On mean flow generation due to oblique reflection of internal waves at a slope

On Mean Flow Generation due to Oblique Reflection of Internal Waves at a Slope

By Takeshi Kataoka and T. R. Akylas

Small-amplitude expansions are utilized to discuss the mean flow induced by the reflection of a weakly nonlinear internal gravity wave beam at a uniform rigid slope, in the case where the beam planes of constant phase meet the slope at an arbitrary direction, not necessarily parallel to the isobaths, and the flow cannot be taken as two-dimensional. Along the vertical, the Eulerian mean flow due to such an oblique reflection is equal and opposite to the Stokes drift so the Lagrangian mean flow vanishes, similar to a two-dimensional reflection. The horizontal Eulerian mean flow, however, is controlled by the mean potential vorticity (PV) and the corresponding Lagrangian mean flow is generally non-zero, in contrast to two-dimensional flow where PV identically vanishes. For an oblique reflection, furthermore, viscous dissipation can trigger generation of horizontal mean flow via irreversible production of mean PV, a phenomenon akin to streaming.

1. Introduction

Internal gravity waves in a continuously stratified fluid are inherently anisotropic due to gravity which provides a preferred direction, and the frequency of a plane

It is a pleasure to contribute this paper to the special issue in honor of Professor Roger Grimshaw on the occasion of his 80th birthday. Professor Grimshaw has been a pioneer in the study of nonlinear waves in fluids. We are grateful to him for all the wonderful things he has taught us about science and for his wisdom, kindness and friendship.

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wave depends solely on the inclination of the wave vector to this direction. As a result, internal waves obey unusual rules of reflection at a sloping boundary: the incident and reflected wave rays make the same angle to the direction of gravity, rather than the normal to the boundary as would be the case for isotropic wave propagation [1]. Furthermore, because the internal wave frequency is independent of the wave vector magnitude, the same reflection rules also apply to wave beams. These are time-harmonic plane waves that propagate at a given angle as determined by the dispersion relation, but whose spatial profile involves a superposition of sinusoidal plane waves with wave vectors of different magnitude but fixed direction [2]. Internal wave beams arise in various geophysical contexts, particularly in connection with the ocean internal tide [3,4] and the generation of atmospheric gravity waves due to thunderstorms [5].

An interesting feature of internal wave beam reflection at a rigid boundary is the possibility of radiating beams at higher harmonics of the incident wave frequency. Such secondary reflected beams were noted in numerical simulations of the internal tide [3] and later explained theoretically [2] as the result of nonlinear interactions between the incident and the primary reflected beam. These interactions occur in the region where the primary-harmonic beams overlap close to the boundary, and give rise to a second-harmonic and a mean disturbance (to leading, quadratic order). If the second harmonic is below the buoyancy frequency, the former of these disturbances then radiates away as a secondary beam at an angle determined by the dispersion relation. The induced mean flow, by contrast, is confined in the interaction region and results in no net transport as the associated Lagrangian mean flow turns out to be zero. The radiation of secondary reflected beams has also been confirmed experimentally [6,7].

In the present paper, we revisit the reflection of an internal wave beam at a sloping boundary using the weakly nonlinear approach of [2]. Similar to the earlier study, the effects of background rotation are not considered. Here, however, the focus is on the case where the beam planes of constant phase meet the slope at an arbitrary direction, not necessarily parallel to the isobaths, so unlike [2] the flow cannot be taken as two-dimensional. Such an oblique reflection can

radiate secondary reflected beams as before, but there are serious ramifications in regard to the induced mean flow. Along the vertical, the Lagrangian mean flow vanishes as in a two-dimensional reflection, but the horizontal mean flow is now tied to the evolution of mean potential vorticity (PV). The induced horizontal mean flow cannot be determined by steady-state analysis alone, and the corresponding Lagrangian mean flow is generally non-zero. A similar ambiguity regarding the induced along-slope mean flow at steady state was found in the special case of obliquely reflecting sinusoidal plane waves [8]. Furthermore, the evolution equation for the mean PV due to an oblique reflection reveals that viscous dissipation can trigger the generation of horizontal mean flow via irreversible production of mean PV [9]. This mechanism is akin to that responsible for 'streaming' due to attenuating waves in acoustics [10] and propagating internal wave beams in the presence of three-dimensional variations [11–13].

2. Formulation and linear solution

The present analysis assumes the same general setting as [2], namely a propagating time-harmonic internal wave beam that reflects at a rigid slope of constant angle α to the horizontal, in a uniformly stratified Boussinesq fluid with (constant) buoyancy frequency N_0. Here, however, the beam is taken to meet the slope at an arbitrary direction, not necessarily parallel to the isobaths, so the flow is no longer two-dimensional (see Fig. 1). To discuss such an oblique reflection, we shall use the coordinate system (ξ, η, ζ) shown in Fig. 2. Specifically, x and y are horizontal coordinates, y being along the slope isobaths, and z is a vertical coordinate pointing upwards, so the plane $z = x \tan\alpha$, $-\infty < y < \infty$, coincides with the slope boundary. Then, (ξ, η, ζ) is defined such that ζ is perpendicular to this plane and ξ, η are in-plane coordinates with η along the direction where the beam planes of constant phase intersect the slope (see Fig. 1); thus, the angle β of η relative to the isobaths (y-direction) is a measure of the reflection obliqueness, with $\beta = 0$ corresponding to normal (two-dimensional) reflection [2].

An advantage of the rotated coordinates (ξ, η, ζ) is that the flow is indepen-

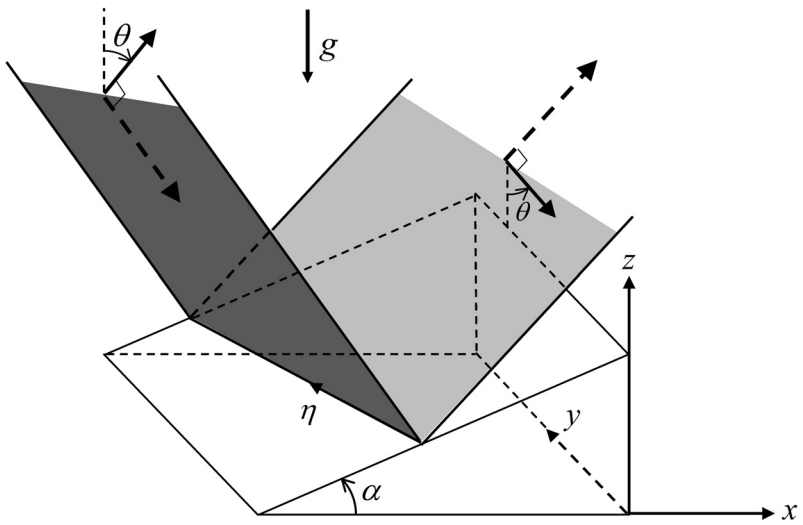

Figure 1. Geometry of obliquely incident and (primary) reflected beam at a slope α. The beam planes of constant phase (shaded) meet the slope along the direction η which is oblique to the isobaths (along y). The propagation direction of each beam (dashed arrow) is inclined to the horizontal by θ, the angle of the wavevector direction (solid arrow) to the vertical, and the beam frequency $\omega = \sin\theta$ in keeping with the dispersion relation.

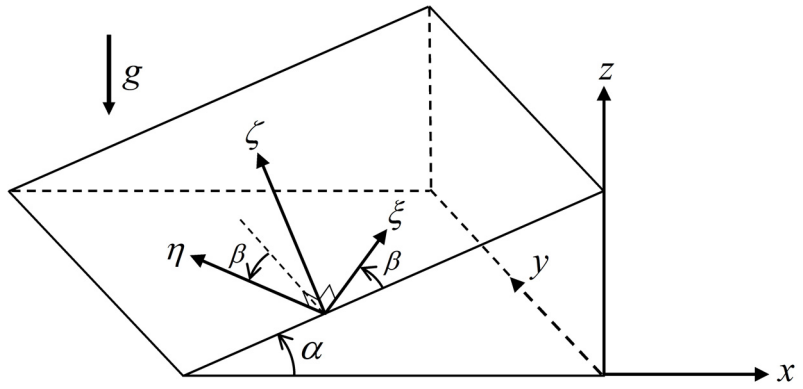

Figure 2. The rotated coordinate system (ξ, η, ζ): ζ is perpendicular to the slope plane $z = x\tan\alpha$, $-\infty < y < \infty$, and ξ, η are in-plane coordinates with η along the direction where the beam planes of constant phase intersect the slope; the angle β that η makes to the isobaths (along y) measures the reflection obliqueness.

dent of η, so incompressibility can be automatically satisfied by introducing a streamfunction $\psi(\xi, \zeta, t)$ for the velocity field (u, w) in the (ξ, ζ) plane:

$$u = \psi_\zeta, \quad w = -\psi_\xi. \tag{1}$$

Then, from mass and momentum conservation, the equations governing ψ, the transverse (along η) velocity $v(\xi, \zeta, t)$ and the density perturbation from hydrostatic equilibrium $\rho(\xi, \zeta, t)$ are

$$\rho_t + J(\rho, \psi) + \psi_\chi + v \sin\alpha \sin\beta = 0, \tag{2a}$$

$$\nabla^2 \psi_t + J(\nabla^2 \psi, \psi) - \rho_\chi = 0, \tag{2b}$$

$$v_t + J(v, \psi) - \rho \sin\alpha \sin\beta = 0, \tag{2c}$$

where

$$J(a, b) = a_\xi b_\zeta - a_\zeta b_\xi, \quad \frac{\partial}{\partial \chi} = \cos\alpha \frac{\partial}{\partial \xi} - \sin\alpha \cos\beta \frac{\partial}{\partial \zeta}, \quad \nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2}.$$

Furthermore, the inviscid boundary condition at the slope is

$$w = 0 \quad (\zeta = 0). \tag{3}$$

(In the equations above and throughout this paper, all variables have been made dimensionless as in [2], employing a characteristic length of the incident wave beam as lengthscale and $1/N_0$ as timescale.)

It will also prove useful for the ensuing analysis to introduce the PV $q(\xi, \zeta, t)$, which in the present setting can be expressed as

$$q = \Omega - J(v, \rho), \tag{4a}$$

where

$$\Omega = -\sin\alpha \sin\beta \nabla^2 \psi + v_\chi \tag{4b}$$

is the vertical vorticity. It can be deduced from equations (2) that in an inviscid fluid q is a materially conserved quantity

$$\frac{Dq}{Dt} = 0, \tag{5}$$

where $D/Dt \equiv \partial/\partial t + J(\,\cdot\,,\psi)$ is the material derivative. In contrast to two-dimensional reflection ($\beta = 0$) where $v \equiv 0$ and hence $q \equiv 0$, for obliquely incident and reflected beams, as discussed later (Sec. 4), PV controls the horizontal mean flow induced by nonlinear interactions.

Assuming weakly nonlinear disturbances, we solve equations (2) subject to the boundary condition (3) using perturbation expansions in an amplitude parameter $\varepsilon \ll 1$:

$$(\psi, \rho, v) = \varepsilon \left(\psi^{(1)}, \rho^{(1)}, v^{(1)}\right) + \varepsilon^2 \left(\psi^{(2)}, \rho^{(2)}, v^{(2)}\right) + \cdots . \tag{6}$$

Upon substituting these expansions into the governing equations, the leading-order problem is

$$\rho_t^{(1)} + \psi_\chi^{(1)} + v^{(1)} \sin\alpha \sin\beta = 0, \tag{7a}$$

$$\nabla^2 \psi_t^{(1)} - \rho_\chi^{(1)} = 0, \tag{7b}$$

$$v_t^{(1)} - \rho^{(1)} \sin\alpha \sin\beta = 0, \tag{7c}$$

$$\psi^{(1)} = 0 \quad (\zeta = 0). \tag{8}$$

Here, the appropriate solution represents the oblique reflection of an infinitesimal-amplitude (linear) beam of frequency $\omega = \sin\theta$, where θ is the beam inclination to the horizontal. Specifically,

$$\left(\psi^{(1)}, \rho^{(1)}, v^{(1)}\right) = (\Psi(\xi,\zeta), R(\xi,\zeta), V(\xi,\zeta))\, \mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}, \tag{9a}$$

where

$$\Psi = \int_0^\infty A(l) \left\{ \exp\left[\mathrm{i}l(\xi + m^{\mathrm{inc}}\zeta)\right] - \exp\left[\mathrm{i}l(\xi + m^{\mathrm{refl}}\zeta)\right] \right\} \mathrm{d}l, \tag{9b}$$

$$R = \frac{-\mathrm{i}\omega\, \Psi_\chi}{\omega^2 - \sin^2\alpha \sin^2\beta}, \quad V = \frac{\sin\alpha \sin\beta\, \Psi_\chi}{\omega^2 - \sin^2\alpha \sin^2\beta}, \tag{9c}$$

and

$$m^{\mathrm{inc}} = \frac{\cos\beta(\omega\, r \cos\theta - \sin\alpha\cos\alpha)}{\omega^2 - \sin^2\alpha},$$

$$m^{\mathrm{refl}} = -\frac{\cos\beta(\omega\, r \cos\theta + \sin\alpha\cos\alpha)}{\omega^2 - \sin^2\alpha}, \tag{9d}$$

with

$$r = \frac{\sqrt{\omega^2 - \sin^2\alpha \sin^2\beta}}{\omega \cos\beta}. \tag{9e}$$

In equation (9b), $A(l)$ is a prescribed function that specifies the incident beam profile; $A(l) = \delta(l - l_0)$, in particular, recovers the oblique reflection of a sinusoidal plane wave [8]. It should be noted that the condition $\omega^2 = \sin^2\theta > \sin^2\alpha \sin^2\beta$, which ensures that r in equation (9e) is real, is always satisfied for an incident beam that is propagating (i.e., it comprises sinusoidal plane waves with real wave vectors pointing in a direction inclined to the vertical by θ, which fixes ω; see Fig. 1), as assumed here. Furthermore, in the limit $\theta \to \alpha$ where the reflected beam lies on the slope, it follows from equations (9 d, e) that $m^{\mathrm{refl}} \to \infty$; thus, the reflected beam thickness tends to zero while the beam velocity field diverges. The possibility of healing this singular behavior by rescaling near the critical angle $\theta = \alpha$ including nonlinear and viscous effects, was examined in [14,15] for the case of two-dimensional reflection. These effects become important when $\theta = \alpha + O(\varepsilon^{2/3})$, and the analysis here assumes that θ is away from this near-critical range.

3. Second harmonic and induced mean flow

We now proceed to compute nonlinear corrections to the linear solution (9). These arise from quadratic interactions in the overlap region of the incident and reflected beams near the slope, as each of these beams separately is an exact nonlinear state [16]. Specifically, the problem governing the $O(\varepsilon^2)$ terms in expansion (6) reads

$$\rho_t^{(2)} + \psi_\chi^{(2)} + v^{(2)} \sin\alpha \sin\beta = F^{(2)}, \tag{10a}$$

$$\nabla^2 \psi_t^{(2)} - \rho_\chi^{(2)} = G^{(2)}, \tag{10b}$$

$$v_t^{(2)} - \rho^{(2)} \sin\alpha \sin\beta = H^{(2)}, \tag{10c}$$

$$\psi^{(2)} = 0 \quad (\zeta = 0), \tag{11}$$

where

$$\left\{ \begin{array}{l} F^{(2)} \\ G^{(2)} \\ H^{(2)} \end{array} \right\} = - \left\{ \begin{array}{l} J(R, \Psi^*) \\ J(\nabla^2\Psi, \Psi^*) \\ J(V, \Psi^*) \end{array} \right\} - \left\{ \begin{array}{l} J(R, \Psi) \\ J(\nabla^2\Psi, \Psi) \\ J(V, \Psi) \end{array} \right\} \mathrm{e}^{-2\mathrm{i}\omega t} + \mathrm{c.c.}. \tag{12}$$

The forcing terms above, as expected, comprise a mean and a second harmonic that result from quadratic interactions of the primary harmonic. Thus, similar to two-dimensional beam reflection [2], the $O(\varepsilon^2)$ correction involves an induced mean flow and a second-harmonic component:

$$\left(\psi^{(2)}, \rho^{(2)}, v^{(2)}\right) = (\Psi_0(\xi,\zeta), R_0(\xi,\zeta), V_0(\xi,\zeta)) + \left\{ (\Psi_2(\xi,\zeta), R_2(\xi,\zeta), V_2(\xi,\zeta))\, \mathrm{e}^{-2\mathrm{i}\omega t} + \text{c.c.} \right\}. \tag{13}$$

Upon substituting expressions (13) into equations (10), the second-harmonic response, which includes a radiating beam if $2\omega < 1$, is qualitatively similar to that found in the case of two-dimensional reflection and can be readily computed by an analogous procedure to that followed in [2]; details are given in the Appendix. This is in sharp contrast to the induced mean flow, however, which cannot be fully determined at this stage. Specifically, equation (10a) specifies the vertical (along z) mean flow velocity,

$$-\Psi_{0\chi} - V_0 \sin\alpha \sin\beta = \frac{-\mathrm{i}\,\omega}{\omega^2 - \sin^2\alpha \sin^2\beta} \frac{\partial J(\Psi, \Psi^*)}{\partial \chi}, \tag{14a}$$

while equations (10 b, c) yield identical information for the mean density perturbation,

$$R_0 = \frac{J(\Psi_\chi, \Psi^*)}{\omega^2 - \sin^2\alpha \sin^2\beta} + \text{c.c.}, \tag{14b}$$

thus leaving the horizontal mean flow undetermined. (Alternatively, equation (14a) expresses V_0 in terms of Ψ_0, which remains undetermined.) It should be noted that this complication does not arise in the case of two-dimensional reflection ($\beta = 0$) where $V_0 \equiv 0$ and Ψ_0 is specified via equation (14a), $\Psi_0 = (\mathrm{i}/\omega)\, J\,(\Psi, \Psi^*)$, also consistent with the slope boundary condition (11).

To compute the $O(\varepsilon^2)$ induced horizontal mean flow, it turns out that one has to carry the perturbation expansions (6) to $O(\varepsilon^4)$. Instead, as the horizontal mean flow is tied to the mean vertical vorticity, it is more instructive to appeal to equation (5) for the PV. The evolution of mean PV is key to determining Ψ_0, as discussed below.

4. Mean PV

Attention is now focused on equations (4) and (5) for the PV q, in order to obtain an equation governing the mean PV and thereby determine the induced horizontal mean flow. To this end, we first compute q by making use of the perturbation expansions (6). Specifically, from equations (4), (9) and (10), it is deduced that the primary and second harmonic carry no PV to leading order, so the dominant contribution to q comes from the mean PV, $\varepsilon^2 Q_0$,

$$q = \varepsilon^2 Q_0 + \varepsilon^3 \left\{ Q_1 \mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.} \right\} + \cdots , \tag{15}$$

where

$$Q_0 = \Omega_0 - \left(J(V, R^*) + \text{c.c.} \right), \tag{16a}$$

with

$$\Omega_0 = -\sin\alpha \sin\beta \, \nabla^2 \Psi_0 + V_{0\chi} \tag{16b}$$

being the mean vertical vorticity. Furthermore, the $O(\varepsilon^3)$ primary-harmonic PV in equation (15) is given by

$$Q_1 = \Omega_1 - \left(J(V_0, R) + J(V, R_0) + J(V_2, R^*) + J(V^*, R_2) \right) \tag{17a}$$

with

$$\Omega_1 = -\sin\alpha \sin\beta \, \nabla^2 \Psi_1 + V_{1\chi}. \tag{17b}$$

Here the vertical vorticity Ω_1 derives from the $O(\varepsilon^3)$ correction to the $O(\varepsilon)$ primary incident and reflected beams in the perturbation expansions (6). In analogy with equation (9a), this correction takes the form

$$\varepsilon^3 \left\{ \left(\Psi_1(\xi, \zeta), R_1(\xi, \zeta), V_1(\xi, \zeta) \right) \mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.} \right\},$$

where Ψ_1, R_1, V_1 satisfy a forced problem obtained by substituting expansions (6) into equations (2) and collecting primary-harmonic terms correct to $O(\varepsilon^3)$. Equations (2 b,c), in particular, yield

$$\mathrm{i}\,\omega \nabla^2 \Psi_1 + R_{1\chi} = G_1, \tag{18a}$$

$$\mathrm{i}\,\omega V_1 + R_1 \sin\alpha \sin\beta = H_1, \tag{18b}$$

where

$$G_1 = J(\nabla^2\Psi_0, \Psi) + J(\nabla^2\Psi, \Psi_0) + J(\nabla^2\Psi_2, \Psi^*) + J(\nabla^2\Psi^*, \Psi_2), \quad (19a)$$

$$H_1 = J(V_0, \Psi) + J(V, \Psi_0) + J(V_2, \Psi^*) + J(V^*, \Psi_2). \quad (19b)$$

Thus, by combining equation (17b) with equations (18) and (19), we find

$$\Omega_1 = -\frac{\mathrm{i}}{\omega}\left(J(\Omega_0, \Psi) + J(V_0, \Psi_\chi) + J(V, \Psi_{0\chi}) + J(V_2, \Psi^*_\chi) + J(V^*, \Psi_{2\chi})\right). \quad (20)$$

Finally, inserting expression (20) into equation (17a) yields

$$\begin{aligned} Q_1 = -\frac{\mathrm{i}}{\omega}J(\Omega_0, \Psi) - J\left(V_0, R + \frac{\mathrm{i}}{\omega}\Psi_\chi\right) - J\left(V, R_0 + \frac{\mathrm{i}}{\omega}\Psi_{0\chi}\right) \\ - J\left(V_2, R^* + \frac{\mathrm{i}}{\omega}\Psi^*_\chi\right) - J\left(V^*, R_2 + \frac{\mathrm{i}}{\omega}\Psi_{2\chi}\right), \end{aligned} \quad (21)$$

which, in view of equations (14) and (A5), can be simplified to

$$\begin{aligned} Q_1 = &-\frac{\mathrm{i}}{\omega}J(\Omega_0, \Psi) \\ &+ \frac{2\sin\alpha\sin\beta}{(\omega^2 - \sin^2\alpha\sin^2\beta)^2}\left\{-J\left[J(\Psi_\chi, \Psi), \Psi^*_\chi\right] + J\left[J(\Psi^*_\chi, \Psi), \Psi_\chi\right]\right\}. \end{aligned} \quad (22)$$

The desired equation for Q_0 is now derived by collecting mean terms in the PV equation (5). Since the primary-harmonic PV is $O(\varepsilon^3)$ according to equation (15), the mean of the convective derivative $J(q, \psi)$ is $O(\varepsilon^4)$ and takes the form

$$\varepsilon^4\left\{J(Q_0, \Psi_0) + (J(Q_1, \Psi^*) + \text{c.c.})\right\}, \quad (23)$$

where Q_1 is given by equation (22). Hence, the mean PV dynamics occurs on the 'slow' time $T = \varepsilon^2 t$ and is governed by the following evolution equation for $Q_0(\xi, \zeta, T)$

$$\frac{\partial Q_0}{\partial T} + J\left(Q_0, \Psi_0\right) + (J(Q_1, \Psi^*) + \text{c.c.}) = 0. \quad (24)$$

(The analysis remains virtually unchanged if the incident and reflected beams also depend on T.)

Equation (24) can be re-cast in a form that is more revealing physically, by first substituting expression (22) for Q_1 and then using the identity$J\left[J(a, b), c\right] =$

$J\left[J(a,c),b\right]-J\left[J(b,c),a\right]$ to simplify the resulting triple Jacobians. After considerable algebra, it transpires that

$$\frac{\partial Q_0}{\partial T}+J\left[Q_0,\ \Psi_0-\frac{\mathrm{i}}{\omega}J(\Psi,\Psi^*)\right]=0. \tag{25}$$

The above form of the mean PV equation shows that Q_0 is convected by a mean flow with streamfunction

$$\overline{\Psi}=\Psi_0-\frac{\mathrm{i}}{\omega}J(\Psi,\Psi^*), \tag{26}$$

which turns out to be the Lagrangian mean flow associated with the (Eulerian) induced mean flow Ψ_0. This can be readily verified by computing the flow particle paths correct to $O(\varepsilon^2)$, from which it follows that the Stokes mean drift streamfunction $\varepsilon^2\Psi^D$ and transverse velocity $\varepsilon^2 V^D$ are

$$\Psi^D=-\frac{\mathrm{i}}{\omega}J(\Psi,\Psi^*), \tag{27a}$$

$$V^D=-\frac{\mathrm{i}}{\omega}\frac{\sin\alpha\sin\beta}{\omega^2-\sin^2\alpha\sin^2\beta}\frac{\partial J\left(\Psi,\Psi^*\right)}{\partial\chi}; \tag{27b}$$

thus, the Lagrangian mean flow (Eulerian mean flow + mean drift) has streamfunction $\overline{\Psi}=\Psi_0+\Psi^D$ and transverse velocity $\overline{V}=V_0+V^D$.

According to equations (27), the vertical (along z) drift is given by

$$-\Psi^D_\chi-V^D\sin\alpha\sin\beta=\frac{\mathrm{i}\,\omega}{\omega^2-\sin^2\alpha\sin^2\beta}\frac{\partial J\left(\Psi,\Psi^*\right)}{\partial\chi}, \tag{28}$$

and cancels the (Eulerian) vertical induced mean flow found earlier (cf. equation (14a)); hence, the Lagrangian vertical mean flow vanishes, as was also found in [8] for obliquely reflecting sinusoidal plane waves. The along-slope (along y) Lagrangian mean flow, $\overline{\Psi}_\zeta\sin\beta+\overline{V}\cos\beta$, is generally non-zero, however, as it depends on Ψ_0, which solves the evolution equation (25) subject to the boundary condition $\Psi_0=0$ on the slope ($\zeta=0$) and an appropriate initial condition at $T=0$. A particular steady-state solution of this problem is

$$\Psi_0=\frac{\mathrm{i}}{\omega}J(\Psi,\Psi^*)=-\Psi^D, \tag{29}$$

in which case not only $\overline{\Psi}=0$, but also $\overline{V}=0$ since $V_0=-V^D$ by virtue of equation (14a); hence, in this instance, the Lagrangian horizontal mean flow is zero as in a two-dimensional reflection ($\beta=0$).

Another steady-state solution of equation (25) is found by setting $Q_0 = 0$. This condition combined with equations (14a) and (16a) yields

$$\left(a\frac{\partial^2}{\partial\xi^2} + 2b\frac{\partial^2}{\partial\xi\partial\zeta} + c\frac{\partial^2}{\partial\zeta^2}\right)\Psi_0 = f_0, \tag{30a}$$

where

$$f_0 = \frac{\mathrm{i}\,\omega}{\omega^2 - \sin^2\alpha\sin^2\beta}\frac{\partial^2 J(\Psi,\Psi^*)}{\partial\chi^2} - \frac{2\,\mathrm{i}\,\omega\sin^2\alpha\sin^2\beta}{(\omega^2 - \sin^2\alpha\sin^2\beta)^2}J(\Psi_\chi,\Psi^*_\chi) \tag{30b}$$

and

$$a = \cos^2\alpha + \sin^2\alpha\sin^2\beta, \quad b = -\sin\alpha\cos\alpha\cos\beta, \quad c = \sin^2\alpha.$$

Since $b^2 - ac = -\sin^2\alpha\sin^2\beta < 0$, equation (30a) is an inhomogeneous elliptic equation for Ψ_0. The appropriate solution is readily found by taking Fourier transform in ξ similar to equation (A2), $\Psi_0(\xi,\zeta) \leftrightarrow \hat{\Psi}_0(l;\zeta)$, and solving the resulting inhomogeneous ordinary differential equation for $\hat{\Psi}_0$ along ζ, subject to the conditions $\hat{\Psi}_0 = 0$ on $\zeta = 0$ and $\hat{\Psi}_0 \to 0$ as $\zeta \to \infty$. The Lagrangian horizontal mean flow corresponding to this steady-state solution Ψ_0 is non-zero.

Finally, it is worth noting that in the special case of obliquely reflecting sinusoidal waves, where the Eulerian induced mean flow as well as the Stokes drift are independent of ξ, the Jacobian in the evolution equation (25) identically vanishes. Thus, in this instance any function of ζ is a solution and, by suitably choosing Ψ_0, either the Eulerian or Lagrangian along-slope mean flow can be made to be zero [8].

5. Streaming

As noted above, the induced horizontal mean flow due to oblique beam reflection is tied to the transport of mean PV, governed by equation (25) which derives from the material conservation of PV in an inviscid fluid, viz. equation (5). Apart from this inviscid mechanism, however, it is known that dissipation may also trigger the generation of horizontal mean flow via irreversible production of mean PV [6]. This type of induced mean flow, referred to as 'streaming', is familiar in acoustics [10] and also plays an important part in the propagation of internal wave beams that feature transverse variations [11–13]. Here, it is

pointed out that when a beam reflects obliquely at a sloping boundary, viscous dissipation can instigate the generation of a horizontal mean flow akin to streaming even in the absense of transverse variations.

Specifically, allowing for viscous dissipation, equations (2 b,c) are modified to

$$\nabla^2\psi_t + J(\nabla^2\psi,\psi) - \rho_\chi - \nu\nabla^4\psi = 0, \tag{31a}$$

$$v_t + J(v,\psi) - \rho\sin\alpha\sin\beta - \nu\nabla^2 v = 0, \tag{31b}$$

and the inviscid PV equation (5) is replaced by

$$\frac{Dq}{Dt} = \nu\left(\nabla^2\Omega - J(\nabla^2 v,\rho)\right), \tag{32}$$

where ν is an inverse Reynolds number.

Now, collecting mean terms in equation (32), as the mean vertical vorticity is $O(\varepsilon^2)$ in view of equations (15) and (16), we take $\nu = \mu\varepsilon^2$, with $\mu = O(1)$, in order for the viscous term to be comparable to the inviscid left-hand side, which is $O(\varepsilon^4)$ as was argued earlier. Thus, the mean-PV transport equation in the presence of viscous dissipation takes the form

$$\frac{\partial Q_0}{\partial T} + J(Q_0,\overline{\Psi}) = \mu\nabla^2 Q_0 + \frac{\mathrm{i}\mu\omega\sin\alpha\sin\beta}{(\omega^2-\sin^2\alpha\sin^2\beta)^2}\left\{\nabla^2 J(\Psi_\chi,\Psi_\chi^*) + 2J\left(\Psi_{\chi\xi},\Psi_{\chi\xi}^*\right) + 2J\left(\Psi_{\chi\zeta},\Psi_{\chi\zeta}^*\right)\right\}. \tag{33}$$

Furthermore, in a viscous fluid, the inviscid boundary condition (3) is replaced by no-slip conditions, which are expected to introduce thin boundary layers near the slope ($\zeta = 0$). Leaving these viscous effects aside, here we shall only comment qualitatively on the right-hand side of equation (33) which accounts for the effect of dissipation on the mean PV. Specifically, the second term on this right-hand side describes the production of mean PV owing to nonlinear interactions in the overlap region of the incident and reflected beams. This process, which is applicable to oblique reflections ($\beta \neq 0$) only, is analogous to the streaming mechanism found for propagating beams in the presence of transverse variations [11–13]. In contrast to the latter situation, however, for an oblique

reflection of a uniform beam, mean-PV production occurs at the same order as mean-PV dissipation, described by the first term on the right-hand side of equation (33); as a result, here resonant (proportional to T) growth of Q_0 is not possible, and the streaming effect may be less pronounced than in the case of a propagating beam with transverse variations.

6. Concluding remarks

Our analysis of oblique reflection of internal wave beams at a sloping boundary has brought out the key role of mean PV in determining the induced horizontal mean flow. Unlike two-dimensional reflections where PV vanishes identically and the induced mean flow is readily found and results in no net transport, the horizontal induced mean flow due to an oblique reflection evolves according to its own dynamics, governed by the mean PV equation. We have identified two possible steady-state solutions of this evolution equation: one for which the associated Lagrangian mean flow vanishes, implying no net transport, and another that corresponds to zero mean PV but for which the associated Lagrangian mean flow is non-zero. In order to decide whether any of these steady states is reached, it would be necessary to compute the evolution of mean PV by solving an appropriate initial-value problem. Since PV is conserved in an inviscid fluid, assuming that PV vanishes initially, the latter steady-state solution is the most physically relevant. Thus, in a high-Reynolds-number environment, oblique reflection of internal waves is expected to induce horizontal mass transport near a sloping boundary.

We have also explored the effect of viscous dissipation on the mean PV dynamics. Apart from the expected damping of mean PV, viscous dissipation in conjunction with nonlinear interactions in the region where the incident and reflected beams overlap, can act as a driving mechanism of horizontal mean flow via the irreversible production of mean PV. This process, which applies to oblique reflections only, is akin to the generation of streaming by attenuating acoustic waves and propagating internal wave beams in the presence of transverse variations. A complete study of viscous effects on the induced mean flow

would require solving the mean PV evolution equation subject to viscous (no-slip) conditions on the slope, a task that is not attempted here. In addition, the present study has assumed that the beam angle of incidence θ is not close to the slope angle α, as the linear solution for the reflected beam is singular when $\theta = \alpha$ and re-scaling becomes necessary in the vicinity of this singularity. For an oblique reflection near the critical angle $\theta = \alpha$, we expect that nonlinear effects on the reflected beam, which in a two-dimensional reflection are minor [14,15], would be enhanced due to the coupling with the induced horizontal mean flow. This problem is left to future studies.

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Appendix

Here, we provide details of the $O(\varepsilon^2)$ second-harmonic response. After substituting expressions (13) into equations (10), the second-harmonic problem can be reduced to a single equation for $\Psi_2(\xi,\zeta)$,

$$\left[(4\omega^2-\sin^2\alpha\sin^2\beta)\nabla^2-\frac{\partial^2}{\partial\chi^2}\right]\Psi_2=f_2, \tag{A1a}$$

where

$$f_2=-3\,\mathrm{i}\,\omega\,J(\nabla^2\Psi,\Psi), \tag{A1b}$$

to be solved subject to the boundary condition (11) on the slope ($\zeta = 0$) and the appropriate radiation condition as $\zeta \to \infty$, specified below.

Upon taking Fourier transform in ξ,

$$\hat{\Psi}_2(l;\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_2(\xi,\zeta)\mathrm{e}^{-\mathrm{i}l\xi}\mathrm{d}\xi, \tag{A2}$$

equation (A1a) can be factored in the form

$$(4\omega^2 - \sin^2\alpha)\left(\frac{\partial}{\partial\zeta} - \mathrm{i}lm_2^{\mathrm{inc}}\right)\left(\frac{\partial}{\partial\zeta} - \mathrm{i}lm_2^{\mathrm{refl}}\right)\hat{\Psi}_2 = \hat{f}_2, \tag{A3a}$$

where

$$\hat{f}_2(l;\zeta) = \begin{cases} 0 & (l<0) \\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} f_2\mathrm{e}^{-\mathrm{i}l\xi}\mathrm{d}\xi & (l>0) \end{cases} \tag{A3b}$$

and

$$m_2^{\mathrm{inc}} = \frac{\cos\beta(2\omega\, r_2\cos\theta_2 - \sin\alpha\cos\alpha)}{4\omega^2 - \sin^2\alpha},$$
$$m_2^{\mathrm{refl}} = -\frac{\cos\beta(2\omega\, r_2\cos\theta_2 + \sin\alpha\cos\alpha)}{4\omega^2 - \sin^2\alpha} \tag{A3c}$$

with

$$r_2 = \frac{\sqrt{4\omega^2 - \sin^2\alpha\sin^2\beta}}{2\omega\cos\beta}, \quad \cos\theta_2 = \sqrt{1-4\omega^2}. \tag{A3d}$$

Equation (A3b) makes explicit that the forcing f_2 in equation (A1b) involves Fourier components with $l > 0$ only, as this is also the case for Ψ (cf. equation (9b)) in keeping with the radiation conditions obeyed by the primary incident and reflected beams. Furthermore, from equations (A3 c,d), it is clear that, for $\omega < 1/2$, m_2^{inc} and m_2^{refl} are real so the second-harmonic response radiates far from the slope. Assuming this to be the case, the appropriate radiation condition, that ensures energy is transported away from the slope, is $\hat{\Psi}_2 \sim \exp\left(\mathrm{i}lm_2^{\mathrm{refl}}\zeta\right)$ ($l > 0$, $\zeta \to \infty$).

Then, for $\omega < 1/2$, upon solving the problem (A3) for $\hat{\Psi}_2$ with the boundary condition $\hat{\Psi}_2 = 0$ ($\zeta = 0$) and the above radiation condition, and inverting the

Fourier transform, the second-harmonic response is

$$\Psi_2 = \int_0^\infty \mathrm{e}^{\mathrm{i}l\xi}\mathrm{d}l \left\{ B_2(l)\mathrm{e}^{\mathrm{i}lm_2^{\mathrm{refl}}\zeta} + \frac{\mathrm{i}}{lr_2 \sin 2\theta_2 \cos\beta} \times \left[\int_\zeta^\infty \hat{f}_2(l;\zeta')\mathrm{e}^{\mathrm{i}lm_2^{\mathrm{inc}}(\zeta-\zeta')}\mathrm{d}\zeta' + \int_0^\zeta \hat{f}_2(l;\zeta')\mathrm{e}^{\mathrm{i}lm_2^{\mathrm{refl}}(\zeta-\zeta')}\mathrm{d}\zeta' \right] \right\}, \quad \text{(A4a)}$$

where

$$B_2(l) = \frac{-\mathrm{i}}{lr_2 \sin 2\theta_2 \cos\beta} \int_0^\infty \hat{f}_2(l;\zeta)\mathrm{e}^{-\mathrm{i}lm_2^{\mathrm{inc}}\zeta}\mathrm{d}\zeta. \quad \text{(A4b)}$$

Furthermore, using equations (10) and (12), the second-harmonic density and transverse velocity amplitudes can be expressed in terms of Ψ_2:

$$R_2 = \frac{-2\mathrm{i}\omega\,\Psi_{2\chi}}{4\omega^2 - \sin^2\alpha \sin^2\beta} + g_2, \quad V_2 = \frac{\sin\alpha \sin\beta\,\Psi_{2\chi}}{4\omega^2 - \sin^2\alpha \sin^2\beta} + h_2, \quad \text{(A5a)}$$

where

$$g_2 = \frac{-(2\omega^2 + \sin^2\alpha \sin^2\beta)J(\Psi_\chi, \Psi)}{(\omega^2 - \sin^2\alpha \sin^2\beta)(4\omega^2 - \sin^2\alpha \sin^2\beta)},$$
$$h_2 = \frac{-3\,\mathrm{i}\,\omega \sin\alpha \sin\beta\, J(\Psi_\chi, \Psi)}{(\omega^2 - \sin^2\alpha \sin^2\beta)(4\omega^2 - \sin^2\alpha \sin^2\beta)}. \quad \text{(A5b)}$$

Equations (A4) and (A5) confirm that for $\omega < 1/2$ the second-harmonic response includes a radiating beam that propagates at the angle $\theta_2 = \cos^{-1}\sqrt{1-4\omega^2}$ to the horizontal. On the other hand, if $\omega > 1/2$, the second-harmonic response is evanescent, $\Psi_2 \to 0$ as $\zeta \to \infty$; in this case m_2^{inc} and m_2^{refl} are complex, and the corresponding expression for Ψ_2 analogous to equation (A4) is straightforward to write down.