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REPRESENTATIONS OF REDUCTIVE GROUPS OVER FINITE RINGS

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Introduction

0.1. In [L, Sec.4] a cohomological construction was given (without proof) for certain representations of a Chevalley group over a finite ring R (arising from the ring of integers in a non-archimedean local field by reduction modulo a power of the maximal ideal); that construction was an extension of the construction of the virtual representations R_T^θ in [DL] for groups over a finite field. One of the aims of this paper is to provide the missing proof. For simplicity we will assume that $R = \mathbf{F}_{q,r} = \mathbf{F}_q[[\epsilon]]/(\epsilon^r)$ (ϵ is an indeterminate, $\mathbf{F}_q$ is a finite field with q elements and $r \ge 1$). The general case requires only minor modifications. On the other hand, we treat possibly twisted groups.

Let $\mathbf{F}$ be an algebraic closure of $\mathbf{F}_q$. Let G be a connected reductive algebraic group defined over $\mathbf{F}$ with a given $\mathbf{F}_q$-rational structure with associated Frobenius map $F : G \to G$.

Using a cohomological method, extending that of [DL], we will construct a family of irreducible representations of the finite group $G(\mathbf{F}_{q,r}), r \ge 1$, attached to a "maximal torus" and a character of it in general position. In the case where $r \ge 2$ and G is split over $\mathbf{F}_q$, the representations that we construct are likely to be the same as those found by Gérardin [G] by a non-cohomological method (induction from a subgroup if r is even; induction from a subgroup in combination with a use of a Weil representation, if r is odd, ≥ 3). In any case, since for $r = 1$, the cohomological construction is the only known construction of the generic representations, it seems natural to seek a cohomological construction which works uniformly for all $r \ge 1$; this is what we do in this paper.

In contrast with the case $r = 1$, for $r \ge 2$ not all irreducible representations of $G(\mathbf{F}_{q,r})$ appear in the virtual representations that we construct. The study of SL_2 with $r = 2$ (see §3) suggests that, to remedy this, one has to consider also virtual representations attached to double cosets with respect to a "Borel subgroup" other than those indexed by the Weyl group.

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0.2. Notation. Let ϵ be an indeterminate. If X is an affine algebraic variety over $\mathbf{F}$ and $r \ge 1$, we set $X_r = X(\mathbf{F}[[\epsilon]]/(\epsilon^r))$. Thus, if X is the set of common zeroes of the polynomials $f_i : \mathbf{F}^N \to \mathbf{F} (i = 1, \dots, m)$, then X_r is the set of all $(x_1, x_2, \dots, x_N) \in (\mathbf{F}[[\epsilon]]/(\epsilon^r))^N$ such that $f_i(x_1, x_2, \dots, x_N)$ (a priori an element of $\mathbf{F}[[\epsilon]]/(\epsilon^r)$) is equal to 0 for $i = 1, \dots, m$. We have $X_1 = X$. For $r = 0$ we set $X_r =$point. Then $X \mapsto X_r$ is a functor from the category of algebraic varieties over $\mathbf{F}$ into itself. If X' is a closed subvariety of X then X'_r is a closed subvariety of X_r. If X is irreducible of dimension d then X_r is irreducible of dimension dr. For any $r \ge r' \ge 0$ we have a canonical morphism $\phi_{r,r'} : X_r \to X_{r'}$. If $r \ge 1$, we have naturally $X \subset X_r$ (using $\mathbf{F} \subset \mathbf{F}[[\epsilon]]/(\epsilon^r)$). If G is an algebraic group over $\mathbf{F}$ then G_r is naturally an algebraic group over $\mathbf{F}$. For any $r \ge r' \ge 0$, $\phi_{r,r'} : G_r \to G_{r'}$ is a homomorphism of algebraic groups hence its kernel, $G_r^{r'}$, is a normal subgroup of G_r. For $r \ge 1$ we have naturally $G \subset G_r$. We have

$$\{1\} = G_r^r \subset G_r^{r-1} \subset \dots G_r^1 \subset G_r^0 = G_r.$$

For $r > r' \ge 0$, we set $G_r^{r',*} = G_r^{r'} - G_r^{r'+1}$. We have a partition

$$G_r = G_r^{0,*} \sqcup G_r^{1,*} \sqcup \dots \sqcup G_r^{r-1,*} \sqcup \{1\}.$$

We fix a prime number l invertible in $\mathbf{F}$. If X is an algebraic variety over $\mathbf{F}$ we write $H_c^j(X)$ istead of $H_c^j(X, \bar{\mathbf{Q}}_l)$.

For a finite group Γ let $\hat{\Gamma} = \mathrm{Hom}(\Gamma, \bar{\mathbf{Q}}_l^*)$.

0.3. If T is a commutative algebraic group over $\mathbf{F}$ with a fixed $\mathbf{F}_q$-structure and with Frobenius map $F : T \to T$ we have a norm map
$N_F^{F^n} : T^{F^n} \to T^F, t \mapsto tF(t)F^2(t) \dots F^{n-1}(t)$.

1. Lemmas

Lemma 1.1. *Let $\mathcal{T}, \mathcal{T}'$ be two commutative, connected algebraic groups over $\mathbf{F}$ with fixed $\mathbf{F}_q$-rational structures with Frobenius maps $F : \mathcal{T} \to \mathcal{T}, F : \mathcal{T}' \to \mathcal{T}'$. Let $f : \mathcal{T} \xrightarrow{\sim} \mathcal{T}'$ be an isomorphism of algebraic groups over $\mathbf{F}$. Let $n \ge 1$ be such that $F^n f = f F^n : \mathcal{T} \to \mathcal{T}'$; thus $f : \mathcal{T}^{F^n} \xrightarrow{\sim} \mathcal{T}'^{F^n}$. Let*

$$H = \{(t, t') \in \mathcal{T} \times \mathcal{T}'; f(F(t)^{-1}t) = F(t')^{-1}t'\}.$$

(A subgroup of $\mathcal{T} \times \mathcal{T}'$ containing $\mathcal{T}^F \times \mathcal{T}'^F$.) Let $\theta \in \widehat{\mathcal{T}^F}$, $\theta' \in \widehat{\mathcal{T}'^F}$ be such that $\theta^{-1} \boxtimes \theta'$ is trivial on $(\mathcal{T}^F \times \mathcal{T}'^F) \cap H^0$. Then $\theta N_F^{F^n} = \theta' N_F^{F^n} f \in \widehat{\mathcal{T}^{F^n}}$.

Setting $t_1 = tF(t) \dots F^{n-1}(t) \in \mathcal{T}$, $\quad t_2 = f(t)F(f(t)) \dots F^{n-1}(f(t)) \in \mathcal{T}'$ for $t \in \mathcal{T}$, we have

$$f(F(t_1)^{-1}t_1) = f(tF^n(t)^{-1}) = f(t)f(F^n(t))^{-1} = f(t)F^n(f(t))^{-1} = F(t_2)^{-1}t_2,$$

so that $(t_1, t_2) \in H$. Now $t \mapsto (t_1, t_2)$ is a morphism $\mathcal{T} \to H$ of algebraic varieties and $\mathcal{T}$ is connected; hence the image of this morphism is contained in H^0. In particular, if $t \in \mathcal{T}^{F^n}$ we have $(N_F^{F^n}(t), N_F^{F^n}(f(t))) \in (\mathcal{T}^F \times \mathcal{T}'^F) \cap H^0$ hence, by assumption, $\theta^{-1}(N_F^{F^n}(t))\theta'(N_F^{F^n}(f(t))) = 1$ for all $t \in \mathcal{T}^{F^n}$. The lemma is proved.

1.2. Let G be a connected reductive algebraic group over $\mathbf{F}$ with a fixed $\mathbf{F}_q$-rational structure with Frobenius map $F : G \to G$. If $r \geq 1$ then $F : G \to G$ induces a homomorphism $F : G_r \to G_r$ which is the Frobenius map for a $\mathbf{F}_q$-rational structure on G_r.

Let T, T' be two F-stable maximal tori of G and let U (resp. U') be the unipotent radical of a Borel subgroup of G that contains T (resp. T'). Note that U, U' are not necessarily defined over $\mathbf{F}_q$. Let $r \geq 2$. Let $\mathcal{T} = T_r^{r-1}, \mathcal{T}' = T_r'^{r-1}$,

$$\Sigma = \{(x, x', y) \in F(U_r) \times F(U'_r) \times G_r; xF(y) = yx'\}.$$

Let $N(T, T') = \{\nu \in G; \nu^{-1}T\nu = T'\}$. Then T acts on $N(T, T')$ by left multiplication and T' acts on $N(T, T')$ by right multiplication. The orbits of T are the same as the orbits of T'; we set $W(T, T') = T\backslash N(T, T') = N(T, T')/T'$ (a finite set). For each $w \in W(T, T')$ we choose a representative $\dot{w}$ in $N(T, T')$. We have $G = \sqcup_{w \in W(T,T')} G_w$ where $G_w = UT\dot{w}U' = U\dot{w}T'U'$.

Let $G_{w,r}$ be the inverse image of G_w under $\phi_{r,1} : G_r \to G$ and let $\Sigma_w = \{(x, x', y) \in \Sigma; y \in G_{w,r}\}$.

Now $T_r^F \times T_r'^F$ acts on Σ by $(t, t') : (x, x', y) \mapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$. This restricts to an action of $T_r^F \times T_r'^F$ on Σ_w for any $w \in W$.

If $\theta \in \widehat{T_r^F}, \theta' \in \widehat{T_r'^F}$ and M is a $T_r^F \times T_r'^F$-module, we shall write $M_{\theta^{-1},\theta'}$ for the subspace of M on which $T_r^F \times T_r'^F$ acts according to $\theta^{-1} \boxtimes \theta'$.

Lemma 1.3. *Assume that* $r \geq 2$. *Let* $w \in W(T, T')$. *Let* $\theta \in \widehat{T_r^F}$, $\theta' \in \widehat{T_r'^F}$. *Assume that* $H_c^j(\Sigma_w)_{\theta^{-1},\theta'} \neq 0$ *for some* $j \in \mathbf{Z}$. *Let* $g = F(\dot{w})^{-1}$ *and let* $n \geq 1$ *be such that* $g \in G^{F^n}$. *Then* $\mathrm{Ad}(g)$ *carries* $\mathcal{T}^{F^n}$ *onto* $\mathcal{T}'^{F^n}$ *and* $\theta|_{\mathcal{T}^F} \circ N_F^{F^n} \in \widehat{\mathcal{T}^{F^n}}$ *to* $\theta'|_{\mathcal{T}'^F} \circ N_F^{F^n} \in \widehat{\mathcal{T}'^{F^n}}$.

By the definition of $G_{w,r}$, the map $U_r \times G_r^1 \times (T_r\dot{w}) \times U'_r \to G_{w,r}$ given by $(u, k, \nu, u') \mapsto uk\nu u'$ is a locally trivial fibration with all fibres isomorphic to a fixed affine space. Hence the map

$$\tilde{\Sigma}_w = \{(x, x', u, u', k, \nu) \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times G_r^1 \times T_r\dot{w};$$
$$xF(u)F(k)F(\nu)F(u') = uk\nu u'x'\} \to \Sigma_w$$

given by $(x, x', u, u', k, \nu) \mapsto (x, x', uk\nu u')$, is a locally trivial fibration with all fibres isomorphic to a fixed affine space. This map is compatible with the $T_r^F \times T_r'^F$ actions where $T_r^F \times T_r'^F$ acts on $\tilde{\Sigma}_w$ by

(a) $\quad (t, t') : (x, x', u, u', k, \nu) \mapsto (txt^{-1}, t'x't'^{-1}, tut^{-1}, t'u't'^{-1}, tkt^{-1}, t\nu t'^{-1}).$

Hence there exists $j' \in \mathbf{Z}$ such that $H_c^{j'}(\tilde{\Sigma}_w)_{\theta^{-1},\theta'} \neq 0$. By the substitution $xF(u) \mapsto x, x'F(u')^{-1} \mapsto x'$, the variety $\tilde{\Sigma}_w$ is rewritten as

(b)

$\{(x, x', u, u', k, \nu) \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times G_r^1 \times T_r\dot{w}; xF(k)F(\nu) = uk\nu u'x'\};$

in these coordinates, the action of $T_r^F \times T_r'^F$ is still given by (a). Let

$$H = \{(t,t') \in \mathcal{T} \times \mathcal{T}'; t'F(t')^{-1} = F(\dot{w})^{-1}tF(t)^{-1}F(\dot{w})\}.$$

(A closed subgroup of $T_r \times T_r'$.) It acts on the variety (b) by the same formula as in (a). (We use the fact that $hk = kh$ for any $h \in G_r^{r-1}, k \in G_r^1$.) By [DL, 6.5], the induced action of H on $H_c^{j'}(\tilde{\Sigma}_w)$ is trivial when restricted to H^0. In particular, the intersection $(T_r^F \times T_r'^F) \cap H^0$ acts trivially on $H_c^{j'}(\tilde{\Sigma}_w)$. Since $H_c^{j'}(\tilde{\Sigma}_w)_{\theta^{-1},\theta'} \neq 0$, it follows that $\theta^{-1} \boxtimes \theta'$ is trivial on $(T_r^F \times T_r'^F) \cap H^0$. Let $g = F(\dot{w})^{-1}$ and let $n \geq 1$ be such that $g \in G^{F^n}$. Then $\mathrm{Ad}(g)$ carries $\mathcal{T}^{F^n}$ onto $\mathcal{T}'^{F^n}$ and (by Lemma 1.1 with $f = \mathrm{Ad}(g)$) it carries $\theta|_{\mathcal{T}^F} \circ N_F^{F^n}$ to $\theta'|_{\mathcal{T}'^F} \circ N_F^{F^n}$. The lemma is proved.

Lemma 1.4. *Assume that $r \geq 2$. Let $\theta \in \widehat{T_r^F}$, $\theta' \in \widehat{T_r'^F}$ be such that*

(a) $$H_c^j(\Sigma)_{\theta^{-1},\theta'} \neq 0$$

for some $j \in \mathbf{Z}$. There exists $n \geq 1$ and $g \in N(T',T)^{F^n}$ such that $\mathrm{Ad}(g)$ carries $\theta|_{\mathcal{T}^F} \circ N_F^{F^n} \in \widehat{\mathcal{T}^{F^n}}$ to $\theta'|_{\mathcal{T}'^F} \circ N_F^{F^n} \in \widehat{\mathcal{T}'^{F^n}}$.

The subvarieties G_w of G have the following property: for some ordering $\leq$ of $W(T,T')$, the unions $\cup_{w' \leq w} G_{w'}$ are closed in G. It follows that the unions $\cup_{w' \leq w} G_{w',r}$ are closed in G_r and the unions $\cup_{w' \leq w} \Sigma_{w'}$ are closed in Σ. The spectral sequence associated to the filtration of Σ by these unions, together with (a), shows that there exists $w \in W(T,T')$ and $j \in \mathbf{Z}$ such that $H_c^j(\Sigma_w)_{\theta^{-1},\theta'} \neq 0$. We can therefore apply Lemma 1.3. The lemma follows.

1.5. Let Φ be the set of characters $\alpha : T \to \mathbf{F}^*$ such that $\alpha \neq 1$ and T acts on some line $L_\alpha \subset LieG$ via α (in the adjoint action); for such α, let G^α be the one dimensional unipotent subgroup of G such that $LieG^\alpha = L_\alpha$. For $\alpha \in \Phi$ there is a unique 1-dimensional torus T^α in T such that T^α is contained in the subgroup of G generated by $G^\alpha, G^{\alpha^{-1}}$. Let $\mathcal{T}^\alpha = (T^\alpha)_r^{r-1}$ (a one dimensional subgroup of $\mathcal{T} = T_r^{r-1}$).

Let $\chi \in \widehat{\mathcal{T}^F}$. We say that χ is *regular* if for any $\alpha \in \Phi$ and any $n \geq 1$ such that $F^n(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$, the restriction of $\chi \circ N_F^{F^n} : \mathcal{T}^{F^n} \to \bar{\mathbf{Q}}_l^*$ to $(\mathcal{T}^\alpha)^{F^n}$ is non-trivial. (It is enough to check that $\chi \circ N_F^{F^n}|_{(\mathcal{T}^\alpha)^{F^n}}$ is non-trivial for any α and for just one n such that $F^n(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$ for all α.)

Let $\theta \in \widehat{T^F}$. We say that θ is *regular* if $\theta|_{\mathcal{T}^F}$ is regular.

1.6. Let T be an F-stable maximal torus of G. Let $U, \tilde{U}, V, \tilde{V}$ be unipotent radicals of Borel subgroups containing T such that $U \cap V = \tilde{U} \cap \tilde{V} = \{1\}$. Let Φ be as in 1.5. Let

$$\Phi^+ = \{\alpha \in \Phi; G^\alpha \subset \tilde{V}\}, \Phi^- = \{\alpha \in \Phi; G^\alpha \subset \tilde{U}\}.$$

Then $\Phi = \Phi^+ \sqcup \Phi^-$ and $\Phi^- = \{\alpha^{-1}; \alpha \in \Phi^+\}$.

For $\alpha \in \Phi^+$ let $ht(\alpha)$ be the largest integer $n \geq 1$ such that $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ with $\alpha_i \in \Phi^+$.

Let $x \in (G^\alpha)_r^b, x' \in (G^{\alpha'})_r^c$ where $\alpha, \alpha' \in \Phi$ and $b, c \in [0, r]$.

(a) If $b + c \geq r$ then $xx' = x'x$.

(b) If $b + c \leq r$ and $\alpha\alpha' \neq 1$ then $xx' = x'xu$ where u is of the form $\prod_{i,i' \geq 1; \alpha^i\alpha'^{i'} \in \Phi} u_{i,i'}$ with $u_{i,i'} \in (G^{\alpha^i\alpha'^{i'}})_r^{b+b'}$.
(The factors in the last product are written in some fixed order. In the special case where $b + c = r - 1$, these factors commute with each other by (a), since $r - 1 + r - 1 \geq r$.)

(c) If $b+c \geq r-1, b+2c \geq r$ and $\alpha\alpha' = 1$ then $xx' = x'x\tau_{x,x'}u$ where $\tau_{x,x'} \in \mathcal{T}^\alpha$ and $u \in (G^\alpha)_r^{r-1}$ are uniquely determined.

Lemma 1.7. *We fix an order on Φ^+. For any $z \in \tilde{V}_r, \beta \in \Phi^+$, define $x_\beta^z \in G_r^\beta$ by $z = \prod_{\beta \in \Phi^+} x_\beta^z$ (factors written using the given order on Φ^+). Let $\alpha \in \Phi^-, a \in [1, r-1]$. Let $z \in \tilde{V}_r^a$ be such that $x_\beta^z \in (G^\beta)_r^{a+1}$ for all $\beta \in \Phi^+$ with $ht(\beta) > ht(\alpha^{-1})$. Let $\xi \in (G^\alpha)_r^{r-a-1}$. Then $\xi z = z\xi\tau_{\xi,z}\omega_{\xi,z}$ where $\tau_{\xi,z} \in \mathcal{T}^\alpha$ and $\omega_{\xi,z} \in \tilde{U}_r^{r-1}$.*

We argue by induction on $N_z = \sharp(\beta \in \Phi^+; x_\beta^z \neq 1)$. If $N_z = 0$ the result is clear. Assume now that $N_z = 1$ so that $z \in G_r^\beta$ with $\beta \in \Phi^+$. If $\alpha\beta = 1$, the result follows from 1.6(c). If $\alpha\beta \neq 1$ and $ht(\beta) > ht(\alpha^{-1})$ then $z \in (G^\beta)_r^{a+1}$ and $\xi z = z\xi$ by 1.6(b). If $\alpha\beta \neq 1$ and $ht(\beta) \leq ht(\alpha^{-1})$ then by 1.6(b) we have $\xi z = z\xi u$ where $u = \prod_{i,i' \geq 1; \alpha^i\beta^{i'} \in \Phi} u_{i,i'}$ with $u_{i,i'} \in (G^{\alpha^i\beta^{i'}})_r^{r-1}$; it is enough to show that if $i, i' \geq 1$, we cannot have $\alpha^i\beta^{i'} \in \Phi^+$. (If $\alpha^i\beta^{i'} \in \Phi^+$ for some $i, i' \geq 1$ then $\alpha\beta \in \Phi^+$ hence $ht(\beta) > ht(\alpha^{-1})$, contradiction.)

Assume now that $N_z \geq 2$. We can write $z = z'z''$ where $z', z'' \in \tilde{V}_r^a$, $N_{z'} < N_z, N_{z''} < N_z$. Using the induction hypothesis we have

$$\xi z = \xi z'z'' = z'\xi\tau_{\xi,z'}\omega_{\xi,z'}z''$$

where $\tau_{\xi,z'} \in \mathcal{T}^\alpha$, $\omega_{\xi,z'} \in \tilde{U}_r^{r-1}$. We have $\omega_{\xi,z'}z'' = z''\omega_{\xi,z'}$ and $\tau_{\xi,z'}z'' = z''\tau_{\xi,z'}$. Using again the induction hypothesis, we have

$$\begin{aligned} z'\xi\tau_{\xi,z'}\omega_{\xi,z'}z'' &= z'\xi\tau_{\xi,z'}z''\omega_{\xi,z'} = z'\xi z''\tau_{\xi,z'}\omega_{\xi,z'} \\ &= z'z''\xi\tau_{\xi,z''}\omega_{\xi,z''}\tau_{\xi,z'}\omega_{\xi,z'} = z\xi\tau_{\xi,z'}\tau_{\xi,z''}\omega_{\xi,z'}\omega_{\xi,z''}. \end{aligned}$$

Thus, $\xi z = z\xi\tau_{\xi,z}\omega_{\xi,z}$ where

$$\tau_{\xi,z} = \tau_{\xi,z'}\tau_{\xi,z''}, \omega_{\xi,z} = \omega_{\xi,z'}\omega_{\xi,z''}.$$

The lemma is proved.

1.8. In the setup of 1.6, let $Z = V \cap \tilde{V}$. Let $\Phi' = \{\beta \in \Phi; G^\beta \subset Z\}$. We have $\Phi' \subset \Phi^+$. Let $\mathcal{X}$ be the set of all subsets $I \subset \Phi'$ such that $I \neq \emptyset$ and $ht : \Phi^+ \to \mathbf{N}$ is constant on I.

To any $z \in Z_r^1 - \{1\}$ we associate a pair (a, I_z) where $a \in [1, r-1]$ and $I_z \in \mathcal{X}$ as follows. We define a by the condition that $z \in Z_r^{a,*}$. If $x_\beta^z \in G^\beta$ are defined as in 1.8 in terms of a fixed order on Φ^+, then $x_\beta^z \in (G^\beta)_r^a$ for all $\beta \in \tilde{\Phi}$ and $x_\beta^z = 1$ for all $\beta \in \Phi^+ - \tilde{\Phi}$. Let I_z be the set of all $\alpha' \in \tilde{\Phi}$ such that $x_{\alpha'}^z \in (G^{\alpha'})_r^{a,*}$ and $x_\beta^z \in (G^\beta)_r^{a+1}$ for all $\beta \in \Phi^+$ such that $ht(\beta) > ht(\alpha')$. It is easy to see, using 1.6(a),(b), that the definition of I_z does not depend on the choice of an order on Φ^+. For $a \in [1, r-1]$ and $I \in \mathcal{X}$ let $Z_r^{a,*,I}$ be the set of all $z \in Z_r^1 - \{1\}$ such that $z \in Z_r^{a,*}, I = I_z$. Thus we have a partition

(a) $$Z_r^1 - \{1\} = \sqcup_{a \in [1,r-1], I \in \mathcal{X}} Z_r^{a,*,I}.$$

Lemma 1.9. *Let $T, T', U, U', r, \mathcal{T}, \mathcal{T}'$ be as in 1.2. Let $\theta \in \widehat{T_r^F}, \theta' \in \widehat{T_r'^F}$. Assume that $\theta'|_{\mathcal{T}^F} = \chi$ is regular. Let Σ be as in 1.2. Then $\sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j(\Sigma)_{\theta^{-1},\theta'}$ is equal to the number of $w \in W(T,T')^F$ such that $Ad(\dot{w}) : T_r'^F \to T_r^F$ carries θ to θ'.*

Using the partition $\Sigma = \sqcup_{w \in W(T,T')} \Sigma_w$ we see that it is enough to prove that $\sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j(\Sigma_w)_{\theta^{-1},\theta'}$ is equal to 1 if $F(w) = w$ and $Ad(\dot{w}) : T_r'^F \to T_r^F$ carries θ to θ' and equals 0, otherwise. We now fix $w \in W(T,T')$. We have

$$\Sigma_w$$
$$= \{(x, x', y) \in F(U_r) \times F(U'_r) \times G_r; xF(y) = yx', y \in U_r G_r^1 \dot{w} T'_r U'_r = U_r Z_r^1 \dot{w} T'_r U'_r\}$$

where $Z = V \cap \dot{w} V' \dot{w}^{-1}$. Here V (resp. V') is the unipotent radical of a Borel subgroup containing T (resp. T') such that $U \cap V = \{1\}$ (resp. $U' \cap V' = \{1\}$. Let

$$\hat{\Sigma}_w = \{(x, x', u, u', z, \tau') \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times Z_r^1 \times T'_r;$$
$$xF(u)F(z)F(\dot{w})F(\tau')F(u') = uz\dot{w}\tau' u' x'\}.$$

The map $\hat{\Sigma}_w \to \Sigma_w$ given by $(x, x', u, u', z, \tau') \mapsto (x, x', uz\dot{w}\tau' u')$ is a locally trivial fibration with all fibres isomorphic to a fixed affine space. This map is compatible with the $T_r^F \times T_r'^F$-actions where $T_r^F \times T_r'^F$ acts on $\hat{\Sigma}_w$ by

(a)
$$(t, t') : (x, x', u, u', z, \tau') \mapsto (txt^{-1}, t'x't'^{-1}, tut^{-1}, t'u't'^{-1}, tzt^{-1}, \dot{w}^{-1} t \dot{w} \tau t'^{-1}).$$

Hence it is enough to show that

$\sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j(\hat{\Sigma}_w)_{\theta^{-1},\theta'}$ is equal to 1 if $F(w) = w$ and $Ad(\dot{w}) : T_r'^F \to T_r^F$ carries θ to θ' and equals 0, otherwise.

By the change of variable $xF(u) \mapsto x, x'F(u')^{-1} \mapsto x'$ we may rewrite $\hat{\Sigma}_w$ as

$$\hat{\Sigma}_w = \{(x, x', u, u', z, \tau') \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times Z_r^1 \times T'_r;$$
$$xF(z)F(\dot{w})F(\tau') = uz\dot{w}\tau' u' x'\}$$

with the $T_r^F \times T_r'^F$-action still given by (a). We have a partition $\hat{\Sigma}_w = \hat{\Sigma}'_w \sqcup \hat{\Sigma}''_w$ where

$$\hat{\Sigma}'_w = \{(x,x',u,u',z,\tau') \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times (Z_r^1 - \{1\}) \times T'_r; \\ xF(z)F(\dot{w})F(\tau') = uz\dot{w}\tau'u'x'\},$$

$$\hat{\Sigma}''_w = \{(x,x',u,u',1,\tau') \in F(U_r) \times F(U'_r) \times U_r \times U'_r \times \{1\} \times T'_r; \\ xF(\dot{w})F(\tau') = u\dot{w}\tau'u'x'\},$$

are stable under the $T_r^F \times T_r'^F$-action. It is then enough to show that

(b) $\sum_{j\in\mathbf{Z}}(-1)^j \dim H_c^j(\hat{\Sigma}''_w)_{\theta^{-1},\theta'}$ is equal to 1 if $F(w) = w$ and $Ad(\dot{w}) : T_r'^F \to T_r^F$ carries θ to θ' and equals 0, otherwise.

(c) $H_c^j(\hat{\Sigma}'_w)_{\theta^{-1},\theta'} = 0$ for all j.

We first prove (c). If M is a $\mathcal{T}'^F$-module we shall write $M_{(\chi)}$ for the subspace of M on which $\mathcal{T}'^F$ acts according to χ. Now $\mathcal{T}'^F$ acts on $\hat{\Sigma}'_w$ by

$$t' : (x,x',u,u',z,\tau') \mapsto (x, t'x't'^{-1}, u, t'u't'^{-1}, z, \tau't'^{-1}).$$

Hence $H_c^j(\hat{\Sigma}'_w)$ becomes a $\mathcal{T}'^F$-module. It is enough to show that $H_c^j(\hat{\Sigma}'_w)_{(\chi)} = 0$.

We shall use the definitions and results in 1.6-1.8 relative to $U, \tilde{U}, V, \tilde{V}$ where $\tilde{U} = \dot{w}U'\dot{w}^{-1}$, $\tilde{V} = \dot{w}V'\dot{w}^{-1}$. The partition 1.8(a) gives rise to a partition $\hat{\Sigma}'_w = \sqcup_{a,I}\hat{\Sigma}_w^{a,I}$ indexed by $a \in [0, r-1], I \in \mathcal{X}$ where

$$\hat{\Sigma}_w^{a,I} = \{(x,x',u,u',z,\tau') \in \hat{\Sigma}'_w; z \in Z_r^{a,*,I}\}.$$

It is easy to see that there is a total order on the set of indices (a, I) such that the union of the $\hat{\Sigma}_w^{a,I}$ for (a, I) less than or equal than some given (a^0, I^0) is closed in $\hat{\Sigma}'_w$. Since the subsets $\hat{\Sigma}_w^{a,I}$ are stable under the action of $\mathcal{T}'^F$, we see that, in order to prove (c), it is enough to show that

(d) $$H_c^j(\hat{\Sigma}_w^{a,I})_{(\chi)} = 0$$

for any fixed a, I as above. We choose $\alpha' \in I$. Let $\alpha = \alpha'^{-1}$. Then $G_r^\alpha \subset U_r \cap \dot{w}U'_r\dot{w}^{-1}$.

For any $z \in Z_r^{a,*}, \xi \in (G^\alpha)_r^{r-a-1}$ we have

$$\xi z = z\xi\tau_{\xi,z}\omega_{\xi,z}$$

where $\tau_{\xi,z} \in \mathcal{T}^\alpha, \omega(\xi,z) \in \dot{w}U_r'^{r-1}\dot{w}^{-1}$ are uniquely determined. (See 1.7.) Moreover, the map $(G^\alpha)_r^{r-a-1} \to \mathcal{T}^\alpha, \xi \mapsto \tau(\xi, z)$ factors through an isomorphism

$$\lambda_z : (G^\alpha)_r^{r-a-1}/(G^\alpha)_r^{r-a} \xrightarrow{\sim} \mathcal{T}^\alpha.$$

Let $\pi : (G^\alpha)_r^{r-a-1} \to (G^\alpha)_r^{r-a-1}/(G^\alpha)_r^{r-a}$ be the canonical homomorphism. We can find a morphism of algebraic varieties

$$\psi : (G^\alpha)_r^{r-a-1}/(G^\alpha)_r^{r-a} \to (G^\alpha)_r^{r-a-1}$$

such that $\pi\psi = 1$ and $\psi(1) = 1$. Let

$$\mathcal{H}' = \{t' \in \mathcal{T}'; t'^{-1}F(t') \in \dot{w}^{-1}\mathcal{T}^\alpha \dot{w}\}.$$

This is a closed subgroup of $\mathcal{T}'$. For any $t' \in \mathcal{H}'$ we define $f_{t'} : \hat{\Sigma}_w^{a,I} \to \hat{\Sigma}_w^{a,I}$ by

$$f_{t'}(x, x', u, u', z, \tau') = (xF(\xi), \hat{x}', u, F(t')^{-1}u'F(t'), z, \tau'F(t'))$$

where

$$\xi = \psi\lambda_z^{-1}(\dot{w}F(t')^{-1}t'\dot{w}^{-1}) \in (G^\alpha)_r^{r-a-1} \subset U_r \cap \dot{w}U'_r\dot{w}^{-1}$$

and $\hat{x}' \in G_r$ is defined by the condition that

$$xF(\xi)F(z)F(\dot{w})F(\tau'F(t')) = uz\dot{w}\tau'F(t')F(t')^{-1}u'F(t')\hat{x}'.$$

In order for this to be well defined we must check that $\hat{x}' \in F(U'_r)$. Thus we must show that

$xF(\xi)F(z)F(\dot{w})F(\tau'F(t')) \in uz\dot{w}\tau'u'F(t')F(U'_r)$

or that

$xF(z)F(\xi)F(\tau_{\xi,z})F(\omega_{\xi,z})F(\dot{w})F(\tau'F(t')) \in uz\dot{w}\tau'u'F(t')F(U'_r)$.

Since $xF(z) = uz\dot{w}\tau'u'x'F(\tau')^{-1}F(\dot{w}^{-1})$, it is enough to show that

$$uz\dot{w}\tau'u'x'F(\tau')^{-1}F(\dot{w}^{-1})F(\xi)F(\tau_{\xi,z})F(\omega_{\xi,z})F(\dot{w})F(\tau'F(t'))$$
$$\in uz\dot{w}\tau'u'F(t')F(U'_r)$$

or that

$x'F(\tau')^{-1}F(\dot{w}^{-1})F(\xi)F(\tau_{\xi,z})F(\omega_{\xi,z})F(\dot{w})F(\tau'F(t')) \in F(t')F(U'_r)$.

Since $x' \in F(U'_r)$, $F(\dot{w}^{-1})F(\omega_{\xi,z})F(\dot{w}) \in F(U'_r)$, it is enough to check that

$F(\tau')^{-1}F(\dot{w}^{-1})F(\xi)F(\tau_{\xi,z})F(\dot{w})F(\tau'F(t')) \in F(t')F(U'_r)$.

Since $F(\dot{w}^{-1})F(\xi)F(\dot{w}) \in F(U'_r)$ it is enough to check that

$F(\tau')^{-1}F(\dot{w}^{-1})F(\tau_{\xi,z})F(\dot{w})F(\tau'F(t')) \in F(t')F(U'_r)$

or that

$F(\dot{w}^{-1})F(\tau_{\xi,z})F(\dot{w})F(F(t')) = F(t')$

or that $\dot{w}^{-1}\tau_{\xi,z}\dot{w} = F(t')^{-1}t'$ or that $\lambda_z(\pi_z(\xi)) = \dot{w}F(t')^{-1}t'\dot{w}^{-1}$. But this is clear.

Thus, $f_{t'} : \hat{\Sigma}_w^{a,I} \to \hat{\Sigma}_w^{a,I}$ is well defined for $t' \in \mathcal{H}'$. It is clearly an isomorphism for any $t' \in \mathcal{H}'$. In particular, it is a well defined isomorphism for any $t' \in \mathcal{H}'^0$. By general principles, the induced map $f_{t'}^* : H_c^j(\hat{\Sigma}_w^{a,I}) \to H_c^j(\hat{\Sigma}_w^{a,I})$ is constant when t' varies in $\mathcal{H}'^0$. In particular, it is constant when t' varies in $\mathcal{T}'^F \cap \mathcal{H}'^0$.

Now $\mathcal{T}'^F \subset \mathcal{H}'$ and for $t' \in \mathcal{T}'^F$, the map $f_{t'}$ coincides with the action of t' in the $\mathcal{T}'^F$-action on $\hat{\Sigma}_w^{a,I}$. (We use that $\psi(1) = 1$.) We see that the induced action of $\mathcal{T}'^F$ on $H_c^j(\hat{\Sigma}_w^{a,I})$ is trivial when restricted to $\mathcal{T}'^F \cap \mathcal{H}'^0$.

We can find $n \geq 1$ such that $F^n(\dot{w}^{-1}\mathcal{T}^\alpha \dot{w}) = \dot{w}^{-1}\mathcal{T}^\alpha \dot{w}$. Then $t' \mapsto t'F(t')F^2(t') \dots F^{n-1}(t')$ is a well defined morphism $\dot{w}^{-1}\mathcal{T}^\alpha \dot{w} \to \mathcal{H}'$. Its image is a connected subgroup of $\mathcal{H}'$ hence is contained in $\mathcal{H}'^0$. If $t' \in (\dot{w}^{-1}\mathcal{T}^\alpha \dot{w})^{F^n}$ then $N_F^{F^n}(t') \in \mathcal{T}'^F$; thus, $N_F^{F^n}(t') \in \mathcal{T}'^F \cap \mathcal{H}'^0$. We see that the action of $N_F^{F^n}(t') \in \mathcal{T}'^F$ on $H_c^j(\hat{\Sigma}_w^{a,I})$ is trivial for any $t' \in (\dot{w}^{-1}\mathcal{T}^\alpha \dot{w})^{F^n}$.

If we assume that $H_c^j(\hat{\Sigma}_w^{a,I})_{(\chi)} \neq 0$, it follows that $t' \mapsto \chi(N_F^{F^n}(t'))$ is the trivial character of $(\dot{w}^{-1}\mathcal{T}^\alpha \dot{w})^{F^n}$. This contradicts our assumption that χ is regular. Thus, (d) holds. Hence (c) holds.

We now prove (b). Let

$$\tilde{H} = \{(t, t') \in T_r \times T'_r; tF(t)^{-1} = F(\dot{w})t'F(t')^{-1}F(\dot{w}^{-1})\}.$$

This is a closed subgroup of $T_r \times T'_r$ containing $T_r^F \times T_r'^F$. Now the action of $T_r^F \times T_r'^F$ on $\hat{\Sigma}''_w$ extends to an action of $\tilde{H}$ given by the same formula. To see this consider $(t, t') \in \tilde{H}$ and $(x, x', u, u', 1, \tau') \in \hat{\Sigma}''_w$. We must show that

$(txt^{-1}, t'x't'^{-1}, tut^{-1}, t'u't'^{-1}, 1, \dot{w}^{-1}t\dot{w}\tau't'^{-1}) \in \hat{\Sigma}''_w$

that is,

$txt^{-1}F(\dot{w})F(\dot{w}^{-1})F(t)F(\dot{w})F(\tau')F(t'^{-1}) = tut^{-1}\dot{w}\dot{w}^{-1}t\dot{w}\tau't'^{-1}t'u't'^{-1}t'x't'^{-1}$

or that

$xt^{-1}F(t)F(\dot{w})F(\tau')F(t'^{-1}) = u\dot{w}\tau'u'x't'^{-1}$

or that $xt^{-1}F(t)F(\dot{w})F(\tau')F(t'^{-1}) = xF(\dot{w})F(\tau')t'^{-1}$

or that $t^{-1}F(t)F(\dot{w})F(t'^{-1}) = F(\dot{w})t'^{-1}$; this is clear. Let T_*, T'_* be the reductive part of T_r, T'_r (thus T_* is a torus isomorphic to T). Let $\tilde{H}_* = \tilde{H} \cap (T_* \times T'_*)$. Then $\tilde{H}_*^0$ is a torus acting on $\hat{\Sigma}''_w$ by restriction of the $\tilde{H}$-action. The fixed point set $(\hat{\Sigma}''_w)^{\tilde{H}_*^0}$ of the $\tilde{H}_*^0$-action is stable under the action of $T_r^F \times T_r'^F$ and by general principles we have

$$\sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j(\hat{\Sigma}''_w)_{\theta^{-1}, \theta'} = \sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j((\hat{\Sigma}''_w)^{\tilde{H}_*^0})_{\theta^{-1}, \theta'}.$$

It is then enough to show that

(e) $\sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j((\hat{\Sigma}''_w)^{\tilde{H}_*^0})_{\theta^{-1}, \theta'}$ is equal to 1 if $F(w) = w$ and $Ad(\dot{w}) : T_r'^F \to T_r^F$ carries θ to θ' and equals 0, otherwise.

Let $(x, x', u, u', 1, \tau) \in (\hat{\Sigma}''_w)^{\tilde{H}_*^0}$. By Lang's theorem the first projection $\tilde{H}_* \to T_*$ is surjective. It follows that the first projection $\tilde{H}_*^0 \to T_*$ is surjective. Similarly the second projection $\tilde{H}_*^0 \to T'_*$ is surjective. Hence for any $t \in T_*, t' \in T'_*$ we have

$txt^{-1} = x, t'x't'^{-1} = x', tut^{-1} = u, t'u't'^{-1} = u'$

hence $x = x' = u = u' = 1$. Thus, $(\hat{\Sigma}''_w)^{\tilde{H}_*^0}$ is contained in

(f) $\{(1, 1, 1, 1, 1, \tau'); \tau' \in T'_r, F(\dot{w}\tau') = \dot{w}\tau'\}$.

The set (f) is clearly contained in the fixed point set of $\tilde{H}$. Note that (f) is empty unless $F(w) = w$. We can therefore assume that $F(w) = w$. In this case, (f) is stable under the action of $\tilde{H}$. In particular it is stable under the action of $\tilde{H}^0_*$. Since (f) is finite and $\tilde{H}^0_*$ is connected, we see that $\tilde{H}^0_*$ must act trivially on (f). Thus, (f) is exactly the fixed point set of $\tilde{H}^0_*$. Hence this fixed point can be identified with $(\dot{w}T'_r)^F$. From this (e) follows easily. The lemma is proved.

2. The main results

2.1. Let G, F be as in 1.2. Let T be an F-stable maximal torus in G and let U be the unipotent radical of a Borel subgroup of G that contains T. (Note that U is not necessarily F-stable.) Let $r \ge 1$. Let $\mathcal{R}(G_r^F)$ be the group of virtual representations of G_r^F over $\bar{\mathbf{Q}}_l$. Let $\langle , \rangle$ be the standard inner product $\mathcal{R}(G_r^F) \times \mathcal{R}(G_r^F) \to \mathbf{Z}$. Let

$$S_{T,U} = \{g \in G_r; g^{-1}F(g) \in F(U_r)\}.$$

The finite group $G_r^F \times T_r^F$ acts on $S_{T,U}$ by $(g_1, t) : g \mapsto g_1 g t^{-1}$. For any $i \in \mathbf{Z}$ we have an induced action of $G_r^F \times T_r^F$ on $H^i_c(S_{T,U})$. For $\theta \in \widehat{T_r^F}$, we denote by $H^i_c(S_{T,U})_\theta$ the subspace of $H^i_c(S_{T,U})$ on which T_r^F acts according to θ. This is a G_r^F-submodule of $H^i_c(S_{T,U})$. Let

$$R^\theta_{T_r,U_r} = \sum_{i \in \mathbf{Z}} (-1)^i H^i_c(S_{T,U})_\theta \in \mathcal{R}(G_r^F).$$

Proposition 2.2. *Assume that $r \ge 2$. Let (T', U', θ') be another triple like T, U, θ. Let $\mathcal{T} = T_r^{r-1}, \mathcal{T}' = T_r'^{r-1}$.*

(a) Let i, i' be integers. Assume that there exists an irreducible G_r^F-module that appears in the G_r^F-module $(H^i_c(S_{T,U})_{\theta^{-1}})^*$ (dual of $H^i_c(S_{T,U})_{\theta^{-1}}$) and in the G_r^F-module $H^{i'}_c(S_{T',U'})_{\theta'}$. There exists $n \ge 1$ and $g \in N(T', T)^{F^n}$ such that $\mathrm{Ad}(g)$ *carries $\theta \circ N_F^{F^n}|_{\mathcal{T}^{F^n}} \in \widehat{\mathcal{T}^{F^n}}$ to $\theta' \circ N_F^{F^n}|_{\mathcal{T}'^{F^n}} \in \widehat{\mathcal{T}'^{F^n}}$.*

(b) Assume that there exists an irreducible G_r^F-module that appears in the virtual G_r^F-module $\sum_i (-1)^i H^i_c(S_{T,U})_\theta$ and in the virtual G_r^F-module $\sum_i (-1)^i H^i_c(S_{T',U'})_{\theta'}$. There exists $n \ge 1$ and $g \in N(T', T)^{F^n}$ such that $\mathrm{Ad}(g)$ *carries $\theta \circ N_F^{F^n}|_{\mathcal{T}^{F^n}} \in \widehat{\mathcal{T}^{F^n}}$ to $\theta' \circ N_F^{F^n}|_{\mathcal{T}'^{F^n}} \in \widehat{\mathcal{T}'^{F^n}}$.*

We prove (a). Consider the free G_r^F-action on $S_{T,U} \times S_{T',U'}$ given by $g_1 : (g, g') \mapsto (g_1 g, g_1 g')$. The map

$$(g, g') \mapsto (x, x', y), x = g^{-1}F(g), x' = g'^{-1}F(g'), y = g^{-1}g'$$

defines an isomorphism of $G_r^F \backslash (S_{T,U} \times S_{T',U'})$ onto Σ (as in 1.2).

The action of $T_r^F \times T_r'^F$ on $S_{T,U} \times S_{T',U'}$ given by right multiplication by t^{-1} on the first factor and by t'^{-1} on the second factor becomes an action of $T_r^F \times T_r'^F$ on Σ given by $(x, x', y) \mapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$. Our assumption implies that the G_r^F-module $H_c^i(S_{T,U})_{\theta^{-1}} \otimes H_c^{i'}(S_{T',U'})_{\theta'}$ contains the unit representation with non-zero multiplicity. Hence the subspace of $H_c^{i+i'}(G_r^F \backslash (S_{T,U} \times S_{T',U'}))$ on which $T_r^F \times T_r'^F$ acts according to $\theta^{-1} \boxtimes \theta'$ is non-zero. Equivalently, $H_c^{i+i'}(\Sigma)_{\theta^{-1},\theta'} \neq 0$. We now use Lemma 1.4; (a) follows.

We prove (b). By general principles we have

$$\sum_i (-1)^i (H_c^i(S_{T,U})_{\theta^{-1}})^* = \sum_i (-1)^i H_c^i(S_{T,U})_\theta.$$

Hence the assumption of (b) implies that the assumption of (a) holds. Hence the conclusion of (a) holds. The proposition is proved.

Proposition 2.3. *We preserve the setup of 2.2. Assume that θ or θ' is regular. The inner product $\langle R^\theta_{T_r,U_r}, R^{\theta'}_{T'_r,U'_r}\rangle$ is equal to the number of $w \in W(T,T')^F$ such that $Ad(\dot{w}) : T_r'^F \to T_r^F$ carries θ to θ'.*

We may assume that θ' is regular. As in the proof of 2.2, we have

$$\begin{aligned} &\langle R^\theta_{T_r,U_r}, R^{\theta'}_{T'_r,U'_r}\rangle \\ &= \sum_{i,i' \in \mathbf{Z}} (-1)^{i+i'} \dim(H_c^i(S_{T,U})_{\theta^{-1}} \otimes H_c^{i'}(S_{T',U'})_\theta)^{G_r^F} \\ &= \sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j(G_r^F \backslash (S_{T,U} \times S_{T',U'}))_{\theta^{-1},\theta'} \\ &= \sum_{j \in \mathbf{Z}} (-1)^j \dim H_c^j(\Sigma)_{\theta^{-1},\theta'} \end{aligned}$$

where $()^{G_r^F}$ denotes the space of G_r^F-invariants. It remains to use 1.9.

Corollary 2.4. *Assume that $r \geq 2$. Let T, U be as in 2.1. Assume that $\theta \in \widehat{T_r^F}$ is regular.*

(a) $R^\theta_{T_r,U_r}$ is independent of the choice of U.

(b) Assume also that the stabilizer of θ in $W(T,T)^F$ is $\{1\}$. Then $R^\theta_{T_r,U_r}$ is $\pm$ an irreducible G_r^F-module.

We prove (a). Let U' be the unipotent radical of another Borel subgroup of G containing T. Let $R = R^\theta_{T_r,U_r}$, $R' = R^\theta_{T_r,U'_r}$. By 2.3 we have

$\langle R, R\rangle = \langle R, R'\rangle = \langle R', R\rangle = \langle R', R'\rangle$.

Hence $\langle R - R', R - R'\rangle = 0$, so that $R = R'$. This proves (a). In the setup of (b), we see from 2.3 that $\langle R^\theta_{T_r,U_r}, R^\theta_{T_r,U_r}\rangle = 1$. This proves (b).

2.5. Assume that $r \geq 2$. Let T be as in 2.1. Assume that $\theta \in \widehat{T_r^F}$ is regular. We set

$R^\theta_{T_r} = R^\theta_{T_r,U_r}$

where U is chosen as in 2.1. (By 2.4(a), this is independent of the choice of U.)

3. An example

3.1. Let $A = \mathbf{F}[[\epsilon]]/(\epsilon^2)$. Define $F : A \to A$ by $F(a_0 + \epsilon a_1) = a_0^q + \epsilon a_1^q$ where $a_0, a_1 \in \mathbf{F}$. Let V be a 2-dimensional $\mathbf{F}$-vector space with a fixed $\mathbf{F}_q$-rational structure with Frobenius map $F : V \to V$. Let $G = SL(V)$. Then G has an $\mathbf{F}_q$-rational structure with Frobenius map $F : G \to G$ such that $F(gv) = F(g)F(v)$ for all $g \in G, v \in V$. Let $V_2 = A \otimes_{\mathbf{F}} V$. Then G_2 (see 0.2) may be identified with the group of all automorphisms of the free A-module V_2 with determinant 1. We regard V as a subset of V_2 by $v \mapsto 1 \otimes v$. Any element of V_2 can be written uniquely in the form $v_0 + \epsilon v_1$ where $v_0, v_1 \in V$. The Frobenius map $F : V_2 \to V_2$ satisfies $F(v_0 + \epsilon v_1) = F(v_0) + \epsilon F(v_1)$ for $v_0, v_1 \in V$.

Let $\widehat{G_2^F}$ be the set of isomorphism classes of irreducible representations of G_2^F over $\bar{\mathbf{Q}}_l$. The objects of $\widehat{G_2^F}$ can be classified using the fact that G_2^F is a semidirect product of G^F and $\mathbf{F}_q^3$.

The table below shows the number of representations in $\widehat{G_2^F}$ of various dimensions assuming that q is odd; the first column indicates the dimension, the second column indicates the number of representations of that dimension.

dim	♯
1	1
q	1
$q+1$	$(q-3)/2$
$(q+1)/2$	2
$q-1$	$(q-1)/2$
$(q-1)/2$	2
q^2+q	$(q-1)^2/2$
q^2-q	$(q^2-1)/2$
$(q^2-1)/2$	$2q$

The analogous table in the case where q is a power of 2 is

dim	♯
1	1
q	1
$q+1$	$(q-2)/2$
$q-1$	$q/2$
q^2+q	$(q-1)(q-2)/2$
$(q^2+q)/2$	$2(q-1)$
q^2-q	$(q^2-q)/2$
$(q^2-q)/2$	$2(q-1)$
q^2-1	q

3.2. Let $\mathcal{B}$ be the set of all A-submodules $L \subset V_2$ such that L is a direct summand of V_2 and L is free of rank 1. Now G_2 acts transitively on $\mathcal{B}$. The diagonal action on $\mathcal{B} \times \mathcal{B}$ has three orbits $\mathcal{O}, \mathcal{O}', \mathcal{O}''$ where

$\mathcal{O} = \{(L, L') \in \mathcal{B} \times \mathcal{B}; L = L'\}$,
$\mathcal{O}' = \{(L, L') \in \mathcal{B} \times \mathcal{B}; L \cap L' = \epsilon L = \epsilon L'\}$,
$\mathcal{O}'' = \{(L, L') \in \mathcal{B} \times \mathcal{B}; L \cap L' = 0\}$.
If $L \in \mathcal{B}$ then $F(L) \in \mathcal{B}$. Thus we obtain a map $F : \mathcal{B} \to \mathcal{B}$, the Frobenius map of a $\mathbf{F}_q$-rational structure on $\mathcal{B}$. Let
$X = \{L \in \mathcal{B}; (L, F(L)) \in \mathcal{O}\}$,
$X' = \{L \in \mathcal{B}; (L, F(L)) \in \mathcal{O}'\}$,
$X'' = \{L \in \mathcal{B}; (L, F(L)) \in \mathcal{O}''\}$.
Then X, X', X'' form a partition of $\mathcal{B}$ into G_2^F-stable subvarieties.

We now define some finite coverings of X, X', X'' as follows. Let e, e' be an $\mathbf{F}$-basis of V such that $F(e) = e, F(e') = e'$. Let $\underline{T}$ be the subgroup of G consisting of the automorphisms $e \mapsto ae, e' \mapsto a^{-1}e'$ with $a \in \mathbf{F}^*$. (An F-stable maximal torus of G.) Let $\underline{U}$ be the subgroup of G consisting of the automorphisms
$e \mapsto e + be', e' \mapsto e'$ with $b \in \mathbf{F}^*$.
Let $\nu \in G$ be such that $\nu(e) = e', \nu(e') = -e$. Let $h \in G$ be such that $he = e, he' = e' + \epsilon e$. Let
$\tilde{X} = \{g \in G_2; g^{-1}F(g) \in \underline{U}_2\}/\underline{U}_2$,
$\tilde{X}' = \{g \in G_2; g^{-1}F(g) \in h\underline{U}_2\}/(\underline{U}_2 \cap h\underline{U}_2h^{-1})$,
$\tilde{X}'' = \{g \in G_2; g^{-1}F(g) \in \nu\underline{U}_2\}$.
(We use the action of $\underline{U}_2$ or $\underline{U}_2 \cap h\underline{U}_2h^{-1}$ on G_2 by right translation.) Then $g \mapsto Age'$ is a well defined morphism $\tilde{X} \to X, \tilde{X}' \to X', \tilde{X}'' \to X''$. This is a finite principal covering with group $\Gamma, \Gamma', \Gamma''$ respectively (acting by right translation) where
$\Gamma = \underline{T}_2^F$ (of order $q^2 - q$)
$\Gamma' = \{x \in \underline{T}_2\underline{U}_2; x^{-1}hF(x) \in h\underline{U}\}/(\underline{U} \cap h\underline{U}h^{-1}) \cong \{\pm 1\} \times \mathbf{F}_q$ (of order $2q$ if q is odd, of order q if q is a power of 2)
$\Gamma'' = \{t \in \underline{T}_2; F(t) = t^{-1}\}$ (of order $q^2 + q$).

For any variety Y with an action of a finite group and any character ω of that finite group, let $H_c^j(Y)_\omega$ denote the subspace of $H_c^j(Y)$ on which the finite group acts according to ω. Thus, for ω in $\hat{\Gamma}$ (resp. $\hat{\Gamma}', \hat{\Gamma}''$), $H_c^j(\tilde{X})_\omega$ (resp. $H_c^j(\tilde{X}')_\omega, H_c^j(\tilde{X}'')_\omega$) is well defined.

3.3. Let $Y = \{g \in G; g^{-1}F(g) \in h\underline{U}_2\}$. We wish to describe Y more explicitly. If $g \in G_2$, the condition that $g \in Y$ is that
$F(g)e = ghue = gh(e + xe') = g(e + xe' + \epsilon xe)$,
$F(g)e' = ghue' = gh(e') = g(e' + \epsilon e)$
for some $x \in A$. Define $a, b, c, d \in A$ by $ge = ae + be', ge' = ce + de'$. The condition that $g \in Y$ is
$F(a)e + F(b)e' = ae + be' + \epsilon xae + \epsilon xbe' + xce + xde'$,
$F(c)e + F(d)e' = ce + de' + \epsilon ae + \epsilon be'$
for some $x \in A$. Thus, we may identify Y with the set of all $(a, b, c, d) \in A^4$ such that

$F(a)=a+\epsilon xa+xc, F(b)=b+\epsilon xb+xd, F(c)=c+\epsilon a, F(d)=d+\epsilon b, ad-bc=1$ for some $x\in A$, or equivalently, such that

$(F(a)-a)(\epsilon b+d)=(F(b)-b)(\epsilon a+c), F(c)=c+\epsilon a, F(d)=d+\epsilon b, ad-bc=1$. Setting $a=a_0+\epsilon a_1, b=b_0+\epsilon b_1, c=c_0+\epsilon c_1, d=d_0+\epsilon d_1$ with $a_i,d_i\in\mathbf{F}$, we see that Y is identified with the set consisting of all $(a_0,b_0,c_0,d_0,a_1,b_1,c_1,d_1)\in\mathbf{F}^8$ such that

(a) $c_0^q=c_0, d_0^q=d_0, c_1^q=c_1+a_0, d_1^q=d_1+b_0$,

(b) $a_0d_0-b_0c_0=1, a_0d_1+a_1d_0-b_0c_1-b_1c_0=0$,

(c) $(a_0^q-a_0)d_0=(b_0^q-b_0)c_0$,

$(a_0^q-a_0)b_0+(a_0^q-a_0)d_1+(a_1^q-a_1)d_0=(b_0^q-b_0)a_0+(b_0^q-b_0)c_1+(b_1^q-b_1)c_0$.

Actually the equations (c) are a consequence of the other equations, hence they can be omitted. The first equation (b) can be written (using (a)):

$(c_1^q-c_1)d_0-(d_1^q-d_1)c_0=1$,

that is,

$(c_1d_0-d_1c_0)^q-(c_1d_0-d_1c_0)=1$.

Setting $f=c_1d_0-d_1c_0$, we see that Y is identified with the set of all

$(a_0,b_0,c_0,d_0,a_1,b_1,c_1,d_1,f)\in\mathbf{F}^9$

such that

$c_0^q=c_0, d_0^q=d_0, c_1^q=c_1+a_0, d_1^q=d_1+b_0$,

$f^q-f=1, c_1d_0-d_1c_0=f, a_0d_1+a_1d_0-b_0c_1-b_1c_0=0$.

Now on Y we have a free right action of $\underline{U}_2^1=\underline{U}_2\cap h\underline{U}_2h^{-1}$, $u:g\mapsto gu$. In terms of coordinates, this is $(a,b,c,d)\mapsto(a+\epsilon xc, b+\epsilon xd, c, d), x\in\mathbf{F}$ or

$$(a_0,b_0,c_0,d_0,a_1,b_1,c_1,d_1,f)\mapsto(a_0,b_0,c_0,d_0,a_1+xc_0,b_1+xd_0,c_1,d_1,f).$$

The set of orbits $Y/\underline{U}_2^1=\tilde{X}'$ may be identified with the set of all

$(a_0,b_0,c_0,d_0,c_1,d_1,f)\in\mathbf{F}^7$

such that

$$c_0^q=c_0, d_0^q=d_0, c_1^q=c_1+a_0, d_1^q=d_1+b_0, f^q-f=1, c_1d_0-d_1c_0=f.$$

We consider the obvious projection of this set to the finite set

$\{(c_0,d_0,f)\in\mathbf{F}^3; c_0^q=c_0, d_0^q=d_0, f^q-f=1, (c_0,d_0)\neq(0,0)\}$.

The fibre of this projection at $(c_0,d_0,f)\in\mathbf{F}^3$ is the affine line $\{(c_1,d_1)\in\mathbf{F}^2; c_1d_0-d_1c_0=f\}$. Thus, $\tilde{X}'$ is a union of $(q^2-1)q$ affine lines. Hence $H_c^j(\tilde{X}')=0$ for $j\neq 2$ and $H_c^2(\tilde{X}')$ is a permutation representation of G_2^F of dimension $(q^2-1)q$. For q odd, it follows easily that, as a G_2^F-module, $H_c^2(\tilde{X}')$ is the direct sum of the $2q$ irreducible representations of degree $(q^2-1)/2$ (each one with multiplicity 1); more precisely, for any $\omega'\in\hat{\Gamma}'$, $H_c^2(\tilde{X}')_{\omega'}$ is irreducible of degree $(q^2-1)/2$ and each irreducible representation of degree $(q^2-1)/2$ is obtained for exactly one ω'.

Similarly, for q a power of 2, $H_c^2(\tilde{X}')$ is the direct sum of the q irreducible representations of degree q^2-1 (each one with multiplicity 1); more precisely, for any $\omega'\in\hat{\Gamma}'$, $H_c^2(\tilde{X}')_{\omega'}$ is irreducible of degree q^2-1 and each irreducible representation of degree q^2-1 is obtained for exactly one ω'.

3.4. Now $\tilde{X}$ is a permutation representation of G_2^F of dimension $q^4 - q^2$ which is easy to analyze. We see that $H_c^j(\tilde{X}) = 0$ for $j \neq 0$ and, for q odd, $H_c^0(\tilde{X})$ is the direct sum of all irreducible representations of degree $q^2 + q$ and $q + 1$ (each one with multiplicity 2), those of degree $1, q, (q+1)/2$ (each one with multiplicity 1) and $H_c^2(\tilde{X}')_{\omega'}$ with $\omega' \in \hat{\Gamma}'$, $\omega'^2 = 1$ (each one with multiplicity $q - 1$). More precisely, if $\omega \in \hat{\Gamma}$ then $H_c^0(\tilde{X})_\omega$ is

irreducible of degree $q^2 + q$ if $\omega|_{\underline{T}_2^1{}^F} \neq 1$;

the direct sum of $\oplus_{\omega' \in \hat{\Gamma}; \omega'^2 = 1} H_c^2(\tilde{X}')_{\omega'}$ with an irreducible representations of degree $q + 1$, if $\omega|_{\underline{T}_2^1{}^F} = 1, \omega^2 \neq 1$;

the direct sum of $\oplus_{\omega' \in \hat{\Gamma}; \omega'^2 = 1} H_c^2(\tilde{X}')_{\omega'}$ with the two irreducible representations of degree $(q + 1)/2$, if $\omega|_{\underline{T}_2^1{}^F} = 1, \omega^2 = 1, \omega \neq 1$;

the direct sum of $\oplus_{\omega' \in \hat{\Gamma}; \omega'^2 = 1} H_c^2(\tilde{X}')_{\omega'}$ with the two irreducible representations of degree 1 and q, if $\omega = 1$.

On the other hand, for q a power of 2, $H_c^0(\tilde{X})$ is the direct sum of all irreducible representations of degree $q^2 + q$ and $q + 1$ (each one with multiplicity 2), those of degree $1, q, (q^2+q)/2$ (each one with multiplicity 1) and $H_c^2(\tilde{X}')_1$ (with multiplicity $q - 1$). More precisely, if $\omega \in \hat{\Gamma}$ then $H_c^0(\tilde{X})_\omega$ is

irreducible of degree $q^2 + q$ if $\omega|_{\underline{T}_2^1{}^F} \neq 1, \omega^2 \neq 1$;

the direct sum of two irreducible representations of degree $(q^2 + q)/2$ if $\omega^2 = 1, \omega \neq 1$;

the direct sum of $H_c^2(\tilde{X}')_1$ with an irreducible representations of degree $q + 1$, if $\omega|_{\underline{T}_2^1{}^F} = 1, \omega \neq 1$;

the direct sum of $H_c^2(\tilde{X}')_1$ with the two irreducible representations of degree 1 and q, if $\omega = 1$.

3.5. Let

$$\mathfrak{S}_0 = \{x_0 \in V; x_0 \wedge F(x_0) = e \wedge e'\}, \quad \mathfrak{S}_{00} = \{x_0 \in \mathfrak{S}_0; F^2(x_0) = -x_0\}.$$

Now G^F acts on $\mathfrak{S}_0$ (restriction of the G-action on V). This restricts to a G^F-action on $\mathfrak{S}_{00}$. We show that this action is simply transitive. If $g \in G^F$ keeps fixed some $x_0 \in \mathfrak{S}_{00}$ then it also keeps fixed $F(x_0)$ hence it must be 1 (recall that $x_0, F(x_0)$ form a basis of V). Thus the G^F-action on $\mathfrak{S}_{00}$ has trivial isotropy. We may identify $\mathfrak{S}_{00}$ with $\{(a, b) \in \mathbf{F}^2; ab^q - a^q b = 1, a^{q^2} = -a, b^{q^2} = -b\}$. For such (a, b) we have automatically $a \neq 0$. We make a change of variable $(a, b) \mapsto (a, c)$ where $c = b/a$. Then $\mathfrak{S}_{00}$ becomes

$\{(a, c) \in \mathbf{F}^2; a^{q+1}(c^q - c) = 1, a^{q^2} = -a, c^{q^2} = c\}$.

The second projection maps this to $\{c \in \mathbf{F}; c^{q^2} = c, c^q \neq c\}$ which has $q^2 - q$ elements. The fibre at c is $\{a \in \mathbf{F}; a^{q+1} = (c^q - c)^{-1}\}$. (For such a we have automatically $a^{q^2} = -a$ since $c^{q^2} = c$.) This fibre has exactly $q + 1$ elements since $(c^q - c)^{-1} \neq 0$. We see that $\sharp(\mathfrak{S}_{00}) = (q + 1)(q^2 - q) = \sharp(G^F)$. It follows that the G^F-action on $\mathfrak{S}_{00}$ is indeed simply transitive.

3.6. We now analyze $\tilde X''$. Let

$$\mathfrak{S}=\{x\in V_2; x\wedge F(x)=e\wedge e'\}.$$

Now G_2^F acts on $\mathfrak{S}$ by $g_1: x\mapsto g_1x$. The map $g\mapsto g(e')$ defines an isomorphism
$\iota:\tilde X''\xrightarrow{\sim}\mathfrak{S}$.
We check that this is a well defined bijection. Let $g\in\tilde X''$. Then $F(g)=g\nu u$ for some $u\in\underline{U}_2$. Let $x=ge'$. Then for some $u\in\underline{U}_2$ we have

$$x\wedge F(x)=(ge')\wedge F(ge')=(ge')\wedge F(g)e'=e'\wedge g^{-1}F(g)e'=e'\wedge\nu ue'$$
$$=e'\wedge\nu e'=e'\wedge(-e)=e\wedge e',$$

hence $x\in\mathfrak{S}$ and ι is well defined. Now let $x\in\mathfrak{S}$. We can find $g\in G_2$ such that $ge'=x$. Then

$$e\wedge e'=x\wedge F(x)=(ge')\wedge F(ge')=(ge')\wedge F(g)e'=e'\wedge g^{-1}F(g)e'.$$

Hence $g^{-1}F(g)e'=-e+be'$ for some $b\in A$. It follows that $g^{-1}F(g)=u'\nu u$ where $u,u'\in\underline{U}_2$. Then $(gu')^{-1}F(gu')=\nu uF(u')$ hence $gu'\in\tilde X''$. Clearly, $\iota(gu')=x$ so that ι is surjective. Now assume that $g,g'\in\tilde X''$ satisfy $\iota(g)=\iota(g')$ that is $ge'=g'e'$. Then $g'=gu'$, $u'\in\underline{U}_2$. We have $g'^{-1}F(g')=\nu u$ with $u\in\underline{U}_2$ hence $u'^{-1}g^{-1}F(g)F(u')=\nu u$. Also, $g^{-1}F(g)=\nu\tilde u$ with $\tilde u\in\underline{U}_2$ hence $u'^{-1}\nu\tilde uF(u')=\nu u$ so that $u'\in\nu\underline{U}_2\nu^{-1}$. Thus, $u'\in\underline{U}_2\cap(\nu\underline{U}_2\nu^{-1})=\{1\}$ hence $u'=1$ and $g'=g$. Thus, ι is injective hence bijective. It commutes with the G_2^F-actions.

Now $\mathfrak{S}$ consists of the elements $x_0+\epsilon x_1$, with $x_0,x_1\in V$ such that
$(x_0+\epsilon x_1)\wedge(F(x_0)+\epsilon F(x_1))=e\wedge e'$,
that is
$x_0\wedge F(x_0)=e\wedge e'$ and $x_1\wedge F(x_0)+x_0\wedge F(x_1)=0$.
We have a morphism
$\kappa:\mathfrak{S}\to\mathfrak{S}_0, x_0+\epsilon x_1\mapsto x_0$.
If $x_0\in\mathfrak{S}_0$ then $\kappa^{-1}(x_0)$ may be identified with
$\{x_1\in V; x_1\wedge F(x_0)+x_0\wedge F(x_1)=0\}$.
Note that $x_0,F(x_0)$ form a basis of V hence $F^2(x_0)=c_0x_0+c_1F(x_0)$ with $c_0,c_1\in\mathbf{F}$. Since $x_0\wedge F(x_0)=e\wedge e'$ is F-stable, we have $x_0\wedge F(x_0)=F(x_0)\wedge F^2(x_0)$ hence $c_0=-1$. Let $\mathfrak{S}_{01}=\mathfrak{S}_0-\mathfrak{S}_{00}$. We have a partition $\mathfrak{S}=\mathfrak{S}_*\cup\mathfrak{S}_{**}$ where $\mathfrak{S}_*=\kappa^{-1}(\mathfrak{S}_{00}),\mathfrak{S}_{**}=\kappa^{-1}(\mathfrak{S}_{01})$ are G_2^F-stable. If $x_0\in\mathfrak{S}_0$, then any $x_1\in V$ can be written uniquely in the form
$x_1=a_0x_0+a_1F(x_0)$
with $a_0,a_1\in\mathbf{F}$. The condition that $x_0+\epsilon x_1\in\kappa^{-1}(x_0)$ is
$(a_0x_0+a_1F(x_0))\wedge F(x_0)+x_0\wedge(a_0^qF(x_0)+a_1^qF^2(x_0))=0$,
that is,
$a_0x_0\wedge F(x_0)+x_0\wedge(a_0^qF(x_0)-a_1^qx_0+a_1^qc_1F(x_0))=0$,
that is,

$a_0x_0 \wedge F(x_0) + a_0^q x_0 \wedge F(x_0) + a_1^q c_1 x_0 \wedge F(x_0) = 0$,

or $a_0 + a_0^q + a_1^q c_1 = 0$.

Thus we may identify $\kappa^{-1}(x_0)$ with $\{(a_0, a_1) \in \mathbf{F}^2; a_0 + a_0^q + a_1^q c_1 = 0\}$. If $c_1 \neq 0$ (that is if $x_0 \in \mathfrak{S}_{01}$) this is isomorphic to the affine line. Thus, κ restricts to an affine line bundle $\mathfrak{S}_{**} \to \mathfrak{S}_{01}$.

Now the action of Γ'' on $\tilde{X}''$ corresponds under ι to the action of $\{\lambda \in A; \lambda F(\lambda) = 1\}$ on $\mathfrak{S}$ by scalar multiplication. Hence the action of $\{t \in \Gamma''; t \in T_2^1\}$ on $\tilde{X}''$ corresponds to the action of $A' = \{\lambda \in A; \lambda F(\lambda) = 1, \lambda \in 1 + \epsilon A\}$ on $\mathfrak{S}$ by scalar multiplication. The action of $1 + \epsilon\lambda_1 \in A'$ (with $\lambda_1 \in \mathbf{F}$) in the coordinates (x_0, a_0, a_1) is $(x_0, a_0, a_1) \mapsto (x_0, a_0 + \lambda_1, a_1)$. Thus it preserves each fibre of κ.

Now $\mathfrak{S}_{**}$ is stable under the action of $\{\lambda \in A; \lambda F(\lambda) = 1\}$ and the restriction of this action to A' preserves each fibre of $\mathfrak{S}_{**} \to \mathfrak{S}_{01}$ (an affine line); hence this group acts trivially on $H_c^j()$ of each such fibre hence it also acts trivially on $H_c^j(\mathfrak{S}_{**})$. Thus, $H_c^j(\mathfrak{S}) \to H_c^j(\mathfrak{S}_*)$ is an isomorphism on the part where $\sum_{\lambda \in A'} \lambda$ acts as 0.

We now study $H_c^j(\mathfrak{S}_*)$. If $x_0 \in \mathfrak{S}_{00}$ then $\kappa^{-1}(x_0)$ may be identified with $\{(a_0, a_1) \in \mathbf{F}^2; a_0 + a_0^q = 0\}$. Thus, $\mathfrak{S}_*$ is an affine line bundle over

$\mathfrak{S}_{00} \times \{a_0 \in \mathbf{F}; a_0 + a_0^q = 0\}$

which is a transitive permutation representation of G_2^F that is explicitly known from 3.5. It follows that $H_c^j(\mathfrak{S}_*) = 0$ for $j \neq 2$ and the part of $H_c^2(\mathfrak{S}_*)$ where $\sum_{\lambda \in A'} \lambda$ acts as 0 is the direct sum of the irreducible representations of degree $q^2 - q$ (each one with multiplicity 2) and of degree $(q^2 - q)/2$ (each one with multiplicity 1); note that the latter representations occur only when q is a power of 2.

We now study the part of $H_c^j(\mathfrak{S})$ where A' acts as 1. This is the same as $H_c^j(A' \backslash \mathfrak{S})$. The map $(x_0, a_0, a_1) \mapsto (x_0, \tilde{a}_0, a_1), \tilde{a}_0 = a_0 + a_0^q$ is an isomorphism of $A' \backslash \mathfrak{S}$ with the set of all $(x_0, \tilde{a}_0, a_1) \in \mathfrak{S}_0 \times \mathbf{F} \times \mathbf{F}$ such that $\tilde{a}_0 + a_1^q c_1 = 0$. (Here c_1 is determined by x_0 as above.) Hence the map $(x_0, a_0, a_1) \mapsto (x_0, a_1)$ is an isomorphism $A' \backslash \mathfrak{S} \xrightarrow{\sim} \mathfrak{S}_0 \times \mathbf{F}$. Thus, $H_c^j(A' \backslash \mathfrak{S}) = H_c^{j-2}(\mathfrak{S}_0)$. Thus, G_2^F acts on $H_c^j(A' \backslash \mathfrak{S})$ through its quotient G^F and that action is explicitly known from the representation theory of G^F.

We see that $H_c^4(\tilde{X}'')$ is the 1 dimensional representation; $H_c^3(\tilde{X}'')$ is the direct sum of all irreducible representations of degree $q-1$ (each one with multiplicity 2) and those of degree $(q-1)/2, q$ (each one with multiplicity 1); $H_c^2(\tilde{X}'')$ is the direct sum of all irreducible representations of degree $q^2 - q$ (each one with multiplicity 2) and of degree $(q^2 - q)/2$ (each one with multiplicity 1); $H_c^j(\tilde{X}'') = 0$ for $j \notin \{2, 3, 4\}$; note that the representations of degree $(q-1)/2$ occur only for q odd, while those of degree $(q^2 - q)/2$ occur only for q a power of 2.

More precisely, if $\omega \in \hat{\Gamma}''$ and q is odd, then

$H_c^4(\tilde{X}'')_\omega$ is irreducible of degree 1 if $\omega = 1$ and is 0 otherwise;

$H_c^3(\tilde{X}'')_\omega$ is irreducible of degree $q-1$ if $\omega|_{\Gamma'' \cap T_2^1} = 1, \omega^2 \neq 1$; it is the direct sum of two irreducible representations of degree $(q-1)/2$ if $\omega|_{\Gamma'' \cap T_2^1} = 1, \omega^2 = 1, \omega \neq 1$;

it it is irreducible of degree q if $\omega = 1$; it is 0 if $\omega|_{\Gamma'' \cap T_2^1} \neq 1$;

$H_c^2(\tilde{X}'')_\omega$ is irreducible of degree $q^2 - q$ if $\omega|_{\Gamma'' \cap T_2^1} \neq 1$ and is 0 otherwise.

Similarly, if $\omega \in \hat{\Gamma}''$ and q is a power of 2, then

$H_c^4(\tilde{X}'')_\omega$ is irreducible of degree 1 if $\omega = 1$ and is 0 otherwise;

$H_c^3(\tilde{X}'')_\omega$ is irreducible of degree $q - 1$ if $\omega|_{\Gamma'' \cap T_2^1} = 1$, $\omega \neq 1$; it is irreducible of degree q if $\omega = 1$; it is 0 if $\omega|_{\Gamma'' \cap T_2^1} \neq 1$;

$H_c^2(\tilde{X}'')_\omega$ is irreducible of degree $q^2 - q$ if $\omega|_{\Gamma'' \cap T_2^1} \neq 1$, $\omega^2 \neq 1$; it is the direct sum of two irreducible representations of degree $(q^2 - q)/2$ if $\omega^2 = 1, \omega \neq 1$; it is 0 otherwise.

3.7. We see that any irreducible representation of G_2^F appears in at least one of the representations $H_c^j(\tilde{X})_\omega, H_c^j(\tilde{X}')_\omega, H_c^j(\tilde{X}'')_\omega$. More precisely, the regular representation of G_2^F is a $\mathbf{Q}$-linear combination of the virtual representations

$$\sum_{j \in \mathbf{Z}} (-1)^j H_c^j(\tilde{X})_\omega, \sum_{j \in \mathbf{Z}} (-1)^j H_c^j(\tilde{X}')_\omega, \sum_{j \in \mathbf{Z}} (-1)^j H_c^j(\tilde{X}'')_\omega.$$

3.8. Let $\gamma \in G$ be such that $\gamma^{-1} F(\gamma) = \nu$. We set $T = \gamma \underline{T} \gamma^{-1}$, $U = \gamma \underline{U} \gamma^{-1}$. Then T is an F-stable maximal torus of G and U is the unipotent radical of a Borel subgroup of G containing T. Hence $S_{T,U}$ is defined (with $r = 2$). Now $g \mapsto g\gamma^{-1}$ defines an isomorphism

$\tilde{X}'' \xrightarrow{\sim} S_{T,U}$

and an isomorphism $\Gamma'' \xrightarrow{\sim} T_2^F$. Also $G_2^F \times \Gamma''$ acts on $\tilde{X}''$ by $(g_1, t) : g \mapsto g_1 g t^{-1}$. This action is compatible with the $G_2^F \times T_2^F$-action on $S_{T,U}$ via the isomorphisms above. We see that the virtual representations $\sum_{j \in \mathbf{Z}} (-1)^j H_c^j(\tilde{X}'')_\omega$ of G_2^F are the same as the virtual representations $R_{T,U}^\theta$.

3.9. We return to the general setup of 1.2. Let $\underline{B}$ be an F-stable Borel subgroup of G with unipotent radical $\underline{U}$. For any $x \in G_r$ let $X_x = \{g \in G_r; g^{-1}F(g) \in x\underline{U}_r\}$. Then G_r^F acts on X_x by left translations hence it acts naturally on $H_c^j(X_x)$. Note that the isomorphism class of the G_r^F-module $H_c^j(X_x)$ depends only on the $(\underline{U}_r, \underline{U}_r)$ double coset of x in G_r. We conjecture that any irreducible representation of G_r^F appears in $\sum_{j \in \mathbf{Z}} (-1)^j H_c^j(X_x)$ for some $x \in G_r$. This holds for $r = 1$ (see [DL]) and also for $G = SL_2, r = 2$, by the results in this section.

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