j)^4} \right),$$

and the third one:

$$[\sum_{i\neq j} \frac{1}{(x_i - x_j)^2}, \sum_m (3x_m^2\partial_m^2 + 6x_m\partial_m)] =$$

$$= -3\sum_{i\neq j} \left(2\frac{-2x_i^2\partial_i + 2x_j^2\partial_j}{(x_i - x_j)^3} + \frac{6x_i^2 + 6x_j^2}{(x_i - x_j)^4} + 2\frac{-2x_i + 2x_j}{(x_i - x_j)^3} \right).$$

So the original expression amounts to:

$$\mathrm{ad}_f^3(X) \sim_\beta$$

$$\sim_\beta 3\sum_i(2x_i\partial_i^3+3\partial_i^2)+3\kappa\sum_{i\neq j}\left(\frac{2x_j^2\partial_j-2x_i^2\partial_i-(x_i+x_j)(x_i\partial_j-x_j\partial_i)}{(x_i-x_j)^3}-3\frac{1}{(x_i-x_j)^2}\right)=$$

$$=3\sum_i(2x_i\partial_i^3+3\partial_i^2)-3\kappa\sum_{i\neq j}\left(\frac{x_i\partial_j+x_j\partial_i+2x_j\partial_j+2x_i\partial_i}{(x_i-x_j)^2}+3\frac{1}{(x_i-x_j)^2}\right).$$

So, now we can finally compute the image of the relation:

$$3[\mathrm{ad}_f^2(X),\mathrm{ad}_f(X)]-2[\mathrm{ad}_f^3(X),X]\sim_\beta$$

$$\sim_\beta 3[3\sum_i(x_i^2\partial_i^2+2x_i\partial_i),\sum_j(x_j^3\partial_j+\frac{3}{2}x_j^2)]-2[3\sum_i(2x_i\partial_i^3+3\partial_i^2),\frac{1}{4}\sum_j x_j^4]-$$

$$-\frac{9}{2}\kappa[\sum_{i\neq j}\frac{x_i^2+x_j^2}{(x_i-x_j)^2},\sum_m x_m^3\partial_m]+6\kappa[\sum_{i\neq j}\frac{x_i\partial_j+x_j\partial_i+2x_i\partial_i+2x_j\partial_j}{(x_i-x_j)^2},\frac{1}{4}\sum_m x_m^4].$$

The part coming from the first two commutators is just the r.h.s. of the relation when $k=0$. It is equal to:

$$-15\cdot 3\sum_i x_i^2\sim 3\cdot 2\cdot 15 b_1,$$

as we would expect since in this case $s_1=1, s_2=0$.

The third commutator gives:

$$[\sum_{i\neq j}\frac{x_i^2+x_j^2}{(x_i-x_j)^2},\sum_m x_m^3\partial_m]=-\sum_{i\neq j}\frac{2x_i^4+2x_j^4}{(x_i-x_j)^2}+\sum_{i\neq j}\frac{2(x_i^2+x_j^2)(x_i^3-x_j^3)}{(x_i-x_j)^3}=$$

$$=2\sum_{i\neq j}\frac{(x_i^2+x_j^2)(x_i^2+x_ix_j+x_j^2)-x_i^4-x_j^4}{(x_i-x_j)^2},$$

and the forth one:

$$[\sum_{i\neq j}\frac{x_i\partial_j+x_j\partial_i+2x_i\partial_i+2x_j\partial_j}{(x_i-x_j)^2},\sum_m x_m^4]=4\sum_{i\neq j}\frac{x_ix_j^3+x_jx_i^3+2x_i^4+2x_j^4}{(x_i-x_j)^2}.$$

If we put together the formulas for the third and forth commutators in the original expression, we get:

$$3[\mathrm{ad}_f^2(X),\mathrm{ad}_f(X)]-2[\mathrm{ad}_f^3(X),X]-3\cdot 15\cdot 2b_1\sim_\beta$$

$$\sim_\beta 3\kappa\sum_{i\neq j}\frac{-3(2x_i^2x_j^2+x_i^3x_j+x_j^3x_i)+2(x_ix_j^3+x_jx_i^3+2x_i^4+2x_j^4)}{(x_i-x_j)^2}=$$

$$=3\kappa\sum_{i\neq j}\frac{4x_i^4+4x_j^4-6x_i^2x_j^2-x_ix_j^3-x_jx_i^3}{(x_i-x_j)^2}=$$

$$=3\kappa\sum_{i\neq j}\frac{4(x_i^2+x_j^2)(x_i-x_j)^2+7x_ix_j(x_i-x_j)^2}{(x_i-x_j)^2}=$$

$$=3\kappa\sum_{i\neq j}(8x_i^2+7x_ix_j)=3(8\kappa(n-1)\sum_i x_i^2+7(\sum_i x_i)^2-7\sum_i x_i^2)\sim_\beta$$

$$\sim_\beta 3\kappa(-16b_1(K-1)+7(a_1^2+2b_1)).$$

Thus we see that:

$$3[\mathrm{ad}_f^2(X),\mathrm{ad}_f(X)]-2[\mathrm{ad}_f^3(X),X]\sim_\beta 3(2(15-\kappa(8K-15))b_1+7\kappa a_1^2)=$$

$$=3((30(1+\kappa(1-K))+14\kappa K)b_1+7\kappa a_1^2).$$

And we can conclude that $s_1=1+k(k+1)(1-K)$ and $s_2=k(k+1)$. □

Remark 5.5.4.4. Note that instead of using the computer calculation from Proposition 5.5.3.2, we could have defined the map on generators by the same formula as β and checked that it satisfies the remaining relations. This is easy to do, in fact the relation we have checked is the most complicated one.

Remark 5.5.4.5. One can think about the isomorphism of Proposition 5.5.4.2 in the following way. For the Lie algebra $\mathbb{C}[x, \partial]$ there exists a standard map:

$$U(\mathbb{C}[x, \partial]) \to S^n\mathbb{C}[x, \partial] = \mathrm{Diff}(\mathbb{C}^n)^{S_n}.$$

One can deform this map to arrive at the map:

$$A_{s_1,s_2} \to \left(\mathrm{Diff}(\mathbb{C}^n)\left[\frac{1}{\prod_{1\leq i<j\leq n}(x_i - x_j)}\right]\right)^{S_n},$$

with $s_1 = 1 + k(k+1)(1-n)$ and $s_2 = k(k+1)$. These maps are given by the formulas in the polynomial representation of the Cherednik algebra, which we used in the proof of Proposition 5.5.4.2. The isomorphism β can be thought of as a certain ultraproduct of these maps.

We have another corollary:

Corollary 5.5.4.6. *The algebra $A_{1+k(k+1)(1-K),k(k+1)}$ is a flat filtered deformation of $U(\mathfrak{po})$.*

Remark 5.5.4.7. Note that via Corollary 5.3.3.5 we can also easily obtain the presentation by generators and relations of DDC-algebras $\mathcal{D}_{1,k,\nu}$.

Remark 5.5.4.8. Also see Section 4.3 of [18], where the group of transformation of parameters k, ν under which the algebra $\mathcal{D}_{1,k,\nu}$ is invariant up to an isomorphism is discussed by P. Etingof.

5.6 Deformed double current algebras for other Γ

In this section we will give an outline of the extensions of the results of the previous section from the DDCA of rank 1 and type A to the rank 1 DDCA of other types, especially type B.

5.6.1 The case of general Γ

In this section we will repeat the construction of Section 5.3 for the DDCA corresponding to arbitrary Γ. Here again for brevity we consider only the case of transcendental ν. Since the construction is literally the same upon changing $\text{Rep}(S_\nu)$ to $\text{Rep}(S_\nu \ltimes \Gamma^\nu)$, we will go over it rather quickly.

First we start with a definition.

Definition 5.6.1.1. The object $H_{t,k,c}(\nu,\Gamma)\mathbf{e} \in \text{Rep}(H_{t,k,c}(\nu,\Gamma))$ is defined to be equal to $\text{Ind}_{S_\nu \ltimes \Gamma^\nu}^{H_{t,k,c}(\nu,\Gamma)}(\Bbbk)$. It follows that $H_{t,k,c}(\nu,\Gamma)\mathbf{e} = \prod_{\mathcal{F}}^{C,r} H_{t_n,k_n,c_n}(n,\Gamma)\mathbf{e}$.

Note that assigning $\deg(V) = 1$ gives us the filtration on $H_{t,k,c}(\nu,\Gamma)\mathbf{e}$ in the same fashion as in the Definition 5.3.4.2 and the remark after it. The same filtration works in finite rank.

Now we can define the DDCA itself:

Definition 5.6.1.2. The DDC algebra $\widetilde{\mathcal{D}}_{t,k,c,\nu}(\Gamma)$ is given by:

$$\widetilde{\mathcal{D}}_{t,k,c,\nu}(\Gamma) := \text{End}_{\text{Rep}(H_{t,k,c}(\nu,\Gamma))}(H_{t,k,c}(\nu,\Gamma)\mathbf{e}) = \text{Hom}_{\text{Rep}(S_\nu \ltimes \Gamma^\nu)}(\mathbb{C}, H_{t,k,c}(\nu,\Gamma)\mathbf{e}).$$

Similarly to Proposition 5.3.1.2, we have:

Proposition 5.6.1.3. *The algebra $\widetilde{\mathcal{D}}_{t,k,c,\nu}(\Gamma)$ can be constructed as the restricted ultraproduct of spherical subalgebras $\prod_{\mathcal{F}}^{r} B_{t_n,k_n,c_n}(n,\Gamma)$ with respect to the filtrations* *mentioned after Definition 5.6.1.1.*

Remark 5.6.1.4. We can also do the same thing in the Deligne categories over $\overline{\mathbb{C}(\nu)}$ and obtain the algebra $\widetilde{\mathcal{D}}^{\text{ext}}_{t,k,c,\nu}(\Gamma)$ over $\overline{\mathbb{C}(\nu)}$.

Remark 5.6.1.5. The analogs of the results of Section 4.1.3 still hold and we can also construct the algebra $\mathcal{D}_{t,k,c}(\Gamma)$ over $\mathbb{C}$, where ν becomes a central element.

Remark 5.6.1.6. Note that we obtain the case of type A if we set $\Gamma = 1$, the trivial group. i.e., we have $\widetilde{\mathcal{D}}_{t,k,\emptyset,\nu}(1) = \widetilde{\mathcal{D}}_{t,k,\nu}$.

5.6.2 The deformed double current algebra of type B

In this section we would like to sketch some results on the presentation of the DDCA in type B by generators and relations akin to the discussion for type A in Section 5.5. Most of the results of this section were obtained through a computer computation.

First of all note that we can obtain the DDCA of type B by taking $\Gamma = \mathbb{Z}/2$.

Definition 5.6.2.1. Denote $\widetilde{\mathcal{D}}_{t,k,c,\nu} := \widetilde{\mathcal{D}}_{t,k,c,\nu}(\mathbb{Z}/2)$. Here c is just a single number, since $\mathbb{Z}/2$ has a single non-trivial conjugacy class. Define $\widetilde{\mathcal{D}}^{\text{ext}}_{t,k,c,\nu}$ and $\mathcal{D}_{t,k,c}$ in the same way.

We saw that $\mathcal{D}_{1,k}$ was a deformation of $U(\mathfrak{po})$. It turns out that a similar statement holds for type B.

Definition 5.6.2.2. By $\mathfrak{po}^+$ denote the Lie subaglebra of $\mathfrak{po}$ given by the linear combinations of even degree monomials. I.e., $\mathfrak{po}^+ = \mathfrak{po}^{\mathbb{Z}/2}$, where $\mathbb{Z}/2$ acts on $\mathfrak{po}$ by $p \mapsto -p$ and $q \mapsto -q$. This Lie algebra has an even grading restricted from the grading of $\mathfrak{po}$, and this grading is also a grading by $\mathfrak{sl}_2$-modules under the adjoint action of $\mathfrak{po}_0$.

It's now easy to see, by similar arguments, that whereas the ultraproduct of type A algebras $\mathbf{e}H_{t,k}(n)\mathbf{e}$ which are isomorphic to $\overline{\mathbb{Q}}[x_1,\ldots,x_n,y_1,\ldots,y_n]^{S_n}$ as vector spaces is a deformation of $U(\mathfrak{po})$, the ultraproduct of type B algebras $\mathbf{e}H_{t,k,c}(n)\mathbf{e}$ which are isomorphic as vector spaces to $\overline{\mathbb{Q}}[x_1,\ldots,x_n,y_1,\ldots,y_n]^{S_n\ltimes(Z/2\mathbb{Z})^n}$ is a deformation of $U(\mathfrak{po}^+)$.

Now one can also provide a presentation of $\mathfrak{po}^+$ similar to Proposition 5.5.2.6. To state such a result we need to give a few definitions.

Definition 5.6.2.3. Denote by $\mathfrak{b}$ the Lie subalgebra of $\mathfrak{po}^+$ given by $\mathfrak{po}^+_{-2}\oplus\mathfrak{po}^+_0$. The Lie subalgebra $\mathfrak{n}$ is given by $\mathfrak{po}^+_2\oplus\mathfrak{po}^+_4\oplus\ldots$. So $\mathfrak{po}^+=\mathfrak{b}\oplus\mathfrak{n}$.[5]

We will need a little more notation:

Definition 5.6.2.4. Fix an isomorpism of $\mathfrak{b}_0$ with $\mathfrak{sl}_2$ given by $e\to b_1=-\frac{q^2}{2}$ and $f\to b_3=\frac{p^2}{2}$. Fix an isomorphism of $\mathfrak{b}_{-2}$ with V_0 with the highest weight vector specified as K.

Fix an isomorphism of $\mathfrak{n}_2$ with V_4 with the highest weight vector specified as $d_1=\frac{q^4}{8}$.

Consider the free Lie algebra $L(\mathfrak{n}_2)$. Consider $\Lambda^2\mathfrak{n}_2\subset L(\mathfrak{n}_2)_4$. As $\mathfrak{sl}_2$-modules we have $\Lambda^2\mathfrak{n}_2\simeq V_6\oplus V_2$. Denote the submodule of $\Lambda^2\mathfrak{n}_2$ isomorphic to V_0 by ϕ_1' and the submodule isomorphic to V_6 by ϕ_2'. Fix an isomorphism of ϕ_1' with V_2 with the highest weight vector specified as $3d_2\wedge d_3-2d_1\wedge d_4$. Fix an isomorphism of ϕ_2' with V_6 with the highest weight vector specified as $g_1=d_2\wedge d_1$. Here $d_i=f^{i-1}d_1$.

Consider $\phi_2'\otimes\mathfrak{n}_2\subset L(\mathfrak{n}_2)_6$. We have $\phi_2'\otimes\mathfrak{n}_2\simeq V_{10}\oplus V_8\oplus V_6\oplus V_4\oplus V_2$ as $\mathfrak{sl}_2$-modules. Denote the submodule isomorphic to V_{10} by ψ_5' and the submodule isomorphic to V_4 by ψ_2'. Fix an isomorphism of ψ_2' with V_4 with the highest weight vector specified as $g_4\otimes d_1-3g_3\otimes d_2+5g_2\otimes d_3-5g_1\otimes d_4$, where $g_i=f^{i-1}g_1$.

[5]The algebras $\mathfrak{b}$ and $\mathfrak{n}$ should not be confused with their analogs from Section 4.

Now we can state how the presentation of $\mathfrak{po}^+$ looks like:

Proposition 5.6.2.5. *(see [23], Table 3.1, relations 2.1, 3.2, 3.3) The Lie algebra $\mathfrak{po}^+$ is generated by $\mathfrak{b} \oplus \mathfrak{n}_2$ with the following set of relations:*

$$\mathfrak{b}_0 \simeq \mathfrak{sl}_2, \ \mathfrak{b}_{-2} \simeq V_0 \textit{ is central}, \ \mathfrak{n}_2 \simeq V_4 \textit{ as an } \mathfrak{sl}_2\textit{-module},$$
$$\phi_1' = 0, \ \psi_5' = 0, \ \psi_2' = 0,$$

where we use fixed isomorphisms from Definition 5.6.2.4.

Via a computer calculation similar to Proposition 5.5.3.2 one can obtain a result about a certain class of flat filtered deformations of $U(\mathfrak{po}^+)$. To state that result we will need one more definition:

Definition 5.6.2.6. Consider the free associative algebra $T(\mathfrak{b} \oplus \mathfrak{n}_2)$. Note that the subspace $S^2\mathfrak{b}_0 \subset T(\mathfrak{b} \oplus \mathfrak{n}_2)$ is isomorphic to $V_4 \oplus V_0$ as a $\mathfrak{sl}_2$-module. Denote the submodule isomorphic to V_4 as α'. Fix an isomorphism of α' with V_4 with the highest weight vector specified by e^2.

Proposition 5.6.2.7. *Suppose U is a flat filtered deformation of $U(\mathfrak{po}^+)$ as an associative algebra (up to an automorphism), such that $U(\mathfrak{b})$ is still a subalgebra of U, and the action of $U(\mathfrak{b})$ on $\mathfrak{n}_2$ is not deformed. Then U is isomorphic to the algebra A_{s_1,s_2,s_3} defined below for some values of s_1, s_2 and s_3. The algebra A_{s_1,s_2,s_3} is generated by $\mathfrak{b}\oplus\mathfrak{n}_2$ with the set of relations given by the first line of Proposition 5.6.2.5 and the following relations, which substitute the last line in Proposition 5.6.2.5:*

$$\phi_1' \simeq 6s_1\mathfrak{b}_0, \ \psi_5' = 0,$$
$$\psi_2' \simeq 24(s_3\alpha' + 12s_2\mathfrak{n}_2),$$

where the notation used is understood in the same way as in Proposition 5.5.3.2.

Now we can state the result about the DDC-algebra of type B.

Proposition 5.6.2.8. *The DDC-algebra $\mathcal{D}_{1,k,c}$ is a flat filtered deformation of $U(\mathfrak{po}^+)$. It is isomorphic to A_{s_1,s_2,s_3} with*

$$s_1 = 4k(k+1)K+\lambda^2-4(k^2+k+1),\ s_2 = 4k(k+1)K+\lambda^2-9(k^2+k+1),\ s_3 = k(k+1),$$

where $\lambda := c + \frac{1}{2}$.

Remark 5.6.2.9. Notice that in the same way as $\mathbb{C}[x, \partial]$ is a flat filtered deformation of $\mathfrak{po}$, Feigin's Lie algebra $\mathfrak{gl}(\lambda) := U(\mathfrak{sl}_2)/(C = \frac{\lambda^2-1}{2})$ (where $C := ef + fe + \frac{h^2}{2}$ is the Casimir) introduced in [22] is a flat filtered deformation of $\mathfrak{po}^+$. More precisely we have $U(\mathfrak{gl}(\lambda)) \simeq \widetilde{\mathcal{D}}_{1,0,\lambda-\frac{1}{2},\nu}$ (for any ν, as this algebra does not depend of ν); indeed, it is easy to see looking at the relations that the deformation $\widetilde{\mathcal{D}}_{1,0,\lambda-\frac{1}{2},\nu}$ arises from the most general deformation of $\mathfrak{po}^+$ as a (filtered) Lie algebra. These relations are given in [23], at the beginning of Table 3.1. For more information about deformations of $\mathfrak{po}^+$ and $\mathfrak{gl}(\lambda)$ see [44].

Note that the parameters s_1, s_2, s_3 in Proposition 5.6.2.8 are not independent: we have

$$s_1 - s_2 = 5(s_3 + 1). \tag{5.4}$$

This is, however, the most general deformation because the parameters s_1, s_2, s_3 are homogeneous of degrees $4, 4, 6$, hence can be rescaled by $s_1 \mapsto z^2 s_1,\ s_2 \mapsto z^2 s_2,\ s_3 \mapsto z^3 s_3$ without changing the algebra. This implies

Corollary 5.6.2.10. *The deformation A_{s_1,s_2,s_3} is flat for all s_1, s_2, s_3.*

Remark 5.6.2.11. For the discussion of the the group of transformation of parameters of $\widetilde{\mathcal{D}}_{1,k,\nu}$ under which the algebra is invariant up to an isomorphism see Section 5.2 of [18], where it is explored by P. Etingof. There the fact that through the interpolation of spherical Cherednik algebras of type B one parameter is lost is also explained.

Appendix A

Appendix: Direct calculation of generators and relations for $\mathfrak{po}$

Here we would like to give a direct proof of Proposition 5.5.2.3. As we mentioned in the main text, we have a surjective map $\pi : L(\mathfrak{n}_1) \to \mathfrak{n}$. Let us denote the ideal generated by ϕ_1, ψ_4, ψ_1 and χ_1 as $I \subset L(\mathfrak{n}_1)$. It's easy to see that this ideal is in the kernel of π. Indeed, to conclude that we only need to know that the generators of $\mathfrak{n}_1$ satisfy the four last relations of Proposition 5.5.2.7, which is a straightforward calculation. Let us denote the quotient $L(\mathfrak{n}_1)/I$ by $\mathfrak{l}$. So we have a surjective map $\pi' : \mathfrak{l} \to \mathfrak{n}$. We only need to prove that π' is injective.

Proposition A.0.0.1. *The map π' just described is an isomorphism.*

Proof. Both algebras have a natural grading given by assigning $\mathfrak{n}_1$ to have degree 1. We will prove that π' is an isomorphism by induction.

It will be easier for us to begin with the induction step. I.e., we will prove that if π' is an isomorphism for all degrees up to $l-2$ (with $l \geq 6$), then it is an isomorphism for $l-1$. We will prove the base of induction (i.e., the fact that π' is an isomorphism for degrees 2, 3 and 4) later, using the general formulas we derived.

So, suppose we know that up to $l-2$ we have $\mathfrak{l}_j \simeq \mathfrak{n}_j \simeq V_{j+2}$. It means that if we want to show that a certain element of $L(\mathfrak{n}_1)$ is in I, we can freely commute elements with total degree $\leq j$ as though they were the elements of $\mathfrak{po}$. Indeed, this will only add to our elements something which is already contained in I. Let us denote the highest weight vector of $\mathfrak{l}_j$ by v_1^{j+2}, which corresponds to q^{j+2} under the above isomorphism (i.e. $\mathfrak{l}_j \simeq \mathfrak{n}_j \simeq V_{j+2}$). We set $v_i^j = f^{i-1}v_1^j$.

Now we know that $\mathfrak{l}_{l-1}$ is a quotient of $\mathfrak{l}_{l-2} \otimes \mathfrak{n}_1 \simeq V_{l+3} \oplus V_{l+1} \oplus V_{l-1} \oplus V_{l-3}$, i.e., we have a surjective map $\xi_{l-1} : V_{l+3} \oplus V_{l+1} \oplus V_{l-1} \oplus V_{l-3} \to \mathfrak{l}_{l-1}$. We only need to prove that $\xi_{l-1}(V_{l+3} \oplus V_{l-1} \oplus V_{l-3}) = 0$.

We would like to describe the highest weight vectors of the simple $\mathfrak{sl}_2$-modules in the decomposition of $\mathfrak{l}_{l-2} \otimes \mathfrak{n}_1$ explicitly. To do so, it is enough to find the vectors of the required weight which are annihilated by the action of e. It is easy to see that the highest weight vector of V_{l+3} is proportional to $v_1^l \otimes c_1$; the highest weight vector of V_{l+1} to $3v_2^l \otimes c_1 - lv_1^l \otimes c_2$; the highest weight vector of V_{l-1} to

$$6v_3^l \otimes c_1 - 4(l-1)v_2^l \otimes c_2 + l(l-1)v_1^l \otimes c_3;$$

the highest weight vector of V_{l-3} to

$$6v_4^l \otimes c_1 - 6(l-2)v_3^l \otimes c_2 + 3(l-2)(l-1)v_2^l \otimes c_3 - l(l-1)(l-2)v_1^l \otimes c_4.$$

Writing the highest weight vectors in this way allows us to write the action of ξ_{l-1} in a straightforward way, i.e., $y \otimes x \in \mathfrak{l}_{l-2} \otimes \mathfrak{n}_1$ is mapped into $\xi_{l-1}(y \otimes x) \to [y, x]$.

Now we need to prove that each of the highest weight vectors corresponding to V_{l+3}, V_{l-1} and V_{l-3} belongs to I, i.e., maps to zero under ξ_{l-1}. Let us start with V_{l+3}.

Now we know that $[v_2^{l-2}, [d_1, c_1]]$ is in I. If we transform this expression using the commutator formulas in $\mathfrak{po}$ for elements of degree less or equal than $l-2$, we will stay in I by the induction assumption. So, in $\mathfrak{l}_{l-1}$ we have:

$$0 = [v_2^{l-2}, [d_1, c_1]] = [[v_2^{l-2}, d_1], c_1] - [[v_2^{l-2}, c_1], d_1] = (l-2)[v_1^l, c_1] - \frac{1}{2}(l-2)[v_1^{l-1}, d_1],$$

where we have calculated $[v_2^{l-2}, d_1]$ and $[v_2^{l-2}, c_1]$ in $\mathfrak{po}$ as we've discussed before the formula. Now we also express $d_1 = [c_2, c_1]$ and get:

$$[v_1^{l-1}, d_1] = [[v_1^{l-1}, c_2], c_1] - [[v_1^{l-1}, c_1], c_2] = -\frac{1}{2}(l-1)[v_1^l, c_1].$$

So we conclude that:

$$0 = (l-2)(l+3)[v_1^l, c_1] = (l-2)(l+3)\xi_{l-1}(v_1^l \otimes c_1),$$

which is proportional to the image of the highest weight vector of V_{l+3}, and since $l \geq 6$, it follows that it is indeed zero.

We use a similar method for two other highest weight vectors. Starting with:

$$0 = [v_1^{l-1}, [c_1, c_4] - [c_2, c_3]],$$

we get:

$$0 = 6[v_3^l, c_1] - 4(l-1)[v_2^l, c_2] + (l-1)l[v_1^l; c_3] = \xi_{l-1}(6v_3^l \otimes c_1 - 4(l-1)v_2^l \otimes c_2 + l(l-1)v_1^l \otimes c_3),$$

which is the highest weight vector of V_{l-1}.

To deal with the highest weight vector of V_{l-3} we start with:

$$0 = [v_2^{l-2}, [d_4, c_1] - 2[d_3, c_2] + 3[d_2, c_3] - 4[d_1, c_4]].$$

After a similar calculation we get:

$$0 = \xi_{l-1}(12(44 - 16l)v_4^l \otimes c_1 + 12(l-2)(11l - 35)v_3^l \otimes c_2 + \tag{A.1}$$
$$+12(l-2)(l-1)(13 - 3l)v_2^l \otimes c_3 + l(l-1)(l-2)(2l - 34)v_1^l \otimes c_4),$$

denote this element of $\mathfrak{l}_{l-2} \otimes \mathfrak{n}_1$ in brackets by α_1. We know that α_1 is of $\mathfrak{sl}_2$-weight $l-3$, and we know that it belongs to the kernel of $\pi' \circ \xi_{l-1}$. Hence it is the element of the submodule isomorphic to $V_{l+3} \oplus V_{l-1} \oplus V_{l-3}$. Denote by α_2 the result of the action by e^3 on the highest weight vector of V_{l+3} and by α_3 the result of the action of e on the highest weight vector of V_{l+1}. We have:

$$\alpha_2 = v_4^l \otimes c_1 + 3v_3^l \otimes c_2 + 3v_2^l \otimes c_3 + v_1^l \otimes c_4,$$

and:

$$\alpha_3 = 6v_4^l \otimes c_1 + (10 - 4l)v_3^l \otimes c_2 + (l-1)(l-4)v_2^l \otimes c_3 + l(l-1)v_1^l \otimes c_4.$$

Now if α_1 is linearly independent of α_2 and α_3, then the highest weight vector of V_{l-3} lies in the linear span of α_i, and since $\xi_{l-1}(\alpha_i) = 0$ it follows that ξ_{l-1} acts on the highest weight vector by zero.

But calculating the roots of the minors of the matrix given by the coordinates of α_i, we see that the common roots are only $l = -1, -2, 5$. So, since in our case $l \geq 6$, we are done, and π' is an isomorphism in the degree $l - 1$.

Now we can prove the base of induction, i.e., the degrees $2, 3, 4$.

Let us begin with $\mathfrak{l}_2$. We have $L(\mathfrak{n}_1) = \Lambda^2\mathfrak{n}_1 = \phi_2 \oplus \phi_1$ ($V_0 \oplus V_4$ as $\mathfrak{sl}_2$-modules) and $I_2 = \phi_1$. So we see that $\mathfrak{l}_2 = \phi_2 \simeq V_4$, which has the same dimension as $\mathfrak{n}_2$. So π' must be an isomorphism in degree 2. Note that it also follows that the minimal set of relations must contain ϕ_1.

Now we deal with $\mathfrak{l}_3$. We have a surjective map

$$\xi_3 : \mathfrak{l}_2 \otimes \mathfrak{n}_1 \simeq V_7 \oplus V_5 \oplus V_3 \oplus V_1 \to \mathfrak{l}_3.$$

Here we have a part of I_3 generated by I_2, i.e., we have $\phi_1 \otimes \mathfrak{n}_1 \simeq V_3 \to I_3$. But the consequence of this relation was exactly calculated by us in the general case, when we used that $0 = [v_1^{l-1}, [c_1, c_4] - [c_2, c_3]]$. As was shown there, this leads to the conclusion that V_3 is in the kernel of ξ_3. However, we cannot kill anything else using only the relation ϕ_1. But $V_7 \oplus V_0$ are precisely ψ_4 and ψ_1, hence they lie in I_3 and in the kernel of ξ_3. So $\mathfrak{l}_3$ has the same dimension as $\mathfrak{n}_3$ and π' is an isomorphism. Note that this also shows that the minimal set of relations must contain ψ_1, ψ_4.

To finish we need to consider $\mathfrak{l}_4$. As before, we have a surjective map

$$\xi_4 : \mathfrak{l}_3 \otimes \mathfrak{n}_1 \simeq V_8 \oplus V_6 \oplus V_4 \oplus V_2 \to \mathfrak{l}_4.$$

The general formulas from the induction step allow us to conclude that $\xi_4(V_8 \oplus V_4) = 0$. Now we need to deal with V_2. However, as we can see from the general formulas, α_1 defined in Equation A.1 becomes linearly dependent with α_2 and α_3 in degree 4. Indeed, it turns out that V_2 does not belong to the ideal generated by ψ_1, ψ_4 and ϕ_0.

We see that all we can generate by ϕ_0 in degree 4 is given by $\phi_0 \otimes \Lambda^2\mathfrak{n}_1 \simeq V_0 \oplus V_4$, so it does not contain anything isomorphic to V_2. All we can generate by ψ_4 is $\psi_4 \otimes \mathfrak{n}_1 \simeq V_{10} \oplus V_8 \oplus V_6 \oplus V_4$, so it does not contain anything isomorphic to V_2. So the only chance to kill V_2 is $\psi_1 \otimes \mathfrak{n}_1 = V_4 \oplus V_2$. But using our calculation (and similar ones) it follows that this doesn't kill V_2 in $\mathfrak{l}_3 \otimes \mathfrak{n}_1$.

But the relation χ_1 takes care of it. So it follows both that $\mathfrak{l}_4$ is isomorphic to $\mathfrak{n}_4$ under π' and that the minimal set of relations must contain χ_1.

$\square$

Bibliography

[1] S. Bouarroudj, P. Grozman, A. Lebedev, and D. Leites. Derivations and central extensions of symmetric modular Lie algebras and superalgebras. *arXiv preprint arXiv:1307.1858*, 2013.

[2] S. Bouarroudj, P. Grozman, A. Lebedev, D. Leites, and I. Shchepochkina. Simple vectorial lie algebras in characteristic 2 and their superizations. *arXiv preprint arXiv:1510.07255*, 2015.

[3] S. Bouarroudj, P. Grozman, and D. Leites. Deforms of symmetric simple modular Lie (super) algebras. *arXiv preprint arXiv:0807.3054*, 2008.

[4] J. Comes and V. Ostrik. On blocks of Deligne's category $\mathrm{Rep}(S_t)$. ***Adv. Math.***, 226(2):1331–1377, 2011.

[5] J. Comes and V. Ostrik. On Deligne's category $\mathrm{Rep}^{ab}(S_d)$. ***Algebra Number Theory***, 8(2):473–496, 2014.

[6] J. Comes and B. Wilson. Deligne's category $\mathrm{Rep}(GL_\delta)$ and representations of general linear supergroups. ***Represent. Theory***, 16(16):568–609, 2012.

[7] P. Deligne. La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel. *Tata Institute of Fundamental Research Publications*, 11:209–273, 2007.

[8] P. Deligne and J. Milne. Tannakian categories. In *Hodge cycles, motives, and Shimura varieties*, pages 101–228. Springer, 1982.

[9] J. Dixon and B. Mortimer. *Permutation groups*, volume 163 of ***Grad. Texts in Math.*** 1996.

[10] I. Entova-Aizenbud. On representations of rational Cherednik algebras of complex rank. ***Represent. Theory***, 18(12):361–407, 2014.

[11] I. Entova-Aizenbud. Schur-Weyl duality for Deligne categories. ***Int. Math. Res. Not. IMRN***, 2015(18):8959–9060, 2014.

[12] I. Entova-Aizenbud, V. Hinich, and V. Serganova. Deligne categories and the limit of categories $\mathrm{Rep}(GL(m|n))$. *International Mathematics Research Notices*, 2020(15):4602–4666, 2020.

[13] P. Etingof. Representation theory in complex rank, I. ***Transform. Groups***, 19(2):359–381, 2014.

[14] P. Etingof. Representation theory in complex rank, II. ***Adv. Math.***, 300:473–504, 2016.

[15] P. Etingof. Semisimple and G-Equivariant Simple Algebras Over Operads. ***Appl. Categ. Structures***, 25(6):965–969, 2017.

[16] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor categories*, volume 205 of ***Math. Surveys Monogr.*** 2016.

[17] P. Etingof and V. Ginzburg. Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. ***Invent. Math.***, 147(2):243–348, 2002.

[18] P. Etingof, D. Kalinov, and E. Rains. New realizations of deformed double current algebras and Deligne categories. *arXiv preprint arXiv:2005.13604*, 2020.

[19] P. Etingof, V. Krylov, I. Losev, and J. Simental. Representations with minimal support for quantized Gieseker varieties. *arXiv preprint arXiv:2002.06741*, 2020.

[20] P. Etingof and X. Ma. Lecture notes on Cherednik algebras. *arXiv preprint arXiv:1001.0432*, 2010.

[21] B. Feigin and D. Fuks. Homology of the Lie algebra of vector fields on the line. ***Funct. Anal. Appl.***, 14(3):201–212, 1980.

[22] B. L. Feigin. The Lie algebras $\mathfrak{gl}(\lambda)$ and cohomologies of Lie algebras of differential operators. ***Russian Math. Surveys***, 43(2):169, 1988.

[23] P. Grozman and D. Leites. Defining relations associated with the principal sl(2)-subalgebras of simple lie algebras. ***In: Dobrushin R., Minlos R., Shubin M. and Vershik A. (eds.) Contemporary Mathematical Physics (F.A. Berezin memorial volume), Amer. Math. Soc. Transl. Ser. 2***, 175:57–68, 1996.

[24] N. Guay. Cherednik algebras and Yangians. ***Int. Math. Res. Not. IMRN***, 2005(57):3551–3593, 2005.

[25] N. Guay. Affine Yangians and deformed double current algebras in type A. ***Adv. Math.***, 211(2):436–484, 2007.

[26] N. Guay. Quantum algebras and symplectic reflection algebras for wreath products. ***Represent. Theory***, 14(4):148–200, 2010.

[27] N. Guay, D. Hernandez, and S. Loktev. Double affine Lie algebras and finite groups. ***Pacific J. Math.***, 243(1):1–41, 2009.

[28] N. Guay and Y. Yang. On deformed double current algebras for simple Lie algebras. ***Math. Res. Lett.***, 25(5):1307–1384, 2017.

[29] B. Hall. *Quantum theory for mathematicians*, volume 267 of ***Grad. Texts in Math.*** 2013.

[30] N. Harman. Deligne categories as limits in rank and characteristic. *arXiv preprint arXiv:1601.03426*, 2016.

[31] N. Harman and D. Kalinov. Classification of simple algebras in the Deligne category $\mathrm{Rep}(S_t)$. ***J. Algebra***, 549:215–248, 2020.

[32] V. Kac. Lie superalgebras. ***Adv. Math.***, 26(1):8–96, 1977.

[33] D. Kalinov. Finite-dimensional representations of Yangians in complex rank. ***Int. Math. Res. Not. IMRN***, 2019.

[34] D. Kalinov. Deformed double current algebras via deligne categories. *arXiv preprint arXiv:2101.08317*, 2021.

[35] D. Kalinov and A. Mandelshtam. Sums of binomial coefficients evaluated at $\alpha \in \overline{\mathbb{Q}}$, and applications. *arXiv preprint arXiv:2010.14494*, 2020.

[36] A. Kleshchev and P.H. Tiep. Small-dimensional projective representations of symmetric and alternating groups. ***Algebra Number Theory***, 6(8):1773–1816, 2012.

[37] F. Knop. A construction of semisimple tensor categories. ***C. R. Math. Acad. Sci. Paris***, 343(1):15–18, 2006.

[38] F. Knop. Tensor envelopes of regular categories. ***Adv. Math.***, 214(2):571–617, 2007.

[39] D. Leites. Towards classification of simple finite dimensional modular lie superalgebras. ***J. Prime Res. Math.***, 3(2):1–10, 2007.

[40] D. Leites, S. Bouarroudj, and P. Grozman. Classification of finite dimensional modular lie superalgebraswith indecomposable Cartan matrix. ***SIGMA Symmetry Integrability Geom. Methods Appl.*** , 5:1–63, 2009.

[41] A. Mathew. Categories parametrized by schemes and representation theory in complex rank. ***J. Algebra***, 381:140–163, 2013.

[42] M. Mori. On representation categories of wreath products in non-integral rank. ***Adv. Math.***, 231(1):1–42, 2012.

[43] I. Musson. *Lie superalgebras and enveloping algebras*, volume 131 of ***Grad. Stud. Math.*** 2012.

[44] G. Post and N. van den Hijligenberg. $\mathfrak{gl}(\lambda)$ and differential operators preserving polynomials. ***Acta Appl. Math.***, 44(1-2):257–268, 1996.

[45] M. P. Purdon. *Potts models and related problems in statistical mechanics*, volume 5 of *Series on Advances in Statistical Mechanics*. World Scientific, 1991.

[46] C. Ryba. The structure of the grothendieck rings of wreath product deligne categories and their generalisations. *International Mathematics Research Notices*, 2018.

[47] C. Ryba. Indecomposable objects of rep_ (glt) in terms of exterior powers of the tautological object and its dual. *Journal of Algebra*, 557:165–171, 2020.

[48] H. Schoutens. *The use of ultraproducts in commutative algebra*, volume 1999 of ***Lecture Notes in Math.*** 2010.

[49] L. Sciarappa. Simple commutative algebras in Deligne's categories $\mathrm{Rep}(S_t)$. *arXiv preprint arXiv:1506.07565*, 2015.

[50] G. Seligman. On automorphisms of Lie algebras of classical type. II. ***Trans. Amer. Math. Soc.***, 94(3):452–482, 1960.

[51] E. Steinitz. Algebraische Theorie der Körper. ***J. Reine Angew. Math.***, 137:167–309, 1910.

[52] H. Strade. Simple Lie algebras over fields of positive characteristic. volume I, 2004.

[53] V. Toledano-Laredo and Y. Yang. The elliptic casimir connection of a simple lie algebra. *arXiv preprint arXiv:1805.12261*, 2018.

[54] A. Utiralova. Olshanski's centralizer construction and Deligne tensor categories. *arXiv preprint arXiv:1901.08370*, 2019.

[55] A. Utiralova. Harish-Chandra bimodules in the Deligne category $\mathrm{Rep}(GL_t)$. *arXiv preprint arXiv:2002.01555*, 2020.

[56] N. Van den Hijligenberg and G. Post. Defining relations for Lie algebras of vector fields. ***Indag. Math. (N.S.)***, 2(2):207–218, 1991.

[57] R. L. Wilson. Classification of generalized Witt algebras over algebraically closed fields. ***Trans. Amer. Math. Soc.***, 153:191–210, 1971.