

DESIGN OF A ROBUST CONTROL SYSTEM FOR  
POSTFAILURE OPERATION

by

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# **DESIGN OF A ROBUST CONTROL SYSTEM FOR POSTFAILURE**

## **OPERATION**

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### **ABSTRACT**

A typical situation in which a feedback system significantly loses its performance is when a sudden and substantial change of the physical process occurs that was not incorporated in the initial controller design. We refer to such changes as failures. The treatment of a failure situation can usually be divided into three phases: 1) failure detection, 2) failure identification, and 3) the redesign of the original controller. In this work we have focussed on the latter.

A systematic approach for redesigning the feedback system under various failure scenarios is presented. More specifically, the problem of controller redesign is formulated as a model matching problem in the presence of uncertainty. The uncertainty is a consequence of the failure identification process itself. Analysis and synthesis methods are derived for the failure situations where either frequency dependent, norm bounded or real parameter uncertainty is present. A measure of the "difference" between the original and the "failed" system is defined. Necessary and sufficient conditions for the existence of a new controller that "recovers" the original system in the given sense are presented.

In the case of real parametric uncertainty, it is shown that performance and stability robustness analysis based on the frequently used Riccati equation approach is no less conservative than if the Small Gain Theorem were applied. A case where real parametric uncertainty can be reflected at the input of the plant is presented. Examples, using a modified model of the F-8 Aircraft, are included to demonstrate the ideas.

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## NOTATION AND DEFINITIONS

$\mathbb{R}$	field of real numbers
$\mathbb{R}_+$	positive real numbers
$\mathbb{R}^n$	space of ordered n-tuples of real numbers
$\mathbb{R}^{n \times m}$	set of n by m matrices with elements in $\mathbb{R}$
$\mathbb{C}, \mathbb{C}^n, \mathbb{C}^{n \times m}$	complex analog of $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n \times m}$
$A^{-1}$	inverse of matrix A
$A^{-R}$	right inverse of matrix A
$x' / A'$	vector / matrix transpose
$A > 0$ ( $< 0$ )	matrix A is positive (negative) definite
$A \geq 0$ ( $\leq 0$ )	matrix A is positive (negative) semi-definite
$\lambda(A)$	an eigenvalue of matrix A
$\lambda_{\max}(A)$	the largest eigenvalue of matrix A
$\overline{\sigma}(A)$	the maximum singular value of matrix A
$\underline{\sigma}(A)$	the minimum singular value of matrix A
$\Lambda(A)$	spectrum of matrix A
$\rho(A)$	the spectral radius of matrix A
$\mu(A)$	the structured singular value of matrix A
$\det(A)$	determinant of matrix A
$G\tilde{(s)}$	$:= G'(-s)$

$j\omega$	point on imaginary axis ( $\omega \in \mathbb{R}$ )
$j\mathbb{R}$	the imaginary axis
$L_2$	Hilbert space of matrix-valued functions which are square integrable on $j\mathbb{R}$ in the sense of the inner product on $j\mathbb{R}$
$H_2$	the functions in $L_2$ that are analytic in the open right half plane, i.e. Hardy space
$L_\infty$	Banach space of matrix-valued functions which are (essentially) bounded on $j\mathbb{R}$
$RL_\infty$	rational functions in $L_\infty$ with real coefficients
$H_\infty$	functions in $L_\infty$ with bounded analytic continuation to right half plane
$RH_\infty$	rational functions in $H_\infty$ with real coefficients
$\ \cdot\ _2$	norm on $L_2$
$\ \cdot\ _\infty$	norm on $L_\infty$
$\ G\ $	the operator norm of $G$
$[A, B, C, D]$	$:= D + C(sI - A)^{-1}B$ , transfer function notation
$\arg\{\cdot\}$	argument
$\text{diag}\{a, \dots, b\}$	diagonal matrix with entries $(a, \dots, b)$
$\text{Im}$	image
$\lim$	limit
$\text{Ker}(\cdot)$	kernel (null space)
$\text{Ker}_R(\cdot)$	right kernel (right null space)

Ric( H )	solution of the Riccati equation associated with the Hamiltonian matrix H
RHP / LHP	Right / Left Half Plane
$\Gamma_G$	Hankel operator associated with $G \in RL_\infty$
$X_-( A )$	stable modal subspace relative to matrix A
$X_+( A )$	unstable modal subspace relative to matrix A
$a \equiv b$	a is defined as b
■	end of proof

### Definition 1

A system  $M(s) \in RH_\infty$  is said to be (strictly) nonexpansive if  $\| M(j\omega) \|_\infty \leq (<) 1$ .

### Definition 2

A subset  $S$  of a vector space  $X$  is convex if  $x, y \in S$  implies

$$S \supset M = \{ z \in X \mid z = \alpha x + (1-\alpha) y, \quad 0 \leq \alpha \leq 1 \}$$

### Definition 3

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be convex if its domain  $D(f)$  is a convex set and for every  $u, v \in D(f)$ ,

$$f(\lambda u + [1-\lambda]v) \leq \lambda f(u) + [1-\lambda] f(v)$$

where  $0 \leq \lambda \leq 1$ .

**Definition 4**

A matrix  $G_i$  in  $RH_\infty$  is inner if  $G_i^* G_i = I$ . A matrix  $G_o$  in  $RH_\infty$  is outer if, for every  $\text{Re}(s) > 0$ ,  $G_o$  has full row rank. Every matrix  $G$  in  $RH_\infty$  has an inner-outer factorization  $G = G_i G_o$ .

**Definition 5**

The lower linear fractional transformation  $H_l(M, K)$  is equal to the closed loop transfer function from  $u_1$  to  $y_1$  in Figure 1,

$$H_l(M, K) = M_{11} + M_{12} K (I - M_{22} K)^{-1} M_{21}$$

**Definition 6**

The upper linear fractional transformation  $H_u(M, K)$  is equal to the closed loop transfer function from  $u_2$  to  $y_2$  in Figure 2,

$$H_u(M, K) = M_{22} + M_{21} K (I - M_{11} K)^{-1} M_{12}$$

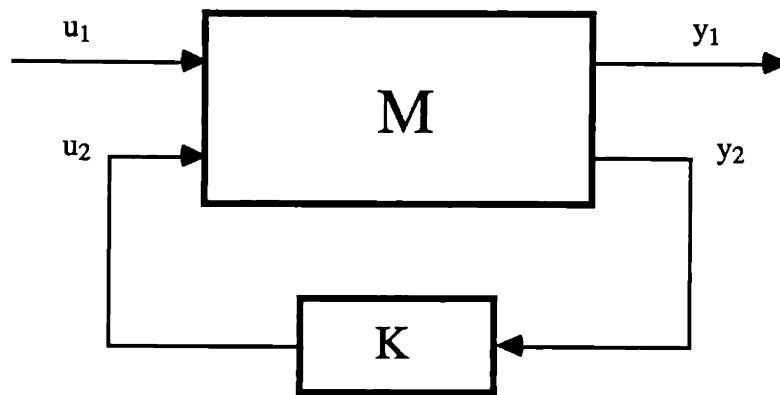


Figure 1 Block Diagram for the Lower Linear Fractional Transformation

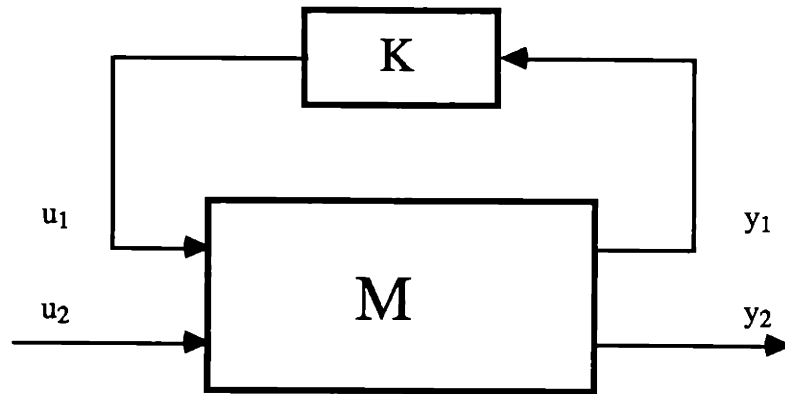


Figure 2 Block Diagram for the Upper Linear Fractional Transformation

# CHAPTER 1

## INTRODUCTION

### 1.1 Motivation

Feedback system design is performed in order to obtain a desired behavior for a physical process, i.e. plant, whose dynamics may be only partially known. A necessary requirement for this procedure to be carried out is the initial derivation of a "simple" mathematical description of these dynamics. Since the model is only an abstraction of the actual process, there is inevitable discrepancy between the plant itself and the model. Therefore, in order to more completely characterize the system for the design phase, the nominal model should be accompanied by modelling uncertainty. If the design is accomplished successfully, the resulting feedback system will be insensitive to certain changes in the environment and the process itself. Unfortunately, this will hold only if the nature and the magnitude of the uncertainty are properly anticipated and taken into account.

A typical situation when the designed control system significantly loses its performance is a sudden and substantial change of the physical process that was not incorporated in its mathematical model. We will refer to such changes as failures. A logical step in dealing with such a problem is the redesign of the existing control system. However, the new feedback system containing the failed plant and the redesigned controller will not necessarily have the same characteristics as the original feedback system.

A treatment of a failure situation can usually be divided into three steps. These are failure detection, failure identification and control system redesign.

It is noted that the failure detection step is critical. It is highly dependent on the process



itself but it is not of primary interest in this thesis. The nature and location of a failure is determined in the failure identification step. It includes modeling of the postfailure plant dynamics which results in its nominal model and the associated uncertainty. Once identification is done, the postfailure system is ready for evaluation and for controller redesign if necessary.

This thesis focuses on developing systematic procedures for controller redesign. A motivation for this is based on the fact that there is no clear answer as to how to quantify the impact of a failure on the original system and how to choose criteria for its control redesign, if needed.

A usual approach to this problem is to base the redesign procedure solely on the new nominal model and uncertainty measure for the failed system. The design criterion is usually identical to the one used for the original system. The procedure consists of varying weighting functions as design parameters until satisfactory performance is obtained. This is often a very tedious and time consuming process of limited practical applicability in postfailure situations. Furthermore, the latter approach completely neglects the existence of the original compensator that delivered the desired performance before the failure occurred.

Therefore, a systematic analysis and design methodology in dealing with the postfailure situation has yet to emerge. An approach will be derived in order to treat the postfailure situation after the failure detection and identification steps have been completed. Such an approach should take advantage of the already existing design of the original, prefailure control system and try to recover its performance as best as possible. Furthermore, it should establish an appropriate measure of divergence of the postfailure system from the original one. Since the available information about the failed plant might vary in its accuracy, the derived methodology will have to guarantee certain robustness properties in the presence of uncertainty.

## 1.2 Previous Work and Related Literature

There is an extensive literature covering failure situations in various technical systems. It is usually highly specialized to the type of system it treats.

The failure detection and identification is probably the most widely covered aspect of the failure treatment in dynamic systems. One of the most frequently cited references is [Willsky76] where a survey of design methods for failure detection is presented. This work is followed by a series of reports on the development of a detection methodology for system failures [LIDS77 - 81], [CW80] and finally [MVW89]. Further contributions are made by Walker [Walker83] and Pattipati et al [Patt84].

An area where the failure detection and identification has been of much interest is in aeronautics, more specifically flight control [Jones73], [Mont82], [Hall85], [LIDS80-83]. A relatively simple redesign approach was proposed by Vander Valde [VV84] in the case of failed sensors and actuators. It is based on the pseudo inverse of the "failed" output and input matrices in the state space description of the system. For the general class of systems, a method for matching the eigenstructure of the prefailure and postfailure systems was introduced. Stability under sensor and actuator failures is also treated in [DG88] and [GK89].

A complex study of failure situations in aircraft, from the control system point of view, is performed by Looze et al [LKWBE85], [WL85]. The possible approaches to the postfailure situations are divided into reconfigurable and restructurable scenarios. The reconfiguration stands for the cases where the failures are anticipated and the suitable control actions are prepared in advance. Once the actual failure occurs, the "failed system" controller is substituted with the one designed for that particular situation. On the other hand, for random failures, the whole procedure has to be done on-line. It is defined as restructurable control action. The failures treated in these papers are restricted to control

surfaces only. The original design was assumed to be done by the LQG/LTR methodology [DS81], [SA87] which was also used for obtaining the postfailure control system. The design parameters were weighting matrices associated with the LQG problem. They had to be varied until the design obtained resembled the original system characteristic. A somewhat different approach was undertaken in [Wagner88]. An advisory system was designed in order to assist pilots of control-impaired aircraft. The aspects of failure-tolerant control that lead to utilization of inherent control redundancy among different controls were discussed.

The approach undertaken in this thesis is based on the criterion that the characteristics of the postfailure compensated system should match those of the original one as closely as possible. The main difference from the above approaches is that the existence of the original compensator is used in the design of the new control system. In this thesis, restructuring is posed as a model matching problem where the error signals of two systems are minimized in a specified sense. Furthermore, the procedure is capable of handling both dynamic and real parameter uncertainties.

The treatment of the dynamic, norm bounded uncertainty is based on the works of Doyle, Stein and others [Doyle82,83,85], [DS81], [DWG82]. Some of the references that have provided efficient methods for designing robust multivariable systems by using state space techniques are [Doyle84], [Fran83], [CDL86], [Fran87]. The " $H_\infty$ " design methodology based on two Riccati equations that is used herein is due to Doyle et al [DGKF88]. This will be used extensively throughout the thesis since performance of the postfailure system will be defined as the value of the " $H_\infty$ " norm of the difference between transfer function of interest of the original and postfailure systems.

Real parameter perturbations are very important in stability and performance analysis and synthesis. The existing approaches for analysis are classified according to the representation of the system. Hence, there are polynomial approaches among which the

best known result is Kharitonov's Theorem [Khar78]. It establishes the conditions on the coefficients of the characteristic polynomial, which are assumed to be independently varying, so that it remains Hurwitz.

Besides the Kharitonov result, the other group of methods for checking the stability of the perturbed characteristic polynomial are those based on the "zero exclusion criterion." They exploit the fact that the roots of polynomials are continuous functions of their coefficients. The assumptions are that the nominal system is stable and that the coefficients of the characteristic polynomial depend on the independent, magnitude bounded, parameters. Then, stability is guaranteed if the polynomial has no roots on the imaginary axis for any value of the varying coefficients. Based on the above criterion, de Gaston and Safonov [DGS88] have defined a nonconservative measure for the largest stability hyperbox in the parameter space. They have used the convexity properties of the image of the hyperbox vertices mapped into the complex plane through the characteristic polynomial of the perturbed system. A more general result was presented by Saeki [Saeki86], where the uncertain parameters could be frequency dependent.

Another group of approaches deal with the presence of real parameter perturbation in the state space representation of a system. They are based on the concept of Lyapunov stability. A similar classification obtains for the system design problem in the presence of real parameter perturbation. The results, except in special cases, are usually ad hoc and very conservative.

A very detailed survey of the literature on parametric uncertainty can be found in [Bhat87] and [Siljak89]. We will mention only those results that deal with real parameter perturbation in state space models. Special attention will be given to the results on robust performance of perturbed systems when the performance is associated with the infinity norm of a particular transfer function.

A sufficient condition for robust stability in the presence of real parameter perturbations in the state space was originally presented in [CP72] and [PTS77]. The "A" matrix of the

perturbed system was given by its nominal value  $A_0$  and the associated error matrix  $E(t)$  that could be time varying. The corresponding Lyapunov equation for the nominal system is

$$A_0' P + P A_0 + 2Q = 0 \quad (1.1)$$

where  $Q > 0$  and  $A_0$  is stable. It was shown that the system remains stable if the maximum singular value of the error matrix  $E$  is strictly smaller than the ratio between the smallest singular value of  $Q$  and the largest singular value of  $P$ . This ratio represents the upper bound on the "two" norm of the admissible perturbation matrix  $E$  and it is clearly dependent on the choice of the matrix  $Q$ . It was shown by Patel and Toda [PT80] that the best choice for  $Q$  in (1.1) is the identity matrix. Furthermore, the structure of the perturbation matrix  $E$  was taken into account by checking the largest magnitude of its elements against the scaled ratio defined above.

The structural information was further taken in consideration by Yedavalli in [Yed85a,b]. It was shown in [YL86] that this approach depends on state space representation of the system. An algorithm for improving the stability bounds based on the state space similarity transformation with diagonal scaling matrices was introduced.

The stability hypersphere in the parameter space of a system given by its state space representation was treated by Keel et al [KBH88]. An assumption was made that the uncertain parameters enter the "A" matrix of the system linearly. Sufficient conditions in the form of upper bounds on the "two" norm of the perturbation vector were derived in order to guarantee stability of the perturbed system. A "robustification" algorithm was introduced where the stability radius was gradually increased by changing the original controller.

For the magnitude bounded time varying and time invariant parameter perturbation, a Riccati equation synthesis approach for linear control systems was developed by Peterson and Hollot [PH86], [Peter87a]. It requires that the uncertain parameters enter the state space representation of the system linearly and that certain "matching" conditions be satisfied. Then, a solution to the introduced Riccati equation exists and the stability of the

closed loop system is guaranteed for the given range of perturbation. This holds since it can be shown that the latter is an upper bound for all Riccati equations corresponding to any point in the perturbation space. Hence, it is said that it contains a Peterson-Hollot bounding function.

In [Peter87b] was shown that the same Riccati equation guarantees the bound on the " $H_\infty$ " norm of the obtained system. Combining these two results led to the disturbance attenuation problem in the presence of real parameter perturbation. For stability analysis only, the different bounding functions are discussed in [BH88]. Synthesis approaches based on Bernstein and Haddad's " $H_2/H_\infty$ " design methodology [BH89], were introduced by [MHB88] and [YBBH89]. The conservatism of these methods is discussed in Chapter 6.

### 1.3 Contribution of Thesis

The main contributions of this thesis are the rigorous mathematical formulation of the postfailure control system redesign problem and the derivation of analysis and synthesis methods for its treatment. These are based on the information about the postfailure plant that is assumed to be obtained in the failure detection and identification steps. In general, it consists of the "failed" plant nominal model and the associated modeling uncertainty. Two different kinds of uncertainty will be studied in this thesis. They are the frequency dependent, norm bounded uncertainty and the real parameter uncertainty. Once the information about the postfailure plant is obtained, the impact of the failure can be assessed and the controller redesign performed if needed.

In this thesis, the basic analysis tools are developed for postfailure control systems.

The impact of a failure is quantified as the "distance" between the "failed" system and the nominal prefailure system. This distance is associated with a suitable measure of the error signal between corresponding outputs of two systems when they are subjected to the same input. We study the case where the input and the output signals of both systems are in  $L_2$ . Therefore, the impact of the failure is given as the " $H_\infty$ " norm value of the difference between the transfer functions corresponding to signals of interest of the original and postfailure systems. It is shown that this approach can be extended to situations where a failure results in loss of an input or output channel of the original plant. In general, this distance has to be evaluated over the uncertainty associated with the model of the postfailure plant.

The criterion whether to redesign the original controller after the failure is then posed as the problem of checking if the obtained performance level denoted as  $\gamma$  exceeds some previously set bound. When redesign is necessary, it is defined herein as the search for a compensator that stabilizes the postfailure plant in the presence of the modeling uncertainty and minimizes or brings the performance index below the previously set value.

The lower bound for the achievable value of the performance index is associated with the nominal model of the postfailure plant with no modeling uncertainty being present. Necessary and sufficient conditions for the "perfect" recovery of the nominal prefailure system in this case are derived. Situations where the postfailure plant is square or nonsquare are treated separately. Special attention is paid to the case where the failure is located only in part of the original plant.

Furthermore, an algorithm for controller redesign in the presence of frequency dependent uncertainty is presented. The algorithm is based on the "D-K" iteration [Doyle83], whose steps are substantially modified. It can be used for minimizing the performance index when the bound on the modelling uncertainty is known. In the case of anticipated failures, it can be used to maximize the uncertainty bound for the fixed value of

the performance index. The importance of this result is based on the fact that the proposed procedure tries to enlarge the bound on the uncertainty with fixed performance specification as much as possible without changing the block diagonal scaling "D" used in the first step of the "D-K" iteration. Therefore, we can possibly achieve the desired value of the above bound without going through approximation of "D" with a stable real rational function and thus there is no need to enlarge the order of the resulting compensator. The characteristics of the algorithm are based on the properties of certain minimization problems with respect to change in the bound of uncertainty.

The application of the algorithm is presented using the augmented model of the F-8 Aircraft. The complete analysis and interpretation of the obtained results is given.

A complete analysis methodology in the state space of the postfailure system with real parameter uncertainty is presented. The latter is assumed to enter the elements of the state space representation linearly and is of bounded magnitude. The significance of this result is that it extends the Lyapunov equation approach of Horisberger and Belanger [HB76] for the stability analysis only, to the stability/performance condition given in the form of a Riccati inequality. Furthermore, it allows the perturbation to enter all the elements of the state space representation of the postfailure system by linearizing the quadratic terms in the Riccati inequalities. Two possible linearization methods are introduced. The linearization based on the properties of positive matrices is used herein in a less conservative way than, for example, in Yeh et al [YBBH89]. The resulting criterion is based on the existence of a matrix  $P$  that simultaneously satisfies a set of Riccati inequalities corresponding to all the vertices of the hyperbox in the parameter space.

Furthermore, a simple transformation is presented for obtaining an equivalent performance condition which is linear in the matrix  $P$  that satisfies the corresponding Riccati inequality. This result is important because it shows that there is no need for constructing a new system whose passivity condition guarantees performance of the



postfailure system as it was done in Boyd and Yang[BY88]. Different algorithms are presented for finding the largest hyperbox in the parameter space where stability and performance specified by the desired value of  $\gamma$  are satisfied.

A very important result is presented in terms of a stability/performance condition based on a single Riccati equation with the suitable bounding function. When the real parametric uncertainty is contained in the "A" matrix only and when the Peterson-Hollot bounding function [PH86] is used, it is shown that the obtained criterion is as conservative as the "Small Gain Theorem" [Zames65] is for the same problem. Therefore, the real uncertainty is treated as frequency dependent in this case.

Furthermore, when the perturbation appears in all elements of the state-space representation of the postfailure system, it is shown that the obtained criterion is even more conservative than if the "Small Gain Theorem" is applied.

The significance of this result becomes even bigger in discussing the conservatism of the proposed synthesis methods, as in [YBBH89], that would result in a compensator that satisfies a single Riccati equation criterion introduced above. Hence, previously presented design methodology for frequency dependent uncertainty applied to the real parameter perturbation design problem can possibly be less conservative than if the single Riccati equation criterion is used.

In Chapter 7, a controller redesign method is proposed for the situations when the perturbed parameters can be reflected to the plant input.

It is important to note that some results obtained in studying the postfailure controller redesign can be extended to a broader class of control problems. The algorithm presented in Chapter 4 can be used in any situation with structured uncertainty where the bound on a single uncertainty is left as a design variable. The stability /performance analysis in based on Riccati equation approach holds for all problems with parametric and frequency dependent uncertainties being present simultaneously.

## 1.4 Organization of Thesis

This thesis is organized into eight chapters. In Chapter 2, the necessary mathematical background for the postfailure control system redesign is provided. The conditions for the existence, uniqueness and definiteness of a solution to the general Riccati equation are presented in detail. The notion of the structured singular value and properties of the " $\mu$ " function are introduced as well as the conditions for nonexpansivity of a given system. An algorithm for " $H_\infty$ " norm minimization based on two Riccati equations developed by Doyle et al [DGKF88] is presented and its individual steps are outlined.

The control redesign problem is formally introduced in Chapter 3. It is shown the the latter can be transformed into a robust model matching problem with structured uncertainty. Performance of the postfailure system is defined. The nominal case, where the model of the failed plant is known exactly, is treated separately. The conditions for perfect recovery of the original system are established for square and nonsquare postfailure plants.

Chapter 4 and Chapter 5 present the control redesign in the presence of the frequency dependent, norm bounded uncertainty. An algorithm for the maximization of the stability margin with fixed performance requirement is introduced in Chapter 4. Properties of the algorithm and a proof of its convergence are presented in the same chapter.

An application of the algorithm is shown in Chapter 5. Augmented model of the longitudinal dynamics of the F-8 Aircraft [SHH77] is used as the benchmark for the evaluation of the latter under the assumption that there was a failure at one of the two actuators.

The postfailure control system analysis in the presence of real parameter perturbation is presented in Chapter 6.

Evaluation of the maximum stability and performance margin of the perturbed system based on the Riccati equation approach is presented. The tradeoff between the numerical

complexity and the conservatism of different approaches is discussed. It is shown that the very frequently used analysis and synthesis methods based on the single Riccati equation and the Peterson-Hollot bounding function [PH86] are in general as conservative as the "Small Gain Theorem" [Zames65].

In Chapter 7, a postfailure redesign method for real parameter perturbations that can be reflected at the input of the plant is presented. Its application is shown at the model of the longitudinal dynamics of the F-8 Aircraft.

Finally, concluding remarks are given in Chapter 8.

## CHAPTER 2

### MATHEMATICAL BACKGROUND

#### 2.1 Introduction

This chapter presents background theory necessary for the postfailure system redesign. The parameterization of all stabilizing controllers for a given plant and Nehari's theorem, which will be used in Chapter 3, are briefly introduced and the specific references are cited. The parameterization will be used for obtaining the postfailure controller that will satisfy stability and performance requirement. Nehari's theorem will provide the conditions for perfect "recovery" of the original system in the sense that will be defined.

Since the postfailure system robustness analysis and synthesis methods that will be presented in this thesis are based on the properties of Riccati equations, their properties are discussed in detail. This specially holds for the existence and uniqueness of solutions to a Riccati equation. They will be used in the formulation of the nonexpansivity conditions for a given system that are also presented herein.

The algorithm for minimization of the " $H_\infty$ " norm in the state space based on the two Riccati equations and the notion of the structured singular value is introduced. It will be used in Chapters 4 and 5.

## 2.2 Parameterization of all Stabilizing Compensator for $G(s)$

### Lemma 2.1 [Fran87]

For each proper real-rational matrix  $G(s)$  there exist eight  $\text{RH}_\infty$  matrices satisfying the equations

$$G(s) = \tilde{M}^{-1}\tilde{N} = NM^{-1} \quad (2.1)$$

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \cdot \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I \quad (2.2)$$

These equation constitute a doubly-coprime factorization of  $G(s)$  where  $N$  and  $M$  are said to be right-coprime and  $\tilde{N}$  and  $\tilde{M}$  are said to be left-coprime. Furthermore,  $M$  and  $\tilde{M}$  can always be chosen as inner matrices.

■

### Lemma 2.2 [Fran87]

Let  $G(s)$  be stabilizable. The set of all compensators  $K(s) \in \text{RL}_\infty$  internally stabilizing  $G(s)$  is parameterized by the formulas

$$K(s) = -[Y - MQ][X - NQ]^{-1} = -[\tilde{X} - Q\tilde{N}][\tilde{Y} - Q\tilde{M}]^{-1}, \quad Q(s) \in \text{RH}_\infty \quad (2.3)$$

■

## 2.3 Nehari's Theorem - The Distance from $R \in L_\infty$ to $H_\infty$ [Fran87]

The distance from some  $R \in L_\infty$  to  $H_\infty$  defined as

$$\text{dist} \{ R, H_\infty \} = \inf_{X \in H_\infty} \| R - X \|_\infty \quad (2.4)$$

is bounded from below by the norm of the Hankel operator  $\Gamma_R$  associated with  $R(s)$ , i.e.

$$\text{dist} \{ R, H_\infty \} \geq \| \Gamma_R \| \quad (2.5)$$

### Theorem 2.1 [Fran87]

There exists a closest  $H_\infty$  matrix  $X$  to a given  $L_\infty$  matrix  $R$ , and  $\| R, H_\infty \|_\infty = \| \Gamma_R \|$ .

■

The time domain interpretation states that the distance from a given noncausal system to the nearest causal one equals the norm of its Hankel operator which, therefore, is a measure of noncausality.

## 2.4 Riccati Equation

In this section we discuss the existence, uniqueness, and properties of solutions to Riccati equations. This is important since the Riccati equation approach is used in the analysis and synthesis of the postfailure robust control systems with both real parameter perturbations and unstructured uncertainties.

### Definition 2.1

Let the matrices  $A$ ,  $R = R'$  and  $Q = Q'$  belong to  $\mathbb{R}^{n \times n}$ . Furthermore, let  $R = R'$  be either positive semidefinite ( $R \geq 0$ ) or negative semidefinite ( $R \leq 0$ ). Then, the matrix  $H \in \mathbb{R}^{2n \times 2n}$  defined as

$$H = \begin{bmatrix} A & R \\ Q & -A' \end{bmatrix} \quad (2.6)$$

is called Hamiltonian matrix.

### Definition 2.2

If there exists a matrix  $P = P' \in \mathbb{R}^{n \times n}$  s.t.

$$[P \ -I] H [I \ P]' = A' P + P A + P R P - Q = 0 \quad (2.7)$$

then (2.7) represents the Riccati equation corresponding to the Hamiltonian matrix  $H$ . The matrix  $P$  is its solution, i.e.  $P = \text{Ric}(H)$ . (2.8)

### Lemma 2.3

The spectrum of a Hamiltonian matrix  $H$  is symmetric with respect to both the imaginary and real axis.

Proof :

Let the spectrum of  $H$  be defined as  $\Lambda(H)$ , meaning that every eigenvalue  $\lambda$  of  $H$  belongs to  $\Lambda(H)$ . Furthermore, let an orthogonal matrix  $J \in \mathbb{R}^{2n \times 2n}$  s.t.  $J' = J^{-1} = -J$  be defined as

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \quad (2.9)$$

Then, it is easy to see that the similarity transformation  $J H J'$  is equal to  $(-H)'$ . Therefore, if an eigenvalue  $\lambda \in \Lambda(H)$  then  $(-\lambda) \in \Lambda(H)$ . This ends the proof. ■

The Riccati equation presented in (2.7) can now be rewritten as :

$$[I \ P] J H [I \ P]' = 0 \quad (2.10)$$

This observation is very important because it leads to the necessary and sufficient conditions for a matrix  $P$  to be the solution to a Riccati equation. The following lemma presented in [Wil71] establishes these conditions.

#### Lemma 2.4

A matrix  $P=P'$  is a solution to the Riccati equation (2.10) if and only if

$$H \operatorname{Im}\{ [I \ P]'\} \in \operatorname{Im}\{ [I \ P]'\} \quad (2.11)$$

The equivalent condition is the existence of a matrix  $L \in \mathbb{R}^{n \times n}$  s.t.

$$H [I \ P]' = [I \ P]' L \quad (2.12)$$

Proof :

A proof was given in [Pot66] and generalized in [Wil71]. ■

The previous lemma establishes the necessary and sufficient conditions for a given matrix  $P$  to be a solution to the Riccati equation (2.10) but it doesn't guarantee its existence. We will now establish the conditions on the matrix  $H$  for the existence of a solution  $P$  to (2.10) that satisfies (2.12).

According to Lemma 2.3, if a Hamiltonian matrix  $H \in \mathbb{R}^{2n \times 2n}$  has no pure imaginary



eigenvalues, then it has "n" stable and "n" unstable eigenvalues. Therefore, its spectrum can be partitioned as  $H_-$  with stable and  $H_+$  with unstable eigenvalues. Then, the eigenvector space of H can be expressed as the direct sum of the following two subspaces :

$X_-(H)$  as the span of all real, possibly generalized, eigenvectors corresponding to eigenvalues of H in  $H_-$

and

$X_+(H)$  as the span of all real, possibly generalized, eigenvectors corresponding to eigenvalues of H in  $H_+$ .

In order to say something more about these subspaces we will look at the Jordan decomposition of a Hamiltonian matrix H. Let the Jordan decomposition of H be given as :

$$H T = T D, \quad T, D \in \mathbb{R}^{2n \times 2n} \quad (2.13)$$

where D is a diagonal matrix with Jordan blocks as its entries and T is a similarity transformation matrix.

Let the matrix D be partitioned as

$$D = \text{diag}\{ D_-, D_+ \} \quad (2.14)$$

where  $D_- \in \mathbb{R}^{n \times n}$  corresponds to the stable and  $D_+ \in \mathbb{R}^{n \times n}$  to the unstable eigenvalues of H. The matrix T can be partitioned in the same manner as

$$T = [ T_-, T_+ ] \quad (2.15)$$

where  $T_-, T_+ \in \mathbb{R}^{2n \times n}$  span  $X_-$  and  $X_+$  respectively. In the case of real, nonrepeated eigenvalues of H, the Jordan decomposition of H will be equal to the eigenvalue decomposition and  $X_-$  and  $X_+$  will be spanned by the corresponding real eigenvectors. In the general case, when H has either complex-conjugate or repeated real eigenvalues,  $X_-$  and  $X_+$  are spanned by the real generalized eigenvectors from  $T_-$  and  $T_+$ .

From the above partition and according to (2.13), we have the following relationship:

$$H T_- = T_- D_- \quad (2.16)$$

or, equivalently,

$$H [ T_1' \ T_2' ]' = [ T_1' \ T_2' ]' D_- \quad (2.17)$$

where  $T_- = [ T_1' \ T_2' ]'$  and  $T_1, T_2 \in \mathbb{R}^{n \times n}$ .

If the matrix  $T_1$  is invertible, the expression in (2.17) can be rewritten as

$$H [ I \ (T_2 T_1^{-1})' ]' = [ I \ (T_2 T_1^{-1})' ]' (T_1 D_- T_1^{-1}) \quad (2.18)$$

By letting  $(T_1 D_- T_1^{-1}) = L$ ,  $(T_2 T_1^{-1}) = P$  and, after checking its symmetry, we can see that (2.18) is equivalent to (2.12) in Lemma 2.3. Therefore, by requiring that  $H$  have no eigenvalues on the imaginary axis and that  $T_1$  be invertible and assuming that  $(T_2 T_1^{-1})$  is symmetric, we have shown that  $(T_2 T_1^{-1}) = P$  is a solution to the Riccati equation (2.7). These conditions are formally formulated by the following lemma [Pot66] and [Mart71].

### Lemma 2.5

Let  $H$ , formulated in (2.6), have no eigenvalues on the imaginary axis and let  $T_1, T_2 \in \mathbb{R}^{n \times n}$  be obtained as in (2.17). Then

- 1)  $T_1' T_2 = (T_1' T_2)'$
- 2) if  $T_1^{-1}$  exists, then
  - i)  $P = T_2 T_1^{-1} = P'$
  - ii)  $P = \text{Ric}(H)$  and it is unique
  - iii)  $[A + RP]$  is stable matrix

Proof :

According to (2.16), we have  $H T_- = T_- D_-$ . By multiplying from the left with  $T_-^{-1}$ ,

we have

$$T'_J H T_- = T'_J T_- D_- \quad (2.19)$$

Since  $J H = (J H)'$ , it is obvious that  $T'_J H T_-$  is symmetric. Therefore, the right hand side term in (2.19),  $T'_J T_- D_-$ , is also symmetric. Its transpose is given as :

$$\begin{aligned} (T'_J T_- D_-)' &= D_-' T_-' J' T_- = \\ &= -D_-' T_-' J T_- \end{aligned} \quad (2.20)$$

Now we have :

$$\begin{aligned} 0 &= T'_J T_- D_- - (T'_J T_- D_-)' = \\ &= (T'_J T_-) D_- + D_-' (T_-' J T_-) \end{aligned} \quad (2.21)$$

The expression in (2.21) is nothing more than a Lyapunov equation. Since  $D_-$  has no eigenvalues on the imaginary axis, the above expression holds iff  $T_-' J T_- = 0$ . This proves the claim in 1).

For the claim i) in 2), let  $P = T_2 T_1^{-1}$ . Then  $P T_1 = T_2$  or, equivalently,  $T_1' P' = T_2'$ . By multiplying it with  $T_1$  from the right, we get  $T_1' P' T_1 = T_2' T_1$ . Since the right hand side term is symmetric, so is  $T_1' P' T_1$ . Therefore,  $P = P'$ .

The proof for ii) that  $P = \text{Ric}(H)$  is identical to the procedure presented in (2.17) and (2.18). We will prove the uniqueness by contradiction. Assume that, besides  $P$ , there is some  $P_1$  s.t.  $P_1 = \text{Ric}(H)$ . Then, according to (2.7) we would have

$$A' P + P A + P R P - Q = 0$$

$$\text{and} \quad A' P_1 + P_1 A + P_1 R P_1 - Q = 0 \quad (2.22)$$

By subtracting these two expressions, we get

$$[A + R P]' (P - P_1) + (P - P_1) [A + R P_1] = 0 \quad (2.23)$$

which is a Sylvester equation.

Since  $P$  and  $P_1$  are stabilizing solutions, then the matrices  $[A + R P]$  and  $[A + R P_1]$  are stable. Therefore,  $\text{Re}[\lambda_i\{ (A+RP) \} + \lambda_j\{ (A+RP_1) \}] \neq 0$  for all  $i, j = 1 : n$ , which is the iff condition for the existence of the unique solution to (2.23) [Bell70]. In this case the solution is equal to zero, implying that  $P = P_1$ .

To prove iii) it is enough to multiply (2.18) with  $[I \ 0]$  on the right. We get :

$$A + R P = T_1 D_- T_1^{-1}$$

Since  $D_-$  is stable so is  $[A + R P]$ . This ends the proof. ■

As we have seen, the condition on the eigenvalues of a Hamiltonian matrix and the invertibility of  $T_1$  are sufficient for the existence of a unique, symmetric and stabilizing solution to (2.7). The necessary as well as sufficient conditions are formulated by the following lemma [Kucera72].

### Lemma 2.6

Let  $R = BB'$  or  $R = -BB'$  without loss of generality. Then, the stabilizability of  $(A, B)$  and the requirement that  $H$  have no pure imaginary eigenvalues are both necessary and sufficient conditions for the existence of the unique  $P = \text{Ric}\{ H \}$  that stabilize  $[A + R P]$ .

Proof :

Necessity :

If  $P$  is a unique stabilizing solution to (2.7), then there is a similarity transformation that transforms  $H$  into

$$\begin{bmatrix} A + R P & R \\ 0 & -[A + R P]' \end{bmatrix} \quad (2.24)$$

Since  $[A + R P]$  is stable, then it is obvious that  $H$  doesn't have eigenvalues on the imaginary axis. Furthermore,  $(A, B)$  is stabilizable since  $[A + (\pm B B') P]$  is stable.

Sufficiency:

It is enough to prove that stabilizability of  $(A, B)$  guarantees existence of  $T_1^{-1}$ . We first show that the kernel of  $T_1$  is invariant with respect to  $D_-$ .

From (2.17), after multiplying it by  $[I \ 0]$  on the left, we have :

$$A T_1 + R T_2 = T_1 D_- \quad (2.25)$$

Let  $x \in \text{Ker } T_1$ . Then

$$R T_2 x = T_1 D_- x \quad (2.26)$$

After a simple algebraic manipulation, we get

$$x' T_2' R T_2 x = x' T_2' T_1 D_- x \quad (2.27)$$

where the term on the left hand side is symmetric, and thus the term  $x' T_2' T_1 D_- x$  will be also. Then, since  $T_2' T_1 = T_1' T_2$  from Lemma 2.5, we have

$$\begin{aligned} x' T_2' R T_2 x &= x' T_2' T_1 D_- x = \\ &= x' D_-' T_2' T_1 x = 0 \end{aligned} \quad (2.28)$$

which implies that  $B' T_2 x = 0$  and  $T_1 D_- x = 0$ . (2.29)

Therefore, if  $x \in \text{Ker } T_1$ , then  $D_- x \in \text{Ker } T_1$  also. It is obvious that  $\text{Ker } T_1$  is " $D_-$ " invariant, and that it is spanned by the subspace of the eigenvectors of  $D_-$ . Then there exists

$$\text{a pair } (\lambda, v) \text{ s.t. } D_- v = \lambda v, \quad \lambda < 0 \quad \text{and} \quad v \in \text{Ker } T_1. \quad (2.30)$$

From (2.17), after multiplying with  $[0 \ I]$  on the left, we have

$$Q T_1 v - A' T_2 v = T_2 D_2 v \quad (2.31)$$

or, equivalently,

$$(A' + \lambda I) T_2 v = 0 \quad (2.32)$$

Therefore,  $(-\lambda)$  is an unstable eigenvalue of  $A$  and  $w = T_2 v$  is a corresponding eigenvector. Then, by using (2.29), we have

$$(T_2 v)' [A' - (-\lambda)I \quad B] = 0 \quad (2.33)$$

implying that  $(-\lambda)$  is uncontrollable. This is a contradiction because  $(A, B)$  was assumed to be stabilizable. Hence,  $\text{Ker } T_1$  is empty and  $T_1^{-1}$  exists. This concludes the proof. ■

### Corollary 2.6.1

If  $R = -BB'$  and  $Q = -C'C$ , the necessary and sufficient conditions for  $P = \text{Ric}(H)$  is the stabilizability of  $(A, B)$  and detectability of  $(C, A)$ . The corresponding Riccati equation is known as the "Control Algebraic Riccati Equation (CARE)". This result and its proof can be found in almost any textbook on Optimal Control.

Until now, we have been concerned with the existence and uniqueness of the "stabilizing" solution  $P$  of the Riccati equation (2.7) s.t.  $[A + RP]$  be a stable matrix. It is important to notice that, under the same conditions, there exists a unique "destabilizing" solution  $P_-$  s.t.  $[A + RP_-]$  has all its eigenvalues in the right half plane. Furthermore, in the case of CARE,  $P$  is either  $(\geq 0)$  or  $(> 0)$  while  $P_-$  is either  $(\leq 0)$  or  $(< 0)$ . These two extreme solutions are not necessarily the only ones. All other solutions  $P_i$  to CARE, if they exist, are ordered s.t.  $P_- \leq P_i \leq P$  and each of them can be obtained as a certain combination of the extreme solutions. They do satisfy the Riccati equation algebraically but do not stabilize or completely destabilize the matrix  $[A + RP]$ . These results are presented in [Wil71].

## 2.5 Nonexpansivity of $M(s)$

Let a stable system  $M$  shown in Figure 2.1 have a state-space representation  $[A, B, C, 0]$ .

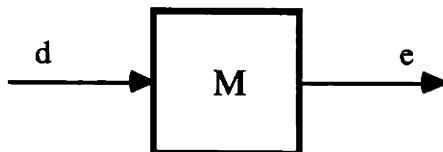


Figure 2.1 Multivariable System  $M(s) \in RH_\infty$

Then the following relationship holds :

$$\|M\|_\infty \leq \gamma \quad \Leftrightarrow \quad \|Md\|_2^2 - \gamma^2 \|d\|_2^2 \leq 0 \quad (2.34)$$

for any  $d \in L_2$  or, equivalently,

$$\|M\|_\infty \leq \gamma \quad \Leftrightarrow \quad \sup_d \int_0^\infty (e'e - \gamma^2 d'd) dt \leq 0 \quad (2.35)$$

### Lemma 2.7

Let the system  $M \in RH_\infty$  be given by its state space representation  $[A, B, C, 0]$ . If  $\exists P = P' \geq 0$  s.t. :

$$A'P + PA' + PBB'P + C'C = 0 \quad (2.36)$$

then  $\|M(s)\|_\infty \leq 1$ .

Proof :

The proof is based on algebraic manipulation of the given Riccati equation provided that its semipositive definite solution  $P$  exists. This was presented in [DGKF88].

Equation (2.36) can be rewritten as

$$(-sI - A')P + P(sI - A) = C'C + PBB'P \quad (2.37)$$

After multiplying (2.37) by  $B'(-sI - A')$  on the left and by  $(sI - A)B$  on the right and by defining

$$M^{\sim}(s) = M'(-s) \quad ; \quad M(s) = C [sI - A]^{-1} B \quad \text{and} \quad (2.38)$$

$$L(s) = B' P [sI - A]^{-1} B \quad (2.39)$$

we get the following equality:

$$(I - M^{\sim} M) = (I - L)^{\sim} (I - L) \quad (2.40)$$

which is Hermitian on the  $j\omega$  axis. The expression in (2.40) corresponds to the "Frequency Domain Equality" in the classical LQG problem.

This, however, implies that

$$I - M^{\sim}(j\omega) M(j\omega) \geq 0 \quad (2.41)$$

or, equivalently,  $\|M(j\omega)\|_{\infty} \leq 1$ . This completes the proof. ■

This result is sufficient only because if  $\|M(j\omega)\|_{\infty} \leq 1$  holds, there is no guarantee that there will exist a matrix  $P$  that satisfies (2.36). The strict nonexpansivity of  $M(s)$  is given by the following conditions.

### Lemma 2.8

Assume that the system  $M(s)$  is asymptotically stable. Then  $\|M\|_{\infty} < 1$  iff the Hamiltonian matrix  $H$  corresponding to the Riccati equation defined in the previous lemma



doesn't have any eigenvalues on the  $j\omega$ -axis.

Proof :

Let  $\|M(j\omega)\|_\infty < 1$ . This is equivalent to the condition of invertibility of  $(I - \tilde{M}M)$  on the  $j\omega$ -axis, i.e.  $G \equiv [I - \tilde{M}(j\omega)M(j\omega)]^{-1} \in RL_\infty$ . Therefore,  $G$  has no poles on the  $j\omega$ -axis. If the state space representation of  $G$  is  $[Ag, Bg, Cg, Dg]$ , by the simple algebraic transformation it can be shown that

$$Ag = H; \quad Bg = [B', 0]'; \quad Cg = [0, B']; \quad Dg = I \quad (2.42)$$

and that  $(Ag, Bg)$  is stabilizable and  $(Cg, Ag)$  is detectable since the system  $M(s)$  is detectable and stabilizable too. Then, the eigenvalues of  $H$  are equal to the poles of  $G(s)$ .

On the other hand, if  $H$  doesn't have any purely imaginary eigenvalues then  $G(j\omega) \in RL_\infty$ . Then it is easy to show that the  $\overline{\sigma}(M)$  cannot be equal to or greater than one at any frequency which implies that  $\|M(j\omega)\|_\infty < 1$ . This completes the proof.

■

The previous two lemmas establish the necessary and sufficient conditions in the state space for strict nonexpansivity of  $M(s)$ .

### Lemma 2.9

The necessary and sufficient condition for  $\|M\|_\infty < 1$  is the existence of  $P = P'$  s.t.:

- i)  $P \geq 0$
  - ii)  $A'P + PA + PBB'P + C'C = 0$
  - iii)  $A + BB'P$  is stable
- (2.43)

Proof :

If there is a  $P=P'$  s.t. the conditions in (2.43) hold,  $\|M\|_\infty < 1$  from Lemma 2.7.

On the other hand, if  $\|M\|_{\infty} < 1$  then the corresponding Hamiltonian matrix doesn't have purely imaginary eigenvalues according to Lemma 2.8. Therefore, Lemma 2.6 guarantees the existence of  $P=P'$  that satisfies (2.43). This concludes the proof. ■

## 2.6 Algorithm for $H_{\infty}$ Design

In this thesis we use the latest results in the state-space synthesis approach to the  $H_{\infty}$  control problem. A new method was derived by Doyle et al [DGKF88], that provides formulas for all controllers solving the " $H_{\infty}$ " minimization problem. They are given as a linear fractional transformation on the free parameter  $Q$  and their computation is based on only two Riccati equations. The dimension of each controller is equal to the dimension of the overall plant plus that of the free parameter. The overall plant  $M_p(s)$  includes the actual plant and all the weighting functions. We now present the algorithm with some preliminary ideas behind it. The detailed proof can be found in [DGKF88].

The main idea behind the algorithm is relatively simple. It states that if the necessary and sufficient conditions for the existence of a stabilizing controller  $K(s)$  that guarantees  $\|M(K)\|_{\infty} < \gamma$  are derived, then the minimum value of the norm of the closed loop transfer function  $M$  can be found by scaling down the value of  $\gamma$  until there is no compensator that can achieve it. The algorithm is iterative and it approaches the minimal solution as closely as we want to, but cannot achieve it because of the strict inequality in the above expression. In order to design such a compensator, conditions guaranteeing the bound on the infinity norm of the closed loop transfer function have to be established.

The goal is to derive a stabilizing compensator that makes the closed loop transfer function satisfy the conditions in (2.43) and then, by introducing a free parameter  $Q$  around it, parameterize all other controllers that satisfy the same conditions.

The algorithm requires the plant transfer function to be given in a (2x2) block form, i.e.

$$M_p = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad (2.44)$$

that corresponds to the following figure.

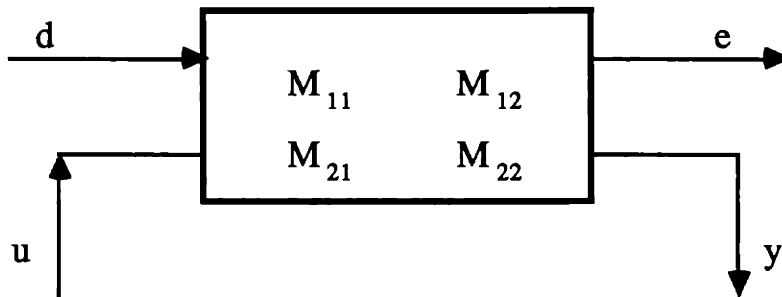


Figure 2.2 Open-Loop System Partitioned According to its Inputs and Outputs

A state-space representation of the plant is given in terms of  $[A, B, C, D]$  where  $C = [C_1, C_2]$ , and

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}; \quad D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \quad (2.45)$$

with the following dynamic and output description :

$$\begin{aligned} \dot{x} &= Ax + B_1d + B_2u \\ e &= C_1x + D_{11}d + D_{12}u \\ y &= C_2x + D_{21}d + D_{22}u \end{aligned} \quad (2.46)$$

The control signal  $u$  is the output and the measurement  $y$  is the input to the controller, while

$d$  stands for external inputs and  $e$  for error signals.

The algorithm is based on the following assumptions :

- i)  $(A, B_1)$  is stabilizable and  $(C_1, A)$  is detectable
- ii)  $(A, B_2)$  is stabilizable and  $(C_2, A)$  is detectable
- iii)  $D_{11} = 0$  and  $D_{22} = 0$
- iv)  $D_{12}$  must have full column rank and  $D_{21}$  full row rank.

### Remark 2.6.1

The assumptions in (iv) are necessary for the appropriate scaling of the control signal  $u$  and measurement  $y$  such that

$$\begin{aligned} D_{12}' D_{12} &= I \\ D_{21} D_{21}' &= I \end{aligned} \quad (2.47)$$

Such a scaled system is used for deriving the minimizing compensator. The role of the scaling will be further explained in the section where the algorithm steps are introduced.

The compensator structure, as a linear fractional transformation on the free parameter  $Q$ , is depicted in Figure 2.3.

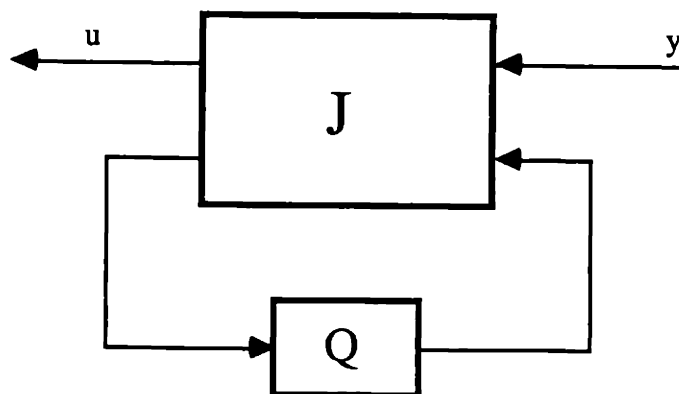


Figure 2.3 Linear Fractional Representation of Compensator with Free Parameter  $Q$

The free parameter  $Q$  is itself an  $H_\infty$  function with the following condition on its norm :

$$\| Q(s) \|_\infty < \gamma \quad (2.48)$$

As it was stated before,  $Q = 0$  is a legitimate choice and it represents the "central" solution to the standard  $H_\infty$  problem.

Let the state-space representation of  $J$  be  $[A_J, B_J, C_J, D_J]$ . Then, for  $\gamma = 1$ , the following holds:

$$A_J = A - K_F C_2 - B_2 K_C + Y_\infty C_1' [ C_1 - D_{12} K_C ]$$

$$B_J = [ K_F \quad K_{F1} ]$$

$$C_J = [ K_C \quad K_{C1} ]'$$

$$D_J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad (2.49)$$

where

$$K_C = [ B_2' X_\infty + D_{12}' C_1 ] [ I - Y_\infty X_\infty ]^{-1}$$

$$K_{C1} = [ -D_{21} B_1' X_\infty - C_2 ] [ I - Y_\infty X_\infty ]^{-1}$$

$$K_F = [ Y_\infty C_2' + B_1 D_{21}' ]$$

$$K_{F1} = [ Y_\infty C_1' D_{21} + B_2 ] \quad (2.50)$$

and  $Y_\infty$  and  $X_\infty$  are real, symmetric solutions to the following Riccati equations, represented in standard state space notation below :

$$X_\infty = \text{Ric} \begin{bmatrix} A - B_2 D_{12}' C_1 & [ B_2 B_2' - B_1 B_1' ] \\ -\tilde{C}_1' \tilde{C}_1 & - [ A - B_2 D_{12}' C_1 ]' \end{bmatrix} = \text{Ric} (H_1) \quad (2.51)$$

$$Y_{\infty} = \text{Ric} \begin{bmatrix} [A - B_1 D_{21}' C_2]' & [C_2' C_2 - C_1 C_1'] \\ -\tilde{B}_1 \tilde{B}_1' & [A - B_1 D_{21}' C_2] \end{bmatrix} = \text{Ric} (H_2) \quad (2.52)$$

with  $\tilde{B}_1 = B_1 [I - D_{21}' D_{21}]$  and  $\tilde{C}_1 = [I - D_{12} D_{12}'] C_1$ .

The above parameterization is valid if the solutions of the Riccati equations satisfy the following conditions :

- i)  $X_{\infty} \geq 0$ ,  $[(A - B_2 D_{12}' C_1) - (B_2 B_2' - B_1 B_1') X_{\infty}]$  stable
  - ii)  $Y_{\infty} \geq 0$ ,  $[(A - B_1 D_{21}' C_2) - Y_{\infty} (C_2' C_2 - C_1' C_1)]$  stable
  - iii)  $\lambda_{\max} (X_{\infty} Y_{\infty}) < 1$
- (2.53)

Stability of the above matrices guarantees the stability of the closed loop system.

A special case of the above parameterization arises when there are no cross product terms between states and control i.e.  $B_1 D_{21}' = 0$ , as well as no correlation between sensor and process noise i.e.  $D_{12}' C_1 = 0$ . In that case the equations in (2.53) assume a simpler form.

### Remark 2.6.2

The described parameterization was derived for  $\gamma = 1$ , but it is also valid for nonunitary values of  $\gamma$ . In that case the external input  $d$  and error signal  $e$  are scaled by  $1/\sqrt{\gamma}$ , which is followed by scaling of the other signals until conditions (2.47) are satisfied.

In order to make the algorithm completely clear, we will now summarize the procedure by briefly stating the steps it consists of.

### 2.6.1 Algorithm for $H_\infty$ Design - Outline

Step 1 Choose the value of the norm by choosing  $\gamma$ .

Step 2 Scale the input  $d$  and output  $e$  by  $1/\sqrt{\gamma}$  by scaling the appropriate matrices in the state space representation. In this manner the system is transformed into one whose infinity norm is possibly less than one.

$$\begin{aligned} B_1 &= 1/\sqrt{\gamma} B_1 & D_{21} &= 1/\sqrt{\gamma} D_{21} \\ C_1 &= 1/\sqrt{\gamma} C_1 & D_{12} &= 1/\sqrt{\gamma} D_{12} \end{aligned} \quad (2.54)$$

Step 3 Scale the control signal  $u$  by  $S_u$  and the measurements  $y$  by  $S_y$  such that the conditions (2.47) are satisfied. Both scaling matrices are square and nonsingular. Then,

$$\begin{aligned} B_2 &= B_2 S_u & C_2 &= S_y C_2 & D_{12} &= D_{12} S_u \\ D_{21} &= S_y D_{21} & D_{22} &= S_y D_{22} S_u \end{aligned} \quad (2.55)$$

Step 4 Solve the two Riccati equations (2.51) and (2.52) and check if the conditions in (2.53) are satisfied. If they are not, increase the value of  $\gamma$  and go back to Step 2. If they are, go to the next step.

Step 5 If looking for the minimal value of the norm, decrease the value of  $\gamma$  and go back to the second step. To get all compensators that achieve the given value of the norm, use formulas in (2.49) and (2.50), in which the input and output matrices of the compensators retain the original scaling.

## 2.7 Structured Singular Value

The "Structured Singular Value" was introduced by Doyle in [Doyle82] as an extension of the robustness analysis techniques based on singular values. The classical analysis setting is presented in the following figure where  $M, \Delta \in \mathbf{RH}_\infty$ , and  $\|\Delta\|_\infty \leq \gamma$ .

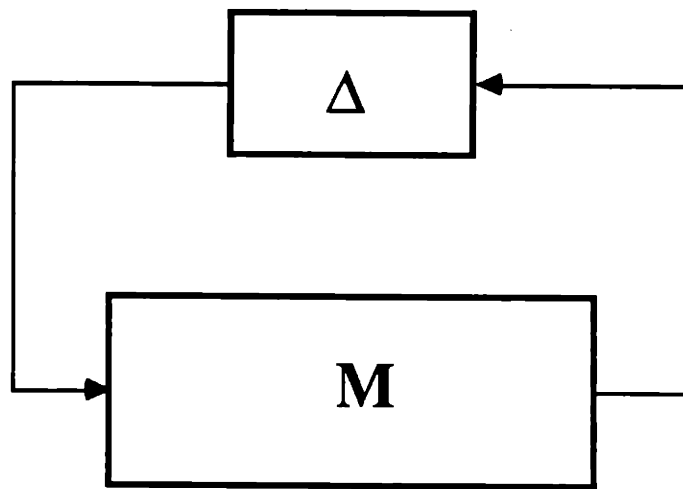


Figure 2.4 Linear Fractional Transformation of System  $M$  with Uncertainty  $\Delta$

It is clear that the stability of the overall system depends on the singularity of  $[I + \Delta M]$ . We assume that  $M(s)$  is a given transfer function and that  $\Delta$  can be any norm bounded operator mapping  $L_2$  signals into  $L_2$  signals. In this case, the necessary and sufficient condition for stability of the overall system is implied by the "Small Gain Theorem (SGT)" [Zames65]

$$\|M(j\omega)\|_\infty \|\Delta(j\omega)\|_\infty < 1. \quad (2.56)$$



The necessity is based on the ability to construct a  $\Delta(j\omega)$  with a bounded norm that destabilizes the overall system if (2.56) doesn't hold.

In the case where the structure of the perturbation matrix  $\Delta(j\omega)$  is specified, meaning that some of its elements are fixed and equal to zero, the above condition becomes conservative. This is so because the uncertainty matrix that we can construct to destabilize the overall system will not necessarily have the desired structure. Therefore, there is a need to come up with a less conservative condition for the stability robustness in the presence of structured, norm bounded uncertainties.

Let  $\Delta_S$  be a set of all possible norm bounded uncertainties  $\Delta$  with a defined structure. By introducing a non-negative real valued function  $\mu(M)$ , the condition (2.56) is changed into the corresponding necessary and sufficient condition for  $\Delta \in \Delta_S$  as follows :

$$\mu\{M(j\omega)\} \gamma < 1 \quad (2.57)$$

at every frequency  $\omega \in [0, \infty)$  while  $\|\Delta(j\omega)\|_\infty \leq \gamma$ .

The above function is defined at a frequency  $\omega$  as:

$$\mu\{M(j\omega)\} = 0 \quad \text{if no } \Delta \in \Delta_S \text{ solves } \det([I + \Delta M(j\omega)]) = 0$$

and

$$\mu\{M(j\omega)\} = \left[ \min_{\Delta \in \Delta_S} \{ \overline{\sigma}(\Delta(j\omega)) \mid \det([I + \Delta M(j\omega)]) = 0 \} \right]^{-1} \quad (2.58)$$

otherwise.

Unfortunately, the introduced function doesn't satisfy the properties of a norm. This makes its usage difficult and, in some instances, reduces its usefulness to the level of a definition only.

Herein we focus our attention to the case where the uncertainty  $\Delta$  has a diagonal structure with the individual norm bounded uncertainties as its entries. Therefore, we have

$$\Delta = \text{diag}\{ \Delta_1, \dots, \Delta_m \} \in \Delta_S \quad (2.59)$$

where  $\Delta_i \in \mathbb{C}^{N_i \times N_i}$  and  $\|\Delta_i(j\omega)\|_\infty \leq \gamma$ . The set of all such matrices,  $\Delta_S$ , is clearly a subset of  $\mathbb{C}^{N \times N}$  where  $N = N_1 + \dots + N_m$ . There are two extreme cases that can be formulated in this setting. These are:

$$i) \Delta_S = \mathbb{C}^{N \times N} \quad \Rightarrow \quad \mu\{M(j\omega)\} = \overline{\sigma}\{M(j\omega)\} \quad (2.60)$$

and

$$ii) \Delta_S = \{ \lambda I \text{ s.t. } \lambda \in \mathbb{C} \} \quad \Rightarrow \quad \mu\{M(j\omega)\} = \rho\{M(j\omega)\} \quad (2.61)$$

where  $\rho\{M(j\omega)\}$  is the spectral radius of  $M(j\omega)$  at the given frequency  $\omega$ .

Let  $U$  be the set of all block diagonal unitary matrices  $U$  whose entries are unitary matrices themselves with dimensions compatible to  $\Delta \in \Delta_S$ . Furthermore, let  $D$  be a set of all block diagonal invertible matrices  $D = \text{diag}\{d_1 I_1, \dots, d_m I_m\}$  where  $d_i \in \mathbb{C}$  and identity matrices  $I_i$  have their dimensions compatible to the corresponding entries of  $\Delta \in \Delta_S$ .

Then, from the definition of the structured singular value, the following holds at every frequency :

$$D \Delta D^{-1} = \Delta \quad (2.62)$$

$$\mu\{D M D^{-1}\} = \mu\{M\}$$

$$\mu\{U M\} = \mu\{M U\} = \mu\{M\}$$

It is shown in [Doyle82] that at every frequency  $\mu\{M(j\omega)\}$  is bounded from below and above as follows:

$$\max_{U \in U} \rho\{M U\} \leq \mu\{M\} \leq \inf_{D \in D} \overline{\sigma}\{D M D^{-1}\} \quad (2.63)$$

Doyle has also shown that the left side inequality is always an equality for any  $M(j\omega)$  and  $U \in U$ . Unfortunately, this optimization problem is not convex and there are no efficient algorithms developed that guarantee convergence to the global maximum.

On the contrary, the function  $\overline{\sigma}\{D M D^{-1}\}$  is convex in  $D$ , [Doyle82]. It was shown that in some special cases the upper bound reduces to equality and that  $\inf \overline{\sigma}\{D M D^{-1}\}$  has no local minimum which is not global. The cases where this holds are those where  $\Delta$

has no repeated blocks,  $m \leq 3$  and  $d_i \in \mathbb{R}_+$ .

Nevertheless, in [Doyle83] it was shown that the upper bound remains reasonably tight even when  $m > 3$ . This has made the optimization in the upper bound a very useful computational tool for evaluating  $\mu\{M\}$ .

In the case of real perturbations, the structured singular value is in general a very conservative estimate of the stability margin. The only exception is the case where  $\Delta$  consists of only one real parameter perturbation  $\delta$  and one unstructured uncertainty block  $\Delta_1$ , i.e.

$$\Delta = \text{diag}\{\delta, \Delta_1\}$$

This was shown in [PD88].

Since the structured singular value defines a stability margin with respect to the norm bounded uncertainty, it is obvious that its minimization is a reasonable criterion in the synthesis of a control system. It can be formulated as follows :

$$\min_{K \in K_s} \sup_{\omega} \mu\{M(K, j\omega, P_0)\} \quad (2.64)$$

where  $K_s$  is a set of all stabilizing compensators for the nominal plant  $P_0(s)$ . One algorithm for the above optimization is performed as the minimization of the upper bound on the structured singular value over all frequencies, i.e.

$$\min_{K \in K_s} \sup_{\omega} \inf_{D \in \mathbf{D}} \overline{\sigma}(D M D^{-1}) \quad (2.65)$$

This optimization problem is convex in  $K$  and  $D$  separately when the other parameter is fixed. Nevertheless, it is not convex in both variables jointly [Doyle83].

If the synthesis objective is to find a compensator that stabilizes the nominal plant and that guarantees  $\mu\{M(K, j\omega, P_0)\} < 1$  at all frequency, the following algorithm can be used:

- 1) pick  $D_i \in \mathbf{D}$
- 2)  $K^* = \arg \{ \min_{K \in K_s} \| D_i M(K) D_i^{-1} \|_\infty \}$
- 3) if  $\mu\{M(K^*, j\omega, P_0)\} < 1$  at all  $\omega$ , stop  
 if not, approximate the obtained  $D$  with the invertible real rational outer transfer function  $D_r(s)$ , let  $D_i = D_r(s)$  and go back to 2)

This algorithm is known as the D-K iteration [Doyle83].

## 2.8 Concluding Remarks

Necessary mathematical background for the analysis and synthesis of the postfailure control system was presented in this chapter. The parameterization of all stabilizing controllers for the given plant was introduced as well as Nehari's theorem that solves the minimum " $L_\infty$ - $H_\infty$ " distance problem. The conditions for the existence, uniqueness and definiteness of the solution to a general Riccati equation were presented. This is important since the robustness properties of the control system with real parameter perturbations will be shown to depend on the existence of a solution to a particular Riccati equation. The state space conditions for the nonexpansivity of the system are introduced. Furthermore, a synthesis method for the infinity norm minimization based on a pair of Riccati equations is presented.

A robust performance condition as well as the robust stability of the postfailure system can be transformed to a robust stability problem with respect to a structured norm bounded uncertainty. The notion of structured singular value was introduced together with its properties.

## CHAPTER 3

### CONTROLLER REDESIGN AFTER A FAILURE

#### 3.1 Introduction

A generalized treatment of a failure situation was conceptualized in Chapter 1. It was divided into three distinct steps. These are the failure detection, failure identification, and redesign of the original controller. The latter is addressed in this thesis.

This chapter introduces a rigorous mathematical formulation of the controller redesign process. It is defined as a minimization problem of the "difference" between the original and postfailure systems over the set of all stabilizing compensators for the failed plant. The difference is associated with the error signal between the corresponding outputs of both systems when subjected to the same input. Therefore, the minimization of the error signal in certain sense will lead to the recovery of the original system.

The introduced methodology depends on the information available about the postfailure plant. This requires knowledge of the "failed plant" model. The modeling of the dynamics after the failure is assumed to be carried out in the failure identification step. This intermediate step of postfailure treatment process results in a nominal model of the failed plant along with its associated uncertainty. The latter is the result of the modeling process itself.

If the information about the postfailure system is provided sufficiently fast, the above method will be used on-line. On the other hand, for the most likely failures, the "failed plant" nominal model can be anticipated in advance. This, in turn, can be used for obtaining

a sequence of compensators each satisfying robustness properties with respect to uncertainties of different magnitudes. Once the failure occurs and the magnitude of the uncertainty estimated, a compensator will have been previously designed to accommodate it.

### **3.2 Control Redesign as a Robustness Problem**

Controller redesign is necessary when the performance of the original system is substantially affected by the failure. The goal then is to come up with a new controller that will "recover" the original system performance as much as possible in the sense defined in section 3.1. The design of the compensator will be based on the nominal model of the failed plant and the magnitude of the associated uncertainty provided from the identification step. If the part of the original plant whose nominal model and the corresponding uncertainty are known has remained unchanged, the identification step will result in the nominal model and uncertainty of the failed part only. The nominal model and the overall uncertainty associated with the postfailure plant as a whole will then be obtained as a combination of those corresponding to the failed part and the unchanged portion of the original plant. With the resulting model, the "redesigned" controller has to guarantee nominal stability, nominal performance and robustness properties. The performance of the postfailure system in this case will be assessed in terms of a "degree" of recovery the original system properties.

Definition of the performance of the original system is arbitrary and it depends on the desired characteristics of a system. It can be variously defined as the measure of sensitivity to certain disturbances, the quality of command following, etc. We assume that the original feedback system was designed to meet all design requirements. This means that its nominal performance is satisfactory, and robustness characteristics are satisfied within an a priori

defined set of modeling uncertainty.

At the same time, any failure in the original plant will most likely change the overall system in the sense that loss in performance may be significant. Hence, one of the goals that any controller redesign method should achieve is to regain as much of the original performance as possible. In that sense, the quality of the redesign procedure can be captured in a measure of the difference between the key characteristics of the two systems. The smaller this difference is, the closer the postfailure system will behave to the original one. This also enables us to characterize the minimal achievable difference between these two systems as a function of the failure.

Since the model of the postfailure plant with the associated uncertainty differs from the original one, we will define the nominal performance of the latter as target for the recovery. Therefore, it is desired that the postfailure system behavior in the presence of its modeling uncertainty be as close as possible to the nominal behavior of the original one.

In general, all performance requirements can be expressed in terms of transfer function characteristics associated with particular signals in the loop. Therefore, we can always evaluate the difference between the two systems by looking at the difference of their corresponding output signals of interest when they are subjected to the same input. Depending on the requirements posed on the original system, these error signals can be evaluated in different ways.

It is easy to see now that the control algorithm redesign can be posed as the problem of minimizing appropriately chosen error signals, as discussed above, in the presence of uncertainty. The setting for the approach is depicted in Figure 3.1.

The original nominal system is given by the plant  $P_p(s)$  and the compensator  $K_p(s)$ . Its performance is associated with  $y_p(s)$ , as the output, and with  $d(s)$  as the input signal. The postfailure system consists of the failed plant given by its nominal model  $P_o(s)$  and the uncertainty  $\Delta$  that are obtained in the failure identification step. The difference between the

corresponding output signals is an error signal  $e(s)$ .

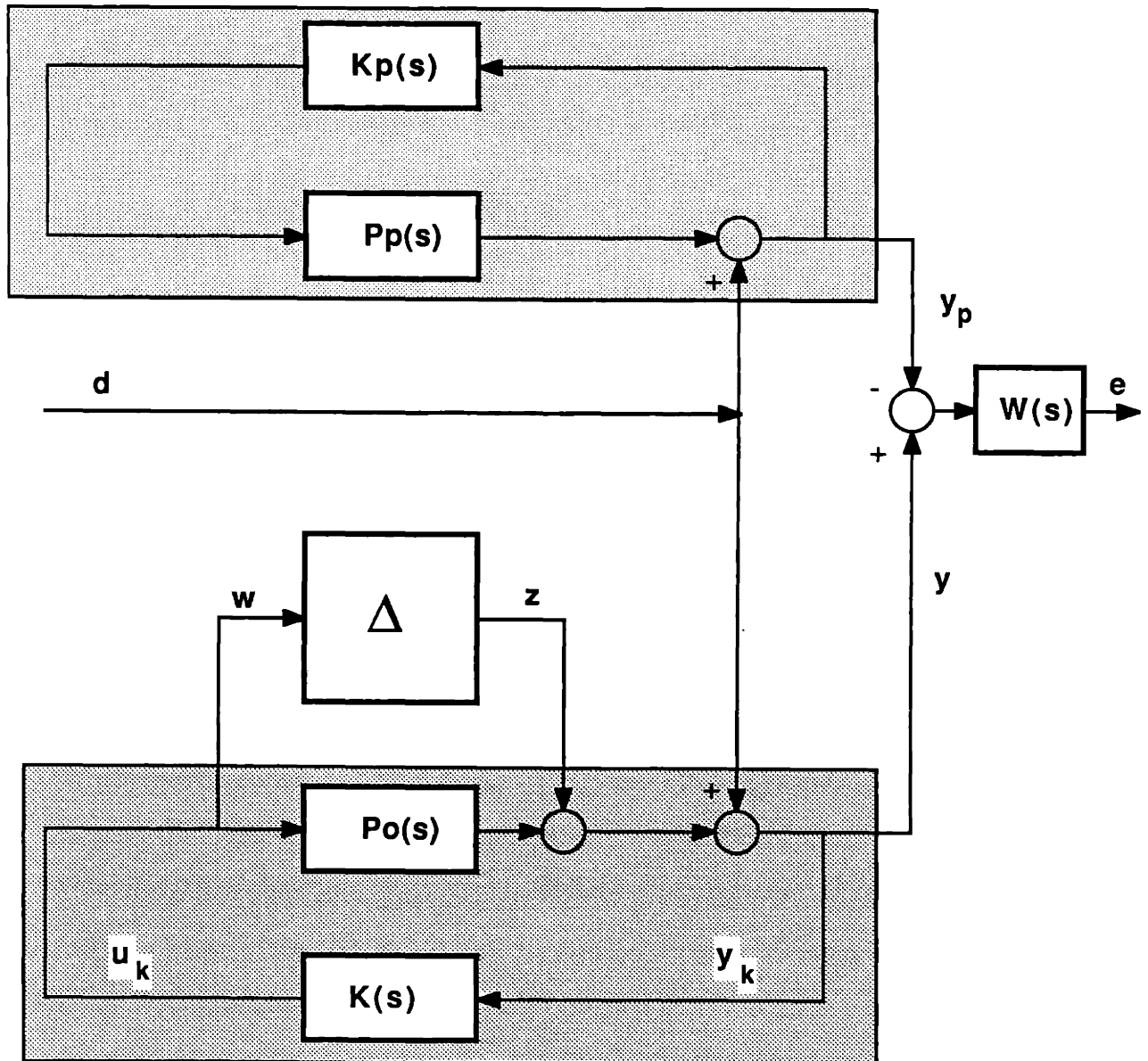


Figure 3.1 Postfailure Error Signal Representation

By rearranging Figure 3.1, we can come up with an equivalent representation of the single



system  $G(s)$  that consists of the overall open loop transfer function  $Mol(s)$ , the uncertainty  $\Delta$ , and a controller  $K(s)$  to be designed. The open loop transfer function  $Mol(s)$  represents, according to Figure 3.1, the mapping from the signals  $[z, d, u_k]$  as inputs to  $[w, (y - y_p), y_k]$  as outputs. Optimization of performance of this overall system corresponds to minimization of its weighted output signal  $e(s)$ , in the desired sense, when subjected to the input  $d$ .

Therefore, the original problem is transformed to the problem of designing a compensator  $K(s)$  that stabilizes the system  $G(s)$  and satisfies the performance requirement associated with the error signal  $e(s)$  in the presence of uncertainty  $\Delta$ . In other words, the redesign controller has to also be "robust" with respect to this uncertainty. Hence, the postfailure controller redesign is posed as a stability and performance robustness problem. The setting for it is represented in Figure 3.2.

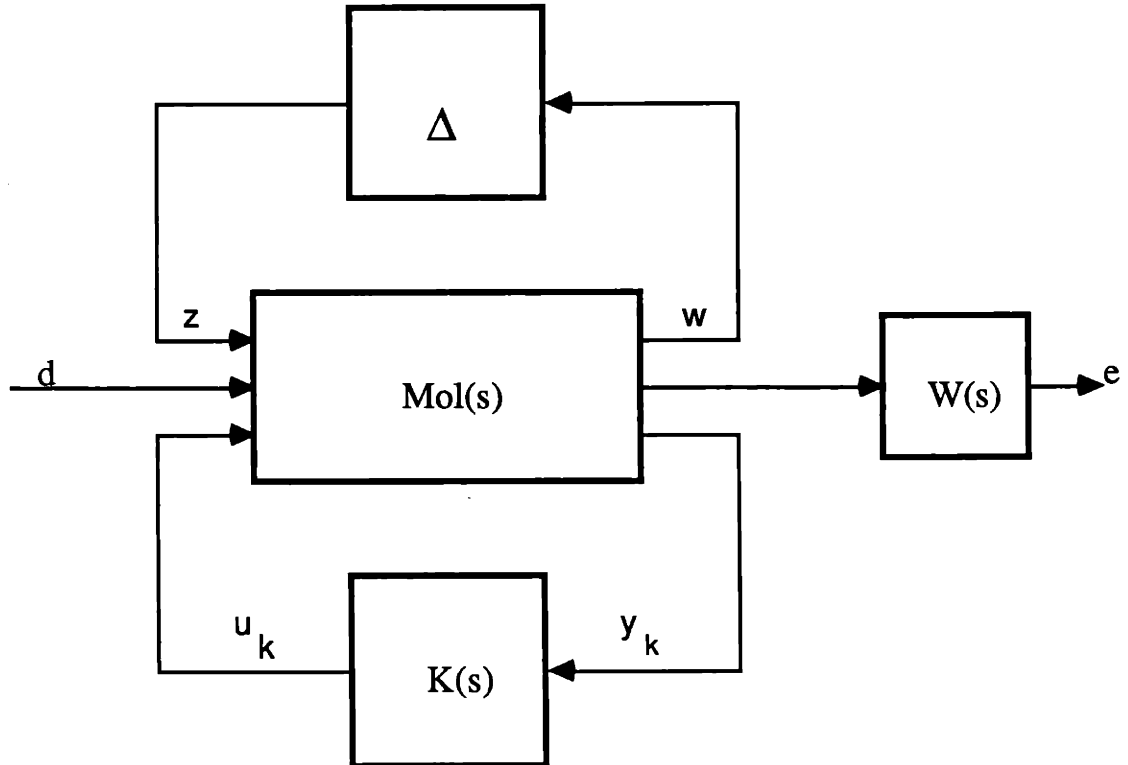


Figure 3.2 Linear Fractional Transformation of Combined Postfailure System

In the following sections we will further define the notion of performance. The input signal  $d$  will be specified together with the requirements on the output signal  $e$ . The importance of the weighting function  $W(s)$  will also be discussed. The best achievable performance in the nominal case, with no uncertainty present, will be derived together with the conditions for its existence.

### 3.3 Performance of the Postfailure System

Performance after the failure, as we have defined it, is captured by the difference between appropriately chosen signals in original and the postfailure system. This difference is expressed by the error signal  $e(s)$  resulting from the common input  $d(s)$  to both systems. Therefore, the necessary steps in defining the performance specifications for the postfailure system are : the choice of the input signal  $d(s)$  and the choice of suitable norm for  $e(s)$ . In general, this will depend on the original prefailure system. If the original system was designed under the assumption that its input was of bounded energy, then the same choice should prevail in the postfailure redesign procedure. The same argument holds for the choice of the norm for  $e(s)$ .

In this thesis we study the case where the input signals belong to the set of all continuous bounded energy signals, i.e.  $d \in L_2$ . Furthermore, we will be interested in minimizing the energy of the error signal. Hence, the performance of the system after the failure is associated with the "two" norm. This choice results from the fact that very often the only available information about the exogenous input signals are that they are of bounded energy.

According to the above definition of performance, the post failure redesign problem is posed as a search for the compensator that will minimize, or keep within given bounds, the

"two" norm of the error signal  $e(s)$ . Obviously, the "best" performance is attained the case where the latter is driven to zero, i.e. when the original system is recovered completely.

In general, there may exist a "frequency preference" range defining different degrees of importance for error signal minimization. This is captured by the suitable choice of a stable weighting function  $W(s)$ , as shown in Figure 3.1.

The performance problem can now stated as follows :

$$\begin{aligned} & \inf_{K \in K_s} \sup_{d \in L_2} \sup_{\Delta \in \Delta_s} \| e \|_2 = \\ & = \inf_{K \in K_s} \sup_{d \in L_2} \sup_{\Delta \in \Delta_s} \| \{ W(s) [M(s, K, \Delta) - M_p(s)] \} d(s) \|_2 = \gamma \end{aligned} \quad (3.1)$$

where  $\| d(s) \|_2 \leq 1$ ,  $K_s$  is the set of all internally stabilizing compensators for the post failure plant  $P(s, \Delta)$ , and  $\Delta_s$  is the set of all admissible uncertainties, i.e.  $\Delta \in \Delta_s$ . The transfer function matrix  $M(s, K, \Delta)$  is the linear, time invariant mapping from the input  $d$  to the output  $y$  of the postfailure system, depicted in Fig 3.1, while  $M_p(s)$  is the corresponding one for the original system. The attainable value  $\gamma$  will be referred to as the performance index of the postfailure system.

The above defined problem is equivalent to

$$\gamma = \inf_{K \in K_s} \sup_{\Delta \in \Delta_s} \| W(s) [ M(s, K, \Delta) - M_p(s) ] \|_\infty \quad (3.2)$$

with the " $H_\infty$ " norm as the induced norm [Fran87].

The choice of transfer functions  $M$  and  $M_p$  depends on the output signals whose difference we are minimizing. The usual transfer functions of interest are the sensitivity  $S(s)$  and the complementary sensitivity  $T(s)$  which, in the case of the original system, are

defined as :

$$S_p(s) = [ I - P_p K_p ]^{-1} \quad (3.3)$$

$$T_p(s) = P_p K_p [ I - P_p K_p ]^{-1} \quad (3.4)$$

The same definition stands for the corresponding postfailure system transfer functions  $S(s, \Delta)$  and  $T(s, \Delta)$  with  $K(s)$  as the compensator and  $P(s, \Delta)$  as the plant. Since these transfer functions are complementary in the sense that  $S(s) + T(s) = I$ , the following equality holds:

$$[S(s, \Delta) - S_p(s)] = - [T(s, \Delta) - T_p(s)] \quad (3.5)$$

Therefore, simultaneous minimization of the unweighted difference of sensitivity and complementary sensitivity transfer functions is redundant. Hence, in the weighted case, it is possible to construct a single stable weighting function  $W(s)$  that will reflect the importance of each difference of transfer functions over the frequency ranges of interest. Consequently, the performance problem can now be stated as

$$\gamma = \inf_{K \in K_s} \sup_{\Delta \in \Delta_s} \| W(s) [ S(s, K, \Delta) - S_p(s) ] \|_{\infty} \quad (3.6)$$

The above defined concept of minimizing the norm of the difference between two transfer function matrices implies that their dimensions are compatible. A problem might arise if the failure results in complete loss of an input or an output of the original plant. Then, the compatibility issue can be resolved by augmenting the input and output matrices in the state space representation of the postfailure system with rows and columns of zeros, respectively, until the original system dimension is matched. The only condition that has to be satisfied then is that the unstable modes of the failed plant be controllable from the control input  $u_k$  and observable from the output  $y_k$ . Therefore, the loss of a control channel or an output will still enable us to approach the problem as in (3.6), provided the above condition holds.

The bound for the achievable performance  $\gamma$ , i.e. the lowest bound for the value of performance index  $\gamma$ , corresponds to the case where no uncertainty is associated with the postfailure plant. The latter is represented as  $P(s, \Delta = 0) = P_0(s)$  as shown in Figure 3.1, while the corresponding nominal value of the performance index is  $\gamma_0$ . The analysis of the nominal postfailure design is carried out in the following section.

### 3.4. Nominal Performance

Nominal performance corresponds to the case where there is no uncertainty associated with the nominal model of the postfailure plant. This situation enables us to minimize the discrepancy between the two systems and, therefore, obtain the "best" performance possible, in the sense already defined. It represents the bound for the achievable performance in the presence of uncertainty.

The nominal performance problem is stated in the following way :

$$\gamma_0 = \inf_{K \in K_s} \| W(s) [ S_0(s) - S_p(s) ] \|_{\infty} \quad (3.7)$$

where  $K_s$  is the set of all internally stabilizing compensators for the nominal postfailure plant  $P_0(s)$  and  $S_0(s) = [ I - P_0 K ]^{-1}$ . From now on we will assume that the unstable modes of the nominal plant are controllable from its control input  $u_k$  and observable from its measured output  $y_k$  as introduced in Figure 3.1. These conditions are necessary and sufficient to guarantee that the set of all internally stabilizing compensators for  $P_0(s)$  is not empty. The loss of a control channel or measured output, as long as it does not violate the above stated conditions, falls under the same framework although the value of  $\gamma_0$  will clearly be affected.

In the classical weighted sensitivity problem [Fran87], it was shown that the minimum

value of the norm was dependent on the poles and zeros of the plant. Furthermore, it was possible to make the norm arbitrarily small when the plant was minimum-phase, regardless of its stability. We show that this doesn't hold in general in the case of the difference between two sensitivity transfer functions. The following lemma establishes the necessary and sufficient conditions for some  $K(s)$  in  $RL_\infty$  to achieve perfect recovery.

### Lemma 3.1

Let  $W(s)$  be an outer function. Then, a compensator  $K(s) \in RL_\infty$  will achieve  $\gamma_0 = 0$  iff

- i)  $P_o K = P_p K_p$
- ii) all RHP poles of  $P_o(s)$  are also RHP poles of  $P_p K_p(s)$
- iii) all left RHP transmission zeros of  $P_o(s)$  are also those of  $P_p K_p(s)$  including locations and directions

Proof :

Let  $S_p(s) = [I + P_p(s) K_p(s)]^{-1}$  be the sensitivity of the original system

and  $S_o(s) = [I + P_o(s) K(s)]^{-1}$  be the sensitivity of the postfailure system.

Each sensitivity transfer function  $S(s) = [I + P K]^{-1}$  satisfies the following "matching" conditions :

$S(s) = 0$  at the location of RHP poles of  $[P(s) K(s)]$

$S(s) = I$  at the location of RHP zeros of  $[P(s) K(s)]$  with the appropriate directions

(3.8)

Sufficiency :

The first condition guarantees that  $S_o(s) = S_p(s)$  and therefore  $\gamma_0 = 0$ . Conditions ii) and iii) assure that internal stability is achieved meaning that there are no unstable pole-zero cancellation between  $P_o(s)$  and  $K(s)$ . This results from the following argument. It is obvious that the same "matching" conditions for both sensitivity transfer functions are

necessary for making  $\gamma_0$  arbitrarily close to zero. Therefore, if the minimum phase postfailure plant has some RHP poles that are not RHP poles of the loop function  $[P_p K_p]$ , the matching conditions will differ unless these poles are cancelled by the compensator  $K(s)$ . This would violate internal stability [Fran87] of the postfailure system and is not admissible. The same argument holds for the RHP left transmission zeros of  $P_o(s)$  that are not those of  $P_p K_p$ .

Necessity :

If  $\gamma_0 = 0$  is achieved by the internally stabilizing compensator  $K(s)$ , we have that  $P_o K = P_p K_p$  since  $W(s)$  has full rank as an outer function. Internal stability guarantees no RHP pole-zero cancellation between  $P_o(s)$  and  $K(s)$ . This is possible only if ii) and iii) hold. This completes the proof.

■

As can be seen from the previous lemma, to achieve perfect recovery of the original system with an internally stabilizing compensator, it was necessary not only to match the RHP poles and zeros of the two loop transfer functions  $P_p K_p$  and  $P_o K$ , but also to assure that there are no unstable pole-zero cancellations between  $P_o(s)$  and  $K(s)$ . We will now discuss the existence and uniqueness of a compensator  $K \in RL_\infty$  that satisfies the conditions in the previous lemma. The results will be different for square and nonsquare plants.

### Corollary 3.1.1

Let  $P_o(s)$  be square and let it satisfy conditions ii) and iii) in Lemma 3.1. Then there exists a unique internally stabilizing compensator  $K$  in  $RL_\infty$  iff  $P_o^{-1} P_p K_p \in RL_\infty$ . Furthermore, the compensator is given as  $K = P_o^{-1} P_p K_p$ .

Proof:

The uniqueness results from the uniqueness of  $Po^{-1}$ . The internal stabilizability and perfect recovery are achieved according to Lemma 3.1. This ends the proof. ■

The case of nonsquare plants is more complicated. From Lemma 1 and its corollary we can see that we can obtain a compensator that achieves perfect recovery if an inverse of  $Po(s)$  exists. Neglecting the internal stabilizability condition for a moment, this can easily be extended to the existence of the right inverse of  $Po$  since  $K=[Po^{-R} PpKp]$  will make  $So(s) = Sp(s)$ . Therefore, we can possibly obtain an internally stabilizing compensator that achieves perfect recovery if the plant has more inputs than the outputs i.e. when it is "flat". Furthermore, since the right inverse is not unique, there might exist infinitely many internally stabilizing compensators that make  $\gamma_0=0$ . But as we will show in Example 3.1, not every right inverse of  $Po$  can be used for obtaining  $K(s)$ .

In the opposite case when the plant has more outputs than inputs, i.e. when it is "tall", the right inverse will not exist and, in general, we will not be able to obtain the compensator. In this case we say the problem is overdetermined. This conclusion is intuitive since we are trying to match the postfailure plant output signals with desired ones that are usually linearly independent. Therefore, we have to be able to independently control each of them. This is possible only if there are as many inputs to the postfailure plant as there are outputs.

The following lemma establishes conditions for obtaining an internally stabilizing  $K(s)$  that achieves  $\gamma_0=0$  for nonsquare plants.

### Lemma 3.2

Let  $Kp(s)$  be an internally stabilizing compensator for a  $Pp(s)$ , let  $W(s)$  be a proper



outer function and let  $P_o(s) = \tilde{M}^{-1}\tilde{N} = NM^{-1}$  be nonsquare. Furthermore, let  $P_o$  be "flat" and have full row rank for all  $\text{Re } s \geq 0$ . Then a possibly nonunique internally stabilizing compensator that achieves perfect recovery can be obtained if

- i) all RHP poles of  $P_o(s)$  are also RHP poles of  $P_pK_p(s)$
- ii)  $\exists N^{-R} \in RH_\infty$  s.t.  $[N^{-R} (X\tilde{M} - Sp) \tilde{M}^{-1}] \in RL_\infty$

Proof :

Sufficiency :

The postfailure plant has the doubly coprime factorization  $P_o(s) = \tilde{M}^{-1}\tilde{N} = NM^{-1}$  with  $\tilde{M}$  being an inner function, which is always possible to achieve. Then, according to (2.2) in Chapter 2, the following holds :

$$W(s) [S_o(s) - Sp(s)] = W(s) [(X\tilde{M} - Sp(s)) - NQ\tilde{M}] \quad (3.9)$$

where  $Q(s) \in RH_\infty$ .

According to ii), the choice of  $Q$  that would make  $\gamma_o = 0$  is

$$Q = N^{-R} (X\tilde{M} - Sp) \tilde{M}^{-1} \quad (3.10)$$

According to i) we have  $(X\tilde{M} - Sp) \tilde{M}^{-1} \in RH_\infty$  which together with ii) guarantees that  $Q(s) \in RH_\infty$ . It is not unique because  $N^{-R}$  is not unique. The internally stabilizing compensators are given by  $K(s) = -[Y - MQ][X - NQ]^{-1}$ . (3.11)

Necessity :

Let  $K(s) \in RL_\infty$  be an internally stabilizing compensator that guarantees perfect recovery for  $P_o(s)$ . Then,  $Q(s) \in RH_\infty$  and  $NQ = (X\tilde{M} - Sp) \tilde{M}^{-1} \in RH_\infty$ . Since  $P_o(s)$  has full row rank,  $N(s)$  has a possibly nonunique stable right inverse which is proper. Therefore, i) and ii) hold. This completes the proof. ■

The previous lemma explains how to obtain an internally stabilizing compensator that

achieves perfect recovery. We now present a simple example to illustrate that not every  $Po^{-R}$  will result in an internally stabilizing compensator  $K = [ Po^{-R} Pp Kp ]$  even if condition i) holds.

### Example 3.1

Let  $Kp$  be an internally stabilizing compensator for  $Pp$  and let  $PpKp = \frac{s+3}{s-2}$ . Furthermore, let the postfailure plant be given as  $Po = [ 1 \quad \frac{s-1}{s-2} ]$ . It is easy to see that the condition i) from Lemma 3.3 is satisfied.

The two simple right inverses of  $Po$  that have full row rank in RHP and the corresponding compensators obtained by  $[Po^{-R}PpKp]$  are :

$$(Po^{-R})_1 = [ 1 \ 0 ]' \quad \Rightarrow \quad (K)_1 = [ \frac{s+3}{s-2} \ 0 ]' \quad (3.12)$$

$$(Po^{-R})_2 = [ 0 \quad \frac{s-2}{s-1} ]' \quad \Rightarrow \quad (K)_2 = [ 0 \quad \frac{s+3}{s-1} ]' \quad (3.13)$$

If these compensators are internally stabilizing, all signals in the closed loop system will be stable including those that correspond to the transfer function  $[ I + K Po ]^{-1}$ . With the above compensators, we have :

$$[ I + (K)_1 Po ]^{-1} = \begin{bmatrix} \frac{s-2}{2s+1} & \frac{-(s-1)(s+3)}{(2s+1)(s-2)} \\ 0 & 1 \end{bmatrix} \notin RH_{\infty} \quad (3.14)$$

$$[ I + (K)_2 Po ]^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{-(s-2)}{(2s+1)(s-1)} & \frac{s-2}{2s+1} \end{bmatrix} \notin RH_{\infty} \quad (3.15)$$

Therefore, internal stability is not achieved. To be able to construct an internally stabilizing compensator  $K(s)$ , we carry out the coprime factorization of  $Po(s)$  that was introduced in Chapter 2. We have the following :

$$M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+2} \end{bmatrix}; \quad N = \left[ 1 \quad \frac{s-1}{s+2} \right]; \quad \tilde{M} = \frac{s-2}{s+2} \quad (3.16)$$

$$X = \frac{s-10}{s+2}; \quad Y = \left[ 0 \quad \frac{-16}{s+2} \right] \quad (3.17)$$

By choosing  $N^{-R} = [1 \ 0]'$  we get

$$Q = \begin{bmatrix} \frac{s^2 - 23s - 14}{(s+2)(2s+1)} \\ 0 \end{bmatrix} \in \text{RH}_\infty; \quad K = \begin{bmatrix} \frac{s^2 - 23s - 14}{(s+2)^2} \\ \frac{16(2s+1)}{(s+2)^2} \end{bmatrix} \quad (3.18)$$

This compensator guarantees internal stability and is not unique. This is so because  $Q$  is not unique due to the nonuniqueness of  $N^{-R}$ . The right inverse of  $P_o$  that achieves this compensator is equal to  $P_o^{-R} = K [P_p K_p]^{-1}$ .

The condition ii) in Lemma 3.2 guarantees that perfect recovery can be achieved with  $K \in \text{RL}_\infty$  as long as the other condition is satisfied. If it doesn't hold, we will show that perfect recovery can be achieved on any compact frequency interval  $[0, \omega^*]$ .

### Lemma 3.3

Let  $K_p(s)$  be an internally stabilizing compensator for  $P_p(s)$ , let  $W(s)$  be a proper outer function and let  $P_o(s) = \tilde{M}^{-1} \tilde{N} = NM^{-1}$  be a "flat" transfer function matrix. Furthermore, let the condition i) from Lemma 3.2 hold but let  $[N^{-R} (X\tilde{M} - S_p) \tilde{M}^{-1}]$  be improper. Then  $\gamma_0$  can be made arbitrarily small and complete recovery can be achieved at

any compact subinterval  $[0, \omega^*]$  of  $j\mathbb{R}$ .

Proof :

Let  $T_1 = [X - \text{Sp}(s)\tilde{M}^{-1}]$  and  $T_2 = N(s)$  with  $T_2^{-R}$  as its stable right inverse. The norm has the following form :

$$\|W [T_1 - T_2 Q]\|_{\infty} \quad (3.19)$$

We can introduce a SISO transfer function  $V(s) \in RH_{\infty}$  as

$$V(s) = 1 / [\tau s + 1]^k \quad \tau, k \in \mathbb{R} \quad (3.20)$$

where  $k$  is chosen s.t.  $[V T_2^{-R} T_1] \in RH_{\infty}$ .

By letting  $Q^* = V T_2^{-R} T_1$ , the norm (3.19) becomes

$$\|W [T_1 - T_2 Q^*]\|_{\infty} = \|W [T_1 - V T_2 T_2^{-R} T_1]\|_{\infty} = \|W (1 - V) T_1\|_{\infty} \quad (3.21)$$

Furthermore, it is bounded from above as follows :

$$\begin{aligned} \|W [T_1 - T_2 Q^*]\|_{\infty} &\leq \|W(j\omega)\|_{\infty} \| [T_1 - T_2 Q^*] \|_{\infty} \leq \\ &\leq \|1 - V(j\omega)\|_{\infty} \|T_1(j\omega)\|_{\infty} \|W(j\omega)\|_{\infty} \end{aligned} \quad (3.22)$$

where  $T_1(\infty) = V(\infty) = 0$  and  $\|W(j\omega)\|_{\infty}$  is a known constant..

Define a characteristic function  $\chi_1(j\omega)$  s.t.

$$\begin{aligned} \chi_1(j\omega) &= 1, \quad |\omega| \leq \omega_1 \\ \chi_1(j\omega) &= 0, \quad |\omega| > \omega_1 \end{aligned} \quad (3.23)$$

By choosing  $\omega_1 > \omega^*$  and from (3.21) and (3.22) we have :

$$\|\chi_1 (T_1 - T_2 Q^*)\|_{\infty} \leq \|\chi_1 [1 - V(j\omega)]\|_{\infty} \|T_1(j\omega)\|_{\infty} = 0 \quad (3.24)$$

$$\|\chi_1 (T_1 - T_2 Q^*)\|_{\infty} \leq \|1 - V(j\omega)\|_{\infty} \|\chi_1 T_1(j\omega)\|_{\infty} = \varepsilon \quad (3.25)$$

where  $\varepsilon$  is an arbitrarily small positive number.

The statement (3.24) is true because we can always choose  $\tau$  in (3.20) s.t.  $\|\chi_1[1 - V(j\omega)]\|_\infty = 0$ . On the other hand, by letting  $\tau = \omega_1$  and making it sufficiently large, (3.25) holds since  $T_1(\infty) = 0$ . This completes the proof.

■

A special class of postfailure situations occur when the change in the original plant can be reflected at the plant input. This usually happens when the failure is located in the actuators. The following lemma defines the necessary and sufficient conditions for perfect recovery in this case.

#### Lemma 3.4

Let the following hold:

- i)  $K_p$  internally stabilizes  $P_p$
  - ii)  $P_o = P_p \eta$  ;  $\eta^{-1} \in RL_\infty$
  - iii)  $W \in RH_\infty$
- (3.26)

Then,  $\|W(s) [S_o(s) - S_p(s)]\|_\infty$  is made equal to zero with internally stabilizing compensator  $K = \eta^{-1} K_p$  iff

- i) all RHP poles of  $\eta$  are exactly matched by either RHP zeros of  $P_p$  or RHP poles of  $K_p$
- ii) all RHP zeros of  $\eta$  are exactly matched by either RHP poles of  $P_p$  or RHP zeros of  $K_p$

Proof :

The above conditions imply that  $\eta(s)$  cannot introduce any RHP poles and transmission zeros that are not those of  $P_p K_p(s)$  and that there is no unstable pole-zero cancellation between  $P_o$  and  $K$ . This, according to Lemma 3.1 guarantees perfect recovery with the

internally stabilizing compensator  $K = \eta^{-1} P_p K_p$ . This completes the proof. ■

### 3.5 Robustness

The generalized failure treatment methodology includes an identification stage that results in a nominal model of the "failed" plant and the uncertainty associated with it. This information is used for the redesign of the original controller which is no longer able to meet the specified requirements.

In the previous section we defined the conditions that the redesigned controller has to satisfy. They consist of nominal stability and performance, treated in detail, and of the robustness requirement that will be explained now.

#### 3.5.1 Robust Stability

The setting for this analysis is represented in Figure 3.2. As shown before, the control algorithm redesign scheme is transformed into an equivalent robustness problem for the augmented plant  $M_o(s)$ . Stabilizing  $M_o$  guarantees the stability of the postfailure system with the same controller  $K$ . The failure identification part is assumed to provide the nominal model of the postfailure plant  $P_o(s)$  and the information about the associated uncertainty. The latter includes the type of perturbation and an estimate of its magnitude, i.e.

$$\Delta \in \Delta_S, \quad \Delta_S - \text{bound on its norm known} \quad (3.27)$$

Then,  $K$  belongs to the set of all stabilizing controllers for the nominal model of the plant that also guarantee stability of the system in the presence of any  $\Delta \in \Delta_S$ .

Let  $G_n$  be the stable nominal closed loop system including the weighting function  $W$ . Then the equivalent representation of the system to the one in Figure 3.2 is given as the upper linear-fractional transformation as shown in Figure 3.3. Stability of the overall system is guaranteed iff the the following condition is satisfied:

$$\det \{ I - G_n(Mol, K, s)\Delta \} \neq 0 \quad \forall \Delta \in \Delta_s \quad \forall s \geq 0 \quad (3.28)$$

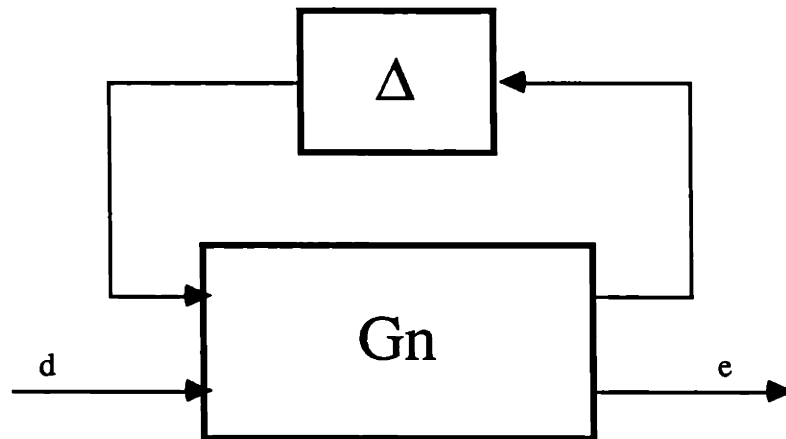


Figure 3.3 Linear Fractional Representation of a System with Uncertainty

The condition in (3.28) holds for all types of perturbation. If  $\Delta$  is specified as unstructured uncertainty, the above condition translates into that given by the "Small Gain Theorem" [Zames65]. In the case of structured uncertainty, consisting of individual unstructured ones, a condition on the "structured singular value" [Doyle82] is introduced.

A very important case arises when  $\Delta$  is specified as real parameter perturbations. Different approaches are being developed for the analysis and synthesis of control systems in the presence of the parameter perturbations. These have been introduced in Chapter 1, in section 1.2.

### 3.5.2 Robust Performance

Performance was previously defined as the measure of difference between the original and the post failure system. It was associated with the value of the infinity norm of a particular transfer function. In the nominal case, the conditions for optimal performance were derived on the basis of achieving the smallest difference between the two systems.

In the presence of uncertainty, the performance requirement is extended from the nominal system only to all systems in Figure 3.3 where  $\Delta \in \Delta_s$ . Therefore, it is defined as follows :

$$\inf_K \sup_{\Delta \in \Delta_s} \| M( G_n, \Delta ) \|_{\infty} \quad (3.29)$$

$$\text{or find a stabilizing } K \text{ s.t. } \sup_{\Delta \in \Delta_s} \| M( G_n, \Delta ) \|_{\infty} < \gamma \quad (3.30)$$

where  $M( G_n, \Delta )$  is the upper linear fractional transformation on  $G_n$  as shown in Figure 3.3. As it was stated before,  $M$  here stands for the weighted difference between the sensitivity transfer functions of the redesigned and original system, i.e.

$$M( G_n, \Delta ) = W [ S( P_o, K, \Delta ) - S_p ] \quad (3.31)$$

A performance requirement of that type can be transformed into a stability condition [Doyle84]. The latter corresponds to an unstructured uncertainty block  $\Delta_p$  that is introduced in addition to the already existing one. Hence, the problem of robust performance becomes a robust stability problem in the presence of the combined, overall structured uncertainty represented as  $\Delta_1 = \text{diag}\{\Delta, \Delta_p\}$ .

If  $\Delta$  is also unstructured uncertainty, then robust performance and stability criteria are associated with the "structured singular value" of the corresponding transfer function that is discussed in Chapter 4.



In the case when  $\Delta$  is given as real parameter perturbations, there are two distinct approaches that can be taken. They will be studied in Chapter 6. The first approach is based on simultaneous conditions for performance and stability. It uses Lyapunov stability theory and the nonexpansivity conditions given in the form of a Riccati equation that are introduced in Chapter 2. The other approach treats performance separately from the stability requirements. A stability margin  $\rho_s$  is evaluated as the measure of the largest stability region in parameter space in a given sense by using one of the methods mentioned in Chapter 1. Equivalently, a performance margin  $\rho_p$  is evaluated in parameter space too. It guarantees that the maximum singular value of the corresponding transfer function stays less than  $\gamma$  over all frequencies. Then the region in parameter space where stability and performance are guaranteed is given as :

$$\rho = \min \{ \rho_s, \rho_p \} \quad (3.32)$$

One of the main difficulties in evaluating  $\rho_p$  by checking the maximum singular value is that the resulting complex function is usually highly nonlinear in the perturbed parameters.

### 3.6 Concluding Remarks

This chapter has laid the foundation for the control algorithm redesign methodology in a generalized failure treatment setting. Redesign of the original controller was approached as the robust stability and performance problem in the presence of the uncertainty associated with the model of the postfailure plant. The latter is provided by the failure identification step. The redesign methodology can be used on-line if the latter information is provided sufficiently fast. For the most likely failures, the postfailure nominal model can be anticipated in advance. Then, a sequence of controllers could be designed each

guaranteeing stability and performance in the presence of uncertainty of increasing magnitudes.

Performance of the postfailure system was defined as a measure of the weighted output error signals. Minimization of the previously defined signals in the given sense was shown to lead to the minimal difference of the postfailure system from the original one.

The nominal case was treated in detail in section 3.4. The bounds on the achievable performance were derived in the absence of uncertainty. The notion of stability robustness and performance was introduced. Two different types of uncertainties were introduced. The frequency dependent, norm bounded uncertainty will be studied in Chapter 4. Redesign in the presence of real parameter uncertainty will be treated in Chapter 6.

## CHAPTER 4

### REDESIGN IN THE PRESENCE OF NORM BOUNDED UNSTRUCTURED UNCERTAINTY

#### 4.1 Introduction

The postfailure controller redesign was introduced in the previous section as a combined stability and performance robustness problem. The compensators to be designed should accommodate uncertainty of different magnitude.

In this section we assume that the identification scheme results in characterization of the frequency dependent uncertainty  $\Delta$  that is associated with the postfailure model of the plant. The uncertainty is modelled as a stable, diagonal transfer function whose entries are unstructured uncertainties present at different locations, with infinity norms bounded by a positive constant  $\alpha$ .

#### 4.2 Problem Formulation

The uncertainty resulting from the failure identification step is given as:

$$\Delta \in RH_{\infty}, \quad \Delta = \text{diag} \{ \Delta_i \}, \quad \|\Delta\|_{\infty} < \alpha, \quad (4.1)$$

A controller that will be designed has to ensure not only stability of the postfailure system but also its performance in the presence of uncertainty. The performance

requirement given by the positive constant  $\gamma$ , previously explained as quantifying the difference from the original system, can be transformed into an additional robust stability condition by adding the uncertainty block  $\Delta_p$  as follows [Doyle84] :

$$\Delta_p \in RH_\infty, \quad \|\Delta_p\|_\infty < \gamma^{-1} \quad (4.2)$$

The closed loop system can now be presented in its upper fractional transformation, as shown in Figure 4.1.

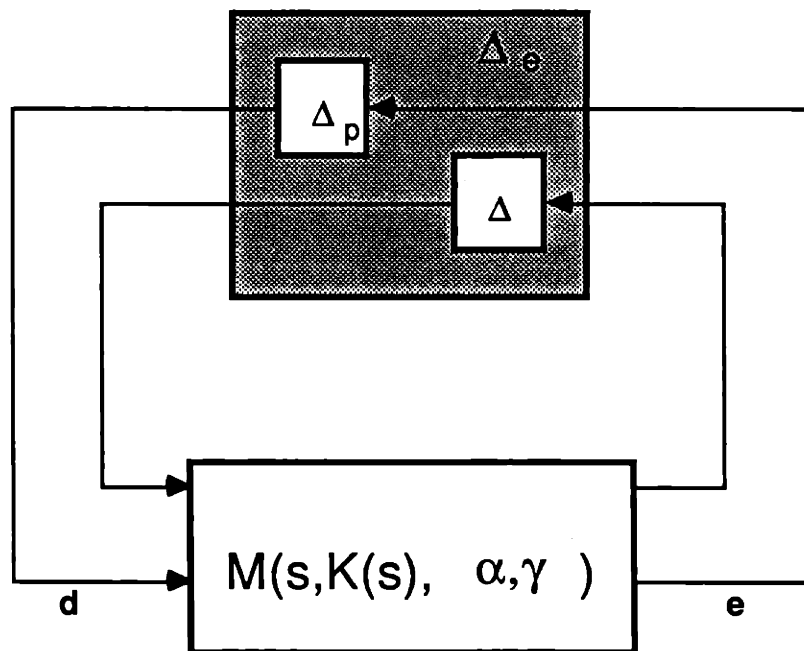


Figure 4.1 Scaled System with Two Block Uncertainty

$M(s)$  is the overall closed loop transfer function, and it can be scaled so that the norms of the uncertainties are smaller than or equal to one. After scaling, it takes the following form :

$$M(s) = M(s, K(s), \alpha, \gamma) \quad (4.3)$$

According to (2.57) in Chapter 2, stability and performance of this system are guaranteed

if:

$$\mu \{ M(s, K(s), \alpha, \gamma) \} \leq 1$$

$$\text{where } \mu \{ M \} = \inf_{D, K} \| D M(j\omega, K(j\omega), \alpha, \gamma) D^{-1} \|_{\infty}, \quad (4.4)$$

$D \in \mathbf{D} = \text{diag} \{ d_1 I_1, \dots, d_m I_m \}$ ,  $d_i, d_i^{-1} \in \mathbf{RH}_{\infty}$ ,  $I_i$  compatible with the dimensions of individual blocks, and  $K(s)$  belongs to the set of all stabilizing compensators for the corresponding plant.

It is easy to see that the scaling factors  $\gamma$  and  $\alpha$  can be used as design parameters. There are two possible situations that may arise regarding these two parameters. They are defined as follows :

i) bound on uncertainty  $\alpha$  known, minimize loss in performance  $\gamma$  by choosing appropriate compensator  $K$ , i.e.

$$\min_{R_+} \gamma \quad \text{s.t.} \quad \inf_{K \in K_s} \sup_{\omega} \mu \{ M(s, K(s), \alpha, \gamma) \} \leq 1 \quad (4.5)$$

ii) bound on performance index  $\gamma$  known, maximize bound on uncertainty  $\alpha$  by choosing appropriate compensator  $K$ , i.e.

$$\max_{R_+} \alpha \quad \text{s.t.} \quad \inf_{K \in K_s} \sup_{\omega} \mu \{ M(s, K(s), \alpha, \gamma) \} \leq 1 \quad (4.6)$$

The first situation corresponds to the case where the postfailure detection and identification are carried out on-line. After the nominal model of the plant and the magnitude of the associated uncertainty become known, we can minimize the loss in performance.

However, for some very likely failures we can anticipate the nominal model of the postfailure plant in advance with the specific magnitude of the uncertainty still unknown. Then a sequence of compensators can be designed according to ii) by relaxing the condition on performance. Since this design is done off-line, the resulting controllers will be stored in memory. When the estimate of the bound on the uncertainty becomes available, the controller with the smallest performance index  $\gamma$  that accommodates it will be chosen. Although the two problems are distinct in nature, their mathematical formulation is identical. We will introduce an algorithm for handling both situations. It will be used for optimizing one of the two above mentioned variables while the other is kept fixed. The algorithm will be based on the "D-K" iteration, introduced in Chapter 2, but a substantial modification will be made. The next section introduces some preliminary results needed for the algorithm.

### 4.3 Preliminary Results

In this section we define certain properties of the system  $M(j\omega, K, \alpha, \gamma)$ , introduced in Figure 4.1, with respect to changes in  $\alpha$  and  $\gamma$ . The value of the performance index will be kept constant while the uncertainty bound  $\alpha$  will be the variable of interest. The results obtained will also hold in the opposite case, where  $\alpha$  is kept constant and  $\gamma$  is allowed to vary.

The system  $M(j\omega, K, \alpha, \gamma) \in RH_\infty$  can be presented in the (2x2) block form with inputs and outputs corresponding to those of the uncertainties  $\Delta$  and  $\Delta_p$ . With  $\gamma$  fixed we have :

$$M(j\omega, K, \alpha) = \begin{bmatrix} \alpha M_{11}(j\omega, K) & M_{12}(j\omega, K) \\ \alpha M_{21}(j\omega, K) & M_{22}(j\omega, K) \end{bmatrix} = [ \alpha m_1 \quad m_2 ] =$$

$$= M(K) \text{diag} \{ \alpha, 1 \} \quad (4.7)$$

where  $K$  is an internally stabilizing compensator,  $M$  is a transfer function matrix with  $m_i$  as its columns and  $\alpha \in \mathbb{R}_+$ . The performance index  $\gamma$  is incorporated in  $M(K)$  and is not shown as a variable since it remains constant.

Let us define the following functions as :

$$\text{i) } f(\alpha) = \inf_{K \in K_s} \| M(K) \text{diag} \{ \alpha I_1, I_2 \} \|_{\infty} \quad (4.8)$$

$$\text{ii) } g(\alpha) = \inf_{D \in \mathbf{D}} \| DM(K)D^{-1} \text{diag} \{ \alpha I_1, I_2 \} \|_{\infty}; \quad K(s) \text{ fixed} \quad (4.9)$$

$$\text{iii) } \mu(\alpha) = \inf_{K \in K_s} \inf_{D \in \mathbf{D}} \| D M(j\omega, K, \alpha) D^{-1} \|_{\infty} \quad (4.10)$$

We now check the properties of these functions with respect to a change in  $\alpha$ .

#### Lemma 4.1

The function  $f(\alpha)$  defined in (4.8) is a nondecreasing function for  $\alpha \geq 0$ .

Proof :

At the fixed frequency  $\omega = \omega_1$ , with the same compensator  $K$  and with  $\varepsilon > 0$ , we have

$$\begin{aligned} M_{\alpha\varepsilon}(j\omega_1, K) &= [\alpha\varepsilon m_1, m_2] = \\ &= [\alpha m_1, m_2] \text{diag} \{ \varepsilon, 1 \} = M_{\alpha}(j\omega_1, K) \text{diag} \{ \varepsilon, 1 \} \end{aligned} \quad (4.11)$$

Using the properties of singular values, the following inequality holds :

with  $\varepsilon \geq 1$

$$\overline{\sigma} \{ M_{\alpha} (j\omega_1, K) \} \leq \overline{\sigma} \{ M_{\alpha\epsilon} (j\omega_1, K) \} \leq \epsilon \overline{\sigma} \{ M_{\alpha} (j\omega_1, K) \}$$

and with  $\epsilon \leq 1$

$$\epsilon \overline{\sigma} \{ M_{\alpha} (j\omega_1, K) \} \leq \overline{\sigma} \{ M_{\alpha\epsilon} (j\omega_1, K) \} \leq \overline{\sigma} \{ M_{\alpha} (j\omega_1, K) \} \quad (4.12)$$

Since  $\omega_1$  is arbitrary and  $K$  is fixed, (4.12) is valid at any frequency and the same inequalities hold for the corresponding infinity norms, i.e.

$$\| M_{\alpha} (j\omega, K) \|_{\infty} \leq \| M_{\alpha\epsilon} (j\omega, K) \|_{\infty} \leq \epsilon \| M_{\alpha} (j\omega, K) \|_{\infty} \quad \text{with } \epsilon \geq 1 \quad \text{and}$$

$$\epsilon \| M_{\alpha} (j\omega, K) \|_{\infty} \leq \| M_{\alpha\epsilon} (j\omega, K) \|_{\infty} \leq \| M_{\alpha} (j\omega, K) \|_{\infty} \quad \text{with } \epsilon \leq 1 \quad (4.13)$$

$$\text{Let } K_1 = \arg \{ \inf_{K \in K_s} \| M(j\omega, K, \alpha) \|_{\infty} \}$$

$$\text{and } K_2 = \arg \{ \inf_{K \in K_s} \| M(j\omega, K, \epsilon \alpha) \|_{\infty} \} \quad (4.14)$$

meaning that they achieve the infimum or that they guarantee the value of the norm to be " $\theta$ " close to the infimum where  $\theta$  is an arbitrarily small number.

From (4.13) and (4.14) we have:

with  $\epsilon \geq 1$

$$\| M_{\alpha} (j\omega, K_1) \|_{\infty} \leq \| M_{\alpha\epsilon} (j\omega, K_2) \|_{\infty} \leq \epsilon \| M_{\alpha} (j\omega, K_1) \|_{\infty}$$

$$\text{or, equivalently, } f(\alpha) \leq f(\epsilon \alpha) \leq \epsilon f(\alpha) \quad (4.15)$$

and with  $\epsilon \leq 1$

$$\epsilon \| M_{\alpha} (j\omega, K_1) \|_{\infty} \leq \| M_{\alpha\epsilon} (j\omega, K_2) \|_{\infty} \leq \| M_{\alpha} (j\omega, K_1) \|_{\infty}$$

$$\text{or, equivalently, } \epsilon f(\alpha) \leq f(\epsilon \alpha) \leq f(\alpha) \quad (4.16)$$

which completes the proof for  $\alpha \in [0, \infty)$ .

■



**Lemma 4.2**

The function  $f(\alpha)$  is uniformly continuous at any compact subinterval of  $[0, \infty)$ .

Proof :

A function is continuous at the point  $\alpha_0$  if the following holds:

$$f(\alpha) \rightarrow f(\alpha_0) \quad \text{when} \quad \alpha \rightarrow \alpha_0. \quad (4.17)$$

Let  $K(s)$  and  $K_0(s)$  be the solution in (4.8) for some  $\alpha$  and  $\alpha_0$  respectively. Then, according to the previous lemma and by letting  $\alpha = \varepsilon \alpha_0$  on  $\mathbb{R}_+$ , we have:

with  $\varepsilon \geq 1$

$$f(\alpha_0) \leq f(\varepsilon \alpha_0) \leq \varepsilon f(\alpha_0) \quad (4.18)$$

or, with  $\varepsilon \leq 1$ ,

$$\varepsilon f(\alpha_0) \leq f(\varepsilon \alpha_0) \leq f(\alpha_0) \quad (4.19)$$

By letting  $\alpha \rightarrow \alpha_0$  or, equivalently,  $\varepsilon \rightarrow 1$ , in the above inequalities, we have

$$\lim_{\varepsilon \rightarrow 1} f(\varepsilon \alpha_0) = \lim_{\alpha \rightarrow \alpha_0} f(\alpha) = f(\alpha_0) \quad (4.20)$$

Since (4.20) holds for every  $\alpha_0 \in (0, \infty)$ , we just need to prove continuity at  $\alpha_0 = 0$ . We have the following inequalities:

$$0 \leq [f(\alpha) - f(0)] = \| M(K_0) \text{diag}\{ \alpha I_1, I_2 \} \|_{\infty} - f(0) \quad (4.21)$$

$$\| M(K_0) \text{diag}\{ \alpha I_1, I_2 \} \|_{\infty} - f(0) \leq \| M(K_0) \text{diag}\{ \alpha I_1, 0 I_2 \} \|_{\infty} \quad (4.22)$$

$$\| M(K_0) \text{diag}\{ \alpha I_1, 0 I_2 \} \|_{\infty} \leq |\alpha| \| M(K_0) \|_{\infty} \quad (4.23)$$

where now  $K_0(s)$  is the minimizing solution for  $\alpha_0=0$ . The first inequality states that the function is nondecreasing at zero too. The triangular inequality (4.22) and the inequality

(4.23) are based on the properties of the infinity norm. From these inequalities we have:

$$0 \leq \lim_{\alpha \rightarrow 0} [f(\alpha) - f(0)] \leq 0 \quad (4.24)$$

which completes the proof that the function is continuous at  $[0, \infty)$ . The uniform continuity on any compact subinterval is then guaranteed by Heine's Theorem [Apos74]. This concludes the proof.

■

#### Corollary 4.2.1

The value of  $\|M(K) \text{diag}\{\alpha I_1, I_2\}\|_{\infty}$  with a fixed compensator  $K(s)$  is a nondecreasing and continuous function of the parameter  $\alpha$  on the interval  $[0, \infty)$ .

#### Lemma 4.3

Let  $D = \text{diag}\{d_1 I_1, \dots, d_j I_j, \dots, d_m I_m\}$  be the "D" scaling introduced in Chapter 2 where  $d_i, d_i^{-1}$  are SISO functions in  $\text{RH}_{\infty}$ , and let  $\Theta = \text{diag}\{I_1, \dots, \alpha I_j, \dots, I_m\}$ . Then the  $D$  and  $\Theta$  scalings commute, i. e.

$$D \Theta M(j\omega) D^{-1} = \Theta D M(j\omega) D^{-1} \quad (4.25)$$

The proof is obvious since both matrices are diagonal and the dimensions are compatible.

■

**Lemma 4.4**

For the fixed compensator  $K(s)$ ,  $\alpha \geq 0$ , and  $\varepsilon \in \mathbb{R}_+$ , the function  $g(\alpha)$  defined in (4.9) is nondecreasing and continuous.

Proof :

At any frequency  $\omega = \omega_1$  and for the two block uncertainty, as stated in Chapter 2, we have

$$\mu\{ M(j\omega) \} = \inf_{D \in \mathbf{D}} \overline{\sigma} \{ D M(j\omega_1) D^{-1} \} \quad (4.26)$$

where the scaling  $D$  was defined previously.

$$\text{Let } D_1 = \arg \left\{ \inf_{D \in \mathbf{D}} \overline{\sigma} \{ D M_\alpha(j\omega_1, K) D^{-1} \} \right\} \quad \text{and}$$

$$D_2 = \arg \left\{ \inf_{D \in \mathbf{D}} \overline{\sigma} \{ D M_{\alpha\varepsilon}(j\omega_1, K) D^{-1} \} \right\} \quad (4.27)$$

where  $D_1$  and  $D_2$  either achieve the infimum or keep the value of  $\overline{\sigma}$  arbitrarily close to it.

From Lemma 4.1 and Lemma 4.3, for the given compensator  $K$  at  $\omega = \omega_1$ , we have :

with  $\varepsilon \geq 1$ ,

$$\overline{\sigma} \{ D_1 M_\alpha D_1^{-1} \} \leq \overline{\sigma} \{ D_2 M_{\alpha\varepsilon} D_2^{-1} \} \leq \varepsilon \overline{\sigma} \{ D_1 M_\alpha D_1^{-1} \}$$

and with  $\varepsilon \leq 1$ ,

$$\varepsilon \overline{\sigma} \{ D_1 M_\alpha D_1^{-1} \} \leq \overline{\sigma} \{ D_2 M_{\alpha\varepsilon} D_2^{-1} \} \leq \overline{\sigma} \{ D_1 M_\alpha D_1^{-1} \} \quad (4.28)$$

The inequalities in (4.28) hold at every frequency. Therefore, they will also hold when the supremum of each entry is achieved over the frequencies. Then the following holds :

with  $\varepsilon \geq 1$ ,

$$g(\alpha) \leq g(\varepsilon \alpha) \leq \varepsilon g(\alpha) \quad (4.29)$$

and with  $\varepsilon \leq 1$ ,

$$\varepsilon g(\alpha_0) \leq g(\varepsilon \alpha_0) \leq g(\alpha_0) \quad (4.30)$$

This shows that the function  $g(\alpha)$  is a nondecreasing function. The proof of continuity is analogous to that presented in Lemma 4.2.

This completes the proof. ■

#### Corollary 4.4.1

With the fixed compensator  $K(s)$ , the supremum over frequency of the " $\mu$ " function corresponding to the system in Figure 4.1, is a continuous and nondecreasing function of the scaling factor  $\alpha$  on  $\mathbb{R}_+$ .

#### Lemma 4.5

With  $\varepsilon > 0$ ,  $D$  as previously defined, and assuming that the solution of the infimization in (4.10) exists for any two  $\alpha$  and  $\varepsilon\alpha$ , the following holds:

with  $\varepsilon \geq 1$ ,

$$\mu(\alpha) \leq \mu(\varepsilon\alpha) \leq \varepsilon \mu(\alpha) \quad (4.31)$$

and with  $\varepsilon \leq 1$ ,

$$\varepsilon \mu(\alpha) \leq \mu(\varepsilon\alpha) \leq \mu(\alpha) \quad (4.32)$$

Proof :

The proof is analogous to those in Lemma 4.1 and Lemma 4.4. ■

The last lemma states that the increase of  $\alpha$  can be performed by the successive minimization of a " $\mu$ " function with  $\alpha$  fixed at each step. This could be done by a sequence of full "D-K" iterations for increasing values of  $\alpha$ . If the obtained minimum value of the " $\mu$ " function with fixed  $\alpha$  is less than one, according to the previous lemma, we could

either find a larger value of  $\alpha$  that makes  $\mu=1$  or we could increase  $\alpha$  up to some previously set upper limit. This search for  $\alpha$  can be performed by bisection with a suitable step size. Unfortunately, the full fledged "D-K" iteration would involve a sequence of approximation of "D" scaling with a real rational stable function and potentially result in substantial increase in order of the compensator before the value of  $\alpha$  is changed. Therefore, the idea is to modify this approach in such a way that the value of the uncertainty bound is increased as much as possible before the "D" scaling is changed. A way to do it is to try to increase  $\alpha$  in the first step of the "D-K" iteration where the scaling "D" is fixed.

The functions  $f(\alpha)$  and  $g(\alpha)$  correspond to the two steps in the "D-K" iteration introduced in Chapter 2. Their properties imply that both the minimum value of the infinity norm and the " $\mu$ " function associated with the same system with the compensator kept fixed are nondecreasing continuous functions of  $\alpha$ . This fact will be used in the algorithm presented in the following section.

#### 4.4 Algorithm Outline

In this section we present an algorithm for enlarging the bound on the uncertainty with the fixed requirement  $\gamma$  on performance. It is based on the previously stated lemmas and properties of the "D-K" iteration [Doyle82] including the fact that the obtained solution will usually correspond to a local optimum only. The algorithm provides a solution to a somewhat different problem than the classical stability/performance robustness one. In the latter, the bounds on the uncertainty and performance are known, and the goal is to find at least one stabilizing compensator for the nominal plant that satisfies both requirements. In comparison, we are looking for the compensator that satisfies one criterion while it optimizes the other.

The objective of the algorithm can be stated as follows :

- for a given  $\gamma$ , find the largest  $\alpha \in \mathbb{R}_+$  s.t.

$$\inf_{D,K} \| D M(j\omega, K(j\omega), \alpha, \gamma) D^{-1} \|_{\infty} \leq 1 \quad (4.33)$$

The outline of the algorithm is shown in the following flowchart where "D<sub>F</sub>" is the diagonal scaling kept fixed in each step A). The detailed explanation will be given on the following page.

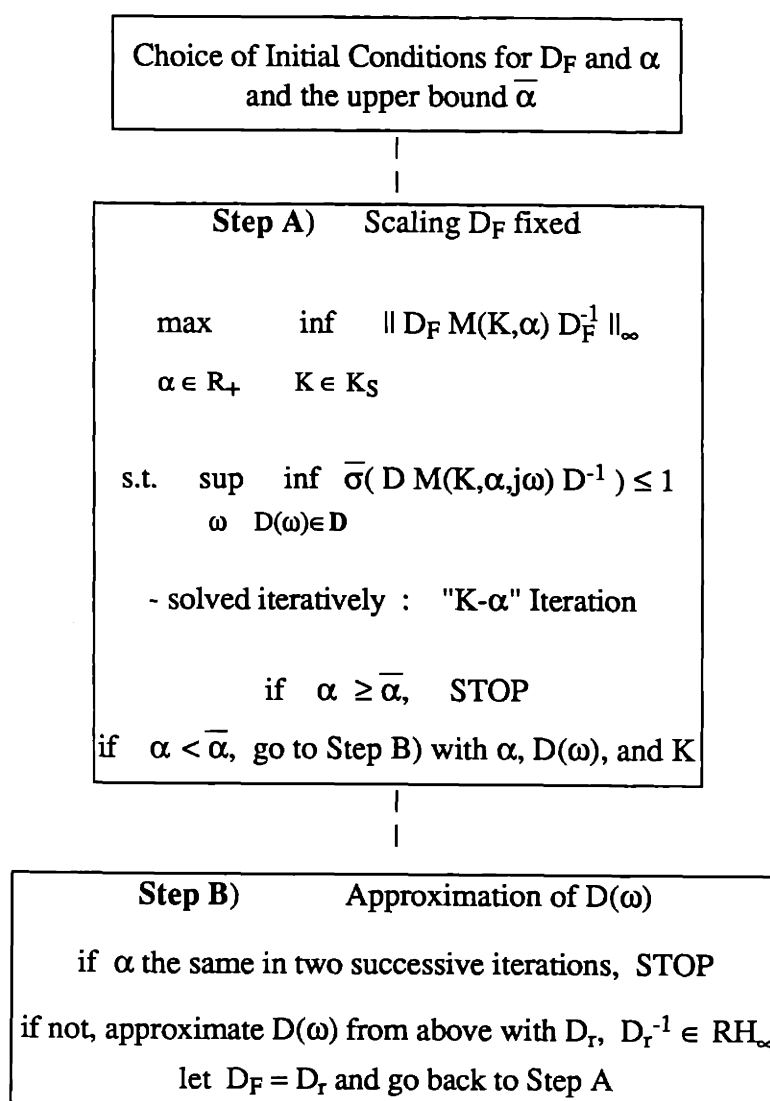


Figure 4.2 Flowchart of the Algorithm

As shown above, the algorithm has two main steps. In the first step we would like to find a pair  $[\alpha, K]$  as the solution to the constrained max-min problem. The constraint is given in the form of the " $\mu$ " function associated with the system  $M(j\omega, K, \alpha)$  which is not allowed to exceed the value of one over frequency. The cost function  $\|D_F M(K, \alpha) D_F^{-1}\|_\infty$  is the upper bound for  $\mu\{M(j\omega, K, \alpha)\}$  and, therefore, it can take a value greater than one. The former is convex in  $K$  [Doyle82] and in  $\alpha$  separately. It is not jointly convex in both variables, which is easy to check.

The constrained problem in Step A) is solved iteratively. The procedure that we will refer to as "K- $\alpha$ " iteration, consists of three substeps. To be able to trace changes of  $\alpha$  and other variables of interest throughout the algorithm, we assign a double index  $(i, j)$  to them where  $i \geq 1$  and  $j \geq 0$ . The index "i" will show how many times step A) has been executed. The "j" index will trace the increase of  $\alpha$  within step A) from the initial value  $\alpha_{1,0}$ . The variables of interest that remain constant within step A) will have the index "i" only. A choice of the initial conditions and stopping of the algorithm will be discussed once the latter is introduced in detail. The detailed representation of both steps is given as follows :

**STEP A)**  $D_{F_i}$  is fixed, the initial choice of  $\alpha_{i,0}$  is available, and the upper bound  $\bar{\alpha}$  is set

A1) start with the value of  $\alpha = \alpha_{i,j}$ ,  $j=0,1,2,..$  and find  $K_{i,j+1}$  that makes

$$\|D_{F_i} M(j\omega, K_{i,j+1}, \alpha_{i,j}) D_{F_i}^{-1}\|_\infty = \inf_{K \in K_s} \|D_{F_i} M(j\omega, K, \alpha_{i,j}) D_{F_i}^{-1}\|_\infty \quad (4.34)$$

A2) with  $K_{i,j+1}$  from above, find  $\mu_s$  and  $D$  s.t.

$$\mu_s = \sup_{\omega} \inf_{D \in D} \bar{\sigma} \{ D M(j\omega, K_{i,j+1}, \alpha_{i,j}) D^{-1} \} = \sup_{\omega} \mu \{ M(j\omega, K_{i,j+1}, \alpha_{i,j}) \} \quad (4.35)$$

A3) if  $\mu_s \leq 1$ , then by intersection find the largest  $\alpha = \alpha_{i,j+1}$  that makes

$$\sup_{\omega} \inf_{D \in \mathbf{D}} \overline{\sigma} \{ D M(j\omega, K_{i,j+1}, \alpha_{i,j+1}) D^{-1} \} \leq 1 \quad (4.36)$$

- if  $\alpha_{i,j} < \alpha_{i,j+1} < \overline{\alpha}$ , then save  $K_f = K_{i,j+1}$  and go back to A1) with  $\alpha_{i,j+1}$
- if  $\alpha_{i,j+1} = \alpha_{i,j}$ , then go to step (B) with :  $D$  from (4.36),  $\alpha^{(i)} = \alpha_{i,j}$ ,  $K_f$ , and  $\mu_{s_i} = \mu_s$
- if  $\alpha_{i,j+1} \geq \overline{\alpha}$ , then save  $\alpha_{\max} = \overline{\alpha}$ ,  $K_f = K$  and STOP the algorithm

if  $\mu_s > 1$ , then go to step (B) with :  $D$  from (4.35),  $\alpha^{(i)} = \alpha_{i,j}$ ,  $K_f$ , and  $\mu_{s_i} = \mu_s$

**STEP B)** if  $\alpha^{(i)} = \alpha^{(i-1)}$  and  $\mu_{s_i} = \mu_{s_{i-1}}$ , save  $\alpha_{\max} = \alpha^{(i)}$ , and the compensator  $K_f$  and STOP.

- if not, approximate  $D(j\omega)$  with the real rational, stable invertible  $D_r(j\omega) \in RH_{\infty}$ , and go back to step (A) with  $D_{F_{i+1}} = D_r$  and  $\alpha_{i+1,0} = \alpha^{(i)}$ . (4.37)

In the next section we will analyze the convergence of the algorithm and discuss its characteristics.

## 4.5 Properties of the Algorithm

The most important properties of the algorithm are presented in this section. They are :



**Property #1)** The sequence  $\{ \alpha^{(i)} \}$ , consisting of the maximum bounds on uncertainty obtained in substep A3) in each "i" iteration, is nondecreasing, meaning that  $\alpha^{(i)} \leq \alpha^{(i+1)}$ .

This property is guaranteed by the construction of the algorithm. We now prove this statement by looking at the different situations that might occur during execution of the algorithm.

Step A) provides the largest bound on the uncertainty while keeping the block diagonal scaling factor  $D_F$  constant. It results in a compensator  $K$  and the largest bound on the uncertainty  $\alpha$  s.t. the corresponding " $\mu$ " function doesn't exceed one over the entire frequency range. Since we do not have direct control over the " $\mu$ " function, the above optimization is carried out by acting on the infinity norm of the corresponding transfer function which is its upper bound.

Let the initial value of  $\alpha$  in the first "i" iteration be  $\alpha_{1,0}$ . Furthermore, let it be chosen such that the infinity norm minimizing compensator  $K=K_{1,1}$ , obtained from (4.34) in A1), will make

$$\mu_s \{ M(j\omega; K_{1,1}; \alpha_{1,0}) \} \leq 1 \quad (4.38)$$

This is evaluated in substep A2). Since the above function is less than or equal to one, the bound on the uncertainty  $\alpha$  can be enlarged just until the inequality in (4.38) is violated. This is possible because the above function is continuous and nondecreasing in  $\alpha$  according to Corollary 4.4.1. The algorithm uses the bisection method that results in the largest value of  $\alpha_i$  that doesn't make " $\mu_s$ " greater than one with the final accuracy that has to be defined in advance. The accuracy will depend on the size of the initial value of  $\alpha$ . The bisection behaves as the new internal loop because after each change of  $\alpha$ , the corresponding " $\mu$ " function has to be evaluated. The process is repeated until a solution is found.

In some cases, the value of the function in (4.38) will not be affected by the increase of  $\alpha$  since it is a nondecreasing function of the latter. Then, the bound on the uncertainty can

be extended indefinitely or to any limit  $\bar{\alpha}$  that we can previously set.

Let the new increased value be  $\alpha_{1,1}$ . The system is then given as  $[D_{F_1}M(K_{1,1}; \alpha_{1,1})D_{F_1}^{-1}]$ .

The compensator  $K_{1,1}$  that makes  $\mu_s\{M(K_{1,1}; \alpha_{1,1})\} = 1$  is not minimizing in the infinity norm sense anymore, i.e.

$$\begin{aligned} \|D_{F_1}M(K_{1,1}; \alpha_{1,1})D_{F_1}^{-1}\|_{\infty} &\geq \inf_{K \in K_s} \|D_{F_1}M(K, \alpha_{1,1})D_{F_1}^{-1}\|_{\infty} = \\ &= \|D_{F_1}M(K_{1,2}; \alpha_{1,1})D_{F_1}^{-1}\|_{\infty} \end{aligned} \quad (4.39)$$

The new minimizing compensator  $K_{1,2}$  is obtained in (4.34) which represents the beginning of a new full iteration within step A).  $K_{1,2}$  minimizes the infinity norm of the system  $[D_{F_1}M(K_{1,2}, \alpha_{1,1})D_{F_1}^{-1}]$ , but it does not necessarily reduce the value of the " $\mu_s$ " function from the one achieved by  $K_{1,1}$  with the same  $\alpha_{1,1}$ . Therefore, we might have the following situation:

$$\mu_s\{M(K_{1,1}; \alpha_{1,1})\} \leq 1 < \mu_s\{M(K_{1,2}; \alpha_{1,1})\} \quad (4.40)$$

regardless of (4.39). This is possible because we don't have full control over the " $\mu$ " function but only on its upper bound, which is the corresponding infinity norm.

In this case we exit step A) and go to step B) with  $\alpha^{(1)} = \alpha_{1,1}$ ,  $K_f = K_{1,1}$ , and the optimal "D" scaling from evaluation of  $\mu_s\{M(K_{1,2}; \alpha_{1,1})\}$ . The idea is to try to further increase  $\alpha$  by changing the scaling  $D_{F_1}$  used in step A). This is done by approximating  $D$  by the stable outer proper transfer function  $D_r$  which is then used in step A) as the new scaling  $D_{F_2}$ .

The algorithm could proceed in a way that the next iteration in step A) doesn't increase the value of  $\alpha$ , i.e.  $\alpha^{(2)} = \alpha^{(1)}$  and that the corresponding " $\mu_s$ " function in (4.35) exceeds one. In this case the algorithm behaves as the ordinary "D-K" iteration. When this happens,

the "D" scaling is approximated by entering step B). This approximation is then used in the new iteration in step A). If in this new step A) the iteration is carried out without further increasing the uncertainty bound  $\alpha$  and without further decreasing " $\mu_s$ ", the algorithm will eventually stop in the following step B) with " $\mu_s$ " greater than one in this case. This will happen because the previous scaling  $D_F$  didn't make any difference. Then,  $\alpha_{\max} = \alpha^{(1)}$  and the compensator that achieves it is  $K_f$ . This shows the importance of saving the compensator that actually achieves  $\alpha_i$  before exiting step A).

In all other cases, the updating of  $D_F$  in step B) will eventually result in making the " $\mu_s$ " function smaller than one in successively executing step A). Then the value of  $\alpha$  will be increased until the largest possible one that doesn't make " $\mu_s$ " exceed one is found. The corresponding "D" scaling will then be approximated in step B) and a new iteration in A) will start.

As shown above, the algorithm enlarges the bound on uncertainty in step A) only. If it is not possible, i.e. if  $\mu_s > 1$  in (4.35), the algorithm behaves as the ordinary "D-K" iteration. The stopping condition is the same as in the latter. The only exception is the case when the previously set upper bound  $\bar{\alpha}$  is reached in substep A3). Therefore, the successive values of  $\alpha$ , obtained during the execution of the algorithm, form an increasing sequence.

**Property #2)** The obtained sequence  $\{ \alpha^{(i)} \}$  is bounded from above.

From the previous discussion about the algorithm construction, it is obvious that the obtained  $\alpha_{\max} \leq \bar{\alpha}$ . Since the sequence is nondecreasing, all its elements are bounded by  $\bar{\alpha}$ .

**Property #3)** The obtained sequence  $\{ \alpha^{(i)} \}$  converges.

This is the consequence of the first two properties. Any nondecreasing sequence that is bounded from above is convergent[Rud87].

**Property #4)** In each step A), the algorithm increases  $\alpha$  as much as possible without changing the block diagonal scaling  $D_F$  and therefore, without increasing the order of compensator.

The increasing of  $\alpha$  is performed in step A) while keeping the scaling  $D_F$  fixed. This is done because we could possibly reach the upper bound  $\bar{\alpha}$  or any other desired level within the first step A), i.e.  $i=1$ , and therefore, without increasing the order of the system. On the other hand, if we would like to do it by the full sequence of "D-K" iterations according to Lemma 4.5, the order of the system would be increased in each "D" step while minimizing " $\mu$ " with the initial value of  $\alpha$  kept fixed.

**Property #5)** The resulting  $\alpha_{\max}$  will represent the largest obtained stability margin only if there is compensator  $K_f$  that actually achieves it.

The algorithm is constructed in such a way that the compensator  $K_f$  is saved in substep A3) only if the previously obtained  $\mu_s$  in (4.35) is smaller than or equal to one. Therefore, there is going to be a compensator  $K_f$  that achieves  $\alpha_{\max}$  only if the initial value of  $\alpha=\alpha_{1,0}$  was increased as in (4.36) at least once during the execution of the algorithm.

In a special case, for a particular choice of  $\alpha_{10}$ , the algorithm might not result in  $\mu_s \leq 1$  in any "i" iteration and no compensator  $K_f$  will be saved. Then the algorithm will behave as the ordinary "D-K" iteration for the system with a fixed  $\alpha=\alpha_{10}$  and it will stop when two successive values of " $\mu_s$ " will remain the same and possibly greater than one. This then would imply that we were not able to come up with a compensator that guarantees the

stability and performance for this magnitude of uncertainty  $\alpha_{\max}$ .

**Property #6)** A sufficient condition for the existence of  $K_f$  that achieves the resulting  $\alpha_{\max}$  is the choice of a sufficiently small  $\alpha_{10}$  such that  $\mu_s \{ M(j\omega; K_{1,1}; \alpha_{10}) \} \leq 1$ .

In property #5 it was shown that a particular choice of  $\alpha_{10}$  could result in algorithm behaving as the ordinary "D-K" iteration and eventually failing to decrease the value of " $\mu_s$ " below one. Furthermore, no compensator  $K_f$  would be saved. Therefore, a suitable choice for  $\alpha_{10}$  is necessary for obtaining the desired behavior of the algorithm.

By choosing  $\alpha_{10}$  as stated above, we can possibly increase the value of the stability margin while the corresponding compensator  $K_f = K_{11}$  will be saved in the following substep A3). We now present a way how to obtain such an initial condition. It can be chosen as follows :

$$\text{i) find } K^* \text{ s.t. } \inf_{K \in K_s} \| M_{22}(K) \|_{\infty} = \| M_{22}(K^*) \|_{\infty} \quad (4.41)$$

where  $M_{22}$  is the (2,2) element of  $M(s)$  presented in (4.7).

$$\text{ii) find the maximum } \alpha \in \mathbb{R}_+ \text{ s.t.}$$

$$\| M_{22}(K^*) \|_{\infty} \leq \sup_{\omega} \mu \{ M(j\omega, K^*, \alpha) \} \leq 1 \quad (4.42)$$

The norm in (4.41) corresponds to the situation when the uncertainty  $\Delta$  is equal to zero. Its value is clearly the lower bound for the " $\mu$ " function in all the cases where  $\Delta$  is nonzero. If it is greater than one, we will not be able to find a compensator that guarantees performance conditions for a given value of  $\gamma$  even in the nominal case. Therefore, the value of the norm in (4.41) being smaller than one is the condition for the "wellposedness" of the

overall problem. If this condition is satisfied, according to Corollary 4.4.1, we can find a sufficiently small  $\alpha = \alpha_{1,0}$  s.t. " $\mu\{M(K^*; \alpha_{1,0})\}$ " doesn't exceed one over frequency.

**Property #7)** The algorithm depends on the initial conditions and the obtained result corresponds to a local optimum.

The algorithm inherits the properties of the standard D-K iteration [Doyle83]. Therefore, the resulting compensator and the bound on uncertainty  $\alpha_{\max}$  will in general correspond to a local optimum. In other words, if the algorithm doesn't obtain a compensator that guarantees robustness properties for some  $\alpha = \alpha_{1,0}$ , this doesn't necessarily mean that such a compensator does not exist.

#### 4.6 Concluding Remarks

In this chapter we have derived an algorithm for increasing the magnitude of the unstructured uncertainty  $\Delta$  while keeping the performance and stability requirements satisfied. The bound on the uncertainty was increased iteratively until its maximum value was reached. The algorithm has two major steps with the first step consisting of an iterative procedure that has three substeps. It was shown that convergence is guaranteed by the construction of the algorithm. The result obtained depends on the initial choice of  $\alpha$  and  $D_F$  and therefore it represents a local optimum.

The algorithm stops either when in two successive iterations the bound on the uncertainty and the values of corresponding " $\mu$ " functions remain unchanged or when the previously set upper bound  $\bar{\alpha}$  is reached. An example will be worked out in Chapter 5.

## CHAPTER 5

### MIMO EXAMPLE FOR CONTROLLER REDESIGN IN THE PRESENCE OF UNSTRUCTURED UNCERTAINTY

#### 5.1 Introduction

In this section we evaluate the control redesign algorithm introduced in Chapter 4. A benchmark for this study is a linearized longitudinal dynamics model of the modified F-8 Aircraft that is augmented with actuator dynamics. We assume that a compensator for the aircraft was designed by the LQG/LTR methodology to guarantee stability and desired performance. This is done without loss of generality since our redesign methodology requires only the stability and the performance of the original system regardless of the design methodology used to achieve it.

Furthermore, we look at the case where one of the two existing actuators fails. The performance and stability of such a "failed" system will be evaluated using the model of the actuator dynamics after failure. By specifying the allowable difference between the original and "failed" systems, we can calculate the bound on the unstructured uncertainty that the original compensator can tolerate. This uncertainty is associated with the newly derived model of the failed actuator.

Next, we will apply the Control Redesign Algorithm to design a compensator that would accommodate uncertainty of a maximal magnitude while keeping the difference between the two systems within desired bounds. We will also perform an analysis as to

how relaxing the requirement on performance influences the increase in the magnitude of uncertainty the redesigned system can tolerate.

## 5.2 Original System - Modified F-8 Aircraft with LQG/LTR Controller

The F-8 is an aircraft used by NASA in their "fly-by-wire" research program [SHH77]. We use its modified equations of motion in the vertical plane, where a flaperon on the wings was included in order to obtain the second control variable in the longitudinal dynamics. The variables that characterize the dynamics of the aircraft are:

$\theta(t)$  = perturbed pitch angle from trim (deg)

$\alpha(t)$  = perturbed angle of attack from trim (deg)

$v(t)$  = perturbation of horizontal velocity (ft/sec)

$q(t)$  = pitch rate (deg/sec)

The other variable that describes dynamics of the aircraft in the vertical plain is the flight path angle  $\gamma$ , defined by

$$\gamma = \theta - \alpha \quad (5.1)$$

The control variables are:

$\delta_e(t)$  = elevator deflection from trim (deg)

$\delta_f(t)$  = flaperon deflection (deg)

The equilibrium flight condition around which the linearized model of the longitudinal dynamics is derived is :

Altitude:	20 000 ft = 6095 m
Speed:	Mach 0.9 = 916.6 ft/sec = 281.58 m/sec
Dynamic Pressure:	550 lbs/ft <sup>2</sup> = 26 429 Pa
Trim Pitch Angle:	2.25 deg.
Trim Angle of Attack:	2.25 deg.
Trim Elevator Angle:	-2.65 deg.



Neglecting the actuator dynamics, let the state vector be defined as

$$\mathbf{x}(t) = [ \theta(t) \quad \gamma(t) \quad q(t) \quad v(t) ]' \quad (5.2)$$

the control vector as

$$\mathbf{u}(t) = [ \delta e(t) \quad \delta f(t) ]' \quad (5.3)$$

and the output vector as

$$\mathbf{y}(t) = [ \theta(t) \quad \gamma(t) ]' \quad (5.4)$$

Then, the state space representation of the linearized model of motion in the vertical plane without the actuator dynamics, is given by the following matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1.5 & -1.5 & 0 & 0.0057 \\ -12 & 12 & -0.8 & -0.0344 \\ -0.8524 & 0.2904 & 0 & -0.014 \end{bmatrix} \quad (5.5)$$

$$\mathbf{B}' = \begin{bmatrix} 0 & 0.16 & -19 & -0.0115 \\ 0 & 0.6 & -2.5 & -0.0087 \end{bmatrix} \quad (5.6)$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (5.7)$$

so that the dynamical equations are:

$$\begin{aligned} \frac{dx}{dt} &= \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C} \mathbf{x}(t) \end{aligned} \quad (5.8)$$

The two actuators corresponding to the elevator and the flaperon are assumed to have the same dynamics described by the following transfer functions:

$$F_e(s) = F_f(s) = 15 [s + 15]^{-1} \quad (5.9)$$

A controller,  $K_p(s)$ , is designed by using the LQG/LTR methodology with matched singular values at both high and low frequencies. By augmenting with integrators in the open loop at the control channels, zero-steady state error in following step commands is guaranteed. The McMillan degree of the compensator is equal to ten, and its dynamics are defined by the position and direction of the associated zeros and poles as follows:

<u>Compensator Poles</u>	<u>Compensator Zeros</u>	(5.10)
-14.515 ± j 39.509	-1.1512 ± j 3.4464	
-23.057 ± j 42.721	-0.0058 ± j 0.0261	
-44.019 ± j 19.697	-15	
-52.508	-15	
-0.0139		
0		
0		

The structure of the original system is presented in Figure 5.1. The singular value plots of the sensitivity, complementary sensitivity and controller transfer functions are shown in Figures 5.2 - 5.4.

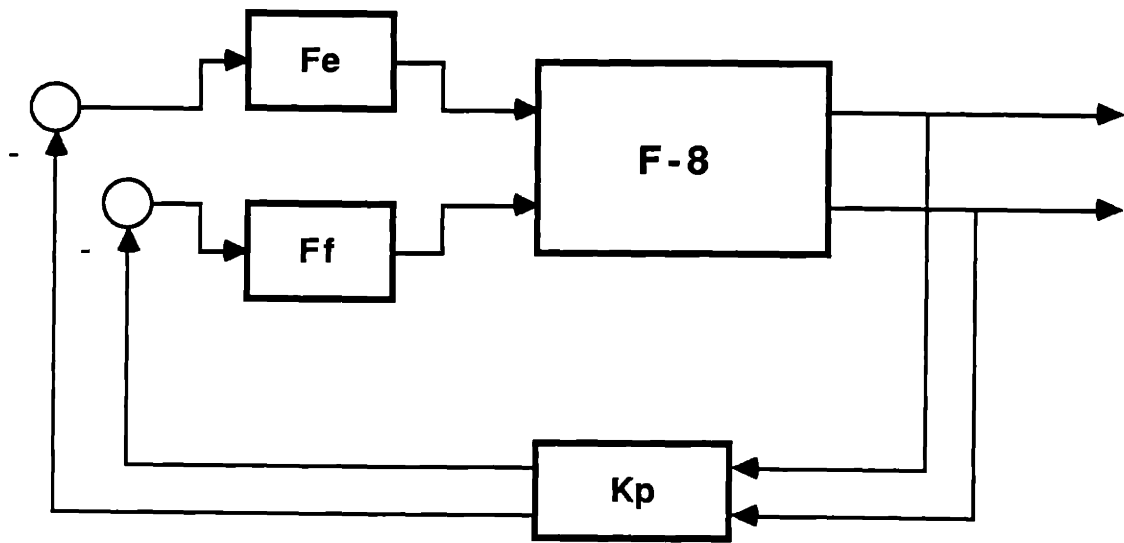


Figure 5.1 Block Diagram Representation of the Original System

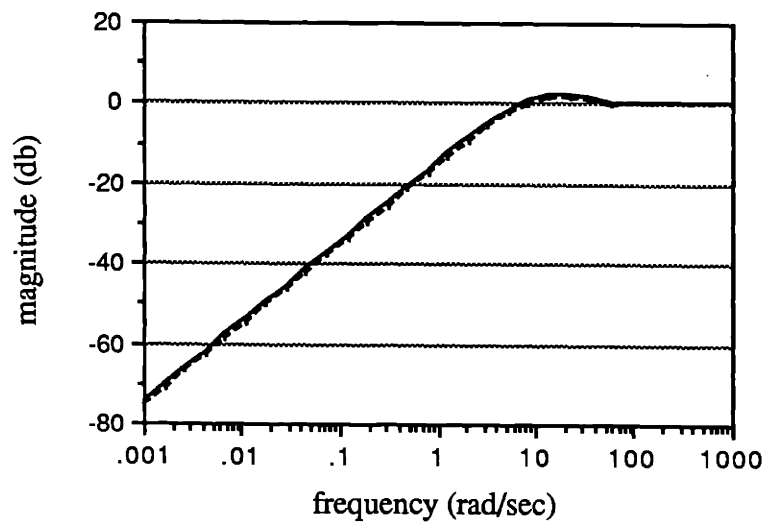


Figure 5.2 Sensitivity Transfer Function of the Original System - S.V. Plot

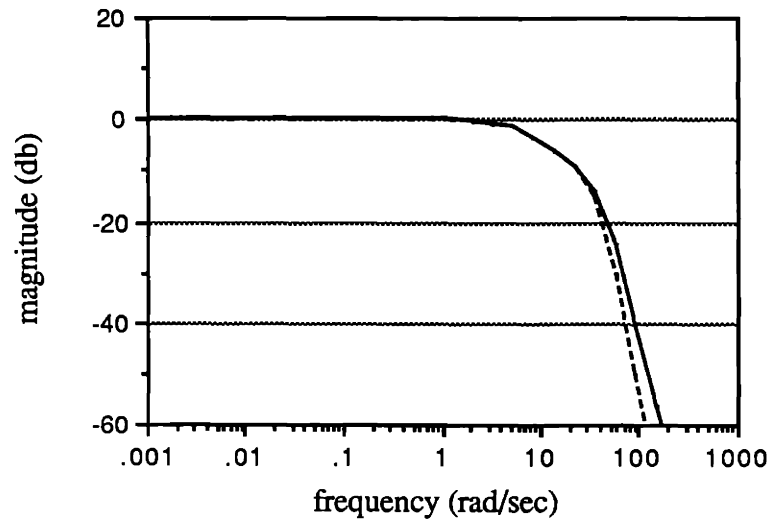


Figure 5.3 Complementary Sensitivity T.F. of the Original System - S.V. Plot

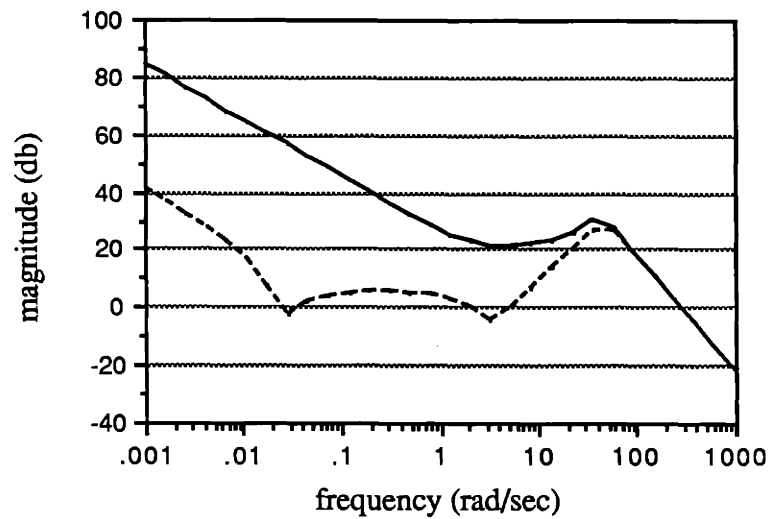


Figure 5.4 Compensator,  $K_p(s)$ , of the Original System - S.V. Plot

### 5.3 System with Failed Actuator and Original Compensator - Analysis

The actuator dynamics were described by the transfer function in (5.9). We assume that the actuator corresponding to the flaperon has failed so that its gain and bandwidth have decreased. From the failure detection and identification processes, we are able to obtain its new model. Let it be given by the nominal transfer function

$$F_{f1} = 3 [s + 12]^{-1} \quad (5.11)$$

and the associated additive uncertainty as will be discussed next. The block diagram of the system after the actuator failure is given in Figure 5.5.

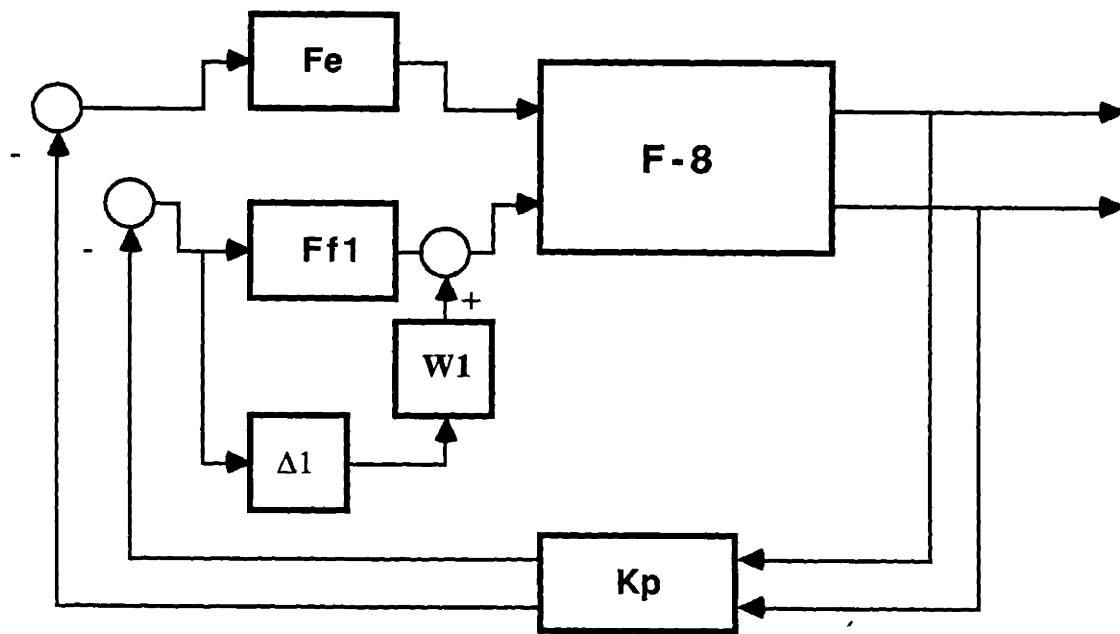


Figure 5.5 Block Diagram of System with Failed Actuator and Original Controller

A possible situation resulting in the actuator failure described by (5.11) may arise as

follows. The actuator is assumed to be a force summing one, meaning that the actual control signal that drives the actuator is a summation of individual signals in redundant control channels. The loss of one or more of these channels will result in the gain reduction in the actuator dynamics. The change in bandwidth can be attributed to the possible oil leakage around the piston of the actuator. The uncertainty in modeling these changes are captured by  $\Delta_1$ .

The uncertainty  $\Delta_1$  is described as unstructured, i.e. norm bounded with  $\|\alpha^{-1}\Delta_1\|_\infty \leq 1$ ,  $\alpha > 0$ . Its frequency dependence is given by the weighting function  $W_1(s)$  that has the following form:

$$W_1(s) = (s+20)^{-1} \quad (5.12)$$

Therefore, it is assumed that the additive uncertainty  $\Delta_1$  presented in Figure 5.5 has the largest impact at low frequencies which is equal to  $\alpha/20$  in magnitude. The overall representation of the failed actuator that includes its nominal model and the multiplicative uncertainty is given in the following figure.

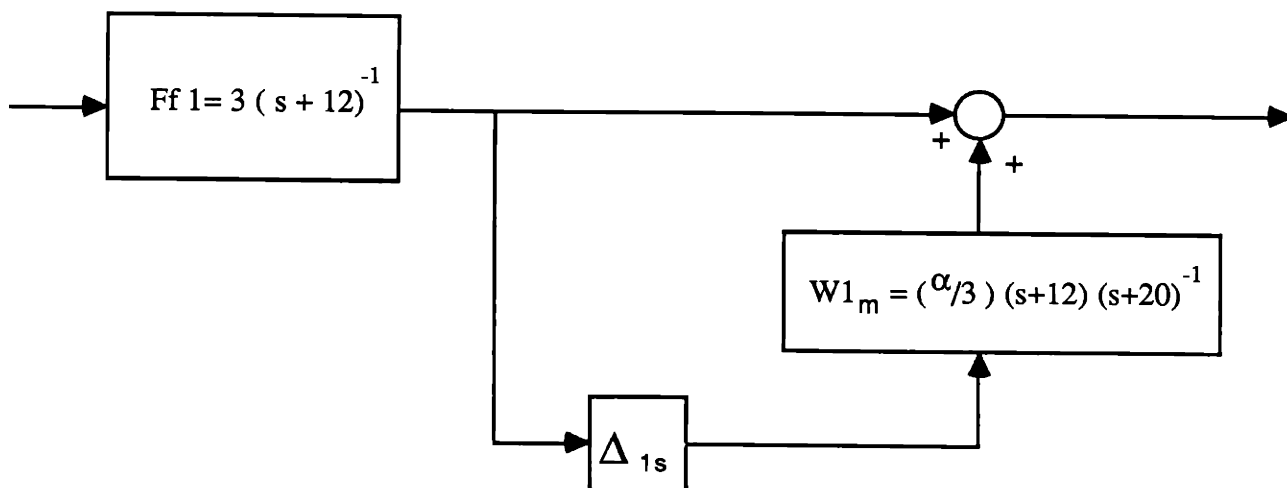


Figure 5.6 Model of the Failed Actuator with the Multiplicative Uncertainty

The scaled uncertainty  $\Delta_{1s}$  has a norm smaller than or equal to one, i.e.  $\|\Delta_{1s}\|_{\infty} \leq 1$ . The weighting function in the multiplicative setting is given as

$$W_{1m}(s) = F_{f1}^{-1} W_1(s) = (\alpha/3) (s+12) (s+20)^{-1} \quad (5.13)$$

where  $\alpha$  is the norm bound on the original uncertainty  $\Delta_1$ .

The magnitude plot of the above defined weighting function is given in Figure 5.7.

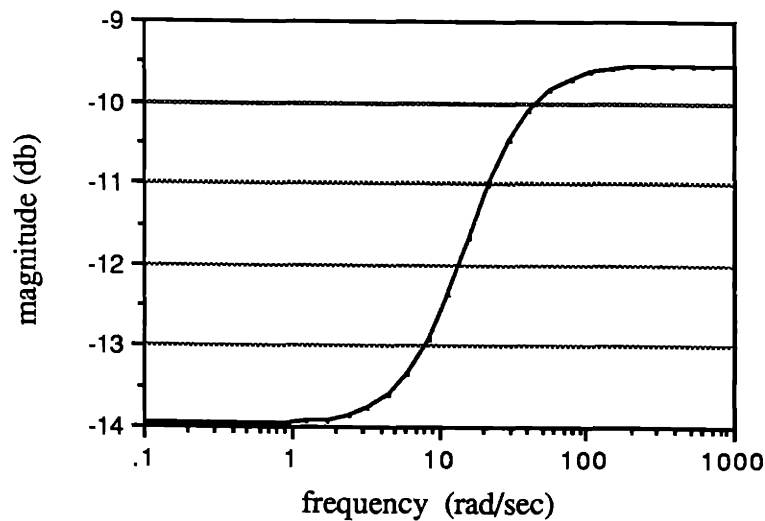


Figure 5.7 Magnitude Plot of the Weighting Function  $(1/\alpha)W_{1m}(s)$

The magnitude of the multiplicative uncertainty at low frequency is equal to  $0.2 \alpha$ . At high frequency it is larger and equals  $0.33 \alpha$ . This representation of uncertainty will be used in the evaluation of the algorithm after the maximum value of  $\alpha$  is obtained.

The controller  $K_p(s)$  was designed to accommodate the original open loop system with unfailed actuators and, therefore, it is reasonable to expect a substantial loss in performance and even instability after the failure has occurred. We now analyze the system with the original controller  $K_p(s)$  after the actuator failure corresponding to the flaperon.

The poles of the closed loop system without uncertainty are :

$$\begin{aligned}
 & -51.45 \\
 & -38.28 \pm j 15.01 \\
 & -16.12 \pm j 36.67 \\
 & -15 \\
 & -11.42.96 \\
 & -8 \\
 & -1.17 \pm j 3.56 \\
 & -1.45 \\
 & -0.0139 \\
 & -0.0058 \pm j 0.027
 \end{aligned}$$

As it is seen, the nominal system with the compensator  $K_p(s)$  and the failed actuator  $F_{f1}(s)$ , remains stable. Nevertheless, the associated sensitivity and complementary sensitivity transfer functions are substantially different, as shown in the following figures.

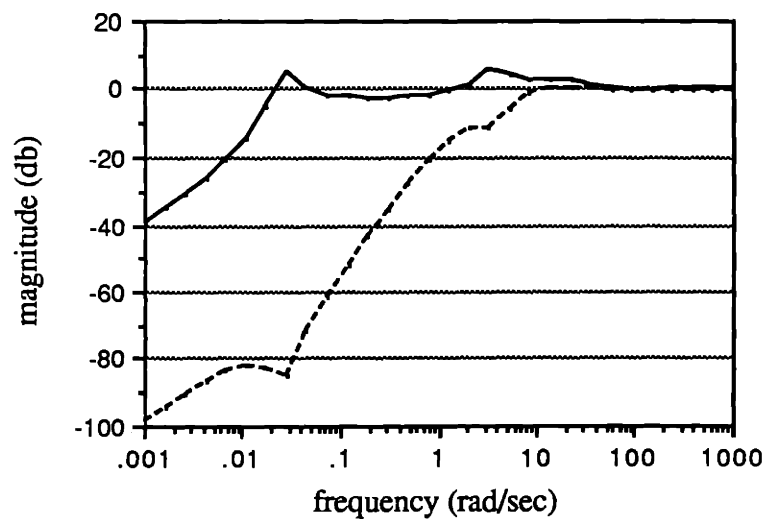


Figure 5.8 Sensitivity T.F. of Nominal System with  $K_p(s)$  and  $F_{f1}(s)$  - S.V. Plot



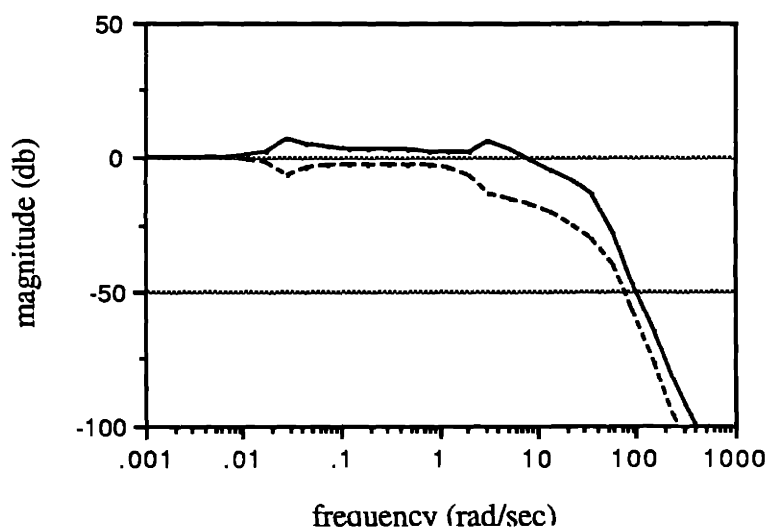


Figure 5.9 Complementary Sensitivity T.F. of Nominal System with  $K_p(s)$  and  $F_{f1}(s)$  - S.V. Plot

Loss in performance is quantified by the value of the performance index  $\gamma$ , and is equal to the norm of the weighted difference between the corresponding sensitivity transfer functions before and after the failure. In the nominal case, it is formulated as

$$\gamma_N = \| W(s) \{ [I + P_o K_p]^{-1} - [I + P_p K_p]^{-1} \} \|_{\infty} = \| W(s) \{ S_o(K_p) - S_p(K_p) \} \|_{\infty} \quad (5.14)$$

where  $P_p(s)$  is the transfer function of the F-8 aircraft with actuators before the failure, i.e. the original plant. Similarly,  $P_o(s)$  describes the postfailure nominal model of the F-8 longitudinal dynamics with actuators included, without any uncertainty present.

The weighting function  $W(s)$  reflects the importance of the change in performance over the frequencies. It is given by the following transfer function:

$$W(s) = (s+5)(s+50) [ (s+10^{-5})(s+100) ]^{-1} \text{diag}\{ 1, 1 \} \quad (5.15)$$

and its singular value plot is shown in Figure 5.10.

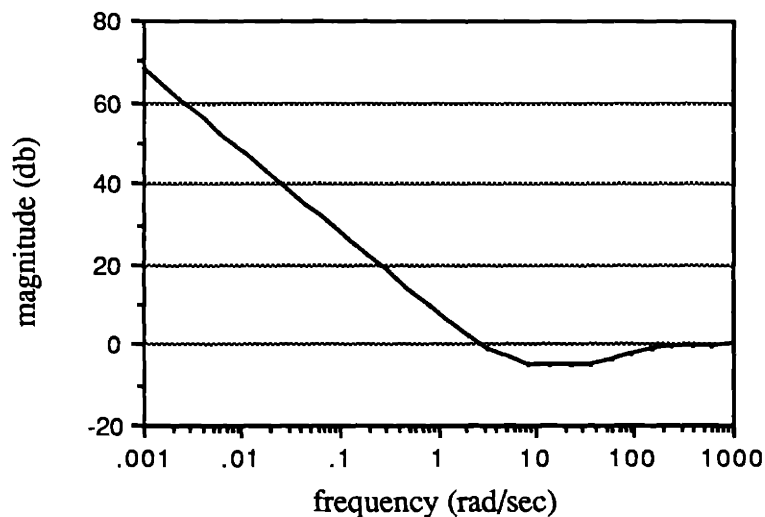


Figure 5.10 Weighting Function  $W(s)$  - S.V. Plot

In this case,  $W(s)$  was chosen to resemble the inverse of the prefailure sensitivity transfer function  $S_p(K_p)$ . The latter was obtained with the compensator  $K_p(s)$  with the augmenting integrators in its output channels. This implies that  $S_p(s)$  has zeros at the origin and, therefore, that its inverse is not in  $RL_\infty$ . To make  $W(s) \in RH_\infty$  approximate the inverse of  $S_p(s)$ , the poles at the origin are shifted to the LHP and hence the term  $(s+10^{-5})^{-1}$  in (5.15) was obtained. Therefore, the expression inside the norm (5.14) approximates the relative error, i.e. :

$$\gamma \sim \| S_p^{-1} \{ S_o(K_p) - S_p(K_p) \} \|_\infty \quad (5.16)$$

The maximum bound on the unstructured uncertainty  $\Delta_1$ , corresponding to Figure 5.5, that would not destabilize the postfailure system with the original compensator  $K_p(s)$  kept unchanged, is found to be equal to 4.66. This is obtained by breaking the loop at the input

and the output of the above mentioned uncertainty and calculating the infinity norm of the corresponding transfer function. Since this value was equal to 0.215, according to the Small Gain Theorem [Zames65], the uncertainty is bounded by the inverse of the latter, i.e.  $(1/0.215) = 4.66$ . On the other hand, the value of the performance index for the nominal case, which was defined by (5.14), is calculated to be  $\gamma_N = 193.95$ . This clearly shows the effect of the actuator failure to the stability robustness properties of the system and to the loss in its performance, in the sense that we have defined it. Therefore, there is a need for changing the compensator in order to minimize the loss in performance and improve the stability robustness of the system after the failure.

#### 5.4 Compensator Redesign

In this section we derive a new compensator  $K(s)$  that accommodates the largest unstructured uncertainty  $\Delta_1$  associated with the failed actuator, while keeping the performance loss within the previously defined bounds. We will also show how the relaxation of the latter influences the resulting stability robustness properties of the system.

The optimization problem we want to solve can be formulated as

$$\begin{aligned} \max_{\alpha \in \mathbb{R}_+} \quad & \text{s.t.} \quad \inf_{K \in K_{s\Delta}} \|W(s) [S(K, \Delta_1) - S_p(K_p)]\|_{\infty} \leq \gamma, \quad \forall \|\Delta_1\|_{\infty} < \alpha \\ & \end{aligned} \quad (5.17)$$

where  $K_{s\Delta}$  defines the family of stabilizing compensators for the plant in the presence of the unstructured uncertainty  $\Delta_1$ . The setting for such a problem is shown in the next figure.

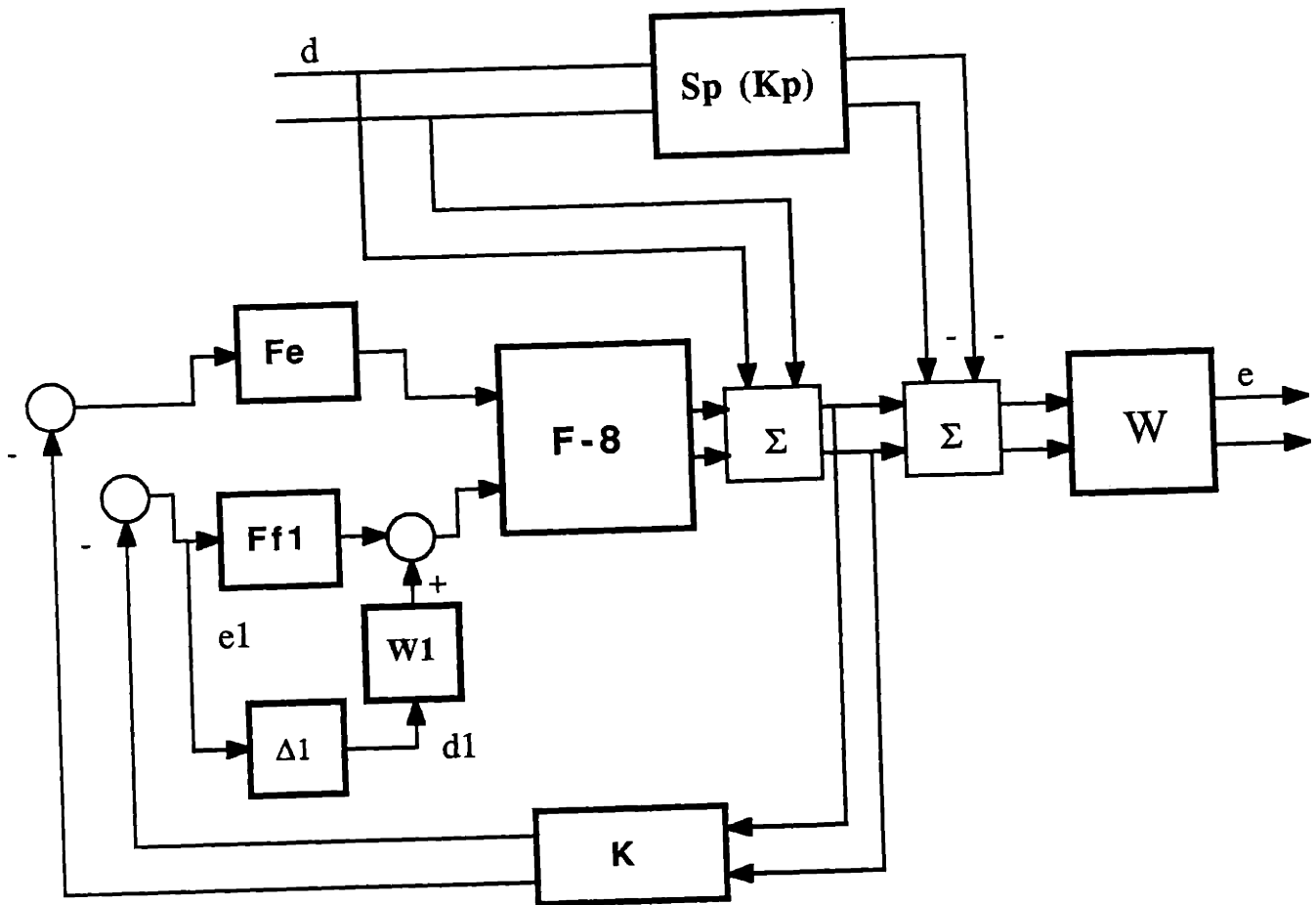


Figure 5.11 Block Diagram Setting for the Compensator Redesign

If the weighting function  $W(s)$  is chosen to approximate the inverse of  $Sp(s)$  as stated in (5.15), the above diagram can be simplified. This is shown in Figure 5.12. The simplification will result in a lower order for the derived compensator if the Doyle-Glover algorithm is used. The latter was introduced in Chapter 2.

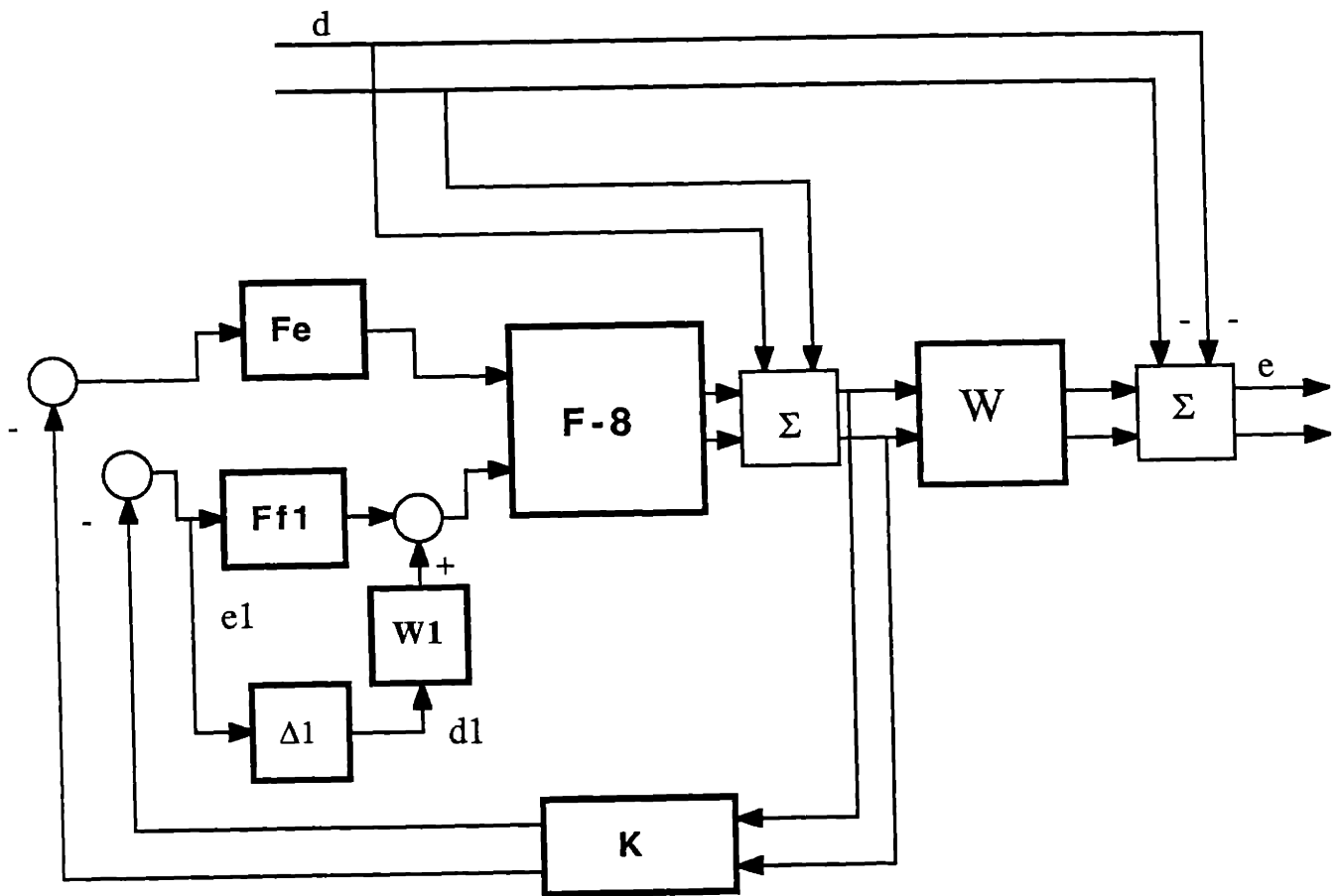


Figure 5.12 Block Diagram Setting for the Compensator Redesign with  $W(s) \sim Sp^{-1}(s)$

There are two extreme situations associated with the value of the performance index  $\gamma$ . The first one arises when the performance loss is of no importance, i.e. when  $\gamma \rightarrow \infty$ . Hence, the problem reduces to finding a compensator that maximizes stability robustness of the postfailure system with respect to the unstructured uncertainty  $\Delta 1$ .

The other situation corresponds to the nominal case where loss in performance is the main issue, while the uncertainty  $\Delta 1$  is assumed to be zero. Consequently, the problem reduces to the minimization of the performance index  $\gamma_N$  which is associated with the nominal postfailure system.

### 5.4.1 Maximization of the Robustness Bound ( $\gamma$ irrelevant )

As we have already stated, an extreme situation occurs when the stability of the postfailure system with respect to the uncertainty  $\Delta_1$  becomes the most important requirement. This usually happens when the identification scheme cannot result in a sufficiently accurate model of the failed system. Then, the prime objective is to maximize  $\alpha$ , i.e. the bound on the norm of the unstructured perturbation  $\Delta_1$ . The formulation of this problem is given as:

$$\min_{K \in K_s} \| H_{e1/d1} ( K ) \|_{\infty} = (\alpha_{\max})^{-1} \quad (5.18)$$

where  $H_{e1/d1}$  is the transfer function from signal  $d_1$  to  $e_1$ , as are defined in Figure 5.11. Since, in our example the postfailure transfer function of the F-8 with actuators remains stable, it is obvious that the open loop situation in this particular control channel will result in an infinite stability margin with respect to  $\Delta_1$  because it makes  $H_{e1/d1}(K) = 0$ . Therefore, a solution to the minimization problem given in (5.18) is any stabilizing compensator  $K(s)$  that satisfies  $\text{diag}\{0,1\}K=0$  including  $K=0$ . The corresponding value of the performance index for the latter is given by:

$$\begin{aligned} \gamma_{\max} &= \| W \{ I - Sp(Kp) \} \|_{\infty} = \\ &= \| W Tp(Kp) \|_{\infty} \sim 10^5 \end{aligned} \quad (5.19)$$

where  $Tp$  stands for the complementary sensitivity transfer function of the the original system. The value  $10^5$  corresponds to the  $W(j\omega=0)$  as was defined in (5.15). If the open loop of the postfailure system hadn't been stable, a compensator different from zero would have been needed to assure stability of the system.

### 5.4.2 Minimization of the Nominal Performance Index $\gamma_N$

The minimal value of the performance index can clearly be achieved in the nominal case where the uncertainty is assumed to be nonexistent. The postfailure nominal transfer function of the aircraft with actuators included,  $P_o(s)$ , can be expressed as follows:

$$P_o(s) = P_p(s) \text{diag}\{ 1, J(s) \} \quad (5.20)$$

where  $J(s) = (3/15) [s + 15] [s + 12]^{-1}$  and  $P_p(s)$  is the corresponding prefailure transfer function. Since  $J(s), J(s)^{-1} \in RH_\infty$  then, according to Lemma 3.3, there exists an internally stabilizing compensator

$$K_o(s) = \text{diag}\{ 1, J(s)^{-1} \} K_p(s) \quad (5.21)$$

that makes the minimal value of  $\gamma_N$  equal to zero. The signal in the second control channel is clearly going to be affected by the magnitude of  $J(s)$  over frequency. When the compensators are obtained for a sufficiently small value of  $\gamma$  in following subsections, we will check if the pattern matches that one expected from equation (5.21).

### 5.4.3 Evaluation of the Minimizing Controller by the Doyle-Glover Method

The first step in the inner loop "j" of the controller redesign algorithm introduced in Chapter 4, is evaluation of the stabilizing controller  $K(s)$  that minimizes the infinity norm of the transfer function from the combined input  $\underline{d} = [d, d1]'$  to the output  $\underline{e} = [e, e1]'$ . These signals are defined in Figure 5.11. The signal  $e1$  is at the same time the weighted control signal  $u_2 = \delta f$ . Since, as it was shown in Chapter 2, the Doyle-Glover algorithm [DGKF88]

requires both control signals to be present in the output, the other control signal has to be included too. Therefore, an augmented output signal  $\underline{e}_a = [e, e_1, e_2]$  is formed, where  $e_2$  represents the weighted control signal  $u_1 = \delta u$ , as shown in Figure 5.13. This requirement is necessary to ensure the properness of the obtained controller.

Hence, the controller that we get from the Doyle-Glover algorithm is minimizing for the infinity norm of the transfer function  $\underline{H}_{e_a/d}$  but not for the  $\underline{H}_{e/d}$  which is the one we are really interested in. Then, there is always going to be an error associated with this scheme that has to be taken into account. It is easy to see that there exists the following relationship between the norms of the two mentioned transfer functions as follows :

$$\| \underline{H}_{e_a/d}(K) \|_{\infty} - \| \underline{H}_{e/d}(K) \|_{\infty} \geq 0 \quad (5.22)$$

where  $K$  is the minimizing controller for the  $\| \underline{H}_{e_a/d}(K) \|_{\infty}$ .

One way to get around this problem is to assign a constant weight,  $\beta$ , on the augmented output signal that corresponds to the control channel  $\delta u$ . As  $\beta$  approaches zero, the compensator  $K$  will eventually approach the minimizing compensator of  $\| \underline{H}_{e_a/d}(K) \|_{\infty}$ . We will choose a sufficiently small value of  $\beta$  so that the difference defined in (5.22) is within desired bounds.

The block diagram representation of the open loop system arranged for the infinity norm minimization is given in Figure 5.13. All the signals are normalized such that the associated uncertainties are of norm smaller than one. The block diagonal scaling, defined in Chapter 2, that will be used in evaluation of the associated " $\mu$ " function will have the following form :

$$D = \text{diag} \{ I_2, \epsilon \} \quad (5.23)$$

The scaling  $\epsilon$  on the output signal  $e_1$  will initially be chosen as a constant, but it will be updated in every step of the "i" iteration, as described in Chapter 4. The weighting function



$W(s)$  is scaled by the inverse of the desired performance index  $\gamma$ . The stability margin  $\alpha$  is scaled in successive steps of the "j" iteration.

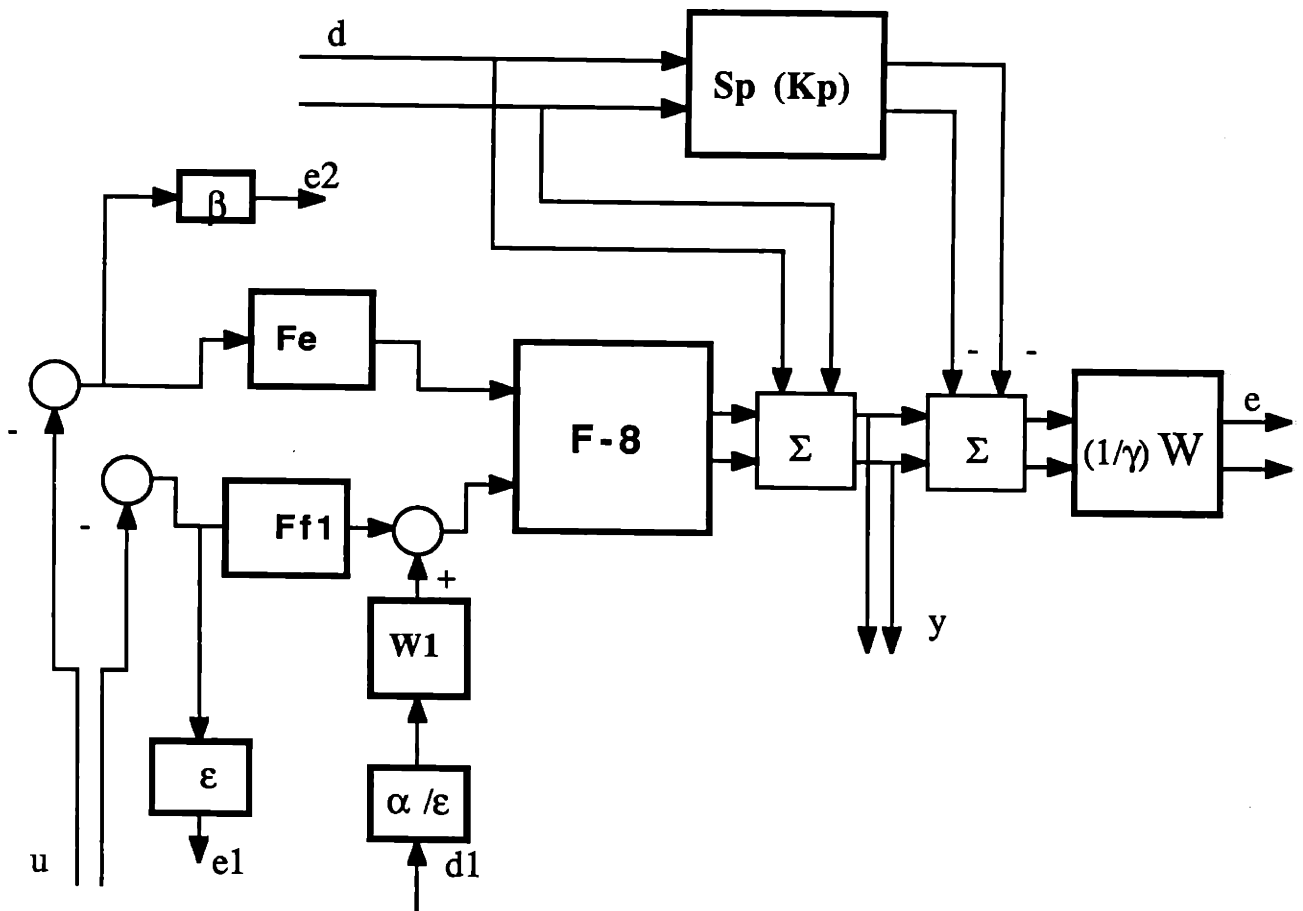


Figure 5.13 Setting for the Infinity Norm Minimization

#### 5.4.4. Compensator Redesign with fixed $\gamma = 5$

We now design a compensator that maximizes the stability margin  $\alpha$ , with the fixed value of the performance index  $\gamma = 5$ . We will show how the compensator  $K_{i,j}$  and  $\alpha_{i,j}$  change through the "i" and "j" iterations that were described in Chapter 4. The diagonal scaling  $D_{F_i}$  remains the same throughout the entire execution of step A. Therefore, it is indexed according to the current value of "i".

Let the initial conditions for  $i = 1$  and  $j = 0$  be:

$$D_{F_1} = I, \quad I \in \mathbb{R}^{3 \times 3} ; \quad \alpha_{1,0} = 0.2 \quad (5.24)$$

where  $\alpha_{1,0}$  was chosen sufficiently small according to the discussion in Chapter 4, section 4.5. The compensator  $K_{1,1}$  is designed to satisfy the following:

$$\min_{K \in K_s} \| D_{F_1} M(\alpha_{1,0}; K) D_{F_1}^{-1} \|_{\infty} = \| D_{F_1} M(\alpha_{1,0}; K_{1,1}) D_{F_1}^{-1} \|_{\infty} \quad (5.25)$$

Its singular value plot is shown in Figure 5.14. The singular value plots of the sensitivity transfer function and the overall postfailure nominal closed loop system transfer function are given in Figure 5.15 and 5.16.

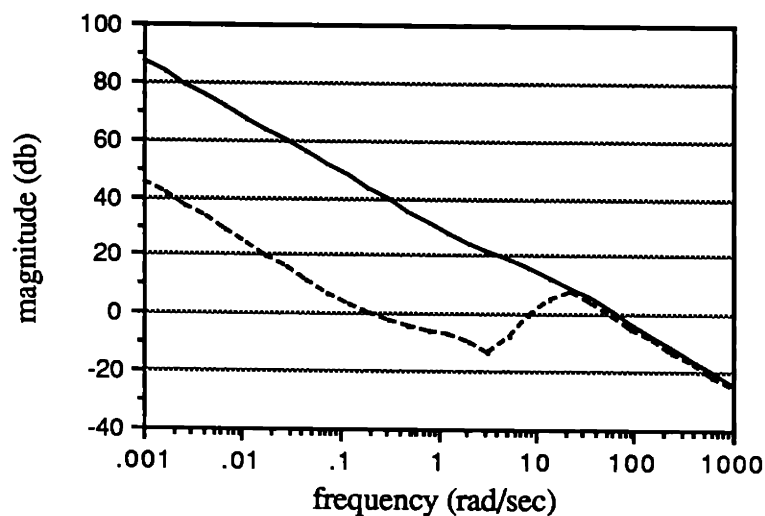


Figure 5.14 Compensator  $K_{1,1}$  - S.V. Plot

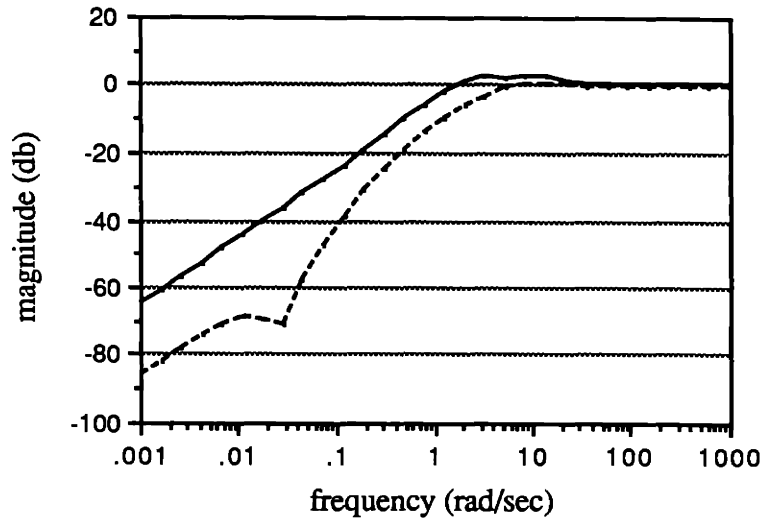


Figure 5.15 Sensitivity T.F. of the Nominal System with  $K_{1,1}$  - S.V. Plot

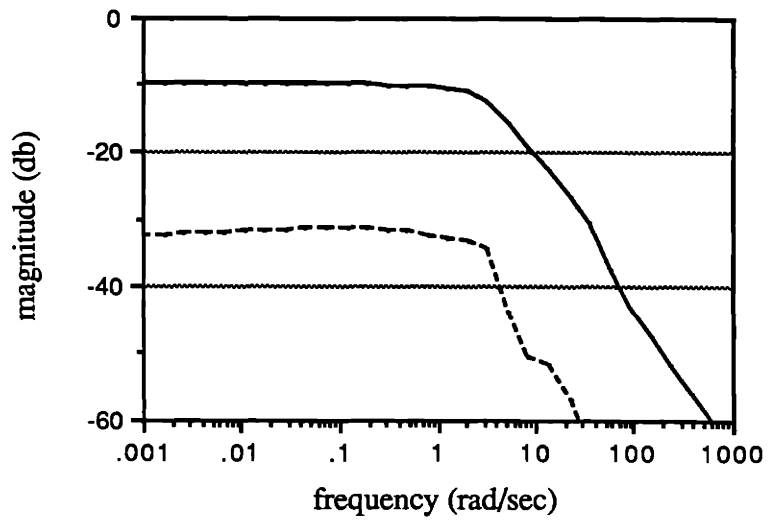


Figure 5.16 Overall T.F. from "d" to "e" with  $K_{1,1}$  - S.V. Plot

The corresponding " $\mu$ " function attains its extreme value at 0.32 over all frequency, i.e.

$$\mu_s \{M(j\omega; \alpha_{1,0}; K_{1,1})\} = 0.32 \quad (5.26)$$

which is shown in Figure 5.17.

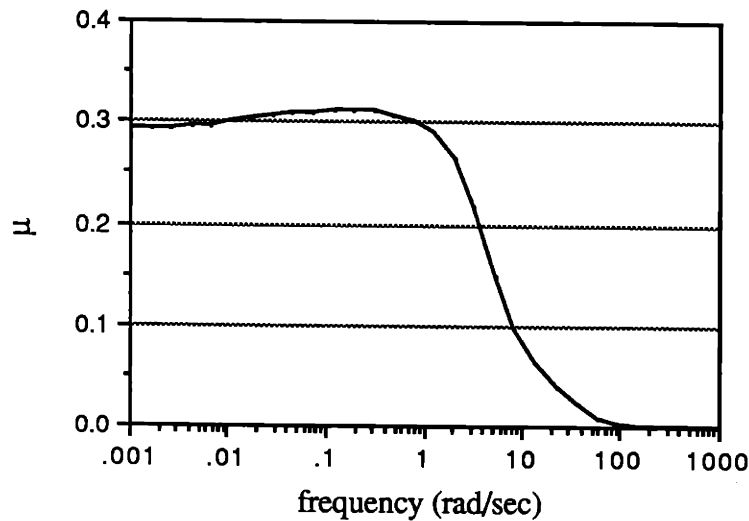


Figure 5.17 " $\mu \{M(j\omega; \alpha_{1,0}; K_{1,1})\}$ " - Frequency Dependence

Since the value of " $\mu$ " is less than one, we have the freedom to increase  $\alpha$ . By the bisection method described in Chapter 4, it is found that

$$\mu_s \{M(j\omega; [12.5 \alpha_{1,0}]; K_{1,1})\} = 1 \quad (5.27)$$

implying that  $\alpha_{1,1} = 12.5 \alpha_{1,0} = 2.5$ . The following step consists of designing a new minimizing compensator  $K_{1,2}$  for  $\alpha_{1,1}$  since the compensator  $K_{1,1}$  is not optimal any more.

Therefore, we have:

$$\min_{K \in K_s} \|D_{F_1} M(j\omega; \alpha_{1,1}; K) D_{F_1}^{-1}\|_{\infty} = \|D_{F_1} M(j\omega; \alpha_{1,1}; K_{1,2}) D_{F_1}^{-1}\|_{\infty} \quad (5.28)$$

The singular value plots of  $K_{1,2}(s)$  and the corresponding closed loop transfer functions together with " $\mu$ " over frequency are given in the following figures.

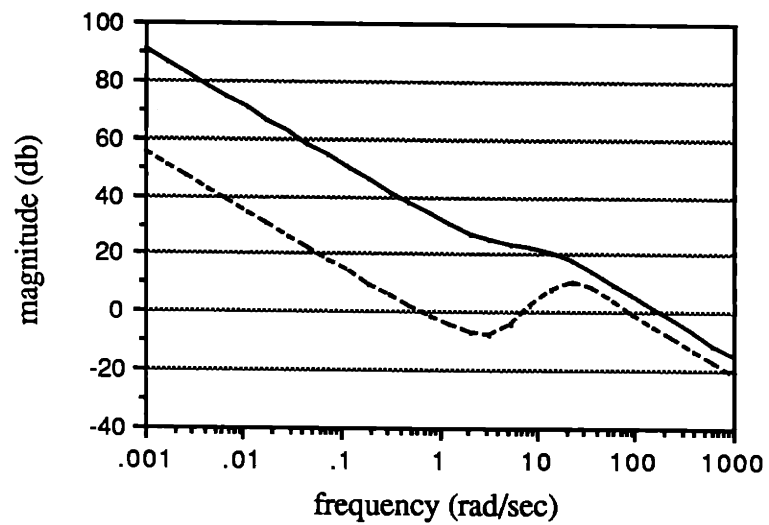


Figure 5.18 Compensator  $K_{1,2}$  - S.V. Plot

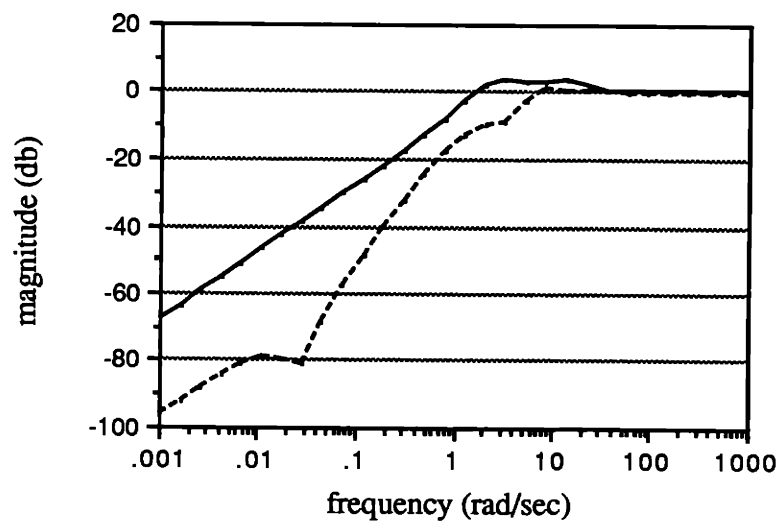


Figure 5.19 Sensitivity T.F. of the Nominal System with  $K_{1,2}$  - S.V. Plot

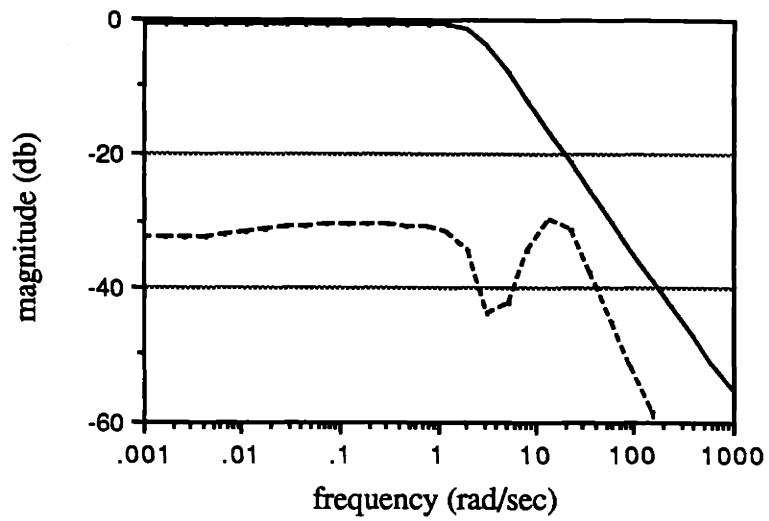


Figure 5.20 Overall T.F. from "d" to "e" with  $K_{1,2}$ - S.V. Plot

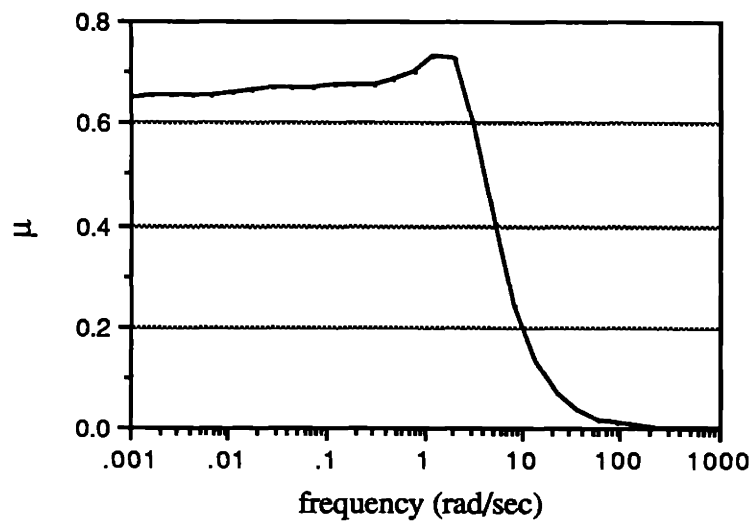


Figure 5.21 " $\mu \{ M(j\omega ; \alpha_{1,1} ; K_{1,2}) \}$ " - Frequency Dependence

From the last figure we can see that the supremum of  $\mu \{ M(j\omega ; \alpha_{1,1} ; K_{1,2}) \}$  over frequency is equal to 0.7, which gives us the opportunity to further increase the value of  $\alpha$ . With  $\alpha_{1,2} = 1.52 \alpha_{1,1} = 3.78$  we have

$$\mu_s \{ M(j\omega ; \alpha_{1,2} ; K_{1,2}) \} = 1 \quad (5.29)$$

The optimizing compensator for  $\alpha_{1,2}$  is  $K_{1,3}$ , and its corresponding plots are given in figures 5.22 to 5.25.

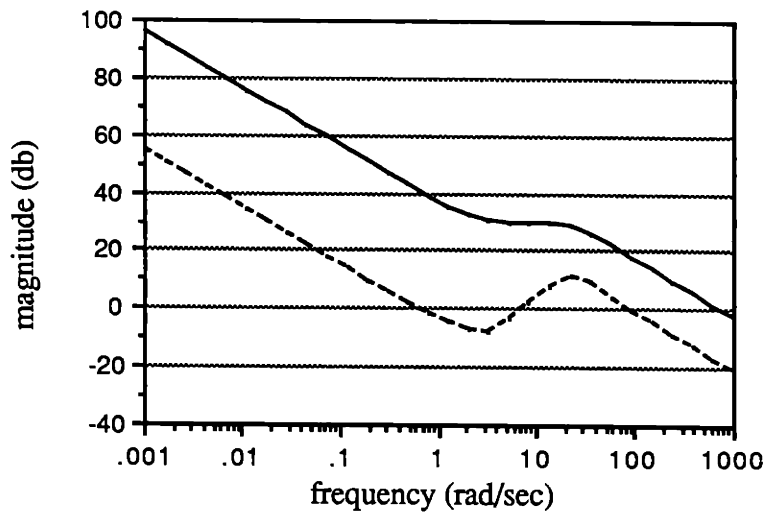


Figure 5.22 Compensator  $K_{1,3}$  - S.V. Plot

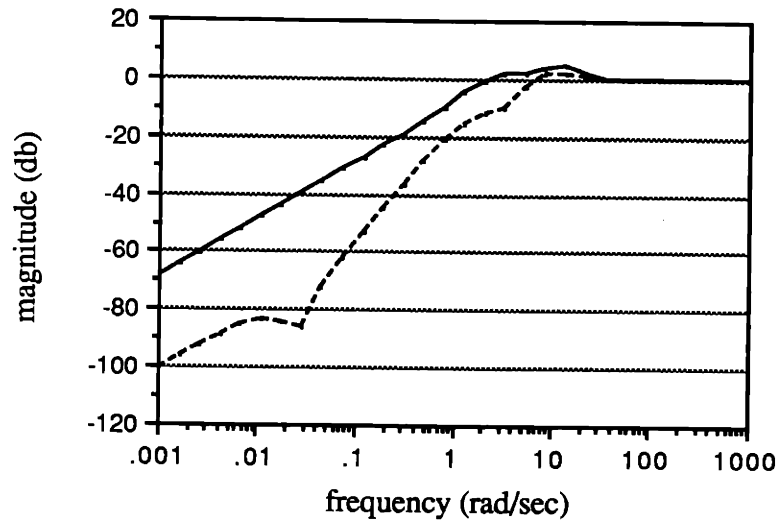


Figure 5.23 Sensitivity T.F. of the Nominal System with  $K_{1,3}$  - S.V. Plot

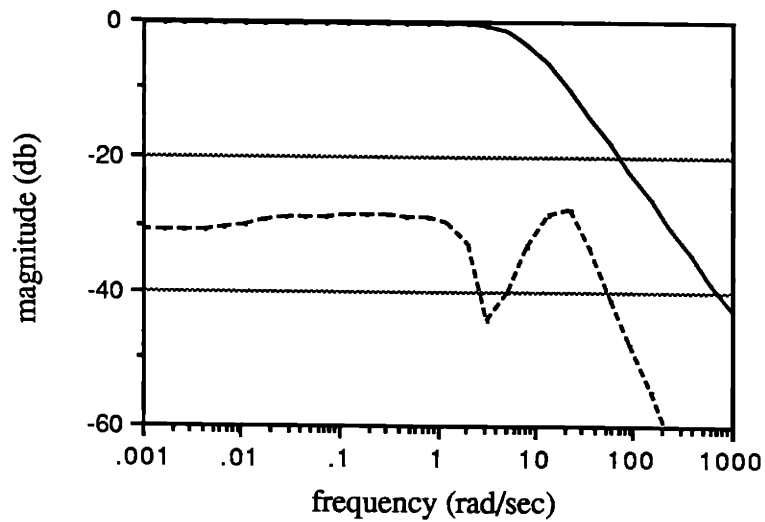


Figure 5.24 Overall T.F. from "d" to "e" with  $K_{1,3}$  - S.V. Plot



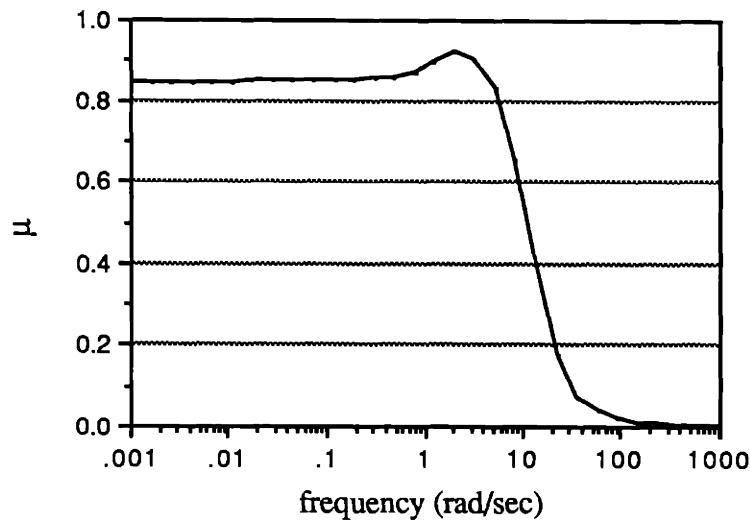


Figure 5.25 " $\mu \{M(j\omega ; \alpha_{1,2} ; K_{1,3})\}$ " - Frequency Dependence

The supremum of the corresponding " $\mu$ " function,  $\mu_s\{M(j\omega ; \alpha_{1,2} ; K_{1,3})\}$ , is equal to 0.9. The value of  $\alpha$  that scales it to one is found to be  $\alpha_{1,3} = 1.1 \alpha_{1,2} = 4.2$ . The following compensator,  $K_{1,4}$ , has the supremum of its corresponding function,  $\mu_s\{M(j\omega ; \alpha_{1,3} ; K_{1,4})\}$ , equal to one. Since " $\mu$ " is a nondecreasing function of  $\alpha$ , we still have to check if  $\alpha$  can be increased further from  $\alpha_{1,3}$  without increasing the value of " $\mu$ " over one. In this particular case, it turns out that there is no change in  $\alpha$  that won't increase the value of " $\mu$ ". Therefore, there are two successive "j" iterations where the value of  $\alpha$  remains unchanged, i.e.  $\alpha_{1,4} = \alpha_{1,3} = 4.2$ . This implies that we have done the best possible while keeping the block diagonal weights fixed to  $D_{F_1}$ . The plots associated with the compensator  $K_{1,4}$  are shown in figures 5.26 to 5.29.

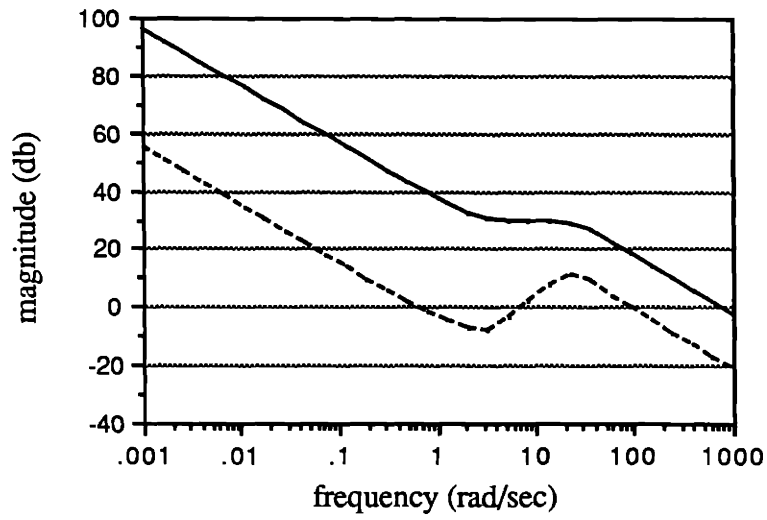


Figure 5.26 Compensator  $K_{1,4}$  - S.V. Plot

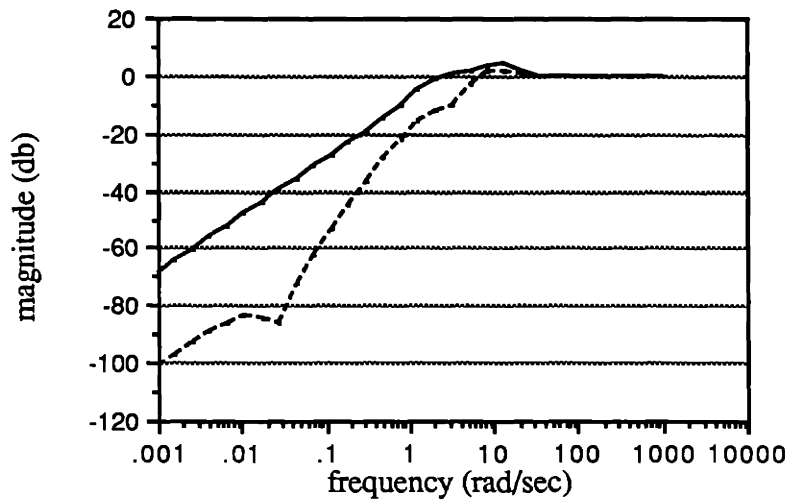


Figure 5.27 Sensitivity T.F. of the Nominal System with  $K_{1,4}$  - S.V. Plot

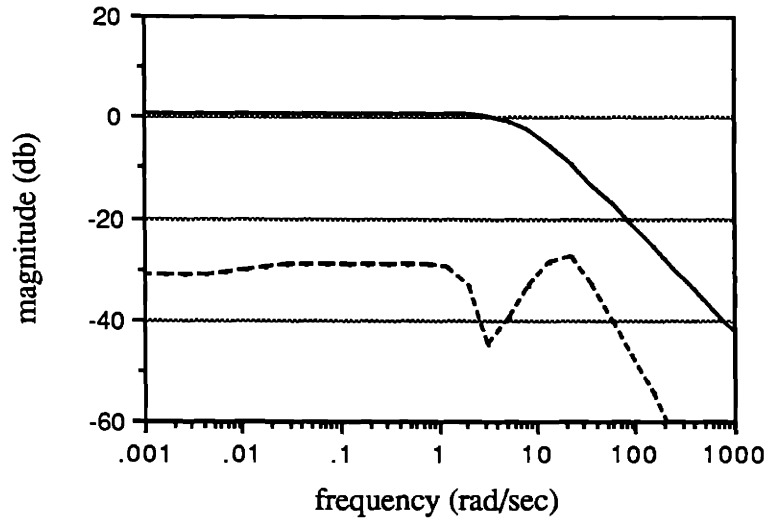


Figure 5.28 Overall T.F. from "d" to "e" with  $K_{1,4}$  - S.V. Plot

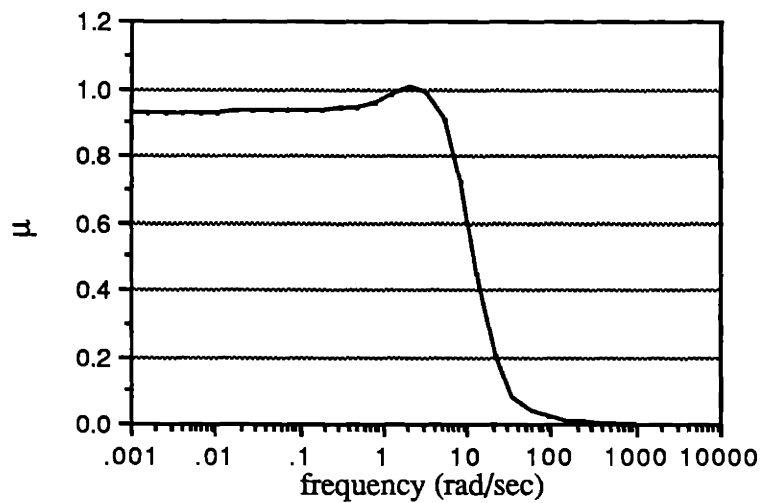


Figure 5.29 " $\mu \{M(j\omega; \alpha_{1,3}; K_{1,4})\}$ " - Frequency Dependence

As we have seen, the "j" iteration for  $i = 1$  has consisted of four steps. It led to the maximum value of  $\alpha = \alpha_{1,4}$  and the minimizing compensator  $K_{1,4}$  that achieves it, while keeping the block diagonal scaling  $D_{F1}$  fixed. The compensator  $K_{1,4}$  is saved as  $K_f(s)$  according to the algorithm outline presented in Chapter 4. In order to do more, we approximate the block diagonal scaling  $D$  that was used for evaluating  $\mu\{M(j\omega; \alpha_{1,4}; K_{1,4})\}$  over frequency, with the invertible outer transfer function  $Dr_1(s)$ . Both of them are shown in figures 5.30 and 5.31.

The new scaling  $D_F$  in the "i = 2" iteration is obtained as  $D_{F2} = Dr_1$ . Therefore, step B) can be looked at as the "D" step in the "D-K" iteration for fixed  $\alpha = \alpha_{1,4}$ . The block diagonal scaling  $Dr_1$  is given as follows:

$$Dr_1 = \text{diag} \{ 1, 1, 0.0119 [(s+1)(s+10^3)] [(s+0.5)(s+10)]^{-1} \} \quad (5.30)$$

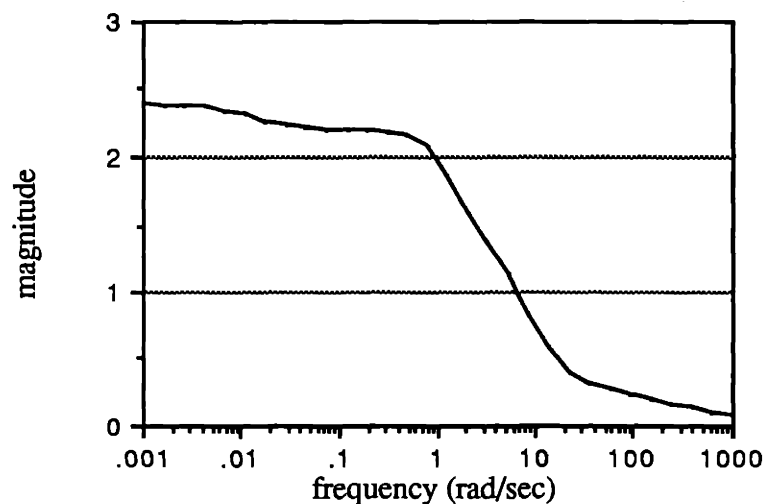


Figure 5.30 "D" Scaling in Evaluation of  $\mu\{M(j\omega; \alpha_{1,4}; K_{1,4})\}$  - Block 2

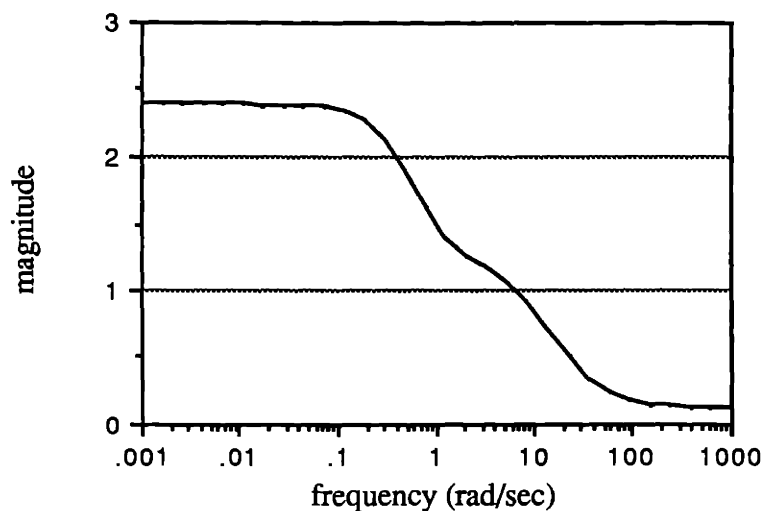


Figure 5.31  $Dr_1$  as the approximation of "D" scaling for  $\alpha_{1,4}$

After the new scaling  $D_{F2}$  is obtained, the "j" iteration is started again with the initial value  $\alpha_{2,0} = \alpha_{1,4} = 4.2$  where the index "i" is kept equal to two. The first compensator obtained in this "i" iteration is  $K_{2,1}$ . The maximum stability margin associated with it,  $\alpha_{2,1}$ , is equal to 4.316. Such a small increase is due to the fact that the " $\mu_s$ " function associated with  $\alpha_{2,0}$  and  $K_{2,1}$  has the value 0.9775, i.e. it is already very close to one. The corresponding plots are given in figures 5.32 - 5.35.

The following compensator  $K_{2,2}$ , that minimizes the transfer function scaled by  $\alpha_{2,1}$ , has the maximum value of the associated " $\mu$ " function equal to one. Therefore, we don't have any more freedom to increase the stability margin without changing the scaling  $D_{F2}$ . The plots associated with  $K_{2,2}$  are given in figures 5.36 - 5.39.

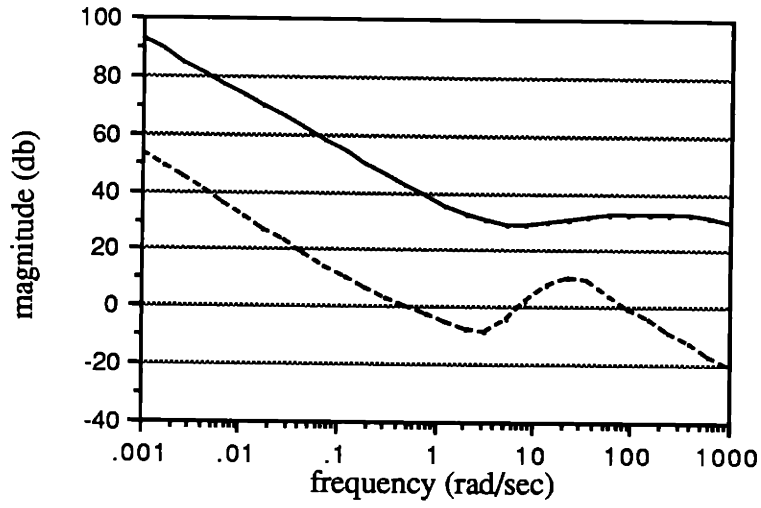


Figure 5.32 Compensator  $K_{2,1}$  - S.V. Plot

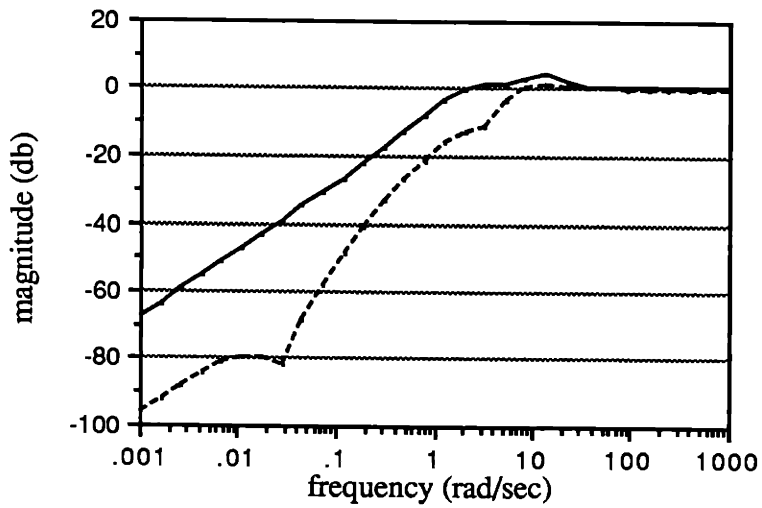


Figure 5.33 Sensitivity T.F. of the Nominal System with  $K_{2,1}$  - S.V. Plot

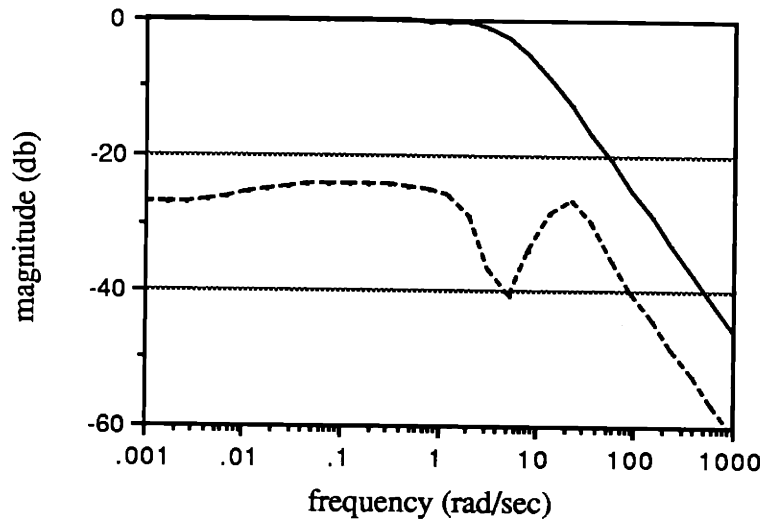


Figure 5.34 Overall T.F. from "d" to "e" with  $K_{2,1}$  - S.V. Plot

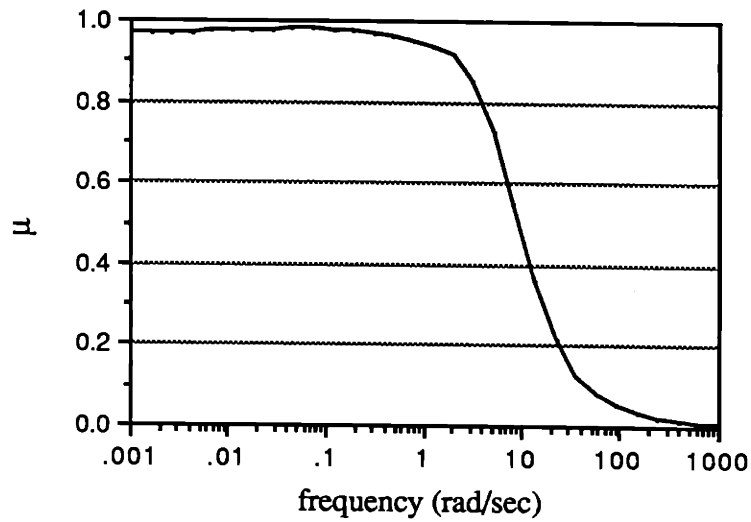


Figure 5.35 " $\mu \{M(j\omega; \alpha_{2,0}; K_{2,1})\}$ " - Frequency Dependence

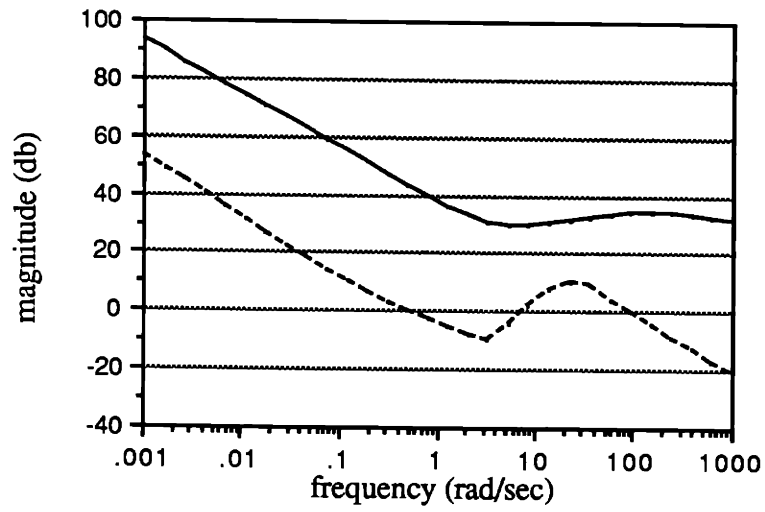


Figure 5.36 Compensator  $K_{2,2}$  - S.V. Plot

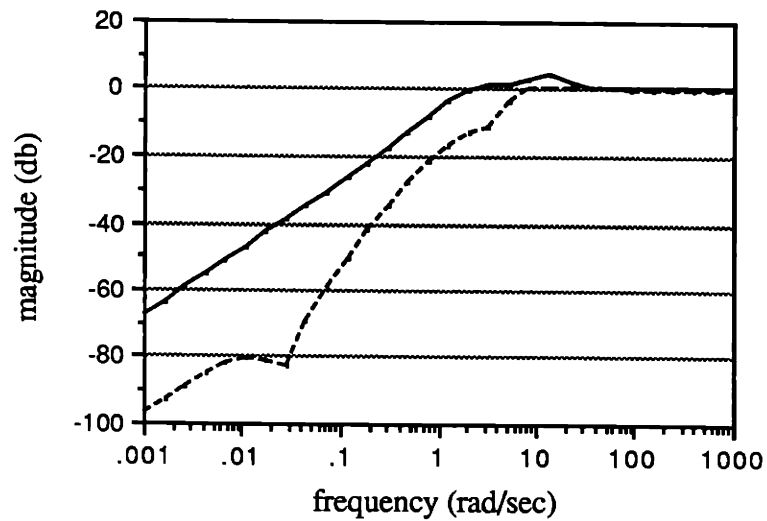
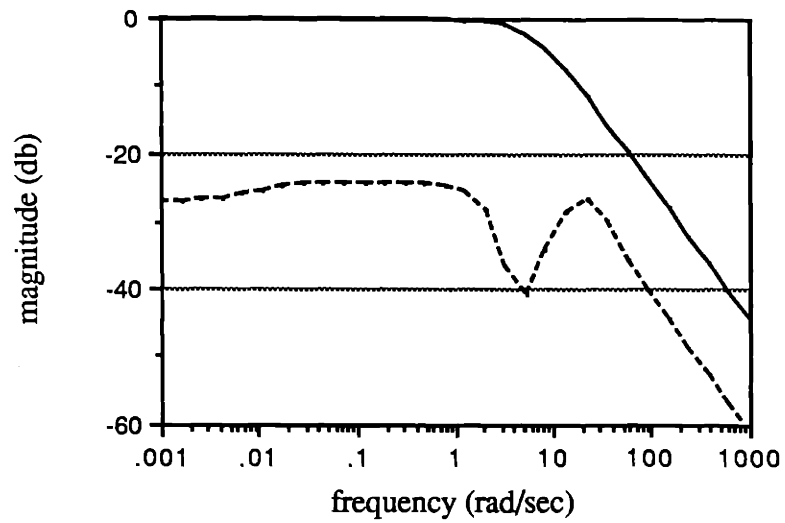
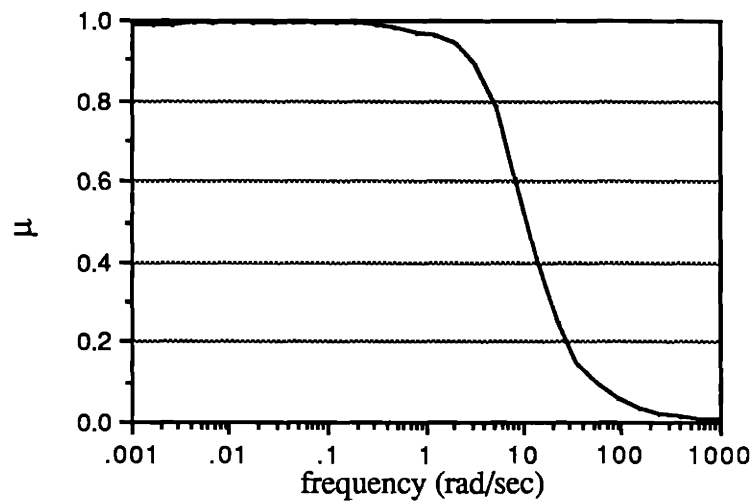


Figure 5.37 Sensitivity T.F. of the Nominal System with  $K_{2,2}$  - S.V. Plot



Figure 5.38 Overall T.F. from "d" to "e" with  $K_{2,2}$  - S.V. PlotFigure 5.39 " $\mu \{M(j\omega ; \alpha_{2,1} ; K_{2,2})\}$ " - Frequency Dependence

The block diagonal scaling  $D$  used to evaluate  $\mu \{M(j\omega; \alpha_{2,1}; K_{2,2})\}$  over frequency is given in figure 5.40.

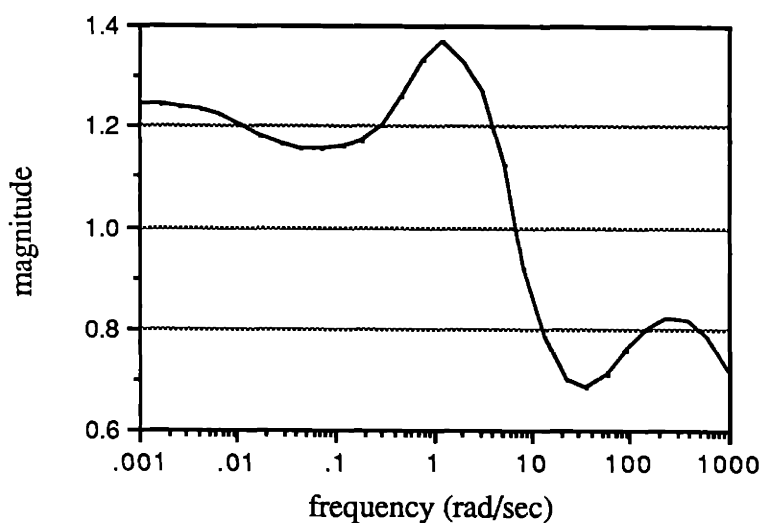


Figure 5.40 "D" Scaling in Evaluation of  $\mu \{M(j\omega; \alpha_{2,1}; K_{2,2})\}$  - Block 2

The value of the second block element fluctuates between 1.25 and 0.75. The most reasonable approximation that we could come up with is equal to a constant of value one. Hence, we cannot improve further the stability margin by changing the block diagonal scaling  $D_{F_2}$ . Therefore, the maximum value of the stability margin for the performance index  $\gamma = 5$  is  $\alpha_{\max} = \alpha_{2,1} = 4.316$  and it is achieved by the minimizing controller  $K_f = K_{2,2}$ . It is important to realize that  $\alpha_{2,1}$  represents only a local extremum, and that it is dependent on the choice of the initial values, accuracy of the approximation of the block diagonal scaling, etc.

### 5.4.5 Compensator Redesign with Fixed $\gamma = 1$

We look at the case when the performance index is set to be equal to one. This is a much tighter condition than in the previous case, but it is still sufficiently slack because the relative error is allowed to be as high as 100%. The initial conditions for the "i = 1" iteration are the same as in the previous case:  $\alpha_{1,0} = 0.2$  and  $D_{F1} = I$ . The first minimizing controller is  $K_{1,1}$  and the corresponding value of the stability margin is  $\alpha_{1,1} = 1.1$ .

The following controller  $K_{1,2}$  guarantees stability and performance for  $\alpha_{1,2} = 1.866$ . This value remains the same in the next "j" iteration making  $\alpha_{1,3} = \alpha_{1,2} = 1.866$  with the minimizing controller  $K_{1,3}$ . In order to improve further, the block diagonal weighting  $D_{F1}$  is replaced with the real rational approximation  $D_{r1}$  of the scaling  $D$  used in evaluation of  $\mu \{M(j\omega; \alpha_{1,2}; K_{1,3})\}$ .

The new "i = 2" loop is started with  $\alpha_{2,0} = \alpha_{1,3} = 1.866$  and  $D_{F2}$  obtained as described above. The first "j" iteration results in compensator  $K_{2,1}$  and  $\alpha_{2,1} = 2.52$ . This value of the stability margin remains the same in the second iteration with the minimizing compensator  $K_{2,2}$ . The best approximation we could obtain over frequency of the block diagonal scaling  $D$  used in evaluating  $\mu \{M(j\omega; \alpha_{2,1}; K_{2,2})\}$  is a constant matrix whose second block has its value very close to one. Therefore, the local maximum of the stability margin for  $\gamma = 1$  is  $\alpha_{\max} = \alpha_{2,1} = 0.126$  and  $K_f = K_{2,2}$ . The associated plots are shown in figures 5.41 - 5.42.

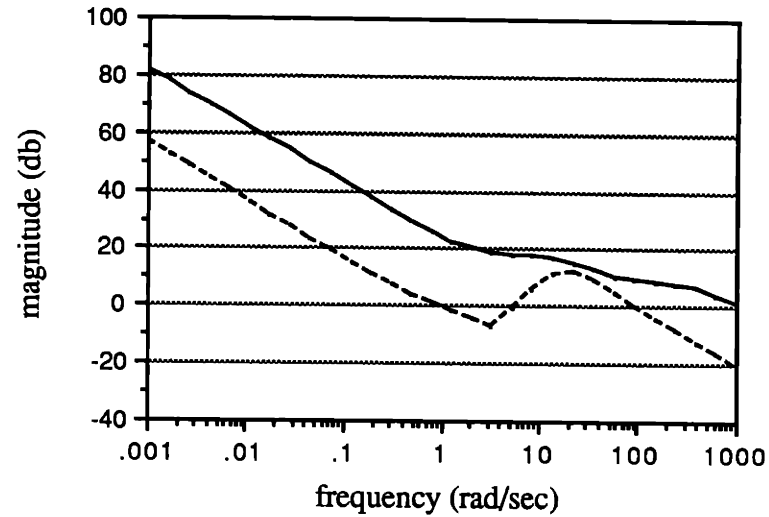


Figure 5.41 Compensator  $K_{2,2}$  - S.V. Plot

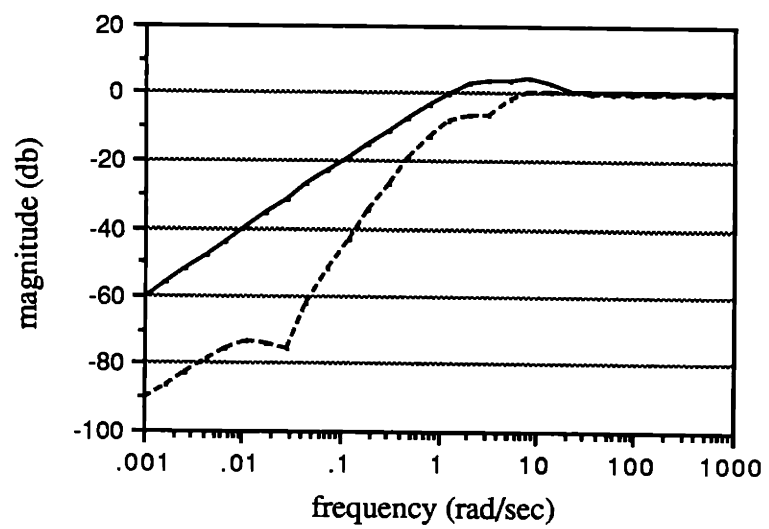


Figure 5.42 Sensitivity T.F. of the Nominal System with  $K_{2,2}$  - S.V. Plot

### 5.4.6 Compensator Redesign with fixed $\gamma = 0.5$

In this case, the maximum value of the performance index  $\gamma$  is set to 0.5 which implies that the maximum allowable relative error cannot exceed 50%. The initial conditions are the same as in the previous two cases:

$$\alpha_{1,0} = 0.2 \quad \text{and} \quad D_{F_1} = I \quad (5.31)$$

The first minimizing compensator  $K_{1,1}$  guarantees the stability margin  $\alpha_{1,1} = 0.22$ . The following compensator,  $K_{1,2}$ , guarantees stability and performance for the same value of  $\alpha$ , i.e.  $\alpha_{1,2} = \alpha_{1,1} = 0.22$ . The best approximation over frequencies of the block diagonal scaling  $D$  used in evaluation of  $\mu \{M(j\omega; \alpha_{1,2}; K_{1,2})\}$  that we could come up with is

$$D_{r_1} = \text{diag} \{ 1, 1, 0.0336 [(s+1)(s+250)] [(s+0.5)(s+5)]^{-1} \} \quad (5.32)$$

The minimizing compensator with  $D_{F_2} = D_{r_1}$  and  $\alpha_{2,0} = \alpha_{1,2} = 0.22$  is  $K_{2,1}$ . The maximum value of the associated " $\mu$ " function over frequency is equal to one and the second block in the associated block scaling  $D$  can be approximated as a constant of the same value. Therefore, in the case when  $\gamma = 0.5$ , we cannot improve more than  $\alpha_{1,2} = \alpha_{1,1} = 0.22$  which was firstly achieved with the minimizing compensator  $K_f = K_{1,2}$ . The corresponding plots for this compensator are given in figures 5.43 and 5.44.

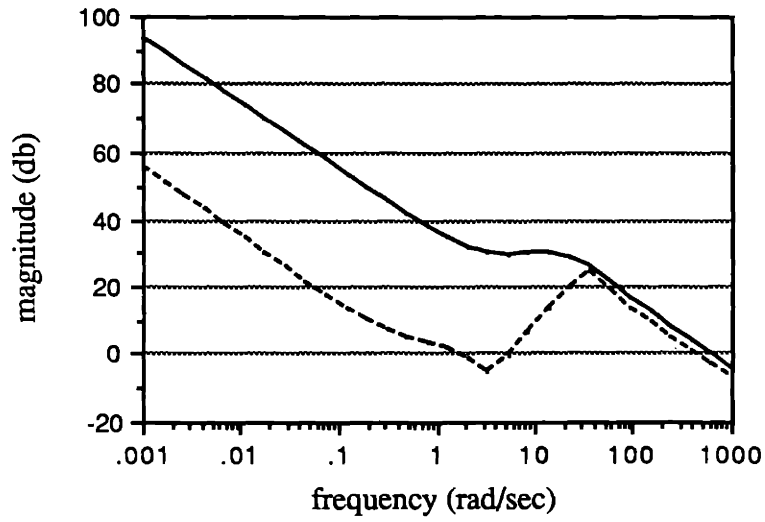


Figure 5.43 Compensator  $K_{1,2}$  - S.V. Plot

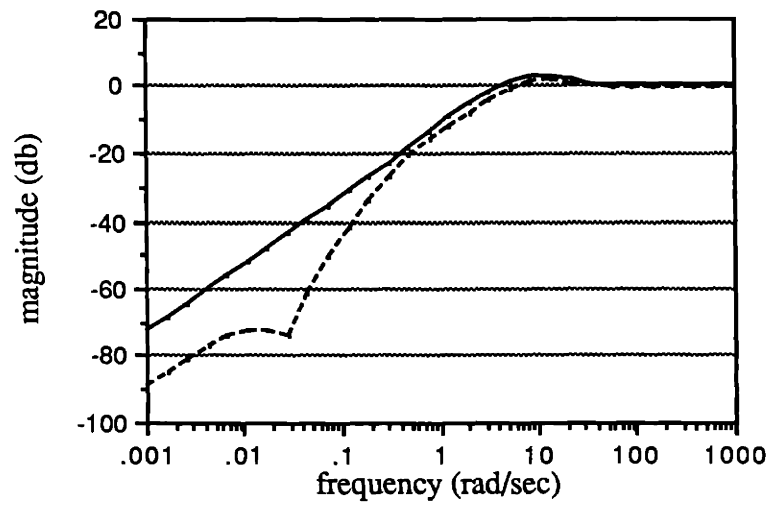


Figure 5.44 Sensitivity T.F. of the Nominal System with  $K_{1,2}$  - S.V. Plot

#### 5.4.7 Change of the Initial Condition - Compensator Redesign with fixed $\gamma = 0.5$

In the previous cases we held the initial conditions  $\alpha_{1,0}=0.2$  and  $D_{F_1}=I$  fixed while  $\gamma$  was varied. In this case we repeat the compensator redesign procedure for  $\gamma = 0.5$  with different initial conditions. The new value of  $D_{F_1}$  is equal to

$$D_{F_1} = \text{diag} \{ [1, 1, 4 [s + 25] [s + 100]]^{-1} \} \quad (5.33)$$

while  $\alpha_{1,0}$  remains the same, i.e.  $\alpha_{1,0}=0.2$ .

A property of the algorithm is that it result in the local optimum. Therefore, we can expect that the resulting  $\alpha_{\max}$  will be different than the one obtained in the previous subsection.

The first obtained compensator  $K_{1,1}$  guaranteed  $\alpha_{1,1}=3.5$   $\alpha_{1,0}= 0.7$ . With the compensator  $K_{1,2}$  we were not able to increase  $\alpha$  further and, therefore, the scaling  $D_{F_1}$  was updated into  $D_{F_2}$ . The following compensator  $K_{2,1}$  resulted in  $\alpha_{2,1} = 1.4$   $\alpha_{2,0}= 0.98$ . The stability margin was further increased to  $\alpha_{2,2} = 1.1$   $\alpha_{2,1} = 1.08$  with  $K_{2,2}$ . This value remained the same with the next obtained compensator. Furthermore, it didn't change with the updated block diagonal scaling. Therefore, the resulting  $\alpha_{\max}$  was  $\alpha_{\max} = \alpha_{2,2} = 1.08$  which is obtained with the compensator  $K_{2,2}$ . The singular value plots of the latter and the resulting sensitivity transfer function are depicted in Figure 5.47 and 5.48.

The above data shows that the change in the initial value of  $D_F$  has significantly increased the resulting bound on the uncertainty  $\alpha_{\max}$ . With  $D_{F_1} = I$  we had  $\alpha_{\max} = 0.22$ , while with  $D_{F_1}$  defined as in (5.33), the resulting  $\alpha_{\max}$  was equal to 1.08. The compensator in the latter case is of higher gain and the higher bandwidth than the one with  $D_{F_1} = I$ . This has shown that the obtained result depends on the choice of initial conditions.

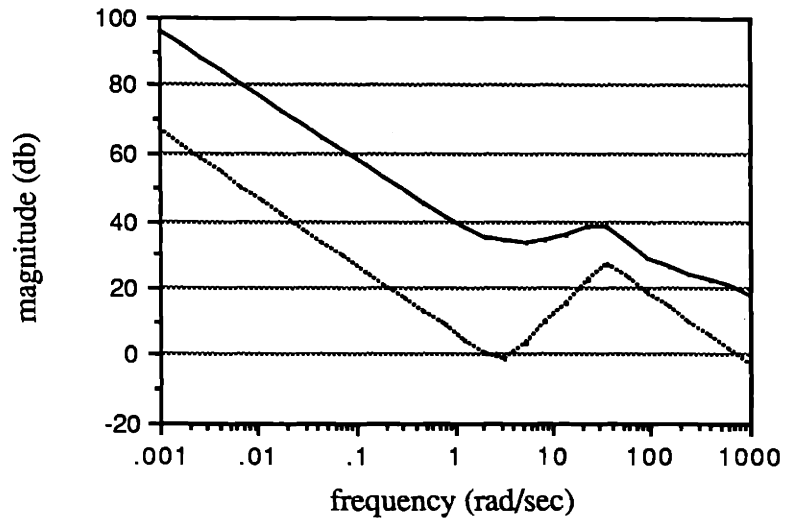


Figure 5.45 Compensator  $K_{2,2}$  - S.V. Plot

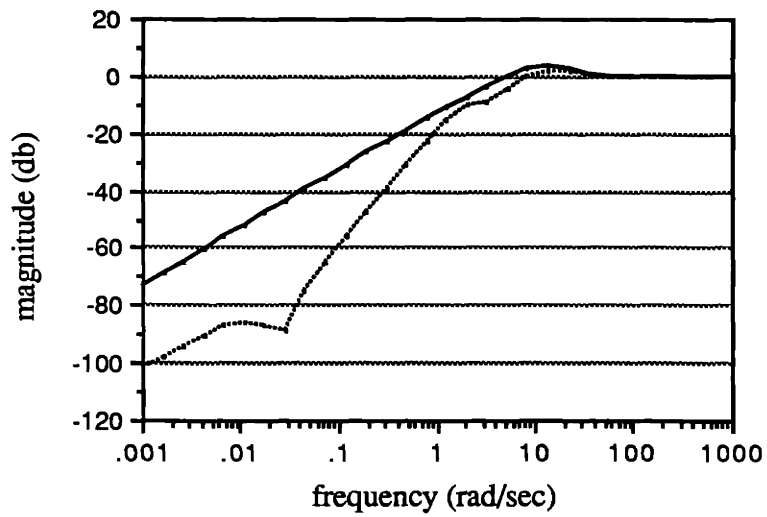


Figure 5.46 Sensitivity T.F. of the Nominal System with  $K_{2,2}$  - S.V. Plot



### 5.5 Interpretation of the Results

The preceding set of simulations have shown that the algorithm results in successive increase in the bound on the uncertainty  $\Delta 1$ . Furthermore, it provides us with the maximum value of the bound that depends on the initial conditions and on the quality of approximation of the block diagonal scaling  $D$ .

The value of the maximum allowable relative error was varied from 500% to 100% and finally to 50%. The resulting bounds on uncertainty with  $D_{F_1} = I$  were 4.32, then 2.52 and 0.22. The dependence between the value of performance index and the bound on the uncertainty is presented in Figure 5.47. It shows that there is clearly a very nonlinear dependence of the stability margin  $\alpha$  on the change of the performance index  $\gamma$ .

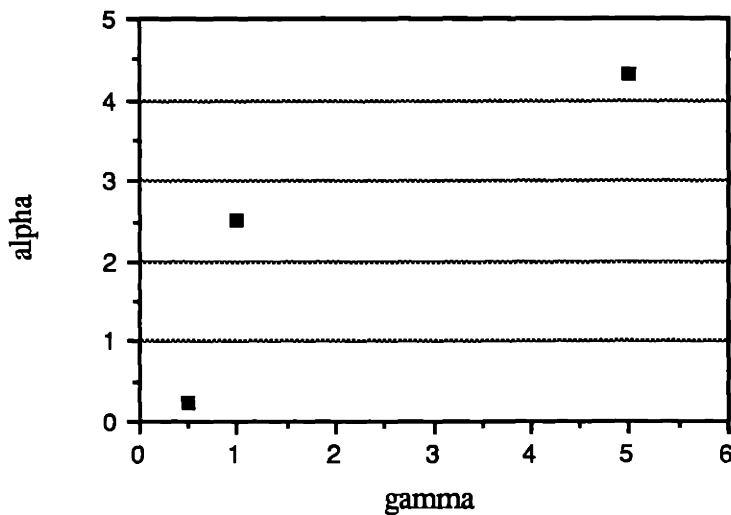


Figure 5.47 Dependence of  $\alpha_{\max}$  on the Performance Index  $\gamma$  with  $D_F = I$

The compensators that guarantee the maximum stability margins are of very high

bandwidth that reduces as the performance conditions are tightened. As  $\gamma$  is decreased, the obtained compensators start resembling the compensator  $K_0(s)$  introduced in section 5.4.2. The latter achieves  $\gamma_N = 0$  and is defined in (5.21) as  $K_0(s) = \text{diag}\{1, J(s)^{-1}\}K_p(s)$ ,  $J(s) = (3/15) [s + 15] [s + 12]^{-1}$ . Its singular value plots are shown in the following figure.

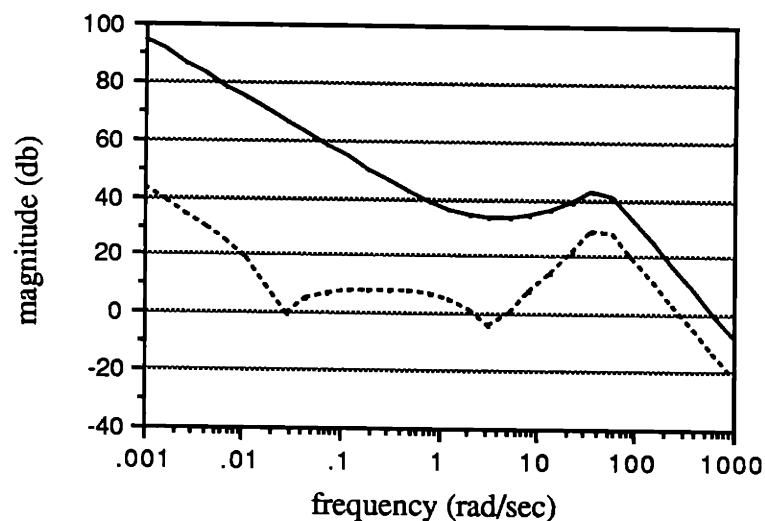


Figure 5.48 The Compensator  $K_0(s)$  that achieves  $\gamma_N = 0$  - S.V. Plot

Therefore, the resulting compensators as  $\gamma \rightarrow 0$  behave as we have expected them to. On the other hand, in the case when  $\gamma \rightarrow \infty$ , an optimal controller is any that keeps the second control channel open, i.e.  $\text{diag}\{0,1\}K=0$ . The latter can tolerate uncertainty of any magnitude. This is possible since the plant is stable.

The largest value of the performance index in the simulation was equal to five, i.e.  $\gamma=5$ . Therefore, the resulting controller had to satisfy this performance specification while using the second control channel as little as possible. When the performance condition was tightened, the control channel with uncertainty had to be used more. This resulted in a



The uncertainty  $\Delta_{1s}$  is scaled such that its norm is smaller than one. The weighting function  $W_{1m}(s)$  was introduced in (5.13) where it was scaled by  $\alpha$ . We have  $\alpha = 4.32$  that was achieved with  $\gamma = 5$ . The corresponding compensator is  $K_f = K_{2,2}$  presented in Figure 5.36.

The equivalent representation to the one in the previous figure is depicted in Figure 5.50. The SISO transfer function  $L(s)$  captures the rest of the system except for the uncertain part.

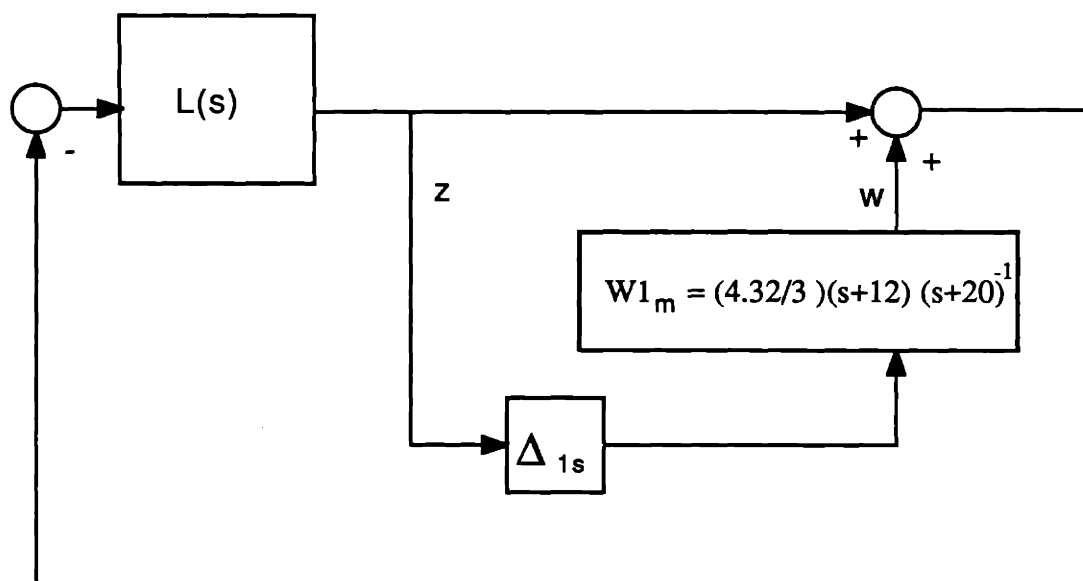


Figure 5.50 Equivalent Postfailure System Representation

We now compare the magnitude plots of  $L(s)$  and  $W_{1m}(s)$ . Since the magnitude of the latter exceeds one after certain frequency as will be shown, we expect that  $L(s)$  rolls off before this actually happens. This condition is needed for stability of the postfailure system. The next figure shows these magnitude plots.

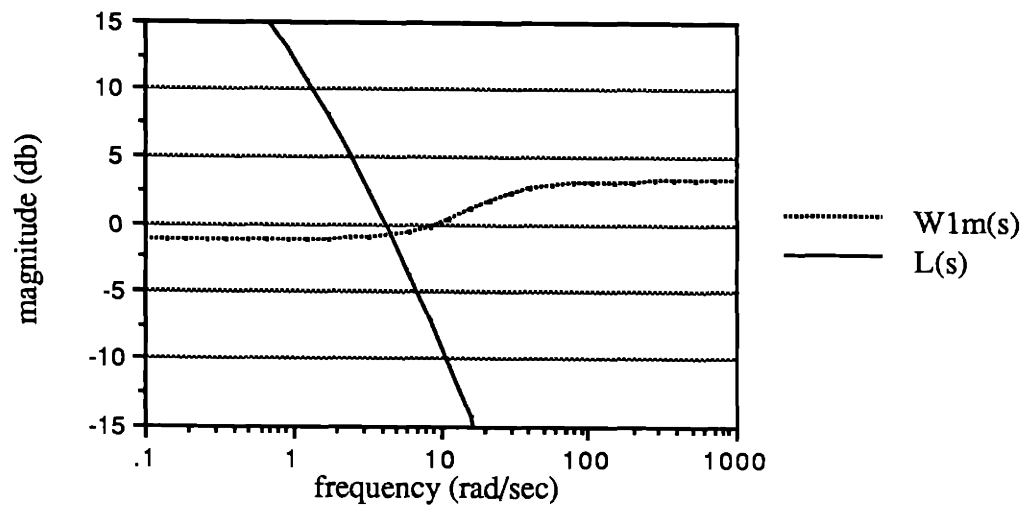


Figure 5.51 Magnitude Plots of  $W_{1m}(s)$  and  $L(s)$

This analysis shows that the choice of the weighting function  $W_1(s)$  has permitted compensators of very high bandwidth, as for example  $K_f = K_{2,2}$  for  $\gamma = 5$ . This is true since the resulting open loop transfer function  $L(s)$  has crossover frequency smaller than the crossover frequency of  $W_{1m}$ . The same figure also shows that  $\alpha$  could be increased further until the system becomes unstable. This is possible since the two crossover frequencies are apart from each other. On the other hand, any increase in  $\alpha$  will violate the performance condition.

In the case of  $\gamma = 1$  and  $\gamma = 0.5$ , the corresponding weighting functions  $W_{1m}(s)$  have magnitudes smaller than one at all frequencies. Therefore, if the performance condition is lifted, there is also potential for further increase of the uncertainty magnitude  $\alpha$  before the system becomes unstable.

We now perform the analysis to see how close we have actually come to violating individual stability and performance conditions with the obtained  $\alpha_{\max}$  and the

corresponding compensators. In order to do this, we look at the system representation shown in Figure 5.11 with the loop broken at the input and at the output of the uncertainty  $\Delta_1$ ,  $\|\Delta_1\|_\infty < 1$ . Let the transfer functions of interest be  $H_{e1/d1}(s)$  from  $d_1$  as the input and  $e_1$  as the output signal, and  $H_{e/d}(s)$  with  $d$  as the input and  $e$  as the output. The first transfer function is scaled by the obtained  $\alpha_{\max}$ . For the different values of  $\gamma$ , with  $D_F=I$  we have :

	$\gamma_N = \ H_{e/d}(s)\ _\infty$	$\alpha_{\max}$	$\ \alpha_{\max} H_{e1/d1}(s)\ _\infty$
$\gamma = 5 \rightarrow$	1.25	4.32	0.89
$\gamma = 1 \rightarrow$	0.39	2.52	0.54
$\gamma = 0.5 \rightarrow$	0.455	0.22	0.1

The stability condition is satisfied when the above scaled function  $\|\alpha_{\max} H_{e1/d1}(s)\|_\infty$  has the infinity norm smaller than or equal to one. The transfer function  $\|H_{e/d}(s)\|_\infty$  corresponds to the nominal performance  $\gamma_N$  with no uncertainty being present. The overall systems corresponding to all three values of  $\gamma$  are designed in such a way that stability and performance are satisfied for all uncertainties whose infinity norms are bounded by  $\alpha_{\max}$ . This means that the following holds :

$$\|\gamma^{-1} H_{e/d}(s, \Delta)\|_\infty \leq 1 \quad \text{for all} \quad \|\Delta\|_\infty \leq \alpha_{\max} \quad (5.34)$$

The inequality in (5.34) guarantees that the above norm will not exceed one but it doesn't give us the information how close we actually get to it with  $\|\Delta\|_\infty \leq \alpha_{\max}$ .

With the relaxed condition on performance as  $\gamma=5$ , we come relatively close to the

stability margin since  $\| \alpha_{\max} H_{e1/d1}(s) \|_{\infty} = 0.89$ . The nominal performance in this case is characterized by  $\gamma_N = \| H_{e/d}(s) \|_{\infty} = 1.25$ . It implies that  $\| H_{e/d}(s, \Delta) \|_{\infty}$  can potentially fluctuate between 1.25 and 5. This is expectable since the emphasis was on stability margin while the performance condition was relaxed.

In the opposite case where  $\gamma=0.5$ , the nominal performance is characterized by  $\gamma_N = \| H_{e/d}(s) \|_{\infty} = 0.455$ . Therefore, it can possibly change a little in the presence of uncertainty since it was very close to 0.5 at the first place. On the contrary, there is a possibility to increase magnitude of uncertainty ten times before the system goes unstable. This holds because  $\| \alpha_{\max} H_{e1/d1}(s) \|_{\infty} = 0.1$ .

When  $\gamma=1$ , we have the situation which is in between the previous two cases. The bound on the uncertainty was increased to  $\alpha_{\max}=2.2$  while the possible fluctuation of the performance specification was also increased. The nominal performance  $\gamma_N$  was given by  $\| H_{e/d}(s) \|_{\infty} = 0.39$  so there was freedom to possibly increase to one with the uncertainty being present.

The above results are consistent with the expected behavior of the resulting systems when  $\gamma$  is varied. For the relaxed specification on performance, the nominal value  $\gamma_N$  is away from the maximum admissible value  $\gamma$ . Therefore, a substantial variation in performance is allowed in the presence of uncertainty in order to increase the magnitude of the latter, i.e.  $\alpha_{\max}$ .

On the contrary, when the performance specification is very tight, as with  $\gamma = 0.5$ , the nominal value  $\gamma_N = 0.455$  is very close to the maximum one and it doesn't leave a possibility for the larger value of  $\alpha_{\max}$ .

The obtained results are dependent on the choice of the initial conditions and the quality of the approximation of scaling "D." This was shown in section 5.4.7 where the compensator redesign was performed with two different initial conditions  $D_{F1}$ . The resulting values of  $\alpha_{\max}$  were 0.22 with  $D_{F1}=I$  and 1.08 with  $D_{F1}$  as in (5.33). In the latter

case we also have  $\|\alpha_{\max} H_{e1/d1}(s)\|_{\infty} = 0.21$  and  $\|H_{e/d}(s)\|_{\infty} = 0.46$ . These results are consistent with the previous discussion.

## 5.6 Concluding Remarks

In this section the control redesign algorithm was evaluated on the augmented model of the F-8 Aircraft. A case is treated where one of the existing two actuators partially fails. Its postfailure dynamics is assumed to be given by the nominal model and the associated unstructured uncertainty. By varying the allowable error between the outputs of the original and postfailure system in the two-norm sense, a sequence of compensators was designed. Each of them maximizes the stability margin with respect to the above mentioned uncertainty.

The algorithm results in compensators that are minimizing in the sense of the " $\infty$ "-norm for the fixed block diagonal weights on the augmented system. Once the maximum bound on the uncertainty is achieved, the block diagonal weights are upgraded in order to reduce the value of the associated " $\mu$ " function.

The algorithm was run with three different values of the performance index  $\gamma$ . The obtained results including the maximum uncertainty bounds and the corresponding compensators were evaluated. It was shown that they follow the expected trend with respect to the successive increasing or decreasing of  $\gamma$ . The tightness of obtained performance bounds to the desired ones are discussed too. It was noted that the results depend on the initial conditions and the quality of approximating the block diagonal scaling "D".



## CHAPTER 6

### REDESIGN IN THE PRESENCE OF REAL PARAMETER UNCERTAINTY

#### 6.1 Introduction

The case of real parameter perturbation is of special interest in the analysis and synthesis of control systems because it arises naturally in many problems. In Chapter 3 it was stated that the failure detection and identification scheme might result in a nominal model of the failed plant with uncertainty associated with some of its real parameters. Once such information is provided after a failure, the closed loop system with the original controller has to be analyzed and evaluated. If the performance of the system has deteriorated substantially, a new compensator has to be designed to recover the performance of the original system to the extent possible.

In this chapter we present the stability and performance analysis of a postfailure system based on the Riccati equation approach. The assumption will be made that all parameter uncertainties are zero at nominal and are bounded in magnitude. We will introduce two analysis methods and discuss the tradeoff between numerical difficulties and conservatism of both.

The first method is based on checking the existence of a matrix  $P$  that satisfies a set of Riccati type inequalities corresponding to the vertices of the hyperbox in parameter space. In this method we will extend the idea of the Lyapunov equation approach of Horisberger

and Belanger [HB76] for stability analysis. The latter was based on the convexity of the Lyapunov equation to a perturbation only in the "A" matrix of the system. Here, we will allow perturbation in all the matrices of the state space representation. The Riccati equation criterion, which is not convex with respect to the perturbation vector, will be rearranged so that convexity is achieved.

Two iterative algorithms for finding the largest hyperbox where stability and performance are guaranteed will be introduced. For a fixed size of the hyperbox, the existence of a symmetric  $P$  that satisfies vertex conditions will be investigated. The first algorithm will use existing optimization techniques for construction of a symmetric matrix  $P$  that satisfies a set of matrix inequalities. For cases where linearity in the matrix  $P$  is needed, a suitable transformation of the Riccati equation criterion will be introduced. The latter will not require construction of the corresponding passive system as in Boyd and Yang [BY88]. The second algorithm will provide a somewhat more conservative answer but will be easier to use.

The other analysis method will be based on a single Riccati equation. We will show that it offers no advantages over the methods that treat parameter perturbation as a frequency dependent norm bounded uncertainty.

Furthermore, we will make a distinction between the situation where stability is checked separately from the performance and the one where they are both checked simultaneously.

## **6.2 Robust Performance in the Presence of Real Parameter Perturbation - Analysis**

A model of the postfailure plant  $P(s)$  is provided in the failure identification step. As explained earlier, it consists of the nominal model and the associated uncertainty. In this

study we consider the case where the uncertainty is present in some of the real parameters of  $P(s)$ . We assume that all uncertain parameters enter the state space representation  $[A_1, B_1, C_1, 0]$  of  $P(s)$  linearly and that they are of bounded magnitude with zero nominal value.

Let the uncertain parameters belong to a set defined as :

$$\Omega = \{ \mathbf{q} \quad \text{s.t.} \quad \mathbf{q} = [q_1, \dots, q_m]' \in \mathbb{R}^m \mid |q_i| \leq r_i, \quad r_i \in \mathbb{R}_+ \} \quad (6.1)$$

s.t.  $A_1 = A_1(\mathbf{q})$ ,  $B_1 = B_1(\mathbf{q})$ , and  $C_1 = C_1(\mathbf{q})$ .

Therefore, the uncertainty belongs to a "hyperbox"  $\Omega$  in the parameter space. In the case of  $m = 2$ , the "hyperbox" is depicted in Figure 6.1.

The controller  $K(s)$  in the postfailure system has to guarantee stability and performance in the presence of parametric uncertainty. The performance requirement in this setting takes the following form :

$$\| M(s, K, \mathbf{q}, \gamma) \|_{\infty} \leq 1 \quad \text{for all} \quad \mathbf{q} \in \Omega \quad (6.2)$$

where  $M = W(s) [S(s, K, \mathbf{q}) - Sp(s)]$  and  $\gamma$  is the performance index, as have already been defined in the previous sections. Without loss in generality we will assume that  $\gamma = 1$ .

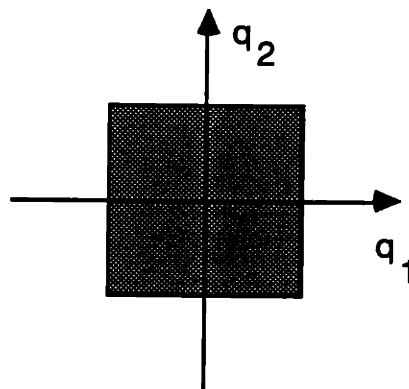


Figure 6.1 Parameter Box in Two-Dimensional Parameter Space

Let the state space representation of  $M(s)$  be given as  $[A(q), B(q), C(q), 0]$ . It is easy to check that its elements will also be linearly dependent on the uncertain parameters  $q_i$  since the state space representation of the postfailure plant  $P(s)$  is. To prove this it is enough to show that the state space representation  $[A_s, B_s, C_s, D_s]$  of  $S(s, q, K, P) = [I + P(q)K]^{-1}$  is linearly dependent on  $q$ . By defining the state space representation of the compensator  $K(s)$  as  $[A_k, B_k, C_k, 0]$ , we have :

$$A_s = \begin{bmatrix} A_1 & B_1 C_k \\ -B_k C_1 & A_k \end{bmatrix}; \quad B_s = \begin{bmatrix} 0 \\ -B_k \end{bmatrix}; \quad C_s = [C_p \quad 0]; \quad D_s = I \quad (6.3)$$

Since  $A_1$ ,  $B_1$ , and  $C_1$  depend linearly on  $q_i$ , the matrices defined in (6.3) will also be linearly dependent on the uncertain parameters.

Therefore, we can define the following :

$$A = A_o + \sum_1^m q_i E_i = A_o + \Delta A \quad (6.4)$$

$$B = B_o + \sum_1^m q_i F_i = B_o + \Delta B \quad (6.5)$$

$$C = C_o + \sum_1^m q_i G_i = C_o + \Delta C \quad (6.6)$$

where  $q_i \in \Omega$  is defined in (6.1) and  $[A_o, B_o, C_o, 0]$  is the state space representation of the nominal system  $M(q=0)$ . The matrices  $E_i$ ,  $F_i$ , and  $G_i$  define the structure of the perturbation with respect to each uncertain parameter  $q_i$ . They can always be scaled in such a way that  $r_i = 1$  for  $i = 1, 2, \dots, m$ .

The nominal system, obtained for  $\mathbf{q} = 0$ , is assumed to be stable and nonexpansive. Any parameter value different from the nominal can potentially destabilize the system or bring its infinity norm above the required bound. There are two possible situations in the analysis of such a problem. The first one is based on separately evaluating the stability and performance, in the given parameter box. The other one assumes simultaneous evaluation of both characteristics of the system. Therefore, we have two possible cases in checking the nonexpansivity of the system:

- 1) stability in the parameter box is known
- 2) stability not known

### 6.2.1. Performance Analysis with Stability in the Parameter Box Known

In this case the stability of the system is guaranteed for all perturbations in the given hyperbox in parameter space. The problem is then formulated in the following way :

$$\text{Let } M(\mathbf{q}) = C(\mathbf{q}) [sI - A(\mathbf{q})]^{-1} B(\mathbf{q}) \in RH_{\infty} \quad \text{for all } \mathbf{q} \in \Omega,$$

then we want  $\|M(\mathbf{q})\|_{\infty} \leq 1$  for all  $\mathbf{q} \in \Omega$ . (6.7)

According to Lemma 2.7, the sufficient condition for the nonexpansivity of the system  $M(\mathbf{q})$  in the hyperbox is given in the following lemma.

#### Lemma 6.1

Let  $M(\mathbf{q}) \in RH_{\infty}$  for all  $\mathbf{q} \in \Omega$ . If  $\exists P=P' \in R^{n \times n}$  s.t.

$$R(\mathbf{q}) = A(\mathbf{q})'P + PA(\mathbf{q}) + PB(\mathbf{q})B(\mathbf{q})'P + C(\mathbf{q})'C(\mathbf{q}) \leq 0 \quad (6.8)$$

then  $\|M(\mathbf{q})\|_{\infty} \leq 1 \quad \forall \mathbf{q} \in \Omega$ .

Proof :

This statement is the extension of the result in Lemma 2.7 to all the points in the hyperbox  $\Omega$  with a common matrix  $P=P'$ .

■

### Corollary 6.1.1

There is a dual formulation for the nonexpansivity condition in (6.8). It requires the existence of a common matrix  $T=T'$  s.t.

$$R_1(\mathbf{q}) = TA(\mathbf{q})' + A(\mathbf{q})T + TC(\mathbf{q})'C(\mathbf{q})T + B(\mathbf{q})B(\mathbf{q})' \leq 0 \quad \forall \mathbf{q} \in \Omega \quad (6.9)$$

The condition in (6.8) represents an infinite dimensional problem because it requires a common  $P=P'$  for all the points in the hyperbox  $\Omega$ . To make this approach useful we have to try to reduce its dimensionality, i.e. to make it finite dimensional. We now show that the existence of a matrix  $P=P'$  s.t.  $R(\mathbf{q})$  is convex with respect to  $\mathbf{q} \in \Omega$  and negative semidefinite at all the vertices of the hyperbox is sufficient to achieve condition 6.8 for all  $\mathbf{q} \in \Omega$ .

### Lemma 6.2

Let the function  $R(\mathbf{q})$  defined in (6.8) be convex with respect to  $\mathbf{q} \in \Omega$  for every  $P=P'$ . Then,  $R(\mathbf{q}) \leq 0 \quad \forall \mathbf{q} \in \Omega$  if  $\exists P=P'$  s.t.  $R(\mathbf{q}_j) \leq 0$ ,  $j = 1, 2, \dots, 2^m$ , where  $\mathbf{q}_j$  corresponds to the "j'th" vertex of the hyperbox  $\Omega$ .

Proof:

The hyperbox  $\Omega$  is a convex set in the parameter space. The function  $R(\mathbf{q})$  is convex over the same set. Therefore, for any  $\mathbf{q} \in \Omega$  we have

$$R(\mathbf{q}) \leq \sum_1^{2^m} \alpha_j R(\mathbf{q}_j) ; \quad \sum_1^{2^m} \alpha_j = 1 ; \quad \alpha_j \geq 0 \quad (6.10)$$

Hence,  $R(\mathbf{q}) \leq 0$  for all  $\mathbf{q} \in \Omega$  since  $R(\mathbf{q}_j) \leq 0$ . This completes the proof. ■

The previous lemma requires that  $R(\mathbf{q})$  is a convex function for every  $P=P'$ . This will hold if  $R(\mathbf{q})$  is linearly dependent on every parameter perturbation  $q_i$ . The overall expression for the former, with the perturbation defined as in (6.4) - (6.6), has the following form :

$$\begin{aligned} R(\mathbf{q}) = R_o + \sum_1^m q_i [ (E_i'P + PE_i) + P(F_i B_o' + B_o F_i')P + (G_i' C_o + C_o' G_i) ] + \\ + P \left( \sum_1^m q_i F_i \right) \left( \sum_1^m q_i F_i' \right) P + \left( \sum_1^m q_i G_i \right) \left( \sum_1^m q_i G_i' \right) \end{aligned} \quad (6.11)$$

where  $R_o = A_o'P + PA_o + PB_o B_o'P + C_o' C_o$ . The last two elements in the above expression are the only ones where the parameter perturbation appears in nonlinear form. If they could be bounded from above with a function linear in  $q_i$ , the resulting  $R(\mathbf{q})$  would be linear and, therefore, convex in  $\mathbf{q}$ . Using the spectral properties of positive matrices we can formulate the following :

$$P \Delta B \Delta B' P \leq P | \Delta B \Delta B' | P \leq \overline{\sigma} ( | \Delta B | )^2 P P = \alpha_B^2 P P$$

and

$$\Delta C' \Delta C \leq | \Delta C' \Delta C | \leq \overline{\sigma} ( | \Delta C | )^2 I = \alpha_C^2 I \quad (6.12)$$

The symbol  $|\Delta C|$  denotes the matrix each element of which is equivalent to the largest magnitude of the corresponding element of  $\Delta C(\mathbf{q})$ . Therefore, it is a fixed matrix and we can always find  $\bar{\sigma}(|\Delta C|)^2 = \alpha_C^2$ . The same is valid for  $\Delta B$ .

The alternative linearization method can be presented as follows :

$$\Delta C' \Delta C = \left( \sum_1^m q_i G_i' \right) \left( \sum_1^m q_i G_i \right) \leq \left( \sum_1^m \bar{\sigma}(G_i) \right)^2 I = \alpha_{C1}^2 I \quad (6.13)$$

where  $|q_i| \leq 1$ . Furthermore,  $\alpha_{B1}$  can be obtained in the analogous way. There is no clear answer which linearization method is more conservative. Therefore, in each individual case both should be applied and the less conservative result adopted.

The function  $R(\mathbf{q})$  can now be bounded from above with some  $R_1(\mathbf{q})$  defined as :

$$\begin{aligned} R_1(\mathbf{q}) = R_o + \sum_1^m q_i [ (E_i' P + P E_i) + P(F_i B_o' + B_o F_i') P + (G_i' C_o + C_o' G_i) ] + \\ + \alpha_B^2 P P + \alpha_C^2 I = R_o + \sum_1^m q_i \Delta R_i + \alpha_B^2 P P + \alpha_C^2 I \geq R(\mathbf{q}) \end{aligned} \quad (6.14)$$

This function is linear in  $\mathbf{q}$  and, hence, convex for every  $P=P'$ . Therefore, we have managed to construct a function that bounds  $R(\mathbf{q})$  from above and is convex in  $\mathbf{q}$ . As it was shown, this construction is not unique. A different but more conservative linearization was conducted in Yeh et al [YBBH89]. By using the result of Lemma 6.2, we can formulate the following.

### Remark 6.2.1

Let the system be stable for every  $\mathbf{q} \in \Omega$ . If  $\exists P=P'$  s.t.  $R_1(\mathbf{q}) \leq 0$  at every vertex of the hyperbox in the parameter space, then the perturbed system  $M(\mathbf{q})$  is nonexpansive for all  $\mathbf{q} \in \Omega$ .



The previous statement establishes the condition for the nonexpansivity of the system in the parameter box in terms of the existence of a symmetric  $P$  that simultaneously satisfies inequalities at vertices by assuming that they are then satisfied at any point in the box. This represents an important result because it reduces an infinite dimensional problem to a finite dimensional one.

Let  $R_1(\mathbf{q})$  at the vertex " $j$ " be given as :

$$R_1(\mathbf{q}_j) = A'(\mathbf{q}_j)P + PA(\mathbf{q}_j) + PB_oB_o'P + C_o'C_o + PW(\mathbf{q}_j)P + Y(\mathbf{q}_j) + \alpha_B^2 P P + \alpha_C^2 I \quad (6.15)$$

$$\text{where } W(\mathbf{q}) = \sum_1^m q_i (F_i B_o' + B_o F_i') \quad \text{and} \quad Y(\mathbf{q}) = \sum_1^m q_i (G_i' C_o + C_o' G_i).$$

The expression in (6.15) can be rewritten in the form of an ordinary Riccati equation as :

$$R_1(\mathbf{q}_j) = A'(\mathbf{q}_j)P + PA(\mathbf{q}_j) + P\tilde{B}(\mathbf{q}_j)\tilde{B}(\mathbf{q}_j)'P + \tilde{C}(\mathbf{q}_j)\tilde{C}(\mathbf{q}_j) \quad (6.16)$$

where  $\tilde{B}(\mathbf{q}_j)$  and  $\tilde{C}(\mathbf{q}_j)$  are augmented input and output matrices. They are defined as follows:

$$\tilde{B}(\mathbf{q}_j) = [ B_o \quad W(\mathbf{q}_j)^{0.5} \quad \alpha_B I ] \quad \text{and} \quad \tilde{C}(\mathbf{q}_j) = \begin{bmatrix} C_o \\ Y(\mathbf{q}_j)^{0.5} \\ \alpha_C I \end{bmatrix} \quad (6.17)$$

The condition defined in Remark 6.1 requires a single matrix  $P=P'$  which simultaneously satisfies  $2^m$  Riccati inequalities. However, there are some situations where the requirement can be further simplified.

### Remark 6.2.2

It is easy to see that if the matrices  $\Delta R_i$  as defined in (6.14) are semidefinite in sign for the given  $P$ , then there is a single Riccati equation corresponding to one of the vertices

whose solution would simultaneously satisfy the remaining inequalities.

Thus, we are led to construction of a single Riccati equation whose solution, if it exists, simultaneously satisfies the inequalities at all vertices. Let  $\Pi(P)$  be s.t.

$$\Pi(P) = \sum_1^m q_i \Delta R_i \quad |q_i| \leq 1 \quad (6.18)$$

where  $P=P' \geq 0$  is the solution of

$$A_0'P + PA_0 + P(B_0B_0' + \alpha_B^2 I)P + (C_0'C_0 + \alpha_C^2 I) + \Pi(P) = 0 \quad (6.19)$$

It is easy to see that such a  $P$  satisfies the inequalities at all points in the parameter box and this fact guarantees nonexpansivity of the system. Such a bounding function  $\Pi(P)$  is introduced in various forms by different authors, as in [PH86] and [BH88]. An obvious advantage of this approach is that the complexity of the problem is reduced to solving a single Riccati equation but at the expense of making the problem significantly more conservative. We will show that the previous formulation is, in general, more conservative than the "Small Gain Theorem" approach for the same problem. Restricting the perturbation to the "A" matrix only, we will show that the Peterson-Hollot bounding function [PH86] will, in our problem setting, lead to a condition as conservative as the one arising from the "Small Gain Theorem." This result is important since the above methodology is frequently used without realization of its conservativeness. An analogous result was independently derived for time varying perturbations contained in the "A" matrix by Khargonekar et al [KPZ87].

There is definitely a trade-off between the numerical complexity in solving a set of Riccati type inequalities and the extreme conservatism in the approach just described. For analysis purposes, we will first pursue the former, less conservative method.

**Remark 6.2.3**

The box  $\Omega$  in parameter space where stability is preserved can be obtained in various ways. One of the nonconservative methods is the algorithm by Safonov and de Gaston [DGS88] that leads to the maximum stability margin. Then, the performance margin can be obtained by gradually scaling down the stability margin, if necessary, until the conditions in Remark 6.2.1 are satisfied.

We now present algorithms for determining the largest hyperbox, with size  $\rho_p$ , where nonexpansivity is preserved under the assumption that the stability bound  $\rho_s$  is already known. The former will be evaluated by successively scaling down the size of a parameter box,  $\rho_s$ , where the stability is preserved. Therefore, the overall combined stability/performance margin  $\rho$  will then simply be equal to the performance one, i.e.

$$\rho = \rho_p \leq \rho_s$$

The objective of the algorithms is to check the existence of  $P = P'$  that simultaneously satisfies the Riccati type matrix inequalities associated with every vertex of the parameter hyperbox in parameter space. This is achieved by establishing an optimization problem where the largest eigenvalue of the overall diagonal matrix, whose entries are the matrices corresponding to each Riccati type inequality, is minimized over the convex set of symmetric matrices  $P$ . If there is such a solution that makes this eigenvalue less than or equal to zero, the nonexpansivity of the system is guaranteed within the given parameter set according to Lemma 6.7.

**6.2.2 Algorithms for Determining the Largest Performance Margin  $\rho_p$** 

Let the size of the hyperbox  $\Omega$  in parameter space where the stability is retained be  $\rho_s$ , i.e.  $|q_i| \leq \rho_s$   $i = 1, 2, \dots, m$ . Furthermore, let the hyperbox  $\Omega_\epsilon$  be obtained by scaling

down  $\Omega$  with some  $0 \leq \epsilon \leq 1$  s.t.  $|q_i| \leq (\rho_s \epsilon)$   $i = 1, 2, \dots, m$ . The sufficient condition for nonexpansivity of the perturbed system in  $\Omega_\epsilon$  can then be expressed as the existence of a  $P=P'$  s.t.

$$f(\epsilon) \equiv \lambda_{\max}[\text{diag} \{ A'(q_j)P + PA(q_j) + P\tilde{B}(q_j)\tilde{B}(q_j)'P + \tilde{C}(q_j)'\tilde{C}(q_j) \}] \leq 0 \quad (6.20)$$

where "j" corresponds to the particular vertex of  $\Omega_\epsilon$ .

The  $f(\epsilon)$ , with a fixed  $P$ , is a nondecreasing and continuous function of  $\epsilon$ . The continuity comes from the fact that the eigenvalues of the real matrices are continuous functions of its parameters [Bell70]. The monotonicity is the consequence of successive enlargement of the hyperbox where the largest eigenvalue is sought for. Therefore, if there is a matrix  $P$  that guarantees the strict inequality in (6.20), we can always enlarge the stability box until zero is achieved in one direction at least.

A possible way to come up with a matrix  $P$  that satisfies inequality (6.20) for the fixed magnitude of perturbation  $q_j(\rho_s \epsilon)$ , is to formulate the following minimization problem

$$\min_{P=P'} \lambda_{\max}[\text{diag} \{ R_1(q_j) \}] \quad j = 1, 2, \dots, 2^m \quad (6.21)$$

The set of all symmetric matrices  $P$  is a convex set. Furthermore, it can be scaled, without loss of generality, to the set of all symmetric matrices  $P$  whose entries are with the bounded magnitude, i.e. each  $|P_{kl}| \leq 1$ ,  $k = 1: n$ ,  $l = 1: n$ . This is possible since the scaling that achieves the above condition doesn't change the input-output properties of the system or the definiteness of  $P$ .

Unfortunately, the function (6.20) is not convex in  $P$ . By restricting the set of all matrices  $P$  to the subset of the positive semidefinite ones, we transform the original problem into a convex one, as the following lemma shows.

**Lemma 6.3**

A matrix  $P = P' \geq 0$  satisfies

$$A'P + PA + P\tilde{B}\tilde{B}'P + \tilde{C}'\tilde{C} \leq 0$$

iff the following holds with  $\alpha \in \mathbb{R}_+$

$$T = \begin{bmatrix} A'P + PA + \alpha^{-1} \tilde{C}'\tilde{C} & P\tilde{B} \\ \tilde{B}'P & -\alpha^{-1} I \end{bmatrix} \leq 0 \quad (6.22)$$

**Proof :**

The matrix  $T$  can be rewritten as

$$\begin{bmatrix} A'P + PA + \alpha P\tilde{B}\tilde{B}'P + \alpha^{-1} \tilde{C}'\tilde{C} & 0' \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\alpha P\tilde{B}\tilde{B}'P & P\tilde{B} \\ \tilde{B}'P & -\alpha^{-1} I \end{bmatrix} \leq 0 \quad (6.23)$$

Let the first matrix be  $T_1$  and the second  $T_2$ , s.t.  $T = T_1 + T_2$ . It is easy to show that  $T_2$  is always negative definite whenever  $P \geq 0$ .

The sufficiency is proved by assuming  $T \leq 0$  which automatically implies  $T_1 \leq 0$ . The other direction is obvious since  $T_1 \leq 0$  and  $T_2 < 0$  imply that their sum, i.e.  $T$ , is negative semidefinite. The scalar  $\alpha$  is just a scaling factor of the inputs and the outputs of the original system  $M = [A, \tilde{B}, \tilde{C}, 0]$  s.t. the scaled system  $[A, \alpha^{1/2} \tilde{B}, \alpha^{-1/2} \tilde{C}, 0]$  has the same infinity norm.

■

The matrix  $T$  can be found in [Wil71] in the form of a Linear Matrix Inequality as well as in [Jac77], where it is used to establish the nonnegativity conditions for constrained and nonquadratic functionals. In both cases it was associated with the Algebraic Riccati Equation (ARE). In Lemma 6.3 its usage is substantially different. The result in the previous lemma is important since it shows that there is no need to construct a passive system corresponding to  $M(s)$  in order to get a criterion convex in  $P=P \geq 0$  as it was done in [BY88].

### Corollary 6.3.1

The matrix  $T$  being negative semidefinite implies the following :

$$[x' \ u'] T [x' \ u'] \leq 0 \quad \text{for every } x \in \mathbb{R}^n, \quad u \in \mathbb{R}^s \quad (6.24)$$

The expression in (6.24) is equivalent to:

$$2 x' P [Ax + \tilde{B}u] \leq u'u - y'y \quad (6.25)$$

where  $dx/dt = Ax + \tilde{B}u$  ;  $y = \tilde{C}x$ .

If the former inequality is integrated from  $t_0 = 0$  to  $t = \tau$  with  $x(t_0) = 0$  and  $u(t_0) = y(t_0) = 0$ , we have :

$$0 \leq V(x) = x' P x \leq \int_0^{\tau} u'u \, dt - \int_0^{\tau} y'y \, dt \quad (6.26)$$

where  $V(x)$  is a Quadratic Lyapunov function associated with the system  $M=[A, \tilde{B}, \tilde{C}, 0]$ . The system is nonexpansive since, at any point in time, the energy at its output is smaller than or equal to the energy at the input .

■

According to the previous lemma and its corollary, the minimization of the maximal eigenvalue of the matrix  $T(q_j)$ , corresponding to a vertex "j", is equivalent to minimization of the associated Riccati inequality. This holds only when  $P=P' \geq 0$ , which represents a stricter condition than the pure symmetry of  $P$  that is needed for the nonexpansivity of the stable system. Nevertheless, the function

$$g(\epsilon, P) = \lambda_{\max} [ \text{diag} \{ T(\epsilon q_j) \} ] \quad (6.27)$$

is a convex function of  $P$  with fixed  $\epsilon$ . Therefore, we have transformed the original problem into a nondifferentiable convex optimization problem. This was done by replacing the Riccati type condition with the augmented Lyapunov condition as it appears in the matrix  $T$ . There are different methods for determining a matrix  $P=P' \geq 0$  that minimizes  $g(\epsilon, P)$ . A very good and detailed survey of the latter can be found in [BY88].

Now we can formulate the algorithm for determining the largest performance and stability bound. The nonexpansivity of the nominal system is assumed.

### I) Min-Max Algorithm

$j = 1 : 2^m$  ---- correspond to vertices of  $\Omega_\epsilon$

- define the following min-max problem

$$\begin{aligned} \min_{P=P'} \quad & \lambda_{\max} [ \text{diag} \{ T(q_j) ; -P \} ] = \\ & = \min_{P=P'} \quad \max_{\|w\|=1} w' [ \text{diag} \{ T(q_j) ; -P \} ] w = \quad (6.28) \\ & = \lambda_{\max}^* (P^*) \end{aligned}$$

where

$$T = \begin{bmatrix} A'P + PA + \tilde{C}'\tilde{C} & P\tilde{B} \\ \tilde{B}'P & -I \end{bmatrix} \quad (6.29)$$

step 0: Let  $\epsilon = 1$  and do the min-max optimization as in (6.28).

If  $\lambda_{\max}^*(\epsilon = 1) \leq 0$ , stop  $\Rightarrow \rho_p = \rho_s$

If  $\lambda_{\max}^*(\epsilon = 1) > 0$ , let  $\epsilon(1) = 0$  and go to step 1.

step 1: With the given  $0 \leq \epsilon(k) \leq 1$ ,  $k=1, 2, \dots$ , solve the min-max problem (6.28)

With the resulting  $\lambda_{\max}^*(\epsilon)$  and  $P^*$  go to step 2.

step 2: Find the largest  $\epsilon(k+1)$  s.t  $\lambda_{\max}(P^*, \epsilon(k+1)) \leq 0$ .

If  $\epsilon(k+1) \geq 1$ , stop  $\Rightarrow \rho_p = \rho_s$

If  $\epsilon(k+1) \leq \epsilon(k)$ , stop  $\Rightarrow \rho_p = \epsilon(k) \rho_s$

If  $\epsilon(k+1) > \epsilon(k)$ , go back to step 1 with  $\epsilon(k+1)$ .

The above algorithm, by its construction, results in a nondecreasing value of  $\epsilon$  in each step. The convergence is assured since we have a monotone increasing sequence that is bounded from above.

The algorithm above enables us to find the largest size hyperbox where the system remains nonexpansive. It consists of a successive search for a matrix  $P=P' \geq 0$  that minimizes the largest eigenvalue corresponding to the vertices of the hyperbox with a fixed



size. In order to exploit existing optimization methods, the original Riccati equation nonexpansivity condition is transformed into an equivalent matrix condition that is convex in  $P$ . This transformation was straightforward and it didn't require any conversion of the original system as it was done in [BY88].

Unfortunately, the minimization involved can be very cumbersome so it makes sense to undertake a somewhat more conservative approach by restricting  $P$  to be a solution of at least one Riccati type equation. Therefore, the performance margin will be obtained by finding a pair  $\rho$  and  $P^*$  s.t. one of the "Riccati" inequalities becomes an equality, i.e.  $P^*$  is the solution of the Riccati equation.

## II) Algorithm 2

We have the following steps:

1) - for each inequality (j) find the largest  $\rho_p(j)$  s.t. the solution  $P(j)=P'(j)$  to the corresponding equality satisfies the other  $(2^m - 1)$  inequalities.

2) - the maximum of  $\rho_p(j)$ ,  $j=1 : 2^m$ , is the performance margin  $\rho_p$

### Remark 6.2.4

There is no a priori indication which Riccati equation should be picked first. Therefore,  $j=1$  is taken as the initial choice.

The advantage of the second algorithm, in spite of its conservatism, is that it is very easy to

implement. Therefore, its results can be used as the conservative estimate for the results of the first algorithm.

### 6.2.3 Simultaneous Stability/Performance Robustness Analysis

The conditions utilized until now have guaranteed nonexpansivity only under the assumption that the system remain stable. However, in the more general case, we do have to incorporate the conditions for stability in the previous result. The following lemma establishes conditions for both stability and performance of the perturbed system.

#### Lemma 6.4

If there is a  $P=P' > 0$  s.t. it simultaneously satisfies the following set of strict inequalities

$$R_1(q_j) = A'(q_j)P + PA(q_j) + P\tilde{B}(q_j)\tilde{B}(q_j)'P + \tilde{C}(q_j)'\tilde{C}(q_j) < 0, \quad j=1: 2^m, \quad (6.30)$$

then :

- i) the system  $M(q)$  is stable for all  $q \in \Omega$
- ii)  $\|M(q)\|_\infty \leq 1$  for all  $q \in \Omega$

Proof :

i) It is based on the Lyapunov stability test [Wonham78]. Since there is a  $P=P' > 0$  and  $G(q) = G(q)' > 0$  s.t. the expression (6.30) can be rewritten as

$$A(q)'P + P A(q) + G(q) = 0$$

for every  $q \in \Omega$ , then  $A(q)$  is asymptotically stable for all  $q \in \Omega$ .

ii) based on Lemma 6.2 and Remark 6.2.1



#### Corollary 6.4.1

The above strict inequalities ( $>$ ) could be relaxed into ( $\geq$ ) if the closed loop system  $M(q)$  would remain detectable for all  $q \in \Omega$ .

The maximal stability/performance box in the parameter space can be evaluated by the min-max algorithm equivalent to the one in section 6.2.2. The initial  $P$  is obtained for  $\varepsilon = 0$ , and the output matrix of the system is augmented to  $[\tilde{C}', \tau I]'$  where  $\tau$  is an arbitrarily small positive scalar. This augmentation is done to assure strict inequality of the Riccati type conditions in (6.30).

### 6.3 Analysis Based on the Single Riccati Equation

As was stated before, a general approach to the stability and performance analysis in the presence of magnitude bounded real parameter perturbation is based on the existence of a matrix  $P=P' \geq 0$  that simultaneously satisfies  $2^m$  Riccati inequalities corresponding to the vertices of the hyperbox in parameter space. Unfortunately, it is computationally cumbersome even for a small number of varying parameters and for low order systems. For that reason, a different approach is usually undertaken. It is based on the derivation of a single Riccati equation whose solution, if it exists, satisfies all the previously mentioned inequalities. It is more conservative since an additional constraint is imposed on the desired matrix  $P$ . Therefore, there is a considerable trade-off between the computational complexity and the conservatism of latter approach.

We first look at the case where the perturbation is present in "A" matrix only. Let  $\Pi(P)$  be s.t.

$$\Pi(P) \geq \sum_1^m q_i (E_i'P + PE_i), \quad |q_i| < 1 \quad (6.31)$$

where  $P=P' \geq 0$  is the solution of

$$A_0'P + PA_0 + PB_0B_0'P + C_0'C_0 + \Pi(P) = 0 \quad (6.32)$$

It is easy to see that such a  $P$  satisfies the inequalities at all points in the parameter box and, consequently, that the performance of the system is guaranteed. Performance was associated with the infinity norm of the difference between the transfer functions of the original and postfailure systems. A bounding function  $\Pi(P)$  is introduced in various forms as in [PH86] and [BH88]. We now study the conservatism of the single Riccati equation approach with the Peterson-Hollot bounds [PH86] for the perturbation in the "A" matrix only. This bounding function is frequently used by different authors.

$$\text{Let} \quad \Pi_1(P) = \sum_1^m | (E_i'P + PE_i) | \quad (6.33)$$

meaning that  $2x' | PE_i | x \geq 0$  for every  $x \in R^n$ . This represents the natural choice for  $\Pi(P)$ , but the corresponding equation is difficult to solve. Hence, the previous form of  $\Pi(P)$  has to be modified in order to obtain a more convenient one.

The matrices  $E_i$  are, or can be transformed to be, of rank one. A possible transformation is a presentation of the matrix  $E_i$  as the sum of rank one matrices. This implies that they can be represented as the product of two vectors, i.e.  $E_i = b_i c_i'$ . Therefore, we have:

$$x' \Pi_1(P) x = 2 \sum_1^m | (x'P b_i) (c_i'x) | \quad (6.34)$$

where the expressions in parentheses are scalar. By using the well known inequality for scalars  $x, y$  and  $z_i \neq 0$ ,  $2 | xy | \leq x^2 / z_i^2 + y^2 z_i^2$ , we have :

$$\Pi(P) = \sum_1^m (1/z_i^2) (P b_i b_i' P) + \sum_1^m z_i^2 (c_i' c_i) \geq \Pi_1(P) \quad (6.35)$$

If  $z_i = 1$  for all  $i = 1: m$ , the following holds:

$$\Pi(P) = P B_1 B_1' P + C_1' C_1 \quad (6.36)$$

After substituting (6.36) into (6.32), the resulting Riccati equation becomes :

$$A_0' P + P A_0 + P [B_0 B_0' + B_1 B_1'] P + [C_0' C_0 + C_1' C_1] = 0 \quad (6.37)$$

which guarantees the nonexpansivity of the two input - two output system in Figure 6.2 with the loop broken around the matrix  $Q$ .

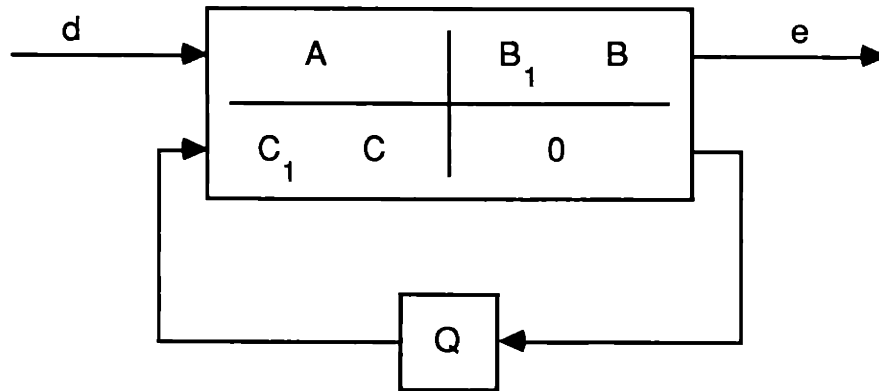


Figure 6.2 The Lower Linear Fractional Representation of Perturbed System

The matrix  $Q$  is a perturbation matrix and it is defined as :

$$Q = \text{diag} \{ q_1 I_1, q_2 I_2, \dots, q_m I_m \} \quad (6.38)$$

where identity matrices  $I_i$  have dimension equal to the rank of the corresponding matrix  $E_i$ .

If  $P=P'>0$  exists as the solution to (6.37), performance and stability conditions are simultaneously satisfied. Nevertheless, the Riccati equation in (6.32) with a bound as in (6.35),

$$A_0'P+PA_0+PB_0B_0'P+C_0'C_0+\sum_1^m (1/z_i^2)(P b_i b_i' P) + \sum_1^m z_i^2(c_i'c_i) = 0 \quad (6.39)$$

can be also viewed as a special case of the  $(m+1)$  block " $\mu$ " problem with scalar scaling factors  $z_i$ . In the case of fixed scaling factors, we can make the following claim.

### Claim 6.3.1

The stability and performance condition for the system in Figure 6.3, given by Riccati equation (6.37), is equivalent to the condition of the "Small Gain Theorem" for the same system.

Proof:

The existence of  $P=P'$  that satisfies (6.37) guarantees that the infinity norm of the augmented system  $M1(s) = [C_0' \ C1']' [sI - A_0]^{-1} [B_0 \ B1]$  is less than or equal to one. This is also the condition of the "Small Gain Theorem (SGT)" for robust stability and performance of the same system since  $\|Q\|_\infty < 1$  and  $\gamma = 1$ . On the other hand, the condition  $\|M1\|_\infty \leq 1$  from "SGT" implies that there is a matrix  $P=P'$  that satisfies equation (6.37) with arbitrarily small error. This completes the proof.

■

The above statement shows how information about real parameter perturbations can be easily lost in the formulation of the stability and performance problem. Furthermore, it shows that the single Riccati equation approach with the Peterson-Hollot bounding function is as conservative as if real parameter perturbations were treated as unstructured uncertainties.

### 6.3.1 Real Parameter Perturbation in "B" and "C" Matrices

In the case where parameter perturbations enter linearly in the "B" and "C" matrices also, an additional step in obtaining a single Riccati equation is needed. Since these matrices appear in the latter in quadratic form, a linearization is necessary in order for Lemma 6.2 to hold. It will be shown that this makes the solution significantly more conservative.

There are different ways to obtain the above described linearization. The approach we took in section 6.2 resulted in the following Riccati equation :

$$\begin{aligned}
R_1(q) = R_o + \sum_1^m q_i [ (E_i'P + PE_i) + P(F_i B_o' + B_o F_i')P + (G_i' C_o + C_o' G_i) ] + \\
+ \alpha_B^2 P P + \alpha_C^2 I
\end{aligned} \tag{6.40}$$

The terms corresponding to the perturbation in the "A" matrix were bounded from above with the Peterson-Hollot bounding function  $\Pi(P)$ . Following the same procedure, and assuming without loss in generality that the matrices  $(F_i B_o)$  and  $(G_i' C_o)$  are of rank one, we can construct the following bounding functions :

$$\Pi_B = P (B_2 B_2' + B_3 B_3') P \geq \sum_1^m q_i P (F_i B_o' + B_o F_i') P ; \quad \forall q \in \Omega \tag{6.41}$$

$$\Pi_C = C_2' C_2 + C_3' C_3 \geq \sum_1^m q_i (G_i' C_o + C_o' G_i) ; \quad \forall q \in \Omega \tag{6.42}$$

The overall single Riccati equation is, therefore, defined as :

$$\begin{aligned}
R_L = R_o + \Pi + \Pi_B + \Pi_C + \alpha_B^2 P P + \alpha_C^2 I = \\
= A_o' P + P A_o + P (B_o B_o' + B_1 B_1' + B_2 B_2' + B_3 B_3' + \alpha_B^2 I) P + \\
+ (C_o' C_o + C_1' C_1 + C_2' C_2 + C_3' C_3 + \alpha_C^2 I) = 0
\end{aligned} \tag{6.43}$$

The existence of  $P=P' \geq 0$  that satisfies (6.43) is the condition for nonexpansivity of the augmented system whose input matrix is  $[ B_o \ B_1 \ B_2 \ B_3 \ \alpha_B I ]$  and the output matrix is  $[ C_o' \ C_1' \ C_2' \ C_3' \ \alpha_C I ]'$ . This at the same time guarantees that the system will be stable and nonexpansive for all  $q \in \Omega$ .

It is obvious that the described method increases the conservatism of the former approach where the perturbation was contained in the "A" matrix only. Therefore, it is even



more conservative than if the "Small Gain Theorem" was used and the real parameter perturbation were treated as frequency dependent uncertainty.

#### 6.4 Concluding Remarks

In this chapter we have introduced methods for analyzing the stability and performance of the postfailure system with real parameter perturbations. The analysis was based on the Riccati equation approach. We have formulated the conditions for the stability and nonexpansivity of the perturbed system that required the existence of a matrix  $P=P'$  that simultaneously satisfies a set of Riccati type inequalities. Algorithms were introduced for obtaining the largest "box" in parameter space where the performance is guaranteed.

Furthermore, a way of constructing an overall single Riccati equation whose solution, if it exists, guarantees stability and performance in the defined region of the parameter space was presented. The Peterson-Hollot bounding function was used in constructing this equation when the perturbation was restricted to the "A" matrix only. We have shown the derived condition to be as conservative as the "Small Gain Theorem" for the same problem. In other words, we have demonstrated that the analysis methodology based on the single Riccati equation and Peterson-Hollot bounds is as conservative as if the parameter perturbation were treated as unstructured uncertainty. In the general case when the perturbation appears in all the elements of the state space representation of the system, we have shown that the single Riccati equation approach can be even more conservative than the "Small Gain Theorem."

## CHAPTER 7

### GAIN RESCHEDULING METHOD

#### 7.1 Introduction

In the previous chapter we presented performance and stability analysis methods based on the Riccati equation approach for systems with real parameter perturbation. The idea was to use the available information about the perturbation in a less conservative way than the classical methods for the unstructured uncertainties would do it. Unfortunately, we have shown that the assumptions we had to make in order to come up with reasonable criteria have resulted in the extensive conservatism. More specifically, the criterion based on a single Riccati equation was even more conservative than if the real perturbation were treated as frequency dependent uncertainty.

Nevertheless, the overall Riccati equation condition that requires the nonexpansivity of the augmented system in (6.43) can also be used in synthesis. By using the algorithm for infinity norm minimization [DGKF88] presented in Chapter 2, we can possibly obtain a compensator that guarantees nonexpansivity of the perturbed system. Hence, it will guarantee stability and performance of the original system in the hyperbox. But due to the conservatism of the criterion, the inability to obtain such a compensator wouldn't necessarily mean that one that achieves performance and stability requirement doesn't exist.

Therefore, we couldn't derive a general synthesis methodology based on the Riccati equation approach that would treat real parameter perturbation less conservatively than as

unstructured uncertainty. In this chapter we present a somewhat different design approach in dealing with real parameter perturbation that can be reflected at the input to the plant. In other words, we restrict our attention to failures which can be treated as gain changes of the original system. We assume that the postfailure value of the gain is not known exactly but belongs to a given set. The redesign methodology based on the gain rescheduling of the original compensator in order to minimize the loss in performance is introduced.

## 7.2 Gain Rescheduling Method

An actuator failure is one of the frequently encountered problems in control systems. In Chapter 5 we have treated a partial actuator failure on the F-8 Aircraft. Due to the failure detection and estimation schemes, we have assumed that the postfailure dynamics of the actuator were described with its new nominal model and the associated unstructured uncertainty. According to the obtained nominal model, the failure has resulted in the change of the gain and the bandwidth of the actuator.

In this chapter we deal with the situation where the failure results only in change of the actuator gain. Furthermore, we assume that the postfailure estimation results in the set of all possible new values of the gain. The "center" point of the set might be chosen as the new nominal point, but any other point in the set might be considered as well.

Let the postfailure plant be

$$P(s) = P_p(s) \text{diag}\{\alpha_1, \dots, \alpha_m\} = P_p(s) Q \quad (7.1)$$

where the set of possible gain values is then given as

$$\Omega = \{ Q \text{ s.t. } Q = \text{diag}\{\alpha_1, \dots, \alpha_m\} \text{ and } r_i \leq \alpha_i \leq s_i < \infty \} \quad (7.2)$$

The design of a postfailure controller was posed in Chapter 3 as the following min-max

problem :

$$\min_{K \in K_s} \max_{Q \in \Omega} \| W \{ S(Q, K) - S_p \} \|_{\infty} \quad (7.3)$$

where  $K_s$  is the set of all internally stabilizing compensators for the whole set  $\Omega$ . The corresponding sensitivity transfer functions before and after the failure are:

$$S_p = [ I + P_p K_p ]^{-1} \quad (7.4)$$

and

$$S(Q) = [ I + P(Q) K ]^{-1} \quad (7.5)$$

It is obvious that if the value of the uncertain gain matrix were known exactly,  $Q=Q_1$  for example, and if none of the control channels were opened, the stabilizing compensator that minimizes (7.3) would be

$$K_1(s) = Q_1^{-1} K_p(s) \quad (7.6)$$

In the general case, the only information we have is the set  $\Omega$ . According to the results in Chapter 6, the available design procedure based on the Riccati equation would treat each real parameter perturbation, i.e. gain, as unstructured uncertainty. In order to avoid the extreme conservatism of that approach, we first try to see if we can keep the structure of the prefailure compensator intact as much as possible while changing its gains only.

Therefore, we restrict the set of all stabilizing compensators  $K_s$  in (7.3) to

$$K_{sr} = \{ K(s) \text{ s.t. } K = \text{diag}\{\beta_1, \dots, \beta_m\} K_p = G K_p, G \in \Gamma \} \quad (7.7)$$

where

$$\Gamma = \{ G \text{ s.t. } G = \text{diag}\{\beta_1, \dots, \beta_m\}, \\ S_p(G, Q) = [ I + P_p Q G K_p ]^{-1} \in RH_{\infty} \quad \forall Q \in \Omega \} \quad (7.8)$$

Then,  $\Gamma$  represents the set of possible gain changes in the prefailure compensator such that the overall system remains internally stable for all the given plant gain variation in  $\Omega$ , i.e.  $\Gamma K_p \in K_s$  for given  $\Omega$ . In order to define  $\Gamma$  we have to evaluate the compact set  $\Lambda$  of the

overall gain perturbation from both plant and compensator that do not destabilize the original system. Let it be given as

$$\Lambda = \{ Z \quad \text{s.t.} \quad Z = \text{diag} \{z_i\}, \text{Sp}(Z) \in \text{RH}_\infty \quad a_i \leq z_i \leq b_i \} \quad (7.9)$$

Therefore,  $\Lambda$  is a compact set of the combined gain changes in both compensator  $K_p$  and plant  $Sp$  that don't destabilize the prefailure system. By knowing  $\Lambda$  and  $\Omega$ , we can always define a set of compensator gain changes  $\Gamma$  that stabilize the system. If  $\Gamma$  is not empty, we have the freedom to optimize the performance criterion. The min-max problem defined in (7.3) can now be redefined as:

$$\min_{G \in \Gamma} \max_{Q \in \Omega} f(Z) \quad (7.10)$$

$$\text{where } f(Z) = \|W \{Sp(Q, G) - Sp\}\|_\infty \text{ and } Z = QG \in \Lambda. \quad (7.11)$$

Since this methodology consists of changing the original compensator gains in order to meet stability and performance specifications for the postfailure system, we will refer to it as the "Gain Rescheduling Method."

If the performance index is specified a priori to some value  $\gamma$ , the problem becomes search for a scaling  $G \in \Gamma$  s.t.

$$\max_{Q \in \Omega} f(Z) \leq \gamma \quad (7.12)$$

In order to solve the above problem, we have to evaluate the set  $\Lambda$  where stability is preserved under the gain perturbation. This can be done in different ways. One of the nonconservative approaches is Safonov and de Gaston's method [DGS88]. Once we get to know the stability region defined by  $\Lambda$ , we can start looking into the properties of the

function  $f(Z)$  over it.

**Lemma 7.1**

Let  $Sp(Z) = [I + Pp Z Kp]^{-1}$ ,  $Z \in \Lambda$  where  $Z$  was defined in (7.9). Then the function

$$f(Z) = \| W (Sp(Z) - Sp) \|_{\infty} \quad ; \quad W(s), Sp(s) \in RH_{\infty} \quad (7.13)$$

is continuous.

Proof :

The expression in (7.13) can be rewritten as:

$$\begin{aligned} f(Z) &= \| W (Sp(Z) - Sp) \|_{\infty} = \\ &= \| W [ Sp ( PpKp - PpZKp) Sp(Z) ] \|_{\infty} = \\ &= \| WSpPp ( I - Z ) KpSp(Z) \|_{\infty} \end{aligned} \quad (7.14)$$

Let  $Z_0 \in \Lambda$  be fixed. The function (7.13) is continuous if  $f(Z) \rightarrow f(Z_0)$  when  $\| Z - Z_0 \| \rightarrow 0$  and  $Z \in \Lambda$ . By using the triangular inequality and properties of the norm, we have

$$\begin{aligned} | f(Z) - f(Z_0) | &\leq \| W Sp(Z_0) Pp ( Z_0 - Z ) Kp Sp(Z) \|_{\infty} \leq \\ &\leq \| W Sp(Z_0) Pp \|_{\infty} \| ( Z_0 - Z ) \| \| Kp Sp(Z) \|_{\infty} \leq \\ &\leq k \| ( Z_0 - Z ) \| \end{aligned} \quad (7.15)$$

where  $k$  is a constant. The above result holds since both  $\| WSp(Z_0)Pp \|_{\infty}$  and  $\| KpSp(Z) \|_{\infty}$  are bounded. This is true because the corresponding transfer functions inside the norms are stable since  $Z, Z_0 \in \Lambda$ . By letting  $\| Z - Z_0 \| \rightarrow 0$  in (7.16) we get  $\lim f(Z) = f(Z_0)$ . This completes the proof. ■

The previous lemma shows the continuity of the infinity norm (7.11) with respect to

changes in gain. Another result that would be useful is its monotonicity. Unfortunately, we can only prove that it has an upper bound which itself is strictly increasing function with respect to the norm of perturbation.

### Lemma 7.2

For all  $Z$  s.t.  $\|Z\| < 1/\|K_o S_o P_o\|_\infty$ , the function  $f(Z)$ , defined in (7.13), is bounded from above with

$$g(Z) = \|Z\| \|W S_p P_p\|_\infty \|K_p S_p\|_\infty [1 - \|Z\| \|K_p S_p P_p\|_\infty]^{-1} \quad (7.16)$$

Furthermore,  $g(Z)$  is monotonically increasing with respect to  $\|Z\|$ .

Proof :

The function  $f(Z)$  can be rewritten as follows:

$$\begin{aligned} f(Z) &= \|W (S_p(Z) - S_p)\|_\infty = \\ &= \|W \{ [I + P_p Z K_p]^{-1} - S_p \}\|_\infty = \\ &= \|W S_p \{ [I + P_p Z K_p S_p]^{-1} - I \}\|_\infty = \\ &= \|W S_p \{ I - P_p Z [I + K_p S_p P_p Z]^{-1} - I \}\|_\infty = \\ &= \|W S_p P_p Z [I + K_p S_p P_p Z]^{-1}\|_\infty \end{aligned} \quad (7.17)$$

When  $\|Z\| \|K_p S_p P_p\|_\infty < 1$ , the stability is guaranteed by the "Small Gain Theorem."

Under this condition and due to the properties of the infinity norm, we have

$$f(Z) \leq \|W S_p P_p\|_\infty \|Z\| [1 - \|Z\| \|K_p S_p P_p\|_\infty]^{-1} = g(Z) \quad (7.18)$$

It is easy to see that the above function is monotonically increasing with respect to  $\|Z\|$ .

This ends the proof. ■

If  $f(Z)$  had been monotonic with respect to the increase of the individual gain, that would have enabled us to solve the previously defined optimization problem by checking the function only on the boundaries of the uncertainty region in  $\Lambda$ . Since this is not the case, we need to evaluate  $f(Z)$  over the whole set  $\Lambda$ . This restricts the application of this approach to the case where this evaluation can be done easily. This includes either the case where only one gain is perturbed or the case when various gains are perturbed simultaneously with the same scaling factor. The cases with more than one gain perturbation can also be treated but that would require extensive computational time and memory size in evaluating  $f(Z)$ . Nevertheless, due to a variety of cases where there is uncertainty in one gain only, and due to the extreme conservatism of the previously described methods, the Gain Rescheduling Method can be useful. This is especially true for the problem defined in (7.10), where a solution, if it exists, requires a simple change in the gain of the original compensator but not a change in its dynamics. This can be done on-line since  $f(Z)$  can be previously evaluated and stored in memory. After a failure has occurred and the estimate of the uncertain gain was obtained, the method can be used. If it doesn't provide a satisfactory solution, the design based on the Riccati equation can be performed.

### **7.3 Example for the "Gain Rescheduling Method"**

In this section we use the "Gain Rescheduling Method" on the model of F-8 Aircraft assuming that a failure has occurred in the actuator corresponding to the flaperon. Furthermore, we assume that the failure has affected the gain of the actuator but not its dynamics. Due to the failure detection and estimation scheme, we have obtained the information about the set  $\Omega$  of possible values of the changed gain but not its exact value.



Let it be given as

$$\Omega = \{ \alpha \text{ s.t. } 0.2 \leq \alpha \leq 0.3 \}$$

(7.19)

This situation is presented in the following figure.

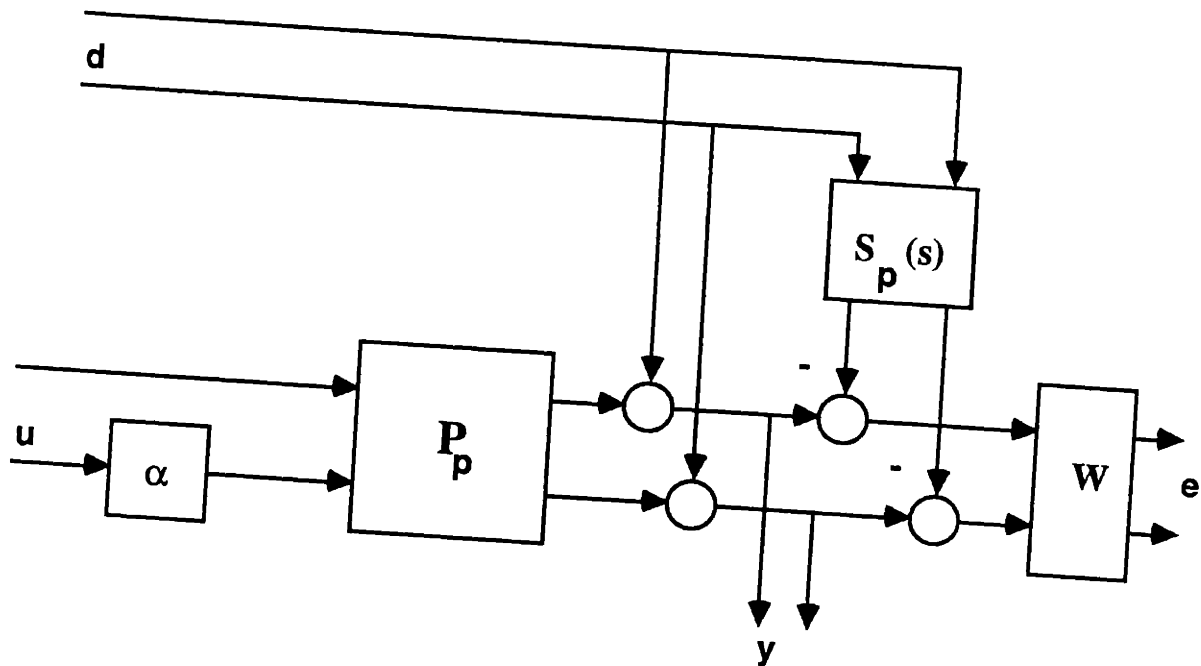


Figure 7.1 Gain Perturbation in Control Channel

To be able to see the impact of this failure, we have to evaluate the stability region  $\Lambda$  with respect to the varying gain in the second control channel and to evaluate the loss of performance over it. We look at the function  $f(z)$  defined in (7.11) where  $z$  represents the overall gain change in the system due to both compensator and plant gain changes. The function has the following form :

$$\begin{aligned}
f(z) &= \| W \{ S_p(z) - S_p \} \|_{\infty} = \\
&= \| W S_p \{ [ I + (z-1) P_p v v' K_p S_p ]^{-1} - I \} \|_{\infty} = \\
&= \| W S_p \{ I - (z-1) \frac{S_p P_p v v' K_p S_p}{1 + (z-1) v' K_p S_p P_p v} - I \} \|_{\infty} = \\
&= \| \frac{(z-1) W S_p P_p v v' K_p S_p}{1 + (z-1) v' K_p S_p P_p v} \|_{\infty} \quad (7.20)
\end{aligned}$$

where  $v' = [ 0 \ 1 ]$  and  $W(s)$  is defined in (5.15).

It is easy to see that stability of the perturbed system depends on the single input - single output transfer function  $[ 1 + (z-1) v' K_p S_p P_p v ]$ . Using the Nyquist type argument, we can show that the stability is guaranteed for  $0.1 < z < 5.5$  which at the same time defines the set  $\Lambda$ . Furthermore, it is easy to see that  $f(1) = 0$  and that, for large  $z$ ,  $f(z)$  approaches the constant value defined by :

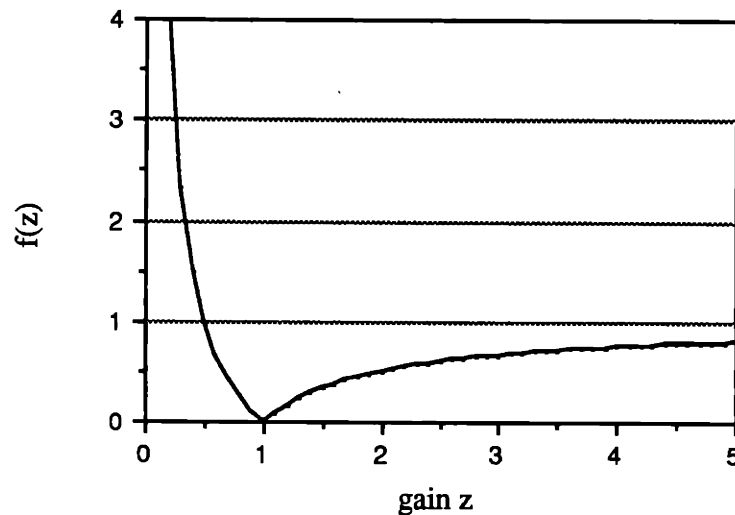
$$\lim_{z \rightarrow \infty} f(z) = \| \frac{W S_p P_p v v' K_p S_p}{v' K_p S_p P_p v} \|_{\infty} \quad (7.21)$$

After defining the set of gain changes that do not destabilize the system, we can evaluate the function  $f(z)$  over it. This is shown in Figure 7.2.

From this figure we can see that the system remains stable in the whole set  $\Omega$  of the uncertain gain  $\alpha$ . Nevertheless, the function  $f(z)$  takes values between  $f(0.2) = 4$  and  $f(0.3) = 2.33$ . This represents significant loss in performance from the original system. Therefore, there is a justification in rescheduling the corresponding gain of the compensator  $K_p(s)$ . The new compensator is now defined as

$$K(s) = \text{diag}\{ 1, \beta \} K_p(s) \quad (7.22)$$

s.t.  $\alpha\beta \in \Lambda \quad \forall \alpha \in \Omega$ .

Figure 7.2 Function  $f(z)$  over  $\Lambda$ 

By choosing  $\beta$  appropriately, we can scale the uncertainty set  $\Omega$  such that its point  $(1/\beta) \in \Omega$  becomes a new nominal point where the loss in performance is equal to zero.

The optimization problem (7.10) is now nothing more than a search for a scaling  $\beta^*$  that guarantees the minimum extreme value of  $f(z)$  over  $\beta^* \Omega$ . According to Figure 7.2,  $f(z)$  in this case behaves as a monotonically increasing function. Therefore, the solution will be the one that makes

$$f(\beta^* 0.2) = f(\beta^* 0.3) \quad (7.23)$$

It turns out that the  $\beta^* = 4.2$ . The uncertainty region is scaled to  $\beta^* \Omega = [0.84 : 1.26]$  and the value of  $f(z)$  at its end points is  $f(0.84) = f(1.26) = 0.2$ . This value at the same time characterizes the largest loss in performance due to the uncertain gain. The scaled uncertainty set includes  $z=1$  which is the nominal point where the loss in performance is equal to zero. It is important to notice that the set is not symmetric around the nominal

point. Therefore, by changing the appropriate gain of the original compensator, we have reduced the maximal loss in performance from the value of 4 to 0.2 only. The scaled uncertainty set and the maximum values of  $f(z)$  over it are shown in the following figure.

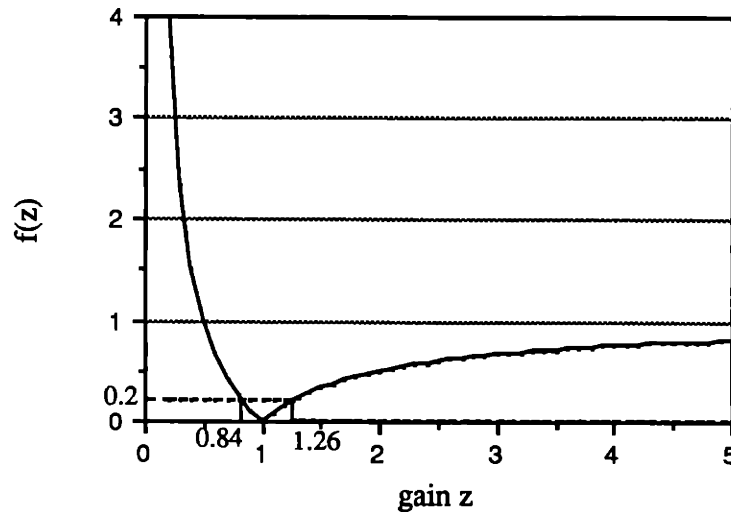


Figure 7.3 Loss in Performance After the Optimal Scaling  $\beta^* = 4.2$

In the above example, the function  $f(z)$  behaved as a monotonically increasing function. This reduced a search for the optimal  $\beta$  to a choice of scaling factor that, on the edges of the scaled set, makes the function have equal values. Furthermore, this case has guaranteed that the scaled set would include the nominal point where the loss in performance was zero.

In a general case, the function  $f(z)$  can have a pattern different from the one encountered above in the sense that it may not be monotonic. Then the optimal scaling would minimize the extreme value of the function that doesn't have to occur at the edges of the set any more.

## 7.4 Concluding Remarks

In this chapter we have presented a design procedure for the postfailure system with real parameter perturbation that can be reflected at the plant input. This was done to avoid the extreme conservatism of the method based on the Riccati equation that was previously described. The assumption was made that the failure estimation and detection scheme would provide the information about the set of the uncertain parameters. Furthermore, the set of all stabilizing compensators was reduced to the one where the dynamics of the compensators was kept equal to that of the original compensator  $K_p$ . The only differences among them were allowed in the output gain matrix.

The loss in performance was defined as the value of the norm of the weighted difference between appropriate transfer functions of the original and the postfailure system. It was shown to be a continuous function of the gain change but not necessarily monotone. The fact that the evaluation of the norm is needed over the whole uncertainty region limits the application of the method. An example based on the model of F-8 Aircraft was treated where the uncertain parameter was treated as the gain change in one of the control channels. The optimum scaling for the corresponding channel of the original controller was found in order to minimize the performance loss over the uncertainty region.

## CHAPTER 8

### CONCLUSION

#### 8.1 Summary

A failure in the existing control system usually leads to significant loss in performance and even instability. Therefore, a redesign of the controller is needed in order to assure stability of the postfailure system and regain the original performance.

The postfailure controller redesign problem was treated in detail herein under the assumption that information about the "failed" plant was available. This information was said to consist of the nominal model of the postfailure plant and the associated uncertainty. The redesign problem was divided in the analysis and synthesis parts.

The analysis of the postfailure system was associated with quantifying the loss in performance due to the failure. This was done by introducing the performance index  $\gamma$ . It represents the value of the norm of the difference between transfer functions associated with the corresponding outputs of the original and the postfailure system subjected to the same input. Since the signals were assumed to be in  $L_2$ , the induced norm on these transfer functions was the " $H_\infty$ " norm. The uncertainty associated with the model of the postfailure plant was taken into account in evaluation of  $\gamma$ . If the obtained value of the performance index over the given set of modeling uncertainty was greater than the previously set bound, the control system had to be redesigned.

The synthesis part was defined as the minimization of the performance index in the presence of uncertainty over the set of all stabilizing compensators for the nominal model of the postfailure plant. Therefore, the overall design was transformed into a model matching

problem in the presence of multiple uncertainty. The cases of dynamic, norm bounded uncertainty and the magnitude bounded real parameter perturbation were treated. This procedure was shown to be applicable on-line if the necessary information from the failure detection and identification steps could be obtained fast enough. For expected failures, an off-line design methodology was proposed. The nominal model of the failed plant was estimated in advance, while the magnitude of the associated uncertainty was left to be evaluated on-line. A sequence of compensators was designed so that each of them had tolerance to uncertainty of different magnitude. Once the failure occurs, there is hopefully going to be a stabilizing compensator to minimize its impact.

The design algorithm derived for the case with unstructured uncertainty was based on the "D-K" iteration [Doyle83]. The latter was modified in order to enlarge the stability margin as much as possible without increasing the order of the resulting compensator. The convergence and properties of the algorithm were presented. The design example demonstrated that the algorithm results in increasing the margin of modeling uncertainty while guaranteeing stability and performance of the postfailure system. Influence of the choice of initial conditions was shown.

With the uncertainty given as highly structured real parameter perturbation in the state space representation of the postfailure plant, the Riccati equation approach was undertaken. The numerical complexity and the conservatism of different algorithms were discussed. In the approach based on a single Riccati equation and the Peterson-Hollot bounding function, it was shown that the conservatism of the latter equals that of the "Small Gain Theorem." In the more general case where the perturbation appears in all elements of the postfailure system state space representation, it was shown that treating the real uncertainty as frequency dependent could be less conservative than if the single Riccati equation approach was used.

Attention is now directed towards future research.

## 8.2 Directions for Future Research

Since the postfailure controller redesign is transformed into a stability and performance robustness problem in the presence of multiple uncertainties, it inherits all the difficulties of the latter. Therefore, progress in the controller redesign methodology is closely connected to progress in the treatment of unstructured uncertainty and real parameter perturbation.

The synthesis methods in the presence of multiple unstructured uncertainty is a difficult problem due its nonconvex nature. The "D-K" iteration usually results in a local solution and depends on the accuracy of approximation of the "D" scaling. These properties are inherited by the algorithm for maximizing the stability margin presented in Chapter 4. Hence, any new design method for multiple unstructured uncertainties is of high interest for the postfailure controller redesign.

It has not been possible to address the problem of simultaneous unstructured and real parameter uncertainties nonconservatively with existing methods. This thesis shows that the Riccati equation approach can be more conservative than if the real uncertainty were treated as frequency dependent. In the case of a single Riccati equation, different bounding functions might guarantee less conservatism than the Peterson-Hollot bounds. In order to decrease the conservatism one might have to resort to computationally intensive numerical methods. Specialized numerical algorithms for simultaneously solving a set of Riccati equations can result in the less conservative results.

An attempt to solve this problem differently might consist of deriving the set of all stabilizing compensators that guarantee a certain value of the infinity norm and, therefore, stability against unstructured uncertainty. This is possible by using the Doyle-Glover algorithm [DGKF88] with the matrix  $Q$  as a free parameter. A compensator that guarantees stability against real uncertainty will then be obtained by choosing the appropriate matrix  $Q$  according to a criterion that has to be developed.



Possible improvements in dealing with postfailure situations can also result by deriving methods for treatment of special cases. These can be failures that occur only in the input or output matrices of the original system state space representation or any other location that enables specialized approaches.

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